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FIBRED LINKS FROM CLOSED BRAIDS

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ABSTRACT. We show that every fibred link with k components can be constructed using a simple d-sheeted cover of S^3 branched over a suitable closed braid, with $d = k$ for $k \geq 3$ and otherwise $d = 3$. The method used is to relate the monodromy homeomorphism of the fibre to a homeomorphism of a disc of which it is a simple d-sheeted cover. It extends the work of Hilden and Birman, who treat the case $k = 1$, **[11]**, **[6]**.

We go on to relate the construction of plumbing a Hopf band on to the fibre, F, of a fibred link, L, (giving a fibre for a new fibred link **[22]**), with the alteration by a Markov move of the closed braid used as branch set in the covering construction for L. We show that a Markov move on the branch set always corresponds to plumbing a Hopf band to the fibre F in some way. Conversely we show how any plumbing of a Hopf band on to F can be realised by first adding trivial components to the branch set which produced L, increasing the degree of the cover correspondingly, and then conjugating the resulting braid and making a Markov move.

1. Overview.

The idea of constructing a fibred knot using a closed braid is rooted in the classical paper of Alexander **[1]**,**[2]**. In recent years it has been fruitfully developed by Goldsmith **[9]** and Birman **[6]**, following the ideas of Hilden **[11]** which we also use substantially in this paper. The work reported here was carried out during a visit of the second author to Zaragoza in 1984, with the support of CAICYT.

1.1 The basic idea.

Take a closed curve or curves C in S^3 which lie as a closed n-braid $\widehat{\beta} = C$ relative to an axis L_β. This means that there is a fibration $p : S^3 - L_\beta \to S^1$ in which C projects regularly to S^1, covering it n times, so that C meets each fibre $p^{-1}\{e^{i\theta}\} = F_\theta$, say, transversely in n points. We shall take L_β to be unknotted, so that $F_\theta \cong D^2$ and C is a closed n-braid in the usual sense.

Take any covering $\pi : M^3 \to S^3$ branched over C. Any choice of representation of C as a closed braid $\widehat{\beta}$ with axis L_β gives rise to an open book decomposition of M^3, with leaves $\pi^{-1}(F_\theta)$ and binding $\pi^{-1}(L_\beta)$. In the cases where C, π are chosen so that $M^3 \cong S^3$ the curves $\pi^{-1}(L_\beta)$ then form a fibred knot or link in the classical sense.

There are infinitely many ways to choose an axis L_β so as to represent C as a closed braid, giving rise to a large selection of inequivalent fibred knots. These knots are all related by the construction of plumbing and deplumbing Hopf bands, (see Theorem C).

Any given fibred link can be constructed in this way for a suitable choice of C and π, see **[12]**. We show (Theorem A) that the construction is still possible under strong restrictions on the degree and type of covering π which is used, although we cannot impose restrictions on the branch set C in general.

In Theorem D we show how to carry out this construction in the case of fibred links which arise from the unknot by plumbing Hopf bands (without deplumbing), while imposing very strong control on the nature of C also.

If we could insist in general that C be always an unlink, for example, then we would be able to deduce Harer's conjecture **[10]** that all fibred links in S^3 are related by plumbing and deplumbing Hopf bands.

1.2 Previous results.

Goldsmith, for example, considers the case where C is the trivial knot, and π is a cyclic covering of some order. An interesting selection of fibred links arises, but by no means all possible ones. She also considers the case of a 'generalised axis', L_β, for a closed curve C, using a fibred knot other than the trivial one as the axis L_β.

Birman **[6]** shows that every open book decomposition of a closed manifold M^3 with *connected* binding, and in particular every fibred *knot* in S^3, arises for some choice of C and representation of C as a closed braid with axis L_β, using suitable 3-sheeted *simple* covers $\pi : M^3 \to S^3$.

The term *simple*, applied to a d-sheeted cover $\pi : M^3 \to S^3$, will be described shortly in more detail. It means that every meridian curve of the branch set C is covered by $d-1$ circles, one of which projects by π as a 2-fold cover, while the rest project homeomorphically. The name is adopted, following Berstein and Edmonds **[3]**, because of the close relation with simple covers in the classical sense of Riemann surfaces.

1.3 Present results.

In this paper we prove:

THEOREM A. *Every fibred link in S^3 with k components can be realised as $\pi^{-1}(L_\beta)$ for a d-sheeted simple cover $\pi : S^3 \to S^3$ branched over some closed braid $\widehat{\beta}$, with $d = \max\{k, 3\}$, where L_β is the axis of $\widehat{\beta}$.*

Our proof shows that in fact *every* open book decomposition of a closed M^3 even with disconnected binding, arises in a similar way. For a given decomposition there is a wide choice of $\widehat{\beta}$ which can be used in the construction.

As an essential part of the proof of Theorem A we prove:

THEOREM B. *Let F be a surface with boundary, and $\pi : F \to D^2$ a d-sheeted simple cover, with $d \geq 3$. Then, up to isotopy fixing ∂F pointwise, every homeomorphism of F which fixes ∂F pointwise covers a homeomorphism of D^2. That is, given a homeomorphism $H' : F \to F$ there exists H isotopic to H' and $h : D^2 \to D^2$ with $\pi \circ H = h \circ \pi$, where H, H' and the isotopy all fix ∂F pointwise.*

In the last section of the paper we use both theorems to study the construction of fibred links by Hopf plumbing. In this construction the fibre F of a given fibred link L and a Hopf band, (a ribbon with a single twist), are plumbed together, using some proper arc in F, to give a surface F' whose boundary L' then forms a fibred link with fibre F', **[22]**.

We establish in THEOREM C a close relation between Hopf plumbings of links and Markov moves on the braids used in the covering construction for the links. It is well-known that any two presentations of a link as a closed braid are related by a sequence of Markov moves. We show in section 5.3 that if we alter the closed braid presentation $\widehat{\beta}$ of the branch set by a single Markov move, increasing braid index, then the fibre F for the fibred link L constructed using a simple cover π is altered by plumbing a Hopf band to F.

Conversely, given $\widehat{\beta}$ and π which construct a fibred link L as in theorem A, we show how to find β' and π' to construct the link L' which arises from L by any given Hopf plumbing.

A direct consequence is the following theorem giving a construction for all fibred links which arise from the trivial knot by a sequence of Hopf plumbings.

THEOREM D. *A fibred link L arises from the trivial knot by a sequence of Hopf plumbings if and only if it can be constructed from the d-sheeted simple cover of S^3 branched over some $\widehat{\beta}$, where the braid β is directly reducible by Markov moves to the trivial braid on $d-1$ strings for some d.*

2. Simple covers of surfaces.

2.1 Introduction.

DEFINITION. A continuous surjective map $\pi : F \to S$ between two surfaces is called a *simple cover with d sheets* if there is a finite set Q in the interior of S, termed the *branch set*, and each $s \in S$ has a disc neighbourhood U over which $\pi : \pi^{-1}(U) \to U$ behaves as follows:

(1) if $s \notin Q$ then $\pi|_{\pi^{-1}(U)}$ is a trivial d-sheeted cover,

(2) if $s \in Q$ then $\pi^{-1}(U)$ has $d-1$ components, one of which is a disc projecting to U as a double cover branched over s, while the others are discs projecting homeomorphically.

DEFINITION. Two covers $\pi_1 : F_1 \to S_1$ and $\pi_2 : F_2 \to S_2$ are *equivalent* if there are homeomorphisms $H : F_1 \to F_2$ and $h : S_1 \to S_2$ such that $\pi_2 \circ H = h \circ \pi_1$. In the special case where $S_2 = S_1$ and $h =$ identity they are called *isomorphic*.

REMARK. The homeomorphism h carries the branch set in S_1 to the branch set in S_2.

The term 'simple cover' and much of the theory in the case $S = S^2$ comes from the classical works of the last century. A modern treatment of this is given in [4]. Simple d-sheeted covers arise classically where F is a non-singular algebraic curve of degree d in $\mathbb{C}^2$ and $\pi : F \to \mathbb{C}$ is the projection of F from a 'generic' point, that is, a point of $\mathbb{C}^2$ through which all tangents to F are simple.

2.2 Covers of a disc.

In our applications we shall only consider the case where S is a disc D^2, and we shall normally work with a fixed branch set $Q = \{q_1, \ldots, q_n\}$. Where ∂F has k components, $\partial F = C_1 \cup \ldots \cup C_k$ say, then a d-sheeted cover $\pi : F \to D^2$ restricts to covers $\pi : C_i \to S^1$ of degree d_i say, with $d_1 + \ldots + d_k = d$. The results of Hurwitz, extended to D^2 as in [4], show that, for a connected surface F, the unordered set of boundary degrees $d_1, \ldots, d_k$ determines any simple cover $\pi : F \to D^2$ up to equivalence, as follows:

THEOREM 2.2.1. *(equivalence theorem)*

Let F and F' be connected surfaces, with $F' \cong F$, and let $\pi : F \to D^2$, $\pi' : F' \to D^2$ be simple covers with degree d_i, resp. d'_i when restricted to the boundary curve C_i, resp. C'_i. If $d_i = d'_i$ for all i then π and π' are equivalent.

PROOF: See section 4.4 for further comment, following [**4**,**4.4**]. □

REMARKS 2.2.2.

1. For a given connected surface F with k boundary components it is possible to construct a simple cover $\pi : F \to D^2$ having any given choice of boundary degrees, except where $k = 1$ and $F \neq D^2$, when we need $d = d_1 \geq 2$.
2. The number of branch points, n, for a simple cover is related to the Euler characteristic $\chi(F)$ and the degree d of the cover by $\chi(F) = d - n$.

For example, when $F = F_{g,k}$, the surface with genus g and k boundary components, we have $2 - 2g - k = d - n$, so that $n = d + 2g + k - 2$.

2.3 Explicit views of a cover.

Given a cover $\pi : F \to D^2$ we may picture it by cutting D^2 along a *splitting complex for* π.

