Voronoi means, moving averages, and power series

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Abstract

We introduce a *non-regular* generalisation of the Nörlund mean, and show its equivalence with a certain moving average. The Abelian and Tauberian theorems establish relations with convergent sequences and certain power series. A strong law of large numbers is also proved.

Keywords: Voronoi means; Nörlund means; Moving averages; Power series; Regular variation; LLN.

1. Introduction

Let the real sequences $\{p_n, q_n, u_n\}_{n=0}^{\infty}$ with $u_n \neq 0$ for $n \geq 0$, be given. The real sequence $\{s_n\}_{n=0}^{\infty}$ has *Voronoi mean*[3] s, written $s_n \to s$ (V, p_n, q_n, u_n), if

$$t_n := \frac{1}{u_n} \sum_{k=0}^{n} p_{n-k} q_k s_k \to s \quad (n \to \infty). \tag{1.1}$$

There are many known special cases of the Voronoi mean. The *generalised Nörlund mean* (N, p_n, q_n) of Borwein [14] is the $(V, p_n, q_n, (p*q)_n)$ mean, with

$$(p * q)_n := \sum_{k=0}^{n} p_{n-k} q_k.$$

Other special cases are:

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[3]Voronoi was the first to introduce the summability method that is now known as the *Nörlund mean* in the Proceedings of the Eleventh Congress of Russian Naturalists and Scientists (in Russian), St. Petersburg, 1902, pp 60-61 (see [23] page 91).

(a) the *Euler method* E_p of order $p \in (0,1)$, which is the Voronoi mean with $p_n = (1-p)^n/n!$, $q_n = p^n/n!$, and $u_n = (p * q)_n$ (see [14]);

(b) the *Nörlund mean* (N, p_n), which is the $(V, p_n, 1, (p * 1)_n)$ mean, and for $k > 0$ and $p_n = \Gamma(n+k)/(\Gamma(n+1)\Gamma(k))$ becomes the *Cesàro mean* (C, k) (see, for example, §4.1 of [23]);

(c) the *weighted mean* or the *discontinuous Riesz mean* $(\overline{N}, q_n)$, which is the $(V, 1, q_n, (1 * q)_n)$ mean, with the further special cases of $q_n = 1$ and $q_n = 1/(n+1)$ giving the Cesàro mean $(C, 1)$ and the logarithmic mean ℓ, respectively (see, for example, §3.8 of [23]);

(d) the *Jajte mean* – the summability method for the law of large numbers (LLN) in [31], which is the $(V, 1, q_n, u_n)$ mean with $\sum_{k=0}^{n} q_k/u_n$ not necessarily converging to 1 as $n \to \infty$;

(e) the *Chow–Lai mean* – the summability method for the LLN in [17], which is the $(V, p_n, 1, u_n)$ mean with $u_n \to \infty$ and $\sum_{n=0}^{\infty} p_n^2 < \infty$.

Recall that a summability method is *regular* if it sums a convergent sequence to its limit (see, for example, page 43 of [23]). The necessary and sufficient conditions for the (V, p_n, q_n, u_n) mean to be regular are (see, for example, Theorem 2 of [23]):

(i) $\sum_{k=0}^{n} |p_{n-k} q_k| < K|u_n|$, with K independent of n;

(ii) $p_{n-k} q_k / u_n \to 0$ as $n \to \infty$ for each $k \geq 0$;

(iii) $\sum_{k=0}^{n} p_{n-k} q_k / u_n \to 1$ as $n \to \infty$.

A consequence of condition (iii) is that a regular (V, p_n, q_n, u_n) mean is *equivalent* to a regular (N, p_n, q_n) mean. Thus, the introduction of a third sequence u_n in (1.1), which is an essential contribution of this paper, gains us nothing *unless the summability method is non-regular.* Moreover, the Jajte mean does not necessarily satisfy (iii), and the Chow-Lai mean never satisfies (iii) (see (d) and (e) above, respectively). For these reasons, we do *not* assume that the triple (p_n, q_n, u_n) necessarily satisfies the regularity conditions (i)–(iii). The non-regular summability methods, apart from their intrinsic interest within summability theory, and far from being peripheral or pathological, are useful in a variety of contexts (see, for example, § 4).

The *Voronoi convolution* of two sequences p_n and q_n, denoted $(p \circ q)_n$, is defined as $(p \circ q)_0 := p_0 q_0$, and for $n \geq 1$ as:

$$(p \circ q)_n := (p * q)_n - (p * q)_{n-1}.$$

The definition of the Voronoi mean (1.1) can now be rewritten as:

$$t_n = \frac{1}{u_n} \sum_{k=0}^{n} (p \circ qs)_k \to s \quad (n \to \infty), \tag{1.2}$$

where $(p \circ qs)_n$ denotes the Voronoi convolution of p_n and $q_n s_n$.

Let the non-zero function u be such that $u(n) := u_n$. The sequence s_n has *continuous* Voronoi mean s, written $s_n \to s$ $(V_x, p_n, q_n, u(x))$, if:

$$t_x := \frac{1}{u(x)} \sum_{0 \leq k \leq x} (p \circ qs)_k \to s \quad (x \to \infty).$$

The formulation (1.2) of the Voronoi mean motivates the introduction of the following summability method. Let $v_0 := u_0$ and

$$v_n := u_n - u_{n-1}, \quad n \geq 1.$$

Also let

$$\begin{aligned} N(x) &:= \sum_{n=0}^{\infty} (p \circ qs)_n x^n, \\ D_u(x) &:= \sum_{n=0}^{\infty} v_n x^n. \end{aligned}$$

If the power series $D_u(x)$ has radius of convergence $R_u \in (0, \infty]$, then s_n is summable to s by the *Voronoi power series*, written $s_n \to s$ $(\mathcal{P}, p_n, q_n, v_n)$ (or, if more appropriate, $(\mathcal{P}, p_n, q_n, D_u(x))$) , if

$$T(x) := \frac{N(x)}{D_u(x)} \to s \quad (x \to R_u -).$$

Three known special cases are (see, for example, [23]):

(α) the *Abel method* A, which is $(\mathcal{P}, 1, 1, 1/(1-x))$ with $R_u = 1$;

(β) the *Borel method* B, which is $(\mathcal{P}, 1, 1/n!, e^x)$ with $R_u = \infty$;

(λ) the *logarithmic method* L, which is $(\mathcal{P}, 1, 1/(1+n), -\log(1-x))$ with $R_u = 1$.

