Nernst Branes from Special Geometry

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Abstract

This thesis is concerned with obtaining black brane solutions to Fayet-Iliopoulos gauged $\mathcal{N} = 2$ supergravity that obey the strong, Planckian version of the third law of black hole mechanics.

We first construct a new two-parameter family of black brane solutions to gauged $\mathcal{N} = 2$ supergravity in four dimensions using time-like dimensional reduction as a solution generating technique. The solutions we obtain have zero entropy density in the zero temperature limit and hence satisfy the strong, Planckian version of the third law of black hole mechanics. Therefore, these 'Nernst branes' could be holographically dual to $(2+1)$-dimensional systems in condensed matter physics where such behaviour is considered generic. Whilst the spacetime interpolates between different hyperscaling-violating Lifshitz geometries and thus correctly captures the scaling behaviour of such condensed matter systems, we observe singular behaviour in both the near horizon and asymptotic regimes.

For the 'very special' class of four-dimensional models under consideration, it is natural to try to resolve such behaviour by lifting the solution to five dimensions. Doing so, we find a family of boosted AdS-Schwarzschild black branes that continue to satisfy the third law. With AdS asymptotics comes access to techniques that allow for a more complete thermodynamic analysis. At the same time, this geometry fits naturally into gauge-gravity duality and resolves all asymptotic singular behaviour, suggesting the four-dimensional solution was unable to access the full degrees of freedom of the system. Interestingly however, the near horizon singularity persists which may suggest that a unique ground state is always accompanied by singular behaviour of the horizon.

Declaration

I hereby declare that all work described in this thesis is the result of my own research unless reference to others is given. None of this material has previously been submitted to this or any other university. All work was carried out in the Theoretical Physics Division of the Department of Mathematical Sciences, University of Liverpool, UK, during the period of October 2012 until July 2016.

To my parents.

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Publication list

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(2) P. Dempster, D. Errington, J. Gutowski and T. Mohaupt, *"Five-dimensional Nernst branes from special geometry,"* JHEP **11** (2016) 114, [arXiv:1609.05062] [2].

There is also one further publication by the author, completed during the period of the degree, that will not be presented in this thesis:

(3) D. Errington, T. Mohaupt and O. Vaughan, *"Non-extremal black hole solutions from the c-map,"* JHEP **05** (2015) 052, [arXiv:1408.0923] [3].

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1 Introduction

String theory is the leading candidate for a quantum theory of gravity and the only viable candidate for a unified theory of spacetime, forces and matter. Supergravity, the supersymmetric version of Einstein's theory of gravity, is the most important tool for investigating gravitational aspects of the theory. Supersymmetry has an important role to play in string theory, with many results only possible because supersymmetry imposes strict constraints. But since supersymmetry, if realised in nature at all, must be a hidden ('broken') symmetry, it is important to understand which results persist if the number of unbroken supersymmetries is reduced.

In this thesis we present new results in $\mathcal{N} = 2$ supergravity, which sits between the highly symmetric, but unrealistic, theories with $\mathcal{N} = 4$ or $\mathcal{N} = 8$ and the less tractable theories with $\mathcal{N} = 1$ supersymmetry. Whilst such $\mathcal{N} = 2$ theories are not realistic, they have rich dynamics and provide an excellent arena in which to construct exact, analytic solutions that allow us to learn more about string theory. One aspect of particular interest is exploring and using the moduli spaces (target manifolds) of these theories, which carry particular structures known as special geometries, along with the technique of dimensional reduction to construct new solutions.

We are interested in finding stationary solutions to $\mathcal{N} = 2$ supergravity theories in four and five dimensions. As such we will make a time-like dimensional reduction as part of the solution generating technique. In the five-dimensional case we will also make an additional space-like reduction such that, regardless of whether we started in four or five dimensions, we always solve the equations of motion in three Euclidean dimensions. We can understand the effect of both space-like and time-like dimensional reduction via a series of maps between the target manifolds of such theories: the r-maps [4], c-maps [5] and q-maps, with the latter being a composition $q = c \circ r$ of an r-map and a c-map. From a physics perspective, these maps describe the relationship between the target manifolds of two rigid or locally supersymmetric field theories.

With local supersymmetry, the target manifold of a five-dimensional $\mathcal{N} = 2$ supergravity theory coupled to n vector multiplets is a projective special real manifold $\mathcal{H} = \bar{M}_n$. By space-like reduction we obtain a four-dimensional theory of $(n + 1)$ vector multiplets, with the extra degrees of freedom coming from the reduction of the gravity multiplet. The relevant scalar geometry is a projective special Kähler manifold $\bar{N}_{2n+2}$. A further space-like reduction gives a three-dimensional supergravity theory, for which the degrees of freedom can be packaged into $(n + 2)$ hypermultiplets, with a quaternionic-Kähler target manifold $\bar{Q}_{4n+8}$ [5], as required by

supersymmetry [6]. This provides us with maps

$$\begin{tikzcd} \bar{M}_n \arrow[r, "\bar{r}"] \arrow[rr, bend right, "\bar{q}"'] & \bar{N}_{2n+2} \arrow[r, "\bar{c}"] & \bar{Q}_{4n+8} \end{tikzcd}$$

where $\bar{r}$, $\bar{c}$ and $\bar{q}$ denote the local, as opposed to the rigid, r-map, c-map and q-map.

Throughout this thesis we shall rely on time-like dimensional reduction as a solution generating technique. The target manifolds of the resulting Euclidean theories are equipped with a split-signature metric, and can be described using para-complex geometry [7, 8]. Indeed, time-like reduction of the five-dimensional theory results in a projective special para-Kähler geometry [7]. Meanwhile, both time-like reduction of the four-dimensional Minkowski theory and space-like reduction of the four-dimensional Euclidean theory give rise to para-quaternionic-Kähler scalar manifolds [8]. Our approach always involves reduction to a Euclidean theory in three dimensions using either the $\bar{c}$-map or $\bar{q}$-map depending on whether we started with a four- or five-dimensional theory, since in three dimensions all degrees of freedom can be dualised into scalars which makes the equations of motion much easier to deal with. Our technique then uses these instanton solutions to the three-dimensional equations of motion as seed solutions that can be dimensionally lifted to provide stationary solitonic solutions, e.g. black holes, to our original supergravity theories [9]. One of the main activities in contemporary string theory is to gain an increased understanding of non-perturbative effects and such black hole solutions represent an important playground for such investigations [10].

One of the most celebrated successes of string theory is the AdS/CFT correspondence [11] which generates a powerful duality between asymptotically AdS supergravity theories in the bulk spacetime and conformal field theories living on the boundary. Recently, this has evolved into the more general notion of a 'gauge-gravity duality' involving non-conformal field theories. Such holographic relationships are examples of strong-weak coupling dualities and thus allow for the translation of non-perturbative field theory calculations into more tractable, perturbative calculations in gravity and vice-versa. Whilst this has enabled the exploration of certain aspects of previously inaccessible regimes of theoretical physics, it remains an active area of research to find exact gravitational duals for strongly coupled field theories. Indeed, there are many examples of strongly coupled systems in condensed matter physics and it is hoped that gauge-gravity duality may allow for a better understanding of these. Significant progress has already been made in this direction, leading to the development of the AdS/CMT correspondence (see [12, 13] and references therein). Further recent progress has been to extend the correspondence to spacetimes which are not asymptotically-AdS but rather exhibit hyperscaling violating and Lifshitz (hvLif) behaviour [14, 15]. These spacetimes correctly capture the non-relativistic and non-conformal nature of condensed matter systems, thus extending the AdS/CMT dictionary.

The central idea in gauge/gravity duality is that each state in the bulk has a corresponding

state in the dual field theory. In particular, black objects are dual to thermal ensembles in the field theory with the same thermodynamic properties (temperature, entropy, chemical potential, etc.) as the bulk spacetime [16,17]. If the near horizon and asymptotic spacetime geometries are both maximally symmetric then one can interpret the radial direction as an RG flow between two conformal field theories, with the near horizon and asymptotic geometries representing the infra-red (IR) and ultra-violet (UV) regimes respectively.

A natural starting point for the correspondence is to consider, for example, charged (Reissner-Nordström) extremal black holes and black branes in AdS [18]. However, like their asymptotically flat cousins they have a large non-zero entropy at zero temperature, thus violating the third law of thermodynamics in its strictest version. We recall that there are at least two versions of the third law: the weak version (or process version) states that it is not possible to reduce the entropy of a system to zero in a finite number of steps, and has been shown to be equivalent to another version by Nernst. The strong version, originally from Planck, states that the entropy of a system should vanish in the zero temperature limit [19]. Whilst the strong version is actually due to Planck, we will be consistent with recent literature and refer to it hereafter as the Nernst Law, despite not being strictly historically correct.

While a non-vanishing entropy for certain classes of extremal black holes is consistent with microstate counting for the corresponding D-brane configuration in string theory [20, 21], this still begs the question of whether one can find other gravitational systems which have a zero entropy or entropy density at zero temperature, and what price we need to pay for such unusual behaviour. Apart from being an interesting question about gravity, finding examples of such systems is relevant for potential dualities between gravity and condensed matter systems where such behaviour is completely generic as we expect crystallisation to freeze out the degrees of freedom, leaving behind a non-degenerate ground state.

Such a task is notoriously difficult in gravitational systems and appears to spoil the almost perfect correspondence between the laws of black hole mechanics and the ordinary laws of thermodynamics, leading some authors to call into question the validity of the Nernst Law for black holes [22]. Developing examples of black objects consistent with the Nernst Law will be the central theme of this thesis.

There already exist so-called 'small black holes' with vanishing entropy in the extremal limit [23], but from the Bekenstein-Hawking formula these must have vanishing horizon area and, given the spherical topology, this corresponds to a divergent curvature on the horizon. Since supergravity is only a valid approximation of string theory providing the curvature remains below the Planck scale, small black holes will be outside the supergravity regime. It was later realised in [24] that a supergravity realisation of Nernst Law behaviour could still be possible (without the need for stringy corrections) by turning to black branes whose Ricci-flat, planar horizon topology avoids this problem altogether. Yet in ungauged supergravity there are certain

no-go theorems [25] forbidding the existence of such horizon topologies and so we must somehow deform the supergravity theory to access them.

Gauged supergravities are deformations of the standard abelian supergravity theories. From a string theory perspective, ungauged four- and five-dimensional supergravity theories come from compactifying ten-dimensional string theory, or eleven-dimensional M-theory, on a six-dimensional internal manifold which is typically a Calabi-Yau threefold. Theories of gauged supergravity appear if we instead perform a flux compactification. This involves switching on some additional tensor fields (fluxes) along the internal manifold that deform it away from a Calabi-Yau geometry [26]. In general, it is difficult to find gauged supergravity solutions in four or five dimensions that are consistent truncations of flux compactifications. However, even without considering a stringy embedding, one can still entertain theories of gauged supergravity in four or five dimensions by promoting some subgroup of global symmetries to a local symmetry [27]. One of the simplest deformations we can consider is a Fayet-Iliopoulos gauging of the R-symmetry group.

Whether or not we consider a stringy embedding, the gauged supergravities in four or five dimensions will typically result in more complicated effective theories that are equipped with non-abelian gauge fields or charged matter fields. As well as introducing new interaction terms into the theory, this gauging also requires a scalar potential be introduced to maintain supersymmetry invariance of the action. Such scalar potentials may support an effective cosmological constant and thus lead to non-asymptotically flat solutions. This has triggered a resurgence of interest in gauged supergravity due to the ability to produce spacetimes that can be used in holography. Furthermore, the scalar potential also allows us to circumvent the no-go theorems and produce non-spherical horizon topologies.

Extremal brane solutions with vanishing entropy density at zero temperature have recently been obtained for a variety of bulk theories in the context of gauged supergravity [24, 28, 29, 30, 31, 32] and could have important applications in extending the dictionary between condensed matter and gravity. They have been dubbed 'Nernst branes' in [24], and it is believed that the corresponding non-extremal solutions exist and satisfy the Nernst Law. In other words, these non-extremal solutions have a finite entropy which goes to zero when the temperature goes to zero while external parameters are kept fixed. Finding such non-extremal solutions is important, since extremal Nernst branes are not completely regular solutions. While all curvature invariants remain finite at the horizon, tidal forces become infinite and scalar fields take infinite values, which suggests a breakdown of the underlying effective field theory [12, 24]. A first step in addressing this issue is to find non-extremal solutions, which can then be studied in the near extremal limit. In this context it is clearly desirable to have completely explicit, analytic solutions. However most results in the literature have to rely on a mixture of analytical and numerical methods. Of course tidal forces may still get very large at the

horizon when one approaches the extremal limit [33], but analytic solutions will enable one to identify the region in parameter space where the solution can be trusted and possibly be mapped to condensed matter systems. Another way to control the near horizon low temperature behaviour is to embed the theory under consideration into a UV-complete theory, for which string theory and its non-perturbative extension M-theory are arguably the best candidates. We shall work directly with four- and five-dimensional $\mathcal{N} = 2$ gauged supergravity, and do not consider the flux compactifications necessary to discuss their stringy origins. As such, we refer the reader to [12, 13, 32] for a further discussion of the possible implications of quantum and string corrections to the zero temperature behaviour and the 'fate' of the Nernst Law.

The outline of this thesis is as follows: in Chapters 2 and 3 we introduce the mathematical and physical concepts required to understand the work that follows. Then, in Chapter 4, we focus on a four-dimensional theory of $\mathcal{N} = 2$ supergravity with a Fayet-Iliopoulos gauging suitable for producing non-asymptotically flat vacua. Using an adaptation of the real formulation of projective special Kähler geometry suited to formulating the $\bar{c}$-map in a symplectically covariant manner, we perform a time-like dimensional reduction and solve the scalar equations of motion in three dimensions. This instanton solution is then lifted to a two-parameter family of four-dimensional Nernst branes that interpolate between different hyperscaling-violating Lifshitz geometries. Analysing their behaviour, we find singularities at either end of the renormalization group flow. Not only do our solutions suffer from the same near horizon tidal forces that had been found previously but, more seriously, in the asymptotic regime there is disagreement between the gravitational and field theoretic descriptions of the thermodynamics and in certain cases there is even a genuine curvature singularity. This suggests we can only trust a holographic duality with condensed matter physics in some finite energy interval, and that we should expect to encounter problems in the deep infra-red and deep ultra-violet. However, the behaviour of the four-dimensional scalar fields strongly indicates that additional degrees of freedom become relevant in the ultra-violet regime and the solution decompactifies. This is consistent with evidence in the literature [34, 35] and, in Chapter 5, we apply the $\bar{q}$-map to a five-dimensional theory of Fayet-Iliopoulos gauged $\mathcal{N} = 2$ supergravity to obtain a family of five-dimensional asymptotically Anti de-Sitter Nernst branes. We are able to check that these are the dimensional lifts of our four-dimensional solutions, and that they continue to satisfy the Nernst Law. The five-dimensional solutions are free of the UV singularities and, with a better understood asymptotic geometry comes a more satisfactory picture of the brane thermodynamics as well as a geometrical understanding of the origins of the UV singularities in four dimensions. The IR singularities persist in five dimensions and we offer some comments on why this might be the case. We end with conclusions and ideas for future work in Chapter 6.

We follow the notations and conventions of [36, 37] except for a difference in sign of the Einstein-Hilbert term that is explained in Appendix A.

2 Preliminary mathematics

In this opening chapter we introduce various mathematical concepts that will be important throughout this thesis. We begin in Section 2.1 with an overview of differential geometry, introducing the elementary material upon which the rest of this chapter builds. In Section 2.2 we will introduce special real manifolds, which are the simplest type of manifolds in special geometry. We then move on to special (para-)Kähler manifolds in Section 2.3, and finish with a discussion of (para-)quaternionic-Kähler manifolds in Section 2.4.

2.1 Differential geometry

2.1.1 Basics of differential geometry

Throughout this section we shall denote by M an n-dimensional differentiable manifold. The set of tangent vectors at a point $p \in M$ is called the tangent space, $T_p\mathcal{M}$. The tangent bundle, TM, assembles all the tangent vectors on M and, as a set, can be viewed as the disjoint union of individual tangent spaces

$$TM = \bigsqcup_{p\in M} T_pM = \bigcup_{p\in M} \{(p, X)|X \in T_pM\}.$$

We can then define $\Gamma(TM)$ to be the set of sections of TM i.e. the set of smooth vector fields on M.

Integral curves

Given a smooth vector field $X \in \Gamma(TM)$, we define an *integral curve*, γ, of X to be a curve on M whose tangent at every point is X. Formally, one can introduce a curve parameter $t \in [0, 1]$ and think of this as the function

$$\gamma : [0, 1] \to M,$$

such that the tangent at a given point $\gamma(t) \in M$ is simply $X_{\gamma(t)}$ i.e. the value of the vector field evaluated at the relevant point. It is clear that the integral curve represents the orbit of a one-parameter group of diffeomorphisms generated by X [38].

Given an integral curve of X through some $p \in M$, we can introduce a local coordinate patch $\mathcal{U} \subset M$ with coordinates $\{x^\mu\}$, such that the integral curve reduces to the following system of

ordinary differential equations in $\mathbb{R}^n$,

$$\frac{dx^\mu}{dt} = X^\mu(x(t)).$$

Definition 1. *A map $\phi : M \to N$ is a* DIFFEOMORPHISM *between two manifolds M and N iff it is one-to-one, onto, smooth and has a smooth inverse.*

Let us now denote by $\Gamma(T^{r,s}M) = \Gamma(\otimes^r T^*M \otimes^s TM)$ the set of smooth tensor fields of rank (r, s) on M. We can then define:

Definition 2. *If $\phi : M \to N$ is a diffeomorphism and $T \in \Gamma(T^{r,s}N)$ is a type (r, s) tensor field on N, the* PULL-BACK *of T is a tensor field $\phi^*(T) \in \Gamma(T^{r,s}M)$ of type (r, s) on M defined by*

$$\phi^*(T)(\eta_1, \ldots, \eta_r, X_1, \ldots, X_s) = T\left((\phi^{-1})^*(\eta_1), \ldots, (\phi^{-1})^*(\eta_r), \phi_*(X_1), \ldots, \phi_*(X_s)\right),$$

for arbitrary $\eta_i \in \Gamma(T^*M), X_j \in \Gamma(TM)$.

Definition 3. *If $\phi : M \to N$ is a diffeomorphism and $T \in \Gamma(T^{r,s}M)$ is a type (r, s) tensor field on M, the* PUSH-FORWARD *of T is a tensor field $\phi_*(T) \in \Gamma(T^{r,s}N)$ of type (r, s) on N defined by*

$$\phi_*(T)(\eta_1, \ldots, \eta_r, X_1, \ldots, X_s) = T(\phi^*(\eta_1), \ldots, \phi^*(\eta_r), (\phi^{-1})_*(X_1), \ldots, (\phi^{-1})_*(X_s)),$$

for arbitrary $\eta_i \in \Gamma(T^*N), X_j \in \Gamma(TN)$.

Definition 4. *On a manifold M, the* LIE DERIVATIVE *of a tensor field $T \in \Gamma(T^{r,s}M)$ with respect to a vector field $\xi \in \Gamma(TM)$ at a point p is*

$$(\mathcal{L}_\xi T)_p = \lim_{t\to 0} \frac{((\psi_{-t})_* T)_p - T_p}{t}, \tag{2.1}$$

where ψ_t is the map sending $p \in M$ to the point parameter distance t along the integral curve of ξ through p. It can be shown that ψ_t is a 1-parameter group of diffeomorphisms on M [39].

Definition 5. *A diffeomorphism $\phi : M \to N$ is a* SYMMETRY TRANSFORMATION *of the tensor field T iff $\phi_*(T) = T$ everywhere. It is clear that the Lie derivative of any tensor field must vanish when evaluated along symmetry transformations.*

From (2.1) that we can compare tensor fields at two distinct points $p, q \in M$ by evaluating the Lie derivative along any integral curve that connects p to q. An alternative way of comparing tensor fields at two different points on a manifold is to use the notion of parallel transport. This requires the existence of an additional structure known as the connection, which we shall now introduce.

2.1.2 Connections on the tangent bundle, TM

Affine connections

An *affine connection* is a map sending a pair of smooth vector fields to another smooth vector field

$$\begin{aligned}\nabla : \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM)\\ (X, Y) &\mapsto \nabla_X Y,\end{aligned}$$

and which satisfies the following properties:

$$\begin{aligned}\nabla_{fX+gY} Z &= f\nabla_X Z + g\nabla_Y Z,\\ \nabla_X(Y + Z) &= \nabla_X Y + \nabla_X Z,\\ \nabla_X(fY) &= (\nabla_X f)Y + f\nabla_X Y \quad \text{(Leibniz rule)},\end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$ and any smooth functions $f, g \in C^\infty(M)$.

The action of the connection on functions is defined by

$$\nabla_X f = X(f) = \pounds_X f, \tag{2.2}$$

where $\pounds_X f$ is the Lie derivative of f along the integral curve of X. On a local coordinate patch $\mathcal{U} \subset M$, we can expand this in a coordinate basis $\{\partial_\mu\}$ of TM as

$$X(f) = X^\mu \frac{\partial f}{\partial x^\mu},$$

which we recognise as the directional derivative of f along the vector field X.

Affine connection in a coordinate basis

Given a coordinate basis for TM, such that $X = X^\mu \partial_\mu$ as seen above, we can expand $\nabla_X Y$ as

$$\begin{aligned}\nabla_X Y &= X^\mu \nabla_\mu (Y^\lambda \partial_\lambda)\\ &= X^\mu (\partial_\mu Y^\lambda)\partial_\lambda + X^\mu Y^\nu (\nabla_\mu \partial_\nu)\\ &= X^\mu \left[\partial_\mu Y^\lambda + \Gamma^\lambda{}_{\mu\nu} Y^\nu\right] \partial_\lambda,\end{aligned} \tag{2.3}$$

where we have made use of the Leibniz rule and defined the *connection coefficients* of the vector field $\nabla_\mu \partial_\nu$ in the chosen coordinate basis as

$$\Gamma^\lambda{}_{\mu\nu} \partial_\lambda := \nabla_\mu(\partial_\nu). \tag{2.4}$$

Affine connection on tensor fields

We can further extend the action of the affine connection to tensors of arbitrary rank by demanding that ∇_X forms a rank preserving map and satisfies

$$\nabla_X(T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2), \tag{2.5}$$

for $T_1 \in \Gamma(T^{r_1,s_1}M), T_2 \in \Gamma(T^{r_2,s_2}M)$. To illustrate how this actually works, let us now compute the action of ∇_X on some smooth one-form $\omega \in \Gamma(T^*M)$. Given some smooth vector field $Y \in \Gamma(TM)$, the product $\omega(Y) \in C^\infty(M)$ represents a smooth function on M. Using the action of ∇_X on functions (2.2), we have

$$\nabla_X(\omega(Y)) = X(\omega(Y)).$$

However, viewing $\omega(Y)$ as a tensor product, the action of ∇_X on tensors (2.5) tells us

$$\nabla_X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y).$$

Combining these two results, we find that the action of ∇_X on smooth covector fields is

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y). \tag{2.6}$$

Parallel transport and geodesics

Given an affine connection on a manifold, we can proceed to define the idea of parallel transport and geodesics as follows. This will be important in later chapters since the Weak Equivalence Principle of General Relativity can be understood by asserting that freely-falling particles travel along geodesics of the spacetime manifold [39].

Definition 6. *Let $\gamma(t)$ be the integral curve generated by the flow of some smooth vector field $V \in \Gamma(TM)$. We say that a rank (r,s) tensor field $T \in \Gamma(T^{r,s}M)$ is* PARALLEL TRANSPORTED *along $\gamma(t)$ if*

$$\nabla_V T = 0.$$

Definition 7. *A* GEODESIC *is a special type of integral curve which parallel transports its own tangent vector i.e. it is an integral curve $\gamma(t)$ generated by $V \in \Gamma(TM)$ satisfying*

$$\nabla_V V = 0,$$

providing t is an affine parameter. Note that such an affine parameter always exists and so we can assume the above definition for all geodesics without loss of generality.

Riemann curvature

If one considers parallel transporting a vector between two points $p \in M$ and $q \in M$ along two different curves c and c', the resulting vectors at q will, in general, differ. This path dependence of parallel transport characterizes the notion of a manifold's intrinsic curvature, and is independent of coordinates. Mathematically, this property is captured by the *Riemann curvature tensor* which measures the failure of a vector to return to its original value when parallel transported around a small closed loop. This is defined as [40]

$$\begin{aligned} R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ (X, Y, Z) &\mapsto \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \end{aligned} \tag{2.7}$$

where $[X, Y] = \pounds_X Y$ is the Lie bracket of vector fields. Expanding in a coordinate basis $\{\partial_\mu\}$ this assumes the form

$$R(X, Y)Z = X^\mu Y^\nu Z^\rho \left[\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\nu \Gamma^\lambda{}_{\mu\rho} + \Gamma^\lambda{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\rho} - \Gamma^\lambda{}_{\nu\sigma}\Gamma^\sigma{}_{\mu\rho}\right] \partial_\lambda. \tag{2.8}$$

Definition 8. *An affine connection ∇ is said to be* FLAT *if $R(X, Y)Z = 0$ for any $X, Y, Z \in \Gamma(TM)$.*

Torsion

Another important tensor field that can be constructed from the affine connection is the torsion tensor. This measures the extent to which a 'parallelogram' formed from small displacement vectors and their parallel transports fails to close on a curved manifold.[1] Torsion is defined as [40]

$$\begin{aligned} T : \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ (X, Y) &\mapsto \nabla_X Y - \nabla_Y X - [X, Y]. \end{aligned} \tag{2.9}$$

This can be written in a coordinate basis $\{\partial_\mu\}$ as

$$T(X, Y) = \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right) X^\mu Y^\nu \partial_\lambda.$$

Definition 9. *An affine connection ∇ is said to be* TORSION-FREE *if $T(X, Y) = 0$ for any $X, Y \in \Gamma(TM)$. In a coordinate basis, this simply implies that the connection components $\Gamma^\lambda{}_{\mu\nu}$ are symmetric in the lower indices.*

[1] Note the important difference between curvature and torsion. Torsion measures the failure of an infinitesimal parallelogram to close, as is made clear by Figure 1 in [41]. In this sense, torsion is related to the translation of TM whilst curvature is related to the rotation of TM and is measured around closed loops (that may need to be closed by addition of $T(X, Y)$ in the presence of torsion).

Geodesic deviation

Now let us consider a one-parameter family of geodesics on M. This is a map

$$\begin{aligned} \gamma : [0,1] \times [0,1] &\to M \\ (s,t) &\mapsto \gamma(s,t), \end{aligned} \tag{2.10}$$

such that for fixed s, $\gamma(s,t)$ is a geodesic with affine parameter t. This identifies s as labelling the individual geodesics. Now suppose $T \in \Gamma(TM)$ is tangent to the geodesics and $S \in \Gamma(TM)$ is tangent to curves of constant t, which are thus parametrized by s. The geodesics occupy a two-dimensional surface in M on which we can use s and t as coordinates. This can be extended to coordinates $(s, t, \dots)$ in a neighbourhood of the surface and thus sets up a coordinate chart in which $S = \frac{\partial}{\partial s}$ and $T = \frac{\partial}{\partial t}$. Importantly, in such a chart, we have $[S, T] = 0$. Focussing on an individual geodesic, the relative velocity of an infinitesimally nearby geodesic in the family, as we move along the geodesic flow, is given by

$$V = \delta s \nabla_T S.$$

The curvature of M can force geodesics in the family to move together or apart. This is captured by the notion of geodesic deviation, and to understand this we require knowledge of the relative acceleration of neighbouring curves along the geodesic flow,

$$A = \nabla_T V = \delta s \nabla_T \nabla_T S.$$

When the affine connection is torsion-free, $\nabla_T S - \nabla_S T = [T, S]$. As we have seen above, $[S, T] = 0$ and so $\nabla_T S = \nabla_S T$. This allows the *geodesic deviation* to be written as

$$A = \delta s \nabla_T \nabla_S T = \nabla_S \nabla_T T + R(T, S)T,$$

where we have used (2.7). Of course, $\nabla_T T = 0$ along geodesics and so, the geodesic deviation is measured simply by the behaviour of the Riemann tensor,

$$A = R(T, S)T,$$

or, in component form,

$$A^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma. \tag{2.11}$$

Holonomy

When a vector field in flat space is parallel transported around a closed loop it will be returned to itself after traversing the loop. However, in general, this is not true for curved manifolds and we have seen that the Riemann curvature tensor directly measures the extent of this failure. We now introduce the related concept of the holonomy group.

Consider the tangent space T_pM at some specific point $p \in M$. We define

$$C_p(M) = \{\gamma : [0,1] \to M | \gamma(0) = \gamma(1) = p\},$$

to be the set of closed loops in $\mathcal{M}$ based at p. Given a connection ∇ on M, we can parallel transport any vector $X \in T_pM$ around some $c(t) \in C_p(M)$ to generate a new vector $X_c \in T_pM$. The Riemann tensor measures the change $\Delta X = X_c - X$ and consequently it is possible to associate to each loop $c(t) \in C_p(M)$ a map

$$P_c : T_pM \to T_pM; \quad X \mapsto X_c.$$

In a coordinate basis $\{\partial_\mu\}$ of T_pM, we can write

$$X_c = P_c X = X h_c = X^\mu (h_c)_\mu{}^\nu \partial_\nu.$$

The set of transformation matrices $\{h_c | c \in C_p(M)\}$ that results from considering all possible loops generates the *holonomy group at* p, $\mathrm{Hol}(\nabla, p) \subset GL(n, \mathbb{R})$. If M is pathwise-connected then $\mathrm{Hol}(\nabla, p) \simeq \mathrm{Hol}(\nabla, q)$ for any given $p, q \in M$. This implies the holonomy group will depend only on the connection, and we can refer to it as $\mathrm{Hol}(\nabla)$. We shall only deal with pathwise-connected manifolds in this thesis.

Lastly, note that manifolds with a flat connection, as per Definition 8, have trivial holonomy since invariance under parallel transport is automatic [42].

2.1.3 Pseudo-Riemannian geometry

The discussion thus far is applicable to any differentiable manifold and requires only the existence of an affine connection. In particular, it does not assume the existence of a metric g. We shall now proceed to examine various properties that appear when the manifold is equipped with a metric. Without requiring definite signature, a pair (M, g) is said to be a *pseudo-Riemannian manifold* of signature (p, q).

Levi-Civita connection

Definition 10. *Let (M, g) be a pseudo-Riemannian manifold. An affine connection ∇ on M is* METRIC COMPATIBLE *if $\nabla g = 0$. It is clear that parallel transport by a metric compatible connection preserves length measurements on M.*

Definition 11. *The presence of a metric singles out a privileged connection, namely the* LEVI-CIVITA CONNECTION *D, which is the unique connection that is both metric compatible and torsion-free:*

$$Dg = 0, \qquad T_D(X, Y) = 0.$$

To obtain the Levi-Civita connection associated to a given metric, one uses the metric compatibility and torsion-free properties to obtain the *Koszul formula* [43]

$$\begin{aligned} 2g(D_X Y, Z) =& Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &+ g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X), \end{aligned} \tag{2.12}$$

where $X, Y, Z \in \Gamma(TM)$. This is sufficient to determine D_X uniquely because the metric is non-degenerate [43]. Specifically, choosing a coordinate basis $\{\partial_\mu\}$ (for which $[\partial_\mu, \partial_\nu] = 0$), the Koszul formula becomes

$$g(D_\rho \partial_\nu, \partial_\sigma) = \frac{1}{2} \left(g_{\nu\sigma,\rho} + g_{\sigma\rho,\nu} - g_{\rho\nu,\sigma} \right),$$

and using the definition of the connection components in (2.4), we find the left hand side to be

$$g\left((\Gamma_D)^\tau{}_{\nu\rho} \partial_\tau, \partial_\sigma \right) = (\Gamma_D)^\tau{}_{\nu\sigma} g_{\tau\sigma},$$

where Γ_D denotes the Levi-Civita connection components. If we then multiply through by an inverse metric we arrive at the familiar expression for the Levi-Civita connection components, otherwise known as the Christoffel symbols

$$(\Gamma_D)^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\nu\sigma,\rho} + g_{\sigma\rho,\nu} - g_{\rho\nu,\sigma} \right). \tag{2.13}$$

We shall work exclusively with the Levi-Civita connection throughout Chapters 4 and 5. Therefore, we shall drop the D subscript and understand that in those chapters $\Gamma^\mu{}_{\nu\rho}$ assumes the form in (2.13).

Orthonormal frames

Given a pseudo-Riemannian manifold (M, g) of signature (p, q) it is possible to introduce an *orthonormal frame* $\{e_a\}$. This is a choice of basis for T_pM that smoothly depends on p and may

or may not be a coordinate basis. The individual frame fields e_a are required to satisfy

$$g(e_a, e_b) = \eta_{ab} \quad \text{with} \quad \eta_{ab} = \begin{pmatrix} -\mathbb{1}_p & 0 \\ 0 & \mathbb{1}_q \end{pmatrix}.$$

Any two different orthonormal frames $\{e_a\}$ and $\{e'_a\}$ are related by $SO(p,q)$ transformations corresponding to a rotation of basis vectors. The frame fields forming an orthonormal frame are related to a coordinate basis $\{\partial_\mu\}$ by volume and orientation preserving $SL(n,\mathbb{R})$ transformations

$$e_a = e_a{}^\mu \partial_\mu.$$

These $SL(n,\mathbb{R})$ transformation matrices $e_a{}^\mu$ are known as *vielbeins* and satisfy

$$g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab}. \tag{2.14}$$

This implies that

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta^{ab} e_a \otimes e_b.$$

Orthonormal frames are physically useful as they embody the inertial (freely falling) frames for which the Equivalence Principle of General Relativity postulates that gravitational effects should be locally absent and spacetime should appear locally flat.[2] Furthermore, it is in fact true that such a frame exists not just at a point but along entire geodesics [44]. As such, we will make use of these coordinates when discussing the local effects (e.g. tidal forces) that freely-falling observers are sensitive to as they traverse their spacetime worldlines (geodesics).

It is also possible to introduce a *dual basis* $\{\theta^a\}$ of T^*M satisfying $\theta^a(e_b) = \delta^a{}_b$. Similar to the orthonormal frame, these are related to the coordinate basis via.

$$\theta^a = e^a{}_\mu dx^\mu, \tag{2.15}$$

where the *inverse vielbein* $e^a{}_\mu$ transformation matrices are defined through the requirement $e^a{}_\mu e_b{}^\mu = \delta^a{}_b$. Of course, one can then rearrange (2.14) to express them as $e^a{}_\mu = g_{\mu\nu}\eta^{ab} e_b{}^\nu$. Further, it is possible to rearrange (2.15) to express the metric g in terms of the dual basis as

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab} \theta^a \otimes \theta^b.$$

[2] This is true at least up to second order curvature contributions, coming from the Riemann tensor [44]. Of course, for a sufficiently small region of spacetime, such second order effects will be negligible and we would be unable to detect any curvature.

Connection 1-form and Cartan's structure equations

Given an affine connection ∇ on a manifold M, it is always possible to express $\nabla_X Y$ in a coordinate basis as seen in (2.3). On a pseudo-Riemannian manifold (M, g), the additional metric structure g allows us to write $\nabla_X Y$ in terms of an orthonormal frame $\{e_a\}$ as

$$\nabla_X Y = X^a \nabla_a (Y^c e_c) = X^a \left[e_a Y^c + \gamma^c{}_{ab} Y^b \right] e_c,$$

where $\gamma^c{}_{ab}$ represent the connection components with respect to the basis $\{e_a\}$ and are defined as

$$\gamma^c{}_{ab} e_c := \nabla_a e_b,$$

and are related to the coordinate basis expression for the connection components by

$$\gamma^c{}_{ab} = e_\lambda{}^c e_a{}^\mu \left(\partial_\mu e_b{}^\lambda + e_b{}^\nu \Gamma^\lambda{}_{\mu\nu} \right).$$

In a similar fashion we can express the components of the torsion tensor T and Riemann curvature tensor R in an orthonormal basis [40].

Let us now define the matrix-valued 1-form $\omega^a{}_b$, called the *connection one-form*, by

$$\omega^a{}_b := \gamma^a{}_{cb} \theta^c. \tag{2.16}$$

This satisfies *Cartan's structure equations*

$$T^a = d\theta^a + \omega^a{}_b \wedge \theta^b, \tag{2.17}$$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \tag{2.18}$$

where we have defined respectively the *torsion* 2*-form* and *curvature* 2*-form* as

$$T^a = \frac{1}{2} T^a{}_{bc} \theta^b \wedge \theta^c, \tag{2.19}$$

$$R^a{}_b = \frac{1}{2} R^a{}_{bcd} \theta^c \wedge \theta^d. \tag{2.20}$$

Isometries

The metric on a pseudo-Riemannian manifold (M, g) is the fundamental structure in General Relativity. Given the fundamental physical role played by symmetries and their consequences, we are particularly interested in finding metric-preserving symmetry transformations. These are known as isometries. Let us now describe how to do this.

We have already seen in (2.2) that the Lie derivative of a function is equivalent to the directional derivative, $\pounds_\xi f = \xi(f)$. This is as expected since saying a function has a certain symmetry

amounts to the vanishing of the directional derivative in a particular direction. Similarly, we can find the following expressions for the Lie derivative of a vector, covector and $(0,2)$ tensor in a coordinate basis $\{\partial_\mu\}$,

$$(\pounds_\xi X)^\mu = [\xi, X]^\mu, \tag{2.21}$$

$$(\pounds_\xi \omega)_\mu = \xi^\nu \nabla_\nu \omega_\mu + \omega_\nu \nabla_\mu \xi^\nu, \tag{2.22}$$

$$(\pounds_\xi T)_{\mu\nu} = \xi^\rho \nabla_\rho T_{\mu\nu} + T_{\mu\rho}\nabla_\nu \xi^\rho + T_{\rho\nu}\nabla_\mu \xi^\rho. \tag{2.23}$$

Definition 12. *Given a pseudo-Riemannian manifold (M,g), a diffeomorphism $\phi : (M,g) \to (M,g)$ satisfying $\phi_*(g) = g$ is a symmetry transformation of the metric and is known as an* ISOMETRY.

Definition 13. *Suppose ψ_t are a 1-parameter group of isometries on the pseudo-Riemannian manifold (M,g), then $\mathcal{L}_\xi g = 0$ from (2.1). Working with the metric compatible Levi-Civita connection D for which $Dg = 0$, we see from (2.23) that isometries are characterised by* KILLING'S EQUATION,

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{2.24}$$

The solutions, ξ, are called KILLING VECTOR FIELDS. *We can think of Killing vectors as the infinitesimal generators of metric symmetries (isometries).*

Berger's classification of Riemannian holonomies

Riemannian manifolds (M,g) have positive definite metric signature. Such manifolds play important roles in string theory; in particular, those with special holonomy admit covariantly constant (parallel) spinors and therefore compactifications involving such manifolds preserve some supersymmetries in the lower dimensional theory e.g. ten-dimensional heterotic string theory compactified over a six-dimensional Calabi-Yau three-fold [37, 45, 46]. Furthermore, throughout this thesis we shall encounter examples of such manifolds as the scalar geometries of supergravity theories and, as such, we find it useful to review Berger's classification theorem.

Theorem 1. *Let (M,g) be a Riemannian manifold of dimension n which is not locally symmetric and whose holonomy representation Hol(D) is irreducible.*[3] *Then its holonomy representation Hol(D) is contained in one of the groups in Table 2.1 (see 10.92 of [42]).*

[3] Locally symmetric means the curvature tensor is parallel i.e. $DR = 0$, whilst a theorem of de Rham shows that an irreducible holonomy representation is equivalent to demanding that M is not a product manifold (10.43 of [42]).

Holonomy	Dimension	Manifold
$SO(n)$	n	Orientable
$U(n)$	$2n$	Kähler
$SU(n)$	$2n$	Calabi-Yau
$Sp(n)$	$4n$	hyperkähler
$Sp(n) \cdot Sp(1)$	$4n$	quaternionic-Kähler
G_2	7	G_2-manifold
Spin(7)	8	Spin(7) manifold

Table 2.1: Berger classification of Riemannian holonomies.

To make Table 2.1 more tangible, we can explicitly prove the first line for an n-dimensional orientable Riemannian manifold (M, g) equipped with Levi-Civita connection D as follows:

Proof: As noted in Definition 10, the metric compatibility of the Levi-Civita connection, $Dg = 0$, ensures the length of a vector X is unchanged when parallel transported. Hence $g_p(P_c X, P_c X) = g_p(X, X)$ for all $X \in T_p M$. Expanding in an orthonormal frame $\{e_a\}$ for $T_p M$, we find

$$\eta_{ab} = h_a{}^c h_b{}^d \eta_{bd},$$

and so $h \in SO(n) \subset GL(n, \mathbb{R})$. Thus the holonomy group is a subgroup $\mathrm{Hol}(D) \subset SO(n)$.

2.2 Special real geometry

Special real geometry appears in the construction of five-dimensional $\mathcal{N} = 2$ supergravity theories. Specifically, it is the geometry describing the consistent supersymmetric coupling of five-dimensional $\mathcal{N} = 2$ vector multiplets to gravity. Following [37, 47, 48, 49] we shall now provide the definitions required to understand the structure of such manifolds.

Definition 14. *A* Hessian manifold (M, g, ∇) *is a pseudo-Riemannian manifold* (M, g) *equipped with a flat, torsion-free connection* ∇ *such that the rank three tensor* ∇g *is completely symmetric.*

Assuming the existence of a flat, torsion-free connection, it is possible to cover M with a set of affine/flat coordinates h^I for which the connection components $\Gamma^I{}_{JK}$ are vanishing. Consequently,

$$\nabla_X Y = X^I (\partial_I Y^J) \partial_J, \quad \text{with } \partial_I = \frac{\partial}{\partial h^I}.$$

Given that $\nabla_I = \partial_I$ in affine (flat) coordinates, the condition that $(\nabla_X g)(Y, Z)$ is totally sym-

metric in X, Y, Z reduces to the requirement

$$\partial_I g_{JK} = \partial_J g_{IK},$$

which implies that the metric components can be described locally as the second derivatives of some real function [50]

$$g_{IJ} = \frac{\partial^2 H(h)}{\partial h^I \partial h^J}. \tag{2.25}$$

The function $H(h)$ is called the *Hesse potential*. It is clear that for any Hessian manifold, there exists a Hesse potential that is unique up to terms linear in h^I.

Definition 15. *An* AFFINE SPECIAL REAL MANIFOLD (M, g, ∇) *is a Hessian manifold whose Hesse potential is a (not necessarily homogeneous) polynomial of degree at most* 3 *[7].*

At this point we can introduce further terminology from the physics literature. The Hessian property provides a flat and torsion-free connection ∇ that we now refer to as a *special connection*, whilst the affine/flat coordinates h^I for which $\nabla_I = \partial_I$ are said to be *special coordinates*. When one constructs a consistent interacting Lagrangian describing rigid 5d, $\mathcal{N} = 2$ vector multiplets, the scalar fields present in these multiplets transpire to parametrize a target manifold with affine special real geometry [7].

Definition 16. *A* $\mathtt{d}$-CONIC HESSIAN MANIFOLD (M, g, ∇, ξ) *is a Hessian manifold* (M, g, ∇) *supported by a vector field* ξ *such that:*

(i) $D\xi = \frac{\mathtt{d}}{2}\mathbb{1}$*, where* D *is the Levi-Civita connection with respect to* g*.*

(ii) $\nabla\xi = \mathbb{1}$*.*

These geometries bear a conical structure that will play an important role when localising supersymmetry in the construction of supergravity theories. Let us follow the treatment of ($\mathtt{d} = 2$)-cones in [51], and the generalization to arbitrary $\mathtt{d}$ in [36]. We begin by analysing each of the above conditions in turn. Firstly, we investigate the effect condition (i) has on the metric by setting $Y = \xi$ in the Koszul formula (2.12):

$$\begin{aligned} 2g(D_X\xi, Z) =& Xg(\xi, Z) + \xi g(X, Z) - Zg(X, \xi) \\ &+ g([X, \xi], Z) - g([X, Z], \xi) - g([\xi, Z], X). \end{aligned}$$

Keeping only the part symmetric in X and Z and using condition (i), we obtain

$$\mathtt{d}g(X, Z) = \xi g(X, Z) + g([X, \xi], Z) + g([Z, \xi], X),$$

or equivalently in local coordinates,

$$\mathtt{d}g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu \xi^\rho + g_{\mu\rho}\partial_\nu \xi^\rho = \pounds_\xi g_{\mu\nu}. \tag{2.26}$$

We recognise this as the equation for a homothetic Killing vector field of weight $\mathtt{d}$, whose orbits generate dilatations that preserve the conformal structure of M.[4]

The part of the Koszul formula (2.12) antisymmetric in X and Z can be obtained by considering the combination $2g(D_X\xi, Z) - 2g(D_Z\xi, X)$. Using definition (2.12), this reduces to

$$D_\mu \xi_\nu = D_\nu \xi_\mu \quad \Rightarrow \quad \xi_\mu = \partial_\mu f, \tag{2.27}$$

for some smooth function $f \in C^\infty(M)$. This shows that the homothetic Killing vector ξ is hypersurface orthogonal to the level sets $f =$ constant.

Using the expression (2.13) for the components of the Levi-Civita connection, we can rewrite (2.26) in terms of the Levi-Civita connection as

$$\pounds_\xi g_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu = \mathtt{d}g_{\mu\nu}. \tag{2.28}$$

Combining with (2.27), we ascertain that the metric can be written as

$$g_{\mu\nu} = \frac{2}{\mathtt{d}} D_\mu \partial_\nu f,$$

which establishes ξ as a closed homothetic Killing vector field [27]. At this point [51] finds it useful to define the function

$$V := g(\xi, \xi) = g^{\mu\nu}\partial_\mu f \partial_\nu f, \tag{2.29}$$

which has derivative

$$\partial_\mu V = \mathtt{d}\partial_\mu f.$$

With an appropriate choice of integration constant we can set $V = \mathtt{d}f$. Now let us choose the function f to be one of the coordinates $x^0 = f$. We see from (2.29) that $g^{00} = V = \mathtt{d}f$, and thus

$$\begin{aligned} \xi = g^{\mu\nu}\xi_\nu \partial_\mu &= g^{\mu\nu}\partial_\nu f \partial_\mu \\ &= g^{00}\frac{\partial}{\partial x^0} \quad \Rightarrow \quad \xi^\mu = \mathtt{d}f\delta^\mu{}_0. \end{aligned} \tag{2.30}$$

[4] A manifold (M, g) admits a conformal Killing vector ξ if and only if $\mathcal{L}_\xi g = \Phi(x^\lambda)g$ for a smooth function $\Phi \in C^\infty(M)$. In the special case where $\Phi = k$ is constant, ξ generates a particular kind of conformal transformation known as a homothety, and ξ is referred to as a homothetic Killing vector of weight k.

Thus $V = g_{\mu\nu}\xi^\mu\xi^\nu = g_{00}(\mathtt{d}f)^2 \Rightarrow g_{00} = \frac{1}{\mathtt{d}f}$ and the metric is

$$\begin{aligned} ds^2 &= g(x^\mu, x^\nu) = g(\xi,\xi) + g(x^i, x^j) \\ &= \frac{df^2}{\mathtt{d}f} + g_{ij}(f, x^k)dx^i dx^j, \end{aligned} \tag{2.31}$$

where $i, j, k = 1, \ldots, n-1$, and no cross term is present since ξ is orthogonal to slices of M with $f = \text{const}$. Now let us introduce the radial coordinate r such that $r^{\mathtt{d}} = \mathtt{d}f$. Changing coordinates we find $\xi = \mathtt{d}f\frac{\partial r}{\partial f}\frac{\partial}{\partial r} = r\frac{\partial}{\partial r}$. Examining the (i,j) component of (2.26), we find the action of the homothety ξ on the metric g_{ij} of $f = \text{const}$ hypersurfaces

$$\xi g_{ij}(r, x^k) = r\frac{\partial}{\partial r}g_{ij}(r, x^k) = \mathtt{d}g_{ij}(r, x^k) \quad \Rightarrow \quad g_{ij}(r, x^k) = r^{\mathtt{d}}\bar{g}_{ij}(x^k).$$

As such the metric decomposes as

$$g = r^{\mathtt{d}-2}dr^2 + r^{\mathtt{d}}\bar{g}_{ij}(x^k)dx^i dx^j. \tag{2.32}$$

For the case $\mathtt{d} = 2$, we recognise this as the Riemannian metric cone over some base manifold B with coordinates x^i and metric $\bar{g}$ i.e. $M = C(B)$ [52].[5] For the extension to $\mathtt{d} \neq 2$ we have the metric of a $\mathtt{d}$-conic manifold, which we denote $M = C_{\mathtt{d}}(B)$.

We now turn our attention to condition (ii) in the definition. This states that $\nabla_X \xi = X$ for any $X \in \Gamma(TM)$. By introducing special coordinates $h^I = (r, rx^i)$ on M this condition becomes (recall that $\nabla_I = \partial_I$ for special coordinates)

$$X^I(\partial_I \xi^J)\partial_J = X^I \partial_I.$$

This equality holds providing $\partial_I \xi^J = \delta^J{}_I$ and consequently,

$$\xi^I = h^I \quad \Rightarrow \quad \xi = h^I \frac{\partial}{\partial h^I}. \tag{2.33}$$

Thus, in the coordinates $h^I = (r, rx^i)$, the homothety becomes an Euler vector on M. Notice the normalization of ξ is fixed by condition (ii) [48]. The Euler vector serves to define the homogeneity of tensor fields on M. In particular, observing that $[\xi, \partial_I] = -\partial_I$, we can plug the definition (2.33) back into the symmetric part of the Koszul formula (2.26) to acquire

$$\xi g_{IJ}(h) = (\mathtt{d} - 2)g_{IJ}(h). \tag{2.34}$$

Condition (ii) implies that on a $\mathtt{d}$-conic Hessian manifold in special coordinates, the components

[5] Of course, in the special case where the Riemannian cone is a cone of opening angle α embedded in Euclidean space, we have a circular base with metric $\bar{g} = dx_1^2 + \cdots + dx_{\dim(M)-1}^2$ and the constraint $x_1^2 + \cdots + x_{\dim(M)}^2 = \sin^2\alpha$ [27].

of the metric g are homogeneous functions of degree $\mathtt{d}-2$. The association of the Euler field ξ with the special coordinates h^I ensures metric homogeneity on the $\mathtt{d}$-cone. Moreover, since we know that $g_{IJ} = \partial^2_{IJ} H$, the Hesse potential for a $\mathtt{d}$-conic Hessian manifold must be homogeneous of degree $\mathtt{d}$ with respect to the special coordinates h^I.

From the perspective of the special coordinates h^I, $\xi = h^I \frac{\partial}{\partial h^I}$ acts as an Euler vector and the coordinates transform as $h^I \mapsto \lambda h^I$. If we instead express the special coordinates as $h^I = (r, rx^i)$, then we have seen that the privileged vector field takes the form $\xi = r\frac{\partial}{\partial r}$. For ξ to continue acting as an Euler vector, we require $h^I = (r, rx^i) \mapsto (\lambda r, \lambda r x^i)$. However, the ratio $x^i = \frac{h^i}{h^0}$ is not involved in this scaling and thus from the perspective of the coordinates (r, x^i), ξ acts as a homothety i.e. $(r, x^i) \mapsto (\lambda r, x^i)$. The projective coordinates x^i are scale invariant and their existence will be important later on when we come to discuss projective manifolds.

Definition 17. *A* CONIC AFFINE SPECIAL REAL MANIFOLD (M, g, ∇, ξ) *is a* 3*-conic Hessian manifold whose Hesse potential is a homogeneous cubic polynomial. In particular,* $H(h) = c_{IJK} h^I h^J h^K$ *for some constants* c_{IJK}.

Conic affine special real (CASR) manifolds are important in physics, appearing as the target manifolds in superconformally invariant theories of rigid $\mathcal{N} = 2$ vector multiplets. In particular, the requirement that the rigid theory be invariant under five-dimensional superconformal transformations forces the Hesse potential to be a homogeneous cubic polynomial [48]. Using the superconformal calculus it is then possible to formulate a five-dimensional theory of local supersymmetry (supergravity) which appears geometrically by taking a certain 'superconformal quotient' of the CASR manifold. We will develop the physics side of this point in more detail in Section 3.1.3 and for the time being focus on how to engineer the quotient of the 3-cone.

We can view a CASR manifold (M, g, ∇, ξ) as a domain $M \subset \mathbb{R}^n$ parametrized by special coordinates h^I with $I = 1, \ldots, n$. The Euler vector ξ induces an $\mathbb{R}^+$ dilatation of the coordinates

$$h^I \to \lambda h^I, \quad \lambda \in \mathbb{R}^+.$$

We therefore want to pick M such that it is invariant under such a scaling in order to preserve the structure of the 3-cone.

Definition 18. *A* PROJECTIVE SPECIAL REAL MANIFOLD $(\mathcal{H}, g_{\mathcal{H}})$ *is a hypersurface* $\mathcal{H} \subset M$ *given by*

$$\mathcal{H} = \{h^I \in M | H(h) = c_{IJK} h^I h^J h^K = 1\}. \tag{2.35}$$

First of all note that the projective special real (PSR) manifold is not itself a Hessian manifold but rather, it appears as a hypersurface within the Hessian CASR manifold which can, of course, be completely recovered from the PSR manifold via. the homothetic $\mathbb{R}^+$ action of ξ.

Let us denote the embedding of $\mathcal{H}$ into M by the map $\iota : \mathcal{H} \hookrightarrow M$. Given a CASR metric g, we can equip $\mathcal{H}$ with a Riemannian metric using the pull-back, $g_{\mathcal{H}} = \iota^*(-\frac{1}{3}g)$. However, when not confined to the PSR hypersurface $\mathcal{H}$, the CASR metric $-\frac{1}{3}g$ has Lorentzian signature $(- + \cdots +)$ with the negative eigendirection along the orbits of ξ [48]. Notice that $\pounds_\xi g = 3g$ and so the homothety does not represent an isometric direction of the CASR 3-cone. In other words, the metric is not preserved along ξ, and we cannot view the CASR as a collection of gauge equivalent level sets, nor construct the PSR manifold by taking a quotient.

Let us proceed by introducing a second metric on M by[6]

$$\begin{aligned} a &= \left(\frac{\partial^2 \tilde{H}}{\partial h^I \partial h^J}\right) dh^I \otimes dh^J \\ &= -2\left(\frac{(ch)_{IJ}}{chhh} - \frac{3}{2}\frac{(chh)_I (chh)_J}{(chhh)^2}\right) dh^I \otimes dh^J, \end{aligned} \tag{2.36}$$

where $\tilde{H} = -\frac{1}{3}\log H$ and $chhh := c_{IJK}h^I h^J h^K, (chh)_I := c_{IJK}h^J h^K$, etc. Immediately one can show that $\pounds_\xi a = 0$, i.e. the metric a is preserved along orbits of ξ which now acts as an isometry. If g has Lorentzian signature then a is necessarily positive definite $(+ + \cdots +)$. This isometry means that $M = \text{CASR}$ can be pictured as a collection of gauge-equivalent level sets, and the gauge-fixing condition (2.35) represents a particular choice of $\mathcal{H} = \text{PSR}$ by determining the scale transformations of the h^I. We should therefore define the PSR manifold by projecting out the action of the isometry ξ. This corresponds to the quotient

$$M = C_3(\mathcal{H}) \quad \Rightarrow \quad \mathcal{H} = M/\mathbb{R}^+,$$

with the quotient metric obtained from a.[7] For the hypersurface embedding $\iota : \mathcal{H} \hookrightarrow M$, the PSR metric can be obtained by the pull-back of the CASR metric,

$$g_{\mathcal{H}} = \iota^*\left(-\frac{1}{3}g\right) = \iota^*(a).$$

Notice that after we pull-back to $\mathcal{H}$, it is impossible to distinguish between the metrics g and a.

This geometry admits a set of projective coordinates ϕ^x that cover the PSR manifold $\mathcal{H}$. Given the embedding $\iota : \mathcal{H} \hookrightarrow M$, the metric $g_{\mathcal{H}}$ can be expressed either in terms of local coordinates ϕ^x on $\mathcal{H}$ or as the pull-back of the CASR metric tensor a as follows

$$g_{\mathcal{H}} = (g_{\mathcal{H}})_{xy} d\phi^x \otimes d\phi^y = \left(a_{IJ}\frac{\partial h^I}{\partial \phi^x}\frac{\partial h^J}{\partial \phi^y}\right)\Bigg|_{H(h)=1} d\phi^x \otimes d\phi^y.$$

[6] We refer the reader to [36] for a treatment of generalised projective special real manifolds where the dimension of the homogeneous Hesse potential is left arbitrary.

[7] This relies on the CASR metric a being a metric product of a one-dimensional factor and the horizontal lift of the PSR metric. In fact, the quotient metric must be obtained from the horizontal lift of the PSR metric. Thus it is easier to treat the PSR as a hypersurface in the CASR and find the metric by pull-back.

Splitting the CASR affine coordinates as $h^I = (h^0, h^x)$, we can obtain a particularly useful set of projective coordinates that are manifestly scale invariant,

$$\phi^x = \frac{h^x}{h^0}.$$

Note that since the Hesse potential is homogeneous degree three, we can write

$$H(h^0, h^1, \dots) = (h^0)^3 H(1, \frac{h^1}{h^0}, \dots) =: (h^0)^3 \hat{H}(\phi^1, \dots),$$

where $\hat{H}$ is a rescaled, non-homogeneous Hesse potential. Note that, since $H(h) = 1$ on the PSR, we can rearrange the above formula to define the coordinate h^0 on the PSR as

$$h^0 = \hat{H}(\phi^1, \dots)^{-\frac{1}{3}}.$$

2.3 Special (para-)Kähler geometry

Special Kähler geometry is relevant for understanding the consistent construction of four-dimensional $\mathcal{N} = 2$ vector multiplets coupled to supergravity. We will encounter both affine special Kähler and projective special Kähler manifolds. Similar to the special real manifolds in the previous section, these appear in the Lagrangian of rigid 4d $\mathcal{N} = 2$ vector multiplets and 4d $\mathcal{N} = 2$ vector multiplets coupled to supergravity respectively [27]. We shall also use this opportunity to introduce the Euclidean versions of these geometries, known as affine special para-Kähler and projective special para-Kähler respectively. Although we will not encounter such geometries when constructing solutions in later chapters, it is nonetheless interesting to examine their properties and summarise how they can materialise from certain dimensional reductions (see Section 3.3.1). In what follows, we introduce an ϵ parameter in order to treat both cases in parallel. This is defined by

$$\epsilon = \begin{cases} -1 & \text{complex manifold}\,, \\ +1 & \text{para-complex manifold}\,. \end{cases}$$

The material presented here simply includes the definitions necessary for a basic understanding, and is based on [40] as well as the treatment of such geometries in the 'Euclidean Supersymmetry' family of papers [7, 53, 54]. Let us begin by introducing ϵ-complex numbers which will play an important role throughout:

Definition 19. *The ring of ϵ-complex numbers $\mathbb{C}_\epsilon := \mathbb{R} \oplus i_\epsilon \mathbb{R}$ is obtained by adjoining to the real numbers a second number line with base unit the ϵ-imaginary number i_ϵ. The ϵ-imaginary*

number satisfies [54]

$$1.i_\epsilon = i_\epsilon.1 = i_\epsilon, \quad \bar{i_\epsilon} = -i_\epsilon, \quad i_\epsilon^2 = \epsilon.$$

In other words, for $\epsilon = -1$, we recover the standard imaginary unit i which squares to -1, whilst for $\epsilon = +1$, we obtain the para-imaginary unit e which squares to $+1$ i.e.

$$i^2 = -1, \qquad e^2 = +1.$$

Definition 20. *An* ALMOST ϵ-COMPLEX MANIFOLD (M, J) *is a manifold equipped with a global tangent space endomorphism $J \in \Gamma(End\, TM)$ such that at each $p \in M$,*

$$J_p^2 = \epsilon Id_{T_pM}.$$

For $\epsilon = -1$, this defines an ALMOST COMPLEX STRUCTURE *on an* ALMOST COMPLEX MANIFOLD*, whilst the case $\epsilon = +1$ defines an* ALMOST PARA-COMPLEX STRUCTURE *on an* ALMOST PARA-COMPLEX MANIFOLD*.*

First note that the existence of such a tensor immediately induces a dual endomorphism of the cotangent space, $J^* \in \Gamma(\text{End } T^*M)$, defined as

$$\begin{aligned} J^* &: T^*M \to T^*M \\ &\omega(X) \mapsto (J^*\omega)(X) = \omega(JX), \quad \text{for any } X \in \Gamma(TM),\, \omega \in \Gamma(T^*M). \end{aligned}$$

Furthermore the existence of an almost ϵ-complex structure allows for a decomposition of the tangent space. To analyse this statement, we shall treat the two situations separately starting with the complex ($\epsilon = -1$) case.

Given an almost complex manifold (M, J), with J defined as above, it is possible to complexify the tangent space at each $p \in M$ via $T_pM \mapsto T_pM^{\mathbb{C}} := T_pM \otimes \mathbb{C}$. Given such a complexification, there is a natural extension of the almost complex structure to an endomorphism $J_p : T_pM \otimes \mathbb{C} \to T_pM \otimes \mathbb{C}$. Since $J_p^2 = -\text{Id}_{T_pM\otimes\mathbb{C}}$, J_p acts as an anti-involution and thus has eigenvalues $\pm i$ in $T_pM \otimes \mathbb{C}$. Consequently, there must exist two *complex eigenspaces* of J_p associated with the eigenvalues $+i$ and $-i$ respectively. As such we find a decomposition of the complexified tangent space into two complimentary subspaces

$$T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^-,$$

where

$$T_pM^{\pm} = \text{Ker}(\text{Id} \pm iJ_p) = \{Z \in T_pM^{\mathbb{C}} | J_pZ = \pm iZ\}.$$

Henceforth we adopt the notation $T_pM^{(1,0)}$ and $T_pM^{(0,1)}$ for these eigenspaces and refer to them

as the holomorphic tangent space and anti-holomorphic tangent space respectively. Note that they are complex conjugate to one another and isomorphic to $\mathbb{C}^d$. Furthermore, given such a decomposition, we can in fact split any vector into a sum of holomorphic and anti-holomorphic pieces. The eigenvalues of the almost complex structure are forced to come in pairs of $\pm i$ ensuring the eigendistributions have equal dimension.

Turning to the case of an almost para-complex manifold (M, J), we have $J_p^2 = \mathrm{Id}_{T_pM}$, meaning J_p now acts as an involution. As such the tangent space now decomposes into two *real eigenspaces*, namely

$$T_pM = T_pM^+ \oplus T_pM^-,$$

where

$$T_pM^\pm = \mathrm{Ker}(\mathrm{Id} \mp J_p) = \{Z \in T_pM | J_pZ = \pm Z\}.$$

Following the conventions of [7], the para-holomorphic tangent space $T_pM^{(1,0)}$ and the anti-para-holomorphic tangent space $T_pM^{(0,1)}$ are para-complex conjugates of one another and isomorphic to $\mathbb{R}^d$. Once more, this allows for a similar decomposition of vectors on M into their para-holomorphic and anti-para-holomorphic parts. In the para-complex case it is not true that eigenvalues of the almost para-complex structure come in pairs and we must impose $\dim(T^+M) = \dim(T^-M)$ by hand [7].

Note that in either case, it is always possible to find coordinates at $p \in M$ such that

$$J_p = \begin{pmatrix} 0 & \epsilon\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Whilst it is always possible to find coordinates at $p \in M$ such that J_p takes the above form, it is not generally possible to find coordinates such that J_p takes this form in an entire neighbourhood of p. If such coordinates exist, they are called *local ϵ-holomorphic coordinates for J* [55]. If M admits local ϵ-holomorphic coordinates around every point, they patch together to form an ϵ-holomorphic atlas on M and J is said to be *integrable*. We shall now explore under what circumstances the almost ϵ-complex structure becomes integrable. We shall review only the key results and refer the reader to [37, 42] for a more in depth discussion.

Theorem 2. ***(Newlander-Nirenberg).*** *The complex eigendistribution $TM^{(1,0)}$ on an almost complex manifold ($\epsilon = -1$) is* INTEGRABLE *if*

$$[\Gamma(TM^{(1,0)}), \Gamma(TM^{(1,0)})] \subset \Gamma(TM^{(1,0)}).$$

Complex conjugation then establishes $TM^{(0,1)}$ as also being an integrable eigendistribution. The almost complex structure J is integrable and referred to as a COMPLEX STRUCTURE.

Theorem 3. ***(Frobenius).*** *Consider the real eigendistributions $T^\pm M$ on an almost para-*

complex manifold ($\epsilon = +1$). The distributions $T^{\pm}M$ are INTEGRABLE *if*

$$[\Gamma(T^{\pm}M), \Gamma(T^{\pm}M)] \subset \Gamma(T^{\pm}M).$$

If both $T^{\pm}M$ are integrable then the almost para-complex structure is integrable and referred to as a PARA-COMPLEX STRUCTURE.

A more convenient means of checking integrability of the ϵ-complex structure is given by the following theorem:

Theorem 4. *The ϵ-complex structure is integrable if the Nijenhuis tensor*

$$N(X,Y) = -J^2(X,Y) + J[JX,Y] + J[X,JY] - [JX,JY],$$

is vanishing for any $X, Y \in \Gamma(TM)$.

The proof of this theorem can be found in Theorem 8.12 of [40] for the complex case and in Proposition 1 of [7] for the para-complex case.

Definition 21. *An almost ϵ-complex manifold (M, J) is an ϵ-*COMPLEX MANIFOLD *if the almost ϵ-complex structure is integrable.*

Definition 22. *A function $f : M \to \mathbb{C}_\epsilon$ is ϵ-*HOLOMORPHIC *on M if*

$$df \circ J = i_\epsilon df,$$

where i_ϵ is the ϵ-imaginary unit and J is an ϵ-complex structure on M.

We now introduce the additional structure of a metric to our ϵ-complex manifold:

Definition 23. *An ϵ-complex pseudo-Riemannian manifold (M, g, J) is an ϵ-*HERMITIAN MANIFOLD *if the metric is compatible with the ϵ-complex structure in the following sense:*

$$g_p(JX, JY) = -\epsilon g_p(X,Y), \quad X, Y \in \Gamma(T_pM).$$

In this case the metric g is said to be an ϵ-Hermitian metric.

Given an ϵ-Hermitian manifold (M, g, J), there always exists a *fundamental* 2*-form* ω defined through the following action:

$$\omega(X,Y) = g(JX,Y), \quad \text{for any } X, Y \in \Gamma(TM).$$

From the decomposition of the tangent space observed earlier we can infer that a Hermitian manifold has metric signature $(2p, 2q)$, whilst a para-Hermitian manifold has metric signature (n, n).

Definition 24. *An ϵ-Hermitian manifold (M, g, J) is an* ϵ-KÄHLER MANIFOLD *if the fundamental 2-form ω is closed, $d\omega = 0$. This is equivalent to the statement $DJ = 0$ where D is again the Levi-Civita connection.*[8] *It is then possible to locally find a set of ϵ-holomorphic coordinates $\{z^a, \bar{z}^b\}$ such that the metric g is represented as [27]*

$$g = Re\left(\frac{\partial^2 K}{\partial z^a \partial \bar{z}^b} dz^a \otimes d\bar{z}^b\right),$$

where K is known as the ϵ-KÄHLER POTENTIAL.