This is a family of disjoint arcs $\{a_j\}$ in D^2 chosen so that

1. the end points of each arc lie on ∂D^2 or on the branch set Q,
2. the set $D^2 - A$ is connected, where $A = \cup a_j$,
3. the cover π restricted to $\pi^{-1}(D^2 - A)$ is trivial.

If we number the d components of $\pi^{-1}(D^2 - A)$ from 1 to d in some way, for example by numbering the points of $\pi^{-1}(*)$ for a point $* \notin A$, then we can reconstruct f and π by reassembling these sheets, given the instructions on how to join the pieces along $\pi^{-1}(a_j)$. So we label the arc a_j with a permutation $\rho_j \in S_d$ to show that as we cross $\pi^{-1}(a_j)$ from sheet i we pass to sheet $\rho_j(i)$. For a general covering we should specify the direction in which we cross a_j, but for a simple cover each of the permutations ρ_j is a transposition.

THEOREM 2.3.1. *Two covers $\pi : F \to D^2$ and $\pi' : F' \to D^2$ giving permutations $\{\rho_j\}$ and $\{\rho'_j\}$ for the same splitting complex which satisfy $\rho'_j = g^{-1}\rho_j g$ for a fixed $g \in S_d$ and all j are isomorphic.*

PROOF: We must construct $H : F \to F'$ with $\pi = \pi' \circ H$. The permutation g shows how to construct H on the sheets of $\pi^{-1}(D^2 - A)$, while the relations between ρ_j and ρ'_j guarantee that H extends continuously across $\pi^{-1}(A)$. □

REMARK. Any set of arcs A in D^2 with $D^2 - A$ connected, and every labelling of the arcs by transpositions $\rho_j \in S_d$ will determine, up to isomorphism, a simple cover π of D^2 by a surface which is connected if and only if the elements ρ_j generate S_d.

FIGURE 2.1

If we have been given π, then the system of arcs shown in figure 2.1 where the branch set $Q = \{q_1, \ldots, q_n\}$ will always be a splitting family, (condition 3 is satisfied since $D^2 - A$ is simply-connected). These are called a *Hurwitz system* of arcs, and the corresponding sequence $(\tau_1, \ldots, \tau_n)$ of transpositions is the *Hurwitz sequence* for π. An explicit view of π constructed from the Hurwitz system is shown in figure 2.2, in the case $d = 4$.

FIGURE 2.2

In general, the tabs containing the branch points $q_1, \ldots, q_n$ can be viewed as neighbourhoods of the splitting arcs $a_1, \ldots, a_n$. The cover π is given by projection on the sheets, and on the attached tabs which are not affected by the transposition. Where $\tau_i = (k\ \ell)$, sheets k and ℓ are joined by a band, drawn here with a half-twist so as to respect orientation on the sheets. The effect of π on this band is to identify points as suggested by a 180° rotation about a horizontal axis normal to the centre of the band. In this identification the horizontal arc

b_i across the band is folded in two and mapped to a_i. Its midpoint is the single critical point which projects to the branch point q_i.

By theorem 2.3.1, every d-sheeted simple cover of D^2 with n branch points at $q_1, \ldots, q_n$ is isomorphic to one drawn as in figure 2.2. We shall find it helpful at some times, however, to use different splitting complexes, so that certain features can be more easily seen in the view of the cover.

2.4 Algebraic data for a covering.

Sufficient data to reconstruct a cover $\pi : F \to D^2$ up to isomorphism can be given directly in terms of D^2 and the branch set Q as follows.

THEOREM 2.4.1.

(a) *Every d-sheeted cover $\pi : F \to D^2$ with branch set $Q \subset D^2$ determines, up to conjugacy in S_d, a homomorphism $\varphi_\pi : \pi_1(D^2 - Q, *) \to S_d$.*

(b) *If π and π' are covers with $\varphi_\pi = \varphi_{\pi'}$ up to conjugacy in S_d then π and π' are isomorphic.*

PROOF OF (a): Number the points of $\pi^{-1}(*)$ from 1 to d. Each loop c in $D^2 - Q$ based at $*$ will induce a permutation on these points as follows. Lift c to a path in F starting from the point i of $\pi^{-1}(*)$ and finishing at the point $\rho(i)$ say. Then $\rho \in S_d$ depends on c only up to homotopy in $D^2 - Q$, over which π is an ordinary cover. We then define $\varphi_\pi(\gamma) = \rho$, where $\gamma = [c] \in \pi_1(D^2 - Q, *)$. Renumbering the points of π will simply alter φ_π by conjugacy. □

REMARKS. Suppose that we have chosen a splitting family of arcs A with $* \notin A$. A choice of numbering of the points $\pi^{-1}(*)$ then numbers the components of $\pi^{-1}(D^2 - A)$ and thus labels each arc by a permutation. For any loop c in $D^2 - Q$ crossing A transversely in a finite number of points, the permutation $\varphi_\pi([c])$ is readily seen to be the product of the permutations from the arcs crossed by c taken in order.

For a simple closed curve c in $D^2 - Q$, the nature of the covering π over c is given by the cycle type of $\varphi_\pi([c])$. If $\pi^{-1}(c)$ has k components then there are k disjoint cycles in the permutation $\varphi_\pi([c])$. The length of each cycle gives the degree of the covering on the corresponding component.

The condition that the cover π be simple is equivalent to the requirement that a meridian loop round each branch point be represented by a transposition in S_d.

The cover π is connected if and only if the image of φ_π acts transitively on the set $\{1, \ldots, d\}$. In general, the orbits of this action correspond to components of F.

PROOF OF (b): Given a cover π choose a splitting family of arcs A and number the points of $\pi^{-1}(*)$. The homomorphism φ_π is then defined. The label of the arc a_j will then be $\varphi_\pi([c_j])$, where c_j is a loop crossing a_j once only, and crossing no other arc. Such a loop can be found, since $D^2 - A$ is connected. Suppose that π' is another cover and that $\varphi_{\pi'}$ is conjugate to φ_π by g. The arcs of A are then labelled as in theorem 2.3.1, and so π and π' are isomorphic.□

REMARKS. Condition 3 for a splitting complex is equivalent to the condition that $\pi_1(D^2 - A, *)$ should map into the kernel of φ_π under the inclusion of $D^2 - A$ in $D^2 - Q$.

By theorem 2.2.1, a connected cover is determined, up to equivalence, by the conjugacy class (cycle type) of $\varphi_\pi([c])$, where c is the boundary of D^2.

2.5 Further explicit views of covers.

We shall now make use of a splitting family A with $D^2 - A$ not simply-connected, in visualising certain covers. The simplest example occurs when we have a 2-sheeted cover of D^2 with just 2 branch points.

Using the view of figure 2.1, we take arcs a_1, a_2 with transpositions $\tau_1, \tau_2 = (1\ 2)$, and the cover appears as in figure 2.3, where the surface F is an annulus.

FIGURE 2.3

As an alternative, we may take a single arc a joining the branch points, labelled $(1\ 2)$. Then each sheet covering $D^2 - A$ is an annulus and we can picture F as in figure 2.4, where π is now given by identifying points of sheets 1 and 2 under a 180° rotation about the horizontal axis shown. The closed curve b is then $\pi^{-1}(a)$.

FIGURE 2.4

The isomorphism between the two explicit covers will not, of course, carry sheet 1 to sheet 1, since the sheets are defined using different choices of A. Under the isomorphism we can see the image of the curve b again covering the arc a.

COVERINGS OF ARCS. Suppose that $\pi : F \to D^2$ is a simple d-sheeted cover.

1 Let a be any arc joining two branch points in D^2. Then π restricted to a neighbourhood U of a is a (disconnected) d-sheeted cover of the disc U. The nature of the cover depends on the element $\varphi_\pi(\partial U) \in S_d$ represented by the boundary curve. This can be read off from a labelled splitting family, and will be the product of two transpositions.

When the element $\varphi_\pi(\partial U)$ is the identity then $\pi^{-1}(U)$ consists of $d-2$ discs projecting homeomorphically to U and one annulus projecting to U as in figure 2.4. In this case $\pi^{-1}(a)$ consists of $d-2$ arcs and one simple closed curve, the core of the annulus. In the other two possible cases $\pi^{-1}(a)$ consists of arcs, and $\pi^{-1}(U)$ is a family of discs, with either one or two of them covering U non-trivially.

2 Let a be an arc with one end on ∂D^2 and the other at a branch point then $\pi^{-1}(a)$ consists of $d-2$ arcs each with one end on ∂F and one *proper* arc b (with both ends on ∂F) which covers a. See for example the arcs $a_1, \ldots, a_5$ in figure 2.2.

EXAMPLE 2.5.1. We now describe some further views of covers extending the example of the annulus in figure 2.4. In these views the surface F appears in a more familiar form than in figure 2.2.

FIGURE 2.5

Construct a covering in which the given set of arcs A forms a splitting family with labels as shown in figure 2.5, i.e. assemble 2 copies of $D^2 - A$ as prescribed by the labels. We then have $g+1$ arcs joining $2g+2$ branch points, each labelled with the transposition $(1\ 2)$, describing a 2-sheeted cover, which we can view in figure 2.6.

FIGURE 2.6

The surface $F = F_{g,2}$, and the cover is viewed, away from the tubes, as projection, for sheet 1, and 180° rotation about the horizontal axis X followed by projection, for sheet 2. On each tube the cover is viewed as in figure 2.4.

FIGURE 2.7

Figure 2.7 shows a similar picture with one fewer branch point, where the branch point at one end is joined to ∂D^2 by a splitting arc labelled $(1\ 2)$. The band joining sheets 1 and 2 corresponds to a neighbourhood of this arc, and the surface is $F_{g,1}$. This view can be readily related to $F_{g,1}$ as shown in figure 2.8, where the curves s_{2k} cover the arcs of the splitting family, while the curves s_{2k-1} also cover arcs joining the branch points.

