Closed Form Solution for the Equations of Motion for Constrained Linear Mechanical Systems and Generalisations: An Algebraic Approach

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Abstract

In this paper, a mathematical methodology is presented for the determination of the solution of motion for linear constrained mechanical systems applicable also to systems with singular coefficients. For mathematical completeness and also to incorporate some other interesting cases, the methodology is formulated for a general class of higher order matrix differential equations. Thus, describing the system in an autoregressive moving average (ARMA) form, the closed form solution is derived in terms of the finite and infinite Jordan pairs of the system's polynomial matrix. The notion of inconsistent initial conditions is considered and an explicit formula for the homogeneous system is given. In this respect, the methodology discussed in the present note provides an alternative view to the problem of computation of the response of complex multi-body systems. Two interesting examples are provided and applications of the equation to such systems are illustrated.

Keywords: Higher Order Linear Systems; Singular Mass Matrix; ARMA Representation; Polynomial matrix; Linear Constrained Mechanical Systems; Multi-Body Dynamics

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1. Introduction

1.1. Motivation

Singular linear systems can occur either in continuous time (modelled using differential equations) or discrete time (using difference equations) frameworks. In engineering literature they are also known as "generalized" or "differential-algebraic" systems and their applications are numerous. In physical sciences and economics, this class of systems play a key-role in the modelling and simulation process of many interesting applications. Examples include, amongst others, the Leontief multi-sector model in economics [37], lumped parameter n-degree-of-freedom systems [16, 47, 52, 54] in mechanics. One field of research, where singular dynamical systems arise naturally, is the field of modelling mechanical systems subject to constraints. Such problems are of fundamental importance in the area of analytical dynamics and the determination of equations of motion of constrained systems, has been the subject of numerous studies dating back to the pioneering works of [36] and [20]. For this type of systems, a very interesting approach has been proposed by Udwadia and Kalaba [52, 53] and by Udwadia and Phohomsiri [55]. According to their approach, additional constraint forces are introduced and eventually, the equations of motion of the constrained mechanical system are obtained. Under this framework, the explicit computation of the constraint forces is *not* always an easy task to perform, especially in complex cases such as multi-body systems, see [14, 16, 40, 48, 49]. In the process of modelling, the mass matrix of the system may end up being singular, as a consequence of the formulation of the unconstrained equations of motion. This can happen either due to the dependence between the generalized coordinates chosen to describe the system or due to occasions where it is possible to assign null mass to a body whose inertia is negligible.

In this paper, there are two main goals. First, by adopting well established tools and methods from the algebraic theory of linear time-invariant systems, we present their application to the computation of the time response of linear mechanical systems subject to constraints, resulting from modeling techniques presented in [55] and [1]. The key results of the algebraic theory and particularly the application of the so called polynomial matrix approach are well established in the control engineering literature, but it is our view that, it is rather obscure in the community of structural/mechanical engineering. Our approach uses the finite and infinite Jordan pairs of the system's polynomial matrix and the Laurent expansion of its inverse. In this setup, the equations of motion can also be handled effectively, and explicit formulas for the computation of the time response in the original generalized coordinate system are derived. Second, the proposed approach is extended to higher order linear differential systems and for a particular class of Apostol-Kolodner equations, see

[3, 30, 31]. This is motivated by the fact that systems of order higher than two may occur when second order interconnected mechanical systems are being considered. For example, a fourth order linear matrix differential equation is encountered when modelling a flexible joint robot with electric motors in the joints [41]. A good collection of such higher order problems that arise in various applications can be found in [6].

1.2. Model Formulation

In this subsection, the mathematical formulation of the linear model is introduced and discussed making the equation of motion for constrained mechanical systems a special case (see Section 5). Thereafter, our attention focuses on higher order linear dynamical systems with a possibly singular leading matrix coefficient which are described by the Eq. (1),

$$A_q x^{(q)}(t) + A_{q-1} x^{(q-1)}(t) + A_{q-2} x^{(q-2)}(t) + \ldots + A_0 x(t) = f(t), \tag{1}$$

where $A_i \in \mathbb{R}^{n\times n}$, $x(t)$ and $f(t)$ are $n \times 1$ vector valued *piecewise - smooth distributions* introduced in [51], whose initial conditions may (or may not) satisfy the admissibility constraints (see Definition 3.2). The use of piecewise - smooth distributions as the *signal space* for the *behavior* of Eq. (1) is wide enough to accommodate both functional and certain types of distributional solutions of Eq. (1). This particular framework incorporates, as a special case, the space of *impulsive-smooth distributions* introduced in [26] and elaborated in [23, 24, 22] and it is fully compatible with the unilateral Laplace transform discussed in [39].

As it has also been discussed in the previous subsection, such types of systems are often encountered in applications from many scientific disciplines, since they can be used to describe effectively a series of interesting phenomena. In the existing literature of higher order linear matrix differential systems with a singular leading matrix, a notable special case of such systems is the one where $A_{q-1} = \cdots = A_1 = \mathbb{O}_{n\times n}$ and $f(t) = \mathbb{O}_{n\times 1}$, i.e.

$$A_q x^{(q)}(t) = A_0 x(t), \tag{2}$$

where systems given by Eq. (2) are called *higher order linear matrix differential equations of Apostol-Kolodner type*; see [30]. The solution of such systems has been originally studied in [3, 35] for $q = 2$ and later in [50, 58] for the general case of $q > 2$. These results have later been extended to the case of singular systems in [30, 31, 45]. Second order Apostol-Kolodner type systems ($q = 2$) have a significant physical interpretation, since they represent linear mechanical systems with no damping.

Instead of the generalized (pseudo) inverse theory [8], a methodology which uses matrix transformations to analyse Eq. (1) and transform the system into an equivalent state space or descriptor form has been extensively used, see [2, 43]. Indeed, in the area of electrical and control engineering, descriptor systems have been the subject of extensive research, see for instance [9, 10, 13, 38, 44]. It should be pointed out that the connection between generalized (pseudo) inverse theory and matrix transformations approach has been explored in [28, 29]. Additionally, in [30], the solution of Eq. (2) is derived through the use of the Weierstrass decomposition of the matrix pencil involved. Thus, the system is decomposed into two subsystems, the so-called slow and fast ones, which in turn can be studied separately. More specifically, the slow subsystem is expressed in state space form through a variable transformation and then solved using known results from the theory of linear systems. The fast subsystem on the other hand is shown to have only the zero solution. These results are combined and the complete solution of the system Eq. (2) is derived for consistent initial conditions. Additionally, the solution is generalised to include the impulsive behaviour of the system at time t_0, which is the result of non-consistent initial conditions.

The analysis of higher order systems through equivalent state space models may not always be desirable. As Antsaklis and Michel [2] comment, the transformation of the system Eq. (1) into an equivalent representation usually involves a change of the internal variables. This may be inconvenient, since it can lead to the loss of the physical meaning of the original variables.

In the present paper, we use a direct method to derive the solution of Eq. (1). Writing the system Eq. (1) as an *Autoregressive (AR) representation* (see [62])

$$\underbrace{\left(A_q \rho^q + \ldots + A_1 \rho + A_0\right)}_{A(\rho)} x(t) = f(t), \tag{3}$$

where $\rho := d/dt$, the solution is presented through the use of the finite and infinite Jordan Pairs of the polynomial matrix $A(\rho)$ and the Laurent expansion of its inverse, as was also studied in [56, Chapter 4], [57]. The algebraic structure of polynomial matrices and the theory of *Jordan pairs* has been studied in the early work of [25] and later in [5, 33] and the references therein. Symbolic and numerical algorithms have also been developed for the computation of the Jordan chains of polynomial matrices in [59].

It should be noted that the analysis of linear systems using algebraic representations has also been the subject of an extensive research in the *behavioral framework* which was first introduced by Willems in his seminal works [61, 62, 63] and later in

[64]. Based on his approach, given as a starting point the solution space (or "behavior") of a system, one seeks to find mathematical models to adequately represent the given trajectories. Several types of models can be employed to obtain this goal, and thus the study of their structural properties, and the possible relations between them is of a great research interest. However, it should be noted that the approach adopted in the present paper follows a rather different path. The mathematical models of constrained mechanical systems are obtained using a modification of the Udwadia and Phohomsiri [55] approach and their resulting behavior is in turn studied in terms of the structural invariants of the matrices involved in the model.

As noted above, an important advantage of the proposed approach is that variable transformations are avoided. Thus, the states of the system retain their physical meaning and the general solution of the system is written in a simple and compact matrix form. Then, using facts from the analytic computation of the matrix exponential found in [4, 11, 15, 42], different analytic formulas of the solution are presented. Additionally, it should be mentioned here that by using this method, it is easier to derive the general solution of a system exhibiting impulsive behavior, which is a result of the *non-consistent* initial conditions. Impulsive behavior is encountered in many physical models where abrupt changes occur in small time. These changes may happen in time that is relatively short compared to the physical progress of the system, and thus they can be considered as instantaneous. Examples include the impacts that can cause impulsive changes to the state variables and in mechanical systems that are inherently impulsive, like nanodevices [65]. Other examples are multimode systems, like mechanical linkages where the modes may relate to the activation and deactivation of different contact forces [21]. Electrical circuits with switches, like the gear shift of a motor vehicle, are also an example. In such systems, switching may lead to an instantaneous change in the state space and the occurrence of an impulse [21]. In overall, it is clear by these examples that the consistency of initial conditions and the impulsive behavior of a system should be carefully considered and not be overlooked.

On the other hand, a drawback of the method presented in this paper is that the computation of the Jordan pairs of a polynomial matrix or the Laurent expansion of its inverse is not a computationally easy task. For large scale systems with high dimensional matrices, existing algorithms may not be suitable. Yet, the analysis presented here can be combined with numerical methods for computing Jordan chains and the Smith form in [59] and [60], to derive a computationally efficient result.

The remaining of the paper is organised as follows. In Section 2, a necessary mathematical background is presented for the better understanding of the main findings. Section 3 presents the solution space of Eq. (1), considering both the cases of inconsistent and consistent initial conditions. In Section 4, new explicit formulas for the homogeneous equation are given and additionally for the case of higher order Apostol-Kolodner systems. Section 5 contains two interesting numerical applications from the area of mechanical engineering. Section 6 concludes the paper and gives new directions for further research.

2. Mathematical Preliminaries

In this section, a necessary brief introduction of the polynomial matrix theory is presented. Specifically, results regarding the finite and infinite zero structure of polynomial matrices, which are used extensively for the computation of the general solution of the system Eq. (1) are summarized. In addition, some formulas for the matrix exponential e^{At} are given, which are used to further simplify the solution of the system. We only focus on the special case where the matrix A is block diagonal, since the general solution includes an exponential term of a matrix in Jordan block diagonal form.

