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The reduction of framed links to ordinary ones

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The main result of this paper is the reduction of the classification of framed links to the classification of ordinary links (Theorem 1). As an application, analogues of Markov's theorem are proved for framed links and 3-manifolds. By a 3-manifold we mean an orientable closed three-dimensional manifold. It is known that every 3-manifold can be obtained by Dehn surgery from S^3 along some framed link (see [1]). The framed links obtained from each other by Fenn-Rourke transformations provide homeomorphic 3-manifolds (see [2]). In analogy with ordinary links, a framed link is the closure of some framed braid determined up to conjugation and Markov transformations (Theorem 3). The group FB_n of framed braids is canonically isomorphic to the group $B_n \leftthreetimes \mathbb{Z}^n$, where B_n is the group of (ordinary) braids (Theorem 2). Therefore, every framed braid yields a 3-manifold and these 3-manifolds are homeomorphic if and only if the braids can be obtained from each other by conjugation and Markov and Fenn-Rourke transformations. In Theorem 4 the geometric Fenn-Rourke transformations are translated into the combinatorial language of framed braids. The author is grateful to L. A. Alanii for several valuable remarks and to V. M. Bukhshtaber for editorial work and for support.

Definition 1. *A framed m-component link* is a collection of m disjoint ribbons in $\mathbb{R}^3$ linked with one another and twisted (each about its median line) an integer number of times. An isotopy of framed links is a collection of homeomorphisms $\phi_t : \mathbb{R}^3 \mapsto \mathbb{R}^3$ depending continuously on $t \in [0,1]$ and such that $\phi_0 = \mathrm{id}$ and ϕ_1 maps one framed link onto the other.

We denote by L_i and F_i the boundaries of the ith ribbon. We obtain a non-orientable link $L_1 \cup F_1 \cup \cdots \cup L_m \cup F_m$. Therefore, to every framed m-component link there corresponds an ordinary non-orientable $2m$-component link.

Theorem 1. *Two framed links are isotopic if and only if the corresponding ordinary links are isotopic.*

Definition 2. A *framed braid* on n strings is a collection of n ribbons in $\mathbb{R}^3$ such that

(2.1) the lower (upper) boundary of the i ribbon is the interval $[(2i,0,0),(2i+1,0,0)] \subset \mathbb{R}^3$ ($[(2i,0,1),(2i+1,0,1)] \subset \mathbb{R}^3$, respectively);
(2.2) every ribbon rises monotonically from the lower boundary to the upper one;
(2.3) every ribbon may be twisted an integer number of times.

By considering the boundaries of the ribbons instead of the ribbons themselves, we obtain an ordinary braid on $2n$ strings. The set FB_n of framed braids inherits the natural group structure from the ordinary braids and it is a subgroup of B_{2n}. There exists a natural action of permutations $s \in S_n$ on

$$\mathbb{Z}^n = \{\bar{f} = (f_1,\ldots,f_n),\ f_i \in \mathbb{Z}\}:\ s(f_1,\ldots,f_n) = (f_{s(1)},\ldots,f_{s(n)}).$$

We define the semi-direct product $B_n \leftthreetimes \mathbb{Z}^n$ as the set of pairs $(\alpha,\bar{f})$ with the group operation (here $s = s(\beta)^{-1}$)

$$(\alpha,\bar{f}) \circ (\beta,\bar{g}) = (\alpha\beta, s(\bar{f}) + \bar{g}).$$

We define the map $\Phi\colon B_n \leftthreetimes \mathbb{Z}^n \to FB_n$ by the formula

$$\Phi(\sigma,\bar{f}) = \Delta(\sigma)\tau_1^{2f_1}\cdots\tau_{2n-1}^{2f_n},$$

where the τ_i are generators of the group $B_{2n} \supset FB_n$, and $\Delta\colon B_n \to B_{2n}$ is the doubling homomorphism defined on the generators σ_i by the formula $\Delta(\sigma_i) = \tau_{2i}\tau_{2i-1}\tau_{2i+1}\tau_{2i}$.

Theorem 2. Φ *is an isomorphism.*

The following is a simple consequence of Markov's theorem (see [3]) on links and Theorem 1.

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Theorem 3. *Upon closure, two framed braids give isotopic framed links if and only if one braid can be obtained from the other by a finite sequence of the following transformations*:

(1) *a conjugation* $\alpha \mapsto \beta\alpha\beta^{-1}$, $\alpha, \beta \in B_n$;
(2) *a Markov transformation* $\alpha \leftrightarrow \alpha\sigma_n^{\pm 1}$, $\alpha \in B_n \subset B_{n+1}$.

Every framed link can be placed in the plane (see [4]) so that every ribbon lies almost entirely in the plane except that a few times it rises above it when it is linked to another ribbon or is twisted about itself, forming loops.

Definition 3. The *Fenn–Rourke transformations* of framed links placed in the plane are additions or removals of an unknotted ± 1-framed ribbon. The ribbons that pass through the latter (they can be placed parallel, side by side) are twisted as a whole in the form of a large loop. Such a loop can be depicted in two ways. The right way is chosen in correspondence with the framing of the added (or removed) ribbon. In particular, we can add or remove a ± 1-framed ribbon situated far from the rest of the link.

Theorem 4. *Two framed braids yield homeomorphic* 3*-manifolds if and only if one can be obtained from the other by a finite sequence of the following transformations*:

(1) *a conjugation* $\alpha \mapsto \beta\alpha\beta^{-1}$, $\alpha, \beta \in B_n$;
(2) *a Markov transformation* $\alpha \leftrightarrow \alpha\sigma_n^{\pm 1}$, $\alpha \in B_n \subset B_{n+1}$;
(3) *a Fenn–Rourke transfromation* $\sigma \leftrightarrow \sigma fr_n^{\pm}(i,j)$, $\sigma \in FB_n \subset FB_{n+1}$, *where we have used the notation*

$$fr^+(i,j) = \Phi(\sigma^+(i,j), n\text{-}tuple\ (1,\dots,1)),$$
$$fr^-(i,j) = \Phi(\sigma^-(i,j), n\text{-}tuple\ (-1,\dots,-1)),$$
$$\sigma^+(i,j) = \alpha(\sigma_n^2)(\sigma_{n-1}\sigma_n^2\sigma_{n-1})\cdots(\sigma_{n-i-j+2}\cdots\sigma_n^2\cdots\sigma_{n-i-j+2})\beta,$$
$$\sigma^-(i,j) = \alpha(\sigma_n^{-2})(\sigma_{n-1}\sigma_n^2\sigma_{n-1})\cdots(\sigma_{n-i-j+2}^{-1}\cdots\sigma_n^{-2}\cdots\sigma_{n-i-j+2}^{-1})\beta,$$
$$\alpha = (\sigma_n\cdots\sigma_{n-j+1})(\sigma_{n-j}^{-1}\cdots\sigma_{n-i-j+1}^{-1}), \quad \beta = (\sigma_{n-i-j+1}^{-1}\cdots\sigma_{n-j}^{-1})(\sigma_{n-j+1}\cdots\sigma_n).$$

We note that Markov transformations take a braid from the subgroup FB_n to the group B_{2n} of all braids. The Fenn–Rourke transformations can be applied only to framed braids in FB_n.

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