FIGURE 2.8

EXAMPLE 2.5.2. An extension of figure 2.6 to the case of a d-sheeted cover, where the splitting family consists of $g+1$ arcs labelled $(1\ 2)$ and further arcs labelled $(i-1\ i)$, one for each i with $3 \leq i \leq d$ gives a view as in figure 2.9 of the surface $F_{g,k}$, with $k = d \geq 3$, where the cover π has degree 1 on each boundary component.

FIGURE 2.9

In this picture, the projection π is realised by turning over alternate sheets, while mapping the connecting tubes as in figure 2.4.

3. Surface homeomorphisms and covers.

In this section we prove the first main theorem, called Theorem B in the introduction. Our proof draws on the ideas of Hilden [**11**] where he treats the case of closed surfaces. The case of surfaces with one boundary component is covered by Birman and Wajnryb [**7**].

THEOREM B. *Let $F = F_{g,k}$ be a surface of genus g, with $k > 0$ boundary components, let $\pi : F \to D^2$ be any simple d-sheeted cover with $d \geq 3$ and let $H : F \to F$ be a homeomorphism fixing ∂F pointwise. Then there is a homeomorphism H' isotopic to H fixing ∂F pointwise and a homeomorphism $h : D^2 \to D^2$ such that $\pi \circ H' = h \circ \pi$.*

OUTLINE OF PROOF: We prove the theorem for generators of the mapping class group, as described in theorem 3.4.1. We use induction on d to reduce to the case $d = k$, for $F_{g,k}$ with $k \geq 3$, and otherwise to the case $d = 3$. Under these conditions any two d-sheeted simple covers of $F_{g,k}$ are equivalent, by theorem 2.2.1, so it is enough to prove the result for one explicit choice of π.

Having made this reduction we prove theorem B for $k = 1$ and 2 simultaneously, by induction on g. We then finish the proof for $k > 2$ and any g by induction on k for each g. □

3.1 Preliminaries.

NOTATION. Write $M(F)$ for the mapping class group of a compact, orientable surface F with boundary, i.e. the group of orientation-preserving homeomorphisms, modulo those isotopic to the identity. The isotopy is not required to fix the boundary pointwise.

Write $P(F) \lhd M(F)$ for the subgroup which does not permute the boundary components of F.

Write $H(F)$ for the group of homeomorphisms which fix ∂F pointwise, modulo those isotopic to the identity fixing ∂F pointwise.

A simple closed curve s in F determines a right-hand Dehn twist about s depending on the orientation of F (not s) and on a choice of annular neighbourhood of s. All choices of annulus give the same element $\tau_s \in H(F)$.

We shall extend the notation, and the idea of Dehn twists, to the case where $Q \subset F$ is a finite set of points in the interior of F. In the applications we shall usually look at $Q \subset D^2$ as the branch set of some cover.

Write $H(F, Q)$ for the set of homeomorphisms of (F, Q) up to isotopy fixing Q setwise and ∂F pointwise.

An arc a in F with both end points in Q determines an element $\tau_a \in H(F, Q)$, thought of as a half-twist about a, as follows. Take a neighbourhood of a, and a homeomorphism of this, of order 2, which carries a to a, exchanging its end points. Extend this over a collar of the boundary of the neighbourhood by a right-hand half Dehn twist, (depending again on the orientation of F), and by the identity over the rest of F. (See [**3**], [**11**]).

REMARK. The natural homomorphism $H(F) \to P(F)$ is surjective; its kernel is generated by Dehn twists about the boundary curves of F. For example, when F is an annulus then $P(F) = \{e\}$ and $H(F) \cong \mathbb{Z}$.

We shall principally be concerned with $H(F)$ in what follows. It is well known that $H(F)$ is generated by Dehn twists about closed curves in F, [**8**]. Smaller generating sets have been given for closed surfaces by Lickorish [**15**] and refined by Humphries [**13**], and the case where $\partial F \neq \phi$ has been discussed by Birman [**5**].

NOTATION. Let $\pi : F \to D^2$ be a simple cover. We say that an element $\eta \in H(F)$ is a *π-cover* if there is a homeomorphism H representing η and a homeomorphism $h : D^2 \to D^2$ such that $\pi \circ H = h \circ \pi$.

Write $H_\pi(F) \subset H(F)$ for the subset consisting of π-covers. Then $H_\pi(F)$ is a subgroup of $H(F)$.

The element h represents an element $\beta \in H(D^2, Q)$ which we say is *covered* by η.

Theorem B can then be restated as saying that $H_\pi(F) = H(F)$ for all (connected) simple covers π with at least 3 sheets.

3.2 Dehn twists.

We shall show how to find sufficiently many Dehn twists in F which are π-covers to form a generating set for the whole of $H(F)$. Some of these are constructed as covers of Dehn twists about closed curves in $D^2 - Q$, or as covers of half-twists $\tau_a \in H(D^2, Q)$.

We now discuss fractional Dehn twists briefly in the context of explicit homeomorphisms of F, rather than the equivalence classes $H(F)$.

LEMMA 3.2.1. *Let A' and A be annuli, and let $\pi : A' \to A$ be a regular n-fold cover. Let $\tau_\alpha : A \to A$ be a Dehn twist to the right through an angle α. Then τ_α is covered by $T : A' \to A'$, where $T = \tau_{\alpha/n}$, i.e. $\pi \circ T = \tau_\alpha \circ \pi$.*

PROOF: We illustrate, without further proof, the case $\alpha = 2\pi$, $n = 3$, in figure 3.1. □

FIGURE 3.1

COROLLARY. A full Dehn twist in A' covers n full twists in A.

REMARK. We always choose the orientations of F and D^2 so that π is orientation preserving.

LEMMA 3.2.2. *(Well-known)*

Let $\pi : F_{0,2} \to D^2$ be the 2-sheeted cover illustrated in figure 2.4. Then $H_\pi(F_{0,2}) = H(F_{0,2})$.

PROOF: We must simply show that the Dehn twist τ_b about the core of the annulus is a π-cover.

Let $A \subset D^2$ be a collar of the boundary. Take $h : D^2 \to D^2$ to be a half Dehn twist on A, fixing ∂D^2, extended by a rigid $180°$ rotation exchanging the branch points on $D^2 - A$, i.e. h represents the element $\tau_a \in H(D^2, Q)$. Each component of $\pi^{-1}(A)$ covers A once, so if we take $H : F_{0,2} \to F_{0,2}$ to be the half Dehn twist on each of these components, and a $180°$ rotation about the vertical axis on the rest of $F_{0,2}$ we have $\pi \circ H = h \circ \pi$. Moreover, H is in total a full Dehn twist about the core b, so $\tau_b \in H_\pi(F_{0,2})$, covering $\tau_a \in H(D^2, Q)$. □

An extension of this result gives the following much-used lemma.

LEMMA 3.2.3. *Let $\pi : F \to D^2$ be a simple cover, with branch set $Q \subset D^2$, and let a be an arc joining two points of Q, whose interior avoids Q. Suppose that $\pi^{-1}(a)$ contains a closed curve b, (see section 2.5). Then $\tau_b \in H_\pi(F)$.*

PROOF: Construct a homeomorphism of (D^2, Q) representing $\tau_a \in H(D^2, Q)$ by using h from the previous lemma on a disc neighbourhood U of a, extended by the identity outside U. This is covered by a homeomorphism H which is the identity outside $\pi^{-1}(U)$, and is a Dehn twist on the component of $\pi^{-1}(U)$ containing b. On each other disc component U_i of $\pi^{-1}(U)$ we construct $H \mid U_i \to U_i$ to cover $h : U \to U$ using the homeomorphism $\pi \mid U_i \to U$. Then H is isotopic to the identity outside a neighbourhood of b, and represents τ_b in $H(F)$, so τ_b covers τ_a. □

We now give a lemma, proved using the techniques introduced by Hilden **[11]**.

LEMMA 3.2.4. *Let $F = F_{g,2}$ with boundary curves C_1, C_2, and let $\pi : F \to D^2$ be a 3-sheeted simple cover. Then τ_{C_1} and τ_{C_2} both lie in $H_\pi(F)$.*

PROOF: If τ_{C_1} is a π-cover and π' is equivalent to π then the twist about the corresponding component to C_1 will be a π'-cover. So it is enough to establish the lemma for one choice of π.

Let π be given by the family of arcs labelled as in figure 3.2, where there are $g+1$ arcs labelled (1 2) and one labelled (2 3). The closed curve b covers the arc a, labelled (2 3), so τ_b covers τ_a, by 3.2.3, and thus $\tau_b \in H_\pi(F)$. Now b is parallel to the boundary component C_1, so $\tau_b = \tau_{C_1}$.

FIGURE 3.2

The boundary ∂D^2 is covered twice by C_2 and once by C_1, so a double Dehn twist in D^2 about ∂D^2 will be covered by $(\tau_{C_1})^2\tau_{C_2}$, by 3.2.1. Thus $(\tau_{C_1})^2\tau_{C_2} \in H_\pi(F)$; since $\tau_{C_1} \in H_\pi(F)$ it follows that $\tau_{C_2} \in H_\pi(F)$ also. □

3.3 Stabilisation of covers.

We now set up the basis for the induction on the number of sheets in the cover, which we shall use in the proof of theorem B. In what follows we suppose that F' is a surface with $F' \subset F$. Define a homomorphism $i : H(F') \to H(F)$, by extending a representative homeomorphism of F' to one of F by the identity on $F - F'$. The subgroup $i(H(F')) \subset H(F)$ is then generated by Dehn twists about curves in F'. Note that if $F'' = F' \cup D$ where D is any disc in F meeting F' along a single arc in $\partial F'$ then $i(H(F')) = i(H(F''))$.