2.1. Matrix Polynomials

Let $\mathbb{R}, \mathbb{C}$ be the field of real and complex numbers respectively, $\mathbb{R}[s]$ the ring of polynomials with coefficients from $\mathbb{R}$ and $\mathbb{R}(s)$ the field of rational functions. By $\mathbb{R}[s]^{p\times m}$, $\mathbb{R}(s)^{p\times m}$, $\mathbb{R}_{pr}(s)^{p\times m}$, we denote the sets of $p\times m$ polynomial, rational and proper rational matrices with real coefficients. For matrices $A_1, A_2, ..., A_n$, their direct sum is denoted by $A_1\oplus A_2\oplus\cdots\oplus A_n$ or by blockdiag$\{A_1, A_2, ..., A_n\}$. A square polynomial matrix is called *regular* if $\exists s\in\mathbb{C}$ such that $det(A(s))\neq 0$.

Consider a regular polynomial matrix

$$A(s) = A_q s^q + A_{q-1}s^{q-1} + ... + A_1 s + A_0, \tag{4}$$

with $A_i\in\mathbb{R}^{r\times r}$ and $A_q\neq 0$.

Definition 2.1. [56, Sections 1.2, 3.3] *A square polynomial matrix $A(s)\in\mathbb{R}[s]^{r\times r}$ is called* unimodular *if* $\det A(s) = c\in\mathbb{R}$, $c\neq 0$ *for all* $s\in\mathbb{C}$. *A rational matrix $A(s)\in\mathbb{R}_{pr}(s)^{r\times r}$ is called* biproper *if* $\lim_{s\to\infty} A(s) = E\in\mathbb{R}^{r\times r}$ *with* $rankE = r$.

Theorem 2.2. [56, Section 1.3] *Let $A(s)$ as in Eq. (4). There exist unimodular matrices $U_L(s)\in\mathbb{R}[s]^{r\times r}$, $U_R(s)\in\mathbb{R}[s]^{r\times r}$ such that*

$$S^{\mathbb{C}}_{A(s)}(s) = U_L(s)A(s)U_R(s) = diag(1, ..., 1, f_z(s), f_{z+1}(s), ..., f_r(s)), \tag{5}$$

with $1\leq z\leq r$ and $f_j(s)/f_{j+1}(s)$ $j = z, z+1, ..., r$. $S^{\mathbb{C}}_{A(s)}(s)$ is called the Smith form *of $A(s)$, where $f_j(s)\in\mathbb{R}[s]$ are the* invariant polynomials *of $A(s)$. The zeros $\lambda_i\in\mathbb{C}$*

of $f_j(s)$*,* $j = z, z+1, ..., r$ *are called* finite zeros *of* $A(s)$*. Assume that* $A(s)$ *has* ℓ *distinct zeros. The partial multiplicities of each zero* $\lambda_i \in \mathbb{C}$*,* $i = 1, ..., \ell$ *satisfy*

$$0 \leq n_{i,z} \leq n_{i,z+1} \leq ... \leq n_{i,r}, \tag{6}$$

i.e.

$$f_j(s) = (s - \lambda_i)^{n_{i,j}} \hat{f}_j(s), \tag{7}$$

$j = z, ..., r$ *with* $\hat{f}_j(\lambda_i) \neq 0$*. The terms* $(s - \lambda_i)^{n_{i,j}}$ *are called* finite elementary divisors *of* $A(s)$ *at* λ_i*. The multiplicity of each zero is* $n_i = \sum_{j=z}^{r} n_{i,j}$*. Denote by* n *the sum of the degrees of the finite elementary divisors of* $A(s)$*, i.e.*

$$n := \deg \left[\prod_{j=z}^{r} f_j(s) \right] = \sum_{i=1}^{\ell} \sum_{j=z}^{r} n_{i,j}. \tag{8}$$

Similarly, we can find $U_L(s) \in \mathbb{R}(s)^{r \times r}$*,* $U_R(s) \in \mathbb{R}(s)^{r \times r}$ *having no poles and zeros at* $s = \lambda_0$ *such that*

$$S_{A(s)}^{\lambda_0}(s) = U_L(s)A(s)U_R(s) = diag(1, ..., 1, (s - \lambda_0)^{n_z}, ..., (s - \lambda_0)^{n_r}), \tag{9}$$

$S_{A(s)}^{\lambda_0}(s)$ *is called the local Smith form of* $A(s)$ *at the point* λ_0*.*

Theorem 2.3. [56, Section 3.3, Corollary 3.54] *Let* $A(s)$ *defined in Eq.* (4)*. There exist biproper matrices* $U_L(s) \in \mathbb{R}_{pr}^{r \times r}(s)$*,* $U_R(s) \in \mathbb{R}_{pr}^{r \times r}(s)$ *such that*

$$U_L(s)A(s)U_R(s) = S_{A(s)}^{\infty}(s) = diag\left(\underbrace{s^{q_1}, s^{q_2}, ..., s^{q_u}}_{u}, \overbrace{\frac{1}{s^{\hat{q}_{u+1}}}, \frac{1}{s^{\hat{q}_{u+2}}}, ..., \frac{1}{s^{\hat{q}_r}}}^{r-u} \right), \tag{10}$$

with $1 \leq u \leq r$*,* $q_1 \geq \ldots \geq q_u \geq 0$ *and* $\hat{q}_r \geq \hat{q}_{r-1} \geq \ldots \hat{q}_{u+1} > 0$*.* $S_{A(s)}^{\infty}(s)$ *is called the* Smith form of $A(s)$ at infinity*. The first* u *terms* $s^{q_1}, ..., s^{q_u}$ *(resp. the latter* $(r-u)$ *terms* $s^{\hat{q}_{u+1}}, ..., s^{\hat{q}_r}$*) are the* poles *(resp.* zeros*)* at $s = \infty$ *of* $A(s)$*. In addition, it holds that* $q_1 = q$*.*

Definition 2.4. [56, Section 4.2.1] *The dual polynomial matrix of* $A(s)$ *is defined as*

$$\tilde{A}(s) := s^q A(\frac{1}{s}) = A_0 s^q + A_1 s^{q-1} + ... + A_q. \tag{11}$$

Theorem 2.5. [56, Section 4.2.1] *Let $\tilde{A}(s)$ as in (11). There exist matrices $\tilde{U}_L(s) \in \mathbb{R}(s)^{r\times r}$, $\tilde{U}_R(s) \in \mathbb{R}(s)^{r\times r}$ having no poles or zeros at $s = 0$, such that*

$$S^0_{\tilde{A}(s)}(s) = \tilde{U}_L(s)\tilde{A}(s)\tilde{U}_R(s) = diag(s^{\mu_1}, \dots, s^{\mu_r}). \tag{12}$$

$S^0_{\tilde{A}(s)}(s)$ is the local Smith form of $\tilde{A}(s)$ at $s = 0$. The terms s^{μ_j} are the finite elementary divisors of $\tilde{A}(s)$ at zero and are called the infinite elementary divisors (i.e.d.) *of $A(s)$.*

The connection between the Smith form at infinity of $A(s)$ and the Smith form at zero of the dual matrix is given in [27], [56, Section 4.2.1]:

$$S^0_{\tilde{A}(s)}(s) = diag\left(1, \underbrace{s^{q-q_2}, \dots, s^{q-q_u}}_{i.p.e.d.}, \underbrace{s^{q+\hat{q}_{u+1}}, \dots, s^{q+\hat{q}_r}}_{i.z.e.d.}\right) = diag(s^{\mu_1}, s^{\mu_2}, \dots, s^{\mu_r}), \tag{13}$$

where by i.p.e.d. and i.z.e.d. we denote the infinite pole and infinite zero elementary divisors respectively. From the above formula it is seen that the orders of the infinite elementary divisors of $A(s)$ are given by

$$\begin{array}{rcll} \mu_1 & = & q - q_1 \overset{q=q_1}{=} 0, & \\ \mu_j & = & q - q_j & j = 2, 3, ..., u, \\ \mu_j & = & q + \hat{q}_j & j = u+1, ..., r. \end{array} \tag{14}$$

We denote by $\bar{\mu}$ the sum of the degrees of the infinite elementary divisors of $A(s)$ i.e.

$$\bar{\mu} := \sum_{j=1}^{r} \mu_j. \tag{15}$$

Theorem 2.6. [12],[25, Chapter 7] *Let $(C_i \in \mathbb{R}^{r\times n_i}, J_i \in \mathbb{R}^{n_i\times n_i})$ be a matrix pair, where J_i is a Jordan matrix corresponding to the zero λ_i. This pair is a* Jordan Pair *of $A(s)$ corresponding to λ_i if and only if it satisfies*

- $\det A(s)$ *has a zero λ_i of multiplicity n_i.*
- $rank \begin{pmatrix} C_i^T & (C_iJ_i)^T & \cdots & (C_iJ_i^{q-1})^T \end{pmatrix}^T = n_i$.
- $\sum_{i=0}^{q} A_iC_iJ_i^i = 0$.

Taking a Jordan Pair for every zero λ_i of $A(s)$ we define a Finite Jordan Pair *of $A(s)$ as*

$$C = \begin{pmatrix} C_1 & \cdots & C_\ell \end{pmatrix} \in \mathbb{R}^{r\times n}, \quad J = J_1 \oplus \cdots \oplus J_\ell \in \mathbb{R}^{n\times n}. \tag{16}$$

Similarly, let $\left(\bar{C}_\infty \in \mathbb{R}^{r\times\bar{\mu}}, \bar{J}_\infty \in \mathbb{R}^{\bar{\mu}\times\bar{\mu}}\right)$ be a matrix pair, where $\bar{J}_\infty$ is a Jordan matrix corresponding to the zero $\lambda_i = 0$. This pair is an Infinite Jordan Pair *of $A(s)$ if and only if it satisfies*

- $\det \tilde{A}(s)$ *has a zero at $\lambda = 0$ of multiplicity $\bar{\mu}$.*
- *rank* $\begin{pmatrix} \bar{C}_\infty^T & (\bar{C}_\infty \bar{J}_\infty)^T & \cdots & (\bar{C}_\infty \bar{J}_\infty^{q-1})^T \end{pmatrix}^T = \bar{\mu}$.
- $\sum_{i=0}^{q} A_i \bar{C}_\infty \bar{J}^{q-i} = 0$.