Definition 25. *An* AFFINE SPECIAL ϵ-KÄHLER MANIFOLD *(M, g, J, ∇) is an ϵ-Kähler manifold (M, g, J) equipped with a flat, torsion-free 'special' connection ∇ such that*

(i) ∇g is completely symmetric

(ii) $\nabla\omega = 0$

Condition (i) tells us the manifold is Hessian with respect to the special connection. Indeed, following Section 2.2, we should be able to cover M with a set of special ϵ-holomorphic coordinates X^I. Condition (ii) can be viewed as ensuring compatibility of the Hessian and Kähler structures such that, on a special coordinate patch of M, there exists a local ϵ-holomorphic function known as the *prepotential*, $F(X)$, such that the ϵ-Kähler potential is given by [7]

$$K(X, \bar{X}) := i_\epsilon(X^I \bar{F}_I - F_I \bar{X}^I),$$

where $F_I = \partial_I F(X)$. The components of the metric are then given by [7]

$$N_{IJ} := \frac{\partial^2 K}{\partial X^I \partial \bar{X}^J} = -i_\epsilon(F_{IJ} - \bar{F}_{IJ}) = -\epsilon 2\mathrm{Im}(F_{IJ}), \tag{2.37}$$

where we use N_{IJ} rather than g_{IJ} for consistency with the literature. Affine special ϵ-Kähler manifolds appear as the scalar geometries in the Lagrangians describing rigid $\mathcal{N} = 2$ vector multiplets in four dimensions carrying either Minkowski ($\epsilon = -1$) or Euclidean ($\epsilon = +1$) signature [7, 27].

Definition 26. *A* CONIC AFFINE SPECIAL ϵ-KÄHLER MANIFOLD *(N, g_N, J, ∇, ξ) is an affine special ϵ-Kähler manifold (N, g_N, J, ∇) supported by a vector field ξ such that:*

(i) $D\xi = \mathbb{1}$, where D is the Levi-Civita connection with respect to g_N

(ii) $\nabla\xi = \mathbb{1}$.

It is clear from comparison with Definition 16 that this is an example of a 2-conic Hessian manifold (a Riemannian metric cone with Hessian structure). We also know from our analysis

[8] See Theorem 8.5 in [40] for a proof of this statement.

of $\mathtt{d}$-conic Hessian manifolds that the vector field ξ is an Euler vector with respect to the special ϵ-holomorphic coordinates X^I on N. Recalling the decomposition of the tangent space on a complex manifold, we have

$$\xi = X^I \frac{\partial}{\partial X^I} + \bar{X}^I \frac{\partial}{\partial \bar{X}^I}.$$

Compared to the CASR manifolds, conic affine special ϵ-Kähler (CASϵK) manifolds come with an additional structure, namely the complex structure J. Its presence creates a second privileged vector field, $J\xi$. With respect to the special ϵ-holomorphic coordinates on N, this assumes the form [3]

$$J\xi = i_\epsilon X^I \frac{\partial}{\partial X^I} - i_\epsilon \bar{X}^I \frac{\partial}{\partial \bar{X}^I}.$$

The vectors $\{\xi, J\xi\}$ are in fact the infinitesimal generators of a $\mathbb{C}^*_\epsilon$ action on the CASϵK manifold, N. For the case with $\epsilon = -1$, the resulting finite transformations are [3]

$$\xi : X^I \mapsto |\lambda| X^I, \quad J\xi : X^I \mapsto e^{i\phi} X^I.$$

It is obvious that ξ generates dilatations (as it did before on the CASR manifold) whilst $J\xi$ generates $U(1)$ transformations. Together, they sweep out a complex cone over the base. For the case with $\epsilon = +1$, we refer the reader to [53, 54] for a review of how these two vectors generate a para-complex cone over the base.

From our earlier discussion of $\mathtt{d}$-conic manifolds, we know from (2.28) that ξ acts homothetically on g_N, whilst a similar calculation reveals $J\xi$ acts isometrically,

$$\pounds_\xi g_N = 2 g_N, \quad \pounds_{J\xi} g_N = 0.$$

We know from (2.34) that a $\mathtt{d}$-conic manifold that is also Hessian has metric components that are homogeneous of degree $\mathtt{d} - 2$. For the 2-conic CASϵK manifold at hand, the metric must be homogeneous degree 0. Given the agreement of Hessian and Kähler structures on the CASϵK, we can deduce that both the Kähler potential and thus the prepotential are homogeneous functions of degree 2 in the special coordinates X^I.

In analogy with our treatment of the CASR manifold in Section 2.2, it is useful to introduce the following rank two tensor field g,

$$g = \mathrm{Re}\left(\frac{\partial^2 \mathcal{K}}{\partial X^I \partial \bar{X}^J} dX^I \otimes d\bar{X}^J \right),$$

with $\mathcal{K}(X, \bar{X}) = -\log K(X, \bar{X})$ and K the ϵ-Kähler potential for the CASϵK metric g_N above. Note from (2.37) that $K = \bar{X} N X$ and so, the components of g can be written as

$$g_{IJ} = \frac{\partial^2 \mathcal{K}}{\partial X^I \partial \bar{X}^J} = -\frac{\partial}{\partial X^I}\left(\frac{1}{K}\right)\frac{\partial K}{\partial \bar{X}^J} - \frac{1}{K}\frac{\partial^2 K}{\partial X^I \partial \bar{X}^J} = -\frac{N_{IJ}}{\bar{X} N X} + \frac{(N\bar{X})_I (NX)_J}{(\bar{X} N X)^2}. \tag{2.38}$$

It is immediate that this has a two-dimensional kernel

$$X^I g_{IJ} = g_{IJ}\bar{X}^J = 0,$$

and hence the rank two tensor g is degenerate along the two-dimensional subspace of N spanned by ξ and $J\xi$ i.e.

$$g(\xi, \cdot) = g(J\xi, \cdot) = 0.$$

An obvious consequence of this degeneracy is that g cannot be a metric on the CASϵK manifold.[9] However, with respect to the tensor field g, both ξ and $J\xi$ act isometrically

$$\pounds_\xi g = \pounds_{J\xi} g = 0.$$

CASϵK manifolds appear in the description of a superconformally invariant theory of rigid $\mathcal{N} = 2$ vector multiplets and, just like the CASR manifold did for the corresponding five-dimensional theory, they will allow us to move between rigid and local theories of 4d, $\mathcal{N} = 2$ supergravity by taking a superconformal quotient. Geometrically, this leaves behind the base manifold of the Riemannian cone as we shall describe below.

Definition 27. *A* PROJECTIVE SPECIAL ϵ-KÄHLER MANIFOLD $(\bar{N}, \bar{g}, \bar{J}, \bar{\nabla})$ *is defined as the quotient manifold $N/\mathbb{C}^*_\epsilon$ of a conic affine special ϵ-Kähler manifold (N, g_N, J, ∇, ξ). The ϵ-Kähler metric $\bar{g}$ on $\bar{N}$ is induced from the (non-metric) rank two tensor field g on N, whilst the ϵ-complex structure $\bar{J}$ and connection $\bar{\nabla}$ are induced directly from J, ∇ on N.*

The projective special ϵ-Kähler (PSϵK) manifold represents a codimension-2 surface within the CASϵK manifold obtained by imposing constraints that fix the action of the homothety ξ and the isometry $J\xi$. The action of the homothety on the CASϵK metric g_N can be fixed by imposing $g_N(\xi, \xi) = \text{const}$. A particularly useful way to restrict this action is to fix the value of the ϵ-Kähler potential for g_N to unity. This defines a smooth hypersurface $S \subset N$ within the Riemannian cone,

$$S = \{K = 1\} \subset N.$$

According to the treatment of Kähler cones in [27], S retains the structure of an ϵ-Kähler

[9] Focussing on the Lorentzian case ($\epsilon = -1$) that will be useful for writing down a four-dimensional supergravity action in Section 3.1.3, we see that the tensor field g we have constructed here is in fact the horizontal lift of the PSK metric, $\bar{g}$ i.e. $g = \pi^*\bar{g}$, which explains the observed degeneracy. The CASϵK metric is then $g_N = \partial^2_{IJ} K dX^I dX^J = \partial^2_{IJ} H dX^I dX^J = \pi^*\bar{g} - \alpha^2 - \beta^2$, where α and β are the 1-forms dual to ξ and $J\xi$, and g has signature $(--+\cdots+)$. However, there are a number of other tensors we are able to construct on N. For example, we can define $\tilde{g} = \partial^2_{IJ}(\log H) = \pi^*\bar{g} + \alpha^2 - \beta^2$ which is a Lorentzian signature $(-+\cdots+)$ metric on the CASϵK with the sign associated to the ξ direction flipped. There is also the metric $\hat{g} = \pi^*\bar{g} + \alpha^2 + \beta^2$ which has positive definite signature $(++\cdots+)$. It is $\hat{g}$ that is related to the vector coupling in four-dimensional supergravity, and that is the true analogue of the CASR metric a that we constructed in the special real case [36].

manifold equipped with a homothetic Killing vector ξ and is called a *Sasakian manifold.*[10] On such a manifold, the Killing vector $J\xi$ generating the $U(1)$ isometry is known as the Reeb vector field. Furthermore, since the Reeb vector remains unconstrained, the Sasakian hypersurface can be envisaged as a $U(1)$ principal bundle over the base manifold, $\bar{N}$. In order to descend from Sasakian to the base we must quotient out the $U(1)$ action, and this can be done by imposing $\mathrm{Im}(X^0) = 0$ which constrains the phase of the special ϵ-holomorphic coordinate, X^0.

Having quotiented the $\mathbb{C}^*_\epsilon$ action, the base manifold $\bar{N}$ is a PSϵK manifold. We now understand the structure of the ϵ-Kähler Riemannian cone

$$N = C(\bar{N}) \quad \Rightarrow \quad \bar{N} = N/\mathbb{C}^*_\epsilon, \text{ or equivalently, } \bar{N} = S/U(1).$$

Labelling the CASϵK manifold coordinates as $X^I = \{X^0, X^1, X^2, \dots\} = \{X^0, X^A\}$, the PS$\epsilon$K manifold can be parametrized by a set of projective coordinates $z^A = \frac{X^A}{X^0}$, subject to the aforementioned constraints on the X^I. Note that since the CASϵK prepotential $F(X)$ is homogeneous of degree 2, we can express it in terms of a rescaled, non-homogeneous prepotential $\mathcal{F}(z)$ using

$$F(X) = F(X^0, X^1, \dots) = (X^0)^2 F\left(1, \frac{X^1}{X^0}, \dots\right) =: (X^0)^2 \mathcal{F}(z^1, \dots).$$

Since the rank two tensor field g is both isometric along, and transverse to, the $\mathbb{C}^*_\epsilon$ action, it can be projected down to produce a natural metric on the quotient space. Indeed, the ϵ-Kähler metric $\bar{g}$ on the PSϵK manifold is

$$\bar{g} = \mathrm{Re}\left(\frac{\partial^2 \mathcal{K}(z,\bar{z})}{\partial z^A \partial \bar{z}^B} dz^A \otimes d\bar{z}^B\right), \tag{2.39}$$

where the ϵ-Kähler potential is[11]

$$\mathcal{K}(z,\bar{z}) = -\log\left(i_\epsilon\left[2(\mathcal{F} - \bar{\mathcal{F}}) - (\mathcal{F}_A + \bar{\mathcal{F}}_A)(z^A - \bar{z}^A)\right]\right),$$

and the coordinates $z^A = \frac{X^A}{X^0}$ are invariant under the action of ξ and $J\xi$. PSϵK manifolds are physically useful for describing the coupling of $\mathcal{N} = 2$ vector multiplets to four-dimensional supergravity with a spacetime signature that is either Lorentzian ($\epsilon = -1$) or Euclidean ($\epsilon = +1$).

The real formulation of affine special Kähler geometry

In the following we restrict ourselves to the complex case ($\epsilon = -1$) as that will be relevant for the work in this thesis. The reader is referred to [36, 54] for details on the $\epsilon = +1$ case.

[10] According to Definition-Theorem 10 in [56] a Sasakian manifold is one whose metric cone is Kähler i.e. $C(S) = \mathbb{R}^+ \times S$ is Kähler, which is clearly the case here.

[11] See Proposition 7 of [53] for an explicit check that this is ϵ-Kähler i.e. a check that the fundamental 2-form is closed.

On any affine special Kähler (ASK) manifold, and consequently any conic affine special Kähler (CASK) manifold , it is possible to make the decomposition

$$X^I := x^I + iu^I(x,y), \quad F_I := y_I + v_I(x,y).$$

We define the *special real coordinates* on the ASK manifold to be

$$q^a := \begin{pmatrix} x^I \\ y_I \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X^I \\ F_I \end{pmatrix}.$$

ASK manifolds are themselves Hessian manifolds and the special real coordinates are ∇-affine coordinates for the special connection ∇. Consequently, the ASK metric can be written as [36]

$$g = \mathrm{Re}\left(N_{IJ} dX^I \otimes d\bar{X}^J\right) = H_{ab} dq^a \otimes dq^b,$$

for some Hesse potential $H(q^a)$. Thus H_{ab} is the real version of the Kähler metric N_{IJ} in (2.37), and correspondingly H_{ab} has negative directions orientated along the $\mathbb{C}^*$ directions of the CASK. The Hesse potential $H(q^a)$ is related to the holomorphic prepotential $F(X^I)$ by a Legendre transformation $(x^I, u_I) := (\mathrm{Re}X^I, \mathrm{Im}X^I) \to (x^I, y_I)$:

$$H(x^I, y_I) = 2\mathrm{Im}F(X^I(x,y)) - 2y_I u^I(x,y).$$

The coordinates (x^I, y_I) form a flat Darboux coordinate system, i.e.

$$\omega = 2dx^I \wedge dy_I = \Omega_{ab} dq^a \wedge dq^b, \quad \nabla dx^I = \nabla dy_I = 0.$$

It is also noteworthy that, on an ASK manifold, the first derivatives of the Hesse potential are related to the imaginary parts of X^I and F_I by

$$H_a := \frac{\partial H}{\partial q^a} = 2\begin{pmatrix} v_I \\ -u^I \end{pmatrix},$$

and form an alternative, 'dual' coordinate system but with respect to a different special connection. Since H is homogeneous of degree two, the two coordinate systems are related by

$$H_a = H_{ab} q^b \Leftrightarrow q^a = H^{ab} H_b.$$

2.4 (para-)Quaternionic-Kähler and hyperkähler geometry

Hyperkähler and quaternionic-Kähler geometries appear in physics when describing $\mathcal{N} = 2$ hypermultiplets coupled to supergravity. The target manifold of rigid (resp. local) $\mathcal{N} = 2$ hypermultiplets possesses a hyperkähler (resp. quaternionic-Kähler) geometry, and the two are related by a superconformal quotient similar to that previously seen for the special real and special Kähler geometries [27]. Their para-quaternionic-Kähler and para-hyperkähler cousins will be relevant for the Euclidean version of such theories. We shall also see how ϵ-quaternionic-Kähler manifolds can appear from dimensional reduction in Sections 3.3.2 and 3.3.3.

Given their mathematical complexity we aim to provide only a brief introduction to their structure, largely following [37]. Readers interested in a more detailed description may refer to the references provided throughout this section.

2.4.1 Symplectic group actions

We have already seen from Table 2.1 that such manifolds are characterised by the appearance of symplectic groups in Berger's holonomy classification. Let us now introduce the symplectic group and its real forms:

Symplectic group

The symplectic group $Sp(n, \mathbb{C}) \subset GL(2n, \mathbb{C})$ is the group of $2n \times 2n$ matrices with complex entries satisfying

$$M^T \Omega M = \Omega,$$

where Ω is the skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}.$$

The symplectic group has two real forms. These are:

1, Compact real form

The compact real form $Sp(n)$ is defined as the intersection

$$Sp(n) := Sp(n, \mathbb{C}) \cap U(2n),$$

of $2n \times 2n$ complex symplectic matrices that are also unitary:

$$M^T \Omega M = \Omega \quad \text{and} \quad M^\dagger M = \mathbb{1}_{2n} \Rightarrow M^\dagger = M^{-1},$$

where $M^\dagger$ denotes the (complex) conjugate transpose of M. As an example, consider the situation with $n = 1$;

$$Sp(1) = Sp(2, \mathbb{C}) \cap U(2) \cong SU(2) \subset SO(4),$$

and since $SU(2)$ is isomorphic to S^3, we see that $Sp(1)$ is a real Lie group.[12]

2, Normal (split) real form

The normal (split) real form $Sp(n, \mathbb{R})$ is defined as the intersection

$$Sp(n, \mathbb{R}) := Sp(n, \mathbb{C}) \cap GL(2n, \mathbb{R}),$$

of $2n \times 2n$ complex symplectic matrices with real entries. Unlike the compact real form, this group is non-compact.

2.4.2 (para-)Hyperkähler manifolds

Definition 28. *An* ALMOST ϵ-HYPERCOMPLEX STRUCTURE $\{J_1, J_2, J_3\}$ *on a manifold M is a triple of pairwise anti-commuting tangent space endomorphisms J_1, J_2, J_3 satisfying the ϵ-quaternionic algebra*

$$J_1^2 = J_2^2 = -\epsilon J_3^2 = \epsilon, \quad J_3 = J_1 J_2.$$

Notice that whilst an almost hypercomplex structure is built from three almost complex structures, an almost para-hypercomplex structure consists of two almost para-complex structures and an almost complex structure. For reference, see [58] for the case $\epsilon = -1$ and [54] for the case $\epsilon = +1$.

Definition 29. *A manifold M is* ALMOST ϵ-QUATERNIONIC *if there exists a sub-bundle $Q \subset End(TM)$ such that for any open neighbourhood $U \subset M$*

$$Q|_U = span\{J_1, J_2, J_3\},$$

where J_i are a basis for almost ϵ-hypercomplex structures on M. Q is said to be an almost ϵ-quaternionic structure on M [59].

It is then possible to give the almost ϵ-quaternionic manifold additional structure:

Definition 30. *A pseudo-Riemannian manifold (M, g, Q) is* ϵ-QUATERNIONIC HERMITIAN *if the basis of almost ϵ-hypercomplex structures are metric compatible [60] i.e. at any $p \in M$,*

$$g_p(JX, JY) = -\epsilon g_p(X, Y), \quad for\ X, Y \in T_pM, J \in Q_p.$$

[12] For an explicit proof of this isomorphism, refer to [57].

We now introduce the concept of ϵ-hyperkähler manifolds which appear in the study of rigid $\mathcal{N} = 2$ hypermultiplets. We will deal with the cases $\epsilon = -1$ (hyperkähler) and $\epsilon = +1$ (para-hyperkähler) separately. In the following, D is the Levi-Civita connection introduced previously.

Definition 31. *A $4n$-dimensional Riemannian manifold (M, g) is* HYPERKÄHLER *if the Riemannian holonomy group $Hol(D)$ is contained within $Sp(n)$.*

Notice that since $Sp(n) = Sp(n, \mathbb{C}) \cap U(2n) \subset U(2n) \subset SO(4n) \subset GL(4n, \mathbb{R})$, it is immediate from Table 2.1 that hyperkähler manifolds are themselves both Kähler and orientable.

Definition 32. *A $4n$-dimensional pseudo-Riemannian manifold (M, g) is* PARA-HYPERKÄHLER *if its holonomy group $Hol(D)$ is contained within $Sp(n, \mathbb{R})$ [61].*

Since $Sp(n, \mathbb{R}) = Sp(n, \mathbb{C}) \cap GL(2n, \mathbb{R}) \subset GL(2n, \mathbb{R}) \subset SO(2n, 2n) \subset GL(4n, \mathbb{R})$, and since the para-unitary group is $U_\pi(2n) = GL(2n, \mathbb{R})$, it is immediate that para-hyperkähler manifolds are themselves both para-Kähler and orientable [7].

Finally, note that both hyperkähler and para-hyperkähler manifolds are Ricci-flat [62, 63].

2.4.3 (para-)Quaternionic-Kähler manifolds

Quaternionic-Kähler manifolds are a more general class of Riemannian manifolds that incorporate their hyperkähler cousins [64]. Physically, quaternionic-Kähler geometries have a crucial role to play in supergravity and string theory where they appear as target spaces for hypermultiplet scalar fields in three-, four- and five-dimensional $\mathcal{N} = 2$ supergravity theories, as shown in [6]. Having seen how the quotienting out of an $\mathbb{R}^+$ or $\mathbb{C}^*$ action provides a geometric manifestation of a rigid/local correspondence for special real and special Kähler geometries, we should expect to find something similar here.

Indeed, it is shown in [65] that a theory of $\mathcal{N} = 2$ hypermultiplets invariant under rigid superconformal transformations forms a hyperkähler cone (HKC) [66, 67] over some base manifold.[13] It is shown in [68] that dividing out an $\mathbb{H}^*$ action from this cone produces a codimension-4 manifold that inherits a quaternionic-Kähler structure. We will not discuss the details of this but merely point out that, by extension of the $\mathbb{C}^*$ quotient discussed previously, fixing the action of the homothety on the hyperkähler manifold will now leave behind a tri-Sasaki-Einstein manifold.[14] The almost hypercomplex structure provides a triplet of Reeb vectors generating a 3-dimensional foliation of the cone which can be projected out to descend to the quaternionic-Kähler leaf space as discussed in [69]. This structure will have an important role to play when we come to gauging the R-symmetry group in Section 3.4.

[13] In light of previous sections, it would be natural to name hyperkähler manifolds as 'affine quaternionic-Kähler manifolds' but, as remarked in [27], the non-standard nomenclature is a consequence of the mathematical community studying such manifolds before their physical significance was realised.

[14] A Sasakian manifold is one whose metric cone is Kähler, whilst a Sasaki-Einstein manifold has a Calabi-Yau metric cone, and a tri-Sasaki-Einstein manifold has a metric cone which is hyperkähler [56].

On the other hand, para-quaternionic-Kähler manifolds appear from a similar procedure starting instead with a para-hyperkähler manifold. Throughout the remainder of this section, we shall treat the $\epsilon = -1$ (quaternionic-Kähler) and $\epsilon = +1$ (para-quaternionic-Kähler) manifolds separately.

Definition 33. *A* QUATERNIONIC-KÄHLER *manifold is a Riemannian manifold* (M, g) *of dimension* $4n > 4$ *with Riemannian holonomy group* $Hol(D) \subset Sp(n) \cdot Sp(1)$ *[70].*

Observe that Definition 33 does not apply to 4-dimensional quaternionic-Kähler manifolds. In this case the holonomy group would reduce to $Sp(1) \cdot Sp(1) \cong SO(4)$ and so any oriented 4-dimensional manifold would be classed as quaternionic-Kähler which is undesirable [64]. Instead, we provide the following stricter definition:

Definition 34. *A* 4*-dimensional Riemannian manifold* (M, g) *of signature* $(4, 0)$ *is quaternionic-Kähler if it is oriented, Einstein and has self-dual Weyl tensor.*

In fact it can be shown that all quaternionic-Kähler manifolds are Einstein [42]. That is, $R_{\mu\nu} = cg_{\mu\nu}$ for some constant c. Furthermore, quaternionic-Kähler manifolds are Ricci-flat with $c = 0$ iff they are locally hyperkähler.[15] This implies a natural division between those quaternionic-Kähler manifolds with positive curvature and those with negative curvature. It is quaternionic-Kähler manifolds of negative curvature that are physically interesting, given their appearance in the description of $\mathcal{N} = 2$ hypermultiplets coupled to supergravity [6].

We can relate quaternionic-Kähler manifolds to the quaternionic structure defined earlier via the following [42]

Theorem 5. *A* $(4n > 4)$*-dimensional Riemannian manifold* (M, g) *is quaternionic-Kähler iff there exists a quaternionic structure* Q *such that* (M, g, Q) *is quaternionic Hermitian and:*

(i) The structure of Q *is preserved under the Levi-Civita connection i.e.* $DJ_\alpha = \sum_\beta k_{\alpha\beta} J_\beta$ *for all* $J_\alpha \in Q$*, and where* $k_{\alpha\beta}$ *is a matrix of one-forms.*

(ii) For any $p \in U_i \cap U_j$*, the quaternionic structures* Q_p *on* U_i *and* U_j *agree.*

Note that quaternionic-Kähler manifolds are, in general, neither Kähler nor even complex. We refer the reader to [36, 37] for a discussion on the practicalities of demonstrating that a manifold is in fact quaternionic-Kähler using the Ambrose-Singer Theorem and the Levi-Civita connection 1-form.

We end this section by introducing para-quaternionic-Kähler manifolds. Let us consider a $4n$-dimensional pseudo-Riemannian manifold (M, g), where g has neutral metric signature $(2n, 2n)$. We then have the definition:

[15] See Theorem 14.45 of [42] for a proof that there don't exist any Ricci-flat quaternionic-Kähler manifolds that are not hyperkähler.

Definition 35. *A* PARA-QUATERNIONIC-KÄHLER *manifold is a pseudo-Riemannian manifold* (M, g) *of dimension* $4n > 4$ *with Riemannian holonomy group* $Hol(D) \subset Sp(n, \mathbb{R}) \cdot Sp(1, \mathbb{R})$

As before, this definition breaks down for the case $n = 1$. We proceed to treat separately the 4-dimensional para-quaternionic-Kähler manifolds and define them by requiring the stricter condition that they be oriented, Einstein and have a self-dual Weyl tensor. Again, the Einstein property generalises to arbitrary n, i.e. $R_{\mu\nu} = cg_{\mu\nu}$ for all para-quaternionic-Kähler manifolds. The difference this time is the split metric signature.

We can also relate para-quaternionic-Kähler manifolds to the para-quaternionic structure defined earlier via the following:

Theorem 6. *A* $(4n > 4)$*-dimensional pseudo-Riemannian manifold* (M, g) *is para-quaternionic-Kähler iff there exists a para-quaternionic structure* $\tilde{Q}$ *such that* $(M, g, \tilde{Q})$ *is para-quaternionic Hermitian and:*

(i) The structure of $\tilde{Q}$ *is preserved under the Levi-Civita connection i.e.* $D\tilde{J}_\alpha = \sum_\beta \tilde{k}_{\alpha\beta}\tilde{J}_\beta$ *for all* $\tilde{J}_\alpha \in Q$*, and where* $\tilde{k}_{\alpha\beta}$ *is a matrix of one-forms.*

(ii) For any $p \in U_i \cap U_j$*, the quaternionic structures* $\tilde{Q}_p$ *on* U_i *and* U_j *agree.*

Note that para-quaternionic-Kähler manifolds are in general neither para-Kähler nor even para-complex. Again, it will not be necessary for us to check explicitly at any point that a manifold is para-quaternionic-Kähler and the reader may again consult [36, 37] for details on how to do this.

3 Preliminary physics

In this chapter we introduce the necessary background physics for understanding the main results of this thesis. We begin in Section 3.1 with an introduction to $\mathcal{N} = 2$ supergravity in four and five dimensions, both from the perspective of the supersymmetry algebra and from the Lagrangian. In Sections 3.2 and 3.3 we introduce the reader to dimensional reduction which will be an important solution generating technique throughout this work. We then use Section 3.4 to explain the procedure of gauging supergravity, and specifically, how to construct the Lagrangian for the particular theory of Fayet-Iliopoulos gauged $\mathcal{N} = 2$ supergravity that we are interested in. In Sections 3.5 and 3.6 we give a short primer on the basic idea of holography and discuss the Anti de-Sitter and hyperscaling-violating Lifshitz spacetimes on which it is based. We conclude this chapter with a review of black holes in Section 3.7, specifically focussing on their horizons and their thermodynamics.

3.1 $\mathcal{N} = 2$ ungauged supergravity

The material presented in this thesis deals exclusively with $\mathcal{N} = 2$ supergravity in four and five dimensions. Such $\mathcal{N} = 2$ theories have eight real supercharges [27]. We shall now explain how to construct an action for such a theory. In particular, we shall provide details on the four-dimensional $\mathcal{N} = 2$ supersymmetry algebra and the structure of its irreducible representations, known as superparticles. We will offer a short review along with suitable references of how this same procedure applies in five dimensions. Afterwards, we use these superparticles to construct various supergravity actions that will be important in later chapters.

3.1.1 Four-dimensional representations of $\mathcal{N} = 2$ supergravity

Recall that the laws of physics are invariant under Poincaré transformations. These transformations consist of translations and Lorentz transformations (rotations and boosts), generated by the operators P_μ and $M_{\mu\nu}$ respectively. These operators satisfy the Poincaré Lie algebra

$$\begin{aligned}
&[P_\mu, P_\nu] = 0\,,\\
&[M_{\mu\nu}, P_\rho] = i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right),\\
&[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}\right).
\end{aligned} \tag{3.1}$$

It was famously proven by Coleman and Mandula [71] that any bosonically generated symmetries of the QFT S-matrix must commute with the Poincaré algebra (3.1) and therefore additional, non-trivial symmetries must be internal with generators that transform as scalars. Golfand and Likhtman showed this no-go theorem can be circumvented by generalizing the Lie algebra to a $\mathbb{Z}_2$-graded Lie algebra, by including new, anti-commuting (fermionic) generators [72,73]. Later, Haag, Lopuszanski and Sohnius presented the most general non-trivial extension of the Poincaré group in four dimensions that includes $4\mathcal{N}$ fermionic 'supersymmetry generators':

$$Q^A_\alpha, \bar{Q}^A_{\dot\alpha} \quad \text{with } \alpha, \dot\alpha = 1, 2 \text{ and } A = 1, \ldots, \mathcal{N},$$

where $\mathcal{N} \in \mathbb{Z}^+$ measures the amount of supersymmetry present. The supersymmetry generators themselves are Weyl spinors, and are related to one another by $\bar{Q}^A_{\dot\alpha} = \epsilon_{\dot\alpha}{}^\beta (Q^A_\beta)^*$ where spinor indices are raised and lowered using $\delta_{\alpha\dot\beta}$. The generators, otherwise known as real *supercharges*, satisfy the following (anti-)commutation relations [74]

$$\begin{aligned}
&[M_{\mu\nu}, Q^A_\alpha] = i(\sigma_{\mu\nu})^\beta_\alpha Q^A_\beta \,, \\
&[M_{\mu\nu}, \bar{Q}^A_{\dot\alpha}] = i(\bar{\sigma}_{\mu\nu})^{\dot\beta}_{\dot\alpha} \bar{Q}^A_{\dot\beta} \,, \\
&[Q^A_\alpha, P_\mu] = [\bar{Q}^A_{\dot\alpha}, P_\mu] = 0 \,, \\
&\{Q^A_\alpha, \bar{Q}^B_{\dot\beta}\} = 2\delta^{AB} (\sigma_\mu)_{\alpha\dot\beta} P^\mu \,, \\
&\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{AB} \,.
\end{aligned} \tag{3.2}$$

The combination of (3.1) and (3.2) is known as the super Poincaré algebra, or superalgebra for short. The σ_μ are the standard Pauli matrices with $\sigma_{\mu\nu} = \frac{1}{4}\sigma_{[\mu}\sigma_{\nu]}$ [75], and $Z^{AB} = -Z^{BA}$ is a complex matrix of so-called *central charges* which commute with all elements of the superalgebra. Notice the dotted (resp. undotted) supercharges transform under Lorentz transformations as two-component Weyl spinors with left (resp. right) handed chiralities. Defining $J_i = \epsilon_{ijk} M_{jk} = \{M_{23}, M_{31}, M_{12}\}$ and $\sigma_i = \{\sigma_{23}, \sigma_{31}, \sigma_{12}\}$ we observe the first identity in (3.2) can be rewritten as

$$[J_i, Q^A_\alpha] = -\frac{1}{2}(\sigma_i)_\alpha{}^\beta Q^A_\beta \,, \tag{3.3}$$

which will be useful later on when we come to discuss multiplet calculus.

The group, K, of automorphisms of the bosonic part of the superalgebra is known as the 'R-symmetry' group. This is the set of transformations $Q^A_\alpha \mapsto K^A{}_B Q^B_\alpha$ that leave (3.1) and (3.2) invariant. When $Z^{AB} = 0$, $K = U(\mathcal{N})$, and when $Z^{AB} \neq 0$, $K \subset U(\mathcal{N})$ [76,77].

Any theory satisfying the symmetries generated by the superalgebra and which has a non-dynamical spacetime metric is said to be a theory of rigid (or global) supersymmetry. Making supersymmetry local naturally results in a gravitational theory due to the presence of a dynam-

ical spacetime metric. Local supersymmetry is otherwise known as supergravity.

In an irreducible representation (irrep) of an algebra, the Casimir operators are proportional to the identity (by Schur's lemma) and thus their eigenvalues can be used to classify the irreps [78]. For the Poincaré algebra in (3.1), such irreps are called *particles* and are classified using the two Casimirs, $P^2 = P^\mu P_\mu$ and $W^2 = W^\mu W_\mu$, where $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}$ is the Pauli-Lubanski pseudovector. For massive representations we boost to the rest frame in which $P^\mu = (-M, 0, 0, 0)$ where it is clear that $P^2 = M^2$ and $W^2 = -M^2 s(s+1)$ where s is the spin.[16] Thus, massive particles can be distinguished from one another by their mass and spin. Irreps of fixed mass are $(2s+1)$-dimensional.

Massless particles are representations for which $P^2 = 0$. We can treat these in a similar fashion by boosting to a frame in which $P^\mu = (E, 0, 0, E)$.[17] The Pauli-Lubanski pseudovector is now proportional to the momentum,

$$W^\mu = M_{12}P^\mu = J_3 P^\mu = \pm s P^\mu \,, \tag{3.4}$$

where J_3 acts as a helicity operator and its eigenvalues $\pm s$ correspond to the helicity of a particle with spin s (whether spin is aligned or anti-aligned with momentum). Since $P^2 = 0$, we should label irreps by their energy, E, which is now Lorentz invariant (unsuitable in massive case as can be changed by boosting). Furthermore, because $W^2 = 0$ also, we should use the eigenvalues of individual W^μ as additional labels; clearly from (3.4) we have $W^\mu = 0$ for $\mu = 1, 2$ whilst $W^\mu = \pm sE$ for $\mu = 0, 3$ and so the second of these is taken to be the helicity [80]. Helicity is Lorentz invariant in the massless case since there it is not possible to reverse the spin alignment by either boosting or rotating frames. It is important to point out that because W^μ is a pseudovector, it is not CPT invariant. In particular, parity transformations reverse the helicity eigenvalue $s \mapsto -s$, meaning that to describe the parity preserving particles we see in nature, one must combine the $+s$ and $-s$ irreps. To recap, massless particles are labelled by their energy and helicity, and, for fixed energy, they form a 2-dimensional irrep when $s > 0$. Meanwhile, the case with $s = 0$, which corresponds to a massless scalar, is already CPT self-conjugate and therefore forms a 1-dimensional irrep [80].

Just as irreps of the Poincaré algebra were called particles, we refer to superalgebra irreps as *superparticles*. It is important to recognise that since the Poincaré algebra is a subalgebra of the full superalgebra, any irrep of the superalgebra will form a representation of the Poincaré algebra. Moreover, this representation will in general be reducible meaning that a superparticle

[16] The second equality follows by noticing $W_\mu P^\mu = 0$ in the chosen frame. This implies $W_0 = 0$ and so $W_\mu = (0, \frac{1}{2}M\epsilon_{i0jk}M^{jk})$. From this we find $W^2 = -(W^i)^2 = -M^2\vec{J}^2$ with the necessary eigenvalues [79]. Of course this is expected since for W^2 to be a Casimir, W^μ must be in the center of the Poincaré group and therefore the W^i are invariant under Poincaré transformations. But having fixed $P^\mu = (M, 0, 0, 0)$, the only transformations that leave it invariant are rotations generated by J^i [80].

[17] There is no 'rest frame' for a single massless particle.

corresponds to a collection of conventional particles [79].

Again, we shall catalogue individual superparticles using Casimir operators. A key difference to the Poincaré algebra is that W^2 is no longer a Casimir; spin is no longer a suitable quantum number since the different particles composing the superparticle are related by the action of fermionic supercharges and therefore have a variety of spins differing by half integer values [76]. Instead, we continue to work with P^2 but replace W^2 by a new Casimir corresponding to *superspin*; we refer to [76] for details on this. The exact structure of the superparticle depends on the value of $\mathcal{N}$, but it can be shown that there is always equal numbers of bosons and fermions present [81]. Since no mass degeneracy is observed in nature we conclude that for supersymmetry to exist at all, it must be broken at some sufficiently high energy scale.

This thesis is concerned with $\mathcal{N} = 2$ theories for which the supercharge anti-commutators reduce to

$$\begin{aligned} \{Q^A_\alpha, \bar{Q}^B_{\dot\beta}\} &= 2\delta^{AB}(\sigma_\mu)_{\alpha\dot\beta}P^\mu\,, \\ \{Q^1_\alpha, Q^2_\beta\} &= -\{Q^2_\alpha, Q^1_\beta\} = 2\epsilon_{\alpha\beta}|Z|\,, \end{aligned} \tag{3.5}$$

where $2|Z| := |Z^{12}|$ and we have made a $U(1)_R$ phase transformation of the supercharges to make the central charges real.[18] As before, there are both massive and massless irreps of the superalgebra. In this thesis we are more concerned with massless representations as they are more relevant for string theory constructions and, as such, we shall focus on these below.

Null or massless irreps

As in the non-supersymmetric case, massless $\mathcal{N} = 2$ irreps are to be labelled by their energy, E, and their helicity, which we now denote by λ. Again we begin the analysis by boosting to a frame in which $P_\mu = (-E, 0, 0, E)$. The superalgebra then tells us

$$\{Q^A_\alpha, Q^B_{\dot\beta}\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta} \delta^{AB}\,.$$

Clearly this implies $Q^A_{\dot 2} = \bar{Q}^A_{\dot 2} = 0$ and using this in the other anti-commutator, $\{Q^A_\alpha, \bar{Q}^B_{\dot\beta}\} = \epsilon_{\alpha\dot\beta}|Z|$, tells us the central charges must all vanish i.e. $Z^{AB} = 0$. From the remaining non-zero half of the supercharges we define the following, correctly normalised, helicity lowering and raising operators respectively

$$a^A = \frac{1}{2\sqrt{E}}Q^A_1\,, \quad \bar{a}^A = \frac{1}{2\sqrt{E}}\bar{Q}^A_{\dot 1}\,.$$

[18] We refer the reader to [27] for a complete analysis of theories with $\mathcal{N} > 2$ extended supersymmetry.

These satisfy the anti-commutation relations for a set of $\mathcal{N}$ creation and $\mathcal{N}$ annihilation operators

$$\{a^A, \bar{a}^B\} = \delta^{AB}\,, \quad \{a^A, a^B\} = 0\,, \quad \{\bar{a}^A, \bar{a}^B\} = 0\,.$$

Acting on any given state, the operators a^A, $\bar{a}^A$ (and hence Q_1^A, $\bar{Q}_{\dot{1}}^A$) act by lowering and raising the helicity by half integer increments respectively. This can be seen from (3.3) as follows

$$\begin{aligned}
&[J_3, Q_1^A] = -\frac{1}{2}(\sigma_3)_1^1 Q_1^A = -\frac{1}{2}Q_1^A\,,\\
\Rightarrow\ & J_3 Q_1^A|E,\lambda\rangle = [J_3, Q_1^A]|E,\lambda\rangle + Q_1^A J_3|E,\lambda\rangle = (\lambda - \frac{1}{2})Q_1^A|E,\lambda\rangle\,,\\
\Rightarrow\ & Q_1^A|E,\lambda\rangle = |E,\lambda - \frac{1}{2}\rangle\,,\\
\text{and similarly,}\quad & [J_3, \bar{Q}_{\dot{1}}^A] = \frac{1}{2}\bar{Q}_{\dot{1}}^A \quad \Rightarrow \quad \bar{Q}_{\dot{1}}^A|E,\lambda\rangle = |E,\lambda + \frac{1}{2}\rangle\,.
\end{aligned}$$

To construct an irrep, we pick a state annihilated by all a^A's. This must be a state of minimal helicity, denoted $|E, \lambda_{\text{min}}\rangle$, or $|\lambda_{\text{min}}\rangle$ for short, and is known as the Clifford vacuum [79]. To construct the superparticle we act on the Clifford vacuum with the creation operators, $\bar{a}^A$, as follows:

helicity	state		no. of states
λ_{min}	$\lvert\lambda_{\text{min}}\rangle$		$1 = \binom{2}{0}$
$\lambda_{\text{min}} + 1/2$	$\bar{a}^1\lvert\lambda_{\text{min}}\rangle$	$\bar{a}^2\lvert\lambda_{\text{min}}\rangle$	$2 = \binom{2}{1}$
$\lambda_{\text{min}} + 1$	$\bar{a}^1\bar{a}^2\lvert\lambda_{\text{min}}\rangle$		$1 = \binom{2}{2}$

This can be extended to the general result that there are $\binom{\mathcal{N}}{k}$ states of helicity $\lambda_{\text{min}} + \frac{k}{2}$. This gives the total number of states as $\sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}} = (2^{\mathcal{N}-1})_{\text{bosons}} + (2^{\mathcal{N}-1})_{\text{fermions}}$. For the case of $\mathcal{N} = 2$ above, we clearly have $2^2 = 4$ particles (2 bosons and 2 fermions) present in each massless irrep [76].

The four states shown above form a basis for $\mathcal{N} = 2$ superalgebra irreps. To avoid entering into the territory of higher spin theories, we limit ourselves to superparticles with $|\lambda| \leq 2$ and then, depending on the initial choice of λ_{min}, we are able to form the following massless

irreps [81][19]

		$\lambda_{\max}$						
		2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1
	2	1						
	$\frac{3}{2}$	2	1					
	1	1	2	1				
helicity	$\frac{1}{2}$		1	2	1			
	0			1	2	1		
	$-\frac{1}{2}$				1	2	1	
	-1					1	2	1
	$-\frac{3}{2}$						1	2
	-2							1

We have already seen that CPT transformations flip the sign of the helicity eigenvalue and so, unless the helicity is symmetrically distributed about 0, superparticles will not in general be CPT invariant. Therefore to construct physical, CPT-invariant irreps as demanded by nature, we must add each superparticle to its CPT conjugate. We shall refer to this CPT-invariant sum of superparticles as a *supermultiplet*.

Let us now list some examples of on-shell $\mathcal{N} = 2, d = 4$ supermultiplets that will be relevant throughout this thesis.

- Hyper multiplet, (χ^I, q^u):

$$\lambda_{\min} = -\frac{1}{2} \quad \Rightarrow \quad \left(-\frac{1}{2}, 0, 0, +\frac{1}{2}\right) \underset{\text{CPT}}{\oplus} \left(-\frac{1}{2}, 0, 0, +\frac{1}{2}\right).$$

  This is where matter sits in an $\mathcal{N} = 2, d = 4$ theory. The degrees of freedom are two Weyl spinors, χ^1, χ^2, and four real scalars, q^1, q^2, q^3, q^4. Despite the symmetric distribution of helicities in $(-\frac{1}{2}, 0, 0, \frac{1}{2})$, the CPT conjugate must still be added for invariance. Technically it is possible to have 'half-hypermultiplets' but they are very rare in practice, and require considerable engineering since they must carry no other quantum numbers in order to be CPT invariant.

[19] An immediate consequence of imposing $|\lambda| \leq 2$ is a restriction on the maximum number of supersymmetries to $\mathcal{N} \leq 8$. We refer the reader to [82, 83, 84] for more details on higher spin theories.

- Vector multiplet, (A_μ, λ^I, z):

$$\lambda_{\min} = -1 \quad \Rightarrow \quad \left(-1, -\frac{1}{2}, -\frac{1}{2}, 0\right) \underset{\text{CPT}}{\oplus} \left(0, +\frac{1}{2}, +\frac{1}{2}, +1\right).$$

  The degrees of freedom are those of a 1-form, A_μ, two Weyl fermions, λ^1, λ^2, and a complex scalar, z. Note that supersymmetry requires the two real scalar degrees of freedom to combine into a complex scalar [85].

- Supergravity multiplet, $(g_{\mu\nu}, \psi^I_\mu, A_\mu)$:

$$\lambda_{\min} = -2 \quad \Rightarrow \quad \left(-2, -\frac{3}{2}, -\frac{3}{2}, -1\right) \underset{\text{CPT}}{\oplus} \left(+1, +\frac{3}{2}, +\frac{3}{2}, +2\right).$$

  The degrees of freedom are that of a graviton, $g_{\mu\nu}$, two Weyl spinors called gravitini, ψ^1_μ, ψ^2_μ, which form a doublet under $SU(2)$ R-symmetry, and a graviphoton 1-form, A_μ.

3.1.2 Five-dimensional representations of $\mathcal{N} = 2$ supergravity

It is crucial to be aware of the changes in spinor representations in different dimensions. An important consequence of this is that $\mathcal{N} = 2$ is in fact the minimal amount of supersymmetry permitted in five dimensions: (3.6) shows there are no spinor representations that allow the fermionic degrees of freedom of the gravitino to match those of the graviton (which has five on-shell bosonic degrees of freedom) and therefore an $\mathcal{N} = 1$ supergravity multiplet is not possible [86].

In five dimensions the minimal spinor representations are symplectic Majorana. In D spacetime dimensions, these spinors have $\frac{1}{2} \times 2^{\lfloor D/2 \rfloor}$ complex degrees of freedom. In five dimensions, this corresponds to four off-shell real degrees of freedom for each fermionic particle. On-shell, the Dirac equation will halve this to two. In order to build correctly balanced $\mathcal{N} = 2$ multiplets, we require a pair of symplectic Majorana spinors, transforming as a doublet under the $SU(2)$ R-symmetry group of the five-dimensional superalgebra.

Armed with the knowledge of this change to spinor representations, we could then proceed with the multiplet calculus as in four dimensions; first constructing the superalgebra, and then considering the possible irreps. For our purposes, we will only be interested in the on-shell massless gravity and vector multiplets. To avoid repeating the analysis of Section 3.1.1, we simply provide a summary of the final results here and check that they are indeed supersymmetric. For this we rely on the following formulae for the on-shell real degrees of freedom of various massless

fields in D dimensions [85]:

$$\begin{aligned} \text{graviton, } g_{\mu\nu}: &\quad \frac{1}{2}D(D-3)\,, \\ \text{fermion, } \lambda: &\quad 2^{\lfloor D/2\rfloor-\epsilon}\,, \\ \text{gravitino, } \psi_\mu: &\quad 2^{\lfloor D/2\rfloor-\epsilon}(D-3)\,, \\ p\text{-form gauge field, } A_{[p]}: &\quad \binom{D-2}{p}\,, \end{aligned} \tag{3.6}$$

where

$$\epsilon = \begin{cases} 0 & \text{for Dirac fermion,} \\ 1 & \text{for (symplectic) Majorana or Weyl fermions.} \end{cases}$$

The $\mathcal{N}=2$ vector multiplet in five dimensions consists of a $U(1)$ gauge field, $\hat{A}_{\hat{\mu}}$, a real scalar, ϕ, and an $SU(2)$ doublet of symplectic Majorana spinors, λ^i [27, 87]. Notice that we have introduced a 'hat' on five-dimensional spacetime indices to distinguish them from those in four dimensions. To verify the field content we can compare the bosonic and fermionic degrees of freedom. On the fermionic side there are $2\times\frac{1}{2}\times 2^{\lfloor\frac{5}{2}\rfloor}=4$ on-shell fermionic degrees of freedom. Accounting for the gauge redundancy and the on-shell condition, the gauge boson contributes three real degrees of freedom whilst the real scalar contributes just one. Consequently, the multiplet is genuinely supersymmetric and, furthermore, this highlights an important difference between vector multiplets in different dimensions: the four-dimensional scalars are complex and the five-dimensional scalars are real.

The $\mathcal{N}=2$ gravity multiplet in five dimensions contains the graviton, $\hat{g}_{\hat{\mu}\hat{\nu}}$, a gauge field, $\hat{\mathcal{A}}_{\hat{\mu}}$, called the graviphoton and an $SU(2)$ doublet of gravitini, $\psi^i_{\hat{\mu}}$ [27, 87]. Again, counting the on-shell degrees of freedom, we have $5+3=8$ on the bosonic side and $2\times 4=8$ on the fermionic side.

Note that since we derived the four-dimensional multiplets explicitly from the multiplet calculus they are guaranteed to be supersymmetric. However, if we wanted to, we could double check this using the above counting of degrees of freedom.

3.1.3 $\mathcal{N}=2$ supergravity actions

In this section we summarise the construction of actions for the coupling of vector multiplets to $\mathcal{N}=2$ supergravity in either four or five dimensions. These shall form the starting point for finding solutions in Chapters 4 and 5. Following the literature we first introduce the action for a rigid superconformal theory, before gauging it to get a locally superconformal theory [77, 88]. We then explain how gauge equivalence can be used to obtain the physical Poincaré supergravity theory. In fact, for practical applications in later chapters, we will prefer to work in the gauge equivalent superconformal formalism. The reason being that in five (resp. four) dimensions it

leads to the linear (resp. symplectic) symmetry of the field equations being manifest at the level of the Lagrangian.

We are also interested in the structure of the Lagrangian for theories describing hypermultiplets coupled to supergravity in three, four or five dimensions, for which the bosonic part is always the same.

In this thesis, we are only interested in bosonic field configurations representing the long-range fields of black holes, gravitational waves etc [89], and we can therefore set all the fermionic degrees of freedom to zero and consider purely bosonic Lagrangians.[20] A consistent truncation is a truncation of the field content for which the solutions of the truncated theory are also solutions of the full untruncated theory. It can be seen by inspection that switching off the fermions is always a consistent truncation.

We shall pay particular attention to the scalar fields in these Lagrangians. The dynamics of the scalar fields are especially interesting: they are encoded in the scalar coupling matrix which generates a so-called non-linear sigma model (see Appendix B.1) that maps from the spacetime to a certain target manifold. As we shall see, the target manifolds that appear all have special geometries of the types discussed in Chapter 2.

Gauge equivalence

Theories with extended supergravity typically involve large numbers of fields. For example, in both four and five dimensions, we have seen that each supercharge has four real components and thus the number of degrees of freedom will grow exponentially with $\mathcal{N}$. Even with $\mathcal{N} = 2$ the construction of such theories can be very complicated. As mentioned above, we rely on a systematic construction using the gauge equivalent superconformal formalism. The superconformal theory has a larger symmetry group, allowing for more stringent requirements on the transformation rules and altogether simpler expressions. For this reason we prefer to work with the superconformal actions when finding solutions, but we bear in mind that a suitable gauge choice will always allow us to recover Poincaré supergravity.

To perform this procedure in detail for $\mathcal{N} = 2$ supergravity would be too much of a digression. Instead, we illustrate the procedure using a simple non-supersymmetric example, and then highlight the key steps for the case involving $\mathcal{N} = 2$ supergravity. For our prototypical example we choose a gauge equivalent, scale-invariant rewriting of the Einstein-Hilbert action. The Einstein-Hilbert action in D dimensions is

$$S_{\text{EH}} = -\int d^D x \frac{1}{2\kappa^2}\sqrt{-g}R\,. \tag{3.7}$$

Let us introduce so-called Weyl rescalings (or dilatations) with parameter $\Lambda(x)$ under which we

[20] We observe only macroscopic bosonic fields in nature. Although technically there is nothing illegal with truncating the bosons and considering only non-vanishing fermions [89].

have the following transformation rules [90]

$$\begin{aligned} \delta g_{\mu\nu} &= -2\Lambda g_{\mu\nu}\,, \\ \delta(\sqrt{-g}) &= -D\Lambda\sqrt{-g}\,, \\ g^{\mu\nu}\delta R_{\mu\nu} &= -2(D-1)\Box\Lambda\,, \end{aligned} \tag{3.8}$$

and $\Box = D^\mu D_\mu$ is the d'Alembertian. Using these, it can be shown [90]

$$\delta S_{\text{EH}} = -\int d^D x \frac{1}{2\kappa^2}\sqrt{-g}\left((2-D)\Lambda R - 2(D-1)\Box\Lambda\right)\,. \tag{3.9}$$

This is clearly not invariant under local scale transformations. We can find a gauge-equivalent, scale-invariant action by introducing a *compensating* real scalar field a, which transforms under dilatations as

$$\delta a = \frac{1}{2}(D-2)\Lambda a\,.$$

Using this and the above transformations (3.8), it is straightforward to check that the action

$$\tilde{S}_{\text{EH}} = -\int d^D x \frac{1}{2}\sqrt{-g}\left(a^2 R - \frac{4(D-1)}{D-2}\partial_\mu a \partial^\mu a\right), \tag{3.10}$$

is invariant under local scale transformations.

Both theories are equivalent, because the extra degree of freedom a is balanced by the additional symmetry. It is important to recognise that the original action (3.7) can be recovered from the new action (3.10), by setting a scale for the theory by means of the dilatation gauge (or D-gauge) fixing condition

$$a(x) = \kappa^{-1}.$$

As we shall see below, the gauge-equivalence between the superconformal and Poincaré supergravity theories is analogous. We must add additional fields, known as *conformal compensators* which reside in *compensating supermultiplets*, to the Poincaré supergravity action to compensate for the action not being invariant under the full superconformal symmetry group.

Five-dimensional vector multiplet action

The bosonic Lagrangian for a rigid superconformal theory of $n_V^{(5)}+1$ vector multiplets is built from the field content described in Section 3.1.2 and is given as [87]

$$\mathcal{L}_5 = -\frac{3}{4}g_{ij}(h)\partial_{\hat{\mu}}h^i\partial^{\hat{\mu}}h^j - \frac{1}{4}g_{ij}(h)\mathcal{F}^i_{\hat{\mu}\hat{\nu}}\mathcal{F}^{j|\hat{\mu}\hat{\nu}} + \frac{\kappa_5}{6\sqrt{6}}c_{ijk}\epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}}\mathcal{F}^i_{\hat{\mu}\hat{\nu}}\mathcal{F}^j_{\hat{\rho}\hat{\sigma}}\mathcal{A}^k_{\hat{\lambda}}\,, \tag{3.11}$$

where we continue the practice of placing hats on five-dimensional spacetime indices $\hat{\mu}, \hat{\nu}, \ldots,$ and at the same time label the vector multiplets using $i, j = 1, \ldots, n_V^{(5)}+1$. We have deliber-

ately picked this number of vector multiplets in anticipation that one of them will later become a conformal compensator. It is explained in [7] that five-dimensional supersymmetry transformations allow the most general Lagrangian to contain a Chern-Simons term, where the coefficients $c_{ijk} = \frac{\partial^3}{\partial h^i \partial h^j \partial h^k} H(h)$ are the third derivatives of a Hesse potential. By construction, the c_{ijk} are totally symmetric, and moreover they are constant due to gauge invariance, so that $H(h)$ is a polynomial of degree at most three [7, 87]. Superconformal invariance requires the Hesse potential to be a homogeneous degree three function of the scalar fields.

There are $n_V^{(5)}+1$ superconformal real scalars which are subject to real scale transformations $h^i \mapsto \lambda h^i$, $\lambda \in \mathbb{R}^+$. The scalar kinetic term itself forms a non-linear sigma model to a CASR manifold M with metric $g_{ij}(h)$ and coordinates h^i. By supersymmetry, the vector coupling is also g_{ij}. As noted in Section (2.2), the CASR metric g has Lorentzian signature $(-+\cdots+)$ with the negative eigendirection along the radial direction of the cone. This is the direction of the homothety ξ which generates the scale transformations.

We can then consider gauging the superconformal symmetry group. To do this we introduce the Weyl multiplet $\left(e_{\hat{\mu}}{}^{\hat{a}}, \omega_{\hat{\mu}}^{\hat{a}\hat{b}}, b_{\hat{\mu}}, f_{\hat{\mu}}{}^{\hat{a}}, \mathcal{V}_{\hat{\mu}i}{}^{j}, \psi_{\hat{\mu}}{}^{i}, \phi_{\hat{\mu}}{}^{i}\right)$ containing the gauge fields for the five-dimensional superconformal symmetry transformations [27, 91]. In order to retain invariance under all of these transformations, we also require a compensating vector multiplet and hypermultiplet. We can use a field dependent linear combination of the $n_V^{(5)}+1$ vector multiplets of (3.11) for this role and formulate the following Lagrangian for $n_V^{(5)}$ physical vector multiplets coupled to $\mathcal{N}=2$ conformal supergravity

$$\begin{aligned} e_5^{-1}\mathcal{L}_5 &= -\frac{1}{2}H(h)R_{(5)} - \frac{3}{4}H(h)g_{ij}(h)\partial_{\hat{\mu}}h^i\partial^{\hat{\mu}}h^j - \frac{1}{4}a_{ij}(h)\mathcal{F}^i_{\hat{\mu}\hat{\nu}}\mathcal{F}^{j|\hat{\mu}\hat{\nu}} \\ &\quad + \frac{\kappa_5}{6\sqrt{6}}e_5^{-1}c_{ijk}\epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}}\mathcal{F}^i_{\hat{\mu}\hat{\nu}}\mathcal{F}^j_{\hat{\rho}\hat{\sigma}}\mathcal{A}^k_{\hat{\lambda}}\,, \end{aligned} \tag{3.12}$$

The Hesse potential is precisely the linear combination of vector multiplet scalars mentioned above, introduced as a dilatation compensator to ensure the action is scale invariant. We remark that the discussion here is a bit simplified since we are ignoring the hypermultiplet contribution to the coefficient of the Einstein-Hilbert term [92]. However, for our purposes, we merely require that (3.12) be gauge equivalent to Poincaré supergravity which is certainly the case, as we shall soon demonstrate.

Importantly, the Weyl multiplet contains the gauge field for translations $e_{\hat{\mu}}^{\hat{a}}$, which is to be identified with the five-dimensional vielbein. It is this gauging of translational symmetry that introduces the Einstein-Hilbert term. In other words, local translations are identified with spacetime diffeomorphisms, producing the gravitational degrees of freedom. We further note that in the process of integrating out the auxiliary fields from the Lagrangian, the vector coupling changes from $g_{ij} = \frac{\partial^2 H}{\partial h^i \partial h^j}$ to a_{ij} as defined in (2.36). The Lagrangian contains $n_V^{(5)}+1$ vector fields and we interpret $n_V^{(5)}$ of them as being from physical vector multiplets, whilst the

$(n_V^{(5)}+1)$'th is the graviphoton which is singled out as being a scalar under the linear group action, $h_i\mathcal{F}^i_{\mu\nu} = T^{\rm GP}_{\mu\nu}$ [77].

Currently, despite the presence of gravity, the theory is still conformally invariant and contains no length scale. This will be introduced later via the Hesse potential $H(h)$ which is playing the role of a conformal compensator in (3.12). The target manifold is still a CASR manifold M with metric g and coordinates h^i. We observed in Section 2.2 that both metrics g and a pull-back to the same PSR metric; however a is a product metric and not a cone metric so we prefer to leave things in terms of g for the time being.[21] More details on our choice of conventions for the Einstein-Hilbert term can be found in Appendix A.

Analogous to the simple example we saw earlier involving the Einstein-Hilbert action, the above superconformal Lagrangian is gauge equivalent to the Lagrangian for a theory of $n_V^{(5)}$ vector multiplets coupled to $\mathcal{N}=2$ Poincaré supergravity in five dimensions. We obtain the physical theory by simply imposing a suitable gauge fixing condition. In this case, we use the following D-gauge condition to set a scale for the theory, thus breaking dilatation symmetry and bringing the Einstein-Hilbert term to its canonical form [92]

$$H(h) = c_{ijk}h^ih^jh^k = \kappa_5^{-2}\,.$$

Applying the D-gauge to (3.12) leads to the following Lagrangian for the physical theory

$$\begin{aligned} e_5^{-1}\mathcal{L}_5 \;=\; & -\frac{1}{2\kappa_5^2}R_{(5)} - \frac{3}{4\kappa_5^2}g_{xy}(\phi)\partial_{\hat\mu}\phi^x\partial^{\hat\mu}\phi^y - \frac{1}{4}a_{ij}(h)\mathcal{F}^i_{\hat\mu\hat\nu}\mathcal{F}^{j|\hat\mu\hat\nu} \\ & + \frac{\kappa_5}{6\sqrt{6}}e_5^{-1}c_{ijk}\epsilon^{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\lambda}\mathcal{F}^i_{\hat\mu\hat\nu}\mathcal{F}^j_{\hat\rho\hat\sigma}\mathcal{A}^k_{\hat\lambda}\,, \end{aligned} \tag{3.13}$$

where $\kappa_5^2 = 8\pi G_5$ and $x,y = 1,\ldots,n_V^{(5)}$. This represents the Lagrangian of $n_V^{(5)}$ vector multiplets coupled to the supergravity multiplet, which is where the graviphoton now resides. The scalar fields ϕ^x parametrize an $n_V^{(5)}$-dimensional PSR target manifold $\mathcal{H}$ with metric obtained by the pull-back of either CASR metric

$$(g_{\mathcal{H}})_{xy} = \iota^*(a)_{xy} = \iota^*(g)_{xy}\,. \tag{3.14}$$

The definition of this metric relies on an embedding $\iota:\mathcal{H}\hookrightarrow M$ where M is the $(n_V^{(5)}+1)$-dimensional CASR target manifold of the locally superconformal theory above. As seen in Section 2.2, the special real coordinates $h^i = (h^0, h^x)$ on the CASR manifold allow us to form projective coordinates on the PSR using the ratios

$$\phi^x = \frac{h^x}{h^0}\,.$$

[21] The conic property is a property of the metric not of the manifold. (M,g) is conical because of the existence of a homothety ξ satisfying $\mathcal{L}_\xi g = 3g$, whereas for the metric a this becomes an isometry $\mathcal{L}_\xi a = 0$.

As discussed in Section 2.2, the PSR manifold $\mathcal{H}$ appears as a particular level set inside the CASR manifold after projecting out the negative eigendirection. As such, both the scalar coupling g_{xy} and the vector coupling a_{ij} have positive definite signature as required by positive definiteness of kinetic energy. Conversely, requiring positive definiteness of physical couplings fixes the CASR metric g to have Lorentzian signature.

It is clear from (2.36) that specifying the Hesse potential $H(h)$, i.e. specifying the number of vector multiplets $n_V^{(5)}$ as well as the coefficients c_{ijk}, fixes the vector kinetic metric a and thus, from (3.14), this completely determines all coupling matrices, and all dynamics of the five-dimensional Lagrangian (3.13). The gauge equivalence of (3.13) and (3.12) implies that after imposing D-gauge the fields h^i only represent $n_V^{(5)}$ rather than $n_V^{(5)}+1$ independent real degrees of freedom. This is seen by observing that,

$$a_{ij}\partial_{\hat{\mu}}h^i\partial^{\hat{\mu}}h^j\Big|_D = g_{xy}\partial_{\hat{\mu}}\phi^x\partial^{\hat{\mu}}\phi^y\,.$$

For practical purposes, it is often more convenient to work 'upstairs' with the locally superconformal Lagrangian; here every object carrying CASR indices i,j behaves tensorially under the natural action of the general linear group. The action of this group extends to the upstairs equations of motion making them easier to work with. We therefore choose to impose the D-gauge condition on (3.12) and work with the following Lagrangian for constructing solutions

$$\begin{aligned}\mathsf{e}_5^{-1}\mathcal{L}_5 = &-\frac{1}{2\kappa_5^2}R_{(5)} - \frac{3}{4\kappa_5^2}a_{ij}(h)\partial_{\hat{\mu}}h^i\partial^{\hat{\mu}}h^j - \frac{1}{4}a_{ij}(h)\mathcal{F}^i_{\hat{\mu}\hat{\nu}}\mathcal{F}^{j|\hat{\mu}\hat{\nu}}\\ &+\frac{\kappa_5}{6\sqrt{6}}\mathsf{e}_5^{-1}c_{ijk}\epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}}\mathcal{F}^i_{\hat{\mu}\hat{\nu}}\mathcal{F}^j_{\hat{\rho}\hat{\sigma}}\mathcal{A}^k_{\hat{\lambda}}\,,\end{aligned} \tag{3.15}$$

where it is understood that the D-gauge is to be imposed. As such, it doesn't matter which scalar coupling we use since, as seen in (3.14), they both pull-back to $(g_{\mathcal{H}})_{xy}$. Therefore, we choose the metric a_{ij} for both to treat the scalars and vectors on an equal footing. It is important to stress that all couplings continue to be completely determined by the Hesse potential.

Four-dimensional vector multiplet action

The coupling of four-dimensional $\mathcal{N}=2$ supergravity to $n_V^{(4)}$ vector multiplets can be found in [27, 93]. Here we use the conventions of [53] and present in parallel the Lorentzian and Euclidean theories by using a parameter ϵ_1 defined as:

$$\epsilon_1 = \begin{cases} -1 & \text{if } D=1+3,\\ +1 & \text{if } D=0+4.\end{cases}$$

Using the superconformal calculus, we follow the same construction as in five dimensions

and begin with the bosonic Lagrangian for a rigid superconformal theory of $n_V^{(4)}+1$ vector multiplets.[22] From the vector multiplet field content described in Section 3.1.1, this is given by [88]

$$\begin{aligned}\mathcal{L}_4 &= i_{\epsilon_1}\left(\partial_\mu F_I \partial^\mu \bar{X}^I - \partial_\mu \bar{F}_I \partial^\mu X^I\right) + \frac{i_{\epsilon_1}}{4} F_{IJ} F^{-I}_{\mu\nu} F^{-J|\mu\nu} - \frac{i_{\epsilon_1}}{4} \bar{F}_{IJ} F^{+I}_{\mu\nu} F^{+J|\mu\nu} \\ &= -N_{IJ}\partial_\mu X^I \partial^\mu \bar{X}^J + \frac{i_{\epsilon_1}}{4} F_{IJ} F^{-I}_{\mu\nu} F^{-J|\mu\nu} - \frac{i_{\epsilon_1}}{4} \bar{F}_{IJ} F^{+I}_{\mu\nu} F^{+J|\mu\nu} \,, \end{aligned} \tag{3.16}$$

where μ,ν are four-dimensional spacetime indices and $I,J=0,\ldots,n_V^{(4)}$ label the vector multiplets. We have again picked $n_V^{(4)}+1$ of them in anticipation that one will become a conformal compensator. F^{+I} and F^{-I} are, respectively, the self-dual and anti-self-dual parts of the field strengths F^I defined such that $\star F^{+I}_{\mu\nu} = -i_{\epsilon_1} F^{+I}_{\mu\nu}$ and $\star F^{-I}_{\mu\nu} = +i_{\epsilon_1} F^{-I}_{\mu\nu}$ [77]. The coefficients $F_I = \frac{\partial F}{\partial X^I}, F_{IJ} = \frac{\partial^2 F}{\partial X^I \partial X^J}$ are derivatives of a holomorphic function of the scalar fields, $F(X)$, known as the prepotential. In four dimensions, superconformal invariance of the Lagrangian requires the prepotential be a homogeneous of degree two function.

Using the chain rule, the first term in (3.16) can be rewritten as a scalar kinetic term with coupling $N_{IJ} = -i_{\epsilon_1}(F_{IJ} - \bar{F}_{IJ})$. As seen in (2.37), N_{IJ} is in fact an ϵ_1-Kähler metric with signature $(--+\cdots+)$ on a target manifold N parametrized by the $n_V^{(4)}+1$ superconformal (para-)complex scalar fields X^I. Since the Kähler potential is related to the prepotential by (2.37), and the prepotential is required to be homogeneous degree two, the scalar manifold N is in fact a CASϵ_1K manifold [88]. As such, the theory is invariant under $\mathbb{C}^*_{\epsilon_1}$ transformations of the scalars.