LEMMA 3.3.1. *Let F be a surface with k boundary components, $C_1 \cup \ldots \cup C_k$, and let $\pi : F \to D^2$ be a $(d+1)$-sheeted simple cover. Then there is a surface $F' \subset F$ such that*

(1) F is the union of F' with a number of discs pasted to F' along single arcs in $\partial F'$,

(2) $\pi \mid F' \to D_1$ is a simple cover of degree d, where $d \geq \begin{cases} k & if k>1 \\ 2 & if k=1. \end{cases}$

PROOF: It is enough to prove the result for a cover equivalent to π. Suppose that π has degrees $(d_1, \ldots, d_k)$ on the components of ∂F. Then $d_1 + \ldots + d_k = d + 1 > k$, so $d_j \geq 2$ for some j. We may suppose that the components have been numbered so that $d_k \geq 2$. Then there exists a covering $\pi' : F \to D^2$ with degrees $(d'_1, \ldots, d'_k)$ where $d'_k = d_k - 1$, and $d'_i = d_i$ otherwise, (see remarks 2.2.2).

Describe π' by some Hurwitz sequence of transpositions $\tau_1, \ldots, \tau_n \in S_d$, where $n = d + 2g + k - 2$. We can suppose that the sheets have been numbered so that the component C_k meets the dth sheet. (Then in the permutation $\tau_1\tau_2 \ldots \tau_n$ which represents ∂D^2 the cycle containing d has length d'_k, corresponding to the boundary component C_k). If we now take a $(d+1)$-sheeted cover with Hurwitz sequence $\tau_1, \ldots, \tau_n, \tau_{n+1}$, where $\tau_{n+1} = (d \ d+1)$, we will have a cover equivalent to π, for it is connected and has $d+1$ sheets, with the right degree on each boundary component.

Replace π by this new cover, illustrated in figure 3.3.

FIGURE 3.3

Then $\pi^{-1}(D_1)$ consists of a disc in sheet $d+1$ together with a connected surface F', and $\pi \mid F' \to D_1$ is a d-sheeted simple cover. The surface F is the union of F' with a number of discs, as required. □

DEFINITION. Where $F' \subset F$ are related as in lemma 3.3.1 we say that the cover $\pi : F \to D^2$ arises from the cover $\pi \mid F' \to D_1$ by *attaching a trivial sheet.*

THEOREM 3.3.2. *It is sufficient to prove theorem B, as formulated in 3.1, in the case where π has degree $d = \max\{3, k\}$. Since any two simple covers of this degree for a given F are equivalent, it is then enough to prove theorem B for a single choice of cover of this degree.*

PROOF: Let $\pi : F \to D^2$ be a simple cover with $r > \max\{3, k\}$ sheets. Choose $F' \subset F$ as in 3.3.1. Then $i(H(F')) = H(F)$, by condition (1).

Set $\pi' = \pi \mid F' \to D_1$. Then $i(H_{\pi'}(F')) \subset H_\pi(F)$, for if $H' : F' \to F'$ covers $h' : D_1 \to D_1$ we can define a π-cover $H : F \to F$ by H' on F', h' on the cover of D_1 in sheet $d+1$ and the identity elsewhere. The extension of H' to F by the identity outside F' is isotopic to H, since they just differ by h' on a disc, so $i([H']) = [H] \in H_\pi(F)$.

If theorem B holds in the case $r = \max\{3, k\}$ then by induction on r we can assume that $H_{\pi'}(F') = H(F')$. Hence $H(F) = i(H(F')) \subset H_\pi(F)$. □

3.4 Generators for $H(F)$.

We draw on existing descriptions for the mapping class groups to establish sets of generators for $H(F)$ which can be used inductively.

THEOREM 3.4.1. *Let $F = F_{g,1}$ and let $F' \cong F_{g-1,2}$ lie as shown in figure 3.4. Then $H(F)$ is generated by $i(H(F'))$ together with twists about ∂F and τ_{s_1}.*

FIGURE 3.4

PROOF: Generators for the mapping class group of F are well-known, [**5**]. □

THEOREM 3.4.2. *Let $F = F_{g,k}$, $k \geq 2$, $g > 0$ and let $F' \cong F_{g,k-1}$ lie as shown in figure 3.5. Then $H(F)$ is generated by $i(H(F'))$ together with twists about curves in ∂F and τ_{s_1}.*

FIGURE 3.5

PROOF: We use Birman's description [**5**] in this case.

Let $F'' \supset F$ be the surface given by filling in one boundary component C_1 with a disc B. We then have homomorphisms $i : H(F') \to H(F)$ and $i : H(F) \to H(F'')$ whose composite is an isomorphism, induced by the inclusion of F' in F''. Then $H(F)$ is generated by $i(H(F'))$ together with the kernel of $i : H(F) \to H(F'')$. This kernel is itself generated by twists about C_1, and 'transport' of C_1 about curves in F'', [**5**]. We now define the term *transport.*

Transport of C_1 about a simple closed curve s of F'' through a point c of B is defined by choosing an annulus in F'' containing B in its interior, and having boundary curves s_1, s_2 parallel to s. Then the homeomorphism $\tau_{s_1}\tau_{s_2}^{-1}$ or its inverse, regarded as an element of $H(F)$, gives transport about s in the sense required, as indicated in figure 3.6.

FIGURE 3.6

Transport about a composite of simple loops is taken to be the composite of the transport homeomorphisms. It depends, up to twists about C_1 only on the homotopy class of the loop used in $\pi_1(F'', c)$.

Now $\pi_1(F'', c)$ can be generated by simple closed curves. In the present case, where we have $g > 0$, we can assume that these are all non-separating curves, one of which, s, is as shown in figure 3.7, while the others each have the form $\varphi(s)$, for some homeomorphism $\varphi : F'' \to F''$, fixed on $\partial F''$ and on $F'' - F'$. Then transport about $\varphi(s)$ is the conjugate in $H(F)$ of transport about s by the element of $i(H(F'))$ determined by φ.

FIGURE 3.7

It is now sufficient to observe that the twists about s_1 and s_2 allow transport of ∂B about s. Together with $i(H(F'))$ and twists about ∂B these twists generate all other transport of C_1, and hence the whole of $H(F)$. □

3.5 Proof of Theorem B.

CASE 1. We start with the result for $F_{g,1}$ and $F_{g,2}$, which we prove by induction on g. Recall that we want to prove that every element of $H(F)$ is a π-cover, where π is a connected d-sheeted simple cover with $d \geq 3$. By theorem 3.3.2 it is enough to show this when $d = 3$. It is then enough to prove for one explicit π, since all others are equivalent.

Suppose, by induction, that theorem B holds for $F_{g-1,1}$ and $F_{g-1,2}$. Now take $\pi : F_{g,1} \to D^2$ given by the Hurwitz sequence $\tau_1, \ldots, \tau_{2g+2}$, where $\tau_i = (1\ 2)$, $i < 2g+2$, and $\tau_{2g+2} = (2\ 3)$. We may picture π as constructed from a splitting complex of arcs as shown in figure 3.8.

FIGURE 3.8

Let $D_1 \subset D^2$ be the subdisc shown, and set $F' = \pi^{-1}(D_1) \cong F_{g-1,2}$. This surface $F' \subset F$ lies as in figure 3.4, except for the addition or deletion of discs meeting ∂F in single arcs, so that theorem 3.4.1 applies equally to this choice of F', with the curve s_1 chosen as shown. Then $H(F)$ is generated by $i(H(F'))$, together with τ_{s_1} and a twist around ∂F. It is now enough to prove that each of these twists lies in $H_\pi(F)$, and that $i(H(F')) \subset H_\pi(F)$.

Since ∂F covers ∂D^2 three times it follows from lemma 3.2.1 that $\tau_{\partial F}$ covers the 3-fold Dehn twist about ∂D^2. We can also see that s_1 covers an arc a joining the first two branch points in D^2, so τ_{s_1} covers τ_a and hence $\tau_{s_1} \in H_\pi(F)$. It remains to show that $i(H(F')) \subset H_\pi(F)$.

Now $\pi' = \pi|_{F'}$ is a simple 3-sheeted cover, and by the induction hypothesis $H_{\pi'}(F') = H(F')$. Clearly $i(H_{\pi'}(F')) \subset H_\pi(F)$, noting that any π'-cover extended by the identity outside F' is a π-cover. The result then follows at once for $F_{g,1}$.

FIGURE 3.9

Now take π and D_1 as in figure 3.9. Then $F \cong F_{g,2}$ and $F' = \pi^{-1}(D_1) \cong F_{g,1}$, where $\pi' = \pi|_{F'}$ is a simple 3-sheeted cover. A similar argument shows that $i(H(F')) \subset H_\pi(F)$, as is τ_{s_1}, and we have already proved that Dehn twists about both curves in ∂F are π-covers, so the result for $F_{g,2}$ follows from theorem 3.4.2.

CASE 2. We now prove theorem B for $F_{g,k}$, $k \geq 3$, $g > 0$, by induction on k for fixed g. Again it is enough to prove it where π is a simple d-sheeted cover with $d = k$ (the minimum possible), and any two such covers are equivalent.

FIGURE 3.10

Take π as in figure 3.10, with Hurwitz sequence $\tau_1, \ldots, \tau_n$, where $\tau_i = (1\ 2), i \leq 2g+2$, and $\tau_{2g+2i-1} = \tau_{2g+2i} = (i\ i+1)$, $2 \leq i \leq k-1$, using splitting arcs as shown. Take the subdisc D_1 which omits the first branch point. Then we have $F' = \pi^{-1}(D_1) \cong F_{g,k-1}$, and $\pi' = \pi|_{F'}$ is a d-sheeted cover. It follows as in case 1 that $i(H(F')) \subset H_\pi(F)$, since $H_{\pi'}(F') = H(F')$ by induction. We also see that $\tau_{s_1} \in H_\pi(F)$, so it is enough to prove that all boundary twists lie in $H_\pi(F)$.

To prove this, consider $F'' = \pi^{-1}(D_2)$ where D_2 omits the last branch point. Again $F'' \cong F_{g,k-1}$, and we have $i(H(F'')) \subset H_\pi(F)$. Now the first $k-2$ boundary curves of F are isotopic to curves in F'', so twists about these lie in $H_\pi(F)$. The boundary curve C_k in F on sheet k is isotopic to the curve s_k, which covers an arc in D^2, so $\tau_{s_k} \in H_\pi(F)$.