The infinite Jordan Pair of $A(s)$ can be constructed by taking a Jordan Pair $\left(\bar{C}_{\infty,j} \in \mathbb{R}^{r\times\bar{\mu}_j}, \bar{J}_{\infty,j} \in \mathbb{R}^{\bar{\mu}_j\times\bar{\mu}_j}\right)$ for different algebraic multiplicities of the zero $\tilde{\lambda}_i = 0$ of $\tilde{A}(s)$ and combining them as

$$\bar{C}_\infty = \begin{pmatrix} C_{\infty,1} & \cdots & C_{\infty,r} \end{pmatrix} \in \mathbb{R}^{r\times\bar{\mu}}, \quad \bar{J}_\infty = J_{\infty,1} \oplus \cdots \oplus J_{\infty,r} \in \mathbb{R}^{\bar{\mu}\times\bar{\mu}}. \tag{17}$$

A method for constructing the Jordan Pairs of a polynomial matrix is given in [25, 34].

Lemma 2.7. [56, Section 4.2.3] *The Laurent expansion of $A(s)^{-1}$ at infinity is given by*

$$A(s)^{-1} = H_{pol}(s) + H_{sp}(s) = H_{\hat{q}_r} s^{\hat{q}_r} + \cdots + H_1 s + H_0 + H_{-1}s^{-1} + H_{-2}s^{-2} + \ldots, \tag{18}$$

where $H_{pol}(s)$, $H_{sp}(s)$ denote the polynomial and the strictly proper part of $A(s)^{-1}$, and $\hat{q}_r$ is the maximum order amongst the orders of zeros at infinity of $A(s)$ in Eq. (10).

Theorem 2.8. [56, Section 4.2.3] *The inverse of $A(s)$ can be decomposed as*

$$A(s)^{-1} = C(sI_n - J)^{-1}B_F + C_\infty(I_\mu - sJ_\infty)^{-1}B_\infty, \tag{19}$$

where the matrix triple (C, J, B_F) and $(C_\infty, J_\infty, B_\infty)$ are the minimal realizations *of the strictly proper and polynomial parts of $A(s)^{-1}$ respectively, given by*

$$H_{sp}(s) = C(sI_n - J)^{-1}B_F \tag{20}$$

and

$$s^{-1}H_{pol}(s^{-1}) = C_\infty(sI_\mu - J_\infty)^{-1}B_\infty \tag{21}$$

or equivalently

$$H_{pol}(s) = C_\infty(I_\mu - sJ_\infty)^{-1}B_\infty \tag{22}$$

with $C \in \mathbb{R}^{r\times n}$, $J \in \mathbb{R}^{n\times n}$ *as in Eq.* (16), $B \in \mathbb{R}^{n\times r}$, $C_\infty \in \mathbb{R}^{r\times\mu}$, $J_\infty \in \mathbb{R}^{\mu\times\mu}$, $B_\infty \in \mathbb{R}^{\mu\times r}$, *where*

$$J_\infty = J_{\infty,r} \oplus J_{\infty,r-1} \oplus \cdots \oplus J_{\infty,u+1}, \tag{23}$$

and $J_{\infty,i}$ *are nilpotent matrices of the form*

$$J_{\infty,i} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{(\hat{q}_i+1)\times(\hat{q}_i+1)},\ i = u+1, ..., r, \tag{24}$$

and $\mu = \sum_{i=u+1}^{r}(\hat{q}_i + 1)$. *It holds that*

$$(sI_n - J)^{-1} = s^{-1}I + s^{-2}J + s^{-3}J^2 + \dots \tag{25}$$

$$(I_\mu - sJ_\infty)^{-1} = I + sJ_\infty + s^2J_\infty^2 + ... + s^{\hat{q}_r}J_\infty^{\hat{q}_r}. \tag{26}$$

By equating the coefficients of the powers of s^i of the last two expressions for $A(s)^{-1}$, we get

$$H_i = C_\infty J_\infty^i B_\infty, \quad i = 0, 1, 2, ..., \hat{q}_r \tag{27a}$$

$$H_{-i} = CJ^{i-1}B, \quad i = 1, 2, \dots \tag{27b}$$

Lemma 2.9. [56, Theorem 4.50] *The coefficients of* $A(s)$ *and* $A(s)^{-1}$ *satisfy the following system of equations:*

$$H_{i-q}A_q + ... + H_iA_0 = \delta_i I_r, \tag{28}$$

where $H_i = 0$ *for* $i > \hat{q}_r$ *and* $\delta_i = 0$, *for* $i \neq 0$ *and* $\delta_0 = 1$. *In particular, the first*

$q+\hat{q}_r$ above equations can be written in matrix form as

$$\left(\begin{array}{ccc} H_{\hat{q}_r} & \dots & 0 \\ \vdots & \ddots & \vdots \\ H_{\hat{q}_r-(q-1)} & \dots & H_{\hat{q}_r} \\ \hline H_{\hat{q}_r-q} & \dots & H_{\hat{q}_r-1} \\ \vdots & \vdots & \vdots \\ H_{-(q-1)} & \dots & H_0 \end{array}\right)\begin{pmatrix} A_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_q \end{pmatrix} = -\left(\begin{array}{ccc} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \hline H_{\hat{q}_r} & \dots & 0 \\ \vdots & \ddots & \vdots \\ H_{\hat{q}_r-(q-1)} & \dots & H_{\hat{q}_r} \\ \vdots & \vdots & \vdots \\ H_1 & \dots & H_q \end{array}\right)\begin{pmatrix} A_0 & \cdots & A_{q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_0 \end{pmatrix}. \quad (29)$$

Remark 2.10. *The fundamental matrix sequence H_i can be effectively computed using the technique proposed in Fragulis et al.* [18].

2.2. Exponential Matrices

Consider a matrix J_i in Jordan form, corresponding to the zero λ_i, i.e.

$$J_i = \begin{pmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}. \quad (30)$$

Lemma 2.11. [11, Lemma 2] *Let $J_i \in \mathbb{R}^{n_i \times n_i}$ be a Jordan matrix corresponding to the zero λ_i. Then*

$$e^{J_i t} = G_i(t), \quad (31)$$

where $G_i(t)$ is the $n_i \times n_i$ matrix function

$$G_i(t) := e^{J_i t} = e^{\lambda_i t} \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \ddots & & \ddots & \vdots \\ 0 & \cdots & \cdots & & 1 \end{pmatrix}, \quad (32)$$

whose elements can be written in compact form as

$$g_{kj}(t) = \begin{cases} e^{\lambda_i t}\frac{t^{j-k}}{(j-k)!} & 1 \le k \le j \le n_i. \\ 0 & \textit{otherwise.} \end{cases} \quad (33)$$

Another useful formula for the matrix exponential that we exploit is the following.

Lemma 2.12. [11, Lemma 4] *Let J be a matrix in Jordan canonical form with ℓ distinct eigenvalues $\lambda_1, ..., \lambda_\ell$ and $M_J(s) = \prod_{i=1}^{\ell}(s-\lambda_i)^{m_i}$ be its minimal polynomial. Then,*

$$e^{Jt} = \sum_{k=0}^{m-1} J^k f_k(t), \tag{34}$$

where the functions $f_k(t)$ satisfy the following equations

$$\sum_{k=i}^{m-1} \binom{k}{i} \lambda_j^{k-i} f_k(t) = \frac{t^i}{i!} e^{\lambda_j t}, \tag{35}$$

with $j = 1, ..., \ell$, $i = 0, ..., m_j - 1$ and $\sum_{i=1}^{\ell} m_i = m = \deg M_J(s)$.

Finally, a formula for the analytic computation of the exponential function of a Jordan matrix, similar to the one in Lemma 2.12, but more detailed is the following.

Lemma 2.13. [4, Theorem 1] *Let J be a matrix in Jordan canonical form with ℓ distinct eigenvalues $\lambda_1, ..., \lambda_\ell$ and $M_J(s) = \prod_{i=1}^{\ell}(s-\lambda_i)^{m_i}$ be its minimal polynomial. Then,*

$$e^{Jt} = \bigoplus_{i=1}^{\ell} \left(\sum_{j=0}^{m_i-1} f_{ij}(t) J_i^j \right), \tag{36}$$

where

$$f_{ij}(t) = \frac{e^{\lambda_i t}}{j!} \left(\sum_{n=j}^{m_i-1} \frac{(-1)^{n-j}}{(n-j)!} \lambda_i^{n-j} t^n \right), \tag{37}$$

with $j = 1, ..., \ell$, $i = 0, ..., m_j - 1$ and $\sum_{i=1}^{\ell} m_i = m = \deg M_J(s)$.

In case where the matrix J_i has complex eigenvalues $\lambda_i = a \pm bi$, with multiplicity $2m_i$, its Jordan form is (see [46])

$$J_i = \begin{pmatrix} D & I_2 & 0 & \cdots & 0 \\ 0 & D & I_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & D & I_2 \\ 0 & \cdots & \cdots & \cdots & D \end{pmatrix} \in \mathbb{R}^{2m_i \times 2m_i}, \tag{38}$$

where

$$D = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \tag{39}$$

and the exponential of this matrix is given by

$$e^{J_i t} = e^{at} \begin{pmatrix} M & Mt & M\frac{t^2}{2} & \cdots & M\frac{t^{m_i-1}}{(m_i-1)!} \\ 0 & M & Mt & \cdots & M\frac{t^{m_i-2}}{(m_i-2)!} \\ 0 & 0 & M & \cdots & \vdots \\ \vdots & \ddots & & \ddots & \vdots \\ 0 & \cdots & \cdots & & M \end{pmatrix}, \tag{40}$$

where

$$M = e^{Dt} = \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}. \tag{41}$$

3. Time Domain Solution

In the present section, the solution of Eq. (1) is analytically presented. First, the system is studied without making any assumption about the consistency of the initial conditions. As it was mentioned in the introduction, non consistency of initial conditions gives rise to impulsive terms in the behavior of the system, a phenomenon that is encountered in many physical systems. In subsection 3.2, we restrict our attention to the case of consistent initial conditions and give a formula for the impulse free solution.

3.1. Arbitrary Initial Conditions

Considering again the system Eq. (1) (or (3)), we take the following results, as a derivation and an assemble of the results in [56] and later in [19, 57] and [32].