We then gauge the superconformal symmetry group which again involves the addition of the Weyl multiplet $\left(e_\mu{}^a, \omega_\mu^{ab}, b_\mu, f_\mu{}^a, \mathcal{V}_{\mu i}{}^j, A_\mu^{U(1)_R}, \psi_\mu{}^i, \phi_\mu{}^i\right)$ as well as a compensating vector multiplet and hypermultiplet. Note that the four-dimensional Weyl multiplet contains a $U(1)_R$ connection that was not present in five dimensions due to the different R-symmetry groups. Using a field dependent linear combination of the $n_V^{(4)}+1$ vector multiplets of (3.16) for this role, we find the following Lagrangian for $n_V^{(4)}$ vector multiplets coupled to $\mathcal{N}=2$ conformal supergravity [77]

$$\begin{aligned}\mathsf{e}_4^{-1}\mathcal{L}_4 &= \frac{1}{2}e^{-\mathcal{K}(X)}R_{(4)} - e^{-\mathcal{K}(X)} g_{IJ}\partial_\mu X^I \partial^\mu \bar{X}^J + \frac{1}{4}e^{-\mathcal{K}(X)}\partial_\mu \mathcal{K}\partial^\mu \mathcal{K} \\ &\quad + \frac{i_{\epsilon_1}}{4}\bar{\mathcal{N}}_{IJ} F^{-I}_{\mu\nu} F^{-J\mu\nu} - \frac{i_{\epsilon_1}}{4}\mathcal{N}_{IJ} F^{+I}_{\mu\nu} F^{+J\mu\nu} \,. \end{aligned} \tag{3.17}$$

As before, gravity has appeared through the identification of local translations with spacetime diffeomorphisms. This introduces the Einstein-Hilbert term, which is non-canonical since it appears multiplied by $e^{-\mathcal{K}(X)} = -N_{IJ}X^I\bar{X}^J = -i_{\epsilon_1}(X^I\bar{F}_I - F_I\bar{X}^I)$ which is the compensator for

[22] Actually, the full superconformal Lagrangian (including fermions) has not been worked out for the Euclidean case but it is shown in [53] that the geometrical quotient can be consistently extended to the para-complex case, meaning we can trust the bosonic Lagrangian.

dilatations formed using a linear combination of vector multiplet scalars [88,94]. See Appendix A for more details on our conventions for the Einstein-Hilbert term.

We have also integrated out the auxiliary $U(1)_R$ gauge field from the superconformal covariant derivative, which generates the second and third terms appearing in (3.17) from the scalar kinetic term in (3.16). Notice that they also come accompanied by the compensator $e^{-\mathcal{K}(X)}$ as shown in [94]. The X^I continue to parametrize the CASϵ_1K manifold N, but the coupling matrix of the scalar kinetic term in (3.17) is now the degenerate CASϵ_1K tensor field g_{IJ} defined earlier in (2.38).

The CASϵ_1K metric N_{IJ} had two negative eigendirections corresponding to the vector fields ξ and $J\xi$ which sweep out a $\mathbb{C}^*_{\epsilon_1}$ cone over a PSϵ_1K base manifold as reviewed in Section 2.3. Thus, integrating out the $U(1)_R$ gauge field results in the scalar coupling changing from the CASϵ_1K metric N_{IJ} to the tensor field g_{IJ} that is projectable onto the PSϵ_1K manifold [88].

Regarding the vector kinetic terms, it is explained in [77] how the integrating out of the auxiliary T-field results in the field strength terms assuming the form of the second line in (3.17) where the gauge field coupling matrix $\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + i_{\epsilon_1}\mathcal{I}_{IJ}$ is then given in terms of the prepotential F and CASϵ_1K metric N (given in (2.37)) as [53]

$$\mathcal{N}_{IJ}(X,\bar{X}) = \bar{F}_{IJ}(\bar{X}) - \epsilon_1 i_{\epsilon_1}\frac{(\bar{N}X)_I(\bar{N}X)_J}{X\bar{N}X}\,. \tag{3.18}$$

Once again, the superconformal Lagrangian (3.17) is gauge equivalent to the Lagrangian for a theory of $n_V^{(4)}$ vector multiplets coupled to Lorentzian/Euclidean $\mathcal{N}=2$ Poincaré supergravity in four dimensions. Again, we obtain the physical theory using an appropriate gauge fixing condition to break the dilatation symmetry. This involves the following D-gauge condition to set a scale for the theory, thus breaking dilatation symmetry and bringing the Einstein-Hilbert term to its canonical form

$$e^{-\mathcal{K}(X)} = -i(X^I\bar{F}_I - F_I\bar{X}^I) = \kappa_4^{-2}. \tag{3.19}$$

Applying the D-gauge to (3.17) and rewriting the gauge field terms using the basis $(F^I, \tilde{F}^I)$ instead of (F^{+I}, F^{-I}) that we were using previously gives the following Lagrangian for the physical theory

$$\begin{aligned}\mathsf{e}_4^{-1}\mathcal{L}_4 &= -\frac{1}{2\kappa_4^2}R_{(4)} - \frac{1}{\kappa_4^2}\bar{g}_{A\bar{B}}(z,\bar{z})\partial_\mu z^A\partial^\mu\bar{z}^{\bar{B}} + \frac{1}{4}F^I_{\mu\nu}\tilde{G}^{\mu\nu}_I \\ &= -\frac{1}{2\kappa_4^2}R_{(4)} - \frac{1}{\kappa_4^2}\bar{g}_{A\bar{B}}(z,\bar{z})\partial_\mu z^A\partial^\mu\bar{z}^{\bar{B}} + \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu}\,,\end{aligned} \tag{3.20}$$

where $\kappa_4^2 = 8\pi G_4$ and $A,B = 1,\ldots,n_V^{(4)}$, and we have introduced the dual (magnetic) field

strengths

$$G_I^{\mu\nu} = \mathcal{R}_{IJ} F^{J|\mu\nu} - \mathcal{I}_{IJ} \tilde{F}^{J|\mu\nu} \,,$$

involving the components of the Hodge dualisation of the (electric) field strength

$$(\star F^I)^{\mu\nu} =: \tilde{F}^I_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{I|\rho\sigma} \,. \tag{3.21}$$

This is the Lagrangian of $n_V^{(4)}$ vector multiplets coupled to the supergravity multiplet, which is where the graviphoton now resides. As in five dimensions, the graviphoton is singled out using a particular linear combination of gauge fields that form the symplectic scalar [88]

$$T^{-\,\mathrm{GP}}_{\mu\nu} = X^I G^-_{I|\mu\nu} - F_I F^{-I}_{\mu\nu} \,.$$

The scalar fields z^A parametrize an $2n_V^{(4)}$-dimensional PSϵ_1K target manifold $\bar{N}$ obtained by the Kähler quotient of the $2(n_V^{(4)}+1)$-dimensional CASϵ_1K target manifold of the locally superconformal theory in (3.17). The quotient induces a natural metric $\bar{g}_{A\bar{B}}$ on $\bar{N}$, whose horizontal lift is g_{IJ} [54],

$$\pi : N \to \bar{N}, \quad \pi^*(\bar{g}_{A\bar{B}}) = g_{IJ} \,.$$

As seen in Section (2.3), the special holomorphic coordinates $X^I = (X^0, X^A)$ on the CASϵ_1K manifold allow us to define special inhomogeneous (or projective) coordinates on the PSϵ_1K manifold using the ratios

$$z^A = \frac{X^A}{X^0} \,.$$

Following conventions in the literature we distinguish between holomorphic indices A and anti-holomorphic indices $\bar{A}$ when working with the physical scalars z^A despite making no such distinction when working with the superconformal scalars X^I [88]. The PSϵ_1K metric $\bar{g}$ is positive definite since it is the projection of the CASϵ_1K tensor field g which was positive definite along the directions horizontal to the $\mathbb{C}^*_{\epsilon_1}$ action that we project out. Conversely, requiring $\bar{g}_{A\bar{B}}$ be positive definite for physicality forces g_{IJ} to be positive definite horizontally and null vertically, as well as N_{IJ} to have signature $(--+\cdots+)$. Furthermore, this particular signature of N_{IJ} is known to make $\mathcal{N}_{IJ}$ positive definite [54] as required for positive definite vector kinetic energy. The gauge equivalence of (3.20) and (3.17) means that after imposing D-gauge the scalar fields X^I only represent $2n_V^{(4)}$ rather than $2(n_V^{(4)}+1)$ independent real degrees of freedom. In other words,

$$g_{IJ} \partial_\mu X^I \partial^\mu \bar{X}^J \Big|_D = \bar{g}_{A\bar{B}} \partial_\mu z^A \partial^\mu \bar{z}^{\bar{B}} \,.$$

For practical purposes, we again prefer to work upstairs with the locally superconformal

Lagrangian

$$\mathsf{e}_4^{-1}\mathcal{L}_4 = -\frac{1}{2\kappa_4^2}R_{(4)} - \frac{1}{\kappa_4^2}g_{IJ}\partial_\mu X^I\partial^\mu\bar{X}^J + \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu}\,. \tag{3.22}$$

We prefer to work with this since, as explained in [77, 88, 90], although the physical Lagrangian (3.20) is not itself invariant under symplectic transformations, the field equations it generates are. These transformations generalise the electric-magnetic duality of Maxwell's theory. It is explained that $(F^I_{\mu\nu}, G_{I|\mu\nu})$ and (X^I, F_I) transform linearly as symplectic vectors, whilst $F_{IJ}, N_{IJ}, \mathcal{N}_{IJ}$ transform fractionally linearly and $i(X^I\bar{F}_I - F_I\bar{X}^I)$ transforms as a symplectic scalar [88]. This means that every object in (3.22) that carries CASϵ_1K indices $I, J, \ldots$ transforms either linearly or fractionally linearly under symplectic transformations. This means there is a natural action of the symplectic group on the upstairs equations of motion that makes them easier to work with.

However, the holomorphic prepotential does not transform as a scalar and the fractional linear transformations of certain objects means we lack full, tensorial behaviour. This is only introduced if we choose to work with the real formulation of special geometry as introduced in Section 2.3. Note that whilst we gauged the full $\mathbb{C}^*_{\epsilon_1}$ action, we have only gauge-fixed local dilatations (using the D-gauge condition to set a scale for the theory) and have left the $U(1)_R$ action unfixed. This amounts to working on the Sasakian manifold $U(1)$ principal bundle over the PSϵ_1K manifold parametrized by the physical scalars. To perform calculations using the genuine physical scalars we must also gauge fix this $U(1)_R$ action. However, it can be shown geometrically that fixing a $U(1)_R$ gauge necessarily requires us to sacrifice manifest symplectic covariance, and we therefore postpone this step and solve the equations of motion upstairs where we have covariant behaviour. Later, when building four-dimensional solutions, we restrict to *purely imaginary field configurations* which forces us to fix a $U(1)_R$ gauge, which we do using the constraint

$$\mathrm{Im}(X^0) = 0\,. \tag{3.23}$$

Regardless of whether we choose to work with the physical Lagrangian (3.20) or the gauge-equivalent version (3.22), it is clear that all couplings and dynamics are fixed once the prepotential is specified.

Hypermultiplet action

Finally, we will also encounter $\mathcal{N} = 2$ hypermultiplets in this thesis. These consist of four real scalar fields and two spinors. This field content is unaffected by dimensional reduction and the only difference is to do with the representations of the spinors in different dimensions. Since we are only concerned with the bosonic fields, we will not be sensitive to this, and thus the bosonic action for n_H hypermultiplets coupled to $\mathcal{N} = 2$ supergravity in five, four or three dimensions

is identical. In four dimensions, the bosonic Lagrangian is

$$\mathsf{e}_4^{-1}\mathcal{L}_4 = -\frac{1}{2\kappa_4^2}R_{(4)} - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - h_{uv}(q)\partial_\mu q^u \partial^\mu q^v \,, \tag{3.24}$$

where $\kappa_4^2 = 8\pi G_4$ and where $R_{(4)}$ and e_4 are the 4-dimensional Ricci scalar and vielbein determinant respectively. We use $X, Y = 1, \ldots, 4n_H$ to label the hypermultiplet scalars, and $\mathcal{F}_{\mu\nu}$ is the field strength of the graviphoton $\mathcal{A}_\mu$. It was demonstrated in [6] that local supersymmetry requires the scalar fields to parametrize a quaternionic-Kähler manifold as discussed in Section 2.4.3.

An interesting observation is that the three-dimensional action can be obtained by dimensional reduction of (3.24). In this case the bosonic degrees of freedom in the four-dimensional gravity multiplet can be repackaged and replaced by an additional hypermultiplet in the three-dimensional description by making use of Hodge duality.

We will not comment on the superconformal hypermultiplet Lagrangian as we did for the vector multiplet Lagrangian (which is more important to the work of later chapters) but refer interested readers to [65].

3.2 Dimensional reduction

We now introduce the technique of dimensional reduction which will play a fundamental role throughout this thesis. In particular, we are interested in the reduction of a $(D+1)$-dimensional Einstein-Maxwell theory with action

$$S = S_{\text{EH}} + S_{\text{gauge}} = \int d^{D+1}x\hat{\mathsf{e}}\left[-\frac{\hat{R}}{2} - \frac{1}{2(p+1)!}\hat{\mathcal{F}}_{\hat{\mu}_1\ldots\hat{\mu}_{p+1}}\hat{\mathcal{F}}^{\hat{\mu}_1\ldots\hat{\mu}_{p+1}}\right], \tag{3.25}$$

where 'hats' refer to $(D+1)$-dimensional quantities, and $\hat{\mathsf{e}} = \det(\hat{e})$ is the determinant of the $(D+1)$-dimensional vielbein. Throughout this section, we continue working in conventions where the Einstein-Hilbert term appears in the Lagrangian with a minus sign, as explained in Appendix A. In what follows, we shall describe the dimensional reduction of the action (3.25) based on [44, 95] and a set of unpublished notes by Ulrich Theis.

3.2.1 The Kaluza-Klein tower

In this thesis we only discuss dimensional reduction over circles S^1 and tori $T^n = S^1 \times \cdots \times S^1$. As such it is sufficient to outline the procedure for an S^1 reduction of the x^0 dimension. Let us first consider the effect on a $(D+1)$-dimensional field $\Phi(x^{\hat{\mu}})$: we can see this by making the

following Fourier expansion

$$\Phi(x^{\hat{\mu}}) = \Phi(x^0, x^\mu) = \sum_n \phi_n(x^\mu) e^{inx^0/R}\,, \tag{3.26}$$

where R is the S^1 radius. Clearly, we have an infinite tower of D-dimensional Fourier modes $\phi_n(x^\mu)$ with masses $|n|/R$. In the standard Kaluza-Klein prescription the 'cylinder condition' is imposed; none of the $(D+1)$-dimensional fields have a dependence on the internal coordinate x^0 [95]. This amounts to truncating the massive spectrum of the D-dimensional theory and retaining only the massless fields, or ϕ_0 in the above example. At first glance, this seems undesirable but notice that as $R \to 0$ all masses diverge except that of ϕ_0, and so this truncation is equivalent to demanding that the S^1 be incredibly small and essentially invisible to the D-dimensional theory. In this case the massive modes will be too heavy to detect without accelerators beyond intergalactic scales [95] and any dependence of the fields on x^0 can be safely neglected. For S^1 reductions it can be shown that this is always a consistent truncation of the $(D+1)$-dimensional theory, but for reductions over general compact manifolds one needs to be careful that interactions between massless modes don't give rise to additional heavy modes [95].

3.2.2 Metric decomposition

Let us now elaborate on the reduction procedure. The first step is to make a suitable ansatz for the $(D+1)$-dimensional fields. The cylinder condition requires that the metric and gauge fields appearing in (3.25) are independent of x^0. We further assume the the $(D+1)$-dimensional metric decomposes as

$$ds^2_{(D+1)} = -\epsilon e^{2\beta\phi}\left(dx^0 + V_\mu dx^\mu\right)^2 + e^{-2\alpha\phi} ds^2_{(D)}\,, \tag{3.27}$$

where α and $\beta \neq 0$ are constants that will be completely fixed by the requirement that we reduce from the $(D+1)$-dimensional Einstein frame to the the D-dimensional Einstein frame.[23] The constant ϵ is used to index the signature of the compact x^0 dimension as follows,

$$\epsilon = \begin{cases} -1 & \text{if } x^0 \text{ space-like,} \\ +1 & \text{if } x^0 \text{ time-like.} \end{cases} \tag{3.28}$$

Looking at the corresponding $(D+1)$-dimensional vielbein

$$\hat{e}_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} e^{\beta\phi} & 0 \\ e^{\beta\phi} V_\mu & e^{-\alpha\phi} e_\mu{}^a \end{pmatrix}, \tag{3.29}$$

[23] The Einstein frame is defined such that the Einstein-Hilbert term in the Lagrangian has a constant coefficient.

we see why the decomposition (3.27) was particularly suited to the underlying symmetries of the theory since it splits the $(D+1)$-dimensional metric degrees of freedom into a Kaluza-Klein scalar, ϕ (often called the dilaton), a Kaluza-Klein vector, V_μ, and a D-dimensional vielbein, ${e_\mu}^a$ [95].

3.2.3 Reduction of Einstein-Hilbert term

The $(D+1)$-dimensional Einstein-Hilbert action is

$$\hat{S}_{EH} = -\int d^{D+1}x\hat{\mathbf{e}}\frac{1}{2}\hat{R}\,,$$

where $\hat{\mathbf{e}}$ is the determinant of the $(D+1)$-dimensional vielbein in (3.29). The $(D+1)$-dimensional Hodge dual of the Ricci scalar is the top form

$$\begin{aligned}\star\hat{R} &= \frac{\hat{\mathbf{e}}}{(D+1)!}\hat{R}\epsilon_{\hat{\mu}_1\ldots\hat{\mu}_{D+1}}dx^{\hat{\mu}_1}\wedge\cdots\wedge dx^{\hat{\mu}_{D+1}}\\ &= \hat{\mathbf{e}}\hat{R}dx^1\wedge\cdots\wedge dx^{D+1} = (-)^t d^{D+1}x\hat{\mathbf{e}}\hat{R}\,,\end{aligned} \tag{3.30}$$

where t is the number of time-like directions in the $(D+1)$-dimensional metric. Working in a dual basis $\{\hat{\theta}^{\hat{a}}\}$, this top form can be decomposed as

$$\star\hat{R} = \frac{1}{(D-1)!}\hat{\varepsilon}_{\hat{a}_1\ldots\hat{a}_{D+1}}\hat{R}^{\hat{a}_1\hat{a}_2}\wedge\hat{\theta}^{\hat{a}_3}\wedge\cdots\wedge\hat{\theta}^{\hat{a}_{D+1}}\,, \tag{3.31}$$

where the curvature two-form is given in terms of the connection 1-form $\hat{\omega}_{\hat{a}\hat{b}}$ by Cartan's second equation (2.18) as

$$\hat{R}_{\hat{a}\hat{b}} = d\hat{\omega}_{\hat{a}\hat{b}} + \hat{\omega}_{\hat{a}}{}^{\hat{c}}\wedge\hat{\omega}_{\hat{c}\hat{b}}\,.$$

The $(D+1)$-dimensional action can be expressed in the language of differential forms as

$$\hat{S}_{EH} = -(-)^t\int\frac{1}{2(D-1)!}\hat{\varepsilon}_{\hat{a}_1\ldots\hat{a}_{D+1}}\hat{R}^{\hat{a}_1\hat{a}_2}\wedge\hat{\theta}^{\hat{a}_3}\wedge\cdots\wedge\hat{\theta}^{\hat{a}_{D+1}}\,.$$

Substituting the non-zero components of the curvature two-form one finds

$$\hat{S}_{EH} = -(-\epsilon)(-)^t\int dx^0\int\left(e^\phi\frac{\star R}{2} + \frac{1}{4}\epsilon e^{3\phi}dV\wedge\star dV + d\star de^\phi\right),$$

where $\star$ represents the D-dimensional Hodge dual operator and ϵ is defined in (3.28) to measure the signature of the compact x^0 direction. Note that the factor $(-\epsilon)(-)^t$ tracks the number of time-like directions in D dimensions. By normalising $\int dx^0 = 1$ and ignoring the boundary term produced from integrating the $d\star de^\phi$ term, this reduces to the D-dimensional action

$$S_{EH} = -(-\epsilon)(-)^t\int e^\phi\frac{\star R}{2} + \frac{1}{4}\epsilon e^{3\phi}dV\wedge\star dV\,. \tag{3.32}$$

We then make a conformal transformation to remove the non-constant e^{ϕ} factor multiplying the Ricci scalar and return to a D-dimensional Einstein frame at the expense of introducing a kinetic term for the scalar ϕ. Writing the final result in terms of an integral over D spacetime coordinates, we have

$$S_{EH} = \int d^D x \mathbf{e} \left(-\frac{R}{2} - \frac{D-1}{2(D-2)} \partial_\mu \phi \partial^\mu \phi + \frac{1}{8} \epsilon e^{\frac{2D-2}{D-2}\phi} V_{\mu\nu} V^{\mu\nu} \right), \tag{3.33}$$

where $\mathbf{e}$ is the determinant of the D-dimensional vielbein. Notice that the $(-\epsilon)(-)^t$ has disappeared when we translate back from differential forms to spacetime coordinates in a similar fashion to (3.30). Furthermore, the conformal rescaling and normalisation procedures above are equivalent to choosing

$$\alpha = \frac{1}{D-2}, \quad \beta = 1, \tag{3.34}$$

in the reduction ansatz (3.27).

To summarise, the reduction of pure $(D+1)$-dimensional gravity renders a D-dimensional system in which certain degrees of freedom of the $(D+1)$-dimensional metric (3.27), known as the Kaluza-Klein scalar, ϕ, and Kaluza-Klein vector, $V = V_\mu dx^\mu$, become coupled to D-dimensional gravity.

3.2.4 Reduction of the gauge field term

Dimensional reduction

The $(D+1)$-dimensional action for a p-form gauge field $\hat{\mathcal{A}}_{[p]}$ is

$$\hat{S}_{\text{gauge}} = \int d^{D+1} x \hat{\mathbf{e}} \left(-\frac{1}{2(p+1)!} \hat{\mathcal{F}}_{\hat{\mu}_1 \ldots \hat{\mu}_{p+1}} \hat{\mathcal{F}}^{\hat{\mu}_1 \ldots \hat{\mu}_{p+1}} \right),$$

where $\hat{\mathcal{F}}_{[p+1]} = d\hat{\mathcal{A}}_{[p]}$ is the corresponding $(p+1)$-form field strength. This action can be rewritten in terms of differential forms as

$$\hat{S}_{\text{gauge}} = (-)^t \int -\frac{1}{2} \hat{\mathcal{F}}_{[p+1]} \wedge \hat{\star} \hat{\mathcal{F}}_{[p+1]} \,.$$

Let us begin by noting that any p-form gauge field in $(D+1)$ dimensions can be decomposed as

$$\begin{aligned} \hat{\mathcal{A}}_{[p]} &= \frac{1}{p!} (\hat{\mathcal{A}}_{[p]})_{\hat{\mu}_1 \ldots \hat{\mu}_p} dx^{\hat{\mu}_1} \wedge \cdots \wedge dx^{\hat{\mu}_p} \\ &= \frac{1}{p!} (\hat{\mathcal{A}}_{[p]})_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} + \frac{p}{p!} (\hat{\mathcal{A}}_{[p]})_{0\nu_1 \ldots \nu_p} dx^0 \wedge dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{p-1}}, \end{aligned} \tag{3.35}$$

where we have split the expression into a sum of terms with and without dx^0. By defining

$$(\mathcal{A}_{[p]})_{\mu_1 \ldots \mu_p} := (\hat{\mathcal{A}}_{[p]})_{\mu_1 \ldots \mu_p}, \quad (\mathcal{A}_{[p-1]})_{\mu_1 \ldots \mu_{p-1}} := (\hat{\mathcal{A}}_{[p]})_{0\mu_1 \ldots \mu_{p-1}},$$

it is then possible to decompose the $(D+1)$-dimensional gauge fields in terms of D-dimensional gauge fields as

$$\hat{\mathcal{A}}_{[p]} = \mathcal{A}_{[p]} + dx^0 \wedge \mathcal{A}_{[p-1]} \,.$$

Using this we can then perform the dimensional reduction itself. A rather lengthy calculation demonstrates how the action can be rewritten as

$$\begin{aligned}\hat{S}_{\text{gauge}} = -(-)^t \int dx^0 \int e^{(2p+2-D)\alpha\phi} \Big[& e^{-(\beta+2\alpha)\phi} \frac{1}{2} \mathcal{F}_{[p]} \wedge \star \mathcal{F}_{[p]} \\ & - \epsilon e^{\beta\phi} \frac{1}{2} \left(\mathcal{F}_{[p+1]} - V \wedge \mathcal{F}_{[p]} \right) \wedge \star \left(\mathcal{F}_{[p+1]} - V \wedge \mathcal{F}_{[p-1]} \right) \Big] \,,\end{aligned}$$

where t is the number of time-like directions in $(D+1)$ dimensions and, to reiterate, $\star$ is the D-dimensional Hodge star operator and we have used $\mathcal{F}_{[p+1]} = d\mathcal{A}_{[p]}$. At this point, we can simply integrate out the x^0 direction (normalised such that $\int dx^0 = 1$) and arrive at an expression for the D-dimensional gauge field action.

However, we notice that this D-dimensional action contains naked gauge fields (we shall see in Section 3.2.5 that the Kaluza-Klein vector becomes a D-dimensional gauge field after reduction). Following the language of [53], we shall refer to the D-dimensional gauge fields $\mathcal{A}_{[p]}, \mathcal{A}_{[p-1]}$ as 'bare gauge fields' since when working them, it is not possible to package all the gauge degrees of freedom into field strengths, and thus gauge invariance of the D-dimensional action is not manifest. However, following [36, 37, 95] we can introduce the p-form gauge field

$$\begin{aligned} & A_{[p]} = \mathcal{A}_{[p]} - V \wedge \mathcal{A}_{[p-1]}, \\ \text{or} \quad & A_\mu = \mathcal{A}_\mu - \mathcal{A}_0 V_\mu, \quad \text{for the particular case of a 1-form.} \end{aligned} \tag{3.36}$$

such that the dimensionally reduced action can be written in terms of manifestly gauge invariant field strengths $F_{[p+1]} = dA_{[p]}$ as,

$$S_{\text{gauge}} = -(-)^t \int e^{(2p-D)\alpha\phi - \beta\phi} \frac{1}{2} \mathcal{F}_{[p]} \wedge \star \mathcal{F}_{[p]} - \epsilon e^{(2p+2-D)\alpha\phi + \beta\phi} \frac{1}{2} F_{[p+1]} \wedge \star F_{[p+1]} \,.$$

As before, we wish to rewrite this in terms of spacetime coordinates. Following (3.30) and expanding the Hodge star, we introduce a factor of $(-\epsilon)(-)^t$ in D dimensions. This will absorb the overall factor of $(-)^t$, and by using $\epsilon^2 = +1$, we arrive at

$$\begin{aligned} S_{\text{gauge}} = \int d^D x \mathbf{e} \Big[& -\frac{1}{2(p+1)!} e^{(2p+2-D)\alpha\phi + \beta\phi} F_{\mu_1 \dots \mu_{p+1}} F^{\mu_1 \dots \mu_{p+1}} \\ & + \frac{1}{2p!} \epsilon e^{(2p-D)\alpha\phi - \beta\phi} \mathcal{F}_{\mu_1 \dots \mu_p} \mathcal{F}^{\mu_1 \dots \mu_p} \Big] \,, \end{aligned} \tag{3.37}$$

where α and β depend on the Einstein-Hilbert reduction and should be fixed according to (3.34).

In summary, the reduction of the $(D+1)$-dimensional action for p-form gauge fields produces

a D-dimensional action for both p-form and $(p-1)$-form gauge fields. For the particular case of $p = 1$ the reduction will generate a gauge field term for A_μ and a scalar kinetic term for $\mathcal{A}_0$, i.e. a kinetic term for the (scalar) components of the $(D+1)$-dimensional gauge fields in the compact direction.

3.2.5 Kaluza-Klein charge quantization

In this section, we examine how dimensional reduction can introduce an electric charge to the lower dimensional theory. This was originally discovered by Kaluza [96] and then developed by Klein [97]. The main idea of Kaluza was to obtain four-dimensional Einstein-Maxwell theory from a five-dimensional theory of pure gravity. In Section 3.2.3 we saw that dimensionally reducing the Einstein-Hilbert term produces a kinetic term for the Kaluza-Klein vector, $V_{\mu\nu}V^{\mu\nu}$. This looks like another field strength term and we shall now investigate how it is able to support an electric charge.

We first consider the following decomposition of the five-dimensional metric

$$ds^2_{(5)} = \hat{g}_{\hat{\mu}\hat{\nu}}dx^{\hat{\mu}}dx^{\hat{\nu}} = (dx^0 + V_\mu dx^\mu)^2 + g_{\mu\nu}dx^\mu dx^\nu, \tag{3.38}$$

from which we find the five-dimensional geodesic equation

$$\ddot{x}^{\hat{\mu}} + \hat{\Gamma}^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}}\dot{x}^{\hat{\nu}}\dot{x}^{\hat{\rho}} = 0\,. \tag{3.39}$$

Because the metric (3.38) is independent of x^0 we can identify $k = \partial_0$ as a Killing vector. Consequently, the following quantity must be conserved along geodesic worldlines,

$$k \cdot \dot{x} = \dot{x}^0 + V_\mu \dot{x}^\mu. \tag{3.40}$$

The four-dimensional equations of motion follow from (3.39) as

$$\begin{aligned} 0 = \ddot{x}^\mu + \hat{\Gamma}^\mu{}_{\hat{\nu}\hat{\rho}}\dot{x}^{\hat{\nu}}\dot{x}^{\hat{\rho}} &= \ddot{x}^\mu + \hat{\Gamma}^\mu{}_{\nu\rho}\dot{x}^\nu\dot{x}^\rho + 2\hat{\Gamma}^\mu{}_{\nu 0}\dot{x}^\nu\dot{x}^0 + \hat{\Gamma}^\mu{}_{00}\dot{x}^0\dot{x}^0 \\ &= \ddot{x}^\mu + \Gamma^\mu{}_{\nu\rho}\dot{x}^\nu\dot{x}^\rho - F^\mu{}_\nu V_\rho \dot{x}^\nu\dot{x}^\rho - F^\mu{}_\nu \dot{x}^\nu\dot{x}^0 \\ &= \ddot{x}^\mu + \Gamma^\mu{}_{\nu\rho}\dot{x}^\nu\dot{x}^\rho - F^\mu{}_\nu \dot{x}^\nu\left(V_\rho\dot{x}^\rho + \dot{x}^0\right), \end{aligned} \tag{3.41}$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ and we expanded the five-dimensional Christoffel symbols according to (43.10) of [44]. To reiterate, this particular piece of the five-dimensional geodesic equation implies

$$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = \left(V_\rho\dot{x}^\rho + \dot{x}^0\right)F^\mu{}_\nu\dot{x}^\nu. \tag{3.42}$$

Notice that we have not said anything yet about dimensional reduction. We have merely taken

the four-dimensional μ components of the 5-dimensional geodesic equation. We can then compare (3.42) with the following four-dimensional equation of motion for a gravitational field with an electromagnetic source

$$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = \frac{Q}{m}F^\mu{}_\nu\dot{x}^\nu. \tag{3.43}$$

Clearly, in order to recover the Lorentz Law we must identify the integral of motion with the ratio of charge to mass as follows [98]

$$\dot{x}^0 + V_\mu\dot{x}^\mu = \frac{Q}{m}\,. \tag{3.44}$$

If we write the four-dimensional line element as $d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu$, where τ is the four-dimensional proper time, we can divide (3.38) by $d\tau^2$ to obtain

$$\frac{ds^2_{(5)}}{d\tau^2} = \left(\dot{x}^0 + V_\mu\dot{x}^\mu\right)^2 + 1 = U^0U_0 + U^\mu U_\mu\,. \tag{3.45}$$

where $U^\mu = \frac{dx^\mu}{d\tau}$ and $U^0 = \frac{dx^0}{d\tau}$ can be thought of as the velocities with respect to the four-dimensional spacetime. Since $\hat{g}_{00} = 1$ from (3.38), we know that $U_0 = U^0$, and (3.45) then identifies the velocity in the x^0 direction, as measured by the four-dimensional spacetime, as being

$$U^0 = \frac{dx^0}{d\tau} = \dot{x}^0 + V_\mu\dot{x}^\mu, \tag{3.46}$$

which we recall from (3.40) as being a constant of motion. We can substitute (3.46) into (3.44) and cross multiply the mass to find that the momentum in the x^0 direction as measured by the four-dimensional metric is constant and given by

$$P^0 = Q\,. \tag{3.47}$$

This appears to demonstrate how the four-dimensional forces of gravity and electromagnetism can be unified via pure gravity in a higher dimension. In particular, diffeomorphism invariance around the compact S^1 generates the $U(1)$ symmetry group of electromagnetism. Furthermore, we gain a geometric understanding of the motion of charged particles: charge corresponds to momentum in the fifth dimension and particles of different charge move differently to one another because of their different initial momenta in the x^0 direction [44].

The above analysis does have some problems such as the reliance on Kaluza's so-called 'cylinder condition' that the metric components be independent of x^0. As noted already, this seems somewhat unnatural and *a priori* there doesn't appear to be any justification for this. Of course, we must employ Klein's resolution that the x^0 dimension is closed, periodic and sufficiently small that we can safely neglect any potential contributions to the metric. This compactification introduces a quantisation of the electric charge. We can see this by making the

identification

$$x^0 \sim x^0 + 2\pi r^0,$$

where r^0 is the parametric length of the S^1. Consequently, we not only find that compactification forces us to truncate the massive modes as discussed in Section 3.2.1, but it also forces the wavefunction of particles moving in the x^0 direction to be standing waves on the circle. The periodicity determines the allowed wavelengths to be [99]

$$\lambda = \frac{2\pi r^0}{N}, \qquad N \in \mathbb{Z}^+.$$

Using the De Broglie relation, the momentum carried by such particles around the x^0 circle is now quantized as follows

$$P^0 = \frac{h}{\lambda} = \frac{hN}{2\pi r^0} = \frac{\hbar N}{r^0} = \frac{N}{r^0}, \qquad N \in \mathbb{Z}^+, \tag{3.48}$$

where we set $\hbar = 1$ by choosing to work in natural units. Recalling (3.47), we see that

$$Q = \frac{N}{r^0}, \qquad N \in \mathbb{Z}^+, \tag{3.49}$$

which explains the quantization of four-dimensional electric charge as integer multiples of a fundamental electric charge, $\frac{1}{r^0}$.

We emphasise that the Einstein-Maxwell theory is not a consistent truncation of higher dimensional General Relativity [44].

3.3 The r-maps, c-maps and q-maps

We now want to apply the dimensional reduction technique to the Lagrangians for vector multiplets coupled to $\mathcal{N} = 2$ supergravity in both five and four dimensions. This will be an important part of our technique for generating black brane solutions in later chapters. In this section, we review the field content and Lagrangian of the dimensionally reduced theory, as well as what happens to the relevant target manifolds under reduction. As the calculations are quite lengthy, we shall just outline the key steps and refer to the literature for full details.

3.3.1 Dimensional reduction of $5d$, $\mathcal{N} = 2$ supergravity and r-maps

We shall first consider the dimensional reduction of a theory of five-dimensional $\mathcal{N} = 2$ supergravity coupled to $n_V^{(5)}$ vector multiplets. At this point we adopt the notation of [53] and denote the Kaluza-Klein scalar and Kaluza-Klein vector by σ and $\mathcal{A}^0$ respectively. Furthermore, since we are starting with $D + 1 = 5$, (3.34) tells us to set $\alpha = \frac{1}{2}, \beta = 1$ in the metric ansatz, such

that (3.27) becomes

$$ds^2_{(5)} = -\epsilon_1 e^{2\sigma}(dx^0 + \mathcal{A}^0)^2 + e^{-\sigma} ds^2_{(4)}\,, \tag{3.50}$$

where we are using a parameter ϵ_1 to track the signature of the x^0 direction that we will compactify over. The dimensional reduction of (3.15) is performed in [53] and the resulting four-dimensional Lagrangian is

$$\begin{aligned}\mathsf{e}_4^{-1}\mathcal{L}_4 = &-\frac{1}{2}R_{(4)} - \frac{3}{4}\partial_\mu\sigma\partial^\mu\sigma + \epsilon_1\frac{1}{8}e^{3\sigma}\mathcal{F}^0_{\mu\nu}\mathcal{F}^{0|\mu\nu} - \frac{3}{4}a_{ij}\partial_\mu h^i\partial^\mu h^j \\ &+ \epsilon_1\frac{1}{2}e^{-2\sigma}a_{ij}\partial_\mu m^i\partial^\mu m^j - \frac{1}{4}e^{\sigma}a_{ij}\mathcal{F}^i_{\mu\nu}\mathcal{F}^{j|\mu\nu} - a_{ij}e^{\sigma}\mathcal{A}^{0|\mu}\partial^\nu m^i\mathcal{F}^j_{\mu\nu} \\ &- \frac{1}{2}e^{\sigma}a_{ij}\partial^\mu m^i\partial_\mu m^j\mathcal{A}^{0|\mu}\mathcal{A}^0_\mu + \frac{1}{2}e^{\sigma}\partial^\mu m^i\partial^\nu m^j\mathcal{A}^0_\mu\mathcal{A}^0_\nu \\ &- \epsilon_1\frac{\mathsf{e}_4^{-1}}{2\sqrt{6}}c_{ijk}m^k\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^i_{\mu\nu}\mathcal{F}^j_{\rho\sigma}\,,\end{aligned} \tag{3.51}$$

where we have defined $m^i := \mathcal{A}^i_0$ as the x^0-components of the five-dimensional gauge fields. The first three terms come from the dimensional reduction of the Einstein-Hilbert term in (3.13) following Section 3.2.3, whilst the fourth comes easily from reducing the scalar kinetic term. The reduction of the gauge field terms in (3.13) follows Section 3.2.4 and leads to the terms on the second and third lines above, whilst the reduction of the Chern-Simons term gives the final piece.

The presence of bare gauge fields not wrapped up in field strengths means the gauge invariance of the reduced Lagrangian (3.51) is not manifest. It is explained in [53] how we can redefine the gauge fields according to (3.36) as

$$A^i_\mu := \sqrt{2}\cdot 6^{-\frac{1}{6}}\left(\mathcal{A}^i_\mu - m^i\mathcal{A}^0_\mu\right), \tag{3.52}$$

and also make a redefinition of the Kaluza-Klein vector via

$$A^0_\mu = -\frac{1}{\sqrt{2}}\mathcal{A}^0_\mu\,, \tag{3.53}$$

such that the reduced Lagrangian can be written in a manifestly gauge invariant way as

$$\mathsf{e}_4^{-1}\mathcal{L}_4 = -\frac{R_{(4)}}{2} - g_{ij}\partial_\mu z^i\partial^\mu\bar{z}^j + \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu}\,, \tag{3.54}$$

where $i = 1, \ldots, n_V^{(5)} + 1$ and $I = (0, i)$. There are clearly $n_V^{(4)} + 1 = n_V^{(5)} + 2$ four-dimensional gauge fields A^I_μ. Equations (3.53) and (3.52) relate these to the five-dimensional fields as follows. A^0_μ is simply a rescaling of the Kaluza-Klein vector according to (3.53), whilst (3.52) provides the relationship between the five-dimensional gauge fields $\mathcal{A}^i_{\hat{\mu}} = (m^i, \mathcal{A}^i_\mu)$ and the four-dimensional gauge fields A^i_μ. Meanwhile the relationship between the four-dimensional and five-dimensional scalar fields can be seen by expanding the ϵ_1-complex four-dimensional scalars as $z^i = x^i + i_{\epsilon_1}y^i$

with

$$x^i = 2 \cdot 6^{-\frac{1}{6}} \mathcal{A}_0^i = 2 \cdot 6^{-\frac{1}{6}} m^i, \qquad y^i = 6^{\frac{1}{3}} e^{\sigma} h^i. \tag{3.55}$$

Clearly the four-dimensional scalars z^i are built from a rescaling of both the five-dimensional superconformal scalars, h^i, and also the components of the five-dimensional gauge fields in the compact direction, $m^i = \mathcal{A}_0^i$. This rescaling of the five-dimensional scalar fields h^i is necessary to make contact with the conventions of four-dimensional supergravity and leads to an alteration of the D-gauge condition, which by homogeneity of the Hesse potential changes from $H(h) = 1$ to $H(y) = 6e^{3\sigma}$. Further, the scalar field coupling is given as

$$g_{ij} = \frac{3}{2}\epsilon_1 \left(\frac{(cy)_{ij}}{cyyy} - \frac{3}{2} \frac{(cyy)_i (cyy)_j}{(cyyy)^2} \right) =: \epsilon_1 \hat{g}_{ij}(y) \,, \tag{3.56}$$

where $cyyy = c_{ijk} y^i y^j y^k$, $(cyy)_i = c_{ijk} y^j y^k$, etc. Later in Section 3.3.3, we will need expressions for various components of the vector couplings $\mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$, as well as the inverse matrix $\mathcal{I}^{IJ}$. These are given in [53] as

$$\begin{aligned}
\mathcal{I}_{00} &= \epsilon_1 (cyyy) \left(\frac{1}{6} + \frac{2}{3} gxx \right) , & \mathcal{R}_{00} &= -\frac{1}{3}(cxxx) \,, \\
\mathcal{I}_{0i} &= -\epsilon_1 \frac{2}{3} (cyyy)(gx)_i \,, & \mathcal{R}_{0i} &= \frac{1}{2}(cxx)_i \,, \\
\mathcal{I}_{ij} &= \epsilon_1 \frac{2}{3} (cyyy) g_{ij} \,, & \mathcal{R}_{ij} &= -(cx)_{ij} \,, \\
\mathcal{I}^{00} &= \epsilon_1 6 (cyyy)^{-1} \,, \\
\mathcal{I}^{0i} &= \epsilon_1 6 (cyyy)^{-1} x^i \,, \\
\mathcal{I}^{ij} &= \epsilon_1 6 (cyyy)^{-1} \left(x^i x^j + \frac{1}{4} g^{ij} \right) .
\end{aligned} \tag{3.57}$$

The Lagrangian (3.54) of the dimensionally reduced theory describes the coupling of $n_V^{(4)} = n_V^{(5)} + 1$ vector multiplets to four-dimensional $\mathcal{N} = 2$ supergravity, as can be seen by comparing with (3.20). Indeed, it can be shown that the scalar fields parametrize a PSϵ_1K manifold $\bar{N}$ of dimension $2n_V^{(4)} = 2(n_V^{(5)} + 1)$, with metric[24]

$$g_{\bar{N}} = -\hat{g}_{ij}(y)(dy^i dy^j - \epsilon_1 dx^i dx^j) \,.$$

An interesting observation is that the dimensional reduction induces a pair of maps

$$\bar{r}^{\epsilon_1} : \mathcal{H} \to \bar{N} \,, \tag{3.58}$$

taking a particular PSR manifold $\mathcal{H}$ of dimension $n_V^{(5)}$ to a $2(n_V^{(5)} + 1)$-dimensional PSϵ_1K manifold $\bar{N}$. These maps are known as the time-like ($\epsilon_1 = -1$) and space-like ($\epsilon_1 = +1$) r-

[24] We will not reproduce this proof here as it will require us to deviate too much from the main focus of generating solutions, but the interested reader may consult [53] for an explicit verification.

maps respectively depending on the signature of the compactified dimension. The set of PSϵ_1K manifolds, $\bar{r}^{\epsilon_1}(\mathcal{H})$, appearing in the image of the r-map are known as *projective very special* ϵ_1*-Kähler* [27].

The Lagrangian (3.54) of such a dimensionally reduced theory is shown in [53] to correspond to a theory completely determined by the 'very special' prepotentials

$$F(X) = -\frac{1}{6}\epsilon_1 c_{ijk}\frac{X^i X^j X^k}{X^0}\,,$$

which only depends on the data c_{ijk} of the five-dimensional theory. As we saw in Section 3.1.3, all couplings of the four-dimensional action are completely determined by the prepotential, and as such, once the c_{ijk} are specified for the five-dimensional theory, this completely fixes the dynamics of the four-dimensional theory.

3.3.2 Dimensional reduction of $4d$, $\mathcal{N}=2$ supergravity and c-maps

Next we want to examine the dimensional reduction of four-dimensional $\mathcal{N}=2$ supergravity coupled to $n_V^{(4)}$ vector multiplets. This time we have $D+1=4$ and so (3.34) sets $\alpha=1, \beta=1$ in the metric ansatz such that (3.27) becomes

$$ds^2_{(4)} = -\epsilon_2 e^{2\phi}(dx^4+B)^2 + e^{-2\phi}ds^2_{(3)}\,, \tag{3.59}$$

where we label the Kaluza-Klein scalar and Kaluza-Klein vector by ϕ and B respectively, and use the parameter ϵ_2 to track the signature of the x^4 direction that we will compactify over. We will also continue using the parameter ϵ_1 as this will allow us to consider in parallel the reduction of a Minkowski ($\epsilon_1=-1$) or Euclidean ($\epsilon_1=+1$) signature four-dimensional theory. To emphasise, we are only using ϵ_1 to label different four-dimensional starting points, and are not at present making any connection to a five-dimensional theory. The number of time-like directions in the three-dimensional theory is then indexed by a new parameter $\epsilon := -\epsilon_1\epsilon_2 = (-1)^t$. Obviously, it is not possible to simultaneously have $\epsilon_1=\epsilon_2=+1$ due to an insufficient number of time-like directions. However, the other three combinations are all valid and are presented in full in [54], which we now review.

We denote the Kaluza-Klein field strength as $H=dB$, and decompose the four-dimensional gauge fields as

$$A^I_\mu = (A^I_m, \zeta^I)\,, \tag{3.60}$$

where m is a three-dimensional space(-time) index, and ζ^I represents the x^4-component of the gauge field. Using this decomposition, the resulting three-dimensional Lagrangian will involve the bare Kaluza-Klein vector B_m, which prevents the abelian gauge symmetry from being manifest [94]. This is exactly the situation we saw in (3.51) and so we follow (3.36) and make the

redefinition

$$A^I_m \mapsto (A^I_m)' := A^I_m - B_m \zeta^I \,, \tag{3.61}$$

to restore manifest gauge invariance of the lower dimensional theory under general coordinate transformations of the compact direction as discussed earlier in (3.36). As explained in [94], this redefinition replaces all problematic terms involving bare gauge fields with the gauge invariant quantity $(F^I_{mn})' + H_{mn}\zeta^I$. Since we shall always make this replacement, we drop the primes from here onwards. The dimensional reduction of (3.20) is carried out in [54] and the resulting three-dimensional Lagrangian is

$$\begin{aligned}
\mathsf{e}_3^{-1}\mathcal{L}_3 = &-\frac{1}{2}R_{(3)} - \partial_m\phi\partial^m\phi + \epsilon_2\frac{1}{8}e^{4\phi}H_{mn}H^{mn} - g_{ij}\partial_m z^i\partial^m\bar{z}^j \\
&+ \frac{1}{4}e^{2\phi}\mathcal{I}_{IJ}\left(F^I_{mn} + H_{mn}\zeta^I\right)\left(F^{J|mn} + H^{mn}\zeta^J\right) \\
&- \epsilon_2\frac{1}{2}e^{-2\phi}\mathcal{I}_{IJ}\partial_m\zeta^I\partial^m\zeta^J - \epsilon_2\frac{1}{2}\mathcal{R}_{IJ}\epsilon^{mnp}\left(F^I_{mn} + H_{mn}\zeta^I\right)\partial_p\zeta^J,
\end{aligned} \tag{3.62}$$

where $F^I_{mn} = \partial_{[m}A^I_{m]}$ are the redefined field strengths associated to the redefined gauge fields (3.61). Again, the first three terms come from the reduction of the Einstein-Hilbert term in (3.20) following Section 3.2.3, whilst the fourth term can easily be seen to descend from the scalar kinetic term. Meanwhile the reduction of the gauge field part of (3.20) follows Section 3.2.4 and leads to the fifth and sixth terms. The seventh term comes from the final term in (3.20): this is not a standard gauge field term and so we must compute its reduction separately. The full details of the reduction are quite complicated and we refer to [54].

In three dimensions gravity has no local dynamics and we have the ability to Hodge dualise vector fields into scalar fields. In particular if $G = G_{mn}dx^m \wedge dx^n$ is a 2-form field strength, we can define

$$G_m = -\frac{1}{2}\epsilon_2\varepsilon_{mnp}G^{np}, \quad G_{mn} = \epsilon_1\varepsilon_{mnp}G^p, \tag{3.63}$$

where G^{np} and G^p are obtained by raising the components of G and $\star G$ respectively. We have chosen this definition such that $\star(\star G) = (-1)^t G$ where t is the number of time-like directions. Recall that $(-1)^t = \epsilon = -\epsilon_1\epsilon_2$, which appears in the three-dimensional identity $\epsilon_{mnp}\epsilon^{mnp} = 2!\epsilon\delta^q{}_p$. Applying this, we can write (3.62) as

$$\begin{aligned}
\mathsf{e}_3^{-1}\mathcal{L}_3 = &-\frac{R_{(3)}}{2} - g_{ij}\partial_m z^i\partial^m\bar{z}^j - \partial_m\phi\partial^m\phi - \epsilon_1\frac{1}{4}e^{4\phi}H_m H^m \\
&+ \epsilon\frac{1}{2}e^{2\phi}\mathcal{I}_{IJ}\left(F^I_m + H_m\zeta^I\right)\left(F^{J|m} + H^m\zeta^J\right) \\
&- \epsilon_2\frac{1}{2}e^{-2\phi}\mathcal{I}_{IJ}\partial_m\zeta^I\partial^m\zeta^J + \mathcal{R}_{IJ}\left(F^I_m + H_m\zeta^I\right)\partial^m\zeta^J.
\end{aligned} \tag{3.64}$$

The dualised field strengths are constrained by the Bianchi identities, $\partial^m F^I_m = \partial^m H_m = 0$, which we encode into the three-dimensional Lagrangian by means of Lagrange multipliers $\tilde{\zeta}_I$

and $\tilde{\phi}$, which are the Hodge duals of the three-dimensional gauge field and Kaluza-Klein vector field respectively. To do this consistently, we follow [5] and add the following term to the Lagrangian in (3.64):

$$\mathsf{e}_3^{-1}\mathcal{L}_{\rm LM} = -\left(F^I_m + H_m\zeta^I\right)\partial^m\tilde{\zeta}_I + \frac{1}{2}H^m\left(\partial_m\tilde{\phi} + \zeta^I\overleftrightarrow{\partial}_m\tilde{\zeta}_I\right). \tag{3.65}$$

The equations of motion for H_m and F^I_m come from the combination of the actions (3.64) and (3.65). They are given by

$$H_m = \epsilon_1 e^{-4\phi}\left(\partial_m\tilde{\phi} + \zeta^I\overleftrightarrow{\partial}_m\tilde{\zeta}_I\right), \tag{3.66}$$

$$F^I_m + H_m\zeta^I = \epsilon e^{-2\phi}\mathcal{I}^{IJ}\left(\partial_m\tilde{\zeta}_J - \mathcal{R}_{JK}\partial_m\zeta^K\right). \tag{3.67}$$

Back-substituting these expressions into the combination of actions (3.64) and (3.65), we can collect like terms and obtain the final expression [54] for the dimensional reduction of (3.20) in terms of the $4(n_V^{(4)}+1)$ scalar fields $\{z^i, \phi, \tilde{\phi}, \zeta^I, \tilde{\zeta}_I\}$,

$$\begin{aligned}\mathsf{e}_3^{-1}\mathcal{L}_3 = &-\frac{R_{(3)}}{2} - g_{ij}\partial_m z^i\partial^m\bar{z}^j - \partial_m\phi\partial^m\phi \\ &+ \epsilon_1 e^{-4\phi}\left(\partial_m\tilde{\phi} + \zeta^I\overleftrightarrow{\partial}_m\tilde{\zeta}_I\right)\left(\partial^m\tilde{\phi} + \zeta^J\overleftrightarrow{\partial}^m\tilde{\zeta}^J\right) - \epsilon_2\frac{1}{2}e^{-2\phi}\mathcal{I}_{IJ}\partial_m\zeta^I\partial^m\zeta^J \\ &- \epsilon\frac{1}{2}e^{-2\phi}\mathcal{I}^{IJ}\left(\partial_m\tilde{\zeta}_I - \mathcal{R}_{IK}\partial_m\zeta^K\right)\left(\partial^m\tilde{\zeta}_J - \mathcal{R}_{JL}\partial^m\zeta^L\right).\end{aligned} \tag{3.68}$$

Generating 4d solutions by dimensional redox

The ability to dualise vectors into scalars makes the equations of motion much easier to deal with in three dimensions. Indeed, computing the dynamics of fields corresponds to finding geodesics on the ϵ-QK target space. This is the crux of our solution generating technique. In Chapter 4 we will build four-dimensional solutions by reducing to three dimensions, solving the equations of motion there and then dimensionally lifting or 'oxidising' the various fields back to four dimensions. To do this it is essential to understand the relationship between the four-dimensional fields and the three-dimensional scalars $\{z^i, \phi, \tilde{\phi}, \zeta^I, \tilde{\zeta}_I\}$ that we will directly solve for. We use the term dimensional redox to refer to this procedure of dimensional reduction followed by oxidation.

Looking at the four-dimensional Lagrangian (3.20), it is clear we will need to express $\{g_{ij}, z^i, F^I, \tilde{F}^I\}$ in terms of the three-dimensional content. Firstly, notice that the four-dimensional scalars z^i also appear in the three-dimensional Lagrangian and so we obtain a solution for them for free when solving the three-dimensional equations of motion. Also, the scalars, ϕ,$\tilde{\phi}$, allow the reconstruction of the four-dimensional metric, $\bar{g}$, according to (3.59). Next recall that were we to reinstate the primes on the redefined three-dimensional gauge fields, these can be

expanded as $(A^I_m)' = A^I_m - B_m\zeta^I$. Meanwhile the four-dimensional gauge fields can be decomposed in terms of bare three-dimensional gauge fields as $A^I_\mu = (A^I_m, \zeta^I) = \left((A^I_m)' + B_m\zeta^I, \zeta^I\right)$. Clearly the component of the gauge field in the compact x^4 direction is given by

$$A^I_4 = \zeta^I,$$

whilst the components of the gauge field in the non-compact directions involve the three-dimensional vectors $(A^I_m)'$ and B_m. Since we will solve for three-dimensional scalars, we must explain how these are related. Dropping the primes once again, we can use (3.63) and (3.67) to see the relation of the field strengths,

$$\begin{aligned}(F)^I_{mn} + H_{mn}\zeta^I &= \epsilon_1\varepsilon_{mnp}\left(F^{I|p} + H^p\zeta^I\right)\\ &= -\epsilon_2 e^{-2\phi}\mathcal{I}^{IJ}\varepsilon_{mnp}\left(\partial^m\tilde{\zeta}_J - \mathcal{R}_{JK}\partial^m\zeta^K\right).\end{aligned} \tag{3.69}$$

The Hodge duals, $\tilde{F}^I$, of the four-dimensional field strengths do not represent independent degrees of freedom and can be found using (3.21). Therefore, by solving the three-dimensional scalar equations of motion, we are able to reconstruct expressions for the four-dimensional fields $\{\bar{g}, z^i, F^I, \tilde{F}^I\}$. This procedure is called dimensional lifting.

c-maps

It is shown in [36, 54] that the Lagrangian (3.68) describes a theory of $n_H = n_V^{(4)} + 1$ hypermultiplets coupled to three-dimensional $\mathcal{N} = 2$ supergravity with either Minkowski ($\epsilon = -1$) or Euclidean ($\epsilon = +1$) signature. Consequently, the scalar target space corresponding to (3.68) is ϵ-quaternionic-Kähler. Again, we see that dimensional reduction induces a family of maps

$$\bar{c}^{(\epsilon_1,\epsilon_2)} : \bar{N} \to \bar{Q}\,. \tag{3.70}$$

Notice that whilst there was a pair of r-maps in (3.58) corresponding to a choice of sign for ϵ_1, we instead have a collection of c-maps since there are now two parameters (physically we are able to choose both the signature of the four-dimensional theory as well as the signature of the compact direction). These maps take a particular PSϵ_1K manifold $\bar{N}$ of dimension $2n_V^{(4)}$ to an ϵ-quaternionic-Kähler manifold $\bar{Q}$ of dimension $4(n_V^{(4)} + 1)$ with metric

$$\begin{aligned}g_{\bar{Q}} = {}& g_{ij}(z,\bar{z})dz^i d\bar{z}^j + (d\phi)^2 - \epsilon_1 e^{-4\phi}\left(d\tilde{\phi} + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)^2\\ &+ \epsilon_2\frac{1}{2}e^{-2\phi}\mathcal{I}_{IJ}d\zeta^I d\zeta^J + \epsilon\frac{1}{2}e^{-2\phi}\mathcal{I}^{IJ}\left(d\tilde{\zeta}_I - \mathcal{R}_{IK}d\zeta^K\right)\left(d\tilde{\zeta}_J - \mathcal{R}_{JL}d\zeta^L\right).\end{aligned} \tag{3.71}$$

The c-maps are known as the spatial c-map ($\epsilon_1 = \epsilon_2 = -1$), the temporal c-map ($\epsilon_1 = -\epsilon_2 = -1$) and the Euclidean c-map ($\epsilon_1 = -\epsilon_2 = +1$), and more details about each of these can be found

in [37, 54]. Notice there is still some dependence on ϵ_1, ϵ_2 in $g_{\bar{Q}}$. Thus, whilst the temporal and Euclidean c-maps both generate a para-quaternionic-Kähler target manifold with $\epsilon = -\epsilon_1\epsilon_2 = +1$, the metrics are not identical.

The set of ϵ-QK manifolds, $\bar{c}^{(\epsilon_1,\epsilon_2)}(\bar{N})$, in the image of the c-map are known as *special ϵ-quaternionic-Kähler* [27]. All of the coupling matrices appearing in the Lagrangian (3.68) of such a dimensionally reduced theory are also present in the Lagrangian (3.20) of the four-dimensional theory. As such, specifying the prepotential $F(X)$ for the four-dimensional theory completely fixes all three-dimensional dynamics.

Adapting the real formulation of special geometry to dimensional reduction

In Chapter 4 we construct four-dimensional black branes by solving three-dimensional scalar equations of motion and dimensionally lifting the results to produce regular four-dimensional solutions. We now present a modified version of the real formulation of special geometry that was first introduced in [94]. This is based on the formulation in Section 2.3 but adapted for dimensional reduction. We shall restrict ourselves to the case $\epsilon_1 = -1, \epsilon_2 = +1$ which is the case relevant for Chapter 4. It is more convenient to formulate the dimensional redox of the four-dimensional theory using real coordinates rather than holomorphic coordinates since they behave tensorially under symplectomorphisms and provide a more transparent parametrization of the para-quaternionic-Kähler target manifold appearing at the three-dimensional level.

We now explain how the real formulation is adjusted for the purposes of dimensional reduction, with the goal of obtaining an expression for the dimensionally reduced Lagrangian (3.68). Before starting, we rewrite the four-dimensional Lagrangian $\mathcal{L}_4(z^A)$ given in (3.20) as $\mathcal{L}_4(X^I)$ as given in (3.22) since we need to work with the superconformal scalars X^I in order to make contact with the existing real formulation introduced in Section 2.3.

Inspired by our treatment of the r-map, we first rescale the complex scalars

$$Y^I := e^{\phi/2} X^I. \tag{3.72}$$

Not only will this simplify the three-dimensional Lagrangian but it also removes the need to impose a D-gauge condition by hand. In particular the D-gauge condition (3.19) is modified to

$$-i(Y^I \bar{F}_I - F_I \bar{Y}^I) = e^{\phi}, \tag{3.73}$$

where $F_I = \frac{\partial F(Y)}{\partial Y^I}$ and we are now working in units where $\kappa_4^2 = 1$. This rescaling promotes the radial direction of the cone from a gauge degree of freedom to a physical one such that the CASK is no longer a collection of gauge-equivalent level sets. Instead all level sets are now physical and (3.73) gives an expression for the Kaluza-Klein scalar e^{ϕ} in terms of the new scalars. We

then make a similar decomposition as before

$$Y^I = x^I + iu^I(x,y), \quad F_I = y_I + iv_I(x,y),$$

and then gather x^I, y_I to form the special real coordinates

$$q^a := \begin{pmatrix} x^I \\ y_I \end{pmatrix} := \mathrm{Re} \begin{pmatrix} Y^I \\ F_I(Y) \end{pmatrix}. \tag{3.74}$$

As we have seen, once we have reduced to three dimensions, it is possible to write the gauge degrees of freedom using scalar fields also. In particular, we define

$$\hat{q}^a := \begin{pmatrix} \frac{1}{2}\zeta^I \\ \frac{1}{2}\tilde{\zeta}_I \end{pmatrix}, \tag{3.75}$$

where ζ^I are the components of the four-dimensional gauge fields A^I_μ along the reduction direction, and $\tilde{\zeta}_I$ are the Hodge-duals of the three-dimensional vector parts. Specifically, these scalars descend from the four-dimensional field strengths as follows:

$$\partial_m \zeta^I := F^I_{m0}\,, \quad \partial_m \tilde{\zeta}_I := G_{I|m0}\,, \tag{3.76}$$

where $G_{I|\mu\nu}$ are defined as

$$G_{I|\mu\nu} := \mathcal{R}_{IJ} F^J_{|\mu\nu} - \mathcal{I}_{IJ} \tilde{F}^J_{\mu\nu}\,.$$

We can make further use of Hodge duality to encode the Kaluza-Klein vector degree of freedom using the scalar field $\tilde{\phi}$ as shown in (3.66), although we will not need this here since we deal only with static configurations.

In terms of rescaled complex scalars Y^I and rescaled real variables q^a, the prepotential $F(Y^I)$ and Hesse potential $H(q^a)$ are related by the Legendre transformation

$$H(x^I, y_I) = 2\,\mathrm{Im}\,F(Y(x,y)) - 2y_I u^I(x,y) = \frac{i}{2}\left(Y^I \bar{F}_I(Y) - F_I(Y)\bar{Y}^I\right) = -\frac{1}{2}e^\phi.$$

We also note that the D-gauge, when expressed in terms of rescaled real scalars, reads

$$-2H(q^a) = e^\phi. \tag{3.77}$$

On the CASK manifold, we again introduce the Hessian metric $H_{ab} = \frac{\partial^2}{\partial q^a \partial q^b} H$ which is the real version of the CASK metric N_{IJ} that we met earlier in Section 3.1.3 and has signature $(--+\cdots+)$ with negative eigendirections along the $\mathbb{C}^*$ directions of the cone. It is also possible

to construct the following tensor field

$$\tilde{H}_{ab} := \frac{\partial^2}{\partial q^a \partial q^b}\tilde{H}\,, \quad \tilde{H} := -\frac{1}{2}\log\left(-2H\right), \tag{3.78}$$

which will be used extensively in this adaptation of the real formulation. This tensor can be interpreted as another metric on the CASK manifold that is related to H_{ab} by flipping the signature along the radial direction generated by the field ξ, combined with a conformal transformation which changes the scale transformation $q^a \to \lambda q^a$, where $\lambda \in \mathbb{R}^{>0}$, from being a homothety to being an isometry. This follows from the obvious fact that while $H_{ab}dq^a dq^b$ is homogeneous of degree 2, $\tilde{H}_{ab}dq^a dq^b$ is homogeneous of degree 0. Note that the metric coefficients H_{ab} and $\tilde{H}_{ab}$ are homogeneous of degrees 0 and -2, respectively. The two real tensors are related by

$$\tilde{H}_{ab} = \frac{1}{(-2H)}\left(H_{ab} - \frac{H_a H_b}{H}\right). \tag{3.79}$$

It will be convenient for us to introduce a set of dual coordinates with respect to the metric $\tilde{H}_{ab}$ defined by

$$q_a := \tilde{H}_a := \frac{\partial \tilde{H}}{\partial q^a} = -\frac{H_a}{2H} = \frac{-1}{H}\begin{pmatrix} v_I \\ -u^I \end{pmatrix}. \tag{3.80}$$

One can show that

$$q_a = -\tilde{H}_{ab}q^b\,, \quad \partial_m q_a = \tilde{H}_{ab}\partial_m q^b\,, \tag{3.81}$$

where we have used that $\tilde{H}_a$ is homogeneous of degree -1 for the first identity and the chain rule for the second. It is also possible to use this metric to lower the index on $\partial_m \hat{q}^a$ to obtain the co-vector field

$$\partial_m \hat{q}_a := \tilde{H}_{ab}\partial_m \hat{q}^b\,. \tag{3.82}$$

It will be important for calculations in Chapter 4 to observe that whilst $\tilde{H}$ is a Hesse potential for $\tilde{H}_{ab}$, the inverse metric $\tilde{H}^{ab}$ has Hesse potential $-\tilde{H}$ [3], i.e.

$$\tilde{H}^{ab} = \frac{\partial q^a}{\partial q_b} = \frac{\partial^2(-\tilde{H})}{\partial q_a \partial q_b}\,. \tag{3.83}$$

A lengthy calculation given in [94] shows the three-dimensional Lagrangian (3.68) can be expressed using the adapted real formulation of special geometry in terms of the above ingredients as

$$\begin{aligned} e_3^{-1}\mathcal{L}_3 \;=\; & -\frac{1}{2}R_{(3)} - \tilde{H}_{ab}\left(\partial_m q^a \partial^m q^b - \partial_m \hat{q}^a \partial^m \hat{q}^b\right) \\ & -\frac{1}{H^2}(q^a \Omega_{ab}\partial_m q^b)^2 + \frac{2}{H^2}(q^a \Omega_{ab}\partial_m \hat{q}^b)^2 \\ & -\frac{1}{4H^2}(\partial_m \tilde{\phi} + 2\hat{q}^a \Omega_{ab}\partial_m \hat{q}^b)^2\,. \end{aligned} \tag{3.84}$$

Although we have packaged the homothetic degree of freedom into the q^a coordinates, the Lagrangian (3.84) still has a $U(1)$ gauge symmetry descending from $\mathbb{C}^*$ transformations of the X^I coordinates. In fact, the coordinates $\{q^a, \hat{q}^a, \tilde{\phi}\}$ represent a $U(1)$ principal bundle over the para-QK target manifold of the physical scalar fields [54]. To descend to the physical submanifold we must gauge fix the $U(1)$ transformations using any suitable constraint. In Chapter 4 we will see how solutions for the three-dimensional fields $q^a, \hat{q}^a, \tilde{\phi}$ can lift to solutions for the four-dimensional degrees of freedom.

3.3.3 Double reduction of $5d$, $\mathcal{N} = 2$ supergravity and q-maps

As we have seen, a particularly nice feature when dimensionally reducing from four dimensions to three dimensions is the ability to Hodge dualise three-dimensional vector fields into scalars as seen in Section 3.3.2. Ultimately, the three-dimensional Lagrangian can then be expressed entirely in terms of scalar fields, making the equations of motion easier to deal with. Later, in Chapter 5, we will do something similar to construct five-dimensional black brane solutions. Again, we will make use of the simplicity in three dimensions by combining the techniques of the previous two sections to dimensionally reduce a theory of $n_V^{(5)}$ vector multiplets coupled to five-dimensional $\mathcal{N} = 2$ supergravity down to three dimensions with the intention of solving the field equations there before lifting back to a five-dimensional interpretation. The reduction itself is done first over the x^0 direction and then over the x^4 direction, which corresponds to making a metric ansatz $M_5 = S^1 \times S^1 \times M_3$, with

$$ds^2_{(5)} = -\epsilon_1 e^{2\sigma}(dx^0 + \mathcal{A}^0)^2 - \epsilon_2 e^{2\phi-\sigma}(dx^4 + B)^2 + e^{-2\phi-\sigma} ds^2_{(3)}\,, \tag{3.85}$$

where $\epsilon_{1,2}$ assume the value -1 for space-like reduction and $+1$ for time-like reduction. The number of time-like directions in the resulting three-dimensional theory is, as before, labelled by the parameter $\epsilon = -\epsilon_1\epsilon_2 = (-1)^t$. The Kaluza-Klein vectors have components $\mathcal{A}^0 = \mathcal{A}^0_4 dx^4 + \mathcal{A}^0_\mu dx^\mu$ and $B = B_\mu dx^\mu$ where $\mu = 1, 2, 3$ are three-dimensional space(-time) indices.