To show that $\tau_{C_{k-1}} \in H_\pi(F)$ it is enough to note that the single twist about ∂D^2 is covered by the product $\tau_{C_1}\tau_{C_2}\ldots\tau_{C_k}$ of all boundary twists, since π is a single cover on each boundary component.

CASE 3. For the final case $F_{0,k}$ we shall prove directly that every Dehn twist is a π-cover, where π is a k-sheeted simple cover. The boundary curve $C_i, i > 1$, is isotopic to the curve s_i, which covers an arc in D^2, so $\tau_{s_i} \in H_\pi(F)$. The remaining boundary twist, τ_{C_1}, is dealt with as before, by considering the cover of a twist about ∂D^2.

FIGURE 3.11

Choose the cover described by k arcs labelled $(1\ 2), (1\ 3), \ldots, (1\ k)$ as in figure 3.11. Then sheets $2, \ldots, k$ consist of collars of boundary components, $C_2, \ldots, C_k$. Sheet 1 looks very like F. Given any simple closed curve s in F, isotop it until it avoids the collars of ∂F, and so lies in sheet 1.

Put $a = \pi(s)$ in D^2. Then $\pi^{-1}(a)$ consists of s together with one curve in each other sheet. A full twist about a is then covered by the product of a full twist about s and one about each of the other curves in $\pi^{-1}(a)$. We already know that all of these other twists are π-covers (they are either boundary twists, or trivial in $H(F)$) so the twist about s will be also.

□

4. Braids and open book decompositions.

In this section we give the connection between closed braids and open-book decompositions which leads directly to our theorem A.

4.1 Open-book decompositions.

DEFINITION. Let $H : F \to F$ be a homeomorphism of a compact oriented surface F, which fixes ∂F pointwise. A 3-manifold M is called the *relative mapping torus* of H if there is a continuous surjective map $p : F \times I \to M$ making exactly the following identifications;

(a) $p(f,1) = p(H(f),0)$ for all $f \in F$,

(b) $p(f,t) = p(f,0)$ for all $t \in I$, when $f \in \partial F$.

The result is an *open-book decomposition* of M, with *leaves* $F_t = p(F \times \{t\})$, and *binding* $L = \partial F_t$ independent of t. The identifications in (a) and (b) determine (M, F_t) up to homeomorphism.

REMARK. Alteration of H by isotopy fixing ∂F pointwise does not alter M. Conversely, given one leaf F_0, say, we can recover the monodromy $H : F_0 \to F_0$, as an element of $H(F_0)$, from the embedding of F_0 in M.

The binding $L \subset M$ acquires an orientation from the surfaces F_t, and can be regarded as a fibred link in M, for we have a natural fibration $p_L : M - L \to S^1$ defined by $p_L(p(f,t)) = e^{2\pi i t}$, with fibres $\mathrm{int}(F_t)$.

Stallings' work on fibred links shows that given the oriented binding L, any choice of coherently oriented spanning surface F of minimal genus is isotopic to F_0 and so itself forms part of an open-book decomposition. The monodromy for F is equivalent to that for F_0, in the sense that $\varphi \circ H = H_0 \circ \varphi$ for some homeomorphism $\varphi : F \to F_0$.

4.2 Braids.

There are several ways of describing the abstract braid group on n strings, [5]. Here we shall use the fact that B_n is isomorphic to the group $H(D^2, Q)$, where Q is a set of n interior points of D^2, (see 3.1). An element $\beta \in H(D^2, Q)$ can be pictured geometrically by taking a representative $h : (D^2, Q) \to (D^2, Q)$ and an isotopy $\bar{h} : D^2 \times I \to D^2 \times I$ from 1_{D^2} to h fixing ∂D^2. The subset $\bar{h}(Q \times I) \subset D^2 \times I$ then appears as n strings of a geometrical braid. Alteration of $\bar{h}$ simply changes the strings by isotopy, fixing their ends.

For example, when $n = 2$ and $\beta = \tau_a \in H(D^2, Q)$, where a is the arc shown in figure 4.1 then β can be pictured as shown in figure 4.2.

DEFINITION. The *closure*, $\widehat{\beta}$, of a geometric braid β is the image of the strings in S^3 when the top and bottom of $D^2 \times I$ containing the braid are identified. The *axis* L_β is, up to isotopy in $S^3 - \widehat{\beta}$, any of the meridian circles $\partial D \times \{t\}$, (see, for example, [19]).

In the present context an alternative equivalent description which relates directly to 4.1 will be more useful.

DEFINITION. Let $h : (D^2, Q) \to (D^2, Q)$ represent a braid β. Choose a continuous map $p : D^2 \times I \to S^3$ which makes exactly the identifications

(a) $p(f, 1) = p(h(f), 0)$ for all $f \in D^2$,

(b) $p(f, t) = p(f, 0)$ for all $t \in I$, when $f \in \partial D^2$.

Then $p(Q \times I)$ is the *closure*, $\widehat{\beta}$, of β, with *axis* $L_\beta = p(S^1 \times \{0\})$.

REMARK. The link $\widehat{\beta} \cup L_\beta$ depends up to isotopy only on β, and not on the choices of h or p.

As in 4.1, one of the leaves $D = D_0$, say, where $D_t = p(D^2 \times \{t\})$, together with the curve $\widehat{\beta} = p(Q \times I)$, is enough to determine $h : (D, Q) \to (D, Q)$ as an element of $H(D, Q)$ i.e. as a braid.

If $L_\beta \cup \widehat{\beta}$ arises also from some other braid $\beta' \in H(D', Q')$ then β' and β are equivalent in the sense that $\varphi \circ h = h' \circ \varphi$ for some homeomorphism $\varphi : (D, Q) \to (D', Q')$ where h and h' represent β and β'.

4.3 Simple covers of 3-manifolds.

DEFINITION. Let M^3, N^3 be closed 3-manifolds. A map $\pi : M \to N$ is called a *simple d-sheeted cover,* with branch set $C \subset N$ if it is locally homeomorphic to the product of an interval with a simple d-sheeted cover of a disc, and the branch points in the products form the set C.

We can now prove our main theorem, giving much tighter bounds on the degree of the cover than the comparable results in [12]. The bounds obtained here are in fact the best possible in general.

THEOREM A. *Let (M, F_t) be any open-book decomposition of a closed manifold M. Then there is a closed braid $C = \widehat{\beta} \subset S^3$ with axis L, and a d-sheeted simple cover $\pi : M \to S^3$ with branch set C such that $F_t = \pi^{-1}(D_t)$ for each t, where $\{D_t\}$ is a family of discs spanning L.*

The binding ∂F_t of the decomposition is then $\pi^{-1}(L)$. If ∂F_t has k components we can choose π so that $d = \max\{3, k\}$.

COROLLARY 4.3.1. *Every fibred link in S^3 can be constructed as $\pi^{-1}(L)$ for some choice of closed braid C and simple cover π branched over C, by applying theorem A to the open-book decomposition of S^3 having the given link as binding.*

PROOF OF THEOREM A: Let $H : F_0 \to F_0$ be the monodromy of the given decomposition. Choose a d-sheeted simple cover $\pi : F_0 \to D^2$ with $d = \max\{3, k\}$. By theorem B the element represented by H in $H(F_0)$ covers a braid $\beta \in H(D^2, Q)$. We may suppose, without altering the decomposition, that H actually covers a representative h of β. Using these representatives, we have maps $p : F_0 \times I \to M$, $p : D^2 \times I \to S^3$ and $\pi \times \mathrm{id} : F_0 \times I \to D^2 \times I$. The composite $p \circ (\pi \times \mathrm{id}) : F_0 \times I \to S^3$ factors through M to give a continuous map $\pi : M \to S^3$ with $\pi \circ p = p \circ (\pi \times \mathrm{id})$. It is not difficult to check that π is a d-sheeted simple cover with branch set $C = p(Q \times I)$; away from $\partial(D^2 \times I)$ it can be compared directly with $\pi \times \mathrm{id}$, while near the boundary it is standard. This cover clearly has the required properties. □

Theorem A shows that when a braid $\beta \in H(D, Q)$ is covered by a homeomorphism $H : F \to F$ under $\pi : F \to D$ then we can extend π to a cover $\pi : M \to S^3$ branched over the closure of β, where M has an open-book decomposition with monodromy H. Given a simple d-sheeted simple cover $\pi : M \to S^3$ branched over C, if we choose $D \subset S^3$ meeting C transversely in $Q = C \cap D$ then $\pi \mid F \to D$ is a simple d-sheeted cover branched over Q, where $F = \pi^{-1}(D)$.

As a converse to theorem A we make precise a theorem of Alexander, [2].

THEOREM 4.3.2. *(Alexander). Let $\pi : M \to S^3$ be a simple d-sheeted cover with branch set C, and let $D \subset S^3$ present C as the closure of a braid $\beta \in H(D, Q)$, i.e. C can be regarded as a closed braid with axis ∂D. Set $F = \pi^{-1}(D)$. Then β is covered under π by a homeomorphism $H : F \to F$, and M has an open-book decomposition with F as one leaf, monodromy H, and binding $\partial F = \pi^{-1}(\partial D)$.*

REMARK. This theorem can be readily extended by replacing D with a surface S presenting C as a generalised closed braid. The cover itself need not be simple, cf. Goldsmith [9], and it is also possible to have a more general manifold N in place of S^3. Our particular concern, however, will be for the case of the theorem, with $M = S^3$.

Before proving theorem 4.3.2, we establish some more general results about covers of 3-manifolds. As in 2.4.1, a simple cover $\pi : M \to N$ determines a homomorphism $\varphi_\pi : \pi_1(N - C, *) \to S_d$, up to conjugacy in S_d, taking every meridian of C to a transposition. Conversely any such homomorphism determines a cover, and two covers with the same homomorphism up to conjugacy are equivalent.

When $N = S^3$ it is natural to prescribe a cover from a diagram of C by giving φ_π in terms of a Wirtinger presentation of $\pi_1(S^3 - C)$, so that each overpass is labelled by a transposition, compatibly arranged at the crossing.

