ALEXANDER POLYNOMIALS OF CLOSED 3-BRAIDS

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1. Introduction

The knots and links which can arise as the closure of 3-string braids, and their relations to the braids which give rise to them have been studied by Murasugi [Mu] and others, including Hartley [H] and more recently Przytycki [P] . Three-braids appear to form a rather special class among braids from some points of view, [M1]; they are, also, the only group of braids for which Burau's representation is known to be faithful [B]. They are, however, varied enough to provide an interesting range of knots and links on which to test a number of conjectures.

In this paper I present a compact formula giving the Alexander polynomial of $\hat{\beta} \cup L_\beta$, and hence $\hat{\beta}$, for the braids $\beta = c^n \sigma_1^{p_1} \sigma_2^{-q_1} \ldots \sigma_1^{p_r} \sigma_2^{-q_r}$, where L_β is the axis of the closed braid $\hat{\beta}$, $c = (\sigma_1\sigma_2)^3$ generates the centre of B_3, and $p_i, q_i > 0$

These braids form Murasugi's class Ω_6, making up the vast majority of 3-string braids, up to conjugacy. The formula, in terms of the indices p_i, q_i and their order of appearance, up to cyclic permutation, enables the number, r, of 'terms' in β

to be read off from the Alexander polynomial, and also the smallest index $m = \min\{p_1, \ldots, q_r\}$ and its multiplicity.

It provides evidence to support the conjecture that for a braid $\beta \in B_n$, the Alexander polynomial of the link $\hat{\beta} \cup L_\beta$, where L_β is the axis of $\hat{\beta}$, determines the link (possibly up to orientation), and hence the conjugacy class of β and its reverse. This Alexander polynomial $\Delta(x, t)$ is just the characteristic polynomial $\det(xI - B(t))$ of the reduced Burau matrix $B(t)$ of β, so the conjecture for $n > 3$ would imply the faithfulness of the corresponding Burau representation.

With considerable conbinatorial ingenuity it might be possible to recover the indices of a 3-string braid β up to cyclic order from the polynomial given in Theorem 3 and so prove the conjecture for 3-braids, but I can see no prospect for a direct attack when $n > 3$.

§2. A formula for the Alexander polynomial of a closed 3-braid

Given any 3-braid β, the Alexander polynomial of the link $\hat{\beta} \cup L_\beta$, the closure of β together with its axis, is given by $\Delta(x, t) = \det(xI - B(t))$, where the variable x refers to the meridian of the axis L_β, and t to all meridians of the oriented closed braid $\hat{\beta}$, and $B(t)$ is the reduced Burau matrix of β, [M2].

In the case of a knot $\hat{\beta}$, its Alexander polynomial can be recovered as $\Delta(1, t)/(1 + t + t^2)$.

<u>Conjecture</u> Given $\Delta(x, t)$ the link $\hat{\beta} \cup L_\beta$ is determined (possibly only up to orientation), and so β is determined up to conjugacy (maybe with reversal or reflection included).

<u>Note</u> $\Delta(x, t) = x^2 - \operatorname{tr} B(t).x + \det B(t)$, so from $\Delta(x, t)$, even given only up to multiples by $\pm x^s t^k$, we can recover $\operatorname{tr} B(t)$ and $\det B(t) = (-t)^{|\beta|}$, where $|\beta|$ = algebraic number of crossings in β.

It will be more convenient to use $s = -t$ in place of t. The Burau matrices for the braid group generators are then

$$\sigma_1(s) = \begin{pmatrix} s & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_2^{-1}(s) = \begin{pmatrix} 1 & 0 \\ 1 & \bar{s} \end{pmatrix}, \qquad (\bar{s} = s^{-1}).$$

Except for a small number of β we can write β up to conjugacy as $\beta = c^n \sigma_1^{p_1} \sigma_2^{-q_1} \dots \sigma_1^{p_r} \sigma_2^{-q_r}$, with $p_i, q_i \geq 1$, and $c = (\sigma_1\sigma_2)^3$ the generator of the centre of B_3, [Mu]. The Burau matrix of c is $s^3 I_2$, so

$$B(s) = s^{3n} M(p_1, q_1) \ldots M(p_r, q_r),$$

where

$$M(p_i, q_i) = \text{Burau matrix of } \sigma_1^{p_i} \sigma_2^{-q_i}$$

$$= \sigma_1(s)^{p_i} (\sigma_2^{-1}(s))^{q_i}$$

$$= s^{-q_i} \begin{pmatrix} s^{p_i} & 1 + s + \ldots s^{p_i - 1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s^{q_i} & 0 \\ s(1 + \ldots + s^{q_i - 1}) & 1 \end{pmatrix}$$

$$= s^{-q_i} C_{p_i}(s) D_{q_i}(s) \quad \text{say.}$$

Then $B(s) = s^{3n - \Sigma q_i} \prod_1^r C_{p_i}(s) D_{q_i}(s)$.

So $\operatorname{tr} B(s) = s^{3n - \Sigma q_i} \operatorname{tr} \prod_1^r C_{p_i}(s) D_{q_i}(s)$.