The resulting three-dimensional theory is derived by plugging the expressions for the couplings $g_{ij}, \mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$ from (3.56) and (3.57) obtained in the reduction from five dimensions to four dimensions into the three-dimensional Lagrangian, (3.68), obtained by reducing the

four-dimensional theory. Doing so, we arrive at the following [37]

$$
\begin{aligned}
\mathsf{e}_3^{-1}\mathcal{L}_3 = &-\frac{R_{(3)}}{2} - \frac{3}{4}a_{ij}(h)\partial_\mu h^i \partial^\mu h^j + \epsilon_1 \frac{3}{4\sigma^2} a_{ij}(h)\partial_\mu x^i \partial^\mu x^j - \frac{3}{4\sigma^2}(\partial\sigma)^2 - \frac{1}{4\phi^2}(\partial\phi)^2 \\
&+ \epsilon_1 \frac{1}{4\phi^2}\left(\partial\tilde{\phi} + \zeta^I \overleftrightarrow{\partial} \tilde{\zeta}_I\right)^2 + \epsilon \frac{\sigma^3}{12\phi}(d\zeta^0)^2 \\
&+ \epsilon_2 \frac{\sigma}{4\phi} a_{ij}(h)\left(\partial\zeta^i - x^i\partial\zeta^0\right)\left(\partial\zeta^j - x^j\partial\zeta^0\right) \\
&+ \epsilon_2 \frac{3}{\sigma^3\phi}\left(\partial\tilde{\zeta}_0 + x^i\partial\tilde{\zeta}_i + \frac{1}{2}(cxx)_i\partial\zeta^i - \frac{1}{6}(cxxx)\partial\zeta^0\right)^2 \\
&+ \epsilon \frac{1}{\sigma\phi} a^{ij}(h)\left(\partial\tilde{\zeta}_i + (cx)_{ik}\partial\zeta^k - \frac{1}{2}(cxx)_i\partial\zeta^0\right) \\
&\times \left(\partial\tilde{\zeta}_j + (cx)_{jl}\partial\zeta^l - \frac{1}{2}(cxx)_j\partial\zeta^0\right). \qquad (3.86)
\end{aligned}
$$

Note that for a space-space reduction with $\epsilon_1 = \epsilon_2 = -1$ the two reductions commute, whilst for space-time or time-space they do not [100]. This is due to differences in signs of various terms in the Lagrangian (3.86), and is related to our earlier observation that the para-quaternionic-Kähler target manifolds generated by the temporal and Euclidean c-maps are not identical.

Generating 5d solutions by dimensional redox

Let us briefly explain the origin of each of the terms here. In doing so, this will help understand the relationship between five-dimensional and three-dimensional fields. Firstly, the scalars σ and ϕ appearing in (3.86) are related to the Kaluza-Klein scalars appearing in the metric ansatz (3.85) by

$$
e^{2\phi} \mapsto \phi\,, \quad 6^{1/3}e^{\sigma} \mapsto \sigma\,. \qquad (3.87)
$$

As such, we see from (3.55) that the scalar fields $z^i = x^i + i_{\epsilon_1} y^i$ decompose into

$$
x^i = 2\cdot 6^{-\frac{1}{6}} m^i = 2\cdot 6^{-\frac{1}{6}}\mathcal{A}_0^i\,, \quad y^i = \sigma h^i\,.
$$

Clearly then the fields x^i appearing in (3.86) represent the components of the five-dimensional gauge fields along the x^0 direction. Meanwhile, y^i is a combination of the five-dimensional Kaluza-Klein scalar σ and the five-dimensional superconformal scalars h^i. By comparing (3.56) and (2.36), we see that $g_{ij}(y) = -\frac{3\epsilon_1}{4\sigma^2}a_{ij}(h)$, and so it is possible to expand the second term

of (3.68) as

$$\begin{aligned}
-g_{ij}\partial_\mu z^i\partial^\mu \bar{z}^j &= -g_{ij}\partial_\mu(x^i + i_{\epsilon_1}y^i)\partial^\mu(x^j - i_{\epsilon_1}y^j) \\
&= -g_{ij}\partial_\mu x^i\partial^\mu x^j - \epsilon_1 g_{ij}\partial_\mu y^i\partial^\mu y^j \\
&= \epsilon_1\frac{3}{4\sigma^2}a_{ij}(h)\partial_\mu x^i\partial^\mu x^j \\
&- \frac{3\epsilon_1^2}{4\sigma^2}a_{ij}(h)\left[h^ih^j\partial_\mu\sigma\partial^\mu\sigma + \sigma^2\partial_\mu h^i\partial^\mu h^j + 2\sigma h^i\partial^\mu\sigma\partial_\mu h^j\right] \\
&= -\frac{3}{4}a_{ij}(h)\partial_\mu h^i\partial^\mu h^j + \epsilon_1\frac{3}{4\sigma^2}\partial_\mu x^i\partial^\mu x^j - \frac{3}{4\sigma^2}(\partial\sigma)^2,
\end{aligned} \tag{3.88}$$

where the final term in the square brackets vanishes since

$$\begin{aligned}
a_{ij}(h)h^i\partial_\mu h^j = \frac{\partial^2\tilde{H}}{\partial h^i\partial h^j}h^i\partial_\mu h^j &= -\frac{\partial\tilde{H}}{\partial h^j}\partial_\mu h^j \\
&= -\partial_\mu\tilde{H} \text{ by chain rule} \\
&= 0 \text{ since } \tilde{H} = -\frac{1}{3}\log H, \text{ with } H = \text{const on the PSR.}
\end{aligned}$$

Equation (3.88) explains the origin of second, third and fourth terms in (3.86). The fifth term in (3.86) is unchanged from (3.68) and represents the kinetic term introduced for the Kaluza-Klein scalar upon reduction of the four-dimensional Einstein Hilbert term. The sixth term in (3.86) is also unchanged and appeared in (3.68) as part of the Lagrange multiplier (3.65) enforcing the various Bianchi identities. The remaining terms in (3.86) come from substituting the relevant components of $\mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$ into (3.68). Notice the decomposition $I = (0, i)$, which explains the regrouping of contracted indices in (3.86).

The Lagrangian (3.86) contains $4n_V^{(5)} + 9$ scalars $\{h^i, x^i, \sigma, \phi, \tilde{\phi}, \zeta^0, \zeta^i, \tilde{\zeta}_0, \tilde{\zeta}_i\}$. Of course the h^i parametrize a CASR manifold and are not the physical five-dimensional scalars. Subject to D-gauge, we can make the replacement

$$-\frac{3}{4}a_{ij}(h)\partial_\mu h^i\partial^\mu h^j\Big|_D = -\frac{3}{4}g_{xy}(\phi)\partial_\mu\phi^x\partial^\mu\phi^y\,,$$

such that (3.86) is built from the scalars $\{\phi^x, x^i, \sigma, \phi, \tilde{\phi}, \zeta^0, \zeta^i, \tilde{\zeta}_0, \tilde{\zeta}_i\}$ and maps to a $4(n_V^{(5)} + 2)$-dimensional ϵ-QK target manifold. Of course, this is to be expected since

$$n_V^{(5)} \xmapsto{\bar{r}^{\epsilon_1}} n_V^{(4)} = n_V^{(5)} + 1 \xmapsto{\bar{c}^{(\epsilon_1,\epsilon_2)}} n_H = 4(n_V^{(4)} + 1) = 4(n_V^{(5)} + 2)\,.$$

As eluded to earlier, solving equations of motion in three dimensions amounts to finding geodesics on the ϵ-QK target manifold since all fields can be dualised into scalars.

Once we have a solution for the three-dimensional scalars, we can dimensionally lift or oxidise back to a five-dimensional solution. Looking at the superconformal Lagrangian (3.15), we need to

be able to express $\{a_{ij}, h^i, \mathcal{F}^i\}$ in terms of the three-dimensional fields $\{h^i, x^i, \sigma, \phi, \tilde{\phi}, \zeta^0, \zeta^i, \tilde{\zeta}_0, \tilde{\zeta}_i\}$ appearing in (3.86). The five-dimensional superconformal scalars h^i appear in the three-dimensional Lagrangian (3.86) and can be ascertained immediately from the three-dimensional equations of motion. We have also seen in (3.55) that the components of the five-dimensional gauge fields along the compact x^0 direction are given by

$$\mathcal{A}_0^i = m^i = \frac{6^{1/6}}{2} x^i \,. \tag{3.89}$$

Rearranging (3.52), the component of the five-dimensional gauge field in the compact x^4 direction can be written as $\mathcal{A}_4^i = \frac{6^{1/6}}{\sqrt{2}} A_4^i + m^i \mathcal{A}_4^0$. Substituting $A_4^I = \zeta^I$ from (3.60) and using (3.89), this can be expressed as

$$\mathcal{A}_4^i = \frac{6^{1/6}}{\sqrt{2}} (\zeta^i - x^i \zeta^0) \,. \tag{3.90}$$

To obtain the remaining three-dimensional components of the five-dimensional gauge fields, we will require knowledge of the Kaluza-Klein vectors. By Hodge dualising (3.66) (resp. (3.67)) and substituting the relevant components of $\mathcal{N}_{IJ}$ from [53], we can establish that the components of the Kaluza-Klein vectors in the non-compact directions are given by (recall $\mu = 1, 2, 3$) [37]

$$H_{\mu\nu} = \varepsilon_{\mu\nu\rho} \frac{1}{\phi^2} \left(\partial^\rho \tilde{\phi} + \zeta^I \partial^\rho \tilde{\zeta}_I - \tilde{\zeta}_I \partial^\rho \zeta^I \right) , \tag{3.91}$$

$$\mathcal{F}^0_{\mu\nu} = -\varepsilon_{\mu\nu\rho} \frac{6\sqrt{2}\epsilon}{\sigma^3 \phi} \left(\partial^\rho \tilde{\zeta}_0 + x^i \partial^\rho \tilde{\zeta}_i + \frac{1}{2} (cxx)_i \partial^\rho \zeta^i - \frac{1}{6} (cxxx) \partial^\rho \zeta^0 \right) - \left[d\zeta^0 \wedge B \right]_{\mu\nu} , \tag{3.92}$$

and [53]

$$\mathcal{A}_4^0 = -\sqrt{2} \zeta^0 . \tag{3.93}$$

Now that we have established the Kaluza-Klein vectors we can work with the Hodge dualisation of the i component in (3.67) to determine the remaining components of the five-dimensional gauge fields in the non-compact directions as follows [37]

$$\begin{aligned} \mathcal{F}^i_{\mu\nu} = & \frac{6^{1/6}\sqrt{2}\epsilon_2}{\sigma\phi} a^{ij}(h) \varepsilon_{\mu\nu\rho} \left(\partial^\rho \tilde{\zeta}_j + (cx)_{jk} \partial^\rho \zeta^k - \frac{1}{2} (cxx)_j \partial^\rho \zeta^0 \right) \\ & + \frac{6^{1/6}}{\sqrt{2}} [(d\zeta^i - x^i d\zeta^0) \wedge B]_{\mu\nu} + 6^{1/6} [dx^i \wedge \mathcal{A}^0]_{\mu\nu} \,. \end{aligned} \tag{3.94}$$

Lastly, the coupling matrix a_{ij} can be reconstructed from the metric degrees of freedom $\phi, \sigma, \mathcal{A}^0, B$ according to (3.85).

So by solving the three-dimensional equations of motion, we can reconstruct the five-dimensional field content $\{a_{ij}, h^i, \mathcal{F}^i\}$. In Chapter 5 we will use this dimensional lift to construct five-dimensional black branes. Again, things will be simpler there since we truncate many of the fields in our theory and so will only be dealing with a subset of the three-dimensional fields.

q-maps

From the arguments above, we see that the double dimensional reduction from five to three dimensions can be understood geometrically as applying the c-map to the image of the r-map. Specifically, this corresponds to a family of composite maps known as the q-maps,

$$\bar{q}^{(\epsilon_1,\epsilon_2)} = \bar{c}^{(\epsilon_1,\epsilon_2)} \circ \bar{r}^{\epsilon_1} : \mathcal{H} \to \bar{Q}\,.$$

This maps an $n_V^{(5)}$-dimensional PSR manifold $\mathcal{H}$ to a $4(n_V^{(5)}+2)$-dimensional ϵ-QK manifold $\bar{Q}$, with metric

$$\begin{aligned}
g_{\bar{Q}}^{(\epsilon_1,\epsilon_2)} &= \frac{3}{4} g_{xy}(\phi) d\phi^x d\phi^y - \epsilon_1 \frac{3}{4\sigma^2} a_{ij}(h) dh^i dh^j + \frac{1}{4\phi^2} d\phi^2 + \frac{3}{4\sigma^2} d\sigma^2 \\
&- \epsilon_1 \frac{1}{4\phi^2} \left(d\tilde{\phi} + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I \right)^2 - \epsilon \frac{\sigma^3}{12\phi} (d\zeta^0)^2 \\
&- \epsilon_2 \frac{\sigma}{4\phi} a_{ij}(h) \left(d\zeta^i - x^i d\zeta^0 \right) \left(d\zeta^j - x^j d\zeta^0 \right) \\
&- \epsilon_2 \frac{3}{\sigma^3 \phi} \left(d\tilde{\zeta}_0 + x^i d\tilde{\zeta}_i + \frac{1}{2}(cxx)_i d\zeta^i - \frac{1}{6}(cxxx) d\zeta^0 \right)^2 \\
&- \epsilon \frac{1}{\sigma\phi} a^{ij}(h) \left(d\tilde{\zeta}_i + (cx)_{ik} d\zeta^k - \frac{1}{2}(cxx)_i d\zeta^0 \right) \\
&\quad \times \left(d\tilde{\zeta}_j + (cx)_{jl} d\zeta^l - \frac{1}{2}(cxx)_j d\zeta^0 \right) .
\end{aligned}$$

The double dimensional reduction from five to three dimensions is fundamental to our technique for generating five-dimensional black brane solutions. As far as we are concerned in this thesis though, the q-map is merely an artefact of this reduction and whilst it leads to some interesting geometrical results, we shall not delve any deeper into this and refer the interested reader to [37,68]. We note that the set of ϵ-QK manifolds, $\bar{Q}(\mathcal{H})$, in the image of the q-map are known as *very special ϵ-quaternionic-Kähler* manifolds [27].

3.4 $\mathcal{N} = 2$ gauged supergravity

In this section we discuss the gauging of Lorentzian signature supergravity to obtain the action for four- and five-dimensional theories of $\mathcal{N} = 2$ Fayet-Iliopoulos (FI) gauged supergravity coupled to n_V vector multiplets. Let us begin by clarifying the difference between ungauged and gauged supergravity according to [27]:

Definition 36. *In* UNGAUGED SUPERGRAVITY *the supersymmetry is made local by gauging the super-Poincaré group. There are no other gauged symmetries and thus all vector fields are abelian and all matter is neutral.*

Definition 37. *In* GAUGED SUPERGRAVITY *the super-Poincaré group is gauged as in the un-*

gauged case but additionally there are either vector fields gauging a Yang-Mills group or there are matter fields charged under a (non-)abelian gauge group.

Although we have so far neglected the hypermultiplet sector, it will now prove fruitful to begin by considering a theory of ungauged $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets and n_H hypermultiplets. In ungauged $\mathcal{N} = 2$ supergravity in either four or five dimensions, the global symmetry group of the physical theory is

$$G = \mathrm{Isom}(M_V) \times \mathrm{Isom}(\mathrm{QK}) \times \mathrm{Aut}(\text{superalgebra}), \tag{3.95}$$

where $\mathrm{Isom}(M_V)$ is the isometry group of the vector multiplet scalar manifold (PSR or PSK depending on dimension) and $\mathrm{Isom}(\mathrm{QK})$ is the isometry group of the quaternionic-Kähler target manifold of the hypermultiplet scalars. The group of superalgebra automorphisms is known as the R-symmetry group.

Gauged $\mathcal{N} = 2$ supergravity is where some subgroup of G is promoted to a local symmetry using the relevant gauging procedure whilst, at the same time, preserving the supersymmetric structure of the theory.[25] From (3.95) we see there are three ways in which supergravity can be gauged. It is known that R-symmetry gaugings can produce asymptotically AdS solutions [27, 101]. For theories of $\mathcal{N} = 2$ supergravity coupled to vector multiplets, the R-symmetry always contains an $SU(2)_R$ factor and, as far as the vector multiplet sector is concerned, this acts only on the fermions. We want to produce non-Minkowski (preferably AdS) vacua and the simplest way to do this is to gauge an abelian $U(1) \subset SU(2)_R$. In pure gauged supergravity this gives rise to a positive cosmological constant and an AdS geometry. With the addition of vector multiplets the gauging produces a positive definite scalar potential that is a generalisation of the cosmological constant. In this case it is possible to specify which linear combination of vector fields is coupled to the fermions using so-called *FI parameters*.

As we shall see in the following subsection, the gauging of a $U(1) \subset SU(2)_R$ R-symmetry can be equivalently described as gauging an isometry of the scalar manifold of the associated superconformal theory. The Lagrangian of the physical theory depends explicitly on the associated moment map, and if we are free to change the moment map by adding a constant, this represents a genuine deformation of the Lagrangian. In $\mathcal{N} = 2$ supergravity with physical hypermultiplets the moment map relevant for gauging a $U(1) \subset SU(2)_R$ R-symmetry is uniquely determined and no such deformation is possible. However, in models without physical hypermultiplets, the moment map is only specified up to a constant, which gives rise to a genuine deformation. This second type of gauging is often referred to as a Fayet-Iliopoulos (FI) gauging in the literature [27, 89, 102, 103]. Details will be given below.

[25] Requiring the preservation of invariance under supersymmetry transformations leads to the introduction of a scalar potential into the gauged supergravity action.

3.4.1 R-symmetry in the superconformal geometry

It is useful to consider the superconformal description of the ungauged theory since the bosonic part of the $\mathcal{N} = 2$ superconformal algebra is a direct sum of the conformal algebra and the R-symmetry algebra [27]. For $\mathcal{N} = 2$ theories, the superconformal calculus requires the introduction of a Weyl multiplet as well as a compensating vector multiplet and a compensating hypermultiplet. The combination of compensating and non-compensating scalars from the n_V+1 vector multiplets and n_H+1 hypermultiplets parametrize a product manifold $M_V^{\rm SC}\times$HKC where $M_V^{\rm SC}$ is either CASR or CASK depending on dimension, and HKC is a hyperkähler cone. Notice that even in theories with $n_H = 0$, there is a non-trivial hypermultiplet sector (at the superconformal level). The geometry $M_V \times$ QK of the physical scalars is obtained as a projective manifold from the embedding manifold of the conformally invariant theory by gauge-fixing as discussed in Section 3.1.3. The first step uses the D-gauge condition to fix the homothetic $\mathbb{R}^+$ action. In this sense, it sets a scale for the theory and breaks the conformal symmetry within the superconformal algebra, leaving behind a (product of) Sasakian structure(s), generated by the various Reeb vectors which capture the residual R-symmetry part of the superconformal algebra. In other words, at the superconformal level the R-symmetry gets absorbed into the geometry and the global symmetry group of ungauged $\mathcal{N} = 2$ supergravity is

$$G^{\rm SC} = {\rm Isom}(M_V^{\rm SC}) \times {\rm Isom}({\rm HKC}) \supset {\rm Aut}({\rm superalgebra}). \tag{3.96}$$

This becomes more clear with Table 3.1. Using the isomorphism $Sp(1) \simeq SU(2)$, we see that after imposing D-gauge, the remaining isometries on the Sasakian structure(s) are precisely the R-symmetry group of superalgebra automorphisms [104]. Indeed, in five dimensions, the R-symmetry group $SU(2)_R$ comes entirely from isometries of the HKC manifold. Meanwhile in four dimensions, the R-symmetry group $U(2)_R = U(1)_R \times SU(2)_R$ comes from a product of isometries on the CASK and on the HKC. In other words, the hypermultiplet scalars are inert under $U(1)_R$ and the vector multiplet scalars are inert under $SU(2)_R$.

D	$M^{\rm SC}$	Cone homotheties/isometries	Sasakian isometries
$D = 5$	CASR $\times$ HKC	$\mathbb{R}^+ \times \mathbb{H}^* = \mathbb{R}^+ \times (\mathbb{R}^+ \times Sp(1))$	$Sp(1)$
$D = 4$	CASK $\times$ HKC	$\mathbb{C}^* \times \mathbb{H}^* = (\mathbb{R}^+ \times U(1)) \times (\mathbb{R}^+ \times Sp(1))$	$U(1) \times Sp(1)$

Table 3.1: Comparison of scalar manifold Sasakian structure(s) in $D = 4, 5$.

In this thesis we are interested in the $SU(2)_R$ symmetry only. We have just seen that this gets wrapped up with the isometries of the QK manifold into the geometry of the HKC manifold. Consequently we shall be interested in gauging symmetry transformations of the superconformal hypermultiplet scalars, corresponding to isometries of the HKC manifold. In fact, we shall focus on the degenerate case of *Fayet-Iliopoulos (FI) gaugings* where $n_H = 0$ and

we gauge the scalars in the compensating hypermultiplet [92]. At this point, we remind the reader that the terminology 'upstairs' refers to the HKC target manifold of the superconformal theory and, at the same time, we introduce the term 'downstairs' for referring to the QK target manifold of the super-Poincaré theory.

3.4.2 $U(1)$ Fayet-Iliopoulos electric gauging in four dimensions

Moment maps for gauged isometries

As eluded to above, we begin by discussing isometries of the HKC manifold. On the $4(n_H+1)$-dimensional HKC manifold we can introduce coordinates q^X and a metric $g = g_{XY} dq^X dq^Y$. The homothetic Killing vector and three $SU(2)$ Killing vectors define four directions that are to be eliminated to descend to the QK manifold. We thus make the decomposition $q^X = \{q^0, q^\alpha, q^u\}$, where q^0 and q^α $(\alpha = 1,2,3)$ parametrize the $\mathbb{H}^*$ directions, and q^u are coordinates on the $4n_H$-dimensional QK manifold.

Supposing k_I are Killing vector fields generating the isometries on the HKC manifold, we can write

$$\delta q^X = \theta^I k_I^X \,.$$

If we further assume that, in order to preserve supersymmetry, these isometries are *tri-holomorphic* then we know

$$\pounds_{k_I} g = 0 \quad \text{and} \quad \pounds_{k_I} \vec{J} = 0 \,, \tag{3.97}$$

where $\vec{J}$ is a vector in $\mathfrak{su}(2)$ space representing the triple of complex structures. In Section 2.4.2 we established that all hyperkähler manifolds are themselves Kähler. In fact, they must be Kähler with respect to any of their complex structures. Therefore, on the HKC manifold there exists a fundamental quaternion-valued 2-form $\vec{\omega} = \omega_1 i + \omega_2 j + \omega_3 k$ that is closed $d\vec{\omega} = 0$. Treating $\vec{\omega}$ as a vector in $\mathfrak{su}(2)$ space, the Hermitian property of Kähler manifolds can be expressed as

$$\vec{\omega}(k_I, \cdot) = g(\vec{J} k_I, \cdot) \,. \tag{3.98}$$

Taking the Lie derivative of (3.98) and using (3.97), we can show $\pounds_{k_I}\vec{\omega} = 0$. From Cartan's magic formula we note that [27]

$$\pounds_{k_I}\vec{\omega} = (d\iota_{k_I} + \iota_{k_I} d)\,\vec{\omega} = d\iota_{k_I}\vec{\omega} \stackrel{!}{=} 0 \,,$$

where we have used the Kähler property $d\vec{\omega} = 0$. The Poincaré Lemma [27] then guarantees, at least locally, that there exist a triple of moment maps $\vec{P}_I$ associated to each of the Killing vectors k_I such that,

$$\iota_{k_I}\vec{\omega} = d\vec{P}_I \quad \Rightarrow \quad \partial_X \vec{P}_I = \vec{J}_X{}^Y k_{IY} \,, \tag{3.99}$$

where in the second equality we have expanded in terms of HKC coordinates q^X and evaluated $\iota_{k_I}\vec{\omega}$ using (3.98). Equation (3.99) demonstrates existence of moment maps for the HKC isometries.

Since k_I^X are tri-holomorphic, they commute with the $\mathbb{H}^*$ action and thus the HKC isometries under consideration must project down to isometries on the QK manifold. The downstairs isometries are described by the same moment maps $\vec{P}_I$ [92]. It is explained in [27] that these moment maps can be expressed in terms of data on the downstairs QK manifold as

$$2n_H\kappa^2\vec{P}_I = -\vec{J}_u{}^v\nabla_v k_I^u\,, \tag{3.100}$$

where $\vec{J}_u{}^v$ are the almost complex structures on the QK manifold, ∇_v is the $SU(2)$ part of the Levi-Civita connection and k_I^u are the components of the HKC Killing vector k_I along the QK projection [27]. Equation (3.100) uniquely determines the moment maps such that they cannot be shifted by arbitrary constants as in the $\mathcal{N}=1$ case [105].

Indeed, if there are no tri-holomorphic isometries ($k_I^u = 0$) then the moment maps are uniquely determined to be zero by (3.100), with the two possible exceptions being when we have no physical hypermultiplets ($n_H = 0$) or when there is rigid supersymmetry ($\kappa = 0$) [104]. In either of these cases there remains a freedom to choose the magnitude of the moment maps despite $k_I^u = 0$, and this is characteristic of FI gaugings. As we shall see shortly, the moment maps themselves enter into the gauged supergravity Lagrangian and it is the freedom to choose their length different from zero that shows they represent genuine deformations.

Since we are interested in $\mathcal{N}=2$ supergravity coupled only to vector multiplets ($\kappa \neq 0, n_H = 0$), our gauging falls into the first of these exceptions. The FI condition $n_H = 0$ implies that the QK manifold is zero-dimensional (a point manifold), and consequently the downstairs Killing vectors must vanish, $k_I^u = 0$. This is clearly a degenerate case of (3.100) allowing constant, non-zero moment maps to exist. In this instance, the $\vec{P}_I$ are $\mathfrak{su}(2)_R$-valued constants [27,104,105,106], and we can expand the moment maps in a basis of Pauli matrices as [107]

$$\vec{P}_I = \sqrt{2}g_{IX}\sigma^X, \tag{3.101}$$

where g_{IX} are components in the basis and the numerical factor is for consistency with the literature [24,102].

Minimal coupling

So far we have just described some isometries on a scalar manifold so let us now explain how these enter into the Lagrangian. Via the non-linear sigma model, the isometries correspond to symmetry transformations of the superconformal scalars. In order to gauge the symmetry transformations of the superconformal hypermultiplet scalars, we introduce the following covariant

derivative [90]

$$D_\mu q^X = \partial_\mu q^X - b_\mu k_D^X + \frac{1}{2}\mathcal{V}_{\mu i}{}^j (k^X)_j{}^i - g A_\mu^I k_I^X(q) + \dots \,, \tag{3.102}$$

where k_D and $(k)_j{}^i$ are the generators of the dilatation and $SU(2)_R$ parts of the $\mathbb{H}^*$ action on the HKC manifold with corresponding connections b_μ and $\mathcal{V}_{\mu i}{}^j$, whilst k_I are the generators for the HKC isometries for which the $n_V + 1$ superconformal vector fields are the connections, and the dots represent additional fermionic terms . We note that it is only possible to gauge an R-symmetry (sub)group $K \subseteq SU(2)_R$ providing $\dim(K) \leq n_V + 1$. Hereafter, the symmetry transformations of the scalars (isometries) are treated as local gauge transformations.

The particular symmetry transformation being gauged is described by a triplet of moment maps which are required to satisfy the so-called equivariance condition. The equivariance condition imposes constraints on the $\vec{P}_I$ such that the gauging is consistent with supersymmetry. At this point we choose to gauge a $U(1)$ symmetry, in which case the equivariance condition is [27]

$$\vec{P}_I \times \vec{P}_J = 0 \quad \Rightarrow \quad \varepsilon^{XYZ} g_{IX} g_{JY} = 0 \,.$$

This affords us the freedom to choose a direction for the $\mathfrak{su}(2)_R$ vector. We choose to align the moment maps with the σ^3 direction in $\mathfrak{su}(2)_R$ space. This is a particularly convenient choice for interpreting the fermion charges, as we shall see later on. Thus,

$$g_{IX} = \begin{pmatrix} 0, & 0, & g_I \end{pmatrix} \,. \tag{3.103}$$

We can then substitute this into (3.101) to obtain

$$\vec{P}_I = \sqrt{2} g_I \sigma^3 \,, \tag{3.104}$$

where g_I are known as the electric FI parameters.

At the superconformal level, the Weyl multiplet contains the connections for all of the superconformal symmetry transformations e.g. the $SU(2)_R$ connection $\mathcal{V}_{\mu i}{}^j$. There are additional symmetry transformations in the theory such as the transformations of the superconformal hypermultiplet scalars q^X (isometries of HKC). We have just gauged such a $U(1)$ symmetry transformation and the relevant connection is a linear combination of the $n_V + 1$ vector fields A_μ^I. At the superconformal level all of these connections are independent, such that all symmetry transformations commute. In particular, the $SU(2)_R$ transformations commute with gauge transformations of the hypermultiplet scalars. This will change when we go downstairs to the physical super-Poincaré theory.

To take the superconformal quotient, we must gauge fix all of the superconformal symmetries. In particular, we impose D-gauge and V-gauge to project out the $\mathbb{H}^*$ action from the HKC manifold. The V-gauge $q^\alpha = 0$ annihilates the phases of the quaternion in the compensating hy-

permultiplet and projects out the $SU(2)_R$ to descend from the tri-Sasakian to the QK manifold. However, the V-gauge is not invariant under the remaining symmetries in the physical theory. To preserve the condition $q^\alpha = 0$, we need to include compensating transformations. This gives rise to the so-called *decomposition law*. By requiring $\delta q^\alpha = 0$ downstairs, it is shown that [27]

$$\vec{\lambda}_{SU(2)_R} = -\vec{\omega}_u \delta q^u - \frac{1}{2}\kappa_4^2 \theta^I \vec{P}_I \,, \tag{3.105}$$

where $\vec{\lambda}_{SU(2)_R}$ and θ^I are the parameters for the $SU(2)_R$ and gauge transformations respectively, whilst $\vec{\omega}_u$ is the connection for diffeomorphisms of the QK manifold. This equation tells us that both QK diffeomorphisms and scalar gauge transformations induce $SU(2)_R$ transformations in the physical theory in order to preserve the V-gauge. It is then clear that gauge transformations of the physical hypermultiplet scalars (QK isometries) induce $SU(2)_R$ transformations in the physical theory. Equivalently, downstairs R-symmetry transformations manifest themselves geometrically as isometries of the upstairs theory. Since gauge transformations no longer commute with $SU(2)_R$ transformations, we expect that the two connections are no longer independent in the super-Poincaré theory. Indeed, the $SU(2)_R$ connection is now related to the connection for scalar gauge transformations by [27]

$$\begin{aligned}
\vec{\mathcal{V}}_\mu &= -\vec{\omega}_u \partial_\mu q^u - \frac{1}{2}\kappa_4^2 A_\mu^I \vec{P}_I \\
&= -\frac{1}{2}\kappa_4^2 A_\mu^I \vec{P}_I \quad \text{since } n_H = 0 \\
&= -\frac{1}{\sqrt{2}}\kappa_4^2 A_\mu^I g_I \sigma^3 \quad \text{using (3.104).}
\end{aligned} \tag{3.106}$$

In ungauged $\mathcal{N} = 2$ supergravity, the gauging of the super-Poincaré group introduces covariant derivatives into the physical action. The covariant derivatives of the gravitino and physical gauginos are given in (21.35) of [27] as

$$\begin{aligned}
D_\mu \psi_{\nu i} &= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{1}{2}i\mathcal{A}_\mu\right)\psi_{\nu i} + \mathcal{V}_{\mu i}{}^j \psi_{\nu j} \\
D_\mu \chi_i^A &= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{1}{2}i\mathcal{A}_\mu\right)\chi_i^A + \mathcal{V}_{\mu i}{}^j \chi_j^A + \Gamma^A_{BC}\chi_i^C \partial_\mu z^B .
\end{aligned} \tag{3.107}$$

Before we gauged the scalar transformations, the $SU(2)_R$ connection was simply $\vec{\mathcal{V}}_\mu = -\vec{\omega}_u \partial_\mu q^u$ and would in fact vanish for $n_H = 0$. However, (3.106) shows that the additional scalar gauging introduces a new term into the connection that deforms the theory in such a way that it no longer commutes with the superconformal quotient (V-gauge) and leaves a trace in the physical theory, as shown by the decomposition law (3.105). This new term serves to couple the physical vector fields to the matter fields. Indeed, substituting (3.106) into (3.107) shows that the fermionic matter becomes minimally coupled to the electric gauge fields. In particular, the gravitino

covariant derivative contains the term

$$D_\mu \psi_{\nu i} \supset -\frac{1}{\sqrt{2}} \kappa_4^2 A_\mu^I g_I \left(\sigma^3\right)_i{}^j \psi_{\nu j} \,. \tag{3.108}$$

Since the Pauli matrix σ^3 has the structure

$$\left(\sigma^3\right)_i{}^j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{3.109}$$

it is then clear that the two gravitini have opposite electric charges. To be precise, with $n_V + 1$ superconformal gauge vectors, the gauge group is $U(1)^{n_V+1}$ and the fermions have charges $\pm g_I$ with respect to each of the gauge fields A_μ^I. Of course, it is always possible to use the FI parameters to specify a privileged gauge vector $A_\mu = A_\mu^I g_I$ which selects a particular $U(1) \subset U(1)^{n_V+1}$ under which the fermions have charge ± 1, whilst they are neutral with respect to all other $U(1)$ factors. Since the charges with respect to this privileged $U(1)$ direction in $U(1)^{n_V+1}$ space coincide with the R-symmetry charges, this FI gauging is a realisation of gauging a $U(1) \subset SU(2)_R$ symmetry.

Scalar potential

The gauging modifies the $SU(2)_R$ connection that appears in the covariant derivatives in the action. The additional terms introduced disturb the supersymmetry invariance of the action. To restore supersymmetry invariance, a scalar potential must be added. According to [27] this is[26]

$$V_4^{\text{electric}}(X, \bar{X}) = -\left[-\frac{1}{2} (\text{Im}\mathcal{N})^{-1|IJ} - 4\kappa^2 X^I \bar{X}^J \right] \vec{P}_I \cdot \vec{P}_J \,. \tag{3.110}$$

The product $\vec{P}_I \cdot \vec{P}_J$ is understood to be a trace of the product of Pauli matrices. Using (3.104), we find $\vec{P}_I \cdot \vec{P}_J = 2 g_I g_J \text{Tr}(\sigma^3 \cdot \sigma^3) = 4 g_I g_J$. This gives

$$V_4^{\text{electric}}(X, \bar{X}) = -4 g_I g_J \left[-\frac{1}{2} (\text{Im}\mathcal{N})^{-1|IJ} - 4\kappa^2 X^I \bar{X}^J \right] . \tag{3.111}$$

Example: pure 4d, $\mathcal{N} = 2$ $U(1)$ FI electrically gauged supergravity

Let us make things more concrete by considering the simple example of a $U(1) \subset SU(2)_R$ FI electric gauging of pure supergravity. Pure supergravity means $n_V = 0$, whilst the FI gauging requires $n_H = 0$. Therefore, at the superconformal level, we have only the gravity multiplet and the compensating vector multiplet and hypermultiplet. The bosonic action is equivalent to Einstein-Maxwell theory. Since we are in four dimensions, the theory is completely determined

[26] Relative to 20.194 of [27], we introduce an overall minus sign into the potential for consistency with our sign conventions. See Appendix B.2 for details.

by the prepotential [27]

$$F = -\frac{i}{2}(X^0)^2.$$

Following (3.19), the D-gauge constraint gives

$$e^{-\mathcal{K}(X)} = -i(X^0\bar{F}_0 - F_0\bar{X}^0) = \kappa_4^{-2} \quad \Rightarrow \quad X^0 = (\sqrt{2}\kappa_4)^{-1}.$$

From this we can use (3.18) to establish the value of $\mathcal{N}_{00}$ as

$$\mathcal{N}_{00} = -i\,.$$

Since $\mathcal{N}_{00} = \mathcal{R}_{00} + i\mathcal{I}_{00}$, we establish that

$$\mathcal{R}_{00} = 0\,, \quad \mathcal{I}_{00} = -1\,,$$

and this can be substituted into (3.20) which, along with the absence of the PSK manifold (since $n_V = 0$), recovers the Einstein-Maxwell Lagrangian as promised.

Furthermore, substituting $(\mathrm{Im}\mathcal{N})^{-1|00} = -1$ and the D-gauge condition $X^0 = (\sqrt{2}\kappa_4)^{-1}$ into (3.111), the scalar potential for pure gauged supergravity becomes

$$\begin{aligned} V_4^{\text{electric}} &= -4g_0^2\left[-\frac{1}{2}(-1) - 4\kappa_4^2(\sqrt{2}\kappa_4)^{-2}\right] \\ &= +6g_0^2\,, \end{aligned} \tag{3.112}$$

which is given in terms of the dimensionful FI parameter g_0. This can be expanded in terms of the dimensionless electric coupling g as $g_0 = \frac{g}{L}$ such that the potential then becomes

$$V_4 = \Lambda_{AdS_4} = +\frac{6g^2}{L^2}\,,$$

and the electric FI gauging of pure supergravity has generated a positive definite scalar potential that will give rise to a negatively curved spacetime geometry, as reviewed in Appendix B.2. Specifically, the potential is identical to the cosmological constant of four-dimensional Anti de-Sitter spacetime, $\Lambda_{\text{AdS}} = \frac{(D-1)(D-2)}{L^2}$ with $D = 4$.

Since $n_V = 0$, there is just a single gauge field (the graviphoton), and the covariant derivative (3.107) of the gravitini becomes

$$D_\mu\psi_{\nu i} = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{1}{2}i\mathcal{A}_\mu\right)\psi_{\nu i} - \frac{1}{\sqrt{2}}\kappa_4^2 A_\mu^{\text{GP}} g_0\vec{\sigma}^3\psi_{\nu j}\,, \tag{3.113}$$

indicating the two gravitini are minimally coupled to the graviphoton with electric charges $\pm g_0$. This corresponds to a vertex of the form $\mathcal{L}_4 \supset \bar{\psi}_{i\mu}\gamma^{\mu\nu\rho}D_\nu\psi^i_\rho$ as shown in Figure 3.2.

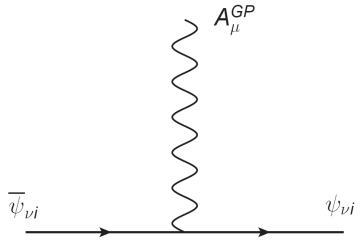


Figure 3.2: Feynman diagram for minimal electric coupling of gravitino to graviphoton in $U(1)$ Fayet-Iliopoulos gauging of pure $\mathcal{N}=2$ supergavity.

3.4.3 $U(1)$ Fayet-Iliopoulos dyonic gauging in four dimensions

As we have seen above, electric Fayet-Iliopoulos gaugings are a natural generalisation of $\mathcal{N}=2$ supergravity coupled to vector multiplets. We can write the scalar potential (3.111) in terms of the superpotential $W=-2g_I X^I$ as

$$V_4^{\text{electric}}(X,\bar{X}) = -\left(N^{IJ}\partial_I W \partial_J \bar{W} - 2\kappa_4^2 |W|^2\right),$$

where we used the identity $N^{IJ} = -\frac{1}{2}\left(\text{Im}\,\mathcal{N}\right)^{-1|IJ} - 2\kappa_4^2 X^I \bar{X}^J$ from (20.189) of [27].

In four dimensions, the electric-magnetic duality allows the existence of magnetic gaugings also. Such magnetic gaugings have been realised in the scalar potentials generated by flux compactifications of string theory on Calabi-Yau three-folds [108]. To engineer from scratch the symplectic extension of the electric gaugings considered in Section 3.4.2 to incorporate dyonic gaugings requires use of the embedding tensor formalism [26,109]. To avoid a lengthy detour, we can motivate the dyonic potential by noting that the superpotential $W=-2g_I X^I$ is essentially 'half' of a symplectic function. Introducing magnetic FI parameters g^I, we can construct the symplectic function $W=2(g^I F_I - g_I X^I)$ which represents the superpotential of a theory with dyonic gaugings. The scalar potential for the dyonic theory would then be

$$V_4^{\text{dyonic}}(X,\bar{X}) = -\left(N^{IJ}\partial_I W \partial_J \bar{W} - 2\kappa_4^2 |W|^2\right), \quad W = 2(g^I F_I - g_I X^I)\,. \tag{3.114}$$

Putting the scalar potential (3.114) together with the superconformal Lagrangian (3.22), we find the full bosonic action for four-dimensional $\mathcal{N}=2$ supergravity with a dyonic $U(1)$ Fayet-Iliopoulos gauging is

$$\begin{aligned}\mathsf{e}_4^{-1}\mathcal{L}_4 = &-\frac{1}{2\kappa_4^2}R_{(4)} - \frac{1}{\kappa_4^2}g_{IJ}\partial_\mu X^I \partial^\mu \bar{X}^J + \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu} \\ &+ V_4^{\text{dyonic}}(X,\bar{X})\,.\end{aligned} \tag{3.115}$$

3.4.4 $U(1)$ Fayet-Iliopoulos electric gauging in five dimensions

In five dimensions, there is no electric-magnetic duality, and therefore magnetic gaugings are disallowed. The five-dimensional electric gauging follows by re-applying the methods used for the four-dimensional case in Section 3.4.2. This is done explicitly in [110, 111] and leads to the following five-dimensional scalar potential

$$V_5(h) = 2 \cdot 6^{-1/3} \left[(chhh)(ch)^{-1|ij} + 3h^i h^j \right] g_i g_j \,. \tag{3.116}$$

Since there is no ambiguity regarding the nature of the gauging, we do not require an 'electric' superscript in five dimensions. Putting the scalar potential (3.116) together with the superconformal Lagrangian (3.15), we find the full action for five-dimensional $\mathcal{N} = 2$ supergravity with an electric $U(1)$ Fayet-Iliopoulos gauging is

$$\begin{aligned} e_5^{-1}\mathcal{L}_5 \;=\; & -\frac{1}{2\kappa^2}R_{(5)} - \frac{3}{4\kappa^2}a_{ij}(h)\partial_{\hat{\mu}}h^i\partial^{\hat{\mu}}h^j - \frac{1}{4}a_{ij}(h)\mathcal{F}^i_{\hat{\mu}\hat{\nu}}\mathcal{F}^{j|\hat{\mu}\hat{\nu}} \\ & + \frac{\kappa}{6\sqrt{6}}e_5^{-1}c_{ijk}\epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}}\mathcal{F}^i_{\hat{\mu}\hat{\nu}}\mathcal{F}^j_{\hat{\rho}\hat{\sigma}}\mathcal{A}^k_{\hat{\lambda}} + V_5(h)\,. \end{aligned} \tag{3.117}$$

To convince ourselves that the the potential in (3.116) is correct, we can show that it consistently reduces to the four-dimensional electric potential (3.111) for very special models when using the Kaluza-Klein ansatz (3.50). Also, see Appendix C for an explicit demonstration that the five-dimensional scalar potential (3.116) is positive definite for the STU-model and therefore able to support solutions with negative spacetime curvature, as reviewed in Appendix B.2.

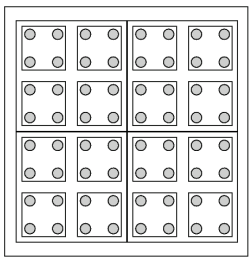
3.5 Holography and Anti de-Sitter space

3.5.1 Holography

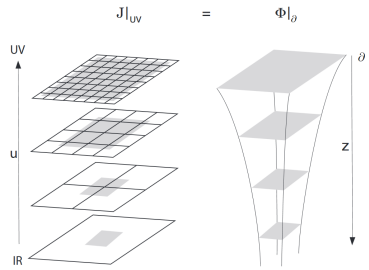
Holographic dualities relate quantum physics of strongly coupled field theories to theories of classical gravity in one higher dimension. Following [112] we will motivate this duality using lattice systems. Consider a non-gravitational theory on a lattice with lattice spacing a. The physics can be described using the Hamiltonian

$$H = \sum_{x,i} J_i(x,a)\mathcal{O}^i(x)\,, \tag{3.118}$$

where x labels the lattice sites and i labels the various operators. J_i denotes the coupling constant (or source) of the operator $\mathcal{O}^i$ at the point x. From its argument, it is clear that the couplings J_i depend on the lattice spacing a. When studying lattice systems, we are interested in how to vary the couplings J_i in order to reach a continuum limit. This requires us to understand the functional dependence of J_i on the regulator (lattice spacing) i.e. the renormalization group.



(a) Kadanoff-Wilson renormalization.



(b) Holographic interpretation.

Figure 3.3: On the left we illustrate the Kadanoff-Wilson coarse-graining of a lattice by doubling the spacing u. On the right we show how different values of u correspond to layers in a higher-dimensional spacetime. Images sourced from [112].

Kadanoff and Wilson proposed a block spin approach whereby we coarse-grain the lattice and replace multiple lattice sites with a single site by averaging over the lattice variables. Doing so, the structure of (3.118) is unchanged but the operators $\mathcal{O}^i$ become weighted differently since their respective couplings have changed. Suppose the lattice spacing is doubled in each step as shown in Figure 3.3. The couplings respond as follows:

$$J_i(x,a) \to J_i(x,2a) \to J_i(x,4a) \to \dots .$$

The couplings manifestly depend on the length scale of the theory and we can write them as $J_i(x,u)$, where $u = (a, 2a, 4a, \dots)$. The evolution of the couplings is determined by the renormalization group flow equations

$$u\frac{\partial}{\partial u} J_i(x,u) = \beta_i \left(J_j(x,u), u\right), \tag{3.119}$$

where β_i is the β-function of the ith coupling. For a weakly coupled system, the β_i's are determined perturbatively. This becomes increasingly difficult at strong coupling, and the holographic proposal is to treat the length scale u as an extra dimension. We should picture the collection of lattices at different values of u as being layers of a new, different higher-dimensional spacetime as shown in Figure 3.3. The couplings are identified with fields in the spacetime

$$J_i(x,u) = \phi_i(x,u) .$$

The dynamics of the fields ϕ_i are governed by a gravitational action. From Figure 3.3 the UV couplings are dual to the bulk fields evaluated on the spacetime boundary, $J_i|_{\text{UV}} = \phi_i|_\partial$, and the UV field theory is said to live on the spacetime boundary. With the extra dimension playing the role of the length scale, running of the coupling simply corresponds to considering different $z = \text{const}$ slices of the spacetime. The dual fields are required to have the same tensor structure as the coupling J_i that they replace, such that $\phi_i \mathcal{O}^i$ is a scalar. Thus spacetime scalars and

vectors are dual to scalar operators and currents in the field theory respectively. Further, the spacetime metric $g_{\mu\nu}$ is identified with the field theory stress tensor $T_{\mu\nu}$. We will see this in Chapter 5 when computing conserved charges using quasilocal techniques.

So far we have been vague about the details of the spacetime since, in general, finding the geometry dual to a particular QFT is very difficult. However, at fixed points of the renormalization group flow, the β-functions vanish and we have a conformal field theory (CFT). This invariance under changes to the length scale makes it straightforward to identify the dual geometry. Considering a D-dimensional QFT, the most general $(D+1)$-dimensional metric with D-dimensional Poincaré invariance is

$$ds^2_{(D+1)} = \Omega^2(z)\left(-dt^2 + d\vec{x}^2 + dz^2\right), \tag{3.120}$$

where z is the coordinate of the extra dimension, $\vec{x} = (x_1, \ldots, x_{D-1})$ and $\Omega(z)$ is undetermined. Since z represents the field theory length scale, conformal invariance amounts to invariance under the transformation

$$(t, \vec{x}) \to \lambda(t, \vec{x}), \qquad z \to \lambda z. \tag{3.121}$$

In order for (3.120) to be invariant under (3.121), the function $\Omega(z)$ must transform as $\Omega(z) \to \lambda^{-1}\Omega(z)$ which fixes it to be

$$\Omega(z) = \frac{L}{z}, \tag{3.122}$$

where L is some constant. Inserting this into (3.120), the spacetime dual of a CFT has the metric

$$ds^2_{(D+1)} = \frac{L^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right), \tag{3.123}$$

which is the line element of $(D+1)$-dimensional Anti de-Sitter space, AdS_{D+1}. L is referred to as the Anti de-Sitter radius. The (conformal) boundary of AdS_{D+1} is located at $z = 0$ as seen in Figure 3.3.

As mentioned earlier, the dynamics of the fields ϕ_i are determined by an action. The AdS metric (3.123) is a solution to the equations of motion of an Einstein-Hilbert action with cosmological constant:

$$S = \frac{1}{2}\int d^{D+1}x\sqrt{-g}\,(-R + 2\Lambda). \tag{3.124}$$

By analysing the resulting Einstein equations, it is possible to produce AdS solutions if the cosmological constant is chosen as

$$\Lambda = \frac{D(D-1)}{2L^2},$$

corresponding to a negative scalar curvature (see Appendix A for sign conventions)

$$R_{\mathrm{AdS}_{(D+1)}} = \frac{D(D+1)}{L^2}. \tag{3.125}$$

The first, and best known, example of holography is the so-called AdS/CFT correspondence relating a four-dimensional $\mathcal{N} = 4$ super Yang-Mills gauge theory with gauge group $SU(N)$ to string theory in $\mathrm{AdS}_5 \times S^5$ [11]. Considering the gravitational and field theory systems don't have the same dimension, an important consistency check is to match their degrees of freedom. This is possible provided we make the identification

$$\frac{1}{4}\left(\frac{L}{l_P}\right)^{D-1} = N^2,$$

relating parameters of the gravitational theory to those of the field theory. In particular, l_P is the Planck length and N^2 is the central charge of the $SU(N)$ CFT [112]. We note that a gravitational theory is (semi-)classical when the coefficient of the action is large. It is shown in [112] that the coefficient contains a factor $L^{D-1}/{l_P}^{D-1}$. Therefore, we can trust our classical AdS dual of the $SU(N)$ gauge theory providing

$$\left(\frac{L}{l_P}\right)^{D-1} \approx N^2 >> 1\,.$$

In other words, holography is valid providing the AdS radius is large in Planck unitsFrom the field theory perspective, this requires a large number of degrees of freedom (or large N). Without this, quantum gravitational corrections will be required.

In fact, the spacetime is only required to be asymptotically AdS such that the boundary isometry group matches the conformal group of a field theory living on the boundary [16]. We can therefore consider placing objects, such as black holes, inside the spacetime providing their gravitational influence is negligible at large distances. Whilst a pure AdS_{D+1} geometry is dual to a CFT vacuum state, black holes are equipped with a certain Hawking temperature such that their presence corresponds to populating the field theory with thermal states. This is precisely what we shall do in Chapter 5. In such a configuration, the near horizon (resp. asymptotic) geometry can be used to probe the infra-red (resp. ultra-violet) behaviour of the field theory.

3.5.2 Anti de-Sitter geometry: an embedded hyperboloid

Anti de-Sitter spacetimes play a central role in holography. We show in Appendix B.2 how the positive definite scalar potentials generated from Fayet-Iliopoulos gaugings of $\mathcal{N} = 2$ supergravity theories can play the role of the cosmological constant needed in (3.124) to produce such geometries. It will be useful for the solutions constructed in Chapter 5 to now review in more detail the construction and properties of this space.

We start by demonstrating how AdS_D is constructed by embedding a D-dimensional hyperboloid in a flat $(D+1)$-dimensional space with two time-like directions, $\mathbb{E}^{2,D-1}$. Thus, AdS_D inherits the structure of the Lorentzian analogue of hyperbolic space. For simplicity, we shall

work with the three-dimensional AdS_3 geometry in the remainder of this section since the properties we demonstrate extend to arbitrary dimensions and, in particular, to AdS_5 which is relevant for Chapter 5. As explained above, the construction begins with a flat four-dimensional space, $\mathbb{E}^{(2,2)}$, with coordinates U, V, X, Y and a signature $(-,-,+,+)$ metric given by

$$ds^2_{\mathbb{E}^{(2,2)}} = -dU^2 - dV^2 + dX^2 + dY^2.$$

AdS_3 is the three-dimensional hypersurface in $\mathbb{E}^{(2,2)}$ given by

$$\mathrm{AdS}_3 = \{x \in \mathbb{E}^{(2,2)} | -U^2 - V^2 + X^2 + Y^2 = -L^2\}\,, \tag{3.126}$$

where L is the AdS radius. This hypersurface in $\mathbb{E}^{(2,2)}$ is in fact a 3-dimensional hyperboloid as shown in Figure 3.4. To investigate the structure of Anti de-Sitter space, we must pull-back the ambient $\mathbb{E}^{(2,2)}$ geometry to this hypersurface. In particular, the metric on AdS_3 will be

$$ds^2_{\mathrm{AdS}_3} = \left(-dU^2 - dV^2 + dX^2 + dY^2\right)\Big|_{\mathrm{AdS}_3}\,.$$

There are several different coordinate systems available to parametrize the hyperboloid. We choose to work with *global coordinates* $\{t, \mu, \theta\}$ since these are sufficient to demonstrate all necessary properties for the remainder of this thesis. Global coordinates are defined via

$$\begin{aligned} U &= L\cosh\mu\sin t\,,\\ V &= L\cosh\mu\cos t\,,\\ X &= L\sinh\mu\sin\theta\,,\\ Y &= L\sinh\mu\cos\theta\,, \end{aligned} \tag{3.127}$$

such that t, θ must be 2π periodic. We are free to choose the coordinate ranges to be

$$t \in [-\pi, \pi)\,, \quad \theta \in [0, 2\pi)\,, \quad \mu \geq 0\,.$$

We can see explicitly in Figure 3.4 the various directions parametrized by these global coordinates. Pulling back, we obtain the metric

$$ds^2_{AdS_3} = L^2\left(-\cosh^2\mu\; dt^2 + d\mu^2 + \sinh^2\mu\; d\theta^2\right). \tag{3.128}$$

It is worth mentioning that this form of the metric can be obtained from the more familiar form $ds^2_{\mathrm{AdS}_3} = -f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2$ with $f(r) = 1 + \frac{r^2}{L^2}$ using the simple substitutions $r = L\sinh\mu$ and $\tau = Lt$.

The AdS_3 hyperboloid has topology $S^1 \times S^1 \times \mathbb{R}$ since the t and θ directions are compact.

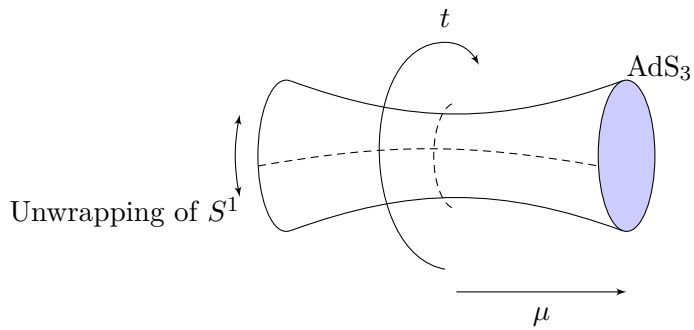


Figure 3.4: AdS_3 hyperboloid with global coordinates t and μ shown explicitly. Each point on this 2-dimensional representation is really an S^1 parametrized by the third and final global coordinate, θ.

The periodicity of the time-like direction allows the existence of closed time-like curves; a simple example of which would be an observer at rest at $\mu = 0$ in Figure 3.4 whose worldline wraps the hyperboloid and returns to the same starting position after $\Delta t = 2\pi$ has elapsed. Closed time-like curves are allowed in General Relativity even though they obviously have nothing to do with normal, macroscopic physics. It is noted in [113] that, in the context of holography, there may well be uses for a compact time direction in modelling physical systems that are periodically excited e.g. resonances. Nonetheless, from a causal perspective, closed time-like curves are clearly pathological and to avoid them one usually goes to the *universal covering space* [114], by 'unwrapping' the hyperboloid's time-like circle as indicated in Figure 3.4. This $S^1 \to \mathbb{R}$ transformation changes the topology to $S^1 \times \mathbb{R}^2$, meaning that the time direction is no longer subject to periodic identification and instead has an infinite range. This will help greatly with visualising the causal structure and allows the Penrose diagram to be drawn as extending infinitely in the time direction. It is worth mentioning that passing to the universal covering space does not affect the asymptotic geometry and so the spacetime remains suitable for use in AdS/CFT.

3.5.3 Conformal compactification

Anti de-Sitter spacetime extends infinitely in the spatial μ direction as well as having a complicated topological and causal structure. To understand this we can use a Penrose diagram, which requires a conformal compactification of the geometry. First of all, note that since light rays propagate along $ds^2 = 0$, we can introduce an overall conformal factor to ds^2 without affecting this [115]. Whilst conformal transformations preserve angles and thus don't affect the causal structure of the space, they do not preserve distances. With a clever choice for the conformal factor, it is possible to bring points at infinity in the original spacetime to a finite distance in the conformally related spacetime. This finiteness allows us to draw a *Penrose diagram* and explore the causal structure in detail. To find a suitable choice of conformal factor, we begin by letting

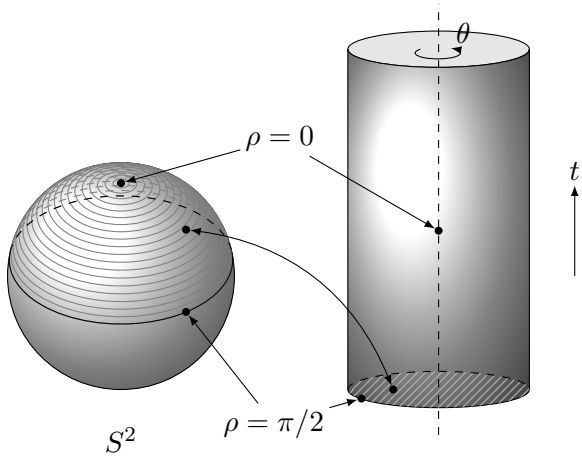


Figure 3.5: Figure showing how the reduced range of the polar coordinate ρ in the S^2 factor of $\widetilde{\mathcal{M}}$ gives rise to half-spheres. Such a half-sphere is topologically equivalent to a disc of radius $\frac{\pi}{2}$ by 'squashing' and allows us to view $\widetilde{\mathcal{M}}$ as a solid cylinder of radius $\frac{\pi}{2}$. Note that this 'squashing' identifies the two shaded regions in the above diagram.

$\sinh\mu = \tan\rho$ such that that the metric becomes

$$\begin{aligned} ds^2_{\mathrm{AdS}_3} &= L^2\left(-\sec^2\rho\, dt^2 + \sec^2\rho\, d\rho^2 + \tan^2\rho\, d\theta^2\right) \\ &= L^2\sec^2\rho\left(-dt^2 + d\rho^2 + \sin^2\rho\, d\theta^2\right), \end{aligned} \tag{3.129}$$

with $t \in (-\infty, \infty)$ after unwrapping the time-like S^1, and with $\rho \in [0, \frac{\pi}{2})$ to match $\sinh\mu \in [0, \infty)$ which follows from $\mu \geq 0$. By choosing the conformal factor $\Omega = \cos\rho$, $\mathcal{M} = \mathrm{AdS}_3$ is conformally related to unphysical spacetime $\widetilde{\mathcal{M}}$ with metric

$$d\tilde{s}^2 = \Omega^2 ds^2 = -dt^2 + d\rho^2 + \sin^2\rho\, d\theta^2\,.$$

This demonstrates AdS_3 is conformal to one half of the 3-dimensional Einstein Static Universe. The reason being that ESU_3 has the standard range $\rho \in [0, \pi]$ for the polar angle ρ appearing in the S^2 factor whilst AdS_3 has the restricted range $\rho \in [0, \frac{\pi}{2})$. The full manifold $\widetilde{\mathcal{M}}$ can be viewed as a solid cylinder with radial coordinate ρ, and a cylindrical boundary at $\rho = \frac{\pi}{2}$. This can be seen explicitly in Figure 3.5 which also demonstrates why surfaces of constant t are half-spheres (discs) with boundary at $\rho = \frac{\pi}{2}$. The points with $\rho = \frac{\pi}{2}$ correspond to $\mu = \infty$ and are not formally part of $\widetilde{\mathcal{M}}$. In fact these points form the *conformal boundary*, $\mathcal{I}$, of AdS_3. On this surface, the metric takes the form

$$d\tilde{s}^2|_{\rho=\frac{\pi}{2}} = -dt^2 + d\theta^2\,,$$

meaning that the conformal boundary is topologically a cylinder, $\mathcal{I} = \mathbb{R} \times S^1$.[27] Given that two-dimensional Minkowski space is conformally isometric to such a Lorentzian cylinder, it is possible to imagine a field theory existing on the boundary of AdS_3. This $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality would be another example of a holographic correspondence [116]. Moreover, the Lorentzian cylinder $\mathcal{I}$ is a time-like surface that unites future and past null infinity as well as spatial infinity [44] i.e.

$$\mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^- \cup i^0 \,.$$

The conformal transformation introduces a new, conformally related measure of distance, $d\tilde{s}^2$, such that infinity in the original AdS_3 now appears at a finite distance in $\widetilde{\mathcal{M}}$. Consequently, it becomes possible to 'travel to conformal infinity' in the unphysical spacetime $\widetilde{\mathcal{M}}$. Thus we must exercise caution when applying statements about causality and geodesic structure made from calculations in $\widetilde{\mathcal{M}}$ to the original $\mathcal{M} = AdS_3$.

3.5.4 Penrose diagrams and geodesics

Following conformal compactification, $\widetilde{\mathcal{M}} \cup \mathcal{I}$ can be viewed as a solid cylinder as seen in Figure 3.5. To make the study of geodesics as simple as possible, we shall suppress all motion in the transverse S^1 direction by setting $d\theta = 0$. This enables the cylinder to be flattened out such that the Penrose diagram can be drawn as an infinite strip, a finite section of which is shown in Figure 3.6. Note that the conformal transformation produces an unphysical metric whose geodesics are of no interest to us since they are equally unphysical. We therefore compute the geodesics with respect to the uncompactified Anti de-Sitter metric in (3.129) to obtain an expression for $\rho(t)$, which can then be plotted on the Penrose diagram. We discuss the null and time-like cases separately below.

Null geodesics

The metric (3.129) is stationary meaning that ∂_t is a global Killing vector field. Thus there exists an integral of motion which we identify with the energy of the massless particle,

$$E = -g_{tt}\dot{t} = L^2 \sec^2\rho\, \dot{t}\,, \quad \text{where } \dot{t} = \frac{dt}{d\lambda} \text{ and } \lambda \text{ is the affine parameter.} \tag{3.130}$$

The null condition, $ds^2 = 0$, becomes

$$0 = L^2 \left(-\frac{\dot{t}^2}{\cos^2\rho} + \frac{\dot{\rho}^2}{\cos^2\rho} \right),$$

[27] The metric on the boundary is only defined up to conformal transformations [116].

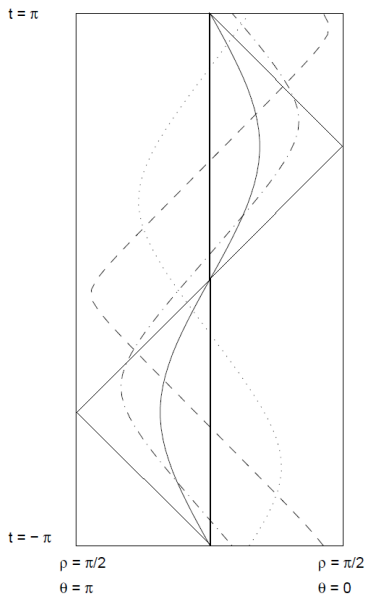


Figure 3.6: Penrose diagram for AdS_3 with a sample of time-like and null geodesic trajectories shown. Because of the 'flattening' of the cylinder, the entire left hand side of the diagram has azimuthal coordinate $\theta = \pi$, meanwhile the right hand side has $\theta = 0$. Image taken from [117] and modified to include null geodesics.

since $\dot{\theta} = 0$ as we only have radial motion. This then gives

$$\dot{\rho} = \pm \dot{t} \tag{3.131}$$

$$\Rightarrow \dot{\rho} = \pm \frac{E}{L^2} \cos^2 \rho \,, \tag{3.132}$$

where we have substituted for $\dot{t}$ from (3.130). Rearranging and integrating we have

$$\tan \rho(\lambda) = \pm \frac{E}{L^2} \lambda \,. \tag{3.133}$$

This gives the radial coordinate, ρ, as a function of the null geodesic's affine parameter, λ. We see that as $\rho \to \frac{\pi}{2}, \lambda \to \infty$, which, according to Figure 3.6, tells us that it takes infinite affine parameter for null geodesics to reach spatial infinity. In other words, AdS_3 is geodesically complete as expected. Of course, to actually draw the null geodesic on the Penrose diagram, we want an expression for the radial coordinate, $\rho(t)$, as a function of coordinate time, t. To do this, we can directly integrate (3.131):

$$d\rho = \pm dt \quad \Rightarrow \rho(t) = \pm(t + t_0) \,, \tag{3.134}$$

where t_0 is a constant of integration and the positive (resp. negative) solutions represent outgoing (resp. ingoing) null geodesics. This explains why null geodesics appear as straight lines at 45° in the Penrose diagram in Figure 3.6.

Dirichlet boundary conditions

Notice from Figure 3.6 that a null geodesic released from $\rho = 0$ will reach the conformal boundary, $\mathcal{I}$, in finite coordinate time, $\Delta t = \frac{\pi}{2}$. One important consequence of $\mathcal{I}$ being a time-like surface is that, given a spatial hypersurface, Σ, there exist points p to the future of Σ such that past-directed null geodesics released from p do not intersect Σ because they instead extend into the conformal boundary. This means Anti de-Sitter space has no Cauchy surfaces, and this lack of global hyperbolicity prevents us from determining the future evolution of initial data on a given hypersurface, Σ. To restore global hyperbolicity, one imposes reflective, Dirichlet, boundary conditions on the conformal boundary, $\mathcal{I}$. Future evolution is now well-defined since null geodesics 'bounce' off $\mathcal{I}$ as seen in Figure 3.6 [44]. This leads to the interesting scenario whereby a light ray can be sent out to the boundary and return in finite proper time from the perspective of an observer at the origin (whose proper time agrees with coordinate time), despite the fact that each leg of the journey requires infinite affine parameter from the perspective of the massless particle. This is reminiscent of the paradoxical time measurements made by a distant observer seeing an object fall into a black hole.

Time-like geodesics

Again, since the metric (3.129) is stationary, ∂_t is a global Killing vector field and there exists a conserved energy,

$$E = -g_{tt}\dot{t} = L^2 \sec^2\rho\, \dot{t}\,, \quad \text{where } \dot{t} = \frac{dt}{d\tau} \text{ and } \tau \text{ is the proper time.} \tag{3.135}$$

With only radial motion ($\dot{\theta} = 0$), the time-like condition $ds^2 = -d\tau^2$ gives

$$-1 = L^2\left(-\frac{\dot{t}^2}{\cos^2\rho} + \frac{\dot{\rho}^2}{\cos^2\rho}\right).$$

Substituting for $\dot{t}$ this gives

$$\frac{\dot{\rho}^2 L^2}{\cos^2\rho} = \frac{E^2\cos^2\rho}{L^2} - 1\,. \tag{3.136}$$

To plot on the Penrose diagram we require an expression for $\rho(t)$. Using that $\dot{\rho} = \frac{d\rho}{dt}\frac{E\cos^2\rho}{L^2}$, we can solve (3.136) to find

$$\sin\rho(t) = \pm\sqrt{1 - \frac{L^2}{E^2}}\sin(t + t_0)\,. \tag{3.137}$$

Time-like geodesics are solutions to (3.137) and appear on the Penrose diagram as sinusoidal waves with different amplitudes but all with period 2π. On the physical AdS_3 hyperboloid, these waves parametrize an elliptic trajectory that can be understood as the intersection of the AdS_3 hyperboloid with a family of totally-time-like 2-planes [118, 119]. Looking in more detail

at equation (3.137) governing the elliptic trajectory, it is immediate that we require $E \geq L$ in order to avoid a complex solution. In fact, $E = L$ corresponds to $\rho(t) = 0$ and indeed, if we repeat the above calculation for the energy of a particle at rest ($\dot{\rho} = \dot{\theta} = 0$), we find $E_{\text{rest}} = L$ indicating that this is in fact the rest energy of a particle in AdS. We also note that letting $E \to \infty$, causes $(\sin\rho)_{\max} \to 1$ and so $\rho_{\max} \to \frac{\pi}{2}$, meaning the test particle will get closer to the boundary and, at the same time, its equation of motion (3.137) will approach

$$\sin\rho(t) \sim \sin(t + t_0) \quad \Rightarrow \rho(t) \sim t + t_0,$$

implying that the elliptic trajectories become more 'zig-zag,' and so the infinite energy limit $E \to \infty$ is the limit in which the time-like trajectory approaches that of a null trajectory. Therefore the limits $E \to L$ and $E \to \infty$ represent respectively the special cases where the elliptic intersection of the 2-plane with the AdS_3 hyperboloid becomes either circular or parabolic. A sample of time-like trajectories with different energies (amplitudes) and different values of t_0 are shown in Figure 3.6. Although the trajectories of massive particles may approach that of a light ray as the energy (mass) increases, they will never coincide since this requires infinite energy and is related to the fact that the Lorentz factor $(1 - \frac{v^2}{c^2})^{-1/2}$ will only diverge in the limit $v \to c$. In [120], Maldacena offers some insight into the inability of massive particles to reach the conformal boundary by looking at how, in the Newtonian limit, a slow-moving particle is subject to a gravitational potential that grows with increasing ρ, eventually becoming an infinite potential wall as the boundary is approached. This wall, produced by the negative spacetime curvature, is responsible for the particle executing an oscillatory motion in the ρ coordinate similar to the harmonic motion of a particle in a box [115].