In [8], we introduced a certain *moving average* summability method, which is equivalent to the logarithmic mean ℓ. Here we introduce its *generalisation* appropriate for the Voronoi mean. If $u : [0,\infty) \to [0,\infty)$ is continuous, strictly increasing, $u(0) = 0$, $u(x) \to \infty$ as $x \to \infty$, and $u(x) \sim u([x])$, where $[\cdot]$ denotes the integer part of x, then for $\lambda \in (1,\infty)$ we define

$$w_\lambda(x) := u^{\leftarrow}(u(x)/\lambda),$$

where $u^{\leftarrow}$ denotes the inverse function of u. In this case, the sequence s_n has *Voronoi moving average* s, written $s_n \to s$ $(\mathcal{V}, p_n, q_n, u_n, \lambda)$, if

$$c_n := \frac{1}{u(n)} \sum_{w_\lambda(n) < k \leq n} (p \circ qs)_k \to (1 - \lambda^{-1})s \quad (n \to \infty). \tag{1.3}$$

We write $s_n \to s$ $(\mathcal{V}_x, p_n, q_n, u(x), \lambda)$ if the limit is taken through a continuous variable. Two known special cases of this method are:

(δ) the *deferred Cesàro mean* $(D, n/\lambda, n)$ of Agnew [1], which is the $(\mathcal{V}, 1, 1, n, \lambda)$ average;

(μ) the *logarithmic moving average* $\mathcal{L}(\lambda)$ of [8], which is the $(\mathcal{V}, 1, 1/(1+n), \log n, \lambda)$ average.

The next section states our results on the properties of the introduced methods, the relations between them, and a law of large numbers. In § 3 we give the proofs, and conclude with some further remarks in the last section.

2. Results

We begin with some necessary and sufficient conditions for the sequence s_n to have a Voronoi mean. Recall $v_n := u_n - u_{n-1}$.

Theorem 1. *Let u_n be a positive and non-decresing sequence such that $u_n \to \infty$ as $n \to \infty$. We have $s_n \to s$ (V, p_n, q_n, u_n) if and only if*

$$(p \circ qs)_n = v_n a_n + b_n, \tag{2.1}$$

where $a_n \to s$ as $n \to \infty$ and $\sum_{n=0}^{\infty} b_n/u_n$ converges.

This is a generalisation of Theorem 6.5 of Bingham and Goldie [10], which was established for the Cesàro mean $(C, 1)$. In [9], we obtain an analogous result for integrals.

The following is a *limitation theorem* for the Voronoi means, and is a generalisation of Theorem 13 of Hardy [23] for the $(\overline{N}, q_n)$ mean.

Theorem 2. *Let $u_{n-1}/u_n = O(1)$. If $s_n \to s$ (V, p_n, q_n, u_n), then*

$$(p \circ qs)_n = sv_n + o(u_n).$$

The ordinary convergence $s_n \to s$ as $n \to \infty$, written $s_n \to s$ (Ω), *always* implies the summability of s_n by a regular method. This is no longer the case if the summability method is non-regular. Our next result gives some necessary and sufficient conditions for $(\Omega) \Rightarrow (V, p_n, q_n, u_n)$. We also give conditions for the converse implication $(V, p_n, q_n, u_n) \Rightarrow (\Omega)$; this is a generalisation of Theorem 2.1 of Móritz and Stadtmüller [46], which was established for $(\overline{N}, q_n) \Rightarrow (\Omega)$.

Let $m_0 := q_0$, and

$$m_n := q_n - q_{n-1}, \quad n \geq 1.$$

If p_n is positive, non-increasing, and $\sum_{n=0}^{\infty} p_n = \infty$, then it can be shown that there exists a real sequence $\{h_n\}_{n=0}^{\infty}$ such that

$$q_n s_n = \sum_{k=0}^{n} h_{n-k} u_k t_k, \quad n \geq 0 \tag{2.2}$$

(see, for example, Kuttner [37], Ishiguro [30]). Following Móritz and Stadtmüller [46], we write

$$\begin{aligned}
U_q &:= \left\{\alpha : \mathbb{N}_0 \to \mathbb{N}_0 \mid \lim_{n\to\infty} \alpha(n) \to \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{q_{\alpha(n)}}{q_n} > 1\right\}, \\
L_q &:= \left\{\beta : \mathbb{N}_0 \to \mathbb{N}_0 \mid \lim_{n\to\infty} \beta(n) \to \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{q_n}{q_{\beta(n)}} > 1\right\}, \\
U_u &:= \left\{\gamma : \mathbb{N}_0 \to \mathbb{N}_0 \mid \lim_{n\to\infty} \gamma(n) \to \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{u_{\gamma(n)}}{u_n} > 1\right\}, \\
L_u &:= \left\{\theta : \mathbb{N}_0 \to \mathbb{N}_0 \mid \lim_{n\to\infty} \theta(n) \to \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{u_n}{u_{\theta(n)}} > 1\right\},
\end{aligned}$$

where $\mathbb{N}_0$ denotes the set of non-negative integers.