Theorem 3.1. *The general solution of Eq.* (1) *(or* (3)*) is given by*

$$x(t) = Ce^{Jt}x_s(0) + \int_0^t Ce^{J(t-\tau)}B_F f(\tau)d\tau + \sum_{i=0}^{\hat{q}_r} C_\infty J_\infty^i B_\infty f^{(i)}(t) - \\ - \sum_{i=0}^{\hat{q}_r - 1} \delta^{(i)}(t) C_\infty J_\infty^{i+1} x_f(0) + f_{imp}^0(t), \tag{42}$$

with

$$x_s(0) = \begin{pmatrix} J^{q-1}B_F & \cdots & B_F \end{pmatrix} \begin{pmatrix} A_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_q \end{pmatrix} \begin{pmatrix} x(0^-) \\ \vdots \\ x^{(q-1)}(0^-) \end{pmatrix}, \qquad (43)$$

$$x_f(0) = \begin{pmatrix} B_\infty & \cdots & J_\infty^{q-1}B_\infty \end{pmatrix} \begin{pmatrix} A_0 & \cdots & A_{q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_0 \end{pmatrix} \begin{pmatrix} x(0^-) \\ \vdots \\ x^{(q-1)}(0^-) \end{pmatrix}, \qquad (44)$$

$$f^0_{imp}(t) = \begin{pmatrix} \delta^{(\hat{q}_r-1)}(t)I_r & \dots & \delta(t)I_r \end{pmatrix} \begin{pmatrix} H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ H_1 & \cdots & H_{\hat{q}_r} \end{pmatrix} \begin{pmatrix} f(0^-) \\ \vdots \\ f^{(\hat{q}_r-1)}(0^-) \end{pmatrix}, \qquad (45)$$

where $\delta(t), \delta'(t), \dots$ *denotes the* Dirac δ-distribution *and its (distributional) derivatives.*

Proof. Let $X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$ and $F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$ denote the unilateral Laplace transform of $x(t)$ and $f(t)$, respectively (see [39, 51]). Taking the Laplace transform of Eq. (1), we get

$$L\{A(\rho)x(t)\} = L\{f(t)\} \Rightarrow A(s)X(s) - X_{in}(s) = F(s), \qquad (46)$$

where $X_{in}(s)$ is the initial conditions vector

$$X_{in}(s) = \begin{pmatrix} s^{q-1}I_r & \cdots & sI_r & I_r \end{pmatrix} \begin{pmatrix} A_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_q \end{pmatrix} \begin{pmatrix} x(0^-) \\ \vdots \\ x^{(q-1)}(0^-) \end{pmatrix}. \qquad (47)$$

Now, going back to Eq. (46), we have

$$A(s)X(s) = X_{in}(s) + F(s) \Rightarrow X(s) = \underbrace{A(s)^{-1}X_{in}(s)}_{X_{hom}(s)} + \underbrace{A(s)^{-1}F(s)}_{X_{dynamic}(s)}. \qquad (48)$$

So the system can be decomposed into its free response $x_{hom}(t) = L^{-1}\{X_{hom}(s)\}$ which is the response of the homogeneous system and is connected to the initial conditions of $x(t)$ and its dynamic response $x_{dynamic}(t) = L^{-1}\{X_{dynamic}(s)\}$ which is the response of the system due to its input $f(t)$. We shall examine these two terms separately. Regarding the homogeneous solution of the system we have

$$X_{hom}(s) = A(s)^{-1}X_{in}(s) = (H_{\hat{q}_r}s^{\hat{q}_r} + \cdots + H_0 + H_{-1}s^{-1} + \dots)X_{in}(s). \qquad (49)$$

After some matrix simplifications, taking into account Eq. (29), the above formula can be decomposed into its polynomial and rational parts

$$X_{hom}(s) = -\begin{pmatrix} s^{\hat{q}_r-1}I_r \dots I_r \end{pmatrix} \begin{pmatrix} H_{\hat{q}_r} & \dots & 0 \\ \vdots & \ddots & \vdots \\ H_{\hat{q}_r-(q-1)} & \dots & H_{\hat{q}_r} \\ \vdots & \vdots & \vdots \\ H_1 & \dots & H_q \end{pmatrix} \begin{pmatrix} A_0 & \cdots & A_{q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_0 \end{pmatrix} \begin{pmatrix} x(0^-) \\ \vdots \\ x^{(q-1)}(0^-) \end{pmatrix}$$

$$+ \begin{pmatrix} s^{-1}I_r & \dots \end{pmatrix} \begin{pmatrix} H_{-q} & \dots & H_{-1} \\ H_{-q-1} & \dots & H_{-2} \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} A_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_q \end{pmatrix} \begin{pmatrix} x(0^-) \\ \vdots \\ x^{(q-1)}(0^-) \end{pmatrix}. \quad (50)$$

It is obvious that by taking the inverse Laplace Transform, the polynomial part $X_{hom}^{pol}(s)$ of $X_{hom}(s)$ gives rise to the impulsive behavior of the system. If however the initial conditions $x(0^-)$, $x^{(1)}(0^-)$, ..., $x^{(q-1)}(0^-)$ satisfy Eq. (3) it is shown in [56, Section 4.2.4] that the polynomial part of $X_{hom}(s)$ is equal to zero, thus the system has no impulsive solutions. In the general case though, we consider $X_{pol}(s) \neq 0$. Taking the inverse Laplace Transform of the above equation, using Eqs. (27a) and (27b), after some simplifications (see [56, Section 4.2]), the general solution of Eq. (1) is given by:

$$x_{hom}(t) = Ce^{Jt}x_s(0) - C_\infty(\delta(t)J_\infty + \delta'(t)J_\infty + ... + \delta^{(\hat{q}_r-1)}(t)J_\infty^{\hat{q}_r})x_f(0). \quad (51)$$

Now regarding the second part of Eq. (48), $X_{dynamic}(s)$, we have

$$X_{dynamic}(s) = A(s)^{-1}F(s) \Rightarrow$$
$$X_{dynamic}(s) = C(sI_n - J)^{-1}B_F F(s) + C_\infty(I_\mu - sJ_\infty)^{-1}B_\infty F(s).$$

Making use of the properties of the Laplace Transform

$$L^{-1}\{(sI - J)^{-1}\} = e^{Jt} \quad (52)$$

and

$$L^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)d\tau, \quad (53)$$

we get

$$L^{-1}\{C(sI_n - J)^{-1}B_F F(s)\} = \int_0^t Ce^{J(t-\tau)}B_F f(\tau)d\tau. \quad (54)$$

On the other hand, from

$$\begin{aligned} L\{f(t)\} &= F(s), \\ L\{f^{(1)}(t)\} &= sF(s) - f(0^-), \\ &\vdots \\ L\{f^{(q)}(t)\} &= s^q F(s) - s^{q-1} f(0^-) - ... - f^{(q-1)}(0^-), \end{aligned} \tag{55}$$

we get

$$\begin{aligned} & L^{-1}\big\{C_\infty (I_\mu - sJ_\infty)^{-1} B_\infty F(s)\big\} = \\ &= L^{-1}\big\{C_\infty (I + sJ_\infty + s^2 {J_\infty}^2 + ... + s^{\hat{q}_r} {J_\infty}^{\hat{q}_r}) B_\infty F(s)\big\} = \\ &= L^{-1}\big\{C_\infty B_\infty F(s) + C_\infty J_\infty B_\infty \big(sF(s) - f(0^-)\big) + ... \\ & ... + C_\infty J_\infty^{\hat{q}_r} B_\infty \big(s^{\hat{q}_r} F(s) - s^{\hat{q}_r - 1} f(0^-) - .... - f^{(\hat{q}_r - 1)}(0^-)\big) + \\ & + \begin{pmatrix} s^{\hat{q}_r - 1} I_r & ... & I_r \end{pmatrix} \begin{pmatrix} H_{\hat{q}_r} & & \\ \vdots & \ddots & \\ H_1 & \cdots & H_{\hat{q}_r} \end{pmatrix} \begin{pmatrix} f(0^-) \\ \vdots \\ f^{(\hat{q}_r - 1)}(0^-) \end{pmatrix} \Bigg\} = \\ &= \sum_{i=0}^{\hat{q}_r} C_\infty J_\infty^i B_\infty f^{(i)}(t) + \begin{pmatrix} \delta^{(\hat{q}_r - 1)}(t) I_r & ... & \delta(t) I_r \end{pmatrix} \begin{pmatrix} H_{\hat{q}_r} & & \\ \vdots & \ddots & \\ H_1 & \cdots & H_{\hat{q}_r} \end{pmatrix} \begin{pmatrix} f(0^-) \\ \vdots \\ f^{(\hat{q}_r - 1)}(0^-) \end{pmatrix} \end{aligned} \tag{56}$$

So overall, the dynamic response of the system is

$$\begin{aligned} x_{dynamic}(t) = & \int_0^t C e^{J(t-\tau)} B_F f(\tau) d\tau + \sum_{i=0}^{\hat{q}_r} C_\infty J_\infty^i B_\infty f^{(i)}(t) + \\ & + \begin{pmatrix} \delta^{(\hat{q}_r - 1)}(t) I_r & ... & \delta(t) I_r \end{pmatrix} \begin{pmatrix} H_{\hat{q}_r} & & \\ \vdots & \ddots & \\ H_1 & \cdots & H_{\hat{q}_r} \end{pmatrix} \begin{pmatrix} f(0^-) \\ \vdots \\ f^{(\hat{q}_r - 1)}(0^-) \end{pmatrix}. \end{aligned} \tag{57}$$

Combining all the equations above, the general solution Eq. (42) is derived. □

3.2. Consistent Initial Conditions

In the following, we continue with the analysis of system Eq. (3) under the assumption of consistent (or admissible) initial conditions.

Definition 3.2. [56] *The set of all initial conditions* $x(0^-), x^{(1)}(0^-), ..., x^{(q-1)}(0^-)$ *and* $f(0^-), ..., f^{(\hat{q}_r - 1)}(0^-)$ *that give rise to smooth functional solutions (equivalently impulse free) of the system* (3) *is called the set of* Admissible Initial Conditions *and is denoted by* $\mathcal{H}_{iu}$.

In accordance to [32], we conclude to the following result.

Theorem 3.3. *The set of all* $\mathcal{H}_{iu}$ *is given by*

$$\mathcal{H}_{iu} = \Big\{ x^{(i)}(0^-),\ i = 0, ..., q-1,\ f^{(i)}(0^-),\ i = 0, ..., \hat{q}_r - 1 : \\ \sum_{i=k}^{\hat{q}_r} H_i \left(f^{(i-k)}(0^-) - \sum_{j=0}^{q-i+k-1} A_j x^{(i+j-k)}(0^-) \right) = 0,\ k = 1, 2, ..., \hat{q}_r \Big\}. \tag{58}$$

Proof. This equation is directly obtained by taking the impulsive part of the general solution Eq. (42) and equating it to the zero vector. □

Under the assumption of consistent initial conditions, the impulsive terms in Eq. (42) reduce to zero, so we conclude to the following theorem.