An explicit construction of a cover can be visualised by use of a 2-dimensional splitting complex, whose 2-cells are labelled by transpositions, prescribing a way to reassemble d copies of the split manifold N. For details of such constructions for general covers branched over C see **[21]**, **[17]**.

LEMMA 4.3.3. *Suppose that $\pi : F \to D$ is a cover with branch set Q, and $\beta \in H(D,Q)$. Choose $* \in \partial D$ and number the points $\pi^{-1}(*)$ to determine $\varphi_\pi : \pi_1(D - Q, *) \to S_d$. Then β induces an isomorphism*

$$\beta_* : \pi_1(D - Q, *) \to \pi_1(D - Q, *),$$

and β is covered by some $H : F \to F$ fixing ∂F if and only if $\varphi_\pi \circ \beta_* = \varphi_\pi$.

PROOF: The map $\pi' = \pi \circ h : F \to D$ is also a cover with branch set Q, where h represents β, and $\varphi_{\pi'} = \varphi_\pi \circ \beta_*$. If $\varphi_{\pi'} = \varphi_\pi$ then the covers are isomorphic and we have $H : F \to F$ with $\pi \circ H = \pi'$ as required.

Conversely if the covers are isomorphic and H fixes ∂F then $\varphi_\pi = \varphi_{\pi'}$. □

REMARK. A given covering $\pi : F \to D$ then determines a 'lifting' subgroup of the braid group $H(D,Q) = \{\beta : \varphi_\pi \circ \beta_* = \varphi_\pi\}$, This subgroup maps homomorphically to $H(F)$, taking β to the element represented by H. An interesting question is to identify the kernel of this homomorphism. Theorem B shows that the image, $H_\pi(F)$, is the whole group for all simple covers of degree ≥ 3

PROOF OF THEOREM 4.3.2: Choose a basepoint $*$ on ∂D and a numbering of $\pi^{-1}(*)$. The map $\varphi_{\pi|_F}$ for $F \to D$ is given from φ_π by $\varphi_{\pi|_F} = \varphi_\pi \circ i_*$, where $i_* : \pi_1(D - Q, *) \to \pi_1(S^3 - C, *)$ is induced by inclusion. Now $\pi_1(S^3 - C, *)$ is generated by the image of $\pi_1(D - Q, *)$, with the relations $i_*\beta_*(g) = i_*(g)$ for all g in $\pi_1(D - Q, *)$. Hence $\varphi_\pi(i_*\beta_*(g)) = \varphi_\pi(i_*(g))$ for all g, so $\varphi_{\pi|_F} \circ \beta_* = \varphi_{\pi|_F}$. Thus β is covered under π by some $H : F \to F$.

The relative mapping torus of H then covers S^3 with branch set C, and defines the same homomorphism φ_π, since this is determined by its effect on $\pi_1(D - Q, *)$, which has been fixed by $\varphi_{\pi|_F}$. Then this mapping torus gives a cover equivalent to M. □

4.4 Braids and surface covers.

At this stage we reformulate the equivalence theorem 2.2.1 in terms of braids. We give an easy corollary which we shall use in section 6.

Take $Q = \{q_1, \ldots, q_n\} \subset D^2$, and choose $*$ on ∂D^2. Write $\pi(\boldsymbol{\tau}) : F(\boldsymbol{\tau}) \to D^2$ for the connected cover given by the Hurwitz sequence $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$, where $\tau_i \in S_d$ generates S_d. The sheets of the cover are numbered by making a choice of numbering on $(\pi(\boldsymbol{\tau}))^{-1}(*)$. In the notation of section 3 we have $\tau_i = \varphi_{\pi(\boldsymbol{\tau})}(g_i)$, where $g_i \in \pi_1(D^2 - Q, *)$ is represented by a loop which meets the arc a_i once, and meets no other splitting arc. The braid group $H(D^2, Q)$ operates on Hurwitz sequences, by defining $\beta(\boldsymbol{\tau}) = \boldsymbol{\tau}'$, where $\tau_i' = \varphi_{\pi(\boldsymbol{\tau})}\beta_*(g_i)$. Then there is an equivalence $H(\beta) : F(\boldsymbol{\tau}) \to F(\boldsymbol{\tau}')$ with $\pi(\boldsymbol{\tau}') \circ H(\beta) = h \circ \pi(\boldsymbol{\tau})$, where h represents β, which preserves the numbering of the sheets above $*$.

REMARK. By 2.3.1 if $\boldsymbol{\tau}$ and $\boldsymbol{\tau}' = g^{-1}\boldsymbol{\tau}g$ are *conjugate* Hurwitz sequences, i.e. there exists g with $g^{-1}\tau_i g = \tau_i'$ for all i, then there is $H(g) : F(\boldsymbol{\tau}) \to F(\boldsymbol{\tau}')$ with $\pi(\boldsymbol{\tau}') \circ H(g) = \pi(\boldsymbol{\tau})$ which permutes the inverse image of $*$ by g.

THEOREM 4.4.1. (Braid version of the equivalence theorem)

$H(D^2, Q)$ acts *transitively* on Hurwitz sequences with connected cover having fixed product $\tau_1\tau_2\ldots\tau_n$.

PROOF: See **[14]**, or **[4]**, cf. also **[18, IV]**. □

PROOF OF THEOREM 2.2.1: Take $\pi = \pi(\boldsymbol{\tau})$ and $\pi' = \pi(\boldsymbol{\tau}')$, where $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$ are Hurwitz sequences whose products have the same cycle type. Their products are then conjugate by some $g \in S_d$. Then $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'' = g^{-1}\boldsymbol{\tau}'g$ have the same product, so by 4.4.1 we have $\beta \in H(D^2, Q)$ and $H(\beta) : F(\boldsymbol{\tau}) \to F(\boldsymbol{\tau}'')$ covering β which does not permute the sheets over $*$. We also have an isomorphism $H(g) : F(\boldsymbol{\tau}'') \to F(\boldsymbol{\tau}')$, so that $H(g) \circ H(\beta) : F(\boldsymbol{\tau}) \to F(\boldsymbol{\tau}')$ is an equivalence between $\pi(\boldsymbol{\tau})$ and $\pi(\boldsymbol{\tau}')$ which covers β and permutes the sheets over $*$ by g. □

COROLLARY 4.4.2. Let $\pi : F \to D^2$ be a simple cover with the same degree on two boundary components C_i and C_j. Then there is an equivalence $H : F \to F$ of π with itself which interchanges C_i and C_j, and fixes the other components of ∂F.

PROOF: Use the proof of 2.2.1 with $\boldsymbol{\tau} = \boldsymbol{\tau}'$. Then the product $\tau_1\tau_2\ldots\tau_n$ has two cycles of the same length, corresponding to the two boundary components C_i and C_j. Take g to interchange these cycles and fix everything else. Then take $H = H(g) \circ H(\beta)$ as above, to give an equivalence of $\pi(\boldsymbol{\tau})$ with itself as required. □

5. Markov moves and Hopf plumbings.

We now look at the case where $C \subset S^3$ and $\pi : M \to S^3$ branched over C are fixed, and we consider the changes in the open-book decomposition of M which arise as we alter the choice of L in relation to C, always assuming that C is a closed braid with axis L.

If we select a disc fibre D spanning L, with $Q = D \cap C$ this determines $\beta \in H(D,Q)$ with C as its closure. Any other β' arising from D' spanning another axis L' is related to β by a sequence of Markov moves, in a sense to be described. The fibres $F = \pi^{-1}(D)$ and $F' = \pi^{-1}(D')$ of the resulting open-book decomposition of M are then shown to be related by a corresponding sequence of Hopf plumbings.

5.1 Markov moves.

CONSTRUCTION 5.1.1. Let $\beta \in H(D,Q)$ be a braid and let a' be an arc in D with one end on Q, the other on ∂D. Extend D to a disc $D' = D \cup D_1$, where $D \cap D_1$ is an arc of ∂D containing the end point of a'. Take $Q' = Q \cup \{q_{n+1}\}$, where $q_{n+1} \in D_1$, and extend a' to an arc a in D' with endpoint q_{n+1}, as illustrated in figure 5.1

FIGURE 5.1

Then the braids $i(\beta)\tau_a^{\pm 1} \in H(D',Q')$ are said to be given from β by a *Markov move on a* (with sign ± 1).

REMARKS. When $Q = \{q_1, \ldots, q_n\}$ is in *standard position*, i.e. the points lie in order on a fixed diameter of D, then $H(D,Q) \xrightarrow{i} H(D',Q')$ is regarded as the standard inclusion of B_n in B_{n+1}, and β is usually written in place of $i(\beta)$. Where a' is the arc along the diameter from q_n to ∂D then $\tau_a = \sigma_n$, the generator of $B_{n+1} \cong H(D',Q')$ which interchanges q_n and q_{n+1}.

Since there is a homeomorphism $g : (D,Q) \to (D,Q)$ carrying any other a' to this arc, it follows that in general $\tau_a = \gamma^{-1}\sigma_n\gamma$, where $\gamma \in H(D,Q) \cong B_n$ is represented by g. In this context the Markov move given above replaces $\beta \in B_n$ by $\beta\gamma^{-1}\sigma_n^{\pm 1}\gamma \in B_{n+1}$.

It is well known that braids related by Markov moves close to isotopic links. Conversely, if the closures of two braids are isotopic then we can pass from one braid to the other, up to equivalence, by a sequence of Markov moves and their inverses, see e.g. [5], [20].

5.2 Hopf plumbing.

DEFINITION. Let $F \subset M^3$ be a surface with boundary, and $b \subset F$ a proper arc. Then $F' \subset M^3$ is given from F by plumbing a Hopf band along b if F' differs from F only in a neighbourhood B of b in M^3 by adding a band R with one full twist (in either sense) as shown in figure 5.2. This is a special case of Murasugi sum of embedded surfaces **[22]**.