Theorem 3. *(i) Let $s_n \to s$ (Ω). Also let (2.2) hold for some sequence h_n, and U_q and L_q be non-empty. Then the necessary and sufficient conditions for $s_n \to s$ (V, p_n, q_n, u_n) are:*

$$\sup_{\alpha\in U_q} \liminf_{n\to\infty} \frac{1}{q_{\alpha(n)} - q_n} \sum_{k=n+1}^{\alpha(n)} [(h \circ ut)_k - m_k t_n] \ \geq\ 0, \tag{2.3}$$

$$\sup_{\beta\in L_q} \liminf_{n\to\infty} \frac{1}{q_n - q_{\beta(n)}} \sum_{k=\beta(n)+1}^{n} [m_k t_n - (h \circ ut)_k] \ \geq\ 0. \tag{2.4}$$

(ii) Let $s_n \to s$ (V, p_n, q_n, u_n). Also let U_u and L_u be non-empty. Then the necessary and sufficient conditions for $s_n \to s$ (Ω) are:

$$\sup_{\gamma\in U_u} \liminf_{n\to\infty} \frac{1}{u_{\gamma(n)} - u_n} \sum_{k=n+1}^{\gamma(n)} [(p \circ qs)_k - v_k s_n] \ \geq\ 0, \tag{2.5}$$

$$\sup_{\theta\in L_u} \liminf_{n\to\infty} \frac{1}{u_n - u_{\theta(n)}} \sum_{k=\theta(n)+1}^{n} [v_k s_n - (p \circ qs)_k] \ \geq\ 0. \tag{2.6}$$

We refer to the *Tauberian conditions* (2.3)-(2.6) as (TCO), and they are *best–possible* for the following equivalence.

Corollary 1. *Let (2.2) hold for some sequence h_n, and U_q, L_q, U_u, L_u, be nonempty. If (TCO) holds, then $(\Omega) \Leftrightarrow (V, p_n, q_n, u_n)$.*

There are many *inclusion* and *equivalence* theorems for (N, p_n), $(\overline{N}, q_n)$, (N, p_n, q_n), and various special cases thereof (see, for example, [15], [18], [23], [29], [30], [33], [34], [38], [39], [48], [49], [50], [51], [54]). Of course, all such results apply to the appropriately specialised Voronoi means. The well-known result *Kronecker's lemma* (see, for example, page 129 of [35]) is an inclusion theorem for Voronoi means:

Theorem K. *Let $\{g_n\}_{n=0}^{\infty}$ be any sequence of non-decreasing positive numbers such that $g_n \to \infty$. If $s_n \to s$ $(V, 1, q_n, 1)$, then $s_n \to 0$ $(V, 1, q_n g_n, g_n)$.*

The following is another inclusion result where the summation to s by one method implies summation to 0 by another method.

Theorem 4. *Let $\{u_n, q_n, \tilde{u}_n, \tilde{q_n}\}_{n=0}^{\infty}$ be positive sequences such that: $u_{n+1}/u_n \to 1$ as $n \to \infty$; $\tilde{u}_n \to \infty$ as $n \to \infty$; and $u_n/q_n \to 1$ $(V, 1, \tilde{q}_n, \tilde{u}_n)$. If $s_n \to s$ $(V, 1, q_n, u_n)$, then $s_n \to 0$ $(V, 1, \tilde{q}_n, \tilde{u}_n)$.*

We now generalise the results of [8], which were established for the logarithmic mean ℓ. Let Λ denote the set of all functions $u : [0, \infty) \to [0, \infty)$ that are continuous, strictly increasing, $u(0) = 0$, $u(x) \to \infty$ as $x \to \infty$, and $u(x) \sim u([x])$. If we write $u_n \in \Lambda$, then we mean $u_n = u(n)$, $n \geq 1$, and $u \in \Lambda$.

Theorem 5. *If $u_n \in \Lambda$, then*

$$(V, p_n, q_n, u_n) \Leftrightarrow (\mathcal{V}, p_n, q_n, u_n, \lambda) \quad \textit{for some (all) } \lambda \in (1, \infty). \tag{2.7}$$

Theorem 6. *Let $u_n \in \Lambda$. If (1.3) holds for all $\lambda \in (1, \infty)$, then it holds uniformly on compact λ-sets of $(1, \infty)$.*

Theorem 7. *If $u \in \Lambda$, then*

$$s_n \to s \quad (V_x, p_n, q_n, u(x)) \quad \Leftrightarrow \quad s_n \to s \quad (\mathcal{V}_x, p_n, q_n, u(x), \lambda) \quad \forall \lambda > 1.$$

Theorem 8. *If $u \in \Lambda$ and*

$$U(x) := \sum_{0 \leq k \leq x} (p \circ qs)_k, \tag{2.8}$$

then the following statements are equivalent:

(i) $U(x) = U_1(x) - U_2(x)$, *with* $U_1(x)$ *satisfying*

$$\lim_{x\to\infty} \frac{U_1(x) - U_1(w_\lambda(x))}{u(x)} = s(1 - \lambda^{-1}), \quad \forall \lambda > 1,$$

and $U_2(x)$ *non-decreasing,*

(ii)

$$\liminf_{\alpha\downarrow 1} \limsup_{x\to\infty} \sup_{\lambda\in[1,\alpha]} \frac{U(x) - U(w_\lambda(x))}{u(x)} < \infty.$$

Corollary 2. *If* $u \in \Lambda$, *then*

$$(V, p_n, q_n, u(n)) \Leftrightarrow (V_x, p_n, q_n, u(x)).$$

The next theorem establishes relations between Voronoi means and Voronoi power series. Some statements require the notions of *slowly* and *regularly* varying functions, for which see [13]. Also note that the notation $f = O_L(\varphi)$ means that f/φ is bounded below.

Theorem 9. *(i) Let* $v_n > 0$; $u_n \to \infty$ *as* $n \to \infty$; *and* $R_u \in (0, \infty)$. *If* $s_n \to s$ (V, p_n, q_n, u_n), *then* $s_n \to s$ $(\mathcal{P}, p_n, q_n, v_n)$.

(ii) Let $\rho \geq -1$; v_n *be regularly varying of index* ρ; *and* $u_n \to \infty$. *If* $(p \circ sq)_n/v_n = O_L(1)$, *then* $s_n \to s$ $(\mathcal{P}, p_n, q_n, v_n)$ *implies* $s_n \to s$ (V, p_n, q_n, u_n).