An interesting consequence of this is to imagine a collection of test particles being simultaneously released from $\rho = 0$. They will travel along different time-like geodesics (elliptical trajectories) depending on their energy (mass) but it is clear from Figure 3.6 that, because of periodicity, they will all return to $\rho = 0$ and simultaneously collide with one another after coordinate time $\Delta t = \pi$ has elapsed [113].

3.6 Hyperscaling-violating Lifshitz spacetimes

Since the discovery of the AdS/CFT correspondence [11], holographic techniques have been extended to non-maximally symmetric spacetimes in attempts to model more realistic quantum field theories, such as those found in condensed matter physics (see e.g. [12]). In this section, we first introduce one of the simplest extensions, *Lifshitz holography*, before discussing the more complicated, *hyperscaling-violating Lifshitz (hvLif) holography*. We introduce some basic properties to help motivate their role in holography, which will be important for Chapter 4. We will not however, give a systematic analysis of geodesics as we did for AdS spacetimes, since

this is a considerably more lengthy procedure in the hvLif case. The interested reader may refer to [121] for details of this calculation.

For consistency with the literature, we will formulate this section using $(D+2)$-dimensional spacetimes and $(D+1)$-dimensional field theories.

3.6.1 Lifshitz holography

As seen in (3.121), conformally invariant systems are invariant under dilatations

$$t \to \lambda t\,, \quad x_i \to \lambda x_i\,, \quad i = 1, \ldots, D\,,$$

where we use x_i to label the spatial coordinates. However, many physical systems exhibit asymmetric scaling behaviour of time and space. Indeed, many systems in condensed matter physics have phase transitions governed by 'Lifshitz fixed points' where the above scale invariance is modified to a so-called *dynamical scale invariance* [122],

$$t \to \lambda^z t\,, \quad x_i \to \lambda x_i\,, \quad i = 1, \ldots, D\,, \tag{3.138}$$

where $z \neq 1$ is the *dynamical critical exponent* and controls the anisotropic scaling of the time direction. As it happens, there exists spacetime geometries with precisely the same scaling behaviour as these $(D+1)$-dimensional field theories. These are the $(D+2)$-dimensional Lifshitz spacetimes, defined by the one-parameter family of metrics,

$$ds^2_{(D+2)} = -\frac{L^2}{r^{2z}}dt^2 + \frac{L^2}{r^2}\left(dr^2 + d\vec{x}^2\right), \tag{3.139}$$

where $d\vec{x}^2 = dx_1^2 + dx_2^2 + \cdots + dx_D^2$. It is a simple exercise to check Lifshitz spacetimes are invariant under the scale transformations (3.138), despite clearly losing invariance under Lorentz (and consequently conformal) transformations. Notice that for $z = 1$, isotropic scaling is restored and (3.139) reduces to the metric on the Poincaré patch of AdS_{D+2}, given in (3.123). The different behaviour of time from space implies that Lifshitz geometries with $z \neq 1$ are dual to non-relativistic field theories.

3.6.2 Hyperscaling-violating Lifshitz holography

A further generalisation of the dictionary between systems of gravity and condensed matter systems comes from considering the following two-parameter class of $(D+2)$-dimensional hvLif spacetimes,

$$ds^2_{(D+2)} = r^{-\frac{2(D-\theta)}{D}}\left(-r^{-2(z-1)}dt^2 + dr^2 + dx^2_{(D)}\right). \tag{3.140}$$

Under the rescalings of the coordinates in (3.138), the metric (3.140) is no longer invariant. In fact, scale invariance is broken to scale covariance:

$$t \to \lambda^z t\,, \quad x_i \to \lambda x_i\,, \quad ds^2_{(D+2)} \to \lambda^{\frac{2\theta}{D}} ds^2_{(D+2)}\,. \tag{3.141}$$

Such metrics are classified according to the two parameters (z, θ). As before, z measures the degree of anisotropy between time and space, whilst $\theta \neq 0$ measures the degree to which scale invariance is broken and maps holographically to the *hyperscaling-violation exponent*. According to [15], holography relates volume elements in the bulk to entropic measures on the boundary and we thus expect $\theta \neq 0$ to modify the scaling behaviour of entropy. Indeed, a system is said to possess hyperscaling behaviour if its thermal entropy scales with the spatial dimension, D, of the boundary, i.e. $S \sim T^D$. In the presence of a non-trivial dynamical critical exponent, this is modified to $S \sim T^{\frac{D}{z}}$ [123]. For the case $\theta \neq 0$, we determine the thermal entropy by the area of the horizon of a non-extremal black brane ($T > 0$). For a horizon located at $r = r_+$, (3.140) tells us $S \sim \int_{r=r_+} d^D x \sqrt{\sigma} \sim r_+^{-(D-\theta)} = r_+^{\theta - D}$. We know from (3.141) that $r^{\theta-D} \sim t^{\frac{\theta-D}{z}}$ and we also know that temperature is an inverse time, $T \sim t^{-1}$. Consequently, we find $S \sim r_+^{\theta-D} \sim T^{\frac{D-\theta}{z}}$ whose deviation from the hyperscaling behaviour $S \sim T^{\frac{D}{z}}$ of a Lifshitz geometry clearly justifies the name.[28] Notice that from the perspective of boundary thermodynamics, $D - \theta$ acts as the effective spatial dimension of the boundary system.

At first glance, hvLif metrics, with their broken scale invariance, may seem rather exotic and unrealistic. But in fact, they have proven themselves very useful in holography for studying certain condensed matter systems known to violate hyperscaling. This has led to the development of the so-called AdS/CMT correspondence [12, 13] and recently, there has been much research [15, 34, 124, 125, 126] on the specific case $\theta = D - 1$. This represents a one-dimensional *hidden Fermi surface* which is thought to play an important role in the holography of compressible states in condensed matter physics [127]. Supergravity embeddings of hyperscaling-violating Lifshitz metrics with $z = 1$ and various values of θ can be found in [34] and references therein.

An important observation of [34] is that,[29]

> *"dimensional reduction of theories admitting scale invariant vacua often leads to theories admitting scale covariant vacua."*

This suggests the UV/IR completion of theories with non-trivial hvLif behaviour may involve oxidation to theories with AdS vacua [35]. Later in this thesis, we will encounter an explicit example of this and will in fact see how the hvLif spacetime inherits certain properties, such as geodesic structure, from the parent AdS spacetime.

[28] We argue for this scaling behaviour using only dimensional analysis here; see [15] for an explicit computation confirming this.

[29] Although [34] doesn't explicitly discuss $z \neq 1$, this is a general comment for any allowed (z, θ). This is important: in Chapter 4 we apply this idea to examples where z is not necessarily equal to one.

We end this section with the physically allowed values of the two parameters (z, θ). In order for the gravitational theory to be physical (i.e. locally attractive [128]), we must demand that the null energy condition holds,

$$T_{\mu\nu}k^\mu k^\nu \geq 0\,,$$

where $k^2 = 0$. Since $G_{\mu\nu} = T_{\mu\nu}$ on shell, this boils down to the following requirements on the parameters of the bulk metric [14],

$$\begin{aligned}(D-\theta)(D(z-1)-\theta) &\geq 0\,,\\ (z-1)(D+z-\theta) &\geq 0\,.\end{aligned} \tag{3.142}$$

A general hvLif metric is physically sensible providing it satisfies (3.142). Some remarks are in order regarding special cases. Firstly, note that scale invariance ($\theta = 0$) restricts $z \geq 1$. Meanwhile, Lorentz invariance ($z = 1$) requires $\theta \leq 0$ or $\theta \geq D$. However, it is noted that the $\theta \geq D$ branch of metrics may well be unstable, at least thermodynamically if not otherwise, and so Lorentz invariant solutions should have $\theta \leq 0$ [14]. When we construct solutions with hvLif metrics in Chapter 4, it will be important to ensure these metrics are compatible with the constraints of the Null Energy Condition in (3.142).

3.7 Black holes

A black hole is defined as a region of spacetime not contained in the past lightcone of future null infinity [77]. Despite the fact that the solutions we construct in Chapters 4 and 5 will be *black branes*, we use this section to review key properties of black holes since there is considerably more literature on this, and many properties extend naturally from the S^{D-2} to $\mathbb{R}^{D-2}$ horizon topology.

3.7.1 Black hole horizons

We begin by reviewing the different horizons that we will encounter in this thesis.

Null hypersurfaces

Let $\mathcal{N}$ be a null hypersurface with normal vector ζ. Then a tangent vector τ, will satisfy $\tau\cdot\zeta = 0$, but because $\mathcal{N}$ is null, $\zeta\cdot\zeta = 0$, meaning ζ itself is also a tangent vector, i.e.

$$\zeta^\mu = \frac{dx^\mu(\lambda)}{d\lambda}, \quad \text{for some null curve } x^\mu(\lambda) \text{ in } \mathcal{N}. \tag{3.143}$$

It is straightforward to demonstrate that $\zeta\cdot D\zeta^\mu \propto \zeta^\mu$, indicating that $x^\mu(\lambda)$ are geodesics of $\mathcal{N}$.

Killing horizons

A Killing horizon is a null hypersurface whose null generators coincide with the orbits of a one parameter group of isometries. Formally, a null hypersurface $\mathcal{N}$ is a *Killing horizon* of a Killing vector field ξ if, on $\mathcal{N}$, ξ is normal to $\mathcal{N}$.

By (3.143) we know that along $\mathcal{N}$, ξ obeys the geodesic equation

$$\xi^\mu D_\mu \xi^\nu = -\kappa \xi^\nu. \tag{3.144}$$

The parameter κ is the surface gravity. Using Killing's equation $D_{(\mu}\xi_{\nu)} = 0$ and Frobenius' Theorem $\xi_{[\mu}D_\nu \xi_{\rho]} = 0$, it is possible to establish the formula for surface gravity [25][30]

$$\kappa^2 = -\frac{1}{2}(D^\mu \xi^\nu)(D_\mu \xi_\nu)\Big|_{\mathcal{N}}. \tag{3.145}$$

It can also be shown that the combination $\xi^\mu \partial_\mu \kappa^2 = 0$ and thus the surface gravity is constant over the horizon [25], except for possible *codimension*-2 *bifurcation surfaces* where the Killing vector ξ vanishes and κ can change sign [77].

The surface gravity is in principle arbitrary since if $\mathcal{N}$ is a Killing horizon of the Killing vector ξ with surface gravity κ, then it is also a Killing horizon of the rescaled Killing vector $c\xi$ with surface gravity $c^2\kappa$, for some $c \in \mathbb{R}$. Thus κ depends not only on $\mathcal{N}$ but also on the normalisation of ξ. However, since $\xi^2 = 0$ on $\mathcal{N}$, there is no natural normalisation here. Instead, for static, asymptotically flat spacetimes, one can avoid such ambiguities by fixing the norm of the time-translation Killing vector field $K = \partial_t$ to be

$$K_\mu K^\mu (r \to \infty) = -1,$$

which in turn fixes the normalisation of the surface gravity [129].

Mathematically, the surface gravity measures the magnitude of the gradient of the norm of the horizon generator, ξ, evaluated at the horizon, $\mathcal{N}$ [130]. Physically, at least for a static and asymptotically flat spacetime, the surface gravity measures the acceleration required by an observer to remain static near the Killing horizon, as measured by an observer at infinity [129].

In this thesis we shall also encounter stationary black hole spacetimes that are non-static. In this case there is still an asymptotic time-translation Killing vector $K = \partial_t$, and we can consider the trajectories of static observers defined to have four-velocities parallel to K. The difference now is that the Killing horizon of K is no longer an event horizon (we will demonstrate below that the event horizon coincides with the static Killing horizon for static spacetimes). For a non-static spacetime, the Killing horizon of K is called the *ergosphere*. Inside the ergosphere there are

[30] Frobenius' Theorem holds since ξ is normal to $\mathcal{N}$.

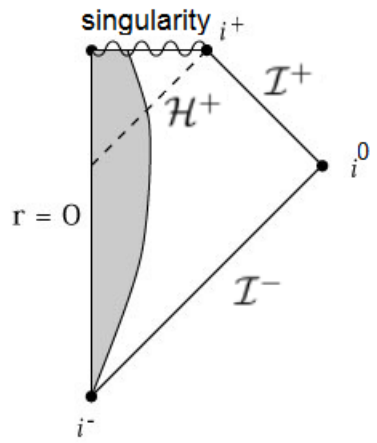


Figure 3.7: Penrose diagram for a spherically-symmetric collapsing star. Future and past null infinity are denoted $\mathcal{I}^+$ and $\mathcal{I}^-$ respectively, spatial infinity is i^0 and the future event horizon is $\mathcal{H}^+$. The stellar interior is shaded grey, the vertical $r = 0$ line is the origin of polar coordinates and the singularity (also at $r = 0$) is labelled accordingly.

'frame dragging' effects that require us to construct a new Killing vector χ that accommodates the motion of the black hole. As we shall see, the Killing horizon of the stationary Killing vector χ agrees with the event horizon. The surface gravity of the black hole will still be given by (3.145) providing we work with χ, and this can be thought of as the acceleration of a stationary observer near the horizon as measured from infinity.

Event horizons

The *future event horizon*, denoted $\mathcal{H}^+$, is defined as the boundary of the topological closure of the causal past of future null infinity, $\mathcal{I}^+$ [25]. The Penrose diagram in Figure 3.7 demonstrates the future event horizon for a stationary (non-evaporating) black hole spacetime. The concept of an event horizon is independent from that of a Killing horizon although the two are closely connected. In particular, it can be shown that for stationary black hole spacetimes, the event horizon is coincident with the stationary Killing horizon [131]. The surface gravity of the event horizon is then given by (3.145), which now defines the strength of the gravitational field on the event horizon. In particular, κ now measures the gravitational acceleration of a stationary particle just outside the horizon. Note that we distinguish the *non-extremal* black holes ($\kappa \neq 0$) from the *extremal* black holes ($\kappa = 0$).

Non-degenerate Killing horizons: non-extremal event horizons

A *non-degenerate Killing horizon* is a Killing horizon $\mathcal{N}$ with $\kappa \neq 0$. If we consider $\kappa \neq 0$ on an orbit of ξ in $\mathcal{N}$ then this orbit only partially coincides with the null generator of $\mathcal{N}$. To see

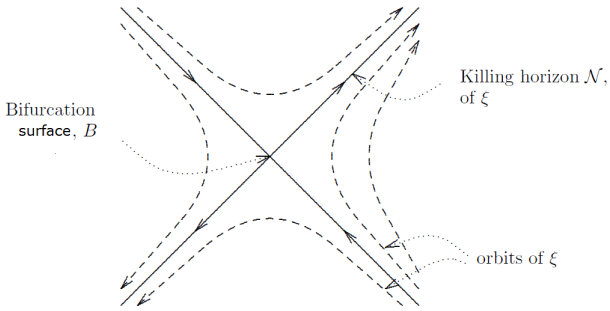


Figure 3.8: A bifurcate Killing horizon. Image sourced from [25].

this, use coordinates on $\mathcal{N}$ such that

$$\xi = \frac{\partial}{\partial \alpha} \quad \text{(except at those points where } \xi = 0\text{)}.$$

Thus α, the group parameter of the isometries generated by ξ, is also one of the coordinates on $\mathcal{N}$. Supposing $\alpha = \alpha(\lambda)$ on one particular orbit of ξ with affine parameter λ, we then have

$$\xi|_{\text{orbit}} = \frac{d\lambda}{d\alpha}\frac{d}{d\lambda} = fl \quad \text{with } f = \frac{d\lambda}{d\alpha} \text{ and } l = \frac{d}{d\lambda} = \frac{dx^\mu(\lambda)}{d\lambda}\partial_\mu. \tag{3.146}$$

We can then write $\kappa = \frac{\partial}{\partial\alpha}\ln|f|$ for orbits on $\mathcal{N}$. This can be integrated to $f = f_0 e^{\kappa\alpha}$, and since α parametrizes a group of isometries, it can be freely shifted by a constant meaning we can choose $f_0 = \pm\kappa$ without loss of generality. We then have

$$f = \frac{d\lambda}{d\alpha} = \pm\kappa e^{\kappa\alpha} \quad \Rightarrow \quad \lambda = \pm e^{\kappa\alpha} + \text{const.} \tag{3.147}$$

Setting the constant to zero, we see that along orbits of ξ in $\mathcal{N}$,

$$\lambda = \pm e^{\kappa\alpha}.$$

Importantly then, as $\alpha \in (-\infty, +\infty)$, the $\kappa \neq 0$ orbit of ξ covers the $\lambda < 0$ and $\lambda > 0$ branches of the null generator. The region where the null generator isn't covered by the $\kappa \neq 0$ orbit is that where $\lambda = 0$. This is a codimension-2 bifurcation surface, B, on which $\xi = 0$ is a fixed point of ξ as seen in Figure 3.8. Note that bifurcate Killing horizons do not form in gravitational collapse: the Penrose diagram in Figure 3.7 does not contain all the necessary regions shown in Figure 3.8 [39].

An argument of Racz and Wald [132] explains that, at B, the flow of ξ is attractive in one direction and repulsive in the other. If we take an arbitrary one-form, A, and evaluate its action

on the Killing vector at the bifurcation surface, we find

$$\langle A, \xi \rangle|_B = A(\xi)|_B = A(0) = 0$$

by linearity of the one-form. Assuming A has the same symmetries as the spacetime, it must have vanishing Lie derivative with respect to the null isometries, $\mathcal{L}_\xi(A) = 0$. This means A is ξ-invariant and consequently $\langle A, \xi \rangle = 0$ across the entire Killing horizon. Outside the horizon we can define a time coordinate t, such that $\xi = \partial_t$. Then the horizon limit of the component A_t of the 1-form is $A_t \to A(\xi) = 0$.

Degenerate Killing horizons: extremal event horizons

A *degenerate Killing horizon* is a Killing horizon $\mathcal{N}$ with $\kappa = 0$. It is clear from (3.144) that for such a horizon, the group parameter is itself an affine parameter for the null generator of $\mathcal{N}$ and therefore covers the entire horizon meaning there is no bifurcation surface [25]. The action of a one-form A on the null vector, ξ, generating the degenerate Killing horizon vanishes by continuity of the previous argument in the limit $\kappa \to 0$. The work of Racz and Wald strongly suggests that all physically relevant Killing horizons are either of bifurcate type or degenerate [132].

Trapping and apparent horizons

The earlier definition of an event horizon depends on the global causal structure; a characterisation that sits nicely in the diffeomorphism invariant framework of General Relativity. Unfortunately however, it requires knowledge of the complete Cauchy evolution of the entire universe which is somewhat impractical for both physical observers and numerical relativists who only have access to finite size laboratories [133]. Consider for example the collapsing star shown in Figure 3.7. The event horizon $\mathcal{H}^+$ extends into a region that is approximately Minkowski, and so it is entirely feasible that an observer could traverse the event horizon unaware of his fate [134].[31]

To deal with this, the idea of a *trapping horizon* was introduced. Trapping horizons depend only on local measurements and rely on the concept of a *trapped surface*. Suppose Σ is a spacelike hypersurface, then a trapped surface on Σ is a closed hypersurface S with the property that both the ingoing and outgoing congruences of future-directed, null geodesics orthogonal to S are converging [135,136]. Since we are considering congruences orthogonal to S, the tangent vectors to the ingoing and outgoing congruences, denoted l^- and l^+ respectively, will both be normal to S. Variations in spatial separation of null geodesics can then be measured using the second fundamental form $\Theta^\pm_{\mu\nu} = \nabla_\mu l^\pm_\nu$, and convergence/divergence is given by the null expansion scalar, $\Theta^\pm = \gamma^{\mu\nu}\Theta^\pm_{\mu\nu}$, where $\gamma_{\mu\nu}$ is the induced metric on Σ [137]. Typically, $\Theta^- < 0$ and

[31] Providing the black hole in question is sufficiently large that tidal forces become negligible.

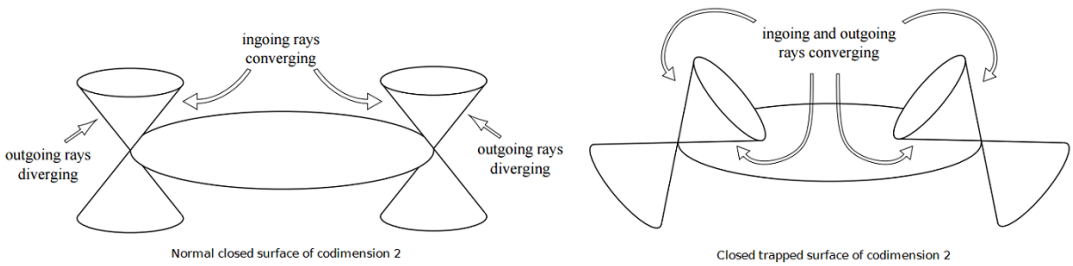


Figure 3.9: Comparing the orientation of light cones on trapped and untrapped codimension-2 surfaces. Image sourced from [135].

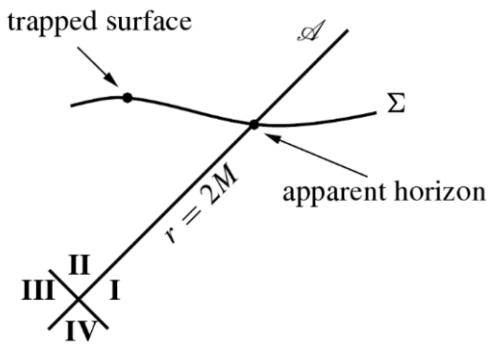


Figure 3.10: Trapped surface ($\Theta^+ < 0$) and apparent horizon ($\Theta^+ = 0$) for a space-like hypersurface Σ in the maximally extended Schwarzschild spacetime. The union of apparent horizons forms the trapping horizon, $\mathscr{A}$. Image taken from [136].

$\Theta^+ > 0$ indicating the ingoing and outgoing future-directed null geodesics emanating from S are converging and diverging respectively. However, in regions of spacetime where the gravitational field is particularly strong it is possible for both $\Theta^- < 0$ and $\Theta^+ < 0$ indicating the presence of a trapped surface, as depicted in Figure 3.9. Suppose $\mathscr{I} \subset \Sigma$ is the union of all trapped surfaces, then its boundary, denoted $\partial\mathscr{I}$, is known as an *apparent horizon*: it has the property of a marginally trapped surface, i.e. $\Theta^+|_{\partial\mathscr{I}} = 0$ [137]. The apparent horizon can be extended toward the future and past of Σ since hypersurfaces to the future and past of Σ will also contain apparent horizons. The union of all such codimension-2 apparent horizons defines a codimension-1 hypersurface, $\mathscr{A}$, known as the *trapping horizon* as seen in Figure 3.10 [136].[32]

It is proven in [38] that the apparent horizon, and consequently the trapping horizon, either coincides with or is contained inside the event horizon, $\mathcal{H}^+$. Establishing the presence of a trapping horizon therefore allows an observer to establish the presence of an event horizon using entirely local measurements of null geodesics. This will be vital for locating the event horizons of the black brane solutions in Chapters 4 and 5. The reason being that, even armed with a complete knowledge of the global geometry, our earlier definition of the event horizon

[32]Note that often the terminology apparent horizon is used in place of trapping horizon despite being technically incorrect.

doesn't easily extend to spacetimes that are not asymptotically flat [138]. For example, we have seen in Section 3.5.3 that for asymptotically AdS solutions, future null infinity is actually part of a time-like surface on the conformal boundary of the spacetime and so defining its causal past is problematic. But since, for stationary spacetimes, the event horizon always coincides with the trapping horizon, we can resolve to locate the trapping horizon instead [136]. Since the entire spacetime is stationary, $\mathscr{A}$ must be a stationary surface and be independent of time. Introducing a radial coordinate r transverse to the horizon, this translates to finding an $r = \text{const}$ hypersurface with a normal vector $n_\mu \propto \partial_\mu r$. Furthermore, on $\mathscr{A}$, we have $\Theta^+ = 0$ meaning the future-directed outgoing null geodesics are confined to $\mathscr{A}$ which implies $\mathscr{A}$ is a null surface and must have a normal vector that is null. We can therefore determine the trapping horizon with the equation $g^{\mu\nu}(\partial_\mu r)(\partial_\nu r) = g^{rr} = 0$. Solving, we find the trapping horizon, and thus the event horizon, is found at $r = r_+$.

3.7.2 Black hole thermodynamics

Laws of black hole mechanics

Let us now review the semi-classical laws of black hole mechanics and their striking resemblance to the laws of thermodynamics. In fact, we regard the laws of black hole mechanics as particular cases of the laws of thermodynamics applied to systems containing black holes. Quantum black hole thermodynamics is beyond the scope of this thesis but we refer the reader to [130] for an overview. The four laws are as follows:

Zeroth Law As mentioned above, the surface gravity, κ, is constant across the black hole event horizon of a stationary black hole [130]. This is reminiscent of the zeroth law of thermodynamics which states that temperature is constant for systems in thermal equilibrium. Hawking made the identification of surface gravity with temperature concrete with the discovery that black holes are not truly black [139]. Instead, quantum fluctuations allow them to emit so-called Hawking radiation and potentially evaporate. The radiation gives the black hole a 'Hawking temperature' [25]

$$T = \frac{\kappa}{2\pi}\frac{\hbar}{k_B} = \frac{\kappa}{2\pi}, \tag{3.148}$$

where we work in units for which both the reduced Planck constant and Boltzmann's constant are set to unity, $\hbar = k_B = 1$.

First Law The 'no-hair theorem' states that the most general stationary black holes, belonging to the Kerr-Newman solutions of Einstein-Maxwell theory, can be completely characterised by their mass M, electric charge Q and angular momentum J [135]. The first law of black hole mechanics describes the change in mass during the interaction of two infinitesimally nearby

stationary black holes as

$$\delta M = \frac{\kappa}{8\pi}\delta A + \mu\delta Q + \Omega\delta J, \tag{3.149}$$

where A is the area of the event horizon, μ is the chemical potential associated to variation of electric charge and Ω is the angular velocity of the horizon. Equation (3.149) has the same structure as the first law of thermodynamics describing energy conservation as

$$\delta M = T\delta S + \mu\delta N - P\delta V. \tag{3.150}$$

An important observation is that since we have identified the Hawking temperature of the black hole with surface gravity in (3.148), we then identify the horizon area with the entropy of the black hole. This leads to the famous Bekenstein-Hawking formula for black hole entropy[33]

$$S = \frac{A}{4}. \tag{3.151}$$

Furthermore, for a grand canonical ensemble, Ω and J represent the pressure and volume, whilst μ and Q are the chemical potential and particle number [77].

Second Law The analogy between area and entropy is further reinforced by Hawking's area theorem. The theorem states that for non-stationary processes e.g. black hole fusion, assuming Cosmic Censorship and the null energy condition, the total area of all event horizons is non-decreasing

$$\delta A \geq 0. \tag{3.152}$$

Immediately we see a connection to the second law of thermodynamics which states that the entropy of an isolated system is always non-decreasing

$$\delta S \geq 0. \tag{3.153}$$

In this instance, the black hole law is stronger than its thermodynamic counterpart since in thermodynamics it is possible to transfer entropy between subsystems but this doesn't apply to black holes since they are unable to bifurcate [140].

Third Law (Nernst Law) In Chapters 4 and 5 we construct black brane solutions that satisfy the third law. As such, it is important to discuss this in some detail below. Unfortunately, this law is not completely understood but of course, this is one of the reasons why it remains an exciting area of research. Below we outline both the 'weak' and 'strong' versions of the third law.

[33] Actually the area formula was proposed before the discovery of Hawking temperature. We present the material here in a pedagogical manner but see [135] for a historical overview.

The weak version of the third law of black hole mechanics was put forward by Israel in 1986, stating that it is impossible to reduce the surface gravity, κ, to zero by a finite series of operations [140, 141]. This inability to produce extremal black holes by physical processes resembles the weak version of the third law of thermodynamics, also known as 'Nernst's Principle' or the 'process version', which asserts:

> *"Any thermodynamic process cannot reach the temperature of absolute zero by a finite number of steps and within a finite time."* [142, 143, 144]

There is also a strong version of the third law of black hole mechanics which states that the black hole entropy tends to zero as the surface gravity is reduced to zero [141]. This resembles Planck's strong version of the third law of thermodynamics:

> *"When temperature falls to absolute zero, the entropy of any pure crystalline substance tends to a universal constant (which can be taken to be zero) [144]*
>
> $$S \to 0 \quad \text{as} \quad T \to 0."$$ (3.154)

Despite Planck being responsible for the strong version of this thermodynamic law, we refer to this version of the third law as the Nernst Law throughout the remainder of this thesis.

In contrast to the other laws of black hole mechanics, the third law has several shortcomings. Whilst there does exist a proof for the weak version, we shall work exclusively with the strong, Planckian version throughout this thesis since it corresponds to systems with a unique ground state and therefore represents generic behaviour in condensed matter physics [12, 13, 32, 145].[34] Unfortunately, at least to date, there exists no rigorous mathematical proof of this. Additionally, there is strong evidence that extremal black holes have non-vanishing entropy thus ruining the correspondence between the laws of black hole mechanics and thermodynamics [20, 146]. This does however beg the question of how widespread this behaviour is amongst black objects and whether it is possible to construct examples that do indeed satisfy the strong version of the third law. The main goal of this thesis is to find such solutions that represent credible candidates for holographic duals to condensed matter systems.

Thermodynamic stability

Having established that black holes emit blackbody radiation and thus behave as genuine thermal systems, a natural question to ask is how they respond to small fluctuations in temperature? In

[34] In [140] an argument was given for the weak version that were it possible to reduce κ to zero in a finite number or steps, then it would leave room to continue the process with further steps and ultimately result in the dynamic production of a naked singularity, thus violating Cosmic Censorship. Later, this was formalised by Israel [141] who proved that a nonextremal black hole cannot lose its trapped surfaces (i.e. it cannot undergo an extremisation procedure) in a finite number of steps.

particular, we imagine placing a black hole in thermal equilibrium with an infinite heat reservoir and studying its reaction to small fluctuations in temperature. The response depends on the *specific heat capacity* of the black hole,

$$C_T = \frac{\partial M}{\partial T}. \tag{3.155}$$

A negative specific heat capacity means mass and temperature are negatively correlated. Then one can imagine placing a black hole of temperature $T_{\rm BH}$ in a heat bath at temperature $T > T_{\rm BH}$. The direction of heat flow is obvious and tells us the black hole must absorb some radiation and thus increase its mass, but this in fact makes the black hole even colder. Consequently the temperature difference increases and heat will continue to flow until all that remains is an infinitely heavy, cold black hole. Likewise, if $T < T_{\rm BH}$, the net heat flow is from the black hole to the reservoir, and as it reduces its mass, it responds by getting hotter. This runaway behaviour ends with evaporation.

On the other hand, a positive specific heat capacity implies mass and temperature are positively correlated. Again, $T > T_{\rm BH}$ results in the black hole absorbing radiation and increasing its mass, but now this is accompanied by an increase in temperature. The temperature difference between the black hole and its surroundings decreases and the system eventually stabilises. Similarly, if $T < T_{\rm BH}$, the black hole radiates some mass away to heat its surroundings and in doing so cools itself in a process that will ultimately reach a thermal equilibrium.

Now consider a black hole in thermal equilibrium with its surroundings, $T = T_{\rm BH}$. If $C_T < 0$, then small, random perturbations in temperature lead to the runaway behaviour described above and the system is said to be *unstable*. Meanwhile, if $C_T > 0$, the system is *stable* against any such fluctuations.

Black holes in asymptotically flat spacetimes have negative specific heat making them thermodynamically unstable. Hawking realised that the only way to enable black holes to survive in thermal equilibrium with their surroundings was to place them in a box, thus restricting the available energy and preventing such runaway behaviour [147, 148]. Whilst this is clearly unrealistic in conventional asymptotically flat spacetimes, it was later discovered that black holes can exist in stable thermal equilibrium in Anti de-Sitter spacetimes [149]. Indeed, we saw in Section 3.5 how AdS behaves naturally like a box due to the presence of an infinite potential wall at asymptotic infinity. An important, and relatively straightforward example of this is found by considering the Hawking-Page phase transition. Excellent reviews of this phenomenon can be found in [148, 150]. To summarise the salient points, we consider an asymptotically AdS_5 spacetime and, using the action for pure gravity plus a cosmological constant, find that there are three solutions to this boundary condition; small black holes, large black holes and thermal AdS_5. The black hole solutions are found by considering the metric of a Schwarzschild-AdS_5

black hole. Using $g^{rr} = 0$ to locate the trapping horizon we find two solutions for the event horizon. The solution with the larger (resp. smaller) event horizon is called a large (resp. small) black hole. It can be shown that the small black holes have negative specific heat capacity making them thermodynamically unstable whilst the opposite is true for their large cousins which remain stable. The third solution to the boundary conditions is thermal AdS_5 which describes a universe filled with radiation. Having ruled out the small black holes as being unstable, Hawking and Page then compared which of the remaining solutions was entropically favoured. The result is strongly temperature dependent: a radiation dominated universe is preferred at low temperatures but as the temperature is increased there is a first order phase transition to a black hole dominated universe, which is preferred at high temperatures.

4 Four-dimensional Nernst branes

This chapter is based on
P. Dempster, D. Errington and T. Mohaupt
"Nernst branes from special geometry,"
JHEP **05** (2015) 079, [arXiv:1501.07863] [1].

We now come to one of the main results of this thesis, namely the construction of four-dimensional Nernst brane solutions [1]. Within the framework of $U(1)$ FI gauged four-dimensional $\mathcal{N} = 2$ supergravity coupled to vector multiplets, extremal Nernst branes have previously been constructed in [24] using a first-order rewriting of the equations of motion, and by considering a specific model: the so-called STU-model. However a similar rewriting for their non-extremal counterparts has so far proven elusive, and the only known examples are five-dimensional [151]. The construction of these relies on deforming the metric of the corresponding five-dimensional extremal solution [152] and imposing suitable consistency conditions. In this chapter we provide a systematic construction of non-extremal Nernst branes in four dimensions by directly solving the second-order equations of motion. Moreover, our results will not only apply to a particular model, but to all models where the prepotential is of the very special type. This gain in generality and systematics should help to expand the AdS/CMT dictionary considerably in the future.

In order to obtain exact, analytic solutions we shall simplify matters by restricting ourselves to very special models that can be obtained by dimensional reduction from five dimensions. Such models have a prepotential of the form

$$F(X) = \frac{f(X^1, \ldots, X^{n_V^{(4)}})}{X^0}, \tag{4.1}$$

where $f = c_{ABC} X^A X^B X^C$, with $A, B, C = 1, \ldots, n_V^{(4)}$, is a homogeneous polynomial of degree three.[35] To avoid a cluttered notation, we shall use n instead of $n_V^{(4)}$ throughout the remainder of this chapter. Assuming an embedding into heterotic or type-II string theory, such prepotentials capture perturbative string effects to leading order in the string coupling. As motivated in the Introduction, we restrict ourselves to static black brane solutions. Apart from this we will

[35] The techniques presented here do not require f to be a polynomial. Therefore the results obtained in this chapter apply to a wider class of prepotentials than just the very special ones. Indeed, they apply to all prepotentials of the form (4.1) with f a homogeneous (not necessarily polynomial) function of degree three.

also impose that the scalar fields take purely imaginary values, as for such 'axion-free' field configurations there is a systematic simplification of the equations of motion. Since we impose stationarity in four dimensions, we can perform a time-like dimensional reduction to obtain an effective three-dimensional Euclidean theory. The degrees of freedom in three dimensions can then be repackaged using the real formulation of special geometry developed in [94], which has been used before to construct solutions to both gauged [102, 103, 153] and ungauged [3, 94] theories of supergravity coupled to vector multiplets.

Solving the three-dimensional equations of motion directly results in an instanton solution depending on a number of integration constants, which are *a priori* undetermined. However, in order that this solution lifts to a *regular* black brane in four dimensions we have to impose suitable regularity conditions. In particular, we require that the four-dimensional solution has a finite entropy density, which happens to simultaneously ensure that the scalar fields take finite values on the horizon. For a given set of charges and FI parameters, we are then left with a two-parameter family of black brane solutions parametrized by a temperature T and chemical potential $|\mu|$, which can both be freely varied. In the limit of zero temperature, we recover the extremal Nernst branes of [24]. Therefore we interpret our solutions as non-extremal (or 'hot') Nernst branes. Indeed, it turns out that the entropy density goes to zero as $T \to 0$ for fixed charges/FI parameters, in agreement with the Nernst Law. Our solutions interpolate between hyperscaling violating Lifshitz geometries with $(z, \theta) = (0, 2)$ at the horizon and $(z, \theta) = (1, -1)$ at infinity. In the zero temperature limit the near horizon geometry changes to $(z, \theta) = (3, 1)$. The presence of hvLif geometries in our solutions further strengthens the possible relationship with condensed matter physics, where hyperscaling violation and spacetime anisotropy frequently occur [12, 14, 123].

This chapter is organised as follows: in Section 4.1 we dimensionally reduce the four-dimensional theory over a time-like circle and rewrite the resulting three-dimensional Lagrangian using the real formulation of special geometry. We then determine the equations of motion for general static field configurations, before concentrating on the case of purely imaginary field configurations. In Section 4.2 we solve the aforementioned equations of motion for the case where we have a single electric charge and some number of electric FI parameters. Having found a solution to the three-dimensional equations of motion we then lift it back to a four-dimensional solution and determine the conditions imposed on the various integration constants by regularity, before carrying out an analysis of the properties of the solution. In Section 4.3 we apply our method to the case where we instead switch on a single magnetic charge and a single magnetic FI parameter, whilst keeping $(n-1)$ of the electric FI parameters. Section 4.4 contains a discussion of our results in the context of holography.

4.1 Dimensional reduction and equations of motion

We start with the bosonic action for four-dimensional $\mathcal{N}=2$ supergravity with a dyonic $U(1)$ FI gauging coupled to n vector multiplets. Taking (3.115) and working in units where $\kappa_4^2 = 8\pi G_4 = 1$, the Lagrangian for this theory is

$$\begin{aligned}\mathsf{e}_4^{-1}\mathcal{L}_4 = -\frac{1}{2}R_{(4)} - g_{IJ}\partial_\mu X^I\partial^\mu \bar{X}^J + \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu} \\ + V_4^{\text{dyonic}}(X,\bar{X}),\end{aligned} \tag{4.2}$$

with the scalar potential given in terms of the dyonic superpotential $W = 2(g^I F_I - g_I X^I)$ as in (3.114). Our goal is to solve the equations of motion at the three-dimensional level where all fields can be Hodge dualised into scalars to aid solving the equations of motion. In Section 3.3.2 we explained how the real formulation of special geometry could be adapted for dimensionally reducing a Lorentzian theory in four dimensions over a time-like circle. Imposing that the background is stationary, we can make the following ansatz for the four-dimensional metric

$$ds^2_{(4)} = -e^{\phi}(dt + B_\mu dx^\mu)^2 + e^{-\phi}ds^2_{(3)}, \tag{4.3}$$

where ϕ, B are the Kaluza-Klein scalar and vector respectively. Performing the dimensional reduction as before, we find the resulting three-dimensional Lagrangian to be given by

$$\begin{aligned} e_3^{-1}\mathcal{L}_3 \;=\; & -\frac{1}{2}R_{(3)} - \tilde{H}_{ab}\left(\partial_\mu q^a\partial^\mu q^b - \partial_\mu\hat{q}^a\partial^\mu\hat{q}^b\right) - \frac{1}{2H}V \\ & -\frac{1}{H^2}(q^a\Omega_{ab}\partial_\mu q^b)^2 + \frac{2}{H^2}(q^a\Omega_{ab}\partial_\mu\hat{q}^b)^2 \\ & -\frac{1}{4H^2}(\partial_\mu\tilde{\phi} + 2\hat{q}^a\Omega_{ab}\partial_\mu\hat{q}^b)^2, \end{aligned} \tag{4.4}$$

which follows from (3.84) and noting that the potential term $+V$ present in (4.2) remains unchanged except for multiplication by a factor $e^{-\phi} = -\frac{1}{2H}$. The scalar potential (3.114) is written in terms of real coordinates in Appendix D.1. Substituting from (D.8), we obtain the following expression for the three-dimensional Lagrangian

$$\begin{aligned} e_3^{-1}\mathcal{L}_3 \;=\; & -\frac{1}{2}R_{(3)} - \tilde{H}_{ab}\left(\partial_\mu q^a\partial^\mu q^b - \partial_\mu\hat{q}^a\partial^\mu\hat{q}^b + g^a g^b\right) \\ & -\frac{1}{H^2}(q^a\Omega_{ab}\partial_\mu q^b)^2 + \frac{2}{H^2}(q^a\Omega_{ab}\partial_\mu\hat{q}^b)^2 \\ & +4(g^a q_a)^2 + \frac{2}{H^2}(q^a\Omega_{ab}g^b)^2 - \frac{1}{4H^2}(\partial_\mu\tilde{\phi} + 2\hat{q}^a\Omega_{ab}\partial_\mu\hat{q}^b)^2, \end{aligned} \tag{4.5}$$

where we have defined $g^a = (g^I, g_I)^T$.

We shall first restrict ourselves to static solutions. This means the Kaluza-Klein vector vanishes i.e. $B_\mu = 0$ in the reduction ansatz (4.3). Using (3.66) and (3.75), this is equivalent to

the final term in (4.5) being absent [94]. The equations of motion for $\hat{q}^a$ are then given by

$$D_\mu\left(\tilde{H}_{ab}\partial^\mu\hat{q}^b\right)+2D_\mu\left(\frac{1}{H^2}q^b\Omega_{ba}(q^c\Omega_{cd}\partial^\mu\hat{q}^d)\right)=0, \tag{4.6}$$

whilst those for q^a read

$$\begin{aligned}&D_\mu\left(\tilde{H}_{ab}\partial^\mu q^b\right)-\frac{1}{2}\partial_a\tilde{H}_{bc}\left(\partial_\mu q^b\partial^\mu q^c-\partial_\mu\hat{q}^b\partial^\mu\hat{q}^c+g^bg^c\right)\\&-\frac{1}{2}\partial_a\left(\frac{1}{H^2}\right)(q^b\Omega_{bc}\partial_\mu q^c)^2+D_\mu\left(\frac{1}{H^2}q^b\Omega_{ba}(q^c\Omega_{cd}\partial^\mu q^d)\right)-\frac{1}{H^2}\Omega_{ab}\partial_\mu q^b(q^c\Omega_{cd}\partial^\mu q^d)\\&+\partial_a\left(\frac{1}{H^2}\right)(q^c\Omega_{cd}\partial^\mu\hat{q}^d)^2+\frac{2}{H^2}\Omega_{ab}\partial_\mu\hat{q}^b(q^c\Omega_{cd}\partial^\mu\hat{q}^d)\\&+4\tilde{H}_{ab}g^b(g^cq_c)+\partial_a\left(\frac{1}{H^2}\right)(q^b\Omega_{bc}g^c)^2+\frac{2}{H^2}\Omega_{ab}g^b(q^c\Omega_{cd}g^d)=0.\end{aligned} \tag{4.7}$$

Finally, the three-dimensional Einstein equations are

$$\begin{aligned}&-\frac{1}{2}R_{(3)\mu\nu}-\tilde{H}_{ab}\left(\partial_\mu q^a\partial_\nu q^b-\partial_\mu\hat{q}^a\partial_\nu\hat{q}^b\right)-\frac{1}{H^2}(q^a\Omega_{ab}\partial_\mu q^b)(q^c\Omega_{cd}\partial_\nu q^d)\\&+\frac{2}{H^2}(q^a\Omega_{ab}\partial_\mu\hat{q}^b)(q^c\Omega_{cd}\partial_\nu\hat{q}^d)+g_{\mu\nu}\left(-\tilde{H}_{ab}g^ag^b+4(g^aq_a)^2+\frac{2}{H^2}(g^a\Omega_{ab}q^b)^2\right)=0.\end{aligned} \tag{4.8}$$

4.1.1 Purely imaginary field configurations

In order to solve the equations of motion (4.6)-(4.8), we make one further simplification and restrict the field content to the so-called *purely imaginary* (PI) field configurations, which we define to be those for which the complex scalars $z^A=\frac{Y^A}{Y^0}=\frac{X^A}{X^0}$ are purely imaginary [3]. Moreover, we restrict ourselves to the subclass of models obtainable from dimensional reduction from five dimensions for which the prepotential assumes the very special form

$$F(Y)=\frac{f(Y^1,\ldots,Y^n)}{Y^0}, \tag{4.9}$$

where the function f is a homogeneous polynomial of degree three and real-valued when evaluated on real fields.[36] To descend from the superconformal description to the physical description we must fix the $\mathbb{C}^*$ action generating the CASK manifold N. This is done by imposing the D-gauge condition (3.73) and by fixing a $U(1)$ gauge. Since we choose to fix the $U(1)$ gauge by taking $\mathrm{Im}Y^0=u^0=0$, the only way to ensure the z^A are purely imaginary is to have purely imaginary scalar fields Y^A, i.e.

$$x^A=0,\qquad A=1,\ldots,n.$$

[36] Superconformal invariance tells us F must be homogeneous of degree two. The very special property imposes that, additionally, it i s the ratio of a homogeneous degree three function of the Y^A, to the homogeneous degree one variable Y^0.

For the very special models, the real parts of z^A exhibit an invariance under the axion-like shift symmetry

$$\text{Re } z^A \mapsto \text{Re } z^A + C^A.$$

Our PI condition models are therefore often referred to as *axion-free* configurations.

The PI condition sets $Y^A = iu^A$ and $Y^0 = x^0$. It is then immediate from (4.9) that $F_0 = \frac{\partial F(Y)}{\partial Y^0} = -\frac{f(Y^1,\ldots,Y^n)}{(Y^0)^2}$ is purely imaginary. Given that $F_0 = y_0 + iv_0$, this corresponds to $y_0 = 0$. To summarise, for the class of models (4.9), the scalar fields q^a take the form [3]

$$(q^a)|_{\text{PI}} = (x^0, 0, \ldots, 0; 0, y_1, \ldots, y_n),$$

and hence we see that $q^a \Omega_{ab} \partial_\mu q^b = 0$. This is the aforementioned simplification of imposing the PI and very special conditions. Following [3] we obtain further simplifications of the equations of motion by extending the PI condition to the scalars $\hat{q}^a$ by imposing

$$(\partial_\mu \hat{q}^a)|_{\text{PI}} = \frac{1}{2}(\partial_\mu \zeta^0, 0, \ldots, 0; 0, \partial_\mu \tilde{\zeta}_1, \ldots, \partial_\mu \tilde{\zeta}_n),$$

which sets also $q^a \Omega_{ab} \partial_\mu \hat{q}^b = 0$. Recall that the quantities $\partial_\mu \zeta^I$ and $\partial_\mu \tilde{\zeta}_I$ encode the four-dimensional field strengths, as seen in (3.76).

In the same way, we extend the PI condition to the FI parameters g^a by imposing

$$(g^a)|_{\text{PI}} = (g^0, 0, \ldots, 0; 0, g_1, \ldots, g_n),$$

which sets $q^a \Omega_{ab} g^b = 0$.

We then find that the equations of motion (4.6), (4.7) and the three-dimensional Einstein equations (4.8) greatly simplify to

$$D_\mu \left(\tilde{H}_{ab} \partial^\mu \hat{q}^b \right) = 0, \tag{4.10}$$

$$D_\mu \left(\tilde{H}_{ab} \partial^\mu q^b \right) - \frac{1}{2} \partial_a \tilde{H}_{bc} \left(\partial_\mu q^b \partial^\mu q^c - \partial_\mu \hat{q}^b \partial^\mu \hat{q}^c + g^b g^c \right) + 4 \tilde{H}_{ab} g^b (g^c q_c) = 0, \tag{4.11}$$

and

$$-\frac{1}{2} R_{(3)\mu\nu} - \tilde{H}_{ab} \left(\partial_\mu q^a \partial_\nu q^b - \partial_\mu \hat{q}^a \partial_\nu \hat{q}^b \right) + g_{\mu\nu} \left(-\tilde{H}_{ab} g^a g^b + 4 (g^a q_a)^2 \right) = 0. \tag{4.12}$$

It turns out to be useful to write the equations of motion in terms of the dual variables q_a and $\hat{q}_a$. Doing so, the equations (4.10)–(4.12) take the simpler form

$$\triangle_{(g)} \hat{q}_a = 0, \tag{4.13}$$

$$\triangle_{(g)} q_a + \frac{1}{2}\partial_a \tilde{H}^{bc} \left(\partial_\mu q_b \partial^\mu q_c - \partial_\mu \hat{q}_b \partial^\mu \hat{q}_c\right) - \frac{1}{2}\partial_a \tilde{H}_{bc} g^b g^c + 4\tilde{H}_{ab} g^b (g^c q_c) = 0, \tag{4.14}$$

and

$$-\frac{1}{2} R_{(3)\mu\nu} - \tilde{H}^{ab} \left(\partial_\mu q_a \partial_\nu q_b - \partial_\mu \hat{q}_a \partial_\nu \hat{q}_b\right) + g_{\mu\nu} \left(-\tilde{H}_{ab} g^a g^b + 4(g^a q_a)^2\right) = 0. \tag{4.15}$$

In the next section we will look for solutions of (4.13)–(4.15) which can be lifted to regular non-extremal black branes in four dimensions.

4.2 Non-extremal black branes

In this section we construct a family of non-extremal black branes in the $\mathcal{N} = 2$ gauged supergravity theory (4.2) with prepotential (4.9). Restricting our attention to the PI configurations described in Section 4.1.1, it is shown in Appendix D.2 that the Hesse potential takes the form

$$H = -\frac{1}{4} \left(-q_0 f(q_1, \ldots, q_n)\right)^{-\frac{1}{2}}. \tag{4.16}$$

For general functions f, the form of the metric $\tilde{H}^{ab}$ (appearing in the equations of motion) is fairly complicated [3]. However, since the field q_0 decouples from the rest, we can use (3.83) to compute

$$\tilde{H}^{00} = \frac{1}{4q_0^2}, \quad q^0 = -\frac{1}{4q_0},$$

and this will be sufficient to find solutions valid for any choice of f. We remark here upon a slight abuse of notation which we will make throughout the remainder of this chapter. Specifically, we denote by q_A with $A = 1, \ldots, n$ those scalar fields which are actually the $(A + n + 1)$'th components of the vector (q_a). The same is true of the components $\tilde{H}_{AB}$ of the metric, which should properly be the $(A + n + 1, B + n + 1)$ components of $\tilde{H}_{ab}$. This notation is convenient since (q^0, q_A) are the remaining non-trivial q^a-fields within our PI ansatz.

For simplicity we will concentrate on solutions which are supported by a single electric charge Q_0 and electric FI parameters $g_1, \ldots, g_n$ in this section. However, as we will see in Section 4.3, the methods introduced in the following can be easily extended to deal also with solutions with a single magnetic charge switched on and sourced by both electric and magnetic FI parameters.

4.2.1 Einstein equations

To construct Nernst solutions that are valid within the supergravity regime, we make a brane-like ansatz for the three-dimensional metric:

$$ds_3^2 = e^{4\psi} d\tau^2 + e^{2\psi}(dx^2 + dy^2), \tag{4.17}$$

where $\psi = \psi(\tau)$ is some function to be determined. We also impose that all fields q_a and $\hat{q}_a$ depend only on τ. The coordinate τ has been chosen such that it is an affine parameter for the curves $C : \tau \mapsto (q^a(\tau), \hat{q}^a(\tau))$ on the scalar target space.[37] Equivalently, the τ-dependent part of the three-dimensional Laplace operator is given by $\frac{\partial^2}{\partial \tau^2}$.

The non-zero components of the Ricci tensor are given by

$$R_{\tau\tau} = 2\ddot{\psi} - 2\dot{\psi}^2, \quad R_{xx} = R_{yy} = e^{-2\psi}\ddot{\psi},$$

where the dot denotes differentiation with respect to τ. With this choice the three-dimensional Einstein equations (4.15) become

$$-\tilde{H}_{ab}g^a g^b + 4(q_a g^a)^2 - \frac{1}{2}e^{-4\psi}\ddot{\psi} = 0, \tag{4.18}$$

for $\mu = \nu \neq \tau$ and

$$\tilde{H}^{ab}\left(\dot{q}_a\dot{q}_b - \dot{\hat{q}}_a\dot{\hat{q}}_b\right) = \dot{\psi}^2 - \frac{1}{2}\ddot{\psi}, \tag{4.19}$$

for $\mu = \nu = \tau$, where we have used (4.18). Equation (4.19) is the Hamiltonian constraint which needs to be imposed on solutions $(q_a(\tau), \hat{q}_a(\tau))$ of the second-order scalar field equations. We remark that since we have consistently reduced the full field equations, we do not need to impose this constraint by hand, but have retained it as a field equation following from an action principle.

4.2.2 Scalar equations of motion

We now turn to the equations of motion for the fields q_a and $\hat{q}_a$. We start with the $\hat{q}_a$ equations of motion, which read simply

$$\ddot{\hat{q}}_a = 0,$$

and can be integrated once to find

$$\dot{\hat{q}}_a = K_a, \tag{4.20}$$

for some constants K_a, which are proportional to the electric and magnetic charges of the solution, $K_a = (-Q_I, P^I)$ [3]. The explicit relations between the $\hat{q}_a$ and the field strengths can be found in (3.75) and (3.76). For the case at hand we only have a single electric charge Q_0, and so the only non-zero component of $\dot{\hat{q}}_a$ is $\dot{\hat{q}}_0 = -Q_0$.

We turn now to the q_a equations of motion (4.14), which become

$$e^{-4\psi}\ddot{q}_a + \frac{1}{2}\partial_a\tilde{H}^{bc}e^{-4\psi}\left(\dot{q}_b\dot{q}_c - \dot{\hat{q}}_b\dot{\hat{q}}_c\right) - \frac{1}{2}\partial_a\tilde{H}_{bc}g^b g^c + 4\tilde{H}_{ab}g^b(q_c g^c) = 0. \tag{4.21}$$

[37] The curves $C(\tau)$, parametrized by q^a and $\hat{q}^a$, are not necessarily geodesics on the target space. As explained in Appendix B.2 it is possible for the scalar potential, present as a result of the FI gauging, to create a gradient force that deforms them away from geodesic motion.

For models (4.9) with the magnetic FI parameter switched off, $g^0 = 0$, on which we concentrate in this section, the q_0 equation of motion decouples from the others. Indeed, using (4.20) with $K_0 = -Q_0$ the q_0 equation of motion becomes

$$\ddot{q}_0 - \frac{\dot{q}_0^2 - Q_0^2}{q_0} = 0. \tag{4.22}$$

This takes precisely the same form as in the ungauged case [3] and can be solved with

$$q_0(\tau) = \pm - \frac{Q_0}{B_0} \sinh\left(B_0\tau + B_0\frac{h_0}{Q_0}\right), \tag{4.23}$$

for some constants B_0 and h_0. Since the solution (4.23) is invariant under $B_0 \to -B_0$, we can take $B_0 \geq 0$ without loss of generality. It will turn out that B_0 acts as a non-extremality parameter for the full solution. Furthermore, as we will see later explicitly, τ naturally takes values $0 \leq \tau < \infty$. Thus in order that $q_0 \neq 0$ for $\tau \geq 0$ we will have to require $\mathrm{sign}(h_0) = \mathrm{sign}(Q_0)$.

The q_A equations of motion, for $A = 1, \ldots, n$, become

$$\begin{aligned} e^{-4\psi}\ddot{q}_A &+ \frac{1}{2}e^{-4\psi}\sum_{B,C=1}^{n}\partial_A\tilde{H}^{BC}\dot{q}_B\dot{q}_C \\ &- \frac{1}{2}\sum_{B,C=1}^{n}(\partial_A\tilde{H}_{BC})g_Bg_C + 4\sum_{B=1}^{n}\tilde{H}_{AB}g_B\left(\sum_{C=1}^{n}q_Cg_C\right) = 0, \end{aligned} \tag{4.24}$$

where we leave the sum explicit here for convenience. Multiplying by q^A and summing over A gives

$$e^{-4\psi}\sum_{A=1}^{n}q^A\ddot{q}_A + e^{-4\psi}\sum_{A,B=1}^{n}\tilde{H}^{AB}\dot{q}_A\dot{q}_B + \sum_{A,B=1}^{n}\tilde{H}_{AB}\,g_Ag_B - 4\left(\sum_{A=1}^{n}g_Aq_A\right)^2 = 0\,, \tag{4.25}$$

where we have made use of the homogeneity properties of the metric $\tilde{H}_{ab}$, viz. $q^a\partial_a\tilde{H}^{bc} = 2\tilde{H}^{bc}$ and $q^a\partial_a\tilde{H}_{bc} = -2\tilde{H}_{bc}$.

One can now compare this equation to (4.18), which for the model at hand becomes

$$-\sum_{A,B=1}^{n}\tilde{H}_{AB}\,g_Ag_B + 4\left(\sum_{A=1}^{n}g_Aq_A\right)^2 - \frac{1}{2}e^{-4\psi}\ddot{\psi} = 0\,.$$

Substituting from this into the last two terms of (4.25) we obtain

$$\sum_{A=1}^{n}q^A\ddot{q}_A + \sum_{A,B=1}^{n}\tilde{H}^{AB}\dot{q}_A\dot{q}_B = \frac{1}{2}\ddot{\psi}\,. \tag{4.26}$$

The left-hand side of this equation can be rewritten as a total derivative

$$\sum_{A=1}^{n} q^A \ddot{q}_A + \sum_{A,B=1}^{n} \tilde{H}^{AB} \dot{q}_A \dot{q}_B = \frac{d}{d\tau}\left(\sum_{A=1}^{n} q^A \dot{q}_A\right),$$

and so we can integrate to find

$$\sum_{A=1}^{n} q^A \dot{q}_A = \frac{1}{2}\dot{\psi} - \frac{1}{4}a_0\,, \tag{4.27}$$

for some integration constant a_0, where we have chosen the factor for later convenience. Now, using the identity $\partial^a \tilde{H} = \tilde{H}^{ab} q_b$ [3] one can show furthermore that

$$\frac{d\tilde{H}}{d\tau} = -q^0 \dot{q}_0 - \sum_{A=1}^{n} q^A \dot{q}_A = \frac{\dot{q}_0}{4q_0} - \sum_{A=1}^{n} q^A \dot{q}_A\,.$$

Substituting this expression into (4.27) gives

$$-2\dot{\psi} + a_0 = 4\frac{d\tilde{H}}{d\tau} - \frac{\dot{q}_0}{q_0}\,.$$

Integrating with respect to τ we have

$$\begin{aligned} -2\psi + a_0\tau + b_0 &= 4\tilde{H} - \log(-q_0) + k \\ &= 4\tilde{H} - \log\left(-4q_0\right) \\ &= -2\log\left(-2H\right) - 2\log\left(2(-q_0)^{1/2}\right) \\ &= -2\log\left(-4H\cdot(-q_0)^{1/2}\right). \end{aligned} \tag{4.28}$$

where we have used the definition of $\tilde{H}$ given in (3.78), and have chosen the definition of the integration constant $k = -\log 4$ for later convenience. In particular, this choice of k leads to a particularly nice expression when we substitute for the Hesse potential (4.16). Doing so, we obtain

$$\log\left(f(q_1,\ldots,q_n)\right) = -2\psi + a_0\tau + b_0. \tag{4.29}$$

Let us now return to the Hamiltonian constraint (4.19) which, upon substituting the expression (4.23), becomes

$$\sum_{A,B=1}^{n} \tilde{H}^{AB} \dot{q}_A \dot{q}_B = \dot{\psi}^2 - \frac{1}{2}\ddot{\psi} - \frac{1}{4}B_0^2. \tag{4.30}$$

So far we have the following picture: the equations of motion for the q_A are given by the set of coupled equations (4.24). The solutions $q_A(\tau)$ of (4.24) should then satisfy the two constraints (4.29) and (4.30).

We proceed by imposing that the q_A are all proportional, which will in turn mean that all of the physical scalar fields z^A are proportional to one another. Specifically, we set $q_A(\tau) = \xi_A q(\tau)$

for some constants ξ_A. In terms of this ansatz, the constraints (4.30) and (the derivative of) (4.29) become

$$3\left(\frac{\dot{q}}{q}\right)^2 = 4\dot{\psi}^2 - 2\ddot{\psi} - B_0^2, \quad 3\left(\frac{\dot{q}}{q}\right) = -2\dot{\psi} + a_0. \tag{4.31}$$

We have made use here of the homogeneity properties of f and the metric $\tilde{H}^{ab}$, as well as the identity $\tilde{H}_{ab}(q)q^a q^b = 1$ [94] which implies, for the models at hand, that

$$\sum_{A,B=1}^{n} \tilde{H}^{AB}(\xi)\xi_A\xi_B = \frac{3}{4}. \tag{4.32}$$

The two equations (4.31) can be combined into a second-order non-linear differential equation for $\psi(\tau)$:

$$\ddot{\psi} - \frac{4}{3}\dot{\psi}^2 - \frac{2}{3}a_0\dot{\psi} + \frac{1}{2}B_0^2 + \frac{1}{6}a_0^2 = 0. \tag{4.33}$$

Introducing the variable

$$y \equiv \exp\left(-\frac{4}{3}\psi - \frac{1}{3}a_0\tau\right),$$

this becomes

$$\ddot{y} - \omega^2 y = 0,$$

for

$$\omega^2 = \frac{2}{3}B_0^2 + \frac{1}{3}a_0^2, \tag{4.34}$$

and hence can be solved by

$$\exp\left(-\frac{4}{3}\psi - \frac{1}{3}a_0\tau\right) = \frac{\alpha}{\omega}\sinh\left(\omega\tau + \omega\beta\right), \tag{4.35}$$

where α and β are integration constants, and we have taken ω to be the positive root without loss of generality. Note that the right hand side should be non-negative for all $\tau > 0$, and hence we should pick $\alpha > 0$ and $\beta \geq 0$. The solution (4.35) now determines the function $\psi(\tau)$ appearing in the metric ansatz in terms of some integration constants, which we will fix based on regularity constraints in Section 4.2.3.

We can now use (4.35) to find an expression for $q(\tau)$. Indeed, differentiating (4.35) with respect to τ and substituting into the second equation in (4.31) we obtain

$$\frac{\dot{q}}{q} = \frac{1}{2}\omega\coth(\omega\tau + \omega\beta) + \frac{1}{2}a_0. \tag{4.36}$$

This can be integrated up to find

$$q(\tau) = \Lambda e^{\frac{1}{2}a_0\tau}\left(\sinh(\omega\tau + \omega\beta)\right)^{\frac{1}{2}}, \tag{4.37}$$

where Λ is an integration constant. Since we have set all of the q_A proportional to each other, we can therefore write

$$q_A = \lambda_A e^{\frac{1}{2}a_0\tau} \left(\sinh(\omega\tau + \omega\beta)\right)^{\frac{1}{2}}, \tag{4.38}$$

for some constants $\xi_A \equiv \lambda_A/\Lambda$. If we now impose

$$q_1 g_1 = q_2 g_2 = \ldots = q_n g_n, \tag{4.39}$$

we find that the q_A equation of motion is satisfied provided the integration constants λ_A are related to the electric FI parameters g_A via (see Appendix D.3)

$$\lambda_A = \pm \frac{3}{8ng_A} \left(\frac{\alpha^3}{\omega}\right)^{\frac{1}{2}}. \tag{4.40}$$

Returning to (4.29) then determines the constant b_0 in terms of α and the FI parameters g_A as

$$e^{b_0} = \pm \left(\frac{3\alpha}{8n}\right)^3 f\left(\frac{1}{g_1}, \ldots, \frac{1}{g_n}\right).$$

Finally, the Kaluza-Klein scalar ϕ appearing in the metric ansatz (4.3) is determined in terms of the q_a via the D-gauge condition (3.77) and the explicit form of the Hesse potential (4.16).

To summarise, we find that the scalars q_a are given by

$$q_0 = \pm - \frac{Q_0}{B_0} \sinh\left(B_0\tau + B_0 \frac{h_0}{Q_0}\right), \tag{4.41}$$

$$q_A = \pm \frac{3}{8ng_A} \left(\frac{\alpha^3}{\omega}\right)^{\frac{1}{2}} e^{\frac{1}{2}a_0\tau} \left(\sinh(\omega\tau + \omega\beta)\right)^{\frac{1}{2}} \quad \text{for} \quad A = 1, \ldots, n, \tag{4.42}$$

whilst the metric degrees of freedom are given by

$$e^{-4\psi} = \left(\frac{\alpha}{\omega}\right)^3 \sinh^3(\omega\tau + \omega\beta) e^{a_0\tau}, \tag{4.43}$$

$$e^{\phi} = \frac{1}{2}(-q_0)^{-\frac{1}{2}} (f(q_1, \ldots, q_n))^{-\frac{1}{2}}. \tag{4.44}$$

The $\pm$ signs in (4.41)–(4.42) should be chosen such that the function e^{ϕ} is well-defined.

At present, having fixed the charge and FI parameter configuration, our solution depends on a choice of model, i.e. a choice of the function f, as well as the six parameters $B_0, h_0, a_0, \omega, \alpha, \beta$.[38] However, we shall see below that many of these parameters can be fixed either by regularity constraints or by exploiting scaling freedoms.

[38] See Section 4.3 for an example of a solution with a different charge and FI parameter configuration.

4.2.3 The Nernst brane solution

In this section we want to look at the conditions on the various integration constants which give rise to regular black brane solutions in four dimensions. In particular, we impose that our solution has finite entropy density, which is the relevant regularity condition for solutions with non-compact horizon.

Let us recall the form of the four-dimensional metric in the τ coordinates:

$$ds_4^2 = -e^{\phi}dt^2 + e^{-\phi+4\psi}d\tau^2 + e^{-\phi+2\psi}(dx^2 + dy^2). \tag{4.45}$$

We will see below that for a suitable choice of integration constants $\tau = \infty$ is an event horizon, while $\tau \to 0$ is the asymptotic regime at infinite distance. The regularity of the solution within the bulk between horizon and infinity depends on the detailed properties of the function f. In particular, when evaluating f on the solution, we require that it has neither zeroes (so that there are in particular no changes of sign of e^{ϕ}) nor poles. Given the experience with similar issues for black hole solutions and domain walls, one expects that such solutions exist for any prepotential arising in string theory upon suitable restriction of the integration constants [154, 155]. In any case, such questions can only be investigated explicitly on a case-by-case basis, while we restrict ourselves to questions that can be answered irrespective of the choice of f.

The position of the event horizon can be found by looking at the value of τ for which the norm of the Killing vector field $k = \partial_t$ vanishes. Since $k^2 = g_{tt} = -e^{\phi} \sim \exp(-\frac{1}{2}B_0\tau - \frac{3}{4}a_0\tau - \frac{3}{4}\omega\tau)$ as $\tau \to \infty$, we can identify the horizon with the limiting value $\tau \to \infty$ provided $a_0 \geq 0$. If $a_0 < 0$ then the position of the horizon will change depending on the relative magnitudes of $|a_0|$ and B_0, and so we will take $a_0 \geq 0$ in what follows.