FIGURE 5.2

It is well known, **[22]**, that F' is a fibre surface if and only if F is a fibre surface, and in this case F' depends up to isotopy only on F and the choice of b up to isotopy within F.

We say also that F arises from F' by *deplumbing* a Hopf band.

HARER'S CONJECTURE. It is an open conjecture of Harer that plumbing and deplumbing Hopf bands, starting from a disc, is sufficient to generate all fibre surfaces in S^3, **[10]**.

REMARK. Plumbing alone (without deplumbing) is not sufficient, **[16]**.

5.3 Their interaction.

We now give details of the connection between Markov moves and Hopf plumbings.

THEOREM 5.3.1. *Let $C \subset S^3$ and let $\pi : M \to S^3$ be a simple cover branched over C. Let C be presented as a closed braid with axis L, with a particular disc fibre D meeting C in Q determining a braid $\beta \in H(D, Q)$. Let a' be an arc in D joining a point of Q to ∂D, and let $D' = D \cup D_1$ be a disc meeting a ball neighbourhood B of a' as in figure 5.3. Take a to be the arc in D' which extends a' as in construction 5.1.1, ending at the point $D_1 \cap C$.*

Then

(a) C is presented by D' as the closure of $\beta' \in H(D', Q')$ where $\beta' = i(\beta)\tau_a$, given from β by a Markov move on a, and

(b) the fibre surface $F' = \pi^{-1}(D')$ is given up to isotopy from $F = \pi^{-1}(D)$ by plumbing a positive Hopf band to F along the proper arc b which covers a'.

FIGURE 5.3

PROOF:

(a) Redrawing the figure inside the ball B as in figure 5.4 shows that C can be positioned as the closure of β' relative to $\partial D'$.

FIGURE 5.4

(b) F and F' only differ inside $\pi^{-1}(B)$, which consists of $d-2$ balls covering B homeomorphically, and one ball, containing the arc b of F, which doubly covers B. In the balls which cover trivially, F and F' differ only by isotopy. The double cover, illustrated in figure 5.5, provides the Hopf band as shown, plumbed along b.

FIGURE 5.5

□

REMARK. A similar figure to 5.5 allows a negative Markov move on β with a corresponding negative Hopf band plumbed to F.

As a corollary, we have

THEOREM C. *The fibres for any two fibred links in M which arise from closed braid presentations of C using a fixed simple cover $\pi : M \to S^3$ branched over C are related by a sequence of plumbing and deplumbing of Hopf bands, and isotopies.*

PROOF: Any two closed braid presentations for C are related by a finite sequence of Markov moves and their inverses, (see section 5.1). □

EXAMPLE 1. For the simple d-sheeted cover of D^2 with Hurwitz sequence $\tau_1, \ldots, \tau_{d-1}$ having $\tau_i = (i\ i+1)$, the covering surface F is homeomorphic to D^2, as shown in figure 5.6.

FIGURE 5.6

The homeomorphism 1_F covers 1_{D^2}, representing the identity braid on $d-1$ strings in $H(D^2, Q)$. Then if we take the d-sheeted cover of S^3 branched over the closure of the identity braid as determined on D^2 by the given sequence, we get an open-book decomposition of the covering manifold M with fibre $F \cong D^2$ and monodromy 1_F. Thus $M \cong S^3$ and we have a d-sheeted simple cover $\pi : S^3 \to S^3$ branched over the trivial link C of $d-1$ components.

By theorem C, any representation of this trivial link as a closed braid with axis L will give, using the same cover π, a fibred link $\pi^{-1}(L)$ in S^3, whose fibre is given from a disc in S^3 by plumbing and deplumbing Hopf bands.

REMARK. We know (Theorem A) that every fibred link in S^3 arises from a simple cover of S^3 over *some* closed braid. If we could show that this closed braid can always be chosen to be the unlink then Harer's conjecture, 5.2, would follow. There is, for example, a lot of freedom in the choice of braid to produce a given monodromy when the map $\pi : F \to D^2$ is given, and further choice of π itself is available by using more sheets.

EXAMPLE 2. As a special case of example 1 we may consider a braid built from the trivial braid on $d-1$ strings by a sequence of Markov moves, each increasing the string index. Call such a braid, or any conjugate, *completely reducible*. Certainly its closure will be the $d-1$ string unlink. Using the d-sheeted cover of S^3 defined in example 1 we get, from each completely reducible braid, a fibred link in S^3 constructed from the disc by plumbing Hopf bands.

In the final section we consider the case where F is a fibre surface which has arisen from some $\beta \in H(D,Q)$ and some cover $\pi : F \to D$. We find conditions on proper arcs b in F, under which plumbing a Hopf band along b can be realised by doing a Markov move on β. In particular, we show that the converse to the construction in example 2 holds, namely:

THEOREM D. *Every fibre surface in S^3 which is given from a disc by a sequence of Hopf plumbings arises, for some d, from the d-sheeted cover of S^3 branched over a completely reducible braid, closing to the unlink on $d-1$ strings.*

REMARK. There is no obvious bound on d in terms of the number of boundary components of the fibre surface, but it is certainly bounded in terms of the number of bands used in the plumbing. We finish with a look at the construction in this way of fibred knots of genus 2, showing that we can always take $d \leq 3$, with $d=2$ in many cases.

6. Realisation of Hopf plunbings.

We suppose that a fibre surface $F \subset M$ has been given, and that we have found a braid $\beta \in H(D,Q)$ and a simple covering $\pi : M \to S^3$ branched over the closed braid C with $F = \pi^{-1}(D)$, and as usual $Q = D \cap C$. We shall give conditions on a proper arc b in F which ensure that we can plumb a Hopf band to F along b by altering β by a Markov move. We shall show that if the conditions do not hold then a simple alteration of β and π (adding a trivial sheet to the cover) can be made so that the conditions are satisfied for the new β and π.

By theorem 4.3.1 we can plumb a Hopf band along b by a Markov move on β if b, after isotopy in F, covers an arc in D under $\pi : F \to D$. Let F have boundary components $C_1, \ldots, C_k$. We look first at the case of arcs joining two different boundary components.

THEOREM 6.1.1. *If $F \subset M$ is a fibre surface constructed from $\beta \in H(D,Q)$ and a covering $\pi : M \to S^3$ of degree ≥ 3, and b is a proper arc in F joining two components of ∂F then we can plumb a Hopf band to F along b by a Markov move on β.*

PROOF: By repeated use of lemma 3.3.1 we can find a surface $F' \subset F$, where F is the union of F' with discs each meeting $\partial F'$ in a single arc, and $\pi \mid F'$ has degree $d = \max\{3, k\}$. The arc b can be isotoped within F to an arc b' lying entirely in F'. It is enough to show that b' covers an arc under $\pi' = \pi \mid F'$. We may then assume without loss that π is the cover of degree d. It is enough to show, for each pair of components C_i and C_j, that one arc joining them covers, because there is a homeomorphism of F fixing ∂F which carries any one such arc to any other (up to isotopy of the arc), and all elements of $H(F)$ are π-covers, by theorem B.

When $k = 2$ (and $d = 3$ for the minimal degree cover) it is easy to find an arc in D whose cover joins the two components; a suitably chosen arc from a Hurwitz family will do.

When $k > 2$, *any* of the Hurwitz arcs will be covered by an arc joining some pair of boundary components, since the minimal degree cover then has degree 1 on each component, (the ends of the covering arc project to the same point on ∂D^2). It only remains to show that *every* pair of components can be so joined. This is now an immediate consequence of 4.4.2. □

REMARK. For $F_{g,2}$ we have shown that every arc joining the components will cover, when the cover π has degree ≥ 3. This can be shown also, when $g = 1$ for the cover π of degree 2, because $H_\pi(F)$ then contains enough homeomorphisms to put such an arc in standard form up to isotopy, noting that we need not fix the boundary pointwise. This is not guaranteed for the degree 2 cover when $g > 1$.

We now look at the case of a proper arc b with both ends in the same boundary component C_1 of a surface F, with a simple cover $\pi : F \to D^2$ of degree $d \geq 3$. For b to cover it is essential that $\pi|_{C_1}$ have degree $d_1 \geq 2$, since the ends of b, after isotopy, must have a common image.

THEOREM 6.1.2. *Let $\pi : F \to D^2$ be a simple cover of degree $d \geq 3$, and let F have boundary components $C_1, \ldots, C_k$. Let b be a proper arc in F with both ends in C_1 and let $d_1 =$* degree $\pi|_{C_1}$. *Then b covers an arc in D^2 so long as*

(1) b does not separate F, and $d_1 \geq 2$, or

(2) b separates $F \cong F_{g,k}$ into two pieces, of genus g_1, g_2 with $g_1 + g_2 = g$, containing respectively k_1, k_2 of the original boundary components $C_2, \ldots, C_k$, and

(a) $d_1 \geq 2$ when $k_1, k_2 \geq 1$,

(b) $d_1 \geq 3$ when $k_1 \geq 1, k_2 = 0$,

(c) $d_1 \geq 4$ in the case $k = 1$, (i.e. $k_1 = k_2 = 0$).

In cases 2 (b) and (c) it is enough to have $d_1 \geq 2, 3$ respectively if the arc b simply cuts off a disc.

PROOF: As in 6.1.1 it is enough to prove where π has minimal degree subject to the conditions on d_1. Thus we can assume that $d_j = 1$ for $j > 1$, and $d_1 = 2, 3$ or 4 as indicated. Any two such minimal degree covers are equivalent, so it is enough to look at one such cover.

If we can prove that one arc b covers then so does the image of b under every element of $H(F)$, using our covering theorem for $H(F)$. Thus it is enough to exhibit just one non-separating covering arc b for one minimal degree cover π to prove (1), and one covering arc b which separates F in the way specified for each case of (2). Since, by 4.4.2, there is a covering homeomorphism $H : F \to F$ realising any permutation of $C_2, \ldots, C_k$, we must simply exhibit a minimal degree cover π and a covering arc b which splits F into F_{g_1,k_1+1} and F_{g_2,k_2+1} for each choice of g_1, g_2, k_1, k_2, in order to prove (2).