(iii) Let $U(x) \geq 0$; $c \geq 0$; $\rho > -1$; $\hat{U}(s) := s\int_0^\infty e^{-sx}U(x)dx$ *converge for* $s > 0$; $\ell(x)$ *be a given slowly varying function;*

$$u(x) := \sum_{0\leq k\leq x} v_k;$$

$u(x) > 0$; $R_u = 1$; *and* $u(x) \sim x^\rho \ell(x)/\Gamma(1+\rho)$ $(x \to \infty)$. *If* $s_n \to c$ $(V_x, p_n, q_n, u(x))$, *then* $s_n \to c$ $(\mathcal{P}, p_n, q_n, D_u(x))$. *Conversely,* $s_n \to c$ $(\mathcal{P}, p_n, q_n, D_u(x))$ *implies* $s_n \to c$ $(V_x, p_n, q_n, u(x))$ *if and only if*

$$\lim_{\lambda\downarrow 1} \liminf_{x\to\infty} \inf_{t\in[1,\lambda]} \frac{1}{x^\rho \ell(x)} \sum_{x<k\leq tx} (p \circ qs)_k \geq 0.$$

(iv) Let $\rho \geq -1$ *and* $\rho \neq 0, 1, ...$; v_n *be regularly varying of index* ρ*; and* $u_n \to \infty$*. If*

$$(p \circ sq)_n/v_n - (p \circ sq)_{n-1}/v_{n-1} = O_L(v_n/u_n),$$

then $s_n \to s$ $(\mathcal{P}, p_n, q_n, v_n)$ *implies* $(p \circ sq)_n/v_n \to s$.

(v) Let $v_n > 0$; $v_n = O(1/n)$, *and* $u_n \to \infty$*. If*

$$(p \circ sq)_n/v_n - (p \circ sq)_{n-1}/v_{n-1} = o(v_n/u_n),$$

then $s_n \to s$ $(\mathcal{P}, p_n, q_n, v_n)$ *implies* $(p \circ sq)_n/v_n \to s$.

Part (i) is an *Abelian* result, and it is a generalization of several known special cases (see, for example, [23], [47]). Part (ii) is Tauberian and is a generalisation of Theorem 4.1 of [32] established for the J–method (see, for example, [23]) and $(\overline{N}, p_n)$. One can extend other closely related Tauberian results, such as those in [36], in a similar way. Part (iii) contains a Tauberian result of best–possible character, and it is a specialization of the Hardy–Littlewood-Karamata theorem for the Laplace–Stieltjes transform. Similar results for Abel and L methods of summation appear in [3] and [8], respectively. Parts (iv) and (v) are a certain generalisation of Theorem 5.3 of [32] and Theorem 1 of [28], respectively, which were established for the J–method and convergence of s_n. Other results of this nature (see, for example, [32], [27], [36]) can be extended similarly.

In [31], Jajte introduced a law of large numbers for the $(V, 1, q_n, u_n)$ summability method. We extend his result by including equivalence relations with other summability methods. Moreover, we generalise the results of [8] on the LLN of Baum–Katz type, which were obtained for the logarithmic mean ℓ.

In Theorem 10 below, we encounter infinite families of summability methods which, while by no means equivalent, *become equivalent in the LLN context*, to the same moment condition. This interesting phenomenon goes back to Chow [16] in 1973 (Euler methods; finite variance) and Lai [40] in 1974 (Cesàro means (C, α), $\alpha \geq 1$; finite mean), and has been developed by, e.g. the first author ([4], [6], [7]).

Let $\phi : [0, \infty) \to [0, \infty)$ be such that:

(i) $\phi(0) = 0$,

(ii) $\phi(x)$ is left-continuous and strictly increasing,

(iii) $\phi(x+1)/\phi(x) \le c$ for some constant $c > 0$,

(iv) for some positive constants a and b it holds that

$$\phi^2(s) \int_s^\infty \frac{dx}{\phi(x)^2} \le as + b, \quad s > 0.$$

Theorem 10. *Let $X, X_1, X_2, ...,$ be a sequence of independent and identically distributed (i.i.d.) random variables, and*

$$m_k := \mathbb{E}[X_k \mathbb{1}_{\{|X_k| \le \phi(k)\}}].$$

Also let $q_0 = 0$.

(a) Let the sets $\Phi_V(\phi)$ and $\widetilde{\Phi}_V(\phi)$ be defined as:

$$\Phi_V(\phi) \; := \; \{(u_n, q_n) : \quad u_n > 0 \text{ and non-decreasing; } q_n > 0\text{; and } u_n/q_n = \phi(n)\},$$

$$\widetilde{\Phi}_V(\phi) \; := \; \{(u_n, q_n) \in \Phi_V(\phi) : v_n \ge \sigma > 0 \text{ and } v_{n+1}v_{n-1} \ge v_n^2\}.$$

The following four statements are equivalent:

(i) $\mathbb{E}\left[\phi^{\leftarrow}(|X|)\right] < \infty$,

(ii) $X_n/\phi(n) \to 0$ *almost surely (a.s.) as* $n \to \infty$,

(iii) $(X_n - m_n) \to 0$ *a.s.* $(V, 1, q_n, u_n)$ *for some (all)* $(u_n, q_n) \in \Phi_V(\phi)$,

(iv) $(X_n - m_n) \to 0$ *a.s.* (V, v_n, q_n, u_n) *for some (all)* $(u_n, v_n q_n) \in \widetilde{\Phi}_V(\phi)$.