Theorem 3.4. *The general solution of Eq.* (4) *for consistent initial conditions is given by*

$$x(t) = Ce^{Jt}x_s(0) + \int_0^t Ce^{J(t-\tau)} B_F f(\tau) d\tau + \sum_{i=0}^{\hat{q}_r} C_\infty J_\infty^i B_\infty f^{(i)}(t). \tag{59}$$

It is obvious that the solution Eq. (59) of Eq. (4) depends on the finite and infinite Jordan pairs of the polynomial matrix $A(\rho)$ and also the matrices B_F, B_∞.

What must be noted though is that the absence of impulsive solutions for $x(t)$ at $t = 0$ does not imply its continuity too. So a system without impulsive solution may still have $x(0^+) \neq x(0^-)$. The conditions under which $x^{(i)}(0^+) = x^{(i)}(0^-)$, $i = 0..., q-1$ were studied in [57] for the homogeneous case only. An extension of this result for the nonhomogeneous system is given in the following theorem.

Theorem 3.5. *If condition*

$$\begin{pmatrix} H_0 & \cdots & H_{q-1} \\ \vdots & & \vdots \\ H_{-q+1} & \cdots & H_0 \end{pmatrix} \begin{pmatrix} A_0 & \cdots & A_{q-1} \\ & \ddots & \vdots \\ & & A_0 \end{pmatrix} \begin{pmatrix} x(0^-) \\ \vdots \\ x^{(q-1)}(0^-) \end{pmatrix} = \\ = \begin{pmatrix} H_0 & \cdots & H_{\hat{q}_r} & & \\ & \ddots & & \ddots & \\ & & H_0 & \cdots & H_{\hat{q}_r} \end{pmatrix} \begin{pmatrix} f(0^+) \\ \vdots \\ f^{(q+\hat{q}_r-1)}(0^+) \end{pmatrix} \tag{60}$$

is satisfied, then the solution $x(t)$ *of Eq.* (1) *and its derivatives* $x^{(1)}(t), ..., x^{(q-1)}(t)$ *will be continuous at* $t = 0$.

Proof. Assume Eq. (60) holds. Taking into account Eq. (28), it can be easily verified that

$$\begin{pmatrix} H_{-q} & \cdots & H_{-1} \\ \vdots & & \vdots \\ H_{-2q-1} & \cdots & H_{-q} \end{pmatrix} \begin{pmatrix} A_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_q \end{pmatrix} + \begin{pmatrix} H_0 & \cdots & H_{q-1} \\ \vdots & & \vdots \\ H_{-q+1} & \cdots & H_0 \end{pmatrix} \begin{pmatrix} A_0 & \cdots & A_{q-1} \\ & \ddots & \vdots \\ & & A_0 \end{pmatrix} = I_{rq} \tag{61}$$

Taking Eq. (59) for $t = 0^+$, and its derivatives up to $(q-1)$, since $\delta(t) = ... = \delta^{(q-1)}(t) = 0$ for $t = 0^+$, we have

$$\begin{pmatrix} x(0^+) \\ \vdots \\ x^{(q-1)}(0^+) \end{pmatrix} = \begin{pmatrix} C \\ \vdots \\ CJ^{q-1} \end{pmatrix} x_s(0) + \\ + \begin{pmatrix} C_\infty B_\infty & \cdots & C_\infty J_\infty^{\hat{q}_r} B_\infty & & \\ & \ddots & & \ddots & \\ & & C_\infty B_\infty & \cdots & J_\infty^{\hat{q}_r} B_\infty \end{pmatrix} \begin{pmatrix} f(0^+) \\ \vdots \\ f^{(q+\hat{q}_r-1)}(0^+) \end{pmatrix}. \tag{62}$$

Substituting $x_s(0)$, making use of Eqs. (27a) and (27b), (62) becomes

$$\begin{pmatrix} x(0^+) \\ \vdots \\ x^{(q-1)}(0^+) \end{pmatrix} = \begin{pmatrix} H_{-q} & \cdots & H_{-1} \\ \vdots & & \vdots \\ H_{-2q-1} & \cdots & H_{-q} \end{pmatrix} \begin{pmatrix} A_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_q \end{pmatrix} \begin{pmatrix} x(0^-) \\ \vdots \\ x^{(q-1)}(0^-) \end{pmatrix} + \\ + \begin{pmatrix} H_0 & \cdots & H_{\hat{q}_r} & & \\ & \ddots & & \ddots & \\ & & H_0 & \cdots & H_{\hat{q}_r} \end{pmatrix} \begin{pmatrix} f(0^+) \\ \vdots \\ f^{(q+\hat{q}_r-1)}(0^+) \end{pmatrix}. \tag{63}$$

Taking into account Eq. (61), the above equation takes the form

$$\begin{pmatrix} x(0^+) \\ \vdots \\ x^{(q-1)}(0^+) \end{pmatrix} = \begin{pmatrix} x(0^-) \\ \vdots \\ x^{(q-1)}(0^-) \end{pmatrix} - \begin{pmatrix} H_0 & \cdots & H_{q-1} \\ \vdots & & \vdots \\ H_{-q+1} & \cdots & H_0 \end{pmatrix} \begin{pmatrix} A_0 & \cdots & A_{q-1} \\ & \ddots & \vdots \\ & & A_0 \end{pmatrix} \begin{pmatrix} x(0^-) \\ \vdots \\ x^{(q-1)}(0^-) \end{pmatrix} \\ + \begin{pmatrix} H_0 & \cdots & H_{\hat{q}_r} & & \\ & \ddots & & \ddots & \\ & & H_0 & \cdots & H_{\hat{q}_r} \end{pmatrix} \begin{pmatrix} f(0^+) \\ \vdots \\ f^{(q+\hat{q}_r-1)}(0^+) \end{pmatrix} \tag{64}$$

Now, in view of Eq. (60), the last to terms of Eq. (64) vanish, hence the solution $x(t)$ of Eq. (1), and its derivatives $x^{(1)}(t), ..., x^{(q-1)}(t)$ are continuous at $t = 0$. $\square$

4. Special Case: Apostol - Kolodner Matrix Differential Equation

4.1. Explicit Formulas

In the case where the system has no input, it is easily seen from Eq. (59) that the solution of the homogeneous Eq. (1) under consistent initial conditions is given by

$$x(t) = Ce^{Jt}x_s(0). \tag{65}$$

In this case, in order to construct the solution of the system, only the knowledge of the finite Jordan pair (C, J) of $A(\rho)$ and B_F is required. Computation of the infinite Jordan pairs is not necessary. Thus, one can make use of the fact that the strictly proper part of the matrix $A(\rho)^{-1}$ is equal to

$$A(\rho)^{-1}_{sp} = C(\rho I - J)^{-1}B_F \tag{66}$$

and so, the above equation can be used to compute B_F. So the computation of the infinite Jordan pair (C_∞, J_∞) can be completely omitted. Now, since the matrices (C, J) constitute the finite Jordan pair of $A(s)$, they have the following form

$$C = [C_1, C_2, ..., C_\ell], \qquad J = J_1 \oplus \cdots \oplus J_\ell, \tag{67}$$

where $C_i \in \mathbb{R}^{n \times n_i}$, $J_i \in \mathbb{R}^{n_i \times n_i}$ and ℓ are the distinct finite zeros λ_i, $i = 1, ..., \ell$ of $A(\rho)$ with $n_i = \sum_{j=z}^{r} n_{i,j}$.

In the special case where the matrix $A(s)$ has $2m$ distinct pairs of complex conjugate zeros and k distinct real zeros, such that $k + 2m = \ell$, then the pair (C, J) is partitioned as

$$C = [C_1, ..., C_k, C_{k+1} ..., C_{k+m}], \qquad J = J_1 \oplus \cdots J_k \oplus J_{k+1} \oplus \cdots J_{k+m}, \tag{68}$$

where $J_1, ..., J_k$ are in the form Eq. (30) and $J_{k+1}, ..., J_{k+m}$ are in the form Eq. (38). Here we assumed, without loss of generality, that the first k zeros are real and the latter $2m$ are complex.

Using this partition, we can derive the following corollaries.

Corollary 4.1. *The general solution Eq. (65) of the homogeneous Eq. (1) is given by*

$$x(t) = \bigoplus_{i=1}^{\ell} C_i e^{J_i t} x_s(0). \tag{69}$$

Using the three different formulas for the matrix exponential presented in Lemmas 2.11, 2.12 and 2.13, we can derive new analytic expressions for the solution of the homogeneous Eq. (1), given in the following three corollaries respectively.

Corollary 4.2. *Using the notation of Lemma 2.11, the general solution of the homogeneous Eq. (1) can be rewritten as*

$$x(t) = \bigoplus_{i=1}^{\ell} \bigoplus_{j=1}^{n_i} C_i G_i(t) x_s(0), \tag{70}$$

Corollary 4.3. *Using the notation of Lemma 2.12, the general solution of the homogeneous Eq. (1) can be rewritten as*

$$x(t) = \sum_{k=0}^{m-1} C J^k f_k(t) x_s(0), \tag{71}$$

where $f_k(t)$'s satisfy Eq. (35).

Corollary 4.4. *Using the notation of Lemma 2.13, the general solution of the homogeneous Eq. (1) can be rewritten as*

$$x(t) = \bigoplus_{i=1}^{\ell} \left(C_i \sum_{j=0}^{m_i-1} f_{ij}(t) J_i^j \right) x_s(0), \tag{72}$$

where $f_{ij}(t)$ satisfy Eq. (37).

4.2. Apostol-Kolodner type

For the special case of Higher order Apostol-Kolodner differential equations

$$F x^{(q)}(t) = G x(t), \tag{73}$$

or equivalently

$$(A_q \rho^q - A_0) x(t) = 0, \tag{74}$$

where $A_q = F$ and $A_0 = G$, the solution formulas are the same. The only difference is the initial condition vectors. Since for these type of systems we have $A_{q-1} = \cdots = A_1 = 0$, $x_s(0)$ and $x_f(0)$ become

$$x_s(0) = \begin{pmatrix} J^{q-1} B_F A_q & \cdots & B_F A_q \end{pmatrix} \begin{pmatrix} x(0^-) \\ \vdots \\ x^{(q-1)}(0^-) \end{pmatrix}, \tag{75}$$

$$x_f(0) = \begin{pmatrix} B_\infty A_0 & \cdots & J_\infty^{q-1} B_\infty A_0 \end{pmatrix} \begin{pmatrix} x(0^-) \\ \vdots \\ x^{(q-1)}(0^-) \end{pmatrix}. \tag{76}$$

See also [30] for necessary comparison.