The area of the horizon is given by

$$\int dxdy\, e^{-\phi+2\psi}\Big|_{\tau\to\infty},$$

which is divergent since the x and y coordinates are non-compact. However, we can still define a finite entropy density s provided the factor $e^{-\phi+2\psi}\Big|_{\tau\to\infty}$ remains finite. From the expressions (4.43)–(4.44) one can show that in this limit we have

$$e^{-\phi+2\psi}\Big|_{\tau\to\infty} \sim \exp\left(\frac{1}{2}B_0\tau + \frac{1}{4}a_0\tau - \frac{3}{4}\omega\tau\right).$$

In order that this be finite and non-zero at the horizon we therefore require

$$\frac{1}{2}B_0 + \frac{1}{4}a_0 = \frac{3}{4}\omega,$$

which turns out, using (4.34), to be equivalent to fixing $a_0 = B_0$. Note that in this case we

likewise have $\omega = B_0$.

We still at this stage have four integration constants h_0, B_0, α, β which are *a priori* yet to be determined. However, note that we can always absorb β into a shift of τ and a redefinition of the constants α and h_0. Indeed, it will be useful to set $\beta = 0$ at this stage so that the asymptotic region of the solution is at $\tau = 0$

Moreover, we see that in the extremal $B_0 \to 0$ limit, the expression (4.35) becomes $e^{-4/3\psi} = \alpha\tau$. Hence, we can scale τ to set $\alpha = 1$. We are therefore left with a two-parameter family of solutions to the three-dimensional equations of motion, parametrized by B_0 and h_0, which we will interpret in terms of thermodynamic quantities in Section 4.2.4.

Before moving on to study properties of the solution, we summarise the results so far: the scalars q_a and $\hat{q}_a$ are given by

$$q_0 = \pm - \frac{Q_0}{B_0} \sinh\left(B_0\tau + B_0\frac{h_0}{Q_0}\right), \tag{4.46}$$

$$q_A = \pm\frac{3}{8ng_A} B_0^{-\frac{1}{2}} e^{\frac{1}{2}B_0\tau} \left(\sinh(B_0\tau)\right)^{\frac{1}{2}} \quad \text{for} \quad A = 1, \ldots, n, \tag{4.47}$$

$$\dot{\hat{q}}_0 = -Q_0, \tag{4.48}$$

whilst the metric degrees of freedom are given by

$$e^{-4\psi} = \frac{1}{B_0^3} \sinh^3(B_0\tau) e^{B_0\tau}, \tag{4.49}$$

$$e^{\phi} = \frac{1}{2}(-q_0)^{-\frac{1}{2}} (f(q_1, \ldots, q_n))^{-\frac{1}{2}}. \tag{4.50}$$

The physical scalar fields $z^A = Y^A/Y^0$ can be determined from the expressions

$$Y^A = -\frac{i}{2} e^{\phi} q_A, \quad Y^0 = -\frac{1}{4q_0}, \tag{4.51}$$

which were obtained in [3]. We find

$$z^A = -i \left(\frac{-q_0 q_A^2}{f(q_1, \ldots, q_n)}\right)^{\frac{1}{2}}. \tag{4.52}$$

Note that for $B_0 \neq 0$, q_0 and q_A all behave as $\exp(B_0\tau)$ when $\tau \to \infty$. We will show in the following section that this implies that the physical scalar fields take finite values on the horizon for $B_0 \neq 0$.

4.2.4 Properties of the Nernst brane solution

We now turn to an analysis of various properties of the solution obtained in Section 4.2.3, postponing a fuller discussion to Section 4.4.

A coordinate change

It is convenient to introduce the radial coordinate ρ via

$$e^{-2B_0\tau} = 1 - \frac{2B_0}{\rho} \equiv W(\rho).$$

With this definition, the asymptotic region is situated at $\rho \to \infty$, while the horizon is at $\rho = 2B_0$. In terms of ρ, we find the expressions

$$q_0 = \pm\frac{\mathcal{H}_0}{W^{1/2}}, \quad \text{and} \quad q_A = \pm\frac{3}{8ng_A}(\rho W)^{-1/2} \quad \text{for} \quad A = 1, \ldots, n,$$

where we have introduced the function[39]

$$\mathcal{H}_0(\rho) = -\left[\frac{Q_0}{B_0}\sinh\left(\frac{B_0 h_0}{Q_0}\right) + \frac{Q_0 e^{-\frac{B_0 h_0}{Q_0}}}{\rho}\right].$$

The physical scalar fields $z^A(\rho)$ then take the form

$$z^A = -i\left(\pm\frac{8n}{3g_A^2} f\left(\frac{1}{g_1}, \ldots, \frac{1}{g_n}\right)^{-1} \rho^{1/2}\mathcal{H}_0\right)^{\frac{1}{2}}. \tag{4.53}$$

Hence, for $h_0 \neq 0$ we find the asymptotic behaviour $z^A \sim \rho^{1/4}$, whilst for $h_0 = 0$ we find $z^A \sim \rho^{-1/4}$.

The four-dimensional line element (4.45) becomes

$$ds_4^2 = -\mathcal{H}^{-\frac{1}{2}} W \rho^{\frac{3}{4}} dt^2 + \mathcal{H}^{\frac{1}{2}} \rho^{-\frac{7}{4}} \frac{d\rho^2}{W} + \mathcal{H}^{\frac{1}{2}} \rho^{\frac{3}{4}} (dx^2 + dy^2), \tag{4.54}$$

where we have found it convenient to define

$$\mathcal{H}(\rho) \equiv \pm 4\left(\frac{3}{8n}\right)^3 f\left(\frac{1}{g_1}, \ldots, \frac{1}{g_n}\right)\mathcal{H}_0(\rho). \tag{4.55}$$

From this form of the metric, it is clear that the limit $B_0 \to 0$ can be achieved simply by setting $W = 1$ and

$$\mathcal{H}_{0|\text{ext}} = -\left(h_0 + \frac{Q_0}{\rho}\right).$$

In this case we reproduce the extremal Nernst brane solutions of [24], albeit in different coordinates. This identifies B_0 as a parameter encoding the non-extremality of the solution.

For $h_0 = 0$, the harmonic function for both the extremal and non-extremal solutions becomes

[39] We follow the sign conventions of [3]. See in particular Section 5.3.1 of [3] for a comparison of conventions for the STU-model.

$\mathcal{H}_0(\rho) = -Q_0/\rho$. The line element (4.54) then becomes

$$ds^2_{4|h_0=0} = -Z^{-\frac{1}{2}} W \rho^{\frac{5}{4}} dt^2 + Z^{\frac{1}{2}} \rho^{-\frac{9}{4}} \frac{d\rho^2}{W} + Z^{\frac{1}{2}} \rho^{\frac{1}{4}} (dx^2 + dy^2), \tag{4.56}$$

where we have defined

$$Z \equiv \pm 4 \left(\frac{3}{8n}\right)^3 Q_0 f\left(\frac{1}{g_1}, \ldots, \frac{1}{g_n}\right),$$

with the sign chosen such that Z is positive. The corresponding extremal solution can be obtained by setting the 'blackening factor' $W = 1$ in (4.56).

We have now explained how we can determine what the black brane metric will look like in all four possible cases depending on whether the two parameters B_0, h_0 are zero or non-zero.

Near horizon behaviour

To investigate the near horizon behaviour of the line element (4.54), we define $r^2 \equiv \rho - 2B_0$ and zoom in on the region $r \approx 0$. We then find that for $B_0 \neq 0$ the near horizon metric looks like

$$\begin{aligned} ds_4^2 \;=\; & -\left(Ze^{\frac{B_0 h_0}{Q_0}}\right)^{-1/2} (2B_0)^{1/4} r^2 dt^2 + 4\left(Ze^{\frac{B_0 h_0}{Q_0}}\right)^{1/2} (2B_0)^{-5/4} dr^2 \\ & + \left(Ze^{\frac{B_0 h_0}{Q_0}}\right)^{1/2} (2B_0)^{1/4} (dx^2 + dy^2), \end{aligned} \tag{4.57}$$

which is the product of a two-dimensional Rindler spacetime with two-dimensional flat space. We also include, for comparison, the near horizon behaviour of the extremal solution which, after putting $\rho = R^{-4}$, becomes

$$ds^2_{4|\text{Ext}} = \frac{1}{R}\left[-\frac{1}{R^4} Z^{-\frac{1}{2}} dt^2 + 16 Z^{\frac{1}{2}} dR^2 + Z^{\frac{1}{2}} (dx^2 + dy^2)\right]. \tag{4.58}$$

By Wick rotating to Euclidean time $t \to t_E = it$ in (4.57) and enforcing regularity of the t_E circle we can read off the temperature

$$4\pi T = Z^{-1/2} (2B_0)^{3/4} e^{-\frac{B_0 h_0}{2Q_0}}. \tag{4.59}$$

We can also read off from (4.57) the entropy density of the solution, which is given by

$$s = Z^{1/2} (2B_0)^{1/4} e^{\frac{B_0 h_0}{2Q_0}}. \tag{4.60}$$

Note that from (4.59) and (4.60) we can eliminate the integration constant B_0 in terms of the thermodynamic quantities s and T via.

$$B_0 = 2\pi s T. \tag{4.61}$$

Asymptotic behaviour

We now turn to a consideration of the asymptotic $\rho \to \infty$ properties of the line element (4.54), which for $h_0 \neq 0$ becomes

$$ds^2_{4|\text{asymp}} = \mathcal{H}(\infty)^{\frac{1}{2}} \rho^{\frac{1}{4}} \left[-\frac{1}{\mathcal{H}(\infty)} \rho^{\frac{1}{2}} dt^2 + \frac{d\rho^2}{\rho^2} + \rho^{\frac{1}{2}} (dx^2 + dy^2) \right].$$

Note that this is the same for both the extremal and non-extremal solutions. Making the coordinate change $\rho = R^{-4}$ then brings this to the form

$$ds^2_{4|\text{asymp}} = \frac{1}{R^3} \left[-\mathcal{H}(\infty)^{-\frac{1}{2}} dt^2 + 16\mathcal{H}(\infty)^{\frac{1}{2}} dR^2 + \mathcal{H}(\infty)^{\frac{1}{2}} (dx^2 + dy^2) \right], \tag{4.62}$$

which, by comparison with (3.123), is conformally AdS_4 with boundary at $R = 0$.

For the case $h_0 = 0$, the asymptotic limit corresponds simply to $W \to 1$ in (4.56), from which we find the asymptotic line element (4.58), after a suitable coordinate redefinition.

Chemical potential

The gauge field strength $F^0_{\tau t}$ is determined from the scalar field $\hat{q}^0$ via (3.76):

$$\dot{A}^0_t = 2\dot{\hat{q}}^0 = 2\tilde{H}^{00}\dot{\hat{q}}_0 = -\frac{Q_0}{2q_0^2}. \tag{4.63}$$

Substituting in the expression (4.46) and integrating with respect to τ gives

$$A^0_t(\tau) = \frac{1}{2}\left(\frac{B_0}{Q_0}\right)\left[\coth\left(B_0\tau + \frac{B_0 h_0}{Q_0}\right) - 1\right], \tag{4.64}$$

where we have chosen the integration constant such that $A_t(\infty) = 0$, i.e. that the gauge fields vanish on the horizon, for reasons discussed in Section 3.7.1. The chemical potential μ is then given by the asymptotic value of A_t [13],

$$\mu \equiv A_t(0) = \frac{1}{2}\left(\frac{B_0}{Q_0}\right)\left[\coth\left(\frac{B_0 h_0}{Q_0}\right) - 1\right], \tag{4.65}$$

which diverges as $h_0 \to 0$. Note that in the extremal limit $B_0 \to 0$ with $h_0 \neq 0$ we get $\mu_{\text{ext}} = 1/(2h_0)$. We note that $\text{sign}(\mu) = \text{sign}(Q_0)$.

Thermodynamics and the Nernst Law

We are now in a position to relate the integration constants B_0 and h_0 appearing in our solution to the thermodynamic quantities s, T and μ. In particular, we can rearrange (4.65) to find

$$e^{\frac{2B_0 h_0}{Q_0}} = 1 + \frac{B_0}{Q_0\mu} = 1 + \frac{2\pi s T}{Q_0\mu},$$

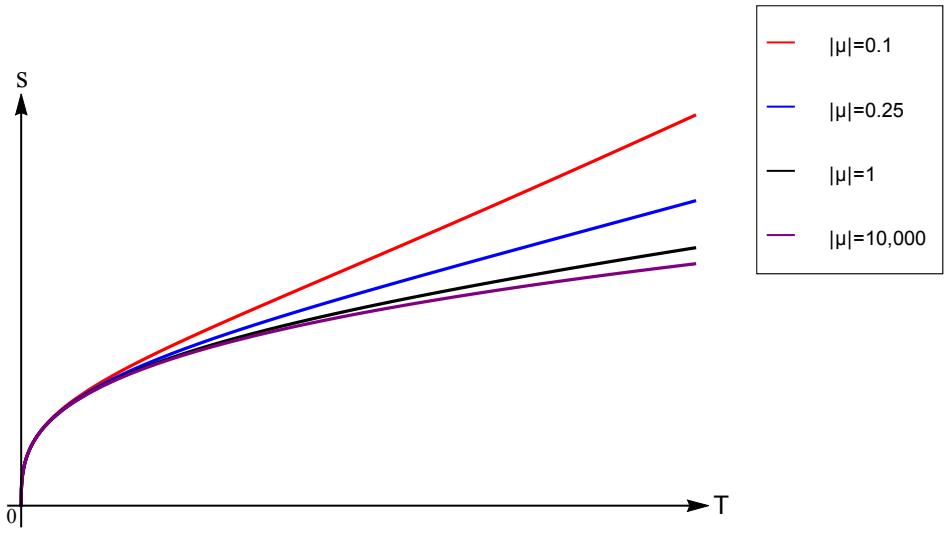


Figure 4.1: Mathematica plot of (4.66), showing how entropy density s varies with temperature T for various values of the chemical potential μ, and with Q_0 and Z fixed.

where we have used (4.61). Returning to (4.60) we then find an *equation of state* determining the entropy density as a function of the electric charge Q_0, FI parameters $g_1, \ldots, g_n$, temperature T and chemical potential μ of the black brane:

$$s^3 = 4\pi Z^2 T \left(1 + \frac{2\pi s T}{Q_0 \mu}\right). \tag{4.66}$$

One consequence of (4.66) is that, if we keep Z, Q_0 and μ fixed and send $T \to 0$, we see that $s \to 0$, which is precisely the strong (Planckian) formulation of the third law of thermodynamics [19]. This identifies the solution constructed in Section 4.2.3 as a non-extremal ('hot') Nernst brane.

We can further analyse (4.66) by looking at the dimensionless ratio T/μ. When T/μ is small, the second term in (4.66) becomes negligible, and we find that the entropy density behaves as $s \sim T^{1/3}$. On the other hand, when T/μ becomes large, the second term in (4.66) dominates, and we find the behaviour $s \sim T$.

In Figure 4.1 we plot equation (4.66) for various values of μ, keeping Q_0 and Z fixed. This shows a) the Nernst Law behaviour $s \to 0$ as $T \to 0$, and b) the crossover from the behaviour $s \sim T^{1/3}$ to $s \sim T$.

4.3 A magnetic black brane

We now turn our attention to a simple reformulation of the procedure in Section 4.2 which for a certain class of prepotentials allows us to construct non-extremal black branes carrying magnetic charge. We will here simply present the supergravity solution, and leave a fuller discussion of the thermodynamics of magnetically-charged black branes for future work.

In particular, we are interested in prepotentials for which one of the fields $Y^1, \ldots, Y^n$ decouples from the others. Without loss of generality, we can assume that Y^1 decouples, and consider prepotentials of the form

$$F(Y) = \left(\frac{Y^1}{Y^0}\right) \tilde{f}(Y^2, \ldots, Y^n),$$

where the function $\tilde{f}$ is homogeneous of degree 2. This class is particularly interesting from the perspective of embedding the model into string theory as it contains the tree-level heterotic prepotentials, which are linear in the heterotic dilaton Y^1/Y^0. We consider black brane solutions which are supported by a single magnetic charge P^1, a magnetic FI parameter g^0, and electric FI parameters $g_2, \ldots, g_n$.

In this case we see that the equations of motion can be solved in precisely the same way as in Section 4.2, with the field q_1 and magnetic charge P^1 playing the role of q_0 and Q_0 in the preceding section. In particular, we have

$$q_1(\tau) = \pm \frac{P^1}{B_0} \sinh\left(B_0 \tau + B_0 \frac{h^1}{P^1}\right),$$

whilst q_0 and $q_2, \ldots, q_n$ take the same form as (4.42) after replacing g_1 with g^0 in the obvious place. Moreover, the function ψ remains unchanged and, since

$$e^{\phi} = \frac{1}{2}(-q_0 q_1 \tilde{f}(q_2, \ldots, q_n))^{-\frac{1}{2}},$$

is symmetric in q_0 and q_1, we find that the line element takes the same form as in Section 4.2. Looking at the near horizon behaviour we again find that regularity of the solution imposes the same relation between the integration constants, $a_0 = B_0$, as before. The entropy density is therefore

$$s = Z^{1/2}(2B_0)^{1/4} e^{\frac{B_0 h^1}{2P^1}},$$

whilst the temperature of the solution is given by

$$4\pi T = Z^{-1/2}(2B_0)^{3/4} e^{-\frac{B_0 h^1}{2P^1}}.$$

4.4 Discussion

In this chapter we have provided a new technique for the construction of non-extremal black brane solutions to large classes of $\mathcal{N} = 2$ $U(1)$ gauged supergravity models in four dimensions, utilising the techniques of time-like dimensional reduction followed by a rewriting of the effective three-dimensional degrees of freedom through the real formulation of special geometry. In Section 4.2 we explicitly constructed a family of non-extremal black branes supported by a single electric charge and an arbitrary number of electric FI parameters. This family of branes

has an entropy density behaving as $s \sim T^{1/3}$ for $T \to 0$, which therefore vanishes at $T = 0$, where we recover the extremal Nernst brane solutions of [24]. We anticipate that such non-extremal Nernst branes will have interesting applications in the context of holography, where they could prove useful in describing dual field theory configurations at finite temperature and chemical potential which satisfy the Nernst Law.

One issue with regards to a holographic interpretation is that our solutions do not fit naturally into the framework of AdS/CMT, since they do not asymptote to AdS_4, but rather conformal AdS_4, as seen in (4.62). The presence of such hyperscaling violating Lifshitz (hvLif) asymptotics means the stress tensor of the dual field theory in the UV would not be scale invariant. Whilst this might appear to be an undesirable property for a field theory, we remarked in Section 3.6 that such behaviour is often found in condensed matter systems and there has been much progress recently in understanding the holographic relationship between hvLif theories of gravity and their dual field theories (see [14, 15, 34] for further details).

By looking at the near horizon and boundary behaviour of our solutions, we see that the Nernst brane interpolates between two hvLif geometries (3.140) with $D = 2$. There are four cases of interest, corresponding to whether h_0 and B_0 are zero or non-zero:

- $h_0 = 0$, $B_0 = 0$: The solution becomes globally hvLif (4.58) with $(z, \theta) = (3, 1)$. It has zero temperature and infinite chemical potential.

- $h_0 = 0$, $B_0 \neq 0$: The solution (4.56) has finite temperature and infinite chemical potential, and interpolates between a near horizon Rindler geometry (4.57), with $(z, \theta) = (0, 2)$, and an asymptotic hvLif geometry with $(z, \theta) = (3, 1)$.

- $h_0 \neq 0$, $B_0 = 0$: The solution has zero temperature and a finite chemical potential. It interpolates between a hvLif geometry with $(z, \theta) = (3, 1)$ at the horizon, and the conformal AdS_4 geometry (4.62) with $(z, \theta) = (1, -1)$ at infinity. This is the Nernst brane solution of [24].

- $h_0 \neq 0$, $B_0 \neq 0$: The solution (4.54) has finite temperature and chemical potential, and interpolates between a near horizon Rindler geometry with $(z, \theta) = (0, 2)$ and the conformal AdS_4 geometry with $(z, \theta) = (1, -1)$ at infinity.

Note that all of these values are consistent with the constraints imposed by the Null Energy Condition (3.142) for hvLif spacetimes. We have therefore found, *analytically*, a family of solutions which interpolate between two hvLif geometries. This family is parametrized by the two integration constants B_0 and h_0, or equivalently by the temperature T and chemical potential μ of the solution, both of which can be freely varied. Both parameters have a distinct effect on the near horizon and asymptotic forms of the solution: while the extremal or zero temperature limit $B_0 \to 0$ changes the near horizon solution from $(z, \theta) = (0, 2)$ to $(z, \theta) = (3, 1)$, the

infinite chemical potential limit $h_0 \to 0$ changes the geometry at infinity from $(z, \theta) = (1, -1)$ to $(z, \theta) = (3, 1)$. If both limits are performed we obtain a global hvLif solution with $(z, \theta) = (3, 1)$ which we interpret as the ground state of the given charge sector. Note that like any Lifshitz solution different from AdS it is not geodesically complete, and that the scalars are non-constant and run off to zero or infinity in the asymptotic regions. However, a similar behaviour can occur for domain wall solutions in gauged supergravity which, for lack of more symmetric solutions, are interpreted as ground states. Sometimes this interpretation can be further justified by an embedding into string theory or M-theory, see for example [156]. While we leave studying the string theory embedding of our solutions for future work, we remark that the interpretation is consistent with a limit where the temperature is zero and the chemical potential infinite.

Since so far solutions interpolating between hvLif geometries have only been found by relying on a mixture of analytical and numerical methods, we have made a significant step forward, and expect that the techniques used and described in this chapter will be useful in making further progress. While we leave searching for a concrete holographic dual of the bulk geometries presented in this chapter to future work, we can already make some interesting observations which shed some light on the properties which such a putative dual theory might possess.

Let us first consider the extremal ($B_0 = 0$) solution with $h_0 = 0$. Since this is the gravitational ground state solution with $(z, \theta) = (3, 1)$, zero temperature and infinite chemical potential, we expect it to be dual to the ground state of a $(2 + 1)$-dimensional QFT with hyperscaling exponent $\theta = 1$ and Lifshitz exponent $z = 3$. We remark that the specific value $\theta = 1$ for a QFT in $d = 2$ space dimensions seems to be required for the description of states with hidden Fermi surfaces as discussed in Section 3.6, although a three-loop calculation gives $z = \frac{3}{2}$ rather than $z = 3$ [15].

Now consider turning on some finite temperature $T > 0$ on the field theory side. A simple scaling argument, reviewed in Section 3.6, relates the entropy density of the thermal state to the temperature as $s \sim T^{\frac{d-\theta}{z}} = T^{1/3}$. We therefore expect that the non-extremal Nernst brane with $h_0 = 0$ in (4.56) provides us with the relevant gravity dual to the $(2+1)$-dimensional QFT with $\theta = 1$ and $z = 3$ at finite temperature. Indeed, taking $|\mu| \to \infty$ in the relation (4.66) we see that the entropy density of the brane solution is related to the temperature as $s \sim T^{1/3}$ which is the expected behaviour from the field theory arguments, and therefore consistent with our tentative interpretation.

We now move on to consider what happens at finite chemical potential $|\mu| < \infty$, which corresponds to $h_0 \neq 0$. In this case, the extremal Nernst brane interpolates between a hvLif geometry with $(z, \theta) = (3, 1)$ at the horizon, and a hvLif with $(z, \theta) = (1, -1)$ at infinity, which is conformal to AdS_4. One possible interpretation is as an RG flow between two QFTs: one with hyperscaling exponent $\theta = -1$ in the UV; and one with hyperscaling exponent $\theta = 1$ and Lifshitz exponent $z = 3$ in the IR. As the gravity solution is smooth, and we do not seem to have a

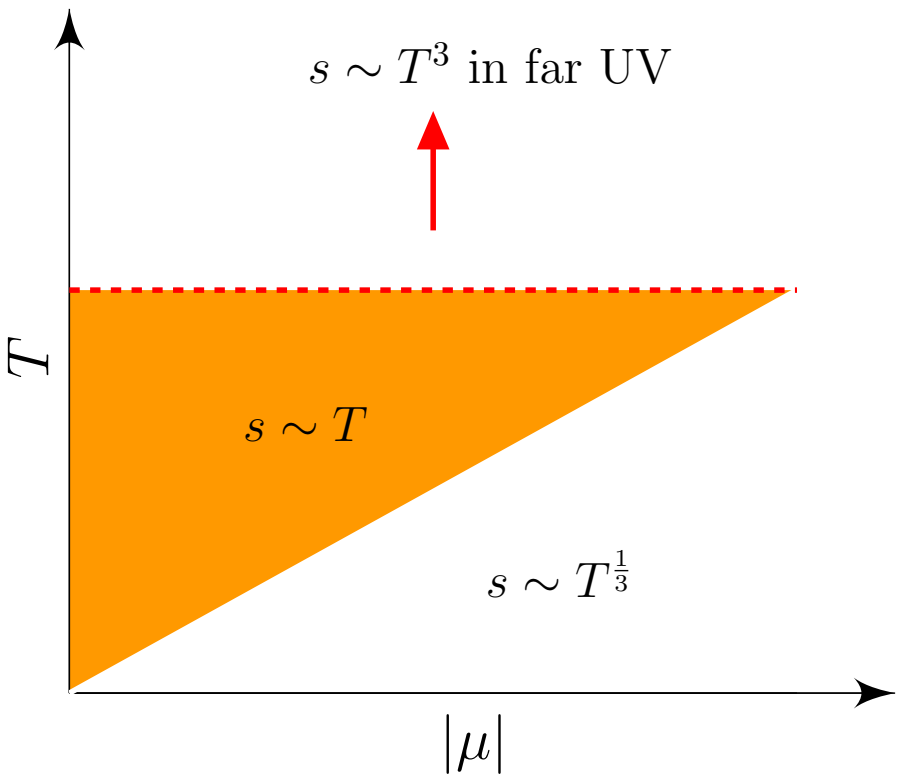


Figure 4.2: The holographic phase diagram for our family of Nernst brane solutions in terms of horizon temperature, T, and chemical potential, μ, which shows a smooth crossover between the two scaling regimes. We have also indicated that we anticipate a different scaling behaviour in the far UV where we don't expect that our supergravity solution accurately describes the tentative dual theory.

natural candidate for an order parameter identifying a phase transition, we think that the more likely interpretation is that the UV 'phase' and the IR 'phase' are related by smooth crossover. For the IR theory we expect that the entropy scales like $s \sim T^{\frac{d-\theta}{z}} = T^{\frac{1}{3}}$, which agrees with the behaviour of the Nernst brane solution for low temperature $\frac{T}{|\mu|} \ll 1$. Adding temperature changes the near horizon geometry, but leaves the asymptotic geometry at infinity unchanged, which is consistent with interpreting these configurations as thermal states. We therefore expect that the IR behaviour is correctly described by the Nernst brane solution, which in turn predicts a scaling $s \sim T$ of the entropy for high temperatures, $\frac{T}{|\mu|} \gg 1$. This however does not agree with the expected scaling of our tentative UV theory with $(z, \theta) = (1, -1)$, which predicts $s \sim T^3$. We also note that the asymptotic UV geometry, while conformal to AdS_4, cannot be interpreted as an alternative ground state of our supergravity theory, because it is not, when taken as a global geometry, part of our family of solutions. If we accept that the UV geometry correctly captures the thermodynamic behaviour then the corresponding UV theory should have a scaling behaviour $s \sim T^3$ $(z = 1, \theta = -1, d = 2)$. The resulting tentative phase diagram is shown in Figure 4.2.

We should also point out some issues with the interpretation of our solutions at either end of the RG flow. First we consider the situation in the deep IR (controlled by the parameter B_0), before looking at the two possible problems in the deep UV (controlled by the parameter h_0):

- Deep IR with $B_0 = 0$: if the temperature is strictly zero, the Nernst brane solution has infinite tidal forces and run-away behaviour of the scalars at the horizon in the extremal limit.[40] This indicates a breakdown of the effective description, and strictly speaking the supergravity solution should not be trusted.

- Deep IR with $B_0 \neq 0$: tidal forces are no longer divergent since the RG flow terminates on the horizon. However, for sufficiently small $B_0 \neq 0$, the tidal forces could be very large and remain significant at the horizon. We should exercise caution when considering solutions at very low temperatures.

- Deep UV with $h_0 \neq 0$: the physical scalar fields $z^A \sim \rho^{1/4}$ run off to infinity in the UV region. Taken together with the mismatch in the scaling of entropy with temperature, this indicates that additional degrees of freedom become relevant. For very special models that can be embedded into five dimensions, the four- and five-dimensional scalar fields are related through multiplication by the Kaluza-Klein scalar field. Since this is the field that controls the size of the compactification circle, the runaway behaviour of the four-dimensional scalars suggests we interpret the UV behaviour as a decompactification limit.

- Deep UV with $h_0 = 0$: in Chapter 5 we shall see that such solutions have an asymptotic curvature singularity. It is noted in [121] that this is a disaster for any possible holographic interpretation since gravity does not decouple at large distances. The refusal of gravity to die off far away from the black brane suggests there must be another factor influencing the geometry. Indeed, in Chapter 5 we will use the fact that the physical scalar fields $z^A \sim \rho^{-1/4}$ run to zero in the deep UV to understand the behaviour of the compactification circle and ultimately resolve this problem.

In order that our four-dimensional Nernst brane be taken seriously holographically, we must acknowledge that the problems at either end of the RG flow prevent us trusting a dual field theory in the deep IR or deep UV. Thus, as in the similar case of the holographic interpretation of hyperscaling violating solutions of Einstein-Maxwell-Dilaton theories [14], the Nernst brane solution is not a valid description of its (tentative) dual over the full range of the energy (radial coordinate) from the UV (infinity) to the IR (horizon), but only over a finite interval outside the horizon.

The incomplete description of the UV behaviour along with the evidence of decompactification suggests embedding the theory into a higher dimensional description. Since the class of

[40] This is observed for the four-dimensional Nernst solutions of [24] and we shall verify it explicitly for our four-dimensional solutions in Chapter 5.

prepotentials that we have considered in this chapter includes those 'very special' prepotentials for which the theory can be uplifted to five dimensions, the most obvious embedding is into five-dimensional supergravity. There are grounds to believe that the dimensional reduction of theories admitting AdS_D vacua would admit vacua with some non-trivial hvLif behaviour [34,35]. Therefore we expect that by lifting our solutions to five dimensions we will obtain new asymptotically AdS_5 finite temperature solutions in $\mathcal{N} = 2$ gauged supergravity which still satisfy the Nernst Law.[41] We remark that AdS_5 asymptotics lead to a scaling of the entropy $s \sim T^{\frac{D-\theta}{z}} = T^3$, $(z = 1, \theta = 0, D = 3)$, which is consistent with our proposed UV theory. This will be the central theme of Chapter 5, where we develop and analyse such five-dimensional Nernst solutions. In five dimensions we are granted access to the full degrees of freedom of the system, and are thus able to understand the origins of the problems facing the four-dimensional Nernst branes covered in this chapter and overcome them accordingly.

[41] Although examples of such asymptotically AdS_5 hot Nernst solutions were constructed in [151], their solutions do not reduce to the finite temperature five-dimensional solutions presented in Chapter 5.

5 Five-dimensional Nernst branes

This chapter is based on
P. Dempster, D. Errington, J. Gutowski and T. Mohaupt
"Five-dimensional Nernst branes from special geometry,"
JHEP **11** (2016) 114, [arXiv:1609.05062] [2].

Having seen strong evidence at the end of Chapter 4 supporting the decompactification of our four-dimensional Nernst brane solutions, we now arrive at another key result of this thesis. In this chapter, we shall construct a family of five-dimensional Nernst brane solutions using the q-map. Importantly, we can check that they consistently reduce to the solutions of Chapter 4, which is to be expected considering the 'very special' models we considered in four dimensions. We will also see that the five-dimensional solutions have AdS_5 asymptotics which gives a clearer picture of the brane thermodynamics as well as resolving some of the four-dimensional singular behaviour.

We begin with the five-dimensional Lagrangian for $\mathcal{N}=2$ supergravity coupled to $n_V^{(5)} = n_V^{(4)}+1$ vector multiplets and with a $U(1)$ Fayet-Iliopoulos gauging. To avoid a cluttered notation we will write $\tilde{n} = n_V^{(5)}$ throughout. In Chapter 4 we used $n = n_V^{(4)}$ and thus there are $\tilde{n} = n+1$ vector multiplets. For completeness, we repeat here the Lagrangian (3.117)

$$\begin{aligned} e_5^{-1}\mathcal{L}_5 \;=\; & -\frac{1}{2}R_{(5)} - \frac{3}{4}a_{ij}(h)\partial_{\hat{\mu}}h^i\partial^{\hat{\mu}}h^j - \frac{1}{4}a_{ij}(h)\mathcal{F}^i_{\hat{\mu}\hat{\nu}}\mathcal{F}^{j|\hat{\mu}\hat{\nu}} \\ & + \frac{1}{6\sqrt{6}}e_5^{-1}c_{ijk}\epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}}\mathcal{F}^i_{\hat{\mu}\hat{\nu}}\mathcal{F}^j_{\hat{\rho}\hat{\sigma}}\mathcal{A}^k_{\hat{\lambda}} + V_5(h), \end{aligned} \tag{5.1}$$

where $i = 1, \ldots, \tilde{n}+1$ and we have set $\kappa_5^2 = 8\pi G_5 = 1$. In five dimensions, the gauging is always electric and the scalar potential is given by (3.116) as

$$V_5(h) = 2\cdot 6^{-1/3}\left[(chhh)(ch)^{-1|ij} + 3h^ih^j\right]g_ig_j. \tag{5.2}$$

This chapter is organised as follows: we first dimensionally reduce the theory to three dimensions in Section 5.1. Then, in Section 5.2, we solve the equations of motion for the three-dimensional theory for general static field configurations, before dimensionally lifting this instanton solution to a family of regular, five-dimensional black branes and analysing their properties. In Section 5.3 we explore the relationship between the five-dimensional Nernst branes we have

just constructed and the four-dimensional solutions of Chapter 4. Lastly, Section 5.4 includes a brief summary of the results of this chapter.

5.1 Dimensional reduction

We want to reduce the five-dimensional Lorentzian theory to three Euclidean dimensions in order to write the equations of motion entirely using scalar fields. Following Section 3.3.3 we make the metric ansatz (3.85) subject to the rescaling (3.87). Since we first reduce over space ($\epsilon_1 = -1$) and then time ($\epsilon_2 = +1$), (3.85) can be expressed as

$$ds^2_{(5)} = 6^{-2/3}\sigma^2 \left(dx^0 + \mathcal{A}^0_4 dx^4\right)^2 - 6^{1/3}\left(\frac{\phi}{\sigma}\right)\left(dx^4\right)^2 + \frac{6^{1/3}}{\sigma\phi} ds^2_{(3)}, \tag{5.3}$$

where all fields depend only on the coordinates of the three-dimensional space. In addition we choose to switch off all of the five-dimensional gauge fields $\mathcal{A}^i = 0$, i.e. we look only for uncharged five-dimensional solutions. The presence of the Kaluza-Klein one-form $\mathcal{A}^0 = \mathcal{A}^0_4 dx^4 \equiv -\sqrt{2}\zeta^0 dx^4$ indicates that we are looking for non-static five-dimensional solutions. Upon compactification of the x^0 circle this will give rise to a non-trivial electric charge for the corresponding four-dimensional solution. Note that whilst the Killing vector $\partial/\partial x^0$ is always space-like in five dimensions, $\partial/\partial x^4$ can be either time-like, space-like, or null, depending on the magnitude of $\mathcal{A}^0_4$. However, after performing the dimensional reduction over x^0, the x^4 direction will always be time-like in four dimensions, and so we are able to use the same dimensional reduction technique as in [157], i.e. we reduce over both a space-like and a time-like direction.

Following (3.86), the resulting three-dimensional action is given by

$$e_3^{-1}\mathcal{L}_3 = -\frac{1}{2}R_{(3)} - \frac{3}{4}a_{ij}(h)\partial_\mu h^i \partial^\mu h^j - \frac{1}{4\phi^2}(\partial\phi)^2 - \frac{3}{4\sigma^2}(\partial\sigma)^2 + \frac{\sigma^3}{12\phi}(\partial\zeta^0)^2 + V_3(h), \tag{5.4}$$

where the three-dimensional scalar potential is given by

$$V_3(h) = \frac{6^{1/3}}{\sigma\phi}V_5(h) = \frac{2}{\sigma\phi}\left[(chhh)(ch)^{-1|ij} + 3h^i h^j\right] g_i g_j. \tag{5.5}$$

In order to solve the equations of motion resulting from (5.4) it is convenient to introduce the variables u, v and y^i via

$$\sigma = u^{-\frac{1}{2}}v^{-\frac{1}{2}}, \quad \phi = u^{\frac{1}{2}}v^{-\frac{3}{2}}, \quad y^i = vh^i, \quad \hat{g}_{ij}(y) = -\frac{3}{4v^2}a_{ij}(h), \tag{5.6}$$

so that the three-dimensional Lagrangian (5.4) becomes

$$e_3^{-1}\mathcal{L}_3 = -\frac{1}{2}R_{(3)} + \hat{g}_{ij}(y)\partial_\mu y^i \partial^\mu y^j - \frac{1}{4u^2}(\partial u)^2 + \frac{1}{12u^2}(\partial\zeta^0)^2 + V_3(y). \tag{5.7}$$

The scalar potential is given in terms of the new fields by

$$V_3(y) = 2\left[(cyyy)(cy)^{-1|ij} + 3y^i y^j\right] g_i g_j \tag{5.8}$$
$$= 3\left[\hat{g}^{ij}(y) + 4y^i y^j\right] g_i g_j \,. \tag{5.9}$$

The explicit steps used in getting to the second line are carried out in Appendix E.1.

We note that the Lagrangian (5.7) has no explicit dependence on the field v appearing in the metric ansatz. However, when taking the rescaled scalar fields y^i as independent variables, the field v can be recovered from the equation

$$v^3 = cyyy,$$

which follows from the hypersurface constraint $chhh = 1$. In terms of the new fields u and v, the five-dimensional metric ansatz (5.3) becomes

$$ds^2_{(5)} = \frac{6^{-2/3}}{uv}\left(dx^0 - \sqrt{2}\,\zeta^0 dx^4\right)^2 - 6^{1/3}\frac{u}{v}(dx^4)^2 + 6^{1/3}v^2 ds^2_{(3)}. \tag{5.10}$$

The rescaling (5.6) can be interpreted as combining the five-dimensional scalar fields parametrizing $\mathcal{H} \subset M$ with the scalar field v, which is a component of the five-dimensional metric. The resulting metric on the combined manifold $\mathcal{H} \times \mathbb{R}^{>0} \simeq \mathcal{H} \times \mathbb{R}$,

$$\hat{g}_{ij}(y) = \frac{3}{2}\left(\frac{(cy)_{ij}}{cyyy} - \frac{3}{2}\frac{(cyy)_i(cyy)_j}{(cyyy)^2}\right), \tag{5.11}$$

is, up to a constant factor, isometric to the positive definite Hessian metric (2.36) on the manifold M. As shown in [47] this metric is isometric to the product metric $g_{\mathcal{H}} + dr^2$ on $\mathcal{H} \times \mathbb{R}$. From (5.7) it is manifest that the scalar manifold $\widehat{Q}$ of our three-dimensional Lagrangian carries a product metric, with the first factor parametrized by y^i and the second factor parametrized by u and ζ^0. The combined scalar manifold has dimension $\tilde{n} + 1 + 2 = \tilde{n} + 3$.

As shown in Section 3.3.3, if we perform the reduction of five-dimensional supergravity with $\tilde{n}$ vector multiplets to three Euclidean dimensions without any truncation, then the resulting scalar manifold is a para-quaternionic-Kähler manifold $\bar{Q}_{PQK}$ of dimension $2(2\tilde{n}+2)+4 = 4\tilde{n}+8$ [54,100]. The submanifold $\widehat{Q} \subset \bar{Q}_{PQK}$ is obtained by a consistent truncation and therefore it is a totally geodesic submanifold of $\bar{Q}_{PQK}$. We remark that $\widehat{Q}$ is a (totally geodesic) submanifold of the up to $(2\tilde{n}+4)$-dimensional totally geodesic para-Kähler manifolds S_{PK} described in [3,157]. These manifolds occurred in cases where it was possible to obtain explicit stationary non-extremal solutions of four- and five-dimensional ungauged supergravity by dimensional reduction over time. As we will see in the following, it is still possible to obtain explicit solutions in the gauged case, where the field equations of the three-dimensional scalars are modified by a scalar

potential. The higher dimensional para-Kähler submanifolds S_{PK} will be relevant when the present work is extended to more general, charged solutions.

5.2 Five-dimensional Nernst branes

5.2.1 Solving the equations of motion

We now turn to the three-dimensional equations of motion coming from (5.7). The equations of motion for y^i, u and ζ^0 read:

$$\triangle_{(g)} y^i + \hat{\Gamma}^i_{jk}(y)\partial_\mu y^j \partial^\mu y^k + 3\hat{\Gamma}^i_{jk}(y)\hat{g}^{jm}(y)\hat{g}^{kn}(y) g_m g_n - 12(y^j g_j)\hat{g}^{ik}(y) g_k = 0, \tag{5.12}$$

$$\triangle_{(g)} u - \frac{1}{u}(\partial u)^2 - \frac{1}{3u}(\partial \zeta^0)^2 = 0, \tag{5.13}$$

$$\triangle_{(g)} \zeta^0 - \frac{2}{u}\partial_\mu u\, \partial^\mu \zeta^0 = 0, \tag{5.14}$$

where we have introduced the Christoffel symbols for the metric $\hat{g}_{ij}(y)$:

$$\hat{\Gamma}^i_{jk}(y) = \frac{1}{2}\hat{g}^{il}(y)\partial_l \hat{g}_{jk}(y).$$

Meanwhile, the Einstein equations read

$$\begin{aligned} &-\frac{1}{2}R_{(3)|\mu\nu} + \hat{g}_{ij}(y)\partial_\mu y^i \partial_\nu y^j - \frac{1}{4u^2}\partial_\mu u\, \partial_\nu u \\ &+ \frac{1}{12u^2}\partial_\mu \zeta^0 \partial_\nu \zeta^0 + 3g_{\mu\nu}\left[\hat{g}^{ij}(y) + 4y^i y^j\right] g_i g_j = 0. \end{aligned} \tag{5.15}$$

We now want to find a three-dimensional instanton solution to the equations of motion (5.12)–(5.15). We make the following ansatz for our three-dimensional line element:

$$ds^2_{(3)} = e^{4\psi} d\tau^2 + e^{2\psi}(dx^2 + dy^2), \tag{5.16}$$

where $\psi = \psi(\tau)$ is some function to be determined, and τ is a radial coordinate which parametrizes the direction orthogonal to the world-volume of the brane. Importantly, this is the same brane-like ansatz for the three-dimensional line element that we considered in (4.17) for the four-dimensional Nernst branes. Moreover we will impose that all of the fields y^i, ζ^0 and u depend only on τ. This coordinate has been chosen such that it is an affine parameter for the curves $C : \tau \mapsto (y^i(\tau), u(\tau), \zeta^0(\tau))$ on the scalar manifold $\widehat{Q} \subset \bar{Q}_{PQK}$.[42]

[42] As explained in Appendix B.2, the curves $C(\tau)$ are not necessarily geodesics on the scalar manifold. This is because it is possible for the scalar potential, produced by the FI gauging, to create a gradient force that deforms them away from geodesic motion.

The Ricci tensor has components

$$R_{\tau\tau} = 2\ddot{\psi} - 2\dot{\psi}^2, \quad R_{xx} = R_{yy} = e^{-2\psi}\ddot{\psi},$$

from which we find that the Einstein equations (5.15) become

$$V_3(y) = \frac{1}{2}e^{-4\psi}\ddot{\psi}, \tag{5.17}$$

for $\mu = \nu \neq \tau$, and

$$-\frac{1}{2}\ddot{\psi} + \dot{\psi}^2 = -\hat{g}_{ij}(y)\dot{y}^i\dot{y}^j + \frac{\dot{u}^2}{4u^2} - \frac{(\dot{\zeta}^0)^2}{12u^2}, \tag{5.18}$$

for $\mu = \nu = \tau$, where we have used (5.17). We will now consider the equations of motion for each of ζ^0, u and y^i in turn.

ζ^0 equation of motion

The equation of motion (5.14) for ζ^0 can be brought to the form

$$\frac{d}{d\tau}\left(\frac{1}{u^2}\dot{\zeta}^0\right) = 0,$$

which is solved by

$$\dot{\zeta}^0 = \sqrt{3}Du^2, \tag{5.19}$$

for some integration constant D, where we have chosen the factor for later convenience. Once we solve the equation of motion for u we will further integrate (5.19) to obtain an expression for the Kaluza-Klein vector $\mathcal{A}^0 = -\sqrt{2}\zeta^0$ appearing in the five-dimensional metric.

u equation of motion

Substituting (5.19) in to the equation of motion (5.13) for u we find

$$\ddot{u} - \frac{1}{u}\dot{u}^2 - D^2u^3 = 0. \tag{5.20}$$

Introducing the variable $\chi = u^{-1}$, this becomes

$$\ddot{\chi} - \frac{\dot{\chi}^2 - D^2}{\chi} = 0\,. \tag{5.21}$$

By differentiation we obtain the necessary condition $\dot{\chi}\ddot{\chi} = \chi\dddot{\chi}$, which can be integrated to $\ddot{\chi} = B_0^2\chi$, where B_0 is a real constant.[43] Parametrizing the general solution as

$$\chi(\tau) = A\cosh(B_0\tau) + \frac{B}{B_0}\sinh(B_0\tau), \tag{5.22}$$

with arbitrary constants A, B, and substituting back into the original equation (5.21) we find the constraint

$$D^2 = B^2 - B_0^2 A^2 \ ,$$

which imposes one relation between the four constants D, A, B, B_0. It will turn out to be useful in what follows to consider A, B_0 and $\Delta := B - B_0 A$ to be the independent quantities, and to write everything in terms of these. In particular, we then have $D^2 = \Delta(\Delta + 2B_0 A)$.

We are also now in a position to further integrate (5.19), which we write as

$$\dot{\zeta}^0 = \pm\frac{\sqrt{3\Delta(\Delta + 2B_0 A)}}{\chi^2}. \tag{5.23}$$

Substituting in (5.22) this can be integrated to find

$$\zeta^0(\tau) = \frac{\sqrt{3}\, B_0\, u(\tau)}{\sqrt{\Delta(\Delta + 2B_0 A)}}\left[A\sinh(B_0\tau) + \frac{B}{B_0}\cosh(B_0\tau)\right] - \zeta^0_\infty, \tag{5.24}$$

for some integration constant ζ^0_∞, which can be fixed by imposing a suitable physicality condition on the solution. Notice that the sign choice in (5.23) isn't present in the above expression for $\zeta^0(\tau)$: we have decided to focus on the positive solution in order to avoid the need to treat both cases in parallel, but of course, it is important to acknowledge the existence of an equivalent negative solution. At this point we anticipate that a horizon, if it exists, will turn out to be located at $\tau \to \infty$. Moreover, as we will show in Section 5.3, upon dimensional reduction we obtain a four-dimensional stationary (in fact static) solution with a Killing horizon, which therefore, for finite temperature, admits an analytic continuation to a bifurcate horizon [132]. In order that the four-dimensional one-form $\mathcal{A}^0(\tau)$ is well defined, it must vanish at the horizon [12, 158], as reviewed in Section 3.7.1.

This leads to

$$\zeta^0_\infty = \frac{\sqrt{3}\, B_0}{\sqrt{\Delta(\Delta + 2B_0 A)}},$$

and therefore the Kaluza-Klein one-form is given by

$$\mathcal{A}^0(\tau) = -\sqrt{\frac{6\Delta}{\Delta + 2B_0 A}}\, u(\tau) e^{-B_0\tau} dx^4. \tag{5.25}$$

[43] Negative B_0^2 would yield a solution periodic in τ, which we discard.

y^i equation of motion

The equation of motion (5.12) for the y^i becomes

$$e^{-4\psi}\ddot{y}^i + e^{-4\psi}\hat{\Gamma}^i_{jk}(y)\dot{y}^j\dot{y}^k + 3\hat{\Gamma}^i_{jk}(y)\hat{g}^{jm}(y)\hat{g}^{kn}(y)\, g_m g_n - 12\,\hat{g}^{ij}(y)\, g_j(y^k g_k) = 0. \tag{5.26}$$

To proceed, we first contract (5.26) with the dual scalar fields $y_i := -\hat{g}_{ij}(y)y^j$ and make use of the identity

$$\hat{\Gamma}^i_{jk}(y)y_i = \frac{1}{2}y_i\hat{g}^{il}(y)\partial_l\hat{g}_{jk}(y) = -\frac{1}{2}y^l\partial_l\hat{g}_{jk}(y) = \hat{g}_{jk}(y),$$

which follows from the fact that $\hat{g}_{ij}(y)$ is homogeneous of degree -2 in the y^i. We thus find

$$e^{-4\psi}\ddot{y}^i y_i + e^{-4\psi}\hat{g}_{ij}(y)\dot{y}^i\dot{y}^j + V_3(y) = 0, \tag{5.27}$$

which upon using (5.17) becomes

$$\ddot{y}^i y_i + \hat{g}_{ij}(y)\dot{y}^i\dot{y}^j = -\frac{1}{2}\ddot{\psi}. \tag{5.28}$$

Given that $\hat{g}_{ij}(y)\dot{y}^j = \dot{y}_i$, we can integrate (5.28) to find

$$\dot{y}^i y_i = -\frac{1}{2}\dot{\psi} + \frac{1}{4}a_0, \tag{5.29}$$

for some integration constant a_0, where the factor has been chosen for later convenience. Writing

$$\dot{y}^i y_i = \frac{3}{4}\frac{(cyy)_i\dot{y}^i}{cyyy} = \frac{1}{4}\frac{d}{d\tau}\left(\log cyyy\right),$$

we can integrate (5.29) further to obtain

$$\log cyyy = -2\psi + a_0\tau + b_0, \tag{5.30}$$

for an integration constant b_0. Again the prefactor has been chosen for later convenience. We now return to the Hamiltonian constraint (5.18). Using (5.22) and (5.19) this becomes:

$$-\frac{1}{2}\ddot{\psi} + \dot{\psi}^2 = \frac{1}{4}B_0^2 - \hat{g}_{ij}(y)\dot{y}^i\dot{y}^j. \tag{5.31}$$

We then have the following picture. The solutions $y^i(\tau)$ to (5.26) should satisfy the constraints (5.29) and (5.31). One way to proceed, which is valid for generic five-dimensional models and analogous to our approach in Chapter 4, is to set all of the y^i proportional to one another, i.e. we put $y^i = \xi^i y$ for some constants ξ^i, which satisfy

$$\hat{g}_{ij}(\xi)\xi^i\xi^j = -\frac{3}{4}.$$

Note that since the (constrained) scalar fields h^i can be recovered from the y^i via $h^i = (cyyy)^{-1/3}y^i$, we see that this ansatz will result in constant five-dimensional scalar fields.

With this assumption, the constraints (5.29) and (5.31) become

$$\frac{3}{4}\left(\frac{\dot{y}}{y}\right)^2 = -\frac{1}{2}\ddot{\psi} + \dot{\psi}^2 - \frac{1}{4}B_0^2, \tag{5.32}$$

$$\frac{3}{4}\left(\frac{\dot{y}}{y}\right) = -\frac{1}{2}\dot{\psi} + \frac{1}{4}a_0. \tag{5.33}$$

Eliminating the quantity $(\dot{y}/y)$ from (5.32)–(5.33) we obtain an equation for the function $\psi(\tau)$:

$$\ddot{\psi} - \frac{4}{3}\dot{\psi}^2 - \frac{2}{3}a_0\dot{\psi} + \frac{1}{2}B_0^2 + \frac{1}{6}a_0^2 = 0.$$

This is precisely the same equation as we found in (4.33) for our four-dimensional solutions, and so can be solved in the same way. Following (4.35) we introduce

$$e^{-4\psi} = \alpha^3 e^{a_0\tau}\left(\frac{\sinh(\omega\tau + \omega\beta)}{\omega}\right)^3, \tag{5.34}$$

for some integration constants α and β, where the quantity ω is given by

$$\omega^2 := \frac{2}{3}B_0^2 + \frac{1}{3}a_0^2. \tag{5.35}$$

We could then differentiate (5.34) and substitute into (5.33) to obtain

$$\frac{\dot{y}}{y} = \frac{1}{2}\omega\coth(\omega\tau + \omega\beta) + \frac{1}{2}a_0.$$

This can then be integrated up to find

$$y(\tau) = \Lambda e^{\frac{1}{2}a_0\tau}\left(\frac{\sinh(\omega\tau + \omega\beta)}{\omega}\right)^{\frac{1}{2}},$$

for some constant Λ, and hence the y^i are given by

$$y^i(\tau) = \lambda^i e^{\frac{1}{2}a_0\tau}\left(\frac{\sinh(\omega\tau + \omega\beta)}{\omega}\right)^{\frac{1}{2}}, \tag{5.36}$$

where we have defined $\lambda^i \equiv \xi^i/\Lambda$. Notice that the solution for the five-dimensional scalars y^i has exactly the same structure as that for the four-dimensional scalars q_A in (4.37).

Finally, we must ensure that the solution (5.36) satisfies the original equations of motion (5.26). This fixes λ^i in terms of the gauging parameters g_i and other integration constants as

$$\lambda^i = \pm\frac{3}{8\tilde{n}g_i}\left(\frac{\alpha^3}{\omega}\right)^{\frac{1}{2}}, \tag{5.37}$$

which has the same structure as the four-dimensional integration constants (4.40). Therefore the function v appearing in the line element (5.10) is given by

$$v(\tau) = (c\lambda\lambda\lambda)^{1/3} e^{\frac{1}{2}a_0\tau} \left(\frac{\sinh(\omega\tau + \omega\beta)}{\omega}\right)^{\frac{1}{2}} . \tag{5.38}$$

The signs in (5.37) should be chosen such that the function $v(\tau)$ is real and positive for all $\tau > 0$.

At this stage we have six independent integration constants $\alpha, \beta, a_0, A, B_0, \Delta$ which are *a priori* yet to be determined. However, following the treatment of our four-dimensional solutions in Section 4.2.3, we choose to set $\beta = 0$ in what follows so that the asymptotic region is at $\tau = 0$ and the near horizon region at $\tau \to \infty$. We can then scale τ to set $\alpha = 1$.

In order for our solution to make sense as a black brane in five dimensions, we need to impose some physicality constraints. In particular, we require that the five-dimensional solution have finite entropy density.[44] Combining the five-dimensional and three-dimensional metric ansätze (5.10) and (5.16) we see that finite entropy density corresponds to a finite value of $v^{3/2}u^{-1/2}e^{2\psi}$ as $\tau \to \infty$ (i.e. at the horizon). To leading order we find

$$v^{3/2}u^{-1/2}e^{2\psi}\Big|_{\tau\to\infty} \sim \exp\left(\frac{1}{4}a_0\tau - \frac{3}{4}\omega\tau + \frac{1}{2}B_0\tau\right).$$

In order that this be finite and non-zero we therefore require $3\omega = a_0 + 2B_0$ which, given (5.35), is equivalent to $a_0 = B_0$, further resulting in $\omega = B_0$. Hence, this physicality constraint further reduces the number of independent integration constants by one.

Before moving on to study properties of the solution, we summarise the story so far. The functions appearing in the five-dimensional line element (5.10) are given by

$$v(\tau) = (c\lambda\lambda\lambda)^{1/3} e^{\frac{1}{2}B_0\tau} \left(\frac{\sinh(B_0\tau)}{B_0}\right)^{\frac{1}{2}} , \tag{5.39}$$

$$u(\tau) = \chi(\tau)^{-1}, \quad \chi(\tau) = A\cosh(B_0\tau) + \frac{B}{B_0}\sinh(B_0\tau), \tag{5.40}$$

$$e^{-4\psi} = e^{B_0\tau}\left(\frac{\sinh(B_0\tau)}{B_0}\right)^3 , \tag{5.41}$$

$$\mathcal{A}^0(\tau) = -\sqrt{\frac{6\Delta}{\Delta + 2B_0A}}\, u(\tau)e^{-B_0\tau}dx^4, \tag{5.42}$$

whilst the scalar fields h^i parametrizing the CASR manifold are constant and given by

$$h^i = \frac{1}{v}y^i = (c\lambda\lambda\lambda)^{-1/3}\lambda^i = \frac{1}{g_i}\left(c_{lmn}g_l^{-1}g_m^{-1}g_n^{-1}\right)^{-1/3} . \tag{5.43}$$

We have therefore found a family of solutions to the equations of motion (5.12)–(5.15) depending on three non-negative parameters B_0, Δ, A. Since the field equations for the three-

[44] Since the coordinates (x, y, x^0) are non-compact the entropy itself will diverge.

dimensional scalars $y^i(\tau), v(\tau), u(\tau)$ are of second order, and our ansatz amounts to three independent scalar fields (since the y^i have been taken to be proportional), we should a priori have expected six independent integration constants. However, as we have seen, physical regularity conditions imposed on the lifted, five-dimensional solution reduces the number of integration constants by one half. This is consistent with physical solutions being uniquely characterised by a system of first order flow equations, despite that the equations of motion are of second order, as has been observed for other types of solutions before [3, 49, 157, 159].

We further remark that since the physical five-dimensional scalar fields have turned out to be constant, their only contribution is to generate an effective cosmological constant as reviewed in Appendix B.2. The value of this cosmological constant is determined by the value of the scalar potential at the corresponding stationary point.[45] Since no five-dimensional gauge fields have been turned on, our solution, which is valid for any five-dimensional vector multiplet theory, can therefore be obtained from an effective action, which only contains the Einstein-Hilbert term together with a cosmological constant, while the gauge fields and scalar fields have been integrated out.

A coordinate change

We introduce the radial coordinate ρ via

$$e^{-2B_0\tau} = 1 - \frac{2B_0}{\rho} \equiv W(\rho), \tag{5.44}$$

so that the near horizon region is at $\rho = 2B_0$, and the asymptotic region is at $\rho \to \infty$. Hence we can use ρ to analytically continue the solution to the region $0 \leq \rho \leq 2B_0$ between the inner and outer horizons. In terms of ρ we find

$$u(\rho) = f(\rho)^{-1} W(\rho)^{1/2}, \qquad f(\rho) = A + \frac{\Delta}{\rho}, \tag{5.45}$$

where we have defined $\Delta := B - B_0 A$. Moreover, we have

$$v(\rho) = (c\lambda\lambda\lambda)^{1/3} (\rho W)^{-1/2}, \qquad e^{4\psi} = \rho^3 W^2, \tag{5.46}$$

and

$$\mathcal{A}^0(\rho) = -\sqrt{\frac{6\Delta}{\Delta + 2B_0 A}} \frac{W(\rho)}{f(\rho)} dx^4. \tag{5.47}$$

Introducing the notation

$$\tilde{\lambda} := \left(\frac{1}{6} c\lambda\lambda\lambda\right)^{1/3},$$

[45] See Appendix C for an example of such a calculation in the case of the STU-model.

the five-dimensional line element (5.10) becomes

$$\begin{aligned} ds^2_{(5)} &= \frac{\rho^{1/2}}{6\tilde{\lambda}} f(\rho) \left(dx^0 - \sqrt{\frac{6\Delta}{\Delta + 2B_0 A}} \frac{W(\rho)}{f(\rho)} dx^4 \right)^2 - \frac{\rho^{1/2} W(\rho)}{\tilde{\lambda} f(\rho)} (dx^4)^2 \\ &\quad + \frac{6\tilde{\lambda}^2 d\rho^2}{\rho^2 W(\rho)} + 6\tilde{\lambda}^2 \rho^{1/2} (dx^2 + dy^2). \end{aligned} \tag{5.48}$$

5.2.2 Properties of the solution

Let us now turn to an investigation of the properties of the solutions constructed in Section 5.2.1, which we recall depend on three independent parameters: A, B_0 and Δ. It is instructive to look at the cases $A > 0$ and $A = 0$ separately. Moreover, we focus first on the situation $B_0 > 0$, and will comment on the $B_0 = 0$ case later.

Solutions with $B_0 > 0$ and $A > 0$

In this situation it is convenient to introduce the notation:

$$\tilde{\Delta} := \frac{\Delta}{2B_0 A}. \tag{5.49}$$

After a suitable scaling of the boundary coordinates, and introducing the new radial coordinate $r := \rho^{1/4}$, we can bring the five-dimensional line element (5.48) to the form

$$\begin{aligned} ds^2_{(5)} &= \frac{r^2}{L^2} f(r) \left(dx^0 - \sqrt{\frac{\tilde{\Delta}}{1 + \tilde{\Delta}}} \frac{W(r)}{f(r)} dx^4 \right)^2 - \frac{r^2 W(r)}{L^2 f(r)} (dx^4)^2 \\ &\quad + \frac{L^2 dr^2}{W(r) r^2} + \frac{r^2}{L^2} (dx^2 + dy^2). \end{aligned} \tag{5.50}$$

Here L is defined by

$$L^2 := 96\tilde{\lambda}^2,$$

and, as we will see, corresponds to the AdS_5 radius, whilst

$$W(r) = 1 - \frac{r_+^4}{r^4}, \qquad f(r) = A + \frac{\Delta}{r^4}, \qquad r_+^4 := 2B_0.$$

In order to interpret our solution, as well as to read off the various thermodynamic quantities associated with it, it is useful to introduce coordinates in terms of which the line element (5.50) becomes manifestly asymptotically AdS_5. We observe that the solution is invariant under the combined parameter rescalings

$$A \to \lambda A\,, \quad \Delta \to \lambda \Delta\,, \quad x^0 \to \frac{x^0}{\sqrt{\lambda}}\,, \quad x^4 \to \sqrt{\lambda} x^4\,, \tag{5.51}$$

where $\lambda > 0$ and B_0 remains invariant. Note that $\tilde{\Delta}$ is invariant, so that for $A > 0$ we obtain a two-parameter family of solutions parametrized by B_0 and $\tilde{\Delta}$. The coordinate transformation

$$t = \frac{1}{\sqrt{A}} x^4, \qquad z = \sqrt{A} x^0 - \sqrt{\frac{\tilde{\Delta}}{A(1+\tilde{\Delta})}}\, x^4, \tag{5.52}$$

absorbs A and brings the metric (5.50) to the form of a boosted AdS-Schwarzschild black brane:

$$ds^2_{(5)} = \frac{L^2 dr^2}{r^2 W} + \frac{r^2}{L^2}\left[-W\left(u_t\, dt + u_z\, dz\right)^2 + \left(u_z\, dt + u_t\, dz\right)^2 + dx^2 + dy^2\right]. \tag{5.53}$$

The constants

$$u_t = \sqrt{1+\tilde{\Delta}}, \quad u_z = \sqrt{\tilde{\Delta}}, \tag{5.54}$$

satisfy $u_t^2 - u_z^2 = 1$ and parametrize a boost along the z-direction. By taking $r \to \infty$ one sees that (5.53) indeed asymptotes to AdS_5 with radius L. We remark that $\tilde{\Delta}$ parametrizes the boost of the brane, while B_0 (or equivalently r_+) will be shown below to be a non-extremality parameter, and therefore related to temperature.

This metric can be further rewritten by making the following co-ordinate transformation:

$$\begin{aligned} r &= e^{l^{-1}\rho}, \qquad x = l y^1, \qquad y = l y^2 \\ t &= \frac{l}{r_+^2}(u_t - u_z) X - l r_+^2 u_z \hat{T}, \qquad z = \frac{l}{r_+^2}(u_t - u_z) X + l r_+^2 u_t \hat{T}\,, \end{aligned} \tag{5.55}$$

to obtain

$$\begin{aligned} ds^2_{(5)} &= e^{-2l^{-1}\rho} dX^2 + e^{2\ell^{-1}\rho}\Big(2 dX d\hat{T} + r_+^4 d\hat{T}^2 + (dy^1)^2 + (dy^2)^2\Big) \\ &+ (1 - r_+^4 e^{-4l^{-1}\rho})^{-1} d\rho^2\,. \end{aligned} \tag{5.56}$$

This metric is the 5-dimensional generalized Carter-Novotný-Horský metric, C_5, constructed in [160].

It is noteworthy that the line element (5.53) can be further simplified by setting $\tilde{r} = r_+ r, \tilde{t} = t/r_+, \tilde{x} = x/r_+, \tilde{y} = y/r_+, \tilde{z} = z/r_+$. This rescaling corresponds to formally setting $r_+ = 1$ in the function W in (5.53), thus fixing the location of the horizon to $r = 1$. However, this reparametrization obscures the fact that r_+ in (5.53) encodes the temperature, which, as we will show later, is defined in a reparametrization invariant way. For this reason we prefer to work with the metric in the form (5.53).

Solutions with $B_0 > 0$ and $A = 0$

Let us now look at the case where we take $A = 0$, so that $f(\rho) = \Delta/\rho$ in (5.48). In this case, after suitably rescaling the boundary coordinates and introducing the radial coordinate r as

before, we find that the five-dimensional line element (5.48) becomes

$$ds^2_{(5)} = \frac{\Delta}{L^2 r^2}\left(dx^0 - \frac{r^4 W(r)}{\Delta}dx^4\right)^2 - \frac{r^6 W(r)}{\Delta L^2}(dx^4)^2 + \frac{L^2 dr^2}{r^2 W(r)} + \frac{r^2}{L^2}(dx^2 + dy^2). \tag{5.57}$$

Making the coordinate redefinition

$$x^4 = \frac{1}{2}(t - z), \qquad x^0 + \frac{r_+^4}{2\Delta}x^4 = t + z,$$

we can bring the metric (5.57) to the form (5.53) of a boosted AdS-Schwarzschild black brane. The boost parameters are given by

$$u_t = \cosh\hat{\beta}, \quad u_z = \sinh\hat{\beta}, \tag{5.58}$$

where the quantity $\hat{\beta}$ is defined via

$$e^{2\hat{\beta}} = \frac{4\Delta}{r_+^4}.$$

Since (5.56) is a rewriting of (5.53), we conclude that the $A = 0$ solution (5.57) is also a 5-dimensional generalized Carter-Novotný-Horský metric, C_5.

As with the $A > 0$ case, we obtain a two-parameter family of black brane solutions. For $A = 0$ the parameters can be taken to be B_0 (equivalently r_+) and Δ. We remark that while both the $A > 0$ and $A = 0$ cases can be mapped to two-parameter families of black branes, the two families cannot be related smoothly by taking $A \to 0$.

Solutions with $B_0 = 0$

If we take $B_0 \to 0$ in (5.48) then the inner and outer horizons coincide, which identifies this limit as the extremal limit. For any value (zero or non-zero) of A we can then bring the metric to the form

$$ds^2_{(5)|\text{Ext}} = \frac{L^2 dr^2}{r^2} + \frac{r^2}{L^2}\left[-dt^2 + dx^2 + dy^2 + dz^2 + \frac{\Delta}{r^4}(dt + dz)^2\right]. \tag{5.59}$$

This solution agrees with the five-dimensional extremal Nernst branes found in [152].[46] We can equivalently obtain this form of the metric from the boosted black brane (5.53) by taking the limits

$$r_+ \to 0, \qquad u_t \to \infty, \qquad u_t^2 r_+^4 \to \Delta = \text{const}.$$

In the extremal limit Δ determines the mass or more precisely, the mass per world-volume, or tension, of the brane. The vacuum AdS_5 solution is obtained by taking the zero mass limit $\Delta \to 0$. Moreover we have seen that B_0 (equivalently r_+) is a non-extremality parameter and

[46] However, the 'heated up' branes of [151] appear to be different from our non-extremal solution.

thus is related to the temperature. The precise expressions for the mass and thermodynamic quantities will be calculated in Section 5.2.3.