(b) Let $\Phi_{uq}(\phi)$ denote the set of pairs (u_n, q_n) such that $u_n \to \infty$ as $n \to \infty$, and there exists a function $h_{uq} : (0, \infty) \to (0, \infty)$ such that:

$$\lim_{x\to\infty} h_{uq}(x) = R_u-, \quad \text{and} \quad D_u(h_{uq}(n))/(q_n h_{uq}^n(n)) \sim \phi(n).$$

If $\phi^{\leftarrow}$ is subadditive, then the following two statements are equivalent:

(i) $\mathbb{E}\left[\phi^{\leftarrow}(|X|)\right] < \infty$,

(ii) $(X_n - m_n) \to 0$ *a.s.* $(\mathcal{P}, 1, q_n, v_n)$ *for some (all)* $(u_n, q_n) \in \Phi_V(\phi) \cap \Phi_{uq}(\phi)$.

(c) Let $\Phi_D(\phi) := \{(u_n, q_n) \in \Phi_V(\phi) : \quad u_n \in \Lambda\}$. *The following two statements are equivalent:*

(i) $\mathbb{E}\left[\phi^{\leftarrow}(|X|)\right] < \infty$,

(ii) $(X_n - m_n) \to 0$ *a.s.* $(\mathcal{V}, 1, q_n, u_n, \lambda)$ *for some (all)* $(u_n, q_n, \lambda) \in \Phi_D(\phi) \times (1, \infty)$.

(d) If $\phi \in \Lambda$ *is regularly varying of index* $\rho > 0$, *then the following three statements are equivalent:*

(i) $\mathbb{E}\left[\phi^{\leftarrow}(|X|)\right] < \infty$,

(ii) $\sum_1^\infty n^{-1}\mathbb{P}[|\sum_{1\le i\le n}(X_i - m_{i+n/(\gamma-1)})| > \phi(n/(\gamma-1))\epsilon] < \infty$ $\forall\epsilon > 0$ *and* $\forall\gamma > 1$,

(iii) $\sum_1^\infty n^{-1}\mathbb{P}[\max_{1\le k\le n}|\sum_{1\le i\le k}(X_i - m_{i+n/(\gamma-1)})| > \phi(n/(\gamma-1))\epsilon] < \infty$ $\forall\epsilon > 0$ *and* $\forall\gamma > 1$.

Corollary 3. *Let* Φ_U *denote the set of sequences* u_n *that are positive, non-decreasing,* $u_n \to \infty$ *as* $n \to \infty$, *and* $R_u = 1$. *Also let* $q_0 = 0$. *If* ϕ *is subadditive, then the following statements are equivalent:*

(i) $\mathbb{E}\left[\phi^{\leftarrow}(|X|)\right] < \infty$,

(ii) $(X_n - m_n) \to 0$ *a.s.* $(\mathcal{P}, 1, u_n/\phi(n), v_n)$ *for some (all)* $u_n \in \Phi_U$.

3. Proofs

Proof of Theorem 1. (*Sufficiency*) Let $\sum_{n=0}^{\infty} b_n/u_n$ converge. Then, by Theorem K above:

$$\frac{1}{u_n}\sum_{k=0}^{n} b_k \to 0 \quad (n \to \infty).$$

If $a_n \to s$ as $n \to \infty$ and (2.1) holds, then

$$\frac{1}{u_n}\sum_{k=0}^{n}(p \circ qs)_k = \frac{1}{u_n}\sum_{k=0}^{n}(v_k a_k + b_k) \to s \quad (n \to \infty).$$

(*Necessity*) Let $s_n \to s\ (V, p_n, q_n, u_n)$. From (1.2) we have:

$$(p \circ qs)_n = t_n u_n - t_{n-1}u_{n-1} = v_n t_{n-1} + u_n(t_n - t_{n-1}). \tag{3.1}$$

If $a_n := t_{n-1}$ and $b_n := u_n(t_n - t_{n-1})$, then (3.1) is the required decomposition of $(p \circ qs)_n$, since $a_n \to s$ and $\sum_{n=0}^{\infty} b_n/u_n$ converges. $\square$

Proof of Theorem 2. Let $s_n \to s\ (V, p_n, q_n, u_n)$. From (1.2) we have:

$$\begin{aligned}(p \circ qs)_n &= t_n u_n - t_{n-1}u_{n-1}\\ &= s(u_n - u_{n-1}) + (t_n - s)u_n - (t_{n-1} - s)u_{n-1} = sv_n + o(u_n).\end{aligned}$$

$\square$

Proof of Theorem 3. We adapt the approach of [46], and prove part (i) only, as the proof of part (ii) follows the same steps.
(*Necessity*) Let $s_n \to s\ (V, p_n, q_n, u_n)$. For any $\alpha \in U_q$ we have:

$$\begin{aligned}\tau_n &:= \frac{1}{q_{\alpha(n)} - q_n}\sum_{k=n+1}^{\alpha(n)}(h \circ ut)_k = \frac{q_{\alpha(n)}s_{\alpha(n)} - q_n s_n}{q_{\alpha(n)} - q_n}\\ &= s_{\alpha(n)} + \frac{1}{\frac{q_{\alpha(n)}}{q_n} - 1}(s_{\alpha(n)} - s_n) \to s \quad (n \to \infty).\end{aligned}$$

It now follows that condition (2.3) must hold:

$$\lim_{n\to\infty}\frac{1}{q_{\alpha(n)} - q_n}\sum_{k=n+1}^{\alpha(n)}[(h \circ ut)_k - m_k t_n] = \lim_{n\to\infty}(\tau_n - t_n) = 0.$$

Similarly, for any $\beta \in L_q$ we have:

$$\begin{aligned}\rho_n &:= \frac{1}{q_n - q_{\beta(n)}} \sum_{k=\beta(n)+1}^{n} (h \circ ut)_k = \frac{q_n s_n - q_{\beta(n)} s_{\beta(n)}}{q_n - q_{\beta(n)}} \\ &= s_n + \frac{1}{\frac{q_n}{q_{\beta(n)}} - 1}(s_n - s_{\beta(n)}) \to s \quad (n \to \infty).\end{aligned}$$