5. Applications to Mechanical Engineering

5.1. 2-DOF Mass Spring Systems

Consider as in [55], a system with two masses m_1 and m_2, connected with springs with stiffness coefficients k_1 and k_2, shown in Figure 1. The system is decomposed into two separate sub-systems, connected by the constraint $q_1 = x_1 + d$, where d is the length of mass m_1, as shown in Figure 2. The equations of the unconstrained system are

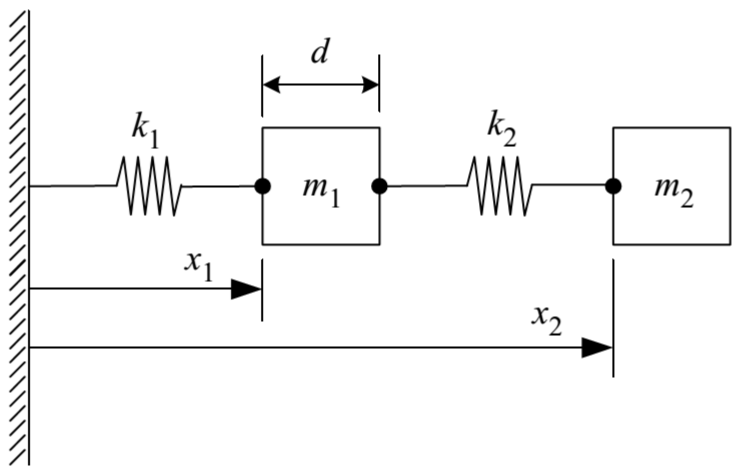

Figure 1: A two degree of freedom system of two masses.

$$\underbrace{\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & m_2 \\ 0 & m_2 & m_2 \end{pmatrix}}_{M} \begin{pmatrix} \ddot{\bar{x}}_1 \\ \ddot{q}_1 \\ \ddot{\bar{q}}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -k1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -k_2 \end{pmatrix}}_{K} \begin{pmatrix} \bar{x}_1 \\ q_1 \\ \bar{q}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -k_1\bar{x}_1 \\ 0 \\ -k_2\bar{q}_2 \end{pmatrix}}_{Q}, \tag{77}$$

where $\bar{x}_1 = x_1 - l_{1,0}$ and $\bar{q}_2 = q_2 - l_{2,0}$ and $l_{1,0}$, $l_{2,0}$ are the unstretched lengths of the springs k_1 and k_2. The connection between the two subsystems is given by the constraint

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} \ddot{\bar{x}}_1 \\ \ddot{q}_1 \\ \ddot{\bar{q}}_2 \end{pmatrix} = \underbrace{0}_{b}. \tag{78}$$

It was shown in [55] that the above system has a unique equation of motion if and only if the matrix $\begin{pmatrix} M^T & A^T \end{pmatrix}^T$ has full rank n. This holds true for m_1, $m_2 > 0$.

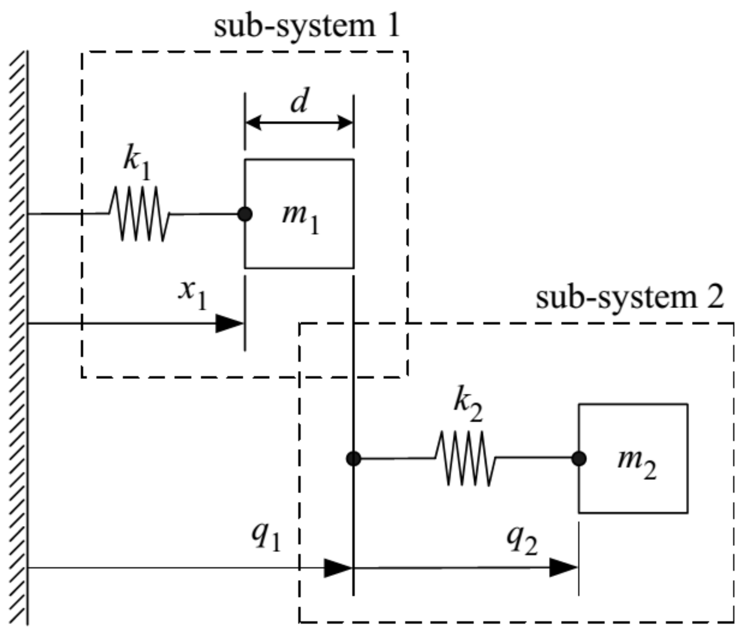

Figure 2: Decomposition using more than two coordinates.

Furthermore, an explicit equation for the acceleration of the system is given through the use of Moore-Penrose matrices. In contrast to that method, we shall implement the procedure proposed in [1], to derive the constrained equations of motion of the system in the form

$$\bar{M}\ddot{q}(t) = \bar{K}q(t), \tag{79}$$

where

$$\bar{M} = \begin{pmatrix} V^T M \\ A \end{pmatrix}, \qquad \bar{K} = \begin{pmatrix} V^T K \\ 0 \end{pmatrix} \tag{80}$$

and V is a matrix whose columns form a basis of A. The matrix V is

$$V = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{81}$$

and the constrained equations of motion are

$$\begin{pmatrix} 0 & m_2 & m_2 \\ m_1 & m_2 & m_2 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \ddot{\bar{x}}_1 \\ \ddot{q}_1 \\ \ddot{\bar{q}}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -k_2 \\ -k_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ q_1 \\ \bar{q}_2 \end{pmatrix}. \tag{82}$$

Since the matrix $\bar{M}$ is invertible for m_1, $m_2 > 0$, the above equations can be written as

$$\begin{pmatrix} \ddot{\bar{x}}_1 \\ \ddot{q}_1 \\ \ddot{\bar{q}}_2 \end{pmatrix} = \begin{pmatrix} \frac{k_2\bar{q}_2 - k_1\bar{x}_1}{m_1} \\ \frac{k_2\bar{q}_2 - k_1\bar{x}_1}{m_1} \\ -\frac{k_2 m_1\bar{q}_2 + k_2 m_2\bar{q}_2 - k_1 m_2\bar{x}_1}{m_1 m_2} \end{pmatrix}, \tag{83}$$

which is the exact same formula for the acceleration obtained in [55]. To further draw this example under the proposed framework, the solution of the constrained system (82) shall be derived. Assuming for simplicity that $m_1 = m_2 = m$ and $k_1 = k_2 = k$ the system can be written as

$$\begin{pmatrix} 0 & m\rho^2 & k + m\rho^2 \\ k + m\rho^2 & m\rho^2 & m\rho^2 \\ \rho^2 & -\rho^2 & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ q_1 \\ \bar{q}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{84}$$

where the system matrix is $A(\rho) = \bar{M}\rho^2 - \bar{K} = A_2\rho^2 + A_0$. The Smith Form of the matrix $A(\rho)$ is

$$S^{\mathbb{C}}_{A(\rho)}(\rho) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\rho^2\left(k^2 + 3km\rho^2 + m^2\rho^4\right)}{m^2} \end{pmatrix}, \tag{85}$$

and following the method presented in [25, 34] we find, after some simplifications, the matrices

$$C = \begin{pmatrix} 0 & 0 & \frac{1}{2}\left(3+\sqrt{5}\right) & 0 & \frac{1}{2}\left(3-\sqrt{5}\right) & 0 \\ 1 & 0 & \frac{1}{2}\left(3+\sqrt{5}\right) & 0 & \frac{1}{2}\left(3-\sqrt{5}\right) & 0 \\ 0 & 0 & -2-\sqrt{5} & 0 & -2+\sqrt{5} & 0 \end{pmatrix}, \tag{86}$$

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{\frac{3k}{m} + \frac{\sqrt{5}k}{m}}}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{\frac{3k}{m} + \frac{\sqrt{5}k}{m}}}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{\frac{3k}{m} - \frac{\sqrt{5}k}{m}}}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{\frac{3k}{m} - \frac{\sqrt{5}k}{m}}}{\sqrt{2}} & 0 \end{pmatrix}, \tag{87}$$

$$B_F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ -\frac{\sqrt{\frac{2}{5(3+\sqrt{5})}}\sqrt{\frac{k}{m}}}{k} & -\frac{\left(-5+\sqrt{5}\right)\sqrt{\frac{k}{6m+2\sqrt{5}m}}}{5k} & \frac{5-3\sqrt{5}}{5\sqrt{2\left(3+\sqrt{5}\right)}\sqrt{\frac{k}{m}}} \\ 0 & 0 & 0 \\ \frac{\sqrt{\frac{2}{5}}}{\sqrt{-\frac{\left(-3+\sqrt{5}\right)k}{m}}m} & \frac{5+\sqrt{5}}{5\sqrt{2}\sqrt{-\frac{\left(-3+\sqrt{5}\right)k}{m}}m} & \frac{5+3\sqrt{5}}{5\sqrt{2}\sqrt{-\frac{\left(-3+\sqrt{5}\right)k}{m}}} \end{pmatrix}. \tag{88}$$

Now, assuming consistent initial displacements $\bar{x}_1(0), q_1(0), \bar{q}_2(0)$ and initial velocities, the general solution of the system is

$$x(t) = Ce^{Jt}x_s(0), \tag{89}$$

where

$$x_s(0) = \begin{pmatrix} JB_FA_2 & B_FA_2 \end{pmatrix} \begin{pmatrix} q(0^-) \\ \dot{q}(0^-) \end{pmatrix}. \tag{90}$$

The displacement of the two masses for a particular choice of the parameters is given in Figure 3.