The extremal solution (5.59) with $\Delta > 0$ looks like a gravitational wave, and indeed, if we make the co-ordinate transformation

$$\begin{aligned} r &= \Delta^{\frac{1}{4}} e^{l^{-1}R}, \qquad x = l\Delta^{-\frac{1}{4}} y^1, \qquad y = l\Delta^{-\frac{1}{4}} y^2 \\ t &= \frac{1}{2} l\Delta^{-\frac{1}{4}}(X - 2\hat{T}), \qquad z = \frac{1}{2} l\Delta^{-\frac{1}{4}}(X + 2\hat{T}) \ , \end{aligned} \tag{5.60}$$

the metric (5.59) becomes

$$ds^2 = e^{-2l^{-1}R} dX^2 + e^{2l^{-1}R}\Big(2dX d\hat{T} + (dy^1)^2 + (dy^2)^2\Big) + dR^2 \ . \tag{5.61}$$

The metric (5.61) is a five-dimensional generalized Kaigorodov metric, K_5, constructed in [160], and is known to describe gravitational waves propagating in AdS_5. The supersymmetry of this solution was investigated in [160], where it was shown that this solution preserves 1/4 of the supersymmetry. Furthermore, after making some appropriate co-ordinate transformations, this solution can be shown to correspond to a class of supersymmetric solutions which appears in the classification of supersymmetric solutions of minimal five-dimensional gauged supergravity constructed in [161]. It is straightforward to show that the null Killing vector which is obtained as a spinor bilinear is given by $\partial_t - \partial_z$, in the co-ordinates of (5.59).

The metric (5.59) displays an interesting scaling behaviour in the limit $r \to 0$. To display it, we introduce coordinates x^-, x^+ by[47]

$$t = x^+ \ , \quad z = x^- - x^+ \ .$$

Then the metric becomes

$$ds^2_{(5)|\text{Ext}} = \frac{l^2 dr^2}{r^2} + \frac{r^2}{l^2}\left[\left(1 + \frac{\Delta}{r^4}\right)(dx^-)^2 - 2dx^- dx^+ + dx^2 + dy^2\right] \ .$$

Dropping terms which are subleading in the 'near horizon limit' $r \to 0$ we obtain

$$ds^2_{(5)|\text{Ext,NH}} = \frac{l^2 dr^2}{r^2} + \frac{r^2}{l^2}\left[\frac{\Delta}{r^4}(dx^-)^2 - 2dx^- dx^+ + dx^2 + dy^2\right] \ . \tag{5.62}$$

This metric is invariant under the scale transformations:

$$x \mapsto \lambda x \ , \quad y \mapsto \lambda y \ , \quad r \mapsto \lambda^{-1} r \ , \quad x^- \mapsto \lambda^{-1} x^- \ , \quad x^+ \mapsto \lambda^3 x^+ \ .$$

[47] For $A = 1$ these coordinates agree with x^0 and x^4 in the extremal limit. Moreover, the near horizon limit preserves the symmetry that allows us to set $A = 1$.

Thus the near horizon metric shows a scaling invariance similar to a Lifshitz metric with scaling exponent $z = 3$ (and no hyperscaling violation, $\theta = 0$). The only difference is that the coordinate x^- has scaling weight -1 rather than $+1$. This type of generalized scaling behaviour was observed in [162, 163, 164], where the metric (5.62) was obtained by taking a particular limit of boosted D3-branes. We will come back to this in Chapter 6, where we discuss the dual field theory interpretation of our solutions.

The boosted black brane

The boosted black brane has similarities with Kerr-like black holes, with the linear momentum related to the boost playing a role analogous to the angular momentum. It is instructive to work this out in some detail, following the discussion of the Kerr solution in [136].

Let us first look for the existence of static observers, who remain at constant (r, x, y, z) and as such have velocities parallel to the Killing vector field ∂_t. Therefore static observers exist in regions where ∂_t is time-like, and the limit of staticity is at the value of r where

$$\begin{aligned} g_{tt} = 0 &\Leftrightarrow -W(r)u_t^2 + u_z^2 = 0 \\ &\Leftrightarrow r^4 = u_t^2 r_+^4 \geq r_+^4, \quad \text{providing } r_+ > 0. \end{aligned}$$

This 'ergosurface' is always located outside the event horizon, with the trivial exception of globally static (unboosted) spacetimes for which $u_t = 1$ and the two surfaces overlap completely. This is different to the rotating case where ergosurface and event horizon always coincide at the north and south pole.

Beyond the limit of staticity there still exist stationary observers which are co-moving (more precisely, but less elegantly 'co-translating') with the brane. Observers which have fixed (r, x, y) and a constant velocity in the z-direction have world lines tangent to Killing vector fields

$$\xi_{(v)} = \partial_t + v\,\partial_z \ ,$$

where the quantity $v = \text{const.}$ will be referred to as the velocity. Such co-moving observers exist in regions where $\xi_{(v)}$ is time-like. Killing vector fields of the form $\xi_{(v)}$ become null for values of r where

$$g_{tt} + 2v g_{tz} + v^2 g_{zz} = 0 \Rightarrow v_\pm = -\frac{g_{tz}}{g_{zz}} \pm \sqrt{\left(\frac{g_{tz}}{g_{zz}}\right)^2 - \frac{g_{tt}}{g_{zz}}} \ .$$

Thus there is a finite range of velocities v, given by $v_- \leq v \leq v_+$, which co-moving observers can attain. Note that at the limit of staticity, where $g_{tt} = 0$, we find that $v_+ = 0$. Therefore v must be negative once the limit of staticity has been passed. The limit for co-moving observers

is reached when $v_- = v_+ =: w$, which happens at the point where

$$g_{tt}g_{zz} - g_{tz}^2 = 0 \ .$$

It is straightforward to verify that this happens at the same value r_+ of r where $W(r_+) = 0$. The limiting velocity w is given by

$$w = -\left.\frac{g_{tz}}{g_{zz}}\right|_{r=r_+} = -\frac{u_z}{u_t} \ , \tag{5.63}$$

and can be interpreted as the boost-velocity of the surface $r = r_+$. Since $W(r_+) = 0$ implies that $g^{rr}(r_+) = 0$, it follows from Section 3.7.1 that on this surface outgoing null congruences have zero expansion. Consequently $r = r_+$ is an apparent horizon, and since the solution is stationary, an event horizon. Moreover this event horizon is a Killing horizon for the vector field $\xi = \partial_t + w\partial_z = \partial_t - \frac{u_z}{u_t}\partial_z$ and we can interpret w as the boost-velocity of this horizon. Observe that the limit of staticity and the limit of stationarity are in general different, and only agree in the unboosted limit $u_z = 0$ where we recover the AdS-Schwarzschild black brane.

We note that there is frame dragging in our solutions, since the metric is non-static for $u_z \neq 0$. Indeed, since the metric coefficients are independent of t and z, the covariant momentum components p_t and p_z are conserved. But even when setting $p_z = 0$, particles have a non-vanishing contravariant momentum component $p^z = g^{zt}p_t \neq 0$ in the z-direction.

The boost velocity of the metric varies between the horizon and infinity. It can be read off by writing the metric in the form

$$ds_{(5)}^2 = -N^2(r)dt^2 + M^2(r)(dz - v(r)dt)^2 + \dots \ ,$$

where the omitted terms involve dx^2, dy^2 and dr^2. An observer at fixed r, x, y is co-moving with the space-time if their velocity is $dz/dt = v$. Bringing the metric (5.53) to the above form one finds

$$v = -\frac{(1-W)u_t u_z}{u_t^2 - W u_z^2},$$

with limits

$$v \xrightarrow[r\to r_+]{} -\frac{u_z}{u_t} = w \geq -1 \ ,$$

and

$$v \xrightarrow[r\to\infty]{} 0 \ .$$

It is straightforward to check that for $u_t > 1$ the boost speed $|v(r)|$ is strictly monotonically increasing from $|v_\infty| = 0$ at infinity to $|v_{\text{horizon}}| = |w| = u_z/u_t \leq 1$ at the horizon. Thus the boost speed is bounded by the speed of light and can only reach it at the horizon and in the

extremal limit. Note that the asymptotic AdS space at infinity is not co-moving. This is different from Kerr-AdS, where the asymptotic AdS space is co-rotating, with implications for the black brane thermodynamics [165, 166, 167]. In particular, we will not need to subtract a background term, corresponding to the motion of the asymptotic AdS space, from our expressions for the boost velocity in order to have quantities satisfying the first law of thermodynamics. We will come back to this later when verifying the first law.

We remark that in the extremal limit, where $r_+ = 0$ and $w = -1$, the limit of staticity is at $r^4 = \Delta$, which is different from zero, unless we take the trivial extremal limit $\Delta = 0$ which brings us to global AdS_5. It can be shown that our solutions with $\Delta > 0$ are 1/4-BPS [2], and thus for $\Delta > 0$ we have BPS solutions which exhibit an ergoregion. Such a situation is not possible for black holes, where the Killing vector field induced by the Killing spinors is the static Killing vector field ∂_t. Killing vector fields resulting from Killing spinors are necessarily globally time-like or null [168]. The vector field ∂_t is time-like at infinity and therefore cannot become space-like in a BPS solution if it is related to the Killing spinors. Consequently, rotating BPS black hole solutions cannot have an ergosphere, and, moreover, must have an event horizon which is non-rotating [169]. This is different for our solution, where the Killing vector field related to the Killing spinor fields is $\partial_t - \partial_z$, and so it is possible for the Killing vector field ∂_t to change signature at finite r. Moreover, the spinor bilinear is null everywhere on the extremal solution meaning our BPS solutions belong to the 'null' (wave-like) rather than the 'time-like' (soliton-like) class [170]. Indeed, the nature of the extremal and BPS limit is different in both cases: for rotating supersymmetric black holes the angular momentum saturates a certain bound, while for boosted black branes we have to perform a double limit where zero temperature is reached at infinite boost. We note that not only does an ergoregion persist in this limit, but also that the horizon is moving with maximal speed. This behaviour is precisely opposite to the one observed for rotating BPS black holes.

5.2.3 Thermodynamics

We now want to turn to an investigation of the thermodynamics of the black brane solutions of Section 5.2.2. The Hawking temperature is related to the surface gravity by $T = \frac{\kappa}{2\pi}$, and the surface gravity κ of a Killing horizon is given by (3.145) as

$$\kappa^2 = -\frac{1}{2}\left(D^\mu \xi^\nu\right)\left(D_\mu \xi_\nu\right)\Big|_{r=r_+} . \tag{5.64}$$

For the boosted black brane (5.53) we evaluate this with $\xi = \partial_t + w\partial_z$ to find the Hawking temperature T:

$$\pi T = \frac{r_+}{L^2\, u_t} . \tag{5.65}$$

We remark that the same result can be obtained by imposing that the Euclidean continuation of the solution does not have a conical singularity at the horizon, see Appendix E.2.

In the zero boost limit $u_t = 1$, $u_z = 0$ we obtain the Hawking temperature of an AdS-Schwarzschild black brane (also known as an AdS-Schwarzschild black hole with planar horizon) [171]. The infinite boost limit u_t, $u_z \to \infty$ is the extremal limit $r_+ \to 0$, where the Hawking temperature becomes zero, $T \to 0$. Since in this limit $w = -u_z/u_t \to -1$, the horizon is moving with the speed of light, which is consistent with the familiar string-theory description of a BPS state as a state with massless excitations moving in one direction only.

Since our solutions are not asymptotically flat, but rather asymptotic to AdS_5, we cannot apply the standard ADM prescription to compute the mass and linear momentum of our branes. Instead, we use the method based on the quasilocal stress tensor [172], see also [173] for a review in the context of the fluid-gravity correspondence. Here we simply present the result, and relegate explicit calculational details to Appendix E.3. To leading order in $1/r$ we find that the quasilocal stress tensor takes the form

$$T_{\mu\nu} = \frac{r_+^4}{2L^3 r^2}\left(\eta_{\mu\nu} + 4u_\mu u_\nu\right) + \dots\,, \tag{5.66}$$

where $\eta_{\mu\nu}$ is the Minkowski metric on $\partial\mathcal{M}_r$ with coordinates (t, x, y, z), and where we have omitted subleading terms. Note that this takes the form of a perfect ultra-relativistic fluid (equation of state $\rho = 3p$, where ρ is the energy density and p is the pressure), with pressure proportional to $r_+^4 \sim T^4$. The proportionality between r_+ and T is the same behaviour as for large AdS-Schwarzschild black holes. In the absence of a boost, it is known that AdS-Schwarzschild black branes behave thermodynamically like large (rather than small) AdS-Schwarzschild black holes [171].

Having obtained the quasilocal stress tensor, mass and linear momentum can be computed as conserved charges associated to the Killing vectors of our solution. Again, the details are relegated to the appendix E.3. The mass, which is the conserved charge associated with time translation invariance, is

$$M = \frac{(4u_t^2 - 1)r_+^4}{2L^5} V_3, \tag{5.67}$$

where $V_3 = \int_\Sigma d^3x$ is the spatial volume of the brane. We can also calculate the momentum in the z-direction, which is the conserved charge associated to z-translation invariance. The result is

$$P_z = -\frac{4r_+^4 u_t u_z}{2L^5} V_3, \tag{5.68}$$

and vanishes as expected in the zero boost limit $u_z = 0$, $u_t = 1$. Notice that these charges satisfy $P_z = M\left(-\frac{4u_t u_z}{4u_t^2 - 1}\right)$, which resembles the motion of a non-relativistic body of mass M, moving at velocity $v_z = -\frac{4u_t u_z}{4u_t^2 - 1}$.

Finally, we calculate the Bekenstein-Hawking entropy of the solution by integrating the pull back of the metric over the horizon. Recalling that we are working in units where $8\pi G_5 = 1$, we find

$$S = \frac{1}{4G_5}\int_{\Sigma_{r=r_+}} d^3x\sqrt{\sigma} = 2\pi\int_{\Sigma_{r=r_+}} d^3x\sqrt{\sigma} = \frac{2\pi r_+^3}{L^3}u_t V_3 \ , \tag{5.69}$$

where σ denotes the pull-back of the metric to the surface $\Sigma_{r=r_+}$.

Using these, we can check that the thermodynamic variables satisfy the first law:

$$\delta M = T\delta S + w\,\delta P_z, \tag{5.70}$$

where the velocity w is given by (5.63). We remark that obtaining (5.70) is a non-trivial consistency check for the correctness of the definition of the thermodynamical quantities, which are initially ambiguous because they require background subtractions corresponding to renormalization of the boundary CFT [172], see also [167] for a discussion in the context of rotating black holes in higher than four dimensions. As noted before, we do not need to apply a background subtraction for the translation velocity w, since the asymptotic AdS_5 background is not co-translating. This is different for AdS-Kerr-type black holes, where the subtraction of the background rotation velocity is crucial for obtaining the correct thermodynamic relations [165, 166, 167]. We also note that T, M, P_z, S, which are all defined in a reparametrization invariant way, depend on the parameter r_+. Therefore r_+ is a physical parameter, despite that it could be absorbed into the coordinates in the metric (5.53). Moreover, without the ability of varying this parameter, one could not obtain the temperature/entropy term in the first law. We refer to Appendix (E.3.2) for further details on this technical point.

The extremal limit of these quantities can be reached by taking $r_+ \to 0$ and $u_t \to \infty$ with $u_t^2 r_+^4 \to \Delta$ fixed. In this case we find that the entropy density $s := S/V_3$ vanishes in the extremal limit, $s \to 0$ as $T \to 0$. Therefore our solutions satisfy the Nernst Law, and can be referred to as Nernst branes. Moreover, since in the extremal case $w = -1$, we find $M = |P_z|$, which is of course the saturation of the BPS bound. As already remarked earlier, in the extremal limit the parameter Δ controls the mass, and $\Delta \to 0$ is the limit where the solution becomes globally AdS_5.

We can eliminate the quantities r_+ and u_t in favour of the thermodynamical variables T and w via

$$u_t = \frac{1}{\sqrt{1-w^2}}, \quad u_z = -\frac{w}{\sqrt{1-w^2}}, \quad r_+ = \frac{L^2(\pi T)}{\sqrt{1-w^2}}.$$

In terms of T and w the mass of the solution is given by

$$M(T,w) = \frac{L^3}{2}V_3\left(\frac{3+w^2}{(1-w^2)^3}\right)(\pi T)^4. \tag{5.71}$$

Hence, we see that the heat capacity

$$C_T \equiv \left.\frac{\partial M}{\partial T}\right|_w > 0, \tag{5.72}$$

is positive, and the solution is thermodynamically stable. This is as expected, at least in the absence of a boost, since it is well known that AdS-Schwarzschild black branes behave thermodynamically like large AdS-Schwarzschild black holes [171]. As we see from (5.71), the introduction of a boost does not introduce thermodynamic instability.

Expressing the entropy in terms of (T, w) we find

$$S(T, w) = 2\pi L^3 V_3 \frac{(\pi T)^3}{(1-w^2)^2}. \tag{5.73}$$

Note that turning off the boost $u_z = 0$, which corresponds to $w = 0$, we have $S \sim T^3$, which is the scaling behaviour expected for an AdS_5-Schwarzschild black brane.

Indeed we can use (5.73) to investigate the behaviour of S as a function of T in both the high temperature and low temperature limits. The limit of high temperature (equivalently small boost velocity) is

$$u_z \to 0, \qquad r_+ \to \infty, \qquad u_z^2 r_+^4 \to \Delta = \text{const.}$$

This corresponds to $|w| \ll 1$, and so we see from (5.73) that $S \sim T^3$. The limit of low temperature (equivalently boost velocity approaching the speed of light) is the extremal limit

$$u_t \to \infty, \qquad r_+ \to 0, \qquad u_t^2 r_+^4 \to \Delta = \text{const.}$$

In this case, one can see that $1 - w^2 \sim T^{4/3}$, and so the entropy scales like $S \sim T^{1/3}$. We will comment further on the thermodynamic properties of our solutions in Section 6.

5.2.4 Curvature properties of five-dimensional Nernst branes

One motivation of the present work is to resolve the singular behaviour of the four-dimensional Nernst branes found in Chapter 4. We will show in section 5.3 that the five-dimensional Nernst branes found above are dimensional lifts of these four-dimensional Nernst branes. To investigate the effect of dimensional lifting on such singularities, we now examine the behaviour of curvature invariants and tidal forces of the five-dimensional solutions. From both the gravitational point of view, and with respect to applications to gauge-gravity dualities, one would like the solutions to have neither naked singularities, nor null singularities (singularities coinciding with a horizon), while the presence of singularities hidden behind horizons is acceptable. In practice, the presence of large curvature invariants or large tidal forces will also be problematic, given that the supergravity action we start with needs to be interpreted as an effective action. Therefore

large curvature invariants or tidal forces are indications that this effective description breaks down due to quantum or, assuming an embedding into string theory, stringy corrections. This would limit the applicability of gauge-gravity dualities to only part of the solution, where the corrections remain sufficiently small.

Curvature invariants

For our five-dimensional metric (5.53) we compute the Kretschmann scalar and Ricci scalar to be

$$K = \frac{2\left(9r_+^8 - 24r_+^4 r^4 + 20r^8\right)}{r^8 L^4}, \qquad R = -\frac{4\left(-5r^4 + 3r_+^4\right)}{r^4 L^2}. \tag{5.74}$$

Note that these only depend on the temperature $T \sim r_+$ and the curvature radius L of the AdS_5 ground state. For the extremal solution ($r_+ = 0$) both curvature invariants take constant values which agree with those for global AdS_5:

$$K_{\mathrm{AdS}_5} = \frac{2D(D-1)}{L^4} = \frac{40}{L^4}, \qquad R_{\mathrm{AdS}_5} = \frac{D(D-1)}{L^2} = \frac{20}{L^2}.$$

For the non-extremal solution the curvature invariants tend to the AdS_5 values asymptotically, but blow up as $r \to 0$. As this is behind the horizon, there are no naked or null singularities related to the curvature invariants of five-dimensional Nernst branes.

Tidal forces

Even if all curvature invariants are finite, there might still be curvature singularities related to infinite tidal forces. Such curvature singularities can be found by computing the components of the Riemann tensor in a 'parallely-propagated-orthonormal-frame' (PPON) associated with the geodesic motion of a freely-falling observer. While such singularities are often considered milder than those associated to curvature invariants, they are nevertheless genuine singularities and have drastic physical effects ('spaghettification') on freely falling observers.

The details of this construction for the five-dimensional extremal solution are relegated to Appendix E.4.1. From Table E.2 we observe that the non-zero components of the Riemann tensor in the PPON all have near horizon behaviour of the form

$$\tilde{R}_{abcd} \sim r^{\alpha} \quad \text{with} \quad \alpha \leq 0\,, \tag{5.75}$$

with $\alpha < 0$ for all but one independent non-vanishing component. Hence, as the observer approaches the horizon of the extremal brane ($r \to 0$) these components will diverge, resulting in infalling observers being subject to infinite tidal forces as can be seen from (2.11). This is the same behaviour that the four-dimensional solutions suffer from, and seems to be the price for having zero entropy. It is an interesting question whether stringy or other corrections could lift

this singularity, and if so, whether it is possible to maintain zero entropy.

Note that we only consider the extremal solution, since non-extremal solutions are manifestly analytic at the horizon $r_+ > 0$. Indeed, in this case, the divergence is transferred to the Cauchy horizon and cloaked by the smooth non-extremal event horizon [174].

5.3 Four-dimensional Nernst branes from dimensional reduction

5.3.1 S^1 bulk evolution

We now want to dimensionally reduce our five-dimensional Nernst branes and compare the resulting four-dimensional spacetimes to those found in Chapter 4. To do so, the space-like direction x^0 is made compact, i.e. we identify $x^0 \sim x^0 + 2\pi r^0$. Clearly then, to understand the four-dimensional properties, it is crucial to first understand the behaviour of the x^0 circle. Writing (5.50) as[48]

$$ds^2_{(5)} = e^{2\sigma}(dx^0 + \mathcal{A}^0_4 dx^4)^2 + e^{-\sigma} ds^2_{(4)},$$

with

$$e^{2\sigma} = \frac{r^2 f(r)}{L^2}, \tag{5.76}$$

we find the four-dimensional line element

$$ds^2_{(4)} = \frac{r}{L}\left\{-\frac{r^2 W(r)}{L^2 f(r)^{1/2}}\, dt^2 + f(r)^{1/2}\frac{L^2\, dr^2}{r^2 W(r)} + \frac{r^2}{L^2} f(r)^{1/2}(dx^2 + dy^2)\right\}, \tag{5.77}$$

after identifying $x^4 \equiv t$. From (5.76) we can read off the behaviour of the physical (geodesic) length R^0_{phys} of the compactification circle:

$$(R^0_{\text{phys}})^2 = (2\pi r^0)^2 e^{2\sigma}(r) = (2\pi r^0)^2 \left(\frac{A r^2}{L^2} + \frac{\Delta}{r^2 L^2}\right) . \tag{5.78}$$

It is obvious that this S^1 does not maintain a constant geodesic size but instead varies dynamically throughout the bulk as shown in Figure 5.1. Notice from (5.78) that for $A > 0$ there are two competing terms, resulting in decompactification both for $r \to \infty$ and for $r \to 0$. The latter decompactification is only reached in the extremal limit, since otherwise we encounter the horizon at $r_+ > 0$. This implies that in the non-extremal case the near horizon solution will still depend on the parameter A, while in the extremal case the near horizon solution becomes independent of A. The insensitivity of the extremal near horizon solution to changes of

[48] As it stands, (5.50) is specialized to the case $A > 0$ since it involves the variable $\tilde{\Delta} = \frac{\Delta}{2B_0 A}$. However, using (5.47), it is possible to write (5.50) in terms of a general Kaluza-Klein vector, valid for both $A > 0$ and $A = 0$. This then allows the reduction of both cases in parallel, leading to (5.77).

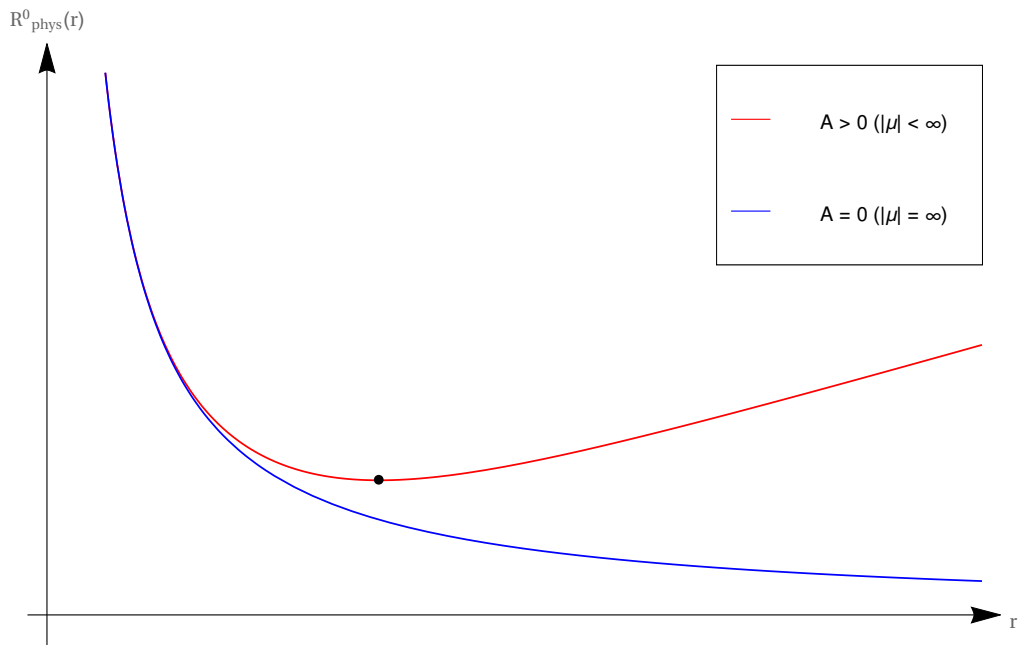


Figure 5.1: Plot showing the evolution of the compactification circle throughout the five-dimensional bulk.

parameters which determine the asymptotic behaviour at infinity, in our case A, can be viewed as a manifestation of the black hole attractor mechanism. Making the solution non-extremal results in the loss of attractor behaviour by making the near horizon solution sensitive to the asymptotic properties of the solution at infinity. A remarkable feature of solutions with $A > 0$ is the existence of a critical point, P_{crit}, where the compactification circle reaches a minimal size at $r^4_{\text{crit}} = \Delta/A$. In contrast, for $A = 0$, this critical point does not exist and so, whilst the circle continues to decompactify as $r \to 0$ in the extremal case, it now shrinks monotonically with increasing r, ultimately becoming a null circle of zero size for $r \to \infty$. This fundamentally different behaviour of the S^1 means we must treat the dimensional reduction of the $A > 0$ and $A = 0$ cases separately in what follows. Additionally, we clearly see that A is the parameter responsible for the asymptotic behaviour at infinity from a five-dimensional point of view. This resembles the role played by the parameter h_0 in the four-dimensional solutions of Chapter 4; this connection will be made manifest in the following subsections.

In the case $A > 0$, the compactification introduces a new continuous parameter, the parametric radius r^0 of the circle. We now observe that the identification $x^0 \simeq x^0 + 2\pi r^0$ breaks the scaling symmetry (5.51), which made the parameter A irrelevant. While the geodesic size of the compactification circle varies with r, for $A > 0$ there is a circle of minimal size at $r^4_{\text{crit}} = \Delta/A$, with geodesic size R^0_{crit} given by

$$(R^0_{\text{crit}})^2 = 8\pi \frac{r_0^2}{L^2} \sqrt{\Delta A} \, .$$

The size of this minimal circle depends only on the combination $r_0^2\sqrt{A}$ and is therefore invariant under any increase in A that is compensated for by a reduction in r_0 and vice-versa. This ability to trade r_0 for A, means A can be used as the physical parameter controlling the minimal circle size, whilst r_0 becomes redundant. It is natural to set $r_0 = \sqrt{A}$, as this is precisely what is needed such that the expression for the four-dimensional charge, Q_0, calculated later in (5.91) is independent of the compactification radius as all four-dimensional quantities should be.

In the case $A = 0$, there is no such invariant length and we can see this in a number of ways. Firstly, the $A \to 0$ limit pushes $r^4_{\text{crit}} = \frac{\Delta}{A} \to \infty$ and so no minimal circle exists. Secondly, with $A = 0$, the geodesic size of the compactification circle is found from (5.78) to be $(R^0_{\text{phys}})^2 = \frac{(2\pi r^0)^2\Delta}{r^2L^2}$ and depends only on Δ; since this is already a parameter of the five-dimensional solution, there is nothing else to be accounted for and no need for additional parameters. One might consider possibly trying to obtain an invariant length from the size of the circle on the horizon, $R^0_{\text{phys}}(r_+)$, which, assuming non-extremality, will at least be finite. However, it is clear from (5.78) that this will be a function of both Δ and r_+, which again are already existing parameters of the five-dimensional $A = 0$ solution.

5.3.2 Dimensional reduction for $A > 0$

Four-dimensional metrics and gauge fields

In Chapter 4 a family of four-dimensional Nernst branes was found, which depend on one electric charge Q_0 and two continuous parameters $B_0^{(4d)}$ and h_0, which can be expressed alternatively in terms of temperature $T^{(4d)}$ and chemical potential μ. It was also observed that the four-dimensional solutions with finite chemical potential exhibited decompactification behaviour in the asymptotic regime, suggesting an interpretation in terms of a five-dimensional geometry. Looking at the behaviour of the compactification circle in Figure 5.1, the natural candidate is the $A > 0$ family of five-dimensional Nernst branes and it is this relationship that we shall now investigate.

We begin by comparing the four-dimensional Nernst brane solutions with finite chemical potential ($h_0 \neq 0$) previously discovered in Chapter 4 to the four-dimensional metric in (5.77) obtained by dimensionally reducing our five-dimensional solution with $A > 0$. Setting $\rho = r^4$ in (4.54) gives:

$$ds^2_{(4)} = -\mathcal{H}^{-1/2}W^{(4d)}r^3dt^2 + \frac{16\mathcal{H}^{1/2}}{W^{(4d)}}\frac{dr^2}{r} + \mathcal{H}^{1/2}r^3(dx^2 + dy^2)\,, \tag{5.79}$$

where $W^{(4d)} = W^{(4d)}(r) = 1 - \frac{2B_0^{(4d)}}{r^4}$ and

$$\mathcal{H}(r) = C\left[\frac{Q_0}{B_0^{(4d)}}\sinh\frac{B_0^{(4d)}h_0}{Q_0} + \frac{Q_0e^{-B_0^{(4d)}h_0/Q_0}}{r^4}\right] =: C\mathcal{H}_0(r)\,. \tag{5.80}$$

Here Q_0 parametrizes the four-dimensional electric charge, whereas the continuous parameters $h_0 \neq 0$ and $B_0^{(4d)} \geq 0$ correspond to chemical potential $|\mu| < \infty$ and temperature $T^{(4d)} \geq 0$. The constant C is determined by the choice of prepotential and gauging of the four-dimensional theory. More precisely, it is determined by the cubic coefficients c_{ijk} and gauge parameters g_i, but since we are assuming this solution can be lifted to five dimensions, these are the same parameters that enter into our five-dimensional theory in (5.1). The precise form of C can be read off from (4.55). At this point we anticipate that the functions $W^{(4d)}$ and W in the four- and five-dimensional solutions can be identified, and hence from now onwards we can use B_0 and T as parameters without any need for 4d superscripts. Since we can no longer rescale the coordinate r, matching the coefficients of dr^2 between the metrics (5.77) and (5.79) fixes the relation between the functions $f(r)$ and $\mathcal{H}(r)$ to be

$$L^2 f = 16^2 \mathcal{H} = 16^2 C \mathcal{H}_0 \,.$$

Then the remaining metric coefficients match if we rescale t, x, y by constant factors involving L.[49] Writing out the functions f and $\mathcal{H}$ and comparing, we obtain:

$$\begin{aligned} 16^2 C \frac{Q_0}{B_0} \sinh \frac{B_0 h_0}{Q_0} &= L^2 A \,, \\ 16^2 C Q_0 e^{-B_0 h_0/Q_0} &= L^2 \Delta \,. \end{aligned} \tag{5.81}$$

While the five-dimensional line element is non-static, the four-dimensional one is static, but as an additional degree of freedom we have a Kaluza-Klein gauge field, given by

$$A_t^0(r) = \zeta^0 = -\frac{1}{\sqrt{2}} \mathcal{A}_4^0 = \frac{\sqrt{6}}{\sqrt{2}} \left(\frac{u_z}{u_t}\right) \frac{W(r)}{f(r)} = -\frac{\sqrt{3} w W(r)}{f(r)} \,, \tag{5.82}$$

where we have used (3.76) and (5.10) for the first two equalities, and then read off $\mathcal{A}_4^0$ from (5.48) for the third equality, and finally substituted $w = -\frac{u_z}{u_t}$.

By matching the expression for $\dot{\zeta}^0$ given by (5.23) with the τ-derivative of (4.64), we can identify the five-dimensional Kaluza-Klein vector with the four-dimensional gauge field providing

$$\begin{aligned} \frac{\sqrt{3\Delta(\Delta + 2B_0 A)}}{A^2} &= -\frac{B_0^2}{2Q_0 \sinh^2 \frac{B_0 h_0}{Q_0}} \,, \\ 1 + \frac{\Delta}{B_0 A} &= \coth \frac{B_0 h_0}{Q_0} \,. \end{aligned} \tag{5.83}$$

[49] Alternatively, we could absorb L into r, but then by comparing the functions W we will conclude that the respective parameters B_0 differ by a factor L^4. Given the relation of B_0 to the position of the event horizon and to temperature, we prefer not to do this.

From this we can find

$$Q_0 = -\frac{1}{6}\sqrt{3\Delta(\Delta+2B_0A)}\,, \tag{5.84}$$

$$h_0 = \frac{Q_0}{B_0}\text{arcoth}\left(1+\frac{\Delta}{B_0A}\right), \tag{5.85}$$

which expresses the four-dimensional parameters Q_0, h_0 in terms of the five-dimensional parameters A, Δ, B_0. Comparing (5.81) to (5.83) we find that these relations are mutually consistent provided that

$$16^2C = -2\sqrt{3}L^2\,. \tag{5.86}$$

This equation relates the overall normalizations of metrics (5.77) and (5.79) and of the underlying vector multiplet actions.

The four-dimensional chemical potential is given by the asymptotic value of the gauge field A_t, which is chosen such that $A_t(r_+) = 0$, as explained in Section 3.7.1. Having matched the five-dimensional Kaluza-Klein vector with the four-dimensional gauge field of Chapter 4, the corresponding expressions for the chemical potential must also match.[50] For reference, we provide the following expression in terms of both four- and five-dimensional parameters,

$$\mu = \frac{1}{2}\frac{B_0}{Q_0}\left[\coth\frac{B_0h_0}{Q_0}-1\right] = \frac{\Delta}{2Q_0A} = -\frac{\sqrt{3}}{A}\sqrt{\frac{\Delta}{\Delta+2B_0A}}, \tag{5.87}$$

where we used (5.83) and (5.84). Notice from (5.84) that $Q_0 < 0$ which then forces $h_0 < 0$ by (5.85), which is consistent with our earlier remark below (4.23) that $\text{sign}(h_0) = \text{sign}(Q_0)$. It is also clear from (5.87) that $\mu < 0$ which is consistent with the earlier remark underneath (4.65) that $\text{sign}(\mu) = \text{sign}(Q_0)$. We then deduce from (5.81) that the four-dimensional constant C must be negative, $C < 0$, which explains the minus sign in (5.86). Furthermore, it is clear from (5.80) that $\mathcal{H}_0(r) < 0$ such that the harmonic function $\mathcal{H}(r) > 0$, which we need in order that the roots of $\mathcal{H}$, which appear in the metric (5.79), are real so as to avoid naked singularities in the four-dimensional solution.

Momentum discretization, charge quantization and parameter counting

Since the reduction is carried out over the x^0 direction, it is instructive to calculate the Killing charge associated to the Killing vector $\partial_0 = \partial/\partial x^0$. For $A > 0$, (5.52) tells us this is related to the Killing vector ∂_z of the five-dimensional spacetime via

$$\partial_0 = \sqrt{A}\partial_z.$$

[50] This is seen explicitly by applying (5.83) to (4.65) and comparing to the asymptotic value of (5.82).

Since the charge associated with ∂_z is the brane momentum (5.68), the Killing charge corresponds to momentum in the x^0 direction, and can be determined as follows:

$$P^0 = \sqrt{A}P_z \simeq -\frac{2}{\sqrt{A}}\sqrt{\Delta(\Delta + 2B_0 A)}, \tag{5.88}$$

where we have divided out the V_3 from P_z to produce a momentum density, as we do with all extensive quantities. Further, we have suppressed factors of the AdS radius L. Notice that $P^0 \leq 0$ and then periodicity of the x^0 direction forces the wavefunction of particles to be standing waves on the S^1 and consequently the momentum must satisfy

$$P^0 \simeq \frac{N}{r^0} = \frac{N}{\sqrt{A}}, \qquad N \in \mathbb{Z}^- \cup \{0\}, \tag{5.89}$$

where we have set $\hbar = 1$ as explained in Section (3.2.5). For fixed $A > 0$, we see that the momentum, P^0, is discretized to a spectrum of allowed values and we can rearrange (5.89) to establish that

$$\sqrt{A}P^0 \simeq N \qquad \Rightarrow -2\sqrt{\Delta(\Delta + 2B_0 A)} \simeq N, \qquad N \in \mathbb{Z}^- \cup \{0\}. \tag{5.90}$$

Given (5.84), we see explicitly that the four-dimensional charge is quantized

$$Q_0 \simeq \sqrt{A}P^0 \simeq N, \qquad N \in \mathbb{Z}^- \cup \{0\}. \tag{5.91}$$

This is just the standard Kaluza-Klein mechanism of Section (3.2.5) at work; given a fixed $A > 0$, the momentum around the circle generates the lower-dimensional electric charge and both are discretized to a spectrum of allowed values. Varying A, which is after all a free parameter of the four-dimensional theory, alters the size of the circle and shifts this momentum spectrum up or down accordingly. It makes sense that Q_0 is independent of $r^0 = \sqrt{A}$, since electric charge is purely four-dimensional and should have no knowledge of its higher dimensional origins. We also observe from (5.84) and (5.91) that our solution is negatively charged but as remarked earlier, a conjugate, positively charged solution does exist and can be manufactured by flipping signs in (5.24) and introducing an overall minus sign to the expression for ζ^0 throughout this chapter. This difference of signs between the gauge field, $A_t^0 = \zeta^0$, and the electric charge, Q_0, is expected. Indeed, it was already present in the previous work on four-dimensional Nernst branes in order to ensure the construction of a symplectically covariant formalism, see e.g. (4.63).

Let us end this discussion by comparing the number of parameters describing the Nernst branes in different dimensions. Five-dimensional Nernst branes are parametrized by three continuous parameters (A, B_0, Δ), but for $A > 0$ we have the scaling symmetry (5.51), which tells us that A is redundant, and that we can parametrize solutions by the two independent and

continuous parameters $(B_0, \tilde{\Delta})$, which then correspond to temperature and boost momentum. Upon compactification a new length scale R^0_{crit} is introduced that destroys the scaling symmetry present in five dimensions. Consequently, the four-dimensional solution picks up an extra parameter: we need to specify the three independent and continuous parameters (B_0, Δ, A) in order to completely define the metric (5.77). In terms of physical parameters, the four-dimensional solution depends on temperature, charge and chemical potential (T, Q_0, μ). These are all independent but, as we have seen, since the five-dimensional momentum has a component in the direction we compactify over, it becomes discrete, which corresponds directly to the discretization of four-dimensional electric charge. As such, the five-dimensional solution involves two independent and continuous thermodynamic parameters whilst the four-dimensional solution relies on three independent parameters, two of which are continuous and one of which is discrete.

5.3.3 Dimensional reduction for $A = 0$

The two-parameter family of four-dimensional Nernst branes found in Chapter 4 exhibits discontinuities in the asymptotic behaviour of both the geometry and the scalar fields when taking the limit $h_0 \to 0$, or equivalently, $|\mu| \to \infty$. This discontinuity can be accounted for by the discontinuous asymptotic behaviour of the compactification circle in the limit $A \to 0$ as seen in Figure 5.1. As such, we now exhibit the connection between the $h_0 = 0$ four-dimensional solution from Chapter 4 and the $A = 0$ five-dimensional solution with one dimension made compact.

To demonstrate this relationship we take the four-dimensional Nernst brane metric (5.79) and set $h_0 = 0$, which reduces the function $\mathcal{H}(r)$ in (5.80) to

$$\mathcal{H}(r) = \frac{CQ_0}{r^4}.$$

Substituting this back into (5.79) gives the following metric

$$ds^2_{(4)} = -C^{1/2}Q_0^{-1/2}W^{(4d)}r^5dt^2 + \frac{16C^{1/2}Q_0^{1/2}dr^2}{W^{(4d)}r^3} + C^{1/2}Q_0^{1/2}r(dx^2 + dy^2). \tag{5.92}$$

On the other hand, the dimensional reduction of the $A = 0$ class of five-dimensional Nernst branes gives

$$ds^2_{(4)} = -\frac{r^5W}{\Delta^{1/2}L^3}dt^2 + \frac{L\Delta^{1/2}}{r^3W}dr^2 + \frac{r\Delta^{1/2}}{L^3}(dx^2 + dy^2), \tag{5.93}$$

where we have used (5.77) with $A = 0$. Again we identify the functions $W^{(4d)}$ and W appearing in the above metrics, which means the parameters B_0 and T will be the same in both cases. As before, this prevents rescaling of the coordinate r and then, by comparing dr^2 terms in (5.92)

and (5.93), we establish the following relationship between four- and five-dimensional quantities

$$16^2 C Q_0 = L^2 \Delta. \tag{5.94}$$

Again, the remaining metric coefficients can be made to match by rescaling t, x, y by constant factors involving L. Following the same procedure as in Section 5.3.2, we match the gauge field and Kaluza-Klein vector by comparing expressions for $\dot{\zeta}^0$. Specifically, we match (5.23) with the τ-derivative of (4.64). The two are equivalent providing

$$Q_0 = -\frac{\Delta}{2\sqrt{3}}, \tag{5.95}$$

which expresses the four-dimensional electric charge in terms of the five-dimensional boost parameter Δ. This is a much simpler expression than in the $A > 0$ case and we observe that this matches the $A \to 0$ limit of (5.84). Considering the discontinuities we have encountered previously when taking $A \to 0$ limits, this seems at first surprising but actually this is completely consistent with Q_0 being a smooth parameter for the four-dimensional solutions of Chapter 4. Having established $Q_0 < 0$, we see from (5.85) that $A \to 0$ corresponds to $h_0 \to 0^-$, and thus from (5.87) that $\mu \to -\infty$. Lastly, we can substitute (5.95) into (5.94) to find the relationship between the overall normalizations of the metrics (5.92) and (5.93),

$$16^2 C = -2\sqrt{3} L^2. \tag{5.96}$$

Clearly this requires $C < 0$ as before and, in fact, is exactly the same relationship as for the $A > 0$ case in (5.86), which is expected since C and L are only sensitive to the four- and five-dimensional multiplet actions respectively, and these are independent of A. Again, since we have matched the gauge fields by comparing $\dot{\zeta}^0$, the chemical potentials must match and this is indeed the case: using the asymptotic value of (5.82) with $A = 0$, we find $\mu = -\infty$ which agrees with the negatively charged, $h_0 = 0$ solutions in Chapter 4.

The parameter counting is also simpler in the $A = 0$ case. Five-dimensional Nernst branes are parametrized by two independent and continuous parameters (B_0, Δ), or equivalently temperature and momentum. However, as we have seen in Section 5.3.1, no new length scale is introduced by the reduction and consequently, the four-dimensional solution obtained via dimensional reduction also depends on exactly two independent parameters: (B_0, Δ) are sufficient to completely determine (5.92) since $A = 0$ is fixed. Using (5.95), these are equivalent to (T, Q_0) with $\mu = -\infty$ is fixed. The difference between the five-dimensional and four-dimensional parameters is that the S^1 again causes charge quantization. This means that whilst both B_0 and Δ are continuous in five dimensions, reducing to four dimensions forces one parameter, namely $Q_0 = -\frac{\Delta}{2\sqrt{3}}$, to become discrete.

One difference between the $A = 0$ solution and the $A > 0$ solution is that for the $A = 0$ solution the compactification circle has no critical size. Therefore we cannot relate the momentum P^0 to the electric charge Q_0 using r_0 as a reference scale, and thus we cannot explicitly see charge quantization in the same way as we did before. This is not a problem since we can relate Q_0 to five-dimensional quantities through (5.95), and moreover, we have seen that the relation between Q_0 and five-dimensional quantities has a well defined limit for $A \to 0$. A related feature of the $A = 0$ solution is that since the compactification circle has no minimum, it contracts to zero size as $r \to \infty$. That means that there is a region in this solution, where the circle has sub-Planckian, or sub-stringy size. While this is problematic for an interpretation as a four-dimensional solution, the lifted five-dimensional solution is simply AdS_5, and can be described consistently within five-dimensional supergravity.

5.3.4 Curvature properties of four-dimensional Nernst branes

We have just demonstrated how the four-dimensional solutions with $A > 0$ and $A = 0$, obtained in Sections 5.3.2 and 5.3.3, exactly match the $h_0 \neq 0$ and $h_0 = 0$ solutions of Chapter 4 respectively. In Chapter 4 these four-dimensional solutions were observed to have hyperscaling-violating Lifshitz geometries and it is known from [121] that such solutions suffer from various curvature singularities. We shall now investigate this explicitly for the metrics (5.77) and (5.93).

Curvature invariants

As with the five-dimensional spacetimes in Section 5.2.4 we can determine the presence of curvature singularities in our four-dimensional solutions by looking at the Kretschmann scalar and Ricci scalar associated to the metrics (5.77) and (5.93). Indeed, since any singular behaviour in the curvature will already be present for the extremal solutions, we will concentrate only on the case $r_+ = 0$. The curvature invariants are:

$$\begin{aligned}
K_4^{A>0} &= \frac{r^2\left(351A^4r^{16} + 1476A^3r^{12}\Delta + 2586A^2r^8\Delta^2 + 1284Ar^4\Delta^3 + 959\Delta^4\right)}{4L^2\left(Ar^4+\Delta\right)^5}, \\
R_4^{A>0} &= \frac{3\left(15A^2r^8 + 34Ar^4\Delta + 15\Delta^2\right)}{2\sqrt{\frac{Ar^4+\Delta}{r^4}}\left(Ar^4+\Delta\right)^2 rL}, \\
K_4^{A=0} &= \frac{959r^2}{4\Delta L^2}, \qquad R_4^{A=0} = \frac{45r}{2\sqrt{\Delta}L}.
\end{aligned} \tag{5.97}$$

For $A > 0$, or equivalently $|\mu| < \infty$, we find that the Ricci scalar behaves as $R \sim r^{-1}$ for large r, and $R \sim r$ for $r \to 0$, whilst the Kretschmann scalar scales as $K \sim r^{-2}$ and $K \sim r^2$ in these respective regions. Hence, the curvature invariants will remain finite along the solution. However, for the $A = 0$ solution we will still have the same behaviour at $r \to 0$, but asymptotically we find $R \sim r$ and $K \sim r^2$. We therefore have a naked curvature singularity as

B_0, h_0	Near horizon		Asymptotic	
	Curvature singularity	Divergent tidal forces	Curvature singularity	Divergent tidal forces
$B_0 = 0, h_0 = 0$	×	✓	✓	×
$B_0 = 0, h_0 \neq 0$	×	✓	×	×
$B_0 > 0, h_0 = 0$	×	×	✓	×
$B_0 > 0, h_0 \neq 0$	×	×	×	×

Table 5.2: Summary of singular behaviour of four-dimensional Nernst brane.

B_0, A	Near horizon		Asymptotic	
	Curvature singularity	Divergent tidal forces	Curvature singularity	Divergent tidal forces
$B_0 = 0, A = 0$	×	✓	×	×
$B_0 = 0, A > 0$	×	✓	×	×
$B_0 > 0, A = 0$	×	×	×	×
$B_0 > 0, A > 0$	×	×	×	×

Table 5.3: Summary of singular behaviour of five-dimensional Nernst brane.

we approach the boundary of the spacetime.

Tidal forces

To investigate whether the four-dimensional solutions of Chapter 4 admit infinite tidal forces in the near horizon regime we follow the analysis of [121], albeit considering a slightly simpler set-up in which the infalling observer is moving only in the radial direction i.e. has zero transverse momentum. The technical details of this procedure are relegated to Appendix E.4.2. The results in Tables E.3 and E.4 show that, for both $A > 0$ and $A = 0$, there exist components of the Riemann tensor, as measured in the PPON, that diverge as $r \to 0$. This indicates that the radially infalling observer will experience infinite tidal forces at the extremal horizon, $r_+ = 0$. As before, tidal forces will remain finite on non-extremal horizons, $r_+ > 0$.

5.3.5 Curing singularities with decompactification

A summary of the singular behaviour of our four- and five-dimensional solutions can be found in Tables 5.2 and 5.3. Notice that since B_0 and A control the near horizon and asymptotic geometries respectively, we can use these to catalogue any singularities.[51] We will now demonstrate explicitly how certain singularities present in the four-dimensional hyperscaling-violating Lifshitz solutions of Section 5.3.4 can be removed by dimensional lifting to the asymptotically Anti de-Sitter solutions of Section 5.2.4.

[51] Given (5.85) it is clear that $A = 0$ matches $h_0 = 0$, and $A > 0$ matches $h_0 \neq 0$.

Curvature invariants

The usual dimensional reduction paradigm relates the five-dimensional Ricci scalar to its four-dimensional counterpart by[52]

$$R_5 \sim e^{\sigma} R_4.$$

Tables 5.2 and 5.3 indicate that the only situation where we encounter a curvature singularity is the asymptotic regime of the four-dimensional solution with $h_0 = 0$, or equivalently $A = 0$ if we are working with five-dimensional parameters. In this instance we have $R_4 \sim r$ from (5.97) whilst $e^{\sigma} \sim 1/r$ from (5.76) resulting in R_5 being asymptotically constant and exactly equal to the value of global AdS_5 as seen in Section 5.2.4. Recalling that the dilaton e^{σ} measures the geodesic length of the x^0 circle, we can now account for the presence of an asymptotic curvature singularity in this class of four-dimensional Nernst branes. Specifically, the four-dimensional $|\mu| = \infty$ asymptotic curvature singularity emerges from a 'bad slicing,' of the parent AdS_5 hyperboloid by a compactification circle that gets pinched at infinity. It was shown previously that the independent four-dimensional scalars are all proportional to each other, see formula (4.52). It was also observed that for infinite chemical potential these scalars approach zero asymptotically. From the five-dimensional point of view, the single profile of the four-dimensional scalars determines the profile of the Kaluza-Klein scalar. Therefore the four-dimensional scalars approaching zero corresponds to the shrinking of the compactification circle. When combining this with the singular behaviour of the four-dimensional metric, we obtain AdS_5.

In the $A > 0$, or equivalently $|\mu| < \infty$ case, the four-dimensional solution of Chapter 4 is asymptotically conformal to AdS_4, or CAdS_4 for short. We see from (5.97) that the curvature invariants of CAdS_4 behave as $R_4 \sim 1/r$ and vanish asymptotically. At the same time, this is compensated by $e^{\sigma} \sim r$ from (5.76), meaning the circle now blows up at large r such that R_5 remains asymptotically constant and equal to R_{AdS_5}. Thus, in this case the asymptotic behaviour of the four-dimensional metric and scalars is reversed compared to the $A = 0$ case, but still leads to the same five-dimensional aysmptotic geometry after lifting.

Tidal forces

As can be seen from Tables 5.2 and 5.3, tidal forces are irrelevant asymptotically and so we are only concerned with the situation near the horizon.[53] It is clear that infinite tidal forces are present at the horizon of the extremal Nernst brane in four dimensions, and are not removed by dimensional lifting. We expect this persistence may well be indicative of a deeper result, namely

[52] A similar argument can be provided for the Kretschmann scalar albeit with the second power of the dilaton, $K_5 \sim e^{2\sigma} K_4$, which is natural given that it is quadratic in the curvature.

[53] Tidal forces only apply to time-like geodesics. In Section 3.5 we saw that outgoing time-like geodesics cannot reach the AdS boundary but instead turn around due to an infinite potential wall.

that infinite tidal forces on the extremal horizon is the price one has to pay in order to obtain solutions that satisfy the third law of Thermodynamics which is, after all, a singular limit of the horizon itself.

5.4 Summary

In this chapter we have constructed a two-parameter family of five-dimensional black branes in FI gauged $\mathcal{N} = 2$ supergravity. We checked that they satisfy the Nernst Law i.e. that they are Nernst branes, and also that they are the dimensional lifts of the four-dimensional solutions in Chapter 4. We also gave a detailed analysis of the five-dimensional thermodynamics and investigated whether the singular behaviour of the four-dimensional solutions is eliminated by dimensional oxidation. We will give a more detailed discussion of these points in Chapter 6.

6 Conclusion and outlook

In this thesis we have constructed black brane solutions of $\mathcal{N} = 2$ gauged supergravity with vanishing entropy density in the extremal limit. Such solutions therefore satisfy the strong version of the Nernst Law of Thermodynamics and are referred to as 'Nernst branes'. Finding such solutions in the supergravity regime is highly non-trivial as it necessarily involves a gauging of the supergravity in order to produce a non-spherical horizon topology. This gauging adds a scalar potential to the Lagrangian, making the equations of motion substantially more difficult to solve.

Motivated by the four-dimensional extremal STU Nernst brane already in the literature [24], we proceeded to construct new Nernst brane solutions in four dimensions. We were able to improve on the existing work by finding non-extremal (or 'hot') Nernst branes for a specific charge and FI parameter configuration. With the presence of temperature we were able to analyse the thermodynamics and find an equation of state from which we could see the limiting behaviour $T \to 0$, and thus provide a more satisfactory demonstration of the Nernst Law. Furthermore, we did not restrict ourselves to a single model but instead solved for an entire class of models, which included the very special models that can be lifted to five dimensions.

The gauging introduced a scalar potential able to play the role of a cosmological constant, thus permitting non-asymptotically flat solutions. Given that the Nernst Law is considered generic for condensed matter systems it was hoped that we could find asymptotically AdS_4 solutions representing prospective gravitational duals. Solving the equations of motion, we found a family of black branes that depend on the two parameters (T, μ), where the temperature T and chemical potential μ controlled the near horizon and asymptotic spacetime geometries respectively. Further analysis revealed that these solutions assumed a hvLif spacetime structure and thus could not be used in the AdS/CMT dictionary. However, recent developments in the literature have revealed that many systems in condensed matter actually exhibit scaling behaviours captured by such metrics, leading to attempts at an exciting new hvLif/CMT dictionary [34].

The particular hvLif horizon geometry depended on whether the temperature was zero or not and similarly at the boundary it depended on whether the chemical potential was finite or infinite. It transpired that $T = 0, \mu = \infty$ was a global hvLif solution that could be interpreted as the ground state, and then varying either T or μ led to a solution that interpolated between different hvLif geometries and thus represented an RG flow between non-conformal field the-

ories in both the UV and IR. However, these four-dimensional solutions suffered from certain problems. At the horizon both the geometry and also a field theory argument implied that the entropy density scaled with temperature as $s \sim T^{1/3}$ which would be consistent with the Nernst Law. However, there were large tidal forces on the horizon that became infinite in the extremal case and suggested we shouldn't trust any duality in the deep IR. Meanwhile, the problems in the deep UV were even more serious: the asymptotic regime saw a disagreement in the predicted entropy density scaling behaviour between the gravitational ($s \sim T$) and field theoretic ($s \sim T^3$) descriptions. Furthermore, in the case of infinite chemical potential there was even an asymptotic curvature singularity destroying any hope of meaningful holography. All of this suggested that our four-dimensional solution could only be trusted in some intermediate energy range.

Further examination of our four-dimensional solutions showed that for any finite value of the chemical potential, the physical scalar fields exhibited runaway behaviour asymptotically. Since we were working within the 'very special' class of models that can be lifted to five dimensions, it was expected that the solution entered a decompactification regime. Performing the dimensional lift, we found a family of boosted black brane solutions. As expected, the oxidation of asymptotically hvLif spacetimes produced asymptotically AdS solutions [34, 35]. Not only did these five-dimensional asymptotically AdS black branes fit more naturally into the gauge-gravity duality via the AdS/CFT correspondence but, at the same time, there is a substantially better understanding of their asymptotic geometry. In particular, the correct counterterms are known for the purpose of holographically renormalizing the stress-tensor and this allowed conserved charges to be computed using quasilocal techniques. Consequently a much more thorough thermodynamic analysis was possible in five dimensions, allowing us to write down a consistent first law of black brane mechanics. We also verified that in the near horizon regime the entropy density continued to scale with temperature as $s \sim T^{1/3}$ and thus vanished in the extremal limit such that our five-dimensional solutions were Nernst branes also. The scaling behaviour of entropy density with temperature in the deep UV as calculated from the geometry was $s \sim T^3$ which matched the field theory prediction that our four-dimensional Nernst branes were unable to match. This suggested that we were not able to see the complete picture with the four-dimensional Nernst branes and that we really must go to five dimensions to access the full degrees of freedom of the system.

Such interpolating behaviour has previously been observed in [160, 164] for certain D3-brane solutions in Type IIB string theory which also have an entropy density that scales as $s \sim T^{1/3}$ in the near horizon regime and $s \sim T^3$ in the asymptotic regime. Indeed, [160] observes that the near horizon metrics of these ten-dimensional D3-brane solutions describe one of the following

Einstein manifolds:

$\mathrm{AdS}_5 \times S^5$	extremal without pp-wave,
$\mathrm{K}_5 \times S^5$	extremal with pp-wave,
$\mathrm{C}_5 \times S^5$	non-extremal,

where K_5 and C_5 are the generalized five-dimensional Kaigorodov and Carter-Novotný-Horský metrics. By compactifying the 5-spheres that foliate the space transverse to the D3-brane, it is clear that the near horizon geometry of the D3-brane is either AdS_5, K_5 or C_5. Remarkably, we are able to show that the near horizon geometries of our five-dimensional Nernst branes exactly match this classification. In particular, for the extremal case, the different near horizon geometries of the D3-brane depend on the presence or absence of a superimposed pp-wave propagating along its worldvolume, and these geometries precisely correspond to those of the extremal five-dimensional Nernst brane in the presence or absence of momentum. Given that we are working with five-dimensional gauged $\mathcal{N} = 2$ supergravity, the dual UV field theory should be a conformally invariant $\mathcal{N} = 1$ field theory. Since we are able to relate our five-dimensional Nernst branes to certain limits of D3-branes, and since the solution is asymptotically AdS_5, we might expect a similar dictionary between geometry and field theory as exists for $\mathcal{N} = 8$ supergravity and $\mathcal{N} = 4$ Super Yang Mills. In particular, it is possible that the dual field theory might be a conformally invariant $\mathcal{N} = 1$ Super Yang Mills theory or a deformation thereof, but without having a higher dimensional embedding which allows one to understand the role of the parameters c_{ijk} and g_i in the gauge theory, we can't say much more. Indeed, it is important to emphasise that FI gauged five-dimensional $\mathcal{N} = 2$ supergravity has so far only been obtained as a consistent truncation of a higher dimensional supergravity in the special cases of pure FI gauged supergravity (no physical vector multiplets) [175] and the STU-model [176, 177].[54] The dimensionally reduced, boosted D3-branes of [160] which match the near horizon geometries of our Nernst branes are themselves solutions of $\mathcal{N} = 8$ supergravity, which is obtained by compactifying Type IIB supergravity on an S^5. In this case the five-dimensional cosmological constant is completely determined by the D3-charge, and we cannot account for the parameters c_{ijk}, g_i of an FI gauged supergravity theory with vector multiplets. Therefore, whilst relating our five-dimensional Nernst branes to boosted D3-branes in Type IIB string theory is helpful for motivating potential holographic interpretations, it should not be considered as evidence of a stringy embedding. It would be interesting to investigate the higher dimensional lift of the two exceptional cases to learn about their dual field theories and try to identify any properties that might hold generically for all field theories in our family of solutions.

[54] Other consistent truncations involve hypermultiplets or massive vector multiplets and consequently have a different gauging to that discussed in Section 3.4 [178, 179].

An interesting feature of our five-dimensional solution is the non-trivial extremal limit, where the boost parameter is sent to infinity, while the momentum density is kept fixed. The resulting extremal near horizon geometry should define a field theory with entropy-temperature relation $S \sim T^{1/3}$. In the context of boosted D-branes and M-branes, the proposed interpretation is a conformal field theory in the infinite momentum frame, which carries a finite momentum density [160]. Moreover, it was proposed in [162, 163, 164, 180] that the compactification of the direction along the boost corresponds to discrete light cone quantisation. In this respect it is interesting to look at the scaling symmetries of the five- and four-dimensional extremal solutions near the horizon. In five dimensions the metric looks like a Lifshitz metric with $z = 3$ and $\theta = 0$, except that the direction along the boost has weight -1 instead of $+1$. Upon reduction to four dimensions, the near horizon geometry, and if we go to infinite chemical potential even the global geometry, is a hyperscaling violating Lifshitz geometry with $z = 3$ and $\theta = 1$ [1]. That is, by reduction over the boost direction one trades the non-trivial scaling of this direction for an overall scaling of the metric. Following [162, 163, 164, 180] we propose to associate a four- and a three-dimensional field theory to the near horizon five- and four-dimensional geometries, respectively, with the three-dimensional theory encoding the zero mode sector of the discrete light cone quantisation of the four-dimensional theory. Both theories are non-relativistic with Lifshitz exponent $z = 3$, and supersymmetric with two supercharges.[55] The four-dimensional theory is scale invariant and arises by deforming a four-dimensional relativistic $\mathcal{N} = 1$ supersymmetric theory by a finite momentum density, while the three-dimensional theory is scale covariant.

Having obtained a tentative holographic interpretation for both the trivial and non-trivial extremal limits of our five-dimensional solutions, we should remember that the non-extremal solutions are simply thought of as being dual to thermal states within these field theories.

The five-dimensional Nernst branes are exactly the dimensional lifts of our four-dimensional Nernst branes, and it is good news for holography that the lifting procedure removes all asymptotic curvature singularities as well as the mismatch between geometrical and thermodynamic scaling relations that was present in the four-dimensional solution. To explain this, it is crucial to understand the variation of the compactification circle along the transverse direction, which from the four-dimensional point of view is encoded in the scalar fields. The apparently singular behaviour of the $\mu = \infty$ four-dimensional geometry arises from a singular slicing of the AdS_5 hyperboloid by a circle of zero size, and is exactly compensated for by the singular behaviour of the four-dimensional scalars, resulting in completely regular five-dimensional asymptotics. Moreover, the compactifcation circle also accounts for the four-dimensional chemical potential, which has no counterpart in the uncompactified five-dimensional solution. In particular, for generic compactifications involving a circle with $A > 0$, we observe that once we decide to make the boost direction compact, the dynamics forces the circle to expand at both ends of

[55] According to the analysis of [181], extremal four-dimensional Nernst branes are BPS.

the RG flow and the resulting minimum introduces a new parameter which we can relate to the chemical potential. As proposed in Section 4.4, we can interpret the apparently singular UV behaviour of four-dimensional Nernst branes as a dynamical decompactification limit, which tells us that the description as a four-dimensional system breaks down and has to be replaced by a five-dimensional one. Interestingly however, the five-dimensional solutions still suffer from infinite tidal forces on the extremal event horizon. The fact that this persists whilst all other four-dimensional problems are resolved by dimensional oxidation may well be indicative of a deeper result, namely that infinite tidal forces on the extremal horizon is the price we have to pay in order to obtain solutions that satisfy the third law of thermodynamics.

From a strictly gravitational point of view, one should still worry about the divergent tidal forces which are present in the extremal limit irrespective of whether we consider four-dimensional or five-dimensional Nernst branes. While sometimes considered to be 'mild,' they represent genuine singular behaviour and make the solution geodesically incomplete. Moreover, they are not cured by stringy α'-corrections [121], and strings probing such singularities get infinitely excited [182]. While at finite temperature there is technically no singularity, near extremality objects falling towards the event horizon will still experience very large tidal forces [33]. This behaviour is, if not an inconsistency, at least a sign that the singularity has physical relevance. Furthermore, the infinite tidal forces are clearly caused by the way the metric complies with the strong version of the third law, namely through a warp factor which scales any finite piece of the world-volume[56] to zero volume. Whilst it is not at all obvious how the divergent tidal forces could be removed in such a way as to preserve the strong version of the third law, we list below some possibilities. Firstly, it would be worthwhile examining whether the tidal forces can be resolved with the inclusion of higher curvature terms. This has previously been investigated in [35, 183]. Experience with small black holes suggests that these would lead to a regular horizon with non-zero area density. Indeed, for small BPS black holes, R^2-corrections remove null curvature singularities, by making the area finite [23]. But as these singularities involve the divergence of some curvature invariant, the implication for tidal forces is unclear. The only example we are aware of where tidal forces have been successfully removed is for the D6-brane of type IIA supergravity, and this happens through an M-theory embedding [184]. However, this does not allow immediate conclusions about the case we are interested in. Secondly, since supergravity theories with eight real supercharges exist in up to six dimensions [27], we could perform a further dimensional lift of our five-dimensional Nernst brane to see if this removes the problem. This also relies on the existence of a string theory embedding since lifting a five-dimensional theory of ungauged supergravity with a very special prepotential to six dimensions is related to either the F-theory limit of the M-theory embedding $\mathrm{M}/CY_3 \to \mathrm{F}/CY_3$,

[56] Here 'finite' refers to the Euclidean metric defined by the coordinates x, y, z, which we use to refer extensive quantities to 'unit world-volume.'

or to the limit $\text{Het}/(K3 \times S^1) \to \text{Het}/K3$ where the ten-dimensional heterotic string theory is compactified over one less dimension. Once again this does not apply to our particular set-up involving gauged supergravity, but it would be interesting to see if a similar lifting to six dimensions exists and whether it can play a role in removing the singular behaviour. Thirdly, we can also approach the problem from the field theory side. For example, in [185] they study the infinite momentum frame CFT dual to a boosted brane and find evidence that the CFT resolves the geometric singularity. In our case it would be interesting to understand the dual four-, or possibly, three-dimensional IR field theory, and to investigate whether it is non-singular, and whether its ground state is unique or degenerate. If it transpires that the ground state is unique, one would need to understand whether this means that (i) tidal forces are acceptable, (ii) they are not, but the dual field theory can be used to construct a 'quantum geometry' of some sort, (iii) or if there is some kind of breakdown of gauge/gravity duality in the extremal limit. Points (i) and (iii) are not necessarily mutually exclusive, since one might invoke the weaker, process version of the third law to assure that the extremal limit cannot be reached by any physical process.

On this last point, it is important to remember that gauge/gravity duality works in both directions and employing holography to investigate the true meaning of tidal forces is only exploiting one direction of the duality. Given a more complete dictionary between both sides of the correspondence, an ambitious, yet important goal, would be to investigate the possibility of designing Nernst branes that are tailored to model specific condensed matter systems. Calculations that are difficult in the strongly coupled gauge theory could then be performed perturbatively in the gravitational theory and translated back to field theory results.

At the same time however, this project has opened up a number of other exciting new research directions. For example, the four-dimensional solution that we construct in detail in Chapter 4 relies on a single electric charge along with an electric FI parameter configuration consistent with the 'purely imaginary' scalar fields, such that the Lagrangian and equations of motion simplify substantially. An interesting extension of our work would be to attempt to find solutions without this restriction and where we have additional charges and FI parameters switched on. We expect that our formalism is particularly suited to finding dyonic solutions, due to its built-in electric-magnetic covariance [3, 94]. Whilst we do not currently have the technology to solve the full three-dimensional equations of motion, it is encouraging that work on static BPS solutions in $U(1)$ gauged supergravity with symmetric scalar target spaces has led to the construction of the general dyonic solution [186, 187, 188, 189]. A simple example of such an extension is the four-dimensional, magnetically-charged Nernst brane constructed in Section 4.3. This solution suffers from the same singular behaviour as its electrically-charged cousin which again suggests a decompactification to five dimensions. However, in five dimensions, only electric FI gaugings of the supergravity are possible and so there is no analogue of the magnetic FI parameter

g^0 that is switched on in the four-dimensional solution. Presumably this parameter arises in the four-dimensional theory via a more complicated compactification of the five-dimensional Nernst brane. It would be extremely interesting to explore this further given its significance for understanding magnetic, or more generally, dyonic, four-dimensional Nernst branes.