It now follows that condition (2.4) must hold:

$$\lim_{n\to\infty} \frac{1}{q_n - q_{\beta(n)}} \sum_{k=\beta(n)+1}^{n} [m_k t_n - (h \circ ut)_k] = \lim_{n\to\infty} (t_n - \rho_n) = 0.$$

(*Sufficiency*) Let the conditions (2.3) and (2.4) hold. For $\varepsilon > 0$, there exists $\alpha \in U_q$ and $\beta \in L_q$ such that:

$$\begin{aligned}-\varepsilon &\leq \liminf_{n\to\infty} \frac{1}{q_{\alpha(n)} - q_n} \sum_{k=n+1}^{\alpha(n)} [(h \circ ut)_k - m_k t_n] \\ &= \liminf_{n\to\infty} (\tau_n - t_n) = s - \limsup_{n\to\infty} t_n, \\ -\varepsilon &\leq \liminf_{n\to\infty} \frac{1}{q_n - q_{\beta(n)}} \sum_{k=\beta(n)+1}^{n} [m_k t_n - (h \circ ut)_k] \\ &= \liminf_{n\to\infty} (t_n - \rho_n) = \liminf_{n\to\infty} t_n - s,\end{aligned}$$

which together imply $t_n \to s$ as $n \to \infty$. $\square$

Proof of Theorem 4. Let $s_n \to s$ $(V, 1, q_n, u_n)$, i.e.

$$t_n = \frac{1}{u_n} \sum_{k=0}^{n} s_k q_k \to s \quad (n \to \infty).$$

We can express the sequence s_n in terms of t_n as:

$$s_0 = \frac{u_0}{q_0} t_0, \quad s_n = \frac{u_n}{q_n}(t_n - u_{n-1} t_{n-1}/u_n) \quad \text{for } n \geq 1.$$

As $u_{n+1}/u_n \to 1$, we have $\hat{t}_n := t_n - u_{n-1}t_{n-1}/u_n \to 0$ as $n \to \infty$. The sequence

$$\tilde{t}_n := \frac{1}{\tilde{u}_n} \sum_{k=0}^{n} s_k \tilde{q}_k = \frac{1}{\tilde{u}_n} \sum_{k=0}^{n} \frac{u_k}{q_k} \tilde{q}_k \hat{t}_k$$

is a linear transformation of the converging sequence $\hat{t}_n$. Moreover, due to assumptions $\tilde{u}_n \to \infty$ as $n \to \infty$, and $u_n/q_n \to 1$ $(V, 1, \tilde{q}_n, \tilde{u}_n)$, it is a regular transformation, and hence the conclusion. $\square$

Proof of Theorem 5. Here we follow closely the approach of [8]. To prove $(V, p_n, q_n, u_n) \Rightarrow (\mathcal{V}, p_n, q_n, u_n, \lambda)$, let $t_n \to s$ (Ω). It is clear from (1.3) that for $n \geq 1$ we have:

$$c_n = t_n - \frac{u_{[w_\lambda(n)]}}{u_n} t_{[w_\lambda(n)]}. \tag{3.2}$$

Thus, the sequence c_n is a *transformation* of the sequence t_n. For each n, the only nonzero coefficients of such a transformation are 1 and $u_{[w_\lambda(n)]}/u_n$. The sum of their absolute values is finite for each n, they shift with n, and their sum tends to $1 - \lambda^{-1}$ as $n \to \infty$. Hence it is a regular transformation.

To prove $(V, p_n, q_n, u_n) \Leftarrow (\mathcal{V}, p_n, q_n, u_n, \lambda)$, let $c_n \to s$ (Ω). From (3.2) it is clear that for each $n \geq 1$ we can write t_n as the following *finite* sum:

$$\begin{aligned} t_n &= c_n + \frac{u_{[w_\lambda(n)]}}{u_n} t_{[w_\lambda(n)]} \\ &= c_n + \frac{u_{[w_\lambda(n)]}}{u_n} \left[c_{[w_\lambda(n)]} + \frac{u_{[w_\lambda([w_\lambda(n)])]}}{u_{[w_\lambda(n)]}} t_{[w_\lambda([w_\lambda(n)])]} \right] \\ &= c_n + \frac{u_{[w_\lambda(n)]}}{u_n} c_{[w_\lambda(n)]} + \frac{u_{[w_\lambda([w_\lambda(n)])]}}{u_n} c_{[w_\lambda([w_\lambda(n)])]} + \ldots. \end{aligned}$$

Thus, the sequence t_n can be seen as a transformation of the sequence c_n with a finite number of nonzero terms. Since these coefficients are either zero or tend to zero with n, and their sum as $n \to \infty$ is $1 + \lambda^{-1} + \lambda^{-2} + \ldots = (1 - \lambda^{-1})^{-1}$, we conclude that it is a regular transformation. $\square$

Proof of Theorem 6. Let (1.3) hold for all $\lambda > 1$. With U defined in (2.8), we can write (1.3) as

$$\frac{U(n) - U(w_\lambda(n))}{u_n} \to (1 - \lambda^{-1}) s \quad (n \to \infty), \quad \forall \lambda > 1, \tag{3.3}$$

which holds also for $\lambda = 1$. Define $\alpha_n := \lambda^{-1} u_n$ and $\widetilde{U}(x) := U(u^{\leftarrow}(x))$, and rewrite (3.3) as

$$\frac{\widetilde{U}(\lambda\alpha_n) - \widetilde{U}(\alpha_n)}{\alpha_n} \to (\lambda - 1)s \quad (\alpha_n \to \infty), \quad \forall \lambda \geq 1. \tag{3.4}$$

Since the linear function x is regularly varying of index 1, the function $\widetilde{U}$ belongs to the de Haan class Π_1 (see Chapter 3 of [13]). Hence the proof of the local uniformity follows from the proof of Theorem 3.1.16 of [13] by using α_n instead of a continuous variable. □