5.2. 2-DOF Mass Spring Systems With Damping And Massless Bodies

Consider as in [7], a lumped parameter system consisting of a mass m with displacement u_3, connected to a wall with linear springs with constants k_1, k_2 and linear viscous dampers with viscous coefficients c_1, c_2, as shown in Figure 4. The system is modelled using two massless joints between the springs and the dampers, with displacements u_1 and u_2 respectively. The equations of motion for the system are

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m \end{pmatrix}}_{M} \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & c_1+c_2 & -c_2 \\ 0 & -c_2 & c_2 \end{pmatrix}}_{K} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} + \underbrace{\begin{pmatrix} k_1+k_2 & 0 & -k_2 \\ 0 & 0 & 0 \\ -k_2 & 0 & k_2 \end{pmatrix}}_{Q} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{91}$$

which can be written in AR form as

$$\begin{pmatrix} k_1+k_2 & 0 & -k_2 \\ 0 & \rho(c_1+c_2) & -\rho c_2 \\ -k_2 & -\rho c_2 & \rho^2 m + \rho c_2 + k_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{92}$$

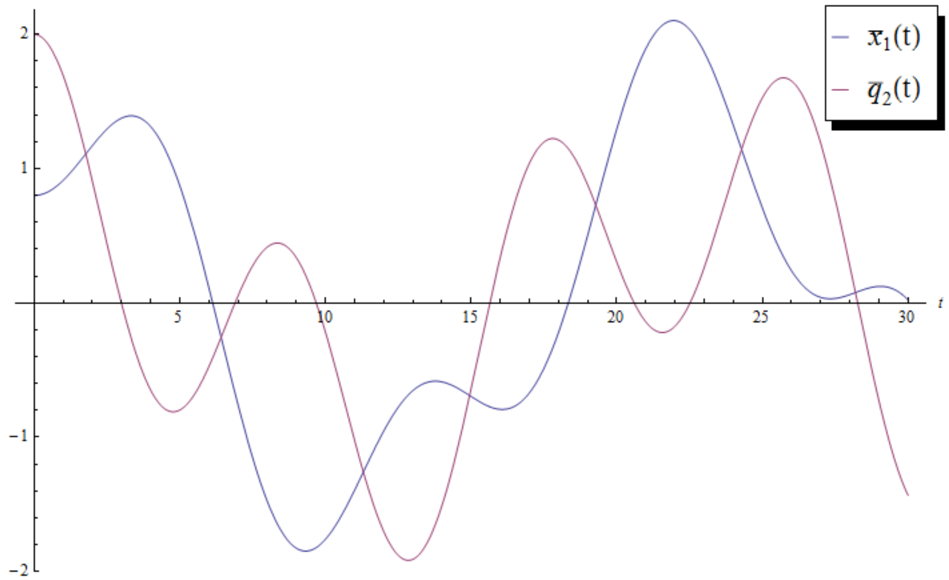

Figure 3: Displacements $\bar{x}_1(t), \bar{q}_2(t)$ of the two masses of the constrained system for $m = 10$ kg, $k = 2$ N/m, $d = 0.2$ m, $l_{1,0} = l_{2,0} = 0.2$ m with initial displacements $x_1(0) = 1$ m, $\bar{x}_1(0) = 0.8$ m, $q_1(0) = 1.2$ m, $q_2(0) = 2.2$ m, $\bar{q}_2(0) = 2$ m and zero initial velocities.

where the system matrix is $A(\rho) = M\rho^2 + K\rho + Q = A_2\rho^2 + A_1\rho + A_0$ and the mass matrix M is singular. Our first step is to find the Jordan pairs of the matrix $A(\rho)$. To simplify the computations, lets assume that $k_1 = 2k_2$ and $c_1 = 2c_2$. The Smith form of the matrix $A(\rho)$ is

$$S^{\mathbb{C}}_{A(\rho)}(\rho) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\rho\left(2k_2+2c_2\rho+3m\rho^2\right)}{3m} \end{pmatrix}, \tag{93}$$

and following the method presented in [25, 34] we find the matrices

$$C = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ -\frac{2k_2}{3m} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 1 \end{pmatrix}, \tag{94}$$

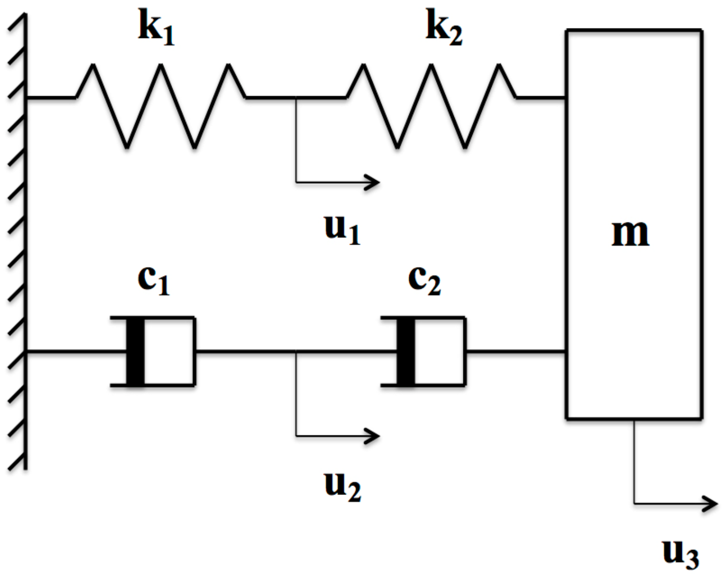

Figure 4: A 1-DOF spring-mass system modelled as a multiple DOF system with massless joints.

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{-c_2-\sqrt{c_2{}^2-6k_2m}}{3m} & 0 \\ 0 & 0 & \frac{-c_2+\sqrt{c_2{}^2-6k_2m}}{3m} \end{pmatrix}, \tag{95}$$

$$B_F = \begin{pmatrix} 0 & -\frac{m}{2c_2k_2} & 0 \\ -\frac{1}{2\sqrt{c_2{}^2-6k_2m}} & -\frac{1}{2\sqrt{c_2{}^2-6k_2m}} & -\frac{3}{2\sqrt{c_2{}^2-6k_2m}} \\ \frac{1}{2\sqrt{c_2{}^2-6k_2m}} & \frac{1}{2\sqrt{c_2{}^2-6k_2m}} & \frac{3}{2\sqrt{c_2{}^2-6k_2m}} \end{pmatrix}, \tag{96}$$

$$H_0 = \begin{pmatrix} \frac{1}{3k_2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_1 = H_2 = ... = 0. \tag{97}$$

Since $H_1 = ... = 0$, the inverse of $A(s)$ is a proper matrix, so from Eq. (42) it is seen that the solution of the system will be impulse free. Moreover, the solution will be continuous at $t = 0$ if Eq. (62) holds, that is

$$\begin{pmatrix} u(0^+) \\ \dot{u}(0^+) \end{pmatrix} = \begin{pmatrix} C \\ CJ \end{pmatrix} \begin{pmatrix} JB_F & B_F \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ A_1 & A_2 \end{pmatrix} \begin{pmatrix} u(0^-) \\ \dot{u}(0^-) \end{pmatrix} \tag{98}$$

Setting $u(0^+) = u(0^-)$, $\dot{u}(0^+) = \dot{u}(0^-)$ in Eq. (98), we obtain the conditions

$$u_1(0^-) = \frac{u_3(0^-)}{3}, \quad \dot{u}_1(0^-) = \dot{u}_2(0^-), \quad \dot{u}_3(0^-) = 3\dot{u}_2(0^-), \tag{99}$$

Now, assuming consistent initial displacements $u_1(0), u_2(0), u_3(0)$ and initial velocities $\dot{u}_1(0), \dot{u}_2(0), \dot{u}_3(0)$, the general solution of the system is

$$x(t) = Ce^{Jt}x_s(0), \tag{100}$$

The displacement of the mass m for specific parameters is given in Figure 5.

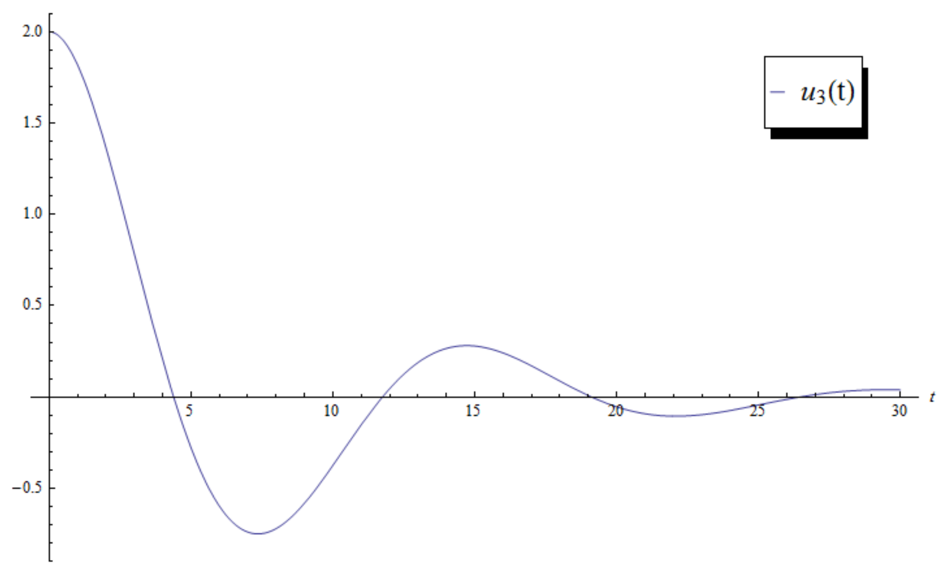

Figure 5: Displacement $u_3(t)$ of the mass for $m = 10$ kg, $k_2 = 3$ N/m, $c_2 = 4$ N s/m with initial displacements $u_1(0) = \frac{2}{3}$, $u_2(0) = 1$ m, $u_3(0) = 2$ m and zero initial velocities.

6. Conclusions

In this paper, using a mathematical approach which considers the finite and infinite Jordan pairs of the matrix $A(\rho)$, the closed form solution of the equations of motion for linear mechanical systems with constrains is derived. The methodology is general enough to accommodate either non-singular or singular leading coefficient matrices. For mathematical completeness and also for the purpose of extending the method to some other interesting cases, the methodology and analysis are presented for the general class of linear systems of order higher than two. The proposed methodology also takes into account the possible inconsistency of initial conditions, so the general solution of the system possibly involves impulsive terms. In addition, we further study the case of homogeneous systems, and propose analytic expressions for the homogeneous solution, namely Eqs. (69), (70), (71), (72), based on different methods for the computation of the matrix exponential. There are many more formulas that could be proposed, since the analytic computation of the matrix exponential is itself the subject of extended study.

A possible direction for further research, could be the extension of these methods to non regular systems, i.e. systems with $\det A(\rho) = 0$ or with $A(\rho)$ non square. An extension of these results to discrete time systems is also possible. Finally, the determination of the stochastic response of linear multi-degree-of-freedom (MDOF) structural systems with singular matrices (see [16, 17]) could be subject of another future extension.

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References

[1] Antoniou, E. N., Pantelous, A. A., Kougioumtzoglou, I., Pirrotta, A., 2016. Response determination of linear dynamical systems with singular matrices: A polynomial matrix theory approach. *Applied Mathematical Modelling*, in press.

[2] Antsaklis, P. J., Michel, A. N., 2006. *Linear systems*. Springer Science & Business Media.

[3] Apostol, T. M., 1975. Explicit formulas for solutions of the second-order matrix differential equation Y" = AY. *American Mathematical Monthly*, 159–162.

[4] Ben Taher, R., Rachidi, M., 2002. Some explicit formulas for the polynomial decomposition of the matrix exponential and applications. *Linear Algebra and its Applications* 350 (1-3), 171–184.

[5] Bernstein, D. S., 2009. *Matrix mathematics. Theory, facts, and formulas*, Second Edition. Princeton, University Press.