Write $\operatorname{tr} \prod_1^r C_{p_i}(s) D_{q_i}(s) = Q(s)$, a polynomial in s.

<u>Theorem 1</u> $\quad Q(s) = 1 + rs \mod s^2$.

<u>Corollary 1</u>. We can find $Q(s)$ from $\operatorname{tr} B(s) = s^{3n - \Sigma q_i} Q(s)$ by multiplying $\operatorname{tr} B(s)$ by s^k to get a polynomial with constant term 1. We then also find $k = 3n - \Sigma q_i$.

<u>Corollary 2</u>. From $\operatorname{tr} B(s)$ we can find r, the number of 'terms' in β of the form $\sigma_1^{p_i} \sigma_2^{-q_i}$.

<u>Theorem 2</u> If $r = 1$, $Q(s)$ determines the unordered pair $\{p_1, q_1\}$. Let $m = \min \{p_1, \ldots, q_r\}$ and let α be the number of these indices equal to m. Then

$(1 - s)^{2r} Q(s) = (1 - s + s^2)^r - \alpha s^{m+1} \mod s^{m+2}$, when $r > 1$.

<u>Corollary 3</u> From $Q(s)$ we can find m and α.

These results follow fairly readily from calculations of C_{p_i} and D_{p_i} mod s^2 or mod s^{m+2}, or from the complete formula for $Q(s)$ given in Theorem 3. I shall construct the formula for $Q(s)$ from a polynomial in s and indeterminates $t_1, \ldots, t_{2r}$ which contains no squares or higher powers of any indeterminate t_k by putting $t_{2i-1} = s^{p_i}$ and $t_{2i} = s^{q_i}$. Each monomial in $\{t_k\}$ has as coefficient a polynomial in s depending on the number of gaps when the indeterminates in the monomial are arranged in order round a circle.

To formulate this explicitly, write the numbers $1, \ldots, 2r$ consecutively round a circle.

<u>Definition</u> For each subset $J \subset \{1, \ldots, 2r\}$ write $c(J)$ for the number of 'blocks' of J on the circle, i.e. the components of the subset of S^1 given by joining all adjacent pairs which lie in J.

Make the convention that $c(J) = 0$ when $J = \emptyset$ and when $J = \{1, \ldots, 2r\}$. Then $c(J)$ counts the number of times you pass from J to its complement J' on making a circuit of S^1, and $c(J') = c(J)$.

For example, the subsets $\{1, 2\}$ and $\{1, 4\}$ of $\{1, 2, 3, 4\}$ have one block, while $\{1, 3\}$ has two.

Write t_J for the monomial $\prod_{k \in J} t_k$ in indeterminates $t_1, \ldots, t_{2r}$. We can now give our explicit formula for $Q(s)$, the trace of the reduce Burau matrix $B(t)$ for $\beta = c^n \sigma_1^{p_1} \sigma_2^{-q_1} \ldots \sigma_1^{p_r} \sigma_2^{-q_r}$, where $s = -t$, normalised to have lowest degree term 1.

Theorem 3 $$Q(s)(1-s)^{2r} = \sum_{J\subset\{1,\ldots,2r\}} (-1)^{|J|} s^{c(J)} (1-s+s^2)^{r-c(J)} t_J,$$

where $t_{2i-1} = s^{p_i}$, $t_{2i} = s^{q_i}$.

Remark The appearance of the numbers $c(J)$ make it possible that the order of the indices $\{p_1, \ldots, q_r\}$ around a circle could be recovered from $Q(s)$ as part of a process for finding the set of indices. In the case $r = 1$ or 2 the index set and, for $r = 2$, the circular order can be recovered from $Q(s)$.

3. Proofs

Proof of Theorem 2

Take $m = \min\{p_i, q_j\}$.

(a) When $m > 1$ then $t_J = 0 \bmod s^{m+2}$, with t_k as in Theorem 3, for all J with $|J| > 1$.

$$\text{Then}\quad Q(s)(1-s)^{2r} = (1-s+s^2)^r - s(1-s+s^2)^{r-1}\sum_1^{2r} t_k \bmod s^{m+2}.$$

Now $\sum_1^{2r} t_k = \alpha s^m \bmod s^{m+1}$, where α is the number of t_k equal to s^m, so $s(1-s+s^2)^{r-1}\sum_1^{2r} t_k = \alpha s^{m+1} \bmod s^{m+2}$, giving Theorem 2 in this case.

(b) When $m = 1$ it is still true that $s^{c(J)}t_J = 0 \bmod s^{m+2}$ for $|J| > 1$, apart from the case $r = 1$, where $c(J) = 0$ when $|J| = 2$, giving a single exception $p_1 = q_1 = 1$, $r = 1$.

The proof of theorem 2 follows as above except in this one simple case, which can be detected in advance from the highest degree term in $Q(s)$, namely $s^{\Sigma p_i + \Sigma q_i}$.