Proof of Theorem 7. From the previous proof it is clear that $s_n \to s$ $(\mathcal{V}_x, p_n, q_n, u(x), \lambda)$ means

$$\frac{\widetilde{U}(\lambda y) - \widetilde{U}(y)}{y} \to (\lambda - 1)s \quad (y \to \infty), \quad \forall \lambda \geq 1, \tag{3.5}$$

where $y := \lambda^{-1} u(x)$. From Theorem 3.2.7 of [13] it now follows that (3.5) holds if and only if $s_n \to s$ $(V_x, p_n, q_n, u(x))$. □

Proof of Theorem 8. This follows from (3.5) and Theorem 3.8.4 of [13]. □

Proof of Corollary 2. From Theorem 7, Theorem 6, and Theorem 5, respectively, it follows that:

$$(V_x, p_n, q_n, u(x)) \Leftrightarrow (\mathcal{V}_x, p_n, q_n, u(x), \lambda) \Leftrightarrow (\mathcal{V}, p_n, q_n, u_n, \lambda) \Leftrightarrow (V, p_n, q_n, u_n).$$

□

Proof of Theorem 9. The following two *equivalence* relations are evident from the definitions of Voronoi mean and Voronoi power series:

$$s_n \to s \quad (V, p_n, q_n, u_n) \quad \Leftrightarrow \quad (p \circ sq)_n / v_n \to s \quad (V, 1, v_n, u_n), \tag{3.6}$$

$$s_n \to s \quad (\mathcal{P}, p_n, q_n, v_n) \quad \Leftrightarrow \quad (p \circ sq)_n / v_n \to s \quad (\mathcal{P}, 1, v_n, v_n). \tag{3.7}$$

(i) Here we follow closely [27] and [47]. The Voronoi power series can be written as

$$\frac{(1-x)\sum_{n=0}^{\infty} u_n t_n x^n}{(1-x)\sum_{n=0}^{\infty} u_n x^n} = \frac{\sum_{n=0}^{\infty} u_n t_n R_u^n (x/R_u)^n}{\sum_{n=0}^{\infty} u_n R_u^n (x/R_u)^n}. \tag{3.8}$$

If $s_n \to s$ (V, p_n, q_n, u_n), then from Theorem 57 of [23] it follows that (3.8) converges to s as $x \to R_u-$.

(ii) Since v_n is regularly varying, it follows that $R_u = 1$. From Theorem 4.1 of [32] we have that $(p \circ sq)_n / v_n \to s \quad (\mathcal{P}, 1, v_n, v_n)$ implies $(p \circ sq)_n / v_n \to s \quad (V, 1, v_n, u_n)$. The conclusion now follows from (3.7) and (3.6).

(iii) Here we apply Theorem 1.7.6 of [13] *twice*. Note that:

$$D_u(e^{-s}) = s \int_0^\infty u(x) e^{-sx} dx.$$

Under our assumptions on $u(x)$, it follows from Theorem 1.7.6 of [13] that $D_u(x) \sim \ell(-1/\log x)/(-\log x)^\rho$ as $x \to 1-$. The conclusions are now immediate from Theorem 1.7.6 of [13] and the fact that $\hat{U}(s) = \sum_{n=0}^\infty (p \circ qs)_n e^{-ns}$.

(iv) Under the stated assumptions, we have that $R_u = 1$ and from Theorem 5.3 of [32] we have that $(p \circ sq)_n / v_n \to s \quad (\mathcal{P}, 1, v_n, v_n)$ implies $(p \circ sq)_n / v_n \to s$. The conclusion now follows from (3.7).

(v) Under the stated assumptions, we have that $R_u = 1$ and from Theorem 1 of [28] we have that $(p \circ sq)_n / v_n \to s \quad (\mathcal{P}, 1, v_n, v_n)$ implies $(p \circ sq)_n / v_n \to s$. The conclusion now follows from (3.7). □

Proof of Theorem 10. (a) The equivalence $(i) \Leftrightarrow (ii)$ is implicit in the proof of the Theorem in [31], whereas $(i) \Leftrightarrow (iii)$ follows from that theorem. The equivalence $(iii) \Leftrightarrow (iv)$ follows from Theorem 3 of [30], from which we know that

$$(V, v_n, q_n, u_n) \Leftrightarrow (V, 1, v_n q_n, u_n).$$

(b) Here we closely follow [8]. From part (a) and the Abelian result of Theorem 9 (i), we have that $(i) \Rightarrow (V, 1, q_n, u_n) \Rightarrow (ii)$. To prove the opposite, note that (ii) implies:

$$\frac{1}{D_u(h_{uq}(m))} \sum_{k=1}^\infty X_k^s q_k h_{uq}^k(m) = 0 \quad (m \to \infty) \quad a.s.,$$

where $X_k^s = X_k - X_k'$, and $\{X_n\}_{n=1}^\infty$ and $\{X_n'\}_{n=1}^\infty$ are i.i.d.. We define

$$\widetilde{X}_m := \frac{1}{D_u(h_{uq}(m))} \sum_{k=1}^m X_k^s q_k h_{uq}^k(m), \quad \widehat{X}_m := \frac{1}{D_u(h_{uq}(m))} \sum_{k=m+1}^\infty X_k^s q_k h_{uq}^k(m).$$