[6] Betcke, T., Higham, N. J., Mehrmann, V., Schröder, C., Tisseur, F., Feb. 2013. NLEVP: A Collection of nonlinear eigenvalue problems. *ACM Transactions on Mathematical Software* 39 (2), 7:1–7:28.

[7] Bhat, S. P., Bernstein, D. S., 1996. Second-order systems with singular mass matrix and an extension of Guyan reduction. *SIAM J. Matrix Anal. Appl.* 17 (3), 649–657.

[8] Campbell, S. L., Meyer, C. D., 2009. *Generalized inverses of linear transformations*. Vol. 56. SIAM.

[9] Campbell, S. L. V., 1980. *Singular systems of differential equations*: Volume 1. Research Notes in Mathematics. Pitman Advanced Pub. Program.

[10] Campbell, S. L. V., 1980. *Singular systems of differential equations*: Volume 2. Research Notes in Mathematics. Pitman Advanced Pub. Program.

[11] Cheng, H.-W., Yau, S. S.-T., 1997. More explicit formulas for the matrix exponential. *Linear Algebra and its Applications* 262, 131–163.

[12] Cohen, N., 1983. Spectral analysis of regular matrix polynomials. *Integral Equations Oper. Theory* 6, 161–183.

[13] Dai, L., 1989. *Singular control systems*. Springer-Verlag New York, Inc.

[14] De Jalon, J. G., Gutierrez-Lopez, M., 2013. Multibody dynamics with redundant constraints and singular mass matrix: existence, uniqueness, and determination of solutions for accelerations and constraint forces. *Multibody System Dynamics* 30 (3), 311–341.

[15] Ding, F., 2013. Computation of matrix exponentials of special matrices. *Applied Mathematics and Computation* 223, 311 – 326.

[16] Fragkoulis, V. C., Kougioumtzoglou, I. A., Pantelous, A. A., 2016. Linear random vibration of structural systems with singular matrices. *ASCE Journal of Engineering Mechanics* 142 (2), 04015081.

[17] Fragkoulis, V. C., Kougioumtzoglou, I. A., Pantelous, A. A., 2016. Statistical linearization of nonlinear systems with singular mass matrices. *ASCE Journal of Engineering Mechanics* 142 (9), 04016063.

[18] Fragulis, G., Mertzios, B., Vardulakis, A., 1991. Computation of the inverse of a polynomial matrix and evaluation of its Laurent expansion. *International Journal of Control* 53 (2), 431–443.

[19] Fragulis, G., Vardulakis, A., 1994. Evaluation of the reachability subspace of general form polynomial matrix descriptions (PMDs). *Kybernetika* 30 (6), 617–628.

[20] Gauß, C. F., 1829. Über ein neues allgemeines Grundgesetz der Mechanik. *Journal für die reine und angewandte Mathematik* 4, 232–235.

[21] Geerts, A., Schumacher, J., 1996. Impulsive-smooth behavior in multimode systems. I: State-space and polynomial representations. *Automatica* 32 (5), 747–758.

[22] Geerts, A. T., Schumacher, J. M., 1996. Impulsive-smooth behavior in multimode systems part i: State-space and polynomial representations. *Automatica* 32 (5), 747–758.

[23] Geerts, T., 1993. Solvability conditions, consistency, and weak consistency for linear differential-algebraic equations and time-invariant singular systems: the general case. *Linear Algebra and its Applications* 181, 111–130.

[24] Geerts, T., 1994. Regularity and singularity in linear-quadratic control subject to implicit continuous-time systems. *Circuits, Systems and Signal Processing* 13 (1), 19–30.

[25] Gohberg, I., Lancaster, P., Rodman, L., 2009. *Matrix polynomials*, reprint of the 1982 original Edition. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM).

[26] Hautus, M. L., 1976. The formal laplace transform for smooth linear systems. In: *Mathematical Systems Theory*. Springer, pp. 29–47.

[27] Hayton, G., Pugh, A., Fretwell, P., 1988. Infinite elementary divisors of a matrix polynomial and implications. *International Journal of Control* 47 (1), 53–64.

[28] Kalogeropoulos, G. I., Karageorgos, A. D., Pantelous, A. A., 2007. On the relation of the drazin inverse and matrix pencil theory methods for the study of generalized linear systems. *Linear Algebra and its Applications* 427 (2), 197–205.

[29] Kalogeropoulos, G. I., Karageorgos, A. D., Pantelous, A. A., 2008. The drazin inverse through the matrix pencil approach and its application to the study of generalized linear systems with rectangular or square coefficient matrices. *Electronic Journal of Linear Algebra* 17 (1), 118–138.

[30] Kalogeropoulos, G. I., Karageorgos, A. D., Pantelous, A. A., 2009. Higher-order linear matrix descriptor differential equations of Apostol–Kolodner type. *Electronic Journal of Differential Equations* 2009 (25), 1–13.

[31] Kalogeropoulos, G. I., Karageorgos, A. D., Pantelous, A. A., 2014. On the solution of higher order linear homogeneous complex σ–α descriptor matrix differential systems of Apostol–Kolodner type. *Journal of the Franklin Institute* 351 (3), 1756–1777.

[32] Karampetakis, N., Vardulakis, A., 1995. On the solution of ARMA-representations. In: *Proceedings of the 3rd IEEE Mediterranean Symposium on New Directions in Control Theory and Applications*. pp. 156–163.

[33] Karampetakis, N., Vologiannidis, S., 2009. On the fundamental matrix of the inverse of a polynomial matrix and applications to ARMA representations. *Linear Algebra and its Applications* 431 (11), 2261–2276.

[34] Karampetakis, N. P., 2004. On the solution space of discrete time AR-representations over a finite time horizon. *Linear Algebra and its Applications* 382, 83–116.

[35] Kolodner, I. I., 1975. On $exp(ta)$ with a satisfying a polynomial. *Journal of Mathematical Analysis and Applications* 52 (3), 514–524.

[36] Lagrange, J. L., 1787. *Mecanique analytique*. Paris: Mme Ve Courcier.

[37] Leontief, W. W., 1986. *Input-output economics*. Oxford University Press on Demand.

[38] Lewis, F. L., 1986. A survey of linear singular systems. *Circuits, Systems and Signal Processing* 5 (1), 3–36.

[39] Lundberg, K. H., Miller, H. R., Trumper, D. L., 2007. Initial conditions, generalized functions, and the laplace transform troubles at the origin. *IEEE control systems* 27 (1), 22–35.

[40] Mariti, L., Belfiore, N., Pennestrì, E., Valentini, P., 2011. Comparison of solution strategies for multibody dynamics equations. *International Journal for Numerical Methods in Engineering* 88 (7), 637–656.

[41] Mehrmann, V., Watkins, D., 2002. Polynomial eigenvalue problems with Hamiltonian structure. *ETNA. Electronic Transactions on Numerical Analysis* [electronic only] 13, 106–118.

[42] Moler, C., Van Loan, C., 2003. Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. *SIAM Review* 45 (1), 3–49.

[43] Ogata, K., 2010. *Modern Control Engineering*. Instrumentation and controls series. Prentice Hall.

[44] Pantelous, A. A., Kalogeropoulos, G. I., 2009. On linear generalized neutral differential delay systems. *Journal of the Franklin Institute* 346 (7), 691–704.

[45] Pantelous, A. A., Karageorgos, A. D., Kalogeropoulos, G. I., 2014. A new approach for second-order linear matrix descriptor differential equations of apostol–kolodner type. *Mathematical Methods in the Applied Sciences* 37 (2), 257–264.

[46] Perko, L., 2013. Differential equations and dynamical systems. Vol. 7. Springer Science & Business Media.

[47] Rao, S., 2011. *Mechanical Vibrations*, 5th Edition. No. v. 978, nos. 0-212813 in Mechanical Vibrations. Prentice Hall.

[48] Schutte, A., Udwadia, F., 2011. New approach to the modeling of complex multibody dynamical systems. *Journal of Applied Mechanics* 78 (2), 021018.

[49] Schutte, A. D., 2010. Permissible control of general constrained mechanical systems. *Journal of the Franklin Institute* 347 (1), 208 – 227.

[50] Taher, R. B., Rachidi, M., 2008. Linear matrix differential equations of higher-order and applications. *Electronic Journal of Differential Equations* 2008 (95), 1–12.

[51] Trenn, S., 2013. Solution concepts for linear daes: a survey. In: *Surveys in Differential-Algebraic Equations I*. Springer, pp. 137–172.

[52] Udwadia, F. E., Kalaba, R. E., 1992. A new perspective on constrained motion. *Proceedings: Mathematical and Physical Sciences*, 407–410.

[53] Udwadia, F. E., Kalaba, R. E., 1993. On motion. *Journal of the Franklin Institute* 330 (3), 571 – 577.

[54] Udwadia, F. E., Kalaba, R. E., 2007. *Analytical dynamics: a new approach.* Cambridge University Press.

[55] Udwadia, F. E., Phohomsiri, P., 2006. Explicit equations of motion for constrained mechanical systems with singular mass matrices and applications to multi-body dynamics. In: *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences.* Vol. 462. The Royal Society, pp. 2097–2117.

[56] Vardulakis, A., 1991. *Linear multivariable control. Algebraic analysis and synthesis methods.* Chichester etc.: John Wiley & Sons.

[57] Vardulakis, A., Antoniou, E., Karampetakis, N., 1999. On the solution and impulsive behaviour of polynomial matrix descriptions of free linear multivariable systems. *International Journal of Control* 72 (3), 215–228.

[58] Verde-Star, L., 1994. Operator identities and the solution of linear matrix difference and differential equations. *Studies in Applied Mathematics* 91 (2), 153–178.

[59] Wilkening, J., 2007. An algorithm for computing Jordan chains and inverting analytic matrix functions. *Linear Algebra and its Applications* 427 (1), 6–25.

[60] Wilkening, J., Yu, J., 2011. A local construction of the Smith normal form of a matrix polynomial. *Journal of Symbolic Computation* 46 (1), 1–22.

[61] Willems, J. C., 1986. From time series to linear system, II. Exact modelling. *Automatica* 22, 675–694.

[62] Willems, J. C., 1986. From time series to linear system-Part I. Finite dimensional linear time invariant systems. *Automatica* 22 (5), 561–580.

[63] Willems, J. C., 1991. Paradigms and puzzles in the theory of dynamical systems. *IEEE Transactions on Automatic Control* 36 (3), 259–294.

[64] Willems, J. C., 2007. The behavioral approach to open and interconnected systems. *IEEE Control Systems* 27 (6), 46–99.

[65] Yang, T., 2001. *Impulsive Control Theory*. Lecture Notes in Control and Information Sciences. Springer Berlin Heidelberg.