It is also worthwhile to point out that our technique of time-like dimensional reduction combined with special geometry that we use to obtain Nernst brane solutions in both Chapters 4 and 5, involves solving the full second order field equations, with the number of integration constants in the solution reflecting this. However, we also see that once we impose regularity of the black brane solutions at the horizon[57] the number of integration constants is reduced by one half, such that the solution satisfies a unique set of first order equations. For BPS and, more generally, extremal solutions, the existence of first order flow equations results from fixed point behaviour implied by the attractor mechanism. Since for non-extremal solutions, scalar flows are terminated on the horizon before reaching any fixed points, the existence of such a first order rewriting of the equations of motion is surprising. However, such behaviour is not unprecedented [3], and has even been interpreted as a remnant of the attractor mechanism [159].[58] For our five-dimensional solutions the scalars are constant, so that the only sense in which we have attractor behaviour is that the scalars sit at a stationary point of the scalar potential. However, from the four- and three-dimensional perspective we have scalar fields which need to exhibit a particular, fine-tuned behaviour at the horizon in order to make the five-dimensional solution regular. This is very similar to attractor behaviour, and the effect of reducing the number of integration constants by one half is the same. Such universal features of scalar dynamics definitely deserve further study.

On a different note, in the usual gauge/gravity set-up, the extra radial dimension is taken to represent the energy scale of the associated field theory. In principle, given the correct dictionary, the radial flow equation of some geometrical quantity should become the RG flow equation for the associated field theory. Given that in field theory there may exist a function, known as the a-function, that behaves monotonically along the RG flow, we could investigate what the corresponding quantity would be on the gravity side. It has been suggested by the authors of [191] that one possibility would be the renormalised entanglement entropy and indeed, their investigations for BPS black branes in four-dimensional $\mathcal{N} = 2$ gauged supergravity find that this quantity decreases monotonically along the RG flow up to a critical point, beyond which it rises and approaches the entropy density of the brane itself. However, given that our extremal Nernst branes have the unusual property of vanishing entropy density, there is an exciting possibility that this monotonicity can be maintained throughout the entire RG flow, leading to a gravitational realisation of the a-function.

[57] This is done for the generic situation with finite temperature.
[58] A related idea seems to be that of 'hot attractors' [190].

Lastly, it would be interesting to also consider the inclusion of hypermultiplets into the theory. It is thought that this will produce Lifshitz rather than hyperscaling-violating Lifshitz geometries in the four-dimensional theory, and since string compactifications with $\mathcal{N} = 2$ supersymmetry always involve at least the universal hypermultiplet, it is expected that this will help with the embedding of these non-relativistic solutions into string theory [174]. Whilst the dual Lifshitz field theory would be non-relativistic, it would still be conformal, and it is hoped that a microstate counting of the black hole entropy would then be possible. This could prove invaluable for understanding the true behaviour of Nernst brane entropy density in the zero temperature limit. However, gauging the hypermultiplets leads to the bosonic field content becoming charged under the gauge group. This means we can no longer dualise vectors into scalars at the three-dimensional level and so we would need to find an alternative means of solving the equations of motion.

In conclusion, we think that the systematic methods and explicit, analytic solutions that we have presented in this thesis will help to make progress towards a classification of solutions in gauged supergravity, as well as to extend and deepen our understanding of the gauge/gravity dictionary. Moreover, our solutions represent an excellent playground for testing a number of other important questions, as outlined above.

A Einstein-Hilbert conventions

Let us explain our choice of conventions for the Einstein-Hilbert term in a little more detail. For this we compare with Appendix A.1 of [27] which provides an excellent overview of all possible combinations of sign choices. Given variables $s_1, s_2, s_3 \in \{\pm 1\}$, [27] tells us that we should first make a choice for the sign of the Minkowski metric

$$\eta_{ab} = s_1 \mathrm{diag}(- + \cdots +).$$

The second sign choice then appears in the Riemann tensor,

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = s_2 \left(\partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\tau} \Gamma^\tau{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\tau} \Gamma^\tau{}_{\mu\sigma} \right).$$

The third sign choice appears in the Einstein equation

$$s_3 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = T_{\mu\nu},$$

or equivalently, in the definition of the Ricci tensor,

$$s_2 s_3 R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}.$$

The signs s_1 and s_3 then determine the sign of the kinetic energies of the scalars and the graviton by entering the Lagrangian as follows

$$\mathsf{e}^{-1} \mathcal{L} = s_1 s_3 \frac{R}{2} - s_1 \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

Comparing with our Lagrangians in e.g. (3.13) and (3.20), we have clearly made the choices

$$s_1 = 1, \quad s_3 = -1.$$

In order to continue defining the Ricci tensor as $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$, we should also choose

$$s_2 = -1.$$

Let us now discuss the consequences of this choice. With these particular conventions, a space-like surface of positive curvature has $\text{sign}(R) = s_1 s_3 = -1$. In other words, spaces with positive curvature have $R < 0$ and vice-versa. Of course this is to be expected if we require the two possible Einstein-Hilbert actions

$$S_{\text{EH}} = -\int d^D x \mathsf{e} \frac{R}{2} \quad \text{and} \quad S'_{\text{EH}} = \int d^D x \mathsf{e} \frac{R'}{2}, \tag{A.1}$$

to represent the same physical information. Notice also that there is an additional minus sign introduced into the Einstein equation such that it now reads

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -T_{\mu\nu}. \tag{A.2}$$

In Chapter 2 we provided the standard differential geometry definition of the Riemann tensor in (2.8). However, we note that in Chapters 3, 4 and 5 the minus sign accompanying the Einstein-Hilbert term in the Lagrangian forces us to redefine this for all physical applications with a minus sign as

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = -\left(\partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\tau}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\tau}\Gamma^{\tau}{}_{\mu\sigma}\right). \tag{A.3}$$

B Non-linear sigma models

B.1 Harmonic maps and target space geodesics

Differential geometry has an important role to play in theories of supergravity. Of course it is well known that we treat spacetime as a differentiable manifold but it is also possible to construct a second geometry using the dynamics of the scalar fields as follows. Consider a matter-coupled supergravity action with schematic form

$$S[\phi, \dots] = \int d^D x \sqrt{g} \left(-\frac{R}{2} - G_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j + \dots \right), \tag{B.1}$$

where the coupling matrix G_{ij} depends on the n scalar fields ϕ^i, and the dots stand for additional terms involving field strengths, fermions etc. Such theories are referred to as gravity-coupled *non-linear sigma models*.

Let us consider the Einstein equations (A.2),

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -T_{\mu\nu},$$

where we have set $8\pi G = 1$ in accordance with the rest of this thesis. By taking a trace and back-substituting, we can reduce this to

$$\frac{1}{2} R_{\mu\nu} - G_{ij}(\phi) \partial_\mu \phi^i \partial_\nu \phi^j = 0. \tag{B.2}$$

Separately, one can apply an Euler-Lagrange procedure to establish the equations of motion for the scalar fields,

$$\triangle_{(g)} \phi^i + \Gamma^i{}_{jk}(\phi) \partial_\mu \phi^j \partial^\mu \phi^k = 0, \tag{B.3}$$

where the connection components $\Gamma^i{}_{jk}(\phi)$ are

$$\Gamma^i{}_{jk}(\phi) = \frac{1}{2} G^{il} \left(\partial_j G_{lk} + \partial_k G_{lj} - \partial_l G_{jk} \right), \tag{B.4}$$

where $\partial_i = \frac{\partial}{\partial \phi^i}$.

We can now imagine that a particular field configuration, or solution to the equations of motion (B.3), denoted $\phi^a(x)$, establishes a mapping between the D-dimensional spacetime $(\mathcal{M}, g)$

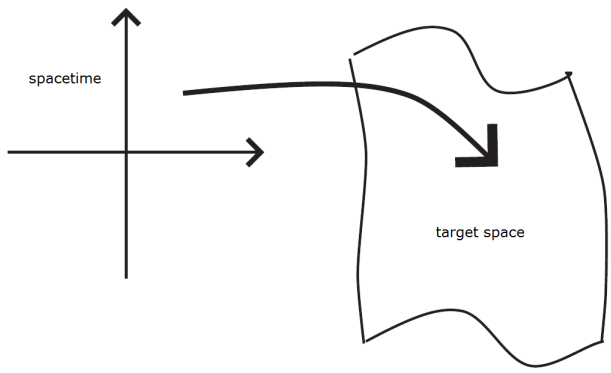


Figure B.1: Scalar fields as harmonic maps from D-dimensional spacetime $(\mathcal{M}, g)$ to an n-dimensional target space (M_n, G) [27].

and an n-dimensional pseudo-Riemannian *target space* (M_n, G). The scalar fields ϕ^i act as coordinates on M_n, and since we assumed $i = 1, \ldots, n$, it must be n-dimensional. Further, the coupling matrix $G_{ij}(\phi)$ of the scalar kinetic term in the action (B.1) assumes the role of a metric on M_n. This explains the form of the connection components (B.4): they are precisely the connection components of the Levi-Civita connection on (M_n, G) as discussed earlier in (2.13).

Any solution to the equations of motion (B.3) determines a *harmonic map* $\phi : \mathcal{M} \to M_n$ as shown in Figure B.1. Such mappings represent geodesics on M_n, and it is clear that knowledge of these geodesics allows the Einstein equation (B.2) to then completely determine the D-dimensional spacetime geometry. This will be a recurring theme when considering supergravity actions in this thesis: we shall see that knowing the scalar field coupling matrix is sufficient to fully determine the spacetime dynamics.

B.2 Potential disturbances to geodesic motion

When dealing with theories of $\mathcal{N} = 2$ gauged supergravity, a scalar potential must be introduced to the Lagrangian (B.1). This potential will disturb the harmonic map such that solutions of the spacetime scalar equations of motion are no longer target space geodesics. To illustrate this point, we shall consider the simple example of a one-dimensional spacetime for which $\phi^i = \phi^i(\tau)$. The Lagrangian will be of the form

$$S[\phi, \ldots] = \int d\tau \sqrt{g} \left(-\frac{R}{2} - G_{ij}(\phi)\dot{\phi}^i \dot{\phi}^j + V(\phi) + \ldots \right), \tag{B.5}$$

where again we neglect the terms not involving scalar fields. Given the negative coefficient for the Einstein-Hilbert term, we pick a positive coefficient for the scalar potential to respect the basic structure of the Lagrangian describing a system with potential in classical mechanics.

Obviously in the case $V(\phi) = 0$, the scalar field equation of motion is $\frac{D}{d\tau}\dot{\phi}^i = 0$ where D is

the Levi-Civita connection, and we recover the geodesic motion of Appendix B.1. In this case, the solutions can be written as

$$\phi^i(\tau) = \phi_0^i = \text{ const.}$$

Such solutions with constant scalars typically correspond to vacuum states of the theory. The target manifold, M_n, represents the space of allowed vacua parametrized by the ϕ_0^i, and is known as the *moduli space of vacua*.

However, for the gauged supergravity models we deal with in this thesis, $V(\phi) \neq 0$, and the scalar field equation of motion becomes[59]

$$\frac{D}{d\tau}\dot{\phi}^i = -G^{ij}(\phi)\partial_j V(\phi) = -\nabla^i V(\phi). \tag{B.6}$$

This means that whenever $V(\phi)$ is a non-constant function of the scalar fields, there exists a gradient force acting on the target manifold trajectory, causing it to deviate from a geodesic path. From (B.6), it is still possible to have vacuum solutions with constant scalars $\dot{\phi}^i = 0$ in the case when

$$\partial_i V(\phi = \phi_0) = 0.$$

This means that the moduli space of vacua is reduced to extrema of the scalar potential, $V(\phi)$. Furthermore, we would typically only be interested in minima so as to ensure physical stability under perturbations of $V(\phi)$. The values of the scalar fields at a particular minimum are usually referred to as *vacuum expectation values* [27].

Let us now assume the existence of such stable minima and consider their effect on the spacetime geometry. To do this, we will need the Einstein equation coming from (B.5). Since $\dot{\phi}^i = 0$, the scalar kinetic term drops out, and after taking a trace and back-substituting, we find

$$R_{\mu\nu}(g) = \frac{2V(\phi_0)}{D-2}g_{\mu\nu} \quad \Rightarrow \quad R = \frac{2D}{D-2}V(\phi_0). \tag{B.7}$$

Not only does this tell us that the spacetimes of vacuum field configurations are Einstein manifolds, but it also indicates how the scalar potential is able to play the role of a cosmological constant and generate non-flat spacetime vacua [101]. These vacua will have positive or negative curvature according to the sign of the scalar potential (see Appendix A for a reminder on our conventions). In this thesis, we are interested in vacua with negative curvature leading to AdS of hvLif asymptotics. Such vacua have $R > 0$ in our conventions, and thus require a positive definite scalar potential.

[59] Note that D is the Levi-Civita connection of the one-dimensional spacetime whilst ∇ represents the gradient operator on the n-dimensional scalar manifold.

C Five-dimensional STU scalar potential

C.1 Inverse CASR metric a^{ij} calculation

Our goal in this appendix is to obtain an expression for the inverse CASR metric $a^{ij}(h)$ for use in the five-dimensional scalar potential, $V_5(h)$. We start by considering the inverse of the matrix $(chhh)^{-1}(ch)_{ij}$, which we shall denote as $(chhh)(ch)^{-1|ij}$. In other words, we can write

$$(chhh)(ch)^{-1|ij}\frac{(ch)_{jk}}{chhh} = \delta^i_k. \tag{C.1}$$

We know that the metric on the CASR manifold is given by (2.36). This can be rearranged to give

$$\frac{(ch)_{jk}}{chhh} = -\frac{1}{2}a_{jk} + \frac{3}{2}\frac{(chh)_j(chh)_k}{(chhh)^2}. \tag{C.2}$$

Substituting (C.2) into (C.1) gives

$$\delta^i_k = (chhh)(ch)^{-1|ij}\left(-\frac{1}{2}a_{jk} + \frac{3}{2}\frac{(chh)_j(chh)_k}{(chhh)^2}\right). \tag{C.3}$$

Now let us introduce the dual scalars h_i via

$$\partial_\mu h_i := a_{ij}\partial_\mu h^j, \qquad h_i = -a_{ij}h^j = -2\left(\frac{(ch)_{ij}}{chhh} - \frac{3}{2}\frac{(chh)_i(chh)_j}{(chhh)^2}\right)h^j = \frac{(chh)_i}{chhh}.$$

We can then write (C.3) as

$$\delta^i_k = (chhh)(ch)^{-1|ij}\left(-\frac{1}{2}a_{jk} + \frac{3}{2}h_jh_k\right). \tag{C.4}$$

In other words, the matrix $(chhh)(ch)^{-1|ij}$ is just the inverse of the term in brackets. Thankfully, the latter is easily invertible. Indeed, we find

$$\left(-\frac{1}{2}a_{ij} + \frac{3}{2}h_ih_j\right)\cdot\left(-2a^{jk} + 3h^jh^k\right) = \delta^i_k.$$

Hence we can write

$$(chhh)(ch)^{-1|ij} = -2a^{ij} + 3h^ih^j, \tag{C.5}$$

and so,

$$a^{ij}(h) = -\frac{1}{2}(chhh)(ch)^{-1|ij} + \frac{3}{2}h^i h^j. \tag{C.6}$$

C.2 Minimising the STU five-dimensional scalar potential

To check the sign of the scalar potential on a vacuum field configuration, we must first find the vacuum expectation values of the scalar fields at the minima and then back-substitute them into the potential. We shall now illustrate how this works for the simple example of the STU-model.

Using (C.5) the scalar potential $V_5(h)$ in (3.116) can be written as

$$V_5(h) = 4 \cdot 6^{-1/3} \left(3h^i h^j - a^{ij}(h)\right) g_i g_j \,. \tag{C.7}$$

We know that the physical theory is subject to the D-gauge constraint $chhh = 1$ which we impose by means of a Lagrange multiplier. Let us define the function

$$F(h, \lambda) := 3(g_i h^i)^2 - a^{ij}(h) g_i g_j + \lambda(chhh - 1)\,, \tag{C.8}$$

such that the equation $\partial_\lambda F = 0$ imposes the D-gauge.

For the STU-model, the only non-zero coefficients c_{ijk} are $c_{012} = 1$. It is therefore possible to explicitly compute the CASR metric $a_{ij} = \partial^2_{ij}\tilde{H} = \partial^2_{ij}(-\frac{1}{3}\log H)$ for the Hesse potential $H = h^0 h^1 h^2$. Inverting, the only non-zero components of the inverse metric are

$$a^{ii} = 3(h^i)^2, \quad i = 0, 1, 2.$$

We can then write

$$F(h, \lambda) = 6(g_0 h^0)(g_1 h^1) + 6(g_0 h^0)(g_2 h^2) + 6(g_1 h^1)(g_2 h^2) + \lambda(h^0 h^1 h^2 - 1)\,. \tag{C.9}$$

Demanding the individual components of the gradient vanish is equivalent to the following

$$\begin{aligned}
\partial_0 F &= 6g_0(g_1 h^1 + g_2 h^2) + \lambda h^1 h^2 \overset{!}{=} 0\,,\\
\partial_1 F &= 6g_1(g_0 h^0 + g_2 h^2) + \lambda h^0 h^2 \overset{!}{=} 0\,,\\
\partial_2 F &= 6g_2(g_0 h^0 + g_1 h^1) + \lambda h^0 h^1 \overset{!}{=} 0\,.
\end{aligned} \tag{C.10}$$

If we multiply these three equations by h^0, h^1 and h^2 respectively, and impose the constraint

$chhh = 1$, we are left with

$$6(g_0h^0)(g_1h^1 + g_2h^2) + \lambda \stackrel{!}{=} 0\,, \tag{C.11}$$

$$6(g_1h^1)(g_0h^0 + g_2h^2) + \lambda \stackrel{!}{=} 0\,, \tag{C.12}$$

$$6(g_2h^2)(g_0h^0 + g_1h^1) + \lambda \stackrel{!}{=} 0\,. \tag{C.13}$$

We can then subtract (C.11) from (C.12) to find

$$(g_2h^2)(g_0h^0 - g_1h^1) \stackrel{!}{=} 0 \quad \Rightarrow \quad g_0h^0 = g_1h^1\,. \tag{C.14}$$

Similarly, subtracting (C.12) from (C.13) gives

$$(g_1h^1)(g_0h^0 - g_2h^2) \stackrel{!}{=} 0, \quad \Rightarrow \quad g_0h^0 = g_2h^2\,. \tag{C.15}$$

We conclude that

$$g_0h^0 = g_1h^1 = g_2h^2 = K, \quad \text{for some } K.$$

Returning to our constraint, we can write

$$h^0h^1h^2 = 1 \quad \Rightarrow \quad \frac{K^3}{g_0g_1g_2} = 1 \quad \Rightarrow \quad K = (g_0g_1g_2)^{1/3}\,.$$

The value of the scalar fields on vacuum field configurations (minima of $V_5(h)$) can then be written as

$$h^i = \frac{1}{g_i}\left(\frac{1}{g_0}\cdot\frac{1}{g_1}\cdot\frac{1}{g_2}\right)^{-1/3}. \tag{C.16}$$

Finally, let us consider the value of the scalar potential on the vacuum field configuration. We have already seen that for the STU-model, the potential reduces to

$$V_5(h) = 4\cdot 6^{-1/3}\left[6(g_0h^0)(g_1h^1) + 6(g_0h^0)(g_2h^2) + 6(g_1h^1)(g_2h^2)\right],$$

and evaluating at the vacua simply corresponds to back-substituting the vacuum expectation value of the scalar fields in (C.16). This gives

$$\begin{aligned} V_5(h) &= 4\cdot 6^{-1/3}\left[6\left(\frac{1}{g_0}\cdot\frac{1}{g_1}\cdot\frac{1}{g_2}\right)^{-2/3} + 6\left(\frac{1}{g_0}\cdot\frac{1}{g_1}\cdot\frac{1}{g_2}\right)^{-2/3} + 6\left(\frac{1}{g_0}\cdot\frac{1}{g_1}\cdot\frac{1}{g_2}\right)^{-2/3}\right] \\ &= 72\cdot 6^{-1/3}\left(\frac{1}{g_0}\cdot\frac{1}{g_1}\cdot\frac{1}{g_2}\right)^{-2/3} > 0\,. \end{aligned} \tag{C.17}$$

Thus the five-dimensional scalar potential $V_5(h)$ is positive definite when evaluated on vacuum field configurations of the STU-model. According to Appendix B.2, this is the correct sign of cosmological constant to produce negatively curved AdS spacetimes.

D

Calculations for Chapter 4

D.1 Real formulation of V_4

In Chapter 4 we need to rewrite the scalar potential $V_4(X, \bar{X})$ and the associated superpotential $W(X)$ in terms of real coordinates. Using the expression $W(X) = 2(g^I F_I - g_I X^I)$ given in (3.114) and that $H_a = H_{ab} q^b$ by homogeneity, it is straightforward to obtain

$$W = W(q^a) = W(x^I, y_I) = 2g^a \left(\Omega_{ab} + \frac{i}{2} H_{ab}\right) q^b = ig^a \left(H_{ab} - 2i\Omega_{ab}\right) q^b, \tag{D.1}$$

where we have defined $(g^a) := (g^I, g_I)^T$.

In order to obtain the potential V_4 as given in (3.114), we must compute the derivatives

$$\partial_I W = \frac{\partial W}{\partial X^I} = \frac{1}{2}\left(\frac{\partial}{\partial x^I} - i\frac{\partial}{\partial u^I}\right) W.$$

Since this derivative involves the real coordinates (x^I, u^I) rather than $(q^a) = (x^I, y_I)^T$, we apply the chain rule to $W(x, y(x, u))$ and compute

$$\left.\frac{\partial W}{\partial x^I}\right|_u = \left.\frac{\partial W}{\partial x^I}\right|_y + \left.\frac{\partial y_J}{\partial x^I}\frac{\partial W}{\partial y_J}\right|_x, \quad \text{and} \quad \left.\frac{\partial W}{\partial u^I}\right|_x = \left.\frac{\partial y_J}{\partial u^I}\frac{\partial W}{\partial y_J}\right|_x.$$

After decomposing the second derivatives of the prepotential F into real and imaginary parts (including a conventional factor of 2) by $2F_{IJ} = R_{IJ} + iN_{IJ}$, one can apply the chain rule to show that

$$\frac{\partial y_J}{\partial x^I} = \frac{1}{2}\left(F_{IJ} + \bar{F}_{IJ}\right) = \frac{1}{2} R_{IJ},$$

and read from [94] that

$$\frac{\partial y_J}{\partial u^I} = -\frac{1}{2} N_{IJ}.$$

Combining this, we find

$$\frac{\partial W}{\partial X^I} = \frac{1}{2}\left(\frac{\partial W}{\partial x^I} + F_{IJ}\frac{\partial W}{\partial y_J}\right), \quad \frac{\partial \bar{W}}{\partial \bar{X}^I} = \frac{1}{2}\left(\frac{\partial \bar{W}}{\partial x^I} + \bar{F}_{IJ}\frac{\partial \bar{W}}{\partial y_J}\right).$$

Finally, we can put all of this together to obtain

$$N^{IJ}\partial_I W \partial_J \bar{W} = \frac{1}{4} W_a \left(H^{ab} + \frac{i}{2}\Omega^{ab}\right) \bar{W}_b, \tag{D.2}$$

where $(W_a) = \left(\frac{\partial W}{\partial x^I}, \frac{\partial W}{\partial y_J}\right)^T$, H^{ab} is the inverse Hessian metric on the CASK manifold (see [94]), and Ω^{ab} is the inverse of Ω_{ab}.

Using (D.1), we have that

$$W_a = ig^b \left(H - 2i\Omega\right)_{ba}, \qquad \bar{W}_a = -ig^b \left(H + 2i\Omega\right)_{ba}. \tag{D.3}$$

This can be substituted into (D.2), which after simplification becomes

$$N^{IJ}\partial_I W \partial_J \bar{W} = H_{ab} g^a g^b, \tag{D.4}$$

where we have used the identity[60] $H_{ab}\Omega^{bc}H_{cd} = -4\Omega_{ad}$ [94].

Applying (D.1) and (D.4) to (3.114) the scalar potential can be written purely in terms of real coordinates as follows:

$$\begin{aligned} V_4 &= -g^a H_{ab} g^b + 2\kappa_4^2 g^a \left(H_{ac} - 2i\Omega_{ac}\right) q^c g^b \left(H_{bd} + 2i\Omega_{bd}\right) q^d \\ &= -g^a g^b \left[H_{ab} - 2\kappa_4^2 H_a H_b - 8\kappa_4^2 \left(\Omega q\right)_a \left(\Omega q\right)_b\right], \end{aligned} \tag{D.5}$$

where we have used homogeneity $H_a = H_{ab}q^b$. Lastly, we substitute the D-gauge condition $-2H = \kappa_4^{-2}$ into (D.5) to obtain

$$V_4 = -g^a g^b \left[H_{ab} + \frac{H_a H_b + 4\left(\Omega q\right)_a \left(\Omega q\right)_b}{H}\right]. \tag{D.6}$$

This gives an expression for the dyonic scalar potential in terms of special real coordinates and has not appeared in the literature previously.

Finally, we would also like an expression for the scalar potential in terms of variables adapted to dimensional reduction. Since the expression in the square brackets of (D.6) is homogeneous of degree zero, it remains invariant if we rescale the real coordinates q^a by $e^{\phi/2}$. This corresponds to the rescaling $X^I \mapsto Y^I = e^{\phi/2}X^I$ that we imposed in (3.72) as the first step of adapting the real formulation to dimensional reduction. To express V_4 in terms of the tensor $\tilde{H}_{ab}$ we use the relation (3.79) to write

$$V_4 = +2Hg^a g^b \left[\tilde{H}_{ab} - \frac{H_a H_b}{H^2} - 2\frac{\left(\Omega q\right)_a \left(\Omega q\right)_b}{H^2}\right]. \tag{D.7}$$

The expression in the square brackets of (D.7) is now homogeneous of degree -2. Finally, we use (3.80) to re-write V_4 in terms of the dual coordinates q_a, and rearrange (D.7) to take into account that upon dimensional reduction the scalar potential in the Lagrangian gets multiplied

[60] This is the standard relation between the metric and Kähler form of a Kähler manifold. The numerical factor is due to conventional choices.

by $e^{-\phi} = -\frac{1}{2H}$. This gives

$$-\frac{1}{2H}V_4 = -g^a g^b \left[\tilde{H}_{ab} - 4q_a q_b - 2\frac{(\Omega q)_a (\Omega q)_b}{H^2}\right]. \tag{D.8}$$

D.2 Hesse potential for PI configurations

In this section we shall follow [3, 94] to obtain the Hesse potential for the very special models with prepotentials of the form

$$F(Y) = \frac{f(Y^1, \ldots, Y^n)}{Y^0},$$

subject to the PI condition. We can compute

$$F_0 = -\frac{f(Y^1, \ldots, Y^n)}{(Y^0)^2}, \qquad F_A = \frac{f_A(Y^1, \ldots, Y^n)}{Y^0}.$$

Restricting to the purely imaginary field configuration introduced in Section 4.1.1, these are

$$F|_{\text{PI}} = \frac{f(iu^1, \ldots, iu^n)}{x^0} = -i\frac{f(u)}{x^0}.$$

and

$$F_0|_{\text{PI}} = iv_0 = i\frac{f(u)}{(x^0)^2}, \qquad F_A|_{\text{PI}} = y_A = -\frac{f_A(u)}{x^0}, \tag{D.9}$$

where we have introduced the notation $f(u) := f(u^1, \ldots, u^n)$, and made use of the fact that f and f_A are real homogeneous functions of degree three and two respectively. From now onwards, we understand that Y^I, F_I are subject to the PI constraints and therefore drop the explicit 'PI' label in the following.

The relation for F_0 can be used to solve for x^0 as a function of the dual coordinates as follows

$$(x^0)^2 = \frac{f(u)}{v_0} \quad \Rightarrow \quad x^0 = -\sqrt{\frac{f(u)}{v_0}}, \tag{D.10}$$

where we must pick the negative root in order to ensure the Hesse potential is strictly negative definite [94] .

We now obtain $H(u,v)|_{\text{PI}}$ by first evaluating (3.73) subject to the PI condition. We have

$$\begin{aligned}
e^{\phi} &= -2x^0 v_0 + 2u^A y_A \\
&= -2\left(-\sqrt{\frac{f(u)}{v_0}}\right) v_0 + 2u^A \left(-\frac{f_A(u)}{x^0}\right) \quad \text{using (D.9) and (D.10)} \\
&= 2\sqrt{v_0 f(u)} - 2\cdot 3\frac{f(u)}{x^0} \quad \text{by homogeneity of } f(u) \\
&= 2\sqrt{v_0 f(u)} - 6\left(-\sqrt{\frac{v_0}{f(u)}}\right) f(u) \quad \text{using (D.10) again} \\
&= 8\sqrt{v_0 f(u)}.
\end{aligned}$$

Using $-2H = e^{\phi}$, this rearranges to give

$$H(u,v)|_{\text{PI}} = -4\sqrt{v_0 f(u^1,\ldots,u^n)}. \tag{D.11}$$

Now recall that the dual real variables are defined as

$$q_0 = -\frac{1}{H}v_0, \qquad q_A = \frac{1}{H}u^A,$$

and so we can write

$$\begin{aligned}
H &= -4\sqrt{(-Hq_0)H^3 f(q_1,\ldots,q_n)} \quad \text{by homogeneity of } f \\
&= -4H^2\sqrt{(-q_0)f(q_1,\ldots,q_n)} \\
\Rightarrow H(q_0,q_A) &= -\frac{1}{4}\left((-q_0)f(q_1,\ldots,q_n)\right)^{-\frac{1}{2}}.
\end{aligned} \tag{D.12}$$

Note the slight abuse of notation above. As commented on in more detail at the beginning of Section 4.2, we denote by q_A with $A = 1,\ldots,n$ the scalar fields which are actually the $(A+n+1)$'th components of the vector (q_a).

D.3 Determining the integration constants λ_A

We have chosen to set the scalar fields proportional to one another

$$q_A(\tau) = \lambda_A Q(\tau),$$

where, from (4.38), we know

$$Q(\tau) = e^{\frac{1}{2}a_0\tau}\left(\sinh\left(\omega\tau + \omega\beta\right)\right)^{\frac{1}{2}}.$$

We will be able to determine the integration constants λ_A by analysing the q_A equation of motion (4.25) which we rewrite here for convenience:

$$e^{-4\psi}\sum_{A=1}^{n} q^A\ddot{q}_A + e^{-4\psi}\sum_{A,B=1}^{n}\tilde{H}^{AB}\dot{q}_A\dot{q}_B + \sum_{A,B=1}^{n}\tilde{H}_{AB}\,g_A g_B - 4\left(\sum_{A=1}^{n} g_A q_A\right)^2 = 0. \tag{D.13}$$

We have seen in (4.39) that we must impose

$$q_1 g_1 = q_2 g_2 = \ldots = q_n g_n = KQ, \tag{D.14}$$

for some constant K.

This means that the final term in (D.13) simply becomes

$$-4\left(\sum_{A=1}^{n} g_A q_A\right)^2 = -4n^2K^2Q^2. \tag{D.15}$$

Next, recalling that $\tilde{H}^{AB}(q)$ is homogeneous of degree -2 in the q_A , the second term in (D.13) is

$$\begin{aligned}\sum_{A,B=1}^{n}\tilde{H}^{AB}(q)\dot{q}_A\dot{q}_B &= \sum_{A,B=1}^{n}\tilde{H}^{AB}(\lambda)\lambda_A\lambda_B\left(\frac{\dot{Q}}{Q}\right)^2\\ &= \frac{3}{4}\left(\frac{\dot{Q}}{Q}\right)^2,\end{aligned} \tag{D.16}$$

where on the first line $\tilde{H}^{AB}(\lambda)$ is a matrix of integration constants after extracting an overall factor of $\frac{1}{(Q(\tau))^2}$, and on the second line we have made use of the identity (4.32).

Meanwhile, by a similar process, the first term in (D.13) can be written as

$$-\sum_{A,B=1}^{n}\tilde{H}^{AB}(q)q_A\ddot{q}_B = -\sum_{A,B=1}^{n}\tilde{H}^{AB}(\lambda)\lambda_A\lambda_B\left(\frac{\ddot{Q}}{Q}\right) = -\frac{3}{4}\frac{\ddot{Q}}{Q}. \tag{D.17}$$

Using (D.17) and (D.16) we see that the first two terms of (D.13) can be expressed together as

$$\begin{aligned}-\frac{3}{4}\frac{\ddot{Q}}{Q} + \frac{3}{4}\left(\frac{\dot{Q}}{Q}\right)^2 = -\frac{3}{4}\frac{d}{d\tau}\left(\frac{\dot{Q}}{Q}\right) &= -\frac{3}{4}\frac{d}{d\tau}\left[\frac{1}{2}\omega\coth\left(\omega\tau+\omega\beta\right)+\frac{1}{2}a_0\right]\\ &= \frac{\frac{3}{8}\omega^2}{\sinh^2\left(\omega\tau+\omega\beta\right)},\end{aligned} \tag{D.18}$$

where in the first line we substitute the result of (4.36), and in the second line we compute the derivative. We have so far neglected the factor of $e^{-4\psi}$ present in the first two terms of (D.13).

Substituting from (4.43), we see that the full expression for these first two terms is

$$\begin{aligned} e^{-4\psi}\sum_{A=1}^{n} q^A\ddot{q}_A + e^{-4\psi}\sum_{A,B=1}^{n}\tilde{H}^{AB}\dot{q}_A\dot{q}_B &= e^{-4\psi}\frac{\frac{3}{8}\omega^2}{\sinh^2(\omega\tau+\omega\beta)} \\ &= \frac{3}{8}\omega^2\left(\frac{\alpha}{\omega}\right)^3 e^{a_0\tau}\sinh(\omega\tau+\omega\beta) \\ &= \frac{3\alpha^3}{8\omega}Q^2. \end{aligned} \tag{D.19}$$

The third term in (D.13) is very easy to deal with. By homogeneity, we have

$$\sum_{A,B=1}^{n}\tilde{H}_{AB}(q)g_Ag_B = Q^2\sum_{A,B=1}^{n}\tilde{H}_{AB}(\lambda)g_Ag_B. \tag{D.20}$$

We can now substitute the results of (D.19), (D.20) and (D.15) to rewrite (D.13) as

$$\frac{3\alpha^3}{8\omega}Q^2 + Q^2\sum_{A,B=1}^{n}\tilde{H}_{AB}(\lambda)g_Ag_B - 4n^2K^2Q^2 = 0. \tag{D.21}$$

Now we know from (4.32) that

$$\sum_{A,B=1}^{n}\tilde{H}^{AB}(\lambda)\lambda_A\lambda_B = \frac{3}{4},$$

and if we define $\lambda^A := -\tilde{H}^{AB}(\lambda)\lambda_B$, then we have

$$\sum_{A=1}^{n}\lambda^A\lambda_A = -\frac{3}{4}. \tag{D.22}$$

We can also show

$$\sum_{A,B=1}^{n}\tilde{H}_{AB}(\lambda)\lambda^A\lambda^B = \frac{3}{4} \quad\Rightarrow\quad 1 = \frac{4}{3}\sum_{A,B=1}^{n}\tilde{H}_{AB}(\lambda)\lambda^A\lambda^B\,. \tag{D.23}$$

Multiplying the first and third terms in (D.21) by (D.23), we get

$$\sum_{A,B=1}^{n}\tilde{H}_{AB}(\lambda)\left[\frac{\alpha^3}{2\omega}\lambda^A\lambda^B + g_Ag_B - \frac{16}{3}n^2K^2\lambda^A\lambda^B\right] = 0. \tag{D.24}$$

To acquire a model independent solution (that doesn't depend on the structure of $\tilde{H}_{ab}$), we should require that the bit inside the square brackets vanishes. Thus,

$$g_Ag_B = \left(\frac{16n^2K^2}{3} - \frac{\alpha^3}{2\omega}\right)\lambda^A\lambda^B.$$

Multiplying through by $\lambda_A \lambda_B$ and summing over $A, B = 1, \ldots, n$, we find

$$n^2 K^2 = \left(\frac{16 n^2 K^2}{3} - \frac{\alpha^3}{2\omega} \right) \left(-\frac{3}{4} \right)^2 ,$$

where we have used (D.14) to determine $\lambda_A g_A = nK$ on the left hand side, and we have substituted from (D.22) on the right hand side. This can of course be simplified to

$$n^2 K^2 = 3 n^2 K^2 - \frac{9\alpha^3}{32\omega} \quad \Rightarrow \quad n^2 K^2 = \frac{9\alpha^3}{64\omega} \quad \Rightarrow \quad K = \pm \frac{3}{8n} \left(\frac{\alpha^3}{\omega} \right)^{\frac{1}{2}} .$$

Rearranging, we obtain the following expression

$$\lambda_A = \pm \frac{3}{8 n g_A} \left(\frac{\alpha^3}{\omega} \right)^{\frac{1}{2}} . \tag{D.25}$$

The integration constants must fulfil this constraint in order for the q_A equation of motion to be satisfied.

E Calculations for Chapter 5

E.1 Rewriting V_3

Our goal in this appendix is to obtain a workable expression for the scalar potential V_3 appearing in (5.7). Let us concentrate on the term $(cyyy)(cy)^{-1|ij}$. This is to be interpreted as the matrix inverse to $(cyyy)^{-1}(cy)_{ij}$ in the sense that

$$(cyyy)(cy)^{-1|ij}\frac{(cy)_{jk}}{cyyy} = \delta^i_k. \tag{E.1}$$

Now, using the expression (5.11) for $\hat{g}_{ij}(y)$:

$$\hat{g}_{ij}(y) = \frac{3}{2}\left(\frac{(cy)_{ij}}{cyyy} - \frac{3}{2}\frac{(cyy)_i(cyy)_j}{(cyyy)^2}\right),$$

we have

$$\delta^i_k = (cyyy)(cy)^{-1|ij}\left[\frac{2}{3}\hat{g}_{jk}(y) + \frac{3}{2}\frac{(cyy)_j(cyy)_k}{(cyyy)^2}\right]. \tag{E.2}$$

We now introduce the dual scalars y_i via

$$\partial_\mu y_i := \hat{g}_{ij}(y)\partial_\mu y^j, \qquad y_i = -\hat{g}_{ij}(y)y^j = \frac{3}{4}\frac{(cyy)_i}{cyyy}.$$

Hence, (E.2) becomes

$$\delta^i_k = (cyyy)(cy)^{-1|ij}\left[\frac{2}{3}\hat{g}_{jk}(y) + \frac{8}{3}y_j y_k\right]. \tag{E.3}$$

In other words, the quantity $(cyyy)(cy)^{-1|ij}$ is just the inverse of the term in square brackets in (E.3). Thankfully, the latter is easily invertible. Indeed, we find

$$\frac{3}{2}\left[\hat{g}^{ij}(y) + 2y^i y^j\right]\cdot\frac{2}{3}\left[\hat{g}_{jk}(y) + 4y_j y_k\right] = \delta^i_k\,.$$

Hence we can rewrite

$$(cyyy)(cy)^{-1|ij} = \frac{3}{2}\hat{g}^{ij}(y) + 3y^i y^j, \tag{E.4}$$

so that the scalar potential term in (5.8) becomes

$$V_3 = 3\left[\hat{g}^{ij}(y) + 4y^i y^j\right]g_i g_j. \tag{E.5}$$

E.2 Euclideanisation of the boosted black brane

As is well known from the study of Kerr black holes, obtaining the Hawking temperature by Euclidean methods is much more subtle for non-static spacetimes. For this reason, we find it useful to give an explicit demonstration of how this works in the case of boosted (non-static) black branes. The treatment of the linear case given below will be parallel to the analysis of the Kerr black hole in [136].

A Euclidean continuation of the boosted black brane solution (5.53) can be obtained by setting $t = i\tau$ and $u_z = i\beta$, and taking τ and β to be real. Observe that following the standard treatment of the Kerr solution, we not only continue time but also the 'boost parameter' $w = -u_z/u_t$, which is analogous to the angular momentum parameter of the Kerr solution in Boyer-Lindquist coordinates.

The Euclidean section of the boosted black brane in (5.53) is then

$$ds^2_{(5)E} = \frac{L^2}{r^2}\frac{dr^2}{W} + \frac{r^2}{L^2}W(u_t\,d\tau + \beta dz)^2 + \frac{r^2}{L^2}\left((-\beta\,d\tau + u_t\,dz)^2 + dx^2 + dy^2\right) .$$

We now explore the near horizon geometry, adapting a similar calculation of [192] for the Kerr-Newman solution. Introducing the new radial variable R by $R^2 = r - r_+$, the function W has the expansion

$$W = \frac{4}{r_+}R^2 + \dots$$

around the horizon. Expanding up to order R^2, the metric takes the form

$$ds^2_{(5)\mathrm{E,NH}} = \frac{L^2}{r_+}\left(1 - \frac{R^2}{r_+}\right)dR^2 + \frac{4r_+}{L^2}R^2 d\chi^2 + \frac{r_+^2 + 2r_+R^2}{L^2}(d\tilde{z}^2 + dx^2 + dy^2) ,$$

where we have replaced the coordinates τ and z by the new coordinates

$$\chi = u_t\tau + \beta z , \quad \tilde{z} = u_t z - \beta\tau .$$

We remark that, in contradistinction to the Kerr-Newman solution discussed in [192], (i) the coordinate $\tilde{z}$ is linear rather than angular, i.e. we do not need to impose an identification on it; and (ii) the coordinate χ is well defined, since u_t and β are constant, so that $u_t d\tau + \beta dz$ is exact. The horizon is at $R = 0$. The coordinates $x, y, \tilde{z}$ parametrize a three-dimensional plane with a metric which is flat up to corrections of order R^2. This part of the metric is clearly regular for $R \to 0$. The variables R and χ parametrize a surface with metric

$$ds^2_{\mathrm{Cone}} = \frac{L^2}{r_+}\left(\left[1 - \frac{R^2}{r_+}\right]dR^2 + 4R^2\frac{r_+^2}{L^4}d\chi^2\right) ,$$

which is, up to a subleading term of order R^2, the metric of a cone with apex at $R = 0$. Thus χ is an angular variable and the surface parametrized by R and χ is topologically a disk. Imposing the absence of a conical singularity at $R = 0$ fixes the periodicity of χ to be

$$\chi \simeq \chi + 2\pi \frac{L^2}{2r_+} \ .$$

Since the coordinate $\tilde{z}$ is linear (has no identifications) we can determine the periodicities of τ and z from

$$(\chi, \tilde{z}) \simeq \left(\chi + 2\pi \frac{L^2}{2r_+}, \tilde{z}\right) \Leftrightarrow (\tau, z) \simeq (\tau + A, z + B) \ ,$$

with

$$A = 2\pi \, u_t \, \frac{L^2}{2r_+}, \qquad B = 2\pi \, \beta \, \frac{L^2}{2r_+}.$$

The Hawking temperature T is read off from the periodicity of τ by $\tau \simeq \tau + T_H^{-1}$, so that

$$\pi T = \frac{r_+}{L^2 u_t} \ ,$$

which agrees with the result found by computing the surface gravity (5.65).

To interpret the periodicity of z, remember that the boost velocity at the horizon is

$$w = -\frac{u_z}{u_t} = -i \frac{\beta}{u_t} \ .$$

Thus

$$B = iw \frac{1}{T} \ ,$$

so that the identifications take the form

$$(\tau, z) \simeq \left(\tau + T^{-1}, z + iwT^{-1}\right) ,$$

which is analogous to the identification for the Euclidean Kerr solution, see for example [192].

E.3 Quasilocal computation of conserved charges

We use the form of our five-dimensional line element given in (5.53), which can be rewritten as

$$ds^2 = \frac{L^2 dr^2}{r^2 W} + \frac{r^2}{L^2} \left(\eta_{\mu\nu} + \frac{r_+^4}{r^4} u_\mu u_\nu\right) dx^\mu dx^\nu, \tag{E.6}$$

where $u_\mu = (u_t, 0, 0, u_z)$. Note that $u_\mu u^\mu = -1$ so we can interpret this as a velocity vector. Following the procedure of [172] we want to calculate the quasilocal stress tensor $T^{\mu\nu}$ associated with the metric (E.6).

E.3.1 The quasilocal stress tensor

Given a time-like surface $\partial\mathcal{M}_r$ at constant radial distance r we define the metric $\gamma_{\mu\nu}$ on $\partial\mathcal{M}_r$ via the ADM-like decomposition

$$ds^2 = N^2 dr^2 + \gamma_{\mu\nu}(dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr). \tag{E.7}$$

We define the extrinsic curvature $\Theta^{\mu\nu}$ via

$$\Theta^{\mu\nu} := -\frac{1}{2}\left(\nabla^\mu \hat{n}^\nu + \nabla^\nu \hat{n}^\mu\right), \tag{E.8}$$

where $\hat{n}^\mu$ is the outward-pointing normal vector to the surface $\partial\mathcal{M}_r$. For solutions asymptoting to AdS_5 the procedure of [172] tells us that the quasilocal stress tensor is then given by[61]

$$T_{\mu\nu} = \Theta_{\mu\nu}(\gamma) - \Theta(\gamma)\gamma_{\mu\nu} - \frac{3}{L}\gamma_{\mu\nu} - \frac{L}{2}G_{\mu\nu}(\gamma), \tag{E.9}$$

where $\Theta = \gamma_{\mu\nu}\Theta^{\mu\nu}$ is the trace of the extrinsic curvature, and $G_{\mu\nu}$ is the Einstein tensor for $\gamma_{\mu\nu}$.

For the case at hand we see that the metric (E.6) decomposes according to (E.7) with

$$N^2 = \frac{L^2}{r^2 W}, \qquad N^\mu = 0, \qquad \gamma_{\mu\nu}(r) = \frac{r^2}{L^2}\left(\eta_{\mu\nu} + \frac{r_+^4}{r^4}u_\mu u_\nu\right). \tag{E.10}$$

The unit normal vector $\hat{n}^\mu$ to a surface of constant r is given by

$$\hat{n}^\mu = \frac{r}{L}W^{1/2}(r)\delta^{\mu,r},$$

from which we find the extrinsic curvature

$$\Theta_{\mu\nu} = -\frac{r}{2L}\left(1 - \frac{r_+^4}{r^4}\right)^{1/2}\partial_r\gamma_{\mu\nu} = -\frac{r^2}{L^3}\left(1 - \frac{r_+^4}{r^4}\right)^{1/2}\left(\eta_{\mu\nu} - \frac{r_+^4}{r^4}u_\mu u_\nu\right). \tag{E.11}$$

In order to calculate the trace of this we need an expression for the inverse metric $\gamma^{\mu\nu}$, which is given by

$$\gamma^{\mu\nu} = \frac{L^2}{r^2}\left[\eta^{\mu\nu} - \frac{r_+^4}{r^4}\left(1 - \frac{r_+^4}{r^4}\right)^{-1}u^\mu u^\nu\right], \tag{E.12}$$

where $u^\mu = \eta^{\mu\nu}u_\nu$, etc. This can be used to compute the trace of the extrinsic curvature

$$\Theta = \Theta_{\mu\nu}\gamma^{\mu\nu} = -\frac{2}{L}\left(1 - \frac{r_+^4}{r^4}\right)^{1/2}\left[2 + \frac{r_+^4}{r^4}\left(1 - \frac{r_+^4}{r^4}\right)^{-1}\right]. \tag{E.13}$$

Putting all this together, and noting that $G_{\mu\nu}(\gamma) = 0$, we can use (E.9) to find the resulting

[61] We remind the reader that in this thesis we work in units where $8\pi G = 1$.

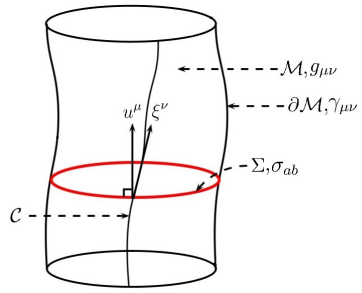


Figure E.1: The time-like worldline C of an observer traversing the spacetime boundary $\partial\mathcal{M}$ and whose tangent velocity vector ξ is not necessarily parallel to the time-like unit normal U to the codimension-2 spatial surface $\Sigma \subset \partial\mathcal{M}$. In fact there exists a one-parameter family of spatial surfaces Σ_t that foliate the boundary $\partial\mathcal{M}$. Image sourced from [193].

gravitational stress-energy tensor induced on the boundary $\partial\mathcal{M}_r$,

$$T_{\mu\nu} = \frac{r_+^4}{2L^3 r^2}\left(\eta_{\mu\nu} + 4u_\mu u_\nu\right) + \dots\,, \tag{E.14}$$

where the dots represent terms which are subleading in the limit $r \to \infty$.

E.3.2 Mass, momentum and conserved charges

The quasilocal stress tensor (E.14) can be used to compute well-defined mass and other conserved charges for the spacetime (E.6). Let Σ be a space-like hypersurface in $\partial\mathcal{M} = \lim_{r\to\infty}\partial\mathcal{M}_r$ and make the ADM decomposition

$$\gamma_{\mu\nu}dx^\mu dx^\nu = -N_\Sigma^2 dt^2 + \sigma_{ab}(dx^a + N_\Sigma^a dt)(dx^b + N_\Sigma^b dt), \tag{E.15}$$

where $\{x^a\}$ are coordinates spanning Σ, which has metric σ_{ab}. Let U be the time-like unit normal to Σ. Then for any isometry of $\gamma_{\mu\nu}$, which we take to be generated by a Killing vector ξ, we can define a conserved charge Q_ξ by

$$Q_\xi = \int_\Sigma d^{d-1}x\,\sqrt{\sigma}\,(U^\mu T_{\mu\nu}\xi^\nu)\,. \tag{E.16}$$

Figure E.1 illustrates what is happening geometrically. In particular, the mass of the solution is given by taking $\xi = \partial_t$, whilst the momentum in the direction x^a is given by taking $\xi = \partial_a$.

For the boosted black brane we can make the ADM decomposition (E.15) of the metric

(E.10) with

$$\sigma_{xx} = \sigma_{yy} = \frac{r^2}{L^2}, \qquad \sigma_{zz} = \frac{r^2}{L^2}\left(1 + \frac{r_+^4}{r^4}u_z^2\right),$$
$$N_\Sigma^z = \frac{r_+^4}{r^4}u_z u_t \left(1 + \frac{r_+^4}{r^4}u_z^2\right)^{-1},$$
$$N_\Sigma^2 = \frac{r_+^8}{L^2 r^6}u_z^2 u_t^2 \left(1 + \frac{r_+^4}{r^4}u_z^2\right)^{-1} + \frac{r^2}{L^2}\left(1 - \frac{r_+^4}{r^4}u_t^2\right).$$

The time-like unit normal to Σ has components

$$U^t = -\frac{L}{r}\left(1 + \frac{r_+^4}{r^4}u_z^2\right)^{1/2}\left(1 - \frac{r_+^4}{r^4}\right)^{-1/2},$$
$$U^z = \frac{L r_+^4}{r^5}u_t u_z \left(1 + \frac{r_+^4}{r^4}u_z^2\right)^{-1/2}\left(1 - \frac{r_+^4}{r^4}\right)^{-1/2}.$$

Using these expressions, as well as the components of the quasilocal stress tensor (E.14), we can calculate the mass and linear momentum associated with the boosted black brane (E.6). Taking $\xi = \partial_t$ and $\xi = \partial_z$ we obtain the expressions (5.67) and (5.68) for the mass and linear momentum respectively.

Finally, let us add some further comments on the fact that r_+, and hence temperature, is a physical parameter despite that it can be absorbed by rescaling coordinates in (5.53). From (E.16),(5.64),(5.69) it is manifest that all quantities entering into the first law are geometric quantities (norms of vector fields, and integrals over submanifolds using the induced metric) which are diffeomorphism invariant. Applying the coordinate transformation $\tilde{r} = r_+ r, \tilde{t} = t/r_+, \tilde{x} = x/r_+, \tilde{y} = y/r_+, \tilde{z} = z/r_+$ to these expressions, it is straightforward to see that the parameter r_+ is not eliminated, but scaled out as an overall prefactor. In particular,

$$\partial_t = r_+ \partial_{\tilde{t}}\,, \quad \partial_z = r_+ \partial_{\tilde{z}}\,,$$

while

$$V_3 = \int_\Sigma dx dy dz = r_+^3 \int_\Sigma d\tilde{x} d\tilde{y} d\tilde{z}\,,$$

so that irrespective of our choice of coordinates we have

$$T \sim r_+, S \sim r_+^3, M \sim r_+^4, P_z \sim r_+^4\,.$$

It is precisely this r_+-dependence of all thermodynamic quantities that gives rise to the correct temperature/entropy tern in the first law. Put differently, when working in the rescaled frame $(\tilde{t}, \tilde{r}, \tilde{x}, \tilde{y}, \tilde{z})$ the parameter r_+ is hidden in the choice of the vector field ξ and the volume element V_3.

E.4 Tidal forces

E.4.1 Five-dimensional tidal forces

In this appendix we shall construct the frame fields describing the parallely-propagated-orthonormal-frame (PPON) associated to an observer freely falling towards the five-dimensional extremal black brane in (5.59). The frame-dragging effects associated to the brane's boost in the z direction mean that an observer who starts falling radially inward from infinity will acquire a velocity in the z direction. We want to pick our first frame field to be the vector field generating the geodesic motion of the observer. To do this, we follow the procedure of [121, 182, 194] and introduce the frame field

$$(e_0)^\mu = \left(\frac{d}{d\tau}\right)^\mu = \dot{t}\,(\partial_t)^\mu + \dot{z}\,(\partial_z)^\mu + \dot{r}\,(\partial_r)^\mu\,, \tag{E.17}$$

where τ is the proper time of our observer or, equivalently, the affine parameter for the geodesic motion, and a dot denotes differentiation with respect to τ. Note that for simplicity, we consider an observer who is not moving in the x and y directions.

It is clear that to obtain e_0, we must first obtain $\dot{t}, \dot{z}$ and $\dot{r}$. To do this, we recall that associated to each of the Killing vector fields $\partial_t, \partial_z, \partial_x, \partial_y$ of (5.59) there is an integral of motion. These conserved quantities are the energy and momenta,

$$E = -g_{t\mu}\dot{x}^\mu = \left(\frac{r^2}{L^2} - \frac{\Delta}{r^2L^2}\right)\dot{t} - \frac{\Delta}{r^2L^2}\dot{z}\,, \tag{E.18}$$

$$p_z = g_{z\mu}\dot{x}^\mu = \left(\frac{r^2}{L^2} + \frac{\Delta}{r^2L^2}\right)\dot{z} + \frac{\Delta}{r^2L^2}\dot{t}\,, \tag{E.19}$$

$$p_x = g_{x\mu}\dot{x}^\mu = \frac{r^2}{L^2}\dot{x} = 0\,, \tag{E.20}$$

$$p_y = g_{y\mu}\dot{x}^\mu = \frac{r^2}{L^2}\dot{y} = 0\,. \tag{E.21}$$

Defining the quantities

$$\alpha := \frac{r^2}{L^2} + \frac{\Delta}{r^2L^2}, \quad \beta := \frac{r^2}{L^2} - \frac{\Delta}{r^2L^2}, \quad \gamma := \frac{\Delta}{r^2L^2},$$

we can simultaneously solve (E.18) and (E.19) to find

$$\dot{t} = \frac{L^4}{r^4}\left(\alpha E + \gamma p_z\right), \tag{E.22}$$

$$\dot{z} = \frac{L^4}{r^4}\left(\beta p_z - \gamma E\right). \tag{E.23}$$

Notice that both of these velocities diverge as we approach the horizon at $r_+ = 0$. This divergence

tells us that this particular coordinate system is not valid beyond the horizon. However, for our current purposes, this is not a problem as we are only interested in tidal forces close to, but outside, the horizon. In order to write down $\hat{e}_0$, we still need to obtain $\dot{r}$. For this we use that $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$ for a time-like observer, which is equivalent to

$$\dot{r} = -\sqrt{-\frac{r^2}{L^2} + \alpha\frac{L^2}{r^2}\left(E - V_+\right)\left(E - V_-\right)}, \tag{E.24}$$

where we have taken the negative root to represent a radially infalling observer and $V_\pm = \frac{1}{\alpha}\left(-\gamma p_z \pm \frac{r^2 p_z}{L^2}\right)$ are the roots of $\alpha E^2 + 2\gamma E p_z - \beta p_z^2 = 0$. Notice that had we instead picked the positive root in (E.24), describing an outgoing time-like geodesic, $\dot{r}$ will become complex for sufficiently large r; this indicates that geodesic cannot reach the boundary but in fact hits a turning point and returns to the bulk [118, 119, 120]. For this reason, we will only be interested in near horizon tidal forces.

We can now substitute the above expressions for $\dot{t}, \dot{z}, \dot{r}$ into (E.17) to obtain the following expression for the first frame field

$$\begin{aligned}(e_0)^\mu = &\frac{L^4}{r^4}\left(\alpha E + \gamma p_z\right)(\partial_t)^\mu + \frac{L^4}{r^4}\left(\beta p_z - \gamma E\right)(\partial_z)^\mu \\ &- \sqrt{-\frac{r^2}{L^2} + \alpha\frac{L^2}{r^2}\left(E - V_+\right)\left(E - V_-\right)}\,(\partial_r)^\mu\,.\end{aligned} \tag{E.25}$$

Whilst the frame field e_0 correctly describes the parallel propagation, it is not correctly normalised. To form an orthonormal basis of frame fields we can apply Gram-Schmidt procedure to the set of linearly independent frame fields $e_a = \{e_0, e_1 = \partial_r, e_2 = \partial_z, e_i = \partial_i\}$ where $i = x, y$. This was done using Maple and returns a basis of frame fields that we shall denote $\{\hat{e}_a\}$ with a hat. These still correctly characterise the parallel propagation but at the same time are fully orthonormal in the sense that they satisfy $g_{\mu\nu}\left(\hat{e}_a\right)^\mu\left(\hat{e}_b\right)^\nu = \eta_{ab}$.

The full expressions for the individual frame fields $\{\hat{e}_a\}$ are quite complicated and not especially illuminating so we omit them here. However, we can then use the frame fields as transformation matrices to obtain the components of the Riemann tensor as measured in the PPON via

$$\tilde{R}_{abcd} = R_{\mu\nu\rho\sigma}\left(\hat{e}_a\right)^\mu\left(\hat{e}_b\right)^\nu\left(\hat{e}_c\right)^\rho\left(\hat{e}_d\right)^\sigma, \tag{E.26}$$

where $R_{\mu\nu\rho\sigma}$ is calculated according to (A.3) in our conventions. The non-zero components of the PPON Riemann tensor are again rather complicated and so, rather than provide full expressions, we instead list their scaling behaviour in the near horizon regime in Table E.2. These components should then be substituted into (2.11) to evaluate tidal forces between geodesics.

Component	Near horizon behaviour
$\tilde{R}_{0101}$	const
$\tilde{R}_{0102}$	r^{-13}
$\tilde{R}_{0112}$	r^{-13}
$\tilde{R}_{0202}$	r^{-13}
$\tilde{R}_{0212}$	r^{-13}
$\tilde{R}_{0i0j}$	$\delta_{ij}r^{-6}$
$\tilde{R}_{0i1j}$	$\delta_{ij}r^{-6}$
$\tilde{R}_{0i2j}$	$\delta_{ij}r^{-6}$
$\tilde{R}_{1212}$	r^{-13}
$\tilde{R}_{1i1j}$	$\delta_{ij}r^{-6}$
$\tilde{R}_{1i2j}$	$\delta_{ij}r^{-6}$
$\tilde{R}_{2i2j}$	$\delta_{ij}r^{-6}$
$\tilde{R}_{ijkl}$	$r^{-3}\left(\delta_{il}\delta_{jk} - \delta ik\delta_{jl}\right)$

Table E.2: Near horizon scaling behaviour of the non-zero components of the five-dimensional Riemann tensor, $\tilde{R}_{abcd}$, as measured in the PPON.

E.4.2 Four-dimensional tidal forces

To investigate the tidal forces present for the four-dimensional extremal Nernst brane solutions of Chapter 4, we must treat the cases with finite and infinite four-dimensional chemical potential separately. The relevant metrics are given by (5.77) and (5.93) respectively. We shall proceed in a similar fashion to Appendix E.4.1 except for the assumption that the infalling observer is now moving only in the radial direction and has no transverse momentum in either the x or y directions. This is slightly different to the analysis of [121] and means the tangent vector for the time-like geodesic on which our radially infalling observer is travelling is given by

$$T^{\mu} = \left(\dot{t}, \dot{r}, \vec{0}\right),$$

where dot denotes differentiation with respect to the observer's proper time, τ.

$A > 0$ tidal forces

The extremal version of (5.77) is given by

$$ds^2_{A>0,\ \mathrm{Ext}} = \frac{r}{l}\left(-\frac{r^2}{l^2\left(1+\frac{\Delta}{Ar^4}\right)^{1/2}}dt^2 + \frac{l^2\left(1+\frac{\Delta}{Ar^4}\right)^{1/2}}{r^2}dr^2 + \frac{r^2}{l^2}\left(1+\frac{\Delta}{Ar^4}\right)^{1/2}\left(dx^2+dy^2\right)\right). \tag{E.27}$$

The energy is again an integral of motion:

$$E = -g_{tt}\dot{t} = \frac{r^3}{l^3\left(1+\frac{\Delta}{Ar^4}\right)^{1/2}}\dot{t} \quad \Rightarrow \quad \dot{t} = \frac{l^3E\left(1+\frac{\Delta}{Ar^4}\right)^{1/2}}{r^3}.$$

For a time-like geodesic we have

$$g_{\mu\nu}T^\mu T^\nu = -1 \quad \Rightarrow \quad \dot{r} = -\frac{1}{l^{1/2}r}\sqrt{l^3E^2 - \frac{r^3}{\left(1+\frac{\Delta}{Ar^4}\right)^{1/2}}},$$

where we pick the negative square root to represent an observer falling radially inwards. We could equally well pick the positive root and consider an outgoing geodesic but $\dot{r}$ will become complex for large r, meaning the geodesic encounters a turning point and is reflected back into the bulk. This is reminiscent of the situation in Appendix E.4.1 and in fact, this inability of time-like geodesics to reach the boundary is an example of a property that hyperscaling-violating Lifshitz spacetimes can inherit from their parent Anti de-Sitter spacetimes. All of this means that we need only focus on the ingoing observer and near horizon tidal forces. Another similarity with Appendix E.4.1 is the divergence of $\dot{t}$ and $\dot{r}$ as $r \to 0$; again this indicates the coordinates are only valid up the horizon which is absolutely fine for the analysis of tidal forces.

Next we align the frame field e_0 with the vector field $\frac{d}{d\tau}$ responsible for generating the integral curve along which the observer is moving:

$$\begin{aligned}(e_0)^\mu = \left(\frac{d}{d\tau}\right)^\mu &= \dot{t}\partial_t^\mu + \dot{r}\partial_r^\mu \\ &= \frac{l^3E\left(1+\frac{\Delta}{Ar^4}\right)^{1/2}}{r^3}\partial_t^\mu - \frac{1}{l^{1/2}r}\sqrt{l^3E^2 - \frac{r^3}{\left(1+\frac{\Delta}{Ar^4}\right)^{1/2}}}\partial_r^\mu.\end{aligned}$$

The observer is moving in the (t, r) directions and so there are two frame fields associated to this: e_0 and e_1. Since the observer isn't moving in any of the x^i $(i \geq 2)$ directions, the frames

Component	Near horizon behaviour
$\tilde{R}_{0101}$	r
$\tilde{R}_{0i0j}$	$\delta_{ij} r^{-4}$
$\tilde{R}_{0i1j}$	$\delta_{ij} r^{-4}$
$\tilde{R}_{1i1j}$	$\delta_{ij} r^{-4}$
$\tilde{R}_{ijkl}$	$r\left(\delta_{il}\delta_{jk} - \delta ik \delta_{jl}\right)$

Table E.3: Near horizon scaling behaviour of the non-zero components of the four-dimensional $A > 0$ Riemann tensor, $\tilde{R}_{abcd}$, as measured in the PPON.

e_i for $i \geq 2$ are just given by the square roots of the inverse metric components i.e.

$$(e_i)^\mu = \frac{l}{r\left(1 + \frac{\Delta}{Ar^4}\right)^{1/4}} \partial_i^\mu.$$

It remains to find the frame e_1 such that the $\{e_a\}$ form a PPON. We have picked e_0 to describe the parallel propagation and so we just need a second frame field, e_1, that is orthonormal to both e_0 and $e_i, i \geq 2$. It follows from simple linear algebra that

$$(e_1)^\mu = -\frac{l^{3/2}\left(1 + \frac{\Delta}{Ar^4}\right)^{1/2}}{r^3} \sqrt{l^3 E^2 - \frac{r^3}{\left(1 + \frac{\Delta}{Ar^4}\right)^{1/2}}} \partial_t^\mu + \frac{lE}{r} \partial_r^\mu.$$

It is interesting to note that in the case of the static four-dimensional metric, the frame fields are already orthonormal whereas in Appendix E.4.1, where the five-dimensional metric is non-static, this is not the case and we had to perform an additional Gram-Schmidt procedure at this point.

We next use Maple to find the components $R_{\mu\nu\rho\sigma}$ of the Riemann tensor in a coordinate basis with lowered indices according to (A.3). These are then multiplied by frame fields to obtain the local tidal forces felt by the observer. We again omit the full expressions and instead list in Table E.3 the scaling behaviour of the non-zero components in the near horizon regime

$A = 0$ tidal forces

Here we repeat the same procedure as above for the $A = 0$ extremal metric. The extremal version of (5.93) is given by

$$ds^2_{A=0,\ \text{Ext}} = -\frac{r^5}{\Delta^{1/2} l^3} dt^2 + \frac{\Delta^{1/2} l}{r^3} dr^2 + \frac{\Delta^{1/2} r}{l^3}\left(dx^2 + dy^2\right). \tag{E.28}$$

The resulting non-zero components of the Riemann tensor as measured in the PPON are given in Table E.4.

Component	Near horizon behaviour
$\tilde{R}_{0101}$	r
$\tilde{R}_{0i0j}$	$\delta_{ij} r^{-4}$
$\tilde{R}_{0i1j}$	$\delta_{ij} r^{-4}$
$\tilde{R}_{1i1j}$	$\delta_{ij} r^{-4}$
$\tilde{R}_{ijkl}$	$r^3 \left(\delta_{il}\delta_{jk} - \delta ik \delta_{jl}\right)$

Table E.4: Near horizon scaling behaviour of the non-zero components of the four-dimensional $A = 0$ Riemann tensor, $\tilde{R}_{abcd}$, as measured in the PPON.

Consistency with existing classification

The near horizon scaling behaviours of the PPON Riemann tensor components in Tables E.3 and E.4 agree. This is consistent with the fact that the parameter A only affects the asymptotic geometry, which is why the metrics (E.27) and (E.28) both take the same form in the small r limit, specifically, a hyperscaling-violating Lifshitz metric with parameters $(z, \theta) = (3, 1)$ as observed in Chapter 4.

It is worthwhile to check the consistency of the results of this appendix with the complete classification of hyperscaling-violating Lifshitz singularities obtained in [121]. It can be shown that our $(z, \theta) = (3, 1)$ geometry is equivalent to a $(n_0, n_1) = (10, 4)$ geometry in their notation. This would place our near horizon metric into Class IV of the analysis in [121], making it is both consistent with the Null Energy Condition and indicative of a null curvature singularity (infinite tidal forces) at $r = 0$.

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