In the case $r = 1$ we have
$$Q(s)(1-s)^2 = (1-s+s^2) - s(s^{p_1} + s^{q_1}) + (1-s+s^2)s^{p_1+q_1}$$

Find the largest degree term $s^{p_1+q_1+2}$, and then remove $(1-s+s^2)(1+s^{p_1+q_1})$ to recover p_1 and q_1.

Proof of Theorem 1 From Theorem 3, $Q(s)(1-2rs) = 1 - rs \bmod s^2$, so $Q(s) = 1 + rs \bmod s^2$.

Proof of Theorem 3 Write $t_{2i-1} = s^{p_i}$, $t_{2i} = s^{q_i}$.

Then $(1 - s)C_{p_i}(s) = \begin{pmatrix} t_{2i-1}(1-s) & 1-t_{2i-1} \\ 0 & 1-s \end{pmatrix} = t_{2i-1}A_1 + B_1$,

where $A_1 = \begin{pmatrix} 1-s & -1 \\ 0 & 0 \end{pmatrix}$ and $B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1-s \end{pmatrix}$.

Similarly $(1-s)D_{q_i} = t_{2i}A_2 + B_2$, where

$$A_2 = \begin{pmatrix} 1-s & 0 \\ -s & 0 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 0 & 0 \\ s & 1-s \end{pmatrix}.$$

Then $Q(s)(1 - s)^{2r} = \mathrm{tr}\left(\prod_{i=1}^{r} (t_{2i-1}A_1 + B_1)(t_{2i}A_2 + B_2)\right)$.

Write M_J for the product of the sequence of matrices A_1, B_1, A_2, B_2 determined explicitly by each subset $J \subset \{1, 2, \ldots, 2r\}$ as $M_J = \prod_1^{2r} C_k$, where $C_k = A_k$ if $k \in J$, and $C_k = B_k$ if $k \notin J$, taking $A_k = A_1$ for k odd, A_2 for k even and similarly for B_k. Then $\prod_{i=1}^{r}(t_{2i-1}A_1 + B_1)(t_{2i}A_2 + B_2) = \sum_{J \subset \{1,\ldots,2r\}} t_J M_J$, and so $Q(s)(1 - s)^{2r} = \sum_J P_J(s)t_J$, where $P_J(s) = \mathrm{tr}\, M_J$. Theorem 3 will then follow from the calculation of $\mathrm{tr}\, M_J$ given in Lemma 4.

Lemma 4 For $J \subset \{1, \ldots, 2r\}$ and M_J as above,

$$\mathrm{tr}\, M_J = (-1)^{|J|} s^{c(J)} (1 - s + s^2)^{r-c(J)}.$$

Proof of Lemma 4 Without altering the trace of M_J we may cyclically permute the matrices $C_1, \ldots, C_{2r}$. In the product we have $c(J)$ blocks of consecutive matrices of type A_k, separated by $c(J)$

blocks of matrices of type B_k, up to cyclic reordering. In the product the subscripts 1 and 2 will alternate.

The proof is by induction on r, based on several straightforward calculations.

(1) $A_1A_2A_1 = vA_1$, $A_2A_1A_2 = vA_2$; $B_1B_2B_1 = vB_1$, $B_2B_1B_2 = vB_2$,

where $v = (1 - s + s^2)$. For example, use $A_1A_2 = v\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$.

(2) $$B_2A_1A_2B_1 = sB_2B_1, \quad B_1A_2A_1B_2 = sB_1B_2 \ ;$$
$$A_1B_2B_1A_2 = sA_1A_2, \quad A_2B_1B_2A_1 = sA_2A_1.$$

Case 1

If any block of J or J' has three or more elements, we can use (1), after cyclic permutation of the matrices C_k if necessary, to give $\mathrm{tr}\, M_J = v\, \mathrm{tr}\, M_K$ where K is the subset of $\{1, \ldots, 2r - 2\}$ given from J by omitting two consecutive numbers in the block of J or J', and renumbering. Then the number of blocks is unchanged, as is the parity of $|J|$, and induction gives
$\mathrm{tr}\, M_K = (-1)^J s^{c(J)} v^{r-1-c(J)}$.

Case 2

If a block of J or J' has two elements, then we can use (2), possibly after cyclic permutation, to give $\mathrm{tr}\, M_J = s\, \mathrm{tr}\, M_K$, where K is given by omitting the two element block and renumbering. Then $c(K) = c(J) - 1$ and again induction gives the result.

Finally, if each block of J and J' has only one element, we have

Case 3

$$M_J = (A_1B_2)^r \qquad \text{or} \qquad M_J = (B_1A_2)^r .$$

Here $c(J) = |J| = r$. Now $A_1B_2 = \begin{pmatrix} -s & -(1-s) \\ 0 & 0 \end{pmatrix}$ and $B_1A_2 = \begin{pmatrix} -s & -s(1-s) \\ 0 & 0 \end{pmatrix}$. A quick calculation confirms that in either case $\operatorname{tr} M_J = (-s)^r$, as required.

The induction is completed by checking the remaining cases with $r = 1$, $M_J = A_1A_2$ and B_1B_2.

This completes the proof of Lemma 4, and of Theorem 3.

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