Then $\widetilde{X}_m + \widehat{X}_m \to 0$ a.s., so in probability. As they are independent and symmetric, from the Lévy inequality (Lemma 2 in V.5 of [20]), $\widehat{X}_m \to 0$ in probability. Since $(\widetilde{X}_1, ..., \widetilde{X}_m)$ and $\widehat{X}_m$ are independent, Lemma 3 of [17] gives $\widetilde{X}_m \to 0$, *a.s.*. Repeating the same argument for

$$\widetilde{X}_m = \frac{1}{D_u(h_{uq}(m))} \sum_{k=1}^{m-1} X_k^s q_k h_{uq}^k(m) + \frac{1}{D_u(h_{uq}(m))} X_m^s q_m h_{uq}^m(m),$$

gives $X_m^s/\phi(m) \to 0$ $(m \to \infty)$ *a.s.*. By the Borel-Cantelli lemma, and the weak symmetrisation inequalities (pp. 257 of [44]),

$$\frac{1}{2}\sum_{k=1}^{\infty} \mathbb{P}\left[\phi^{\leftarrow}(|X-\mu_x|) \geq k\right] = \frac{1}{2}\sum_{k=1}^{\infty} \mathbb{P}\left[|X-\mu_x| \geq \phi(k)\right]$$
$$\leq \sum_{k=1}^{\infty} \mathbb{P}\left[|X^s| \geq \phi(k)\right] < \infty,$$

with μ_x the median of X, and $X^s = X - X'$, with X and X' i.i.d. Since $\phi^{\leftarrow}$ is assumed subadditive, we finally obtain:

$$\mathbb{E}\left[\phi^{\leftarrow}(|X|)\right] \leq \mathbb{E}\left[\phi^{\leftarrow}(|X-\mu_x| + |\mu_x|)\right] \leq \phi^{\leftarrow}(|\mu_x|) + \mathbb{E}\left[\phi^{\leftarrow}(|X-\mu_x|)\right] < \infty.$$

(c) This follows immediately from part (a) and Theorem 5.

(d) Part (a) and Corollary 2 show that (i) is equivalent with

$$\frac{1}{\phi(x)} \sum_{0<i\leq x} (X_i - m_i) \to 0 \quad a.s. \quad (x \to \infty).$$

By Theorem 3.2.7 of [13] this is equivalent to

$$\frac{1}{\phi(x)} \sum_{x<i\leq \gamma x} (X_i - m_i) \to 0 \quad a.s. \quad (x \to \infty) \quad \forall \gamma > 1.$$

The remainder of the proof proceeds identically to that on page 1787 of [8], and is thus omitted. □

Proof of Corollary 3. We have $(u_n, u_n/\phi(n)) \in \Phi_V(\phi)$. Let $g : (0,\infty) \to (0,\infty)$ be any function such that $\lim_{x\to\infty} g(x) = 1-$. Since $D_u(g(n))/g^n(n) \sim u_n$, we also have $(u_n, u_n/\phi(n)) \in \Phi_{uq}(\phi)$. The conclusion now follows from Theorem 10 (b). □

4. Further remarks

We give a brief account of the non-regular summability methods that appear in probability theory, analysis, and number theory.

4.1. LLN

As already mentioned in the introduction, the Chow-Lai laws of large numbers (LLNs) in [17] are not regular. Further results of the same kind were also given by Li et al. [43] (double sequences of random variables). Similarly, the Marcinkiewicz-Zygmund LLN ([45]; [22] §6.7; [5] §3) gives a non-regular summability method for L_p $(0 < p < 2)$ when $p \neq 1$ (that is except, in the Kolmogorov case), as Jajte [31] remarks. Generalising this, Jajte [31] introduces his methods, which include both regular (e.g. Cesàro and logarithmic) and also non-regular methods.

Many extensions of the Kolmogorov strong LLN (SLLN) are known, in which a.s. convergence under a summability method is tied to a moment condition – see e.g. [12], [7], [8] – but here the methods are regular. The main results not included here are the Marcinkiewicz-Zygmund law (above) and the Baum-Katz law ([2], [22] §6.11,12). This has been extensively developed by Lai [41], who introduced the idea of *r-quick convergence* (see also [11]). This is essentially probabilistic, and gives, not a summability method as such, but a convergence concept giving a probabilistic analogue of a summability method – again non-regular.

4.2. Analysis

By a theorem of Leja [42], any regular Nörlund mean sums a power series at at most countably many points outside its circle of convergence. This was extended by K. Stadtmüller to non-regular Nörlund means; her result was developed further with Grosse-Erdmann [21].

Further examples of non-regular summability methods useful in analysis arise in the theory of Fourier series. With $s_n := \sum_{k=0}^{n} a_k$, write

$$\sum a_n = s \quad \text{or} \quad s_n \to s \quad (R, 1) \quad \text{for} \quad \sum_{1}^{\infty} a_n \frac{\sin nh}{nh} \to s \quad (h \downarrow 0),$$

$$\sum a_n = s \quad \text{or} \quad s_n \to s \quad (R_1) \quad \text{for} \quad \frac{2}{\pi} \sum_{1}^{\infty} s_n \frac{\sin nh}{nh} \to s \quad (h \downarrow 0),$$

Neither method is regular, and the two are not comparable. But $(R, 1)$ is *Fourier effective* – sums the Fourier series of any f to f a.e. – which (R_1) is not: there are Fourier series summable (R_1) nowhere [26].

The R here is for the Riemann, and there are Riemann methods of higher order. If one replaces $\sin nh/(nh)$ by its square, one obtains $(R, 2)$, and similarly for (R_2); these methods are regular [25]. These methods reduce to Abel and Cesàro methods; see [23] App. III, §12.16.

4.3. Number theory

The *Ingham summability method I* is defined by saying that

$$s_n \to s \quad (I) \quad \text{if} \quad \frac{1}{x}\sum_{n \le x} n s_n [x/n] \to s \quad (x \to \infty).$$

This method is not regular, but can be used, together with the Wiener-Pitt (Tauberian) theorem, to prove the Prime Number Theorem (PNT), using only the non-vanishing of ζ on the 1-line,

$$\zeta(1 + it) \neq 0 \quad (t \in \mathbb{R}). \tag{4.1}$$

The proof of this goes back to the first proof of the prime number theorem, and has always been recognized as that property of the Riemann zeta function which is most central in the proof of this theorem (Wiener [52] IV.9, [53] §17). Indeed, (4.1) was part of Wiener's motivation in creating his Tauberian theory. The PNT is proved via Ingham's method in Hardy [23] §12.11; Ingham's method is developed further in Hardy [23] App. IV. 4, Erdős and Segal [19].

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