CONSTRUCTION FOR (1). Suppose that $F \cong F_{g,k}$. Let $\pi' : F' \to D'$ be a cover of degree 1 on each boundary component of $F' \cong F_{g-1,k+1}$.

FIGURE 6.1

Construct a cover of $D = D' \cup D_1$ as shown in figure 6.1 using π' to cover D' and adjoining one extra branch point in D_1 with permutation (1 2) say. Then the surface covering D is homeomorphic to F' with an extra band joining sheets 1 and 2, and hence is homeomorphic to F. The cover has degree 2 on one component of the boundary, and the arc a is covered by a proper arc b which crosses the added band, and does not separate F.

REMARK. It is easy to give an explicit Hurwitz sequence for π, but the essential information is its relation to π'.

CONSTRUCTION FOR (2). We shall build a cover of D^2 by F from covers by $F' \cong F_{g_1,k_1+1}$ and $F'' \cong F_{g_2,k_2+1}$, of degrees d', d''. The exact relation of d', d'' to k_1, k_2 depends on the subcases (a), (b) and (c), but the basic construction is the same in each case.

FIGURE 6.2

As shown in figure 6.2 we divide D^2 into three pieces D', D_1, D''. Use the cover $\pi' : F' \to D'$ on sheets $1, \ldots, d'$, and $\pi'' : F'' \to D''$ on sheets $d'+1, \ldots, d = d' + d''$, with the cover on D_1 prescribed by having one branch point labelled with the permutation $(d' \ d'+1)$. The arc a in D_1 is covered by a proper arc across the band which joins sheets d' to $d'+1$, and consequently the covering surface is separated by b into the pieces required.

CASE 2(a). We can find covers π', π'' with degrees $d' = k_1 + 1$, $d'' = k_2 + 1$, so long as $k_1, k_2 \geq 1$, for each choice of g_1, g_2. In this case we then have a cover of D^2 by F of degree $k+1$, since $k = k_1 + k_2 + 1$. This cover has degree 1 on each boundary component except for the one arising from sheets d' and $d'+1$, where the degree is 2. The construction gives a cover of the degree stated, and a covering arc b which separates as required.

CASE 2(b). Where $k_2 = 0$, but $k_1 \neq 0$ we must take $d'' = 2$, unless $g_2 = 0$ also, but we can still take $d' = k_1 + 1$. The total degree d is then $k+2$, having degree 3 on one component and again degree 1 on those not met by the separating arc b. (When $g_2 = 0$ we may take $d'' = 1$).

CASE 2(c). When $k_1 = k_2 = 0$ we take $d' = d'' = 2$, giving a total degree of 4 on the single boundary component, again with a reduction if $g_2 = 0$.

REMARK. The hypothesis that $d \geq 3$, imposed to ensure that all elements of $H(F)$ cover, and hence that other arcs besides the chosen b will cover, is automatically forced except in case (1) when $F \cong F_{g,1}$. Here there is a cover of degree 2 and our construction exhibits a non-separating arc in F which covers. When $g = 1$ we know that all elements of $H(F)$ cover, so all non-separating arcs will cover, but when $g > 1$, there will be non-separating arcs which do *not* cover under the double cover $\pi : F_{g,1} \to D^2$.

As a corollary we get our main plumbing theorem.

THEOREM 6.1.3. *Let $F \subset M^3$ be a fibre surface, presented as $F = \pi^{-1}(D)$ for a simple cover $\pi : M \to S^3$ in which the branch set C represents the closure of a braid $\beta \in H(D,Q)$. Let b be a proper arc in F. Then the fibre surface given by plumbing a Hopf band along β arises by a Markov move on b provided that if both ends of b lie in the same component of ∂F then the degree of π is at least 2 (3 or 4) on this component, as determined by 6.1.2.*

PROOF: The arc b covers, by 6.1.1 or 6.1.2. □

Finally, if F, β are as in theorem 6.1.3, but the degree of π on a component of ∂F is not large enough to satisfy 6.1.2 for a given arc b, it is easy to alter β and π so as to increase this degree as follows. Extend the cover $\pi : F \to D$ to $\pi^* : F^* \to D^*$ by adding a trivial sheet, as in 3.3, to increase the degree on a chosen boundary component of F. The branch set Q^* for π^* then consists of the branch set Q for π, together with one further point.

Suppose that $F \subset M^3$ is a fibre surface with monodromy H, covering $\beta \in H(D,Q)$. Now $i(H(F) = H(F^*)$ and $i[H] \in H(F^*)$ covers $i(\beta) \in H(D^*,Q^*)$. Construct the relative mapping torus M^* for a homeomorphism $H^* : F^* \to F^*$ which covers $i(\beta)$. There is then a covering $\pi^* : M^* \to S^3$ extending π^* on F^*, whose branch set is the closure of the braid $i(\beta)$. This is the closure of the braid $i(\beta)$ together with an extra trivial string.

There is a homeomorphism $j : F \to F^*$, which is the identity outside a neighbourhood of ∂F. Then $[j \circ H \circ j^{-1}] = i[H] = [H^*] \in H(F^*)$. The mapping torus M^* is then equivalent, as an open book, to the manifold M^3 presented as the mapping torus of $H : F \to F$, (section 4.1). The fibre surface $F \subset M^3$ may then be replaced, up to homeomorphism, by $F^* \subset M^*$, while increasing the degree of the cover on the chosen boundary component, and altering the braid β to $i(\beta)$.

Corollary 6.1.4. *Let $F \subset M^3$ be a fibre surface, presented as $F = \pi^{-1}(D)$ for a simple cover $\pi : M^3 \to S^3$ branched over the closure of a braid $\beta \in H(D,Q)$. Then the fibre surface given by plumbing a Hopf band to F arises from the closure of β' which is given from β by a Markov move, possibly after addition of some trivial strings to β.*

□

We have the most satisfying consequence in the shape of

Theorem D. *Every fibre surface F in S^3 which is given from a disc by a sequence of Hopf plumbings arises for some d from the d-sheeted simple cover of S^3 branched over a completely reducible braid, closing to the unlink on $d-1$ strings.*

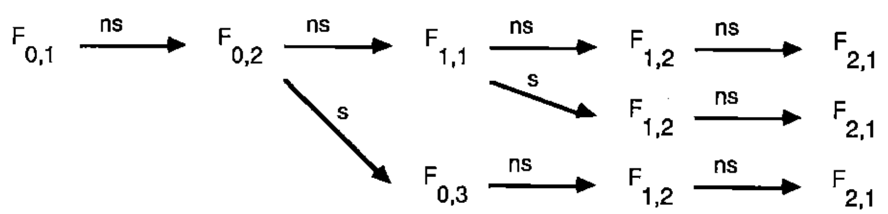
Proof: By induction on the number of bands, using corollary 6.1.4. For if β is completely reducible to the unlink on k strings, say, then β' is completely reducible to the unlink on $k+s$ strings, and the degree of the cover has been increased by s. The induction starts with 0 bands, using the case $F = D^2$ presented as a d-sheeted cover by $\pi : F \to D^2$ with $d-1$ branch points, corresponding to the identity braid on $d-1$ strings, as in section 5, example 1. Here we can even take $d = 2$, although at some stage extra strings may need to be added as the bands are put on, to ensure that the conditions of 6.1.3 are met. □

Remark. It is known that theorem D does not hold in general with $d = 2$, for the fibred knots constructed in this way have special features of their Alexander polynomial which exclude, for example, the connected sum of two trefoils, **[16]**.

6.2 Examples.

We conclude with the examples of genus 2 fibred knots which can be constructed by plumbing Hopf bands, analysing the maximum degree d needed in this case.

Starting from the disc $F_{0,1}$ we must plumb on four Hopf bands to reach $F_{2,1}$. The sequence of intermediate surfaces will depend on the nature of the arcs used at each stage. The diagram below summarises the possible ways, where s and ns are used to indicate whether the arc separates or not.



In the cases listed the only type of separating arc available will cut off a disc, because of the low genus. The eventual need to have only one boundary component ensures that at some steps we may only choose an arc which connects different boundary components.

Start with $F_{0,1}$ represented by the identity braid on 1 string with $d = 2$. Then without increasing d we can plumb on one Hopf band by a Markov move to get $F_{0,2}$ with braid $\sigma_1^{\pm 1}$. The non-separating arc covers when $d = 2$, so we can use another Markov move to get $F_{1,1}$ with braid $\sigma_1^{\pm 1}\sigma_2^{\pm 1}$. (This realises the figure eight or trefoil knot, depending on signs). The separating arc at this stage requires $d = 3$ to cover, so we must pass to $\sigma_1^{\pm 1}$ in B_3 before doing a Markov move. The result, up to conjugacy, is $\sigma_1^{\pm 1}\sigma_3^{\pm 1}$ giving the connected sum of two Hopf links with fibre $F_{0,3}$.

The bottom line of the table can be completed without increase of degree (6.1.2) by two further Markov moves, giving a braid in B_6 which closes to the 2-string unlink.

Returning to the surface $F_{1,1}$, we must increase to $d = 3$ if we want to use the separating arc (6.1.3), and again we can complete with one further Markov move, without further increase of d.

To follow the top line we can pass to $F_{1,2}$ without increase of d, since all non-separating arcs in $F_{1,1}$ cover when $d = 2$. We can continue with degree 2 to $F_{2,1}$ since the arcs joining the two components in $F_{1,2}$ also cover when $d = 2$. The resulting knots are presented by braids in B_5 which close to the unknot.

REMARK. Higher genus knots constructed by plumbing Hopf bands may need a degree d cover where d is roughly the number of bands used, because up to half the arcs used may be separating, of type 2(b) from 6.1.3, being attached in each case to a component on which the degree is 1. The remaining arcs will have to join different components, so they can be added without increasing d. We do not have any explicit examples constructed in this way for which an alternative construction of lower degree can be shown not to exist, so this bound may be rather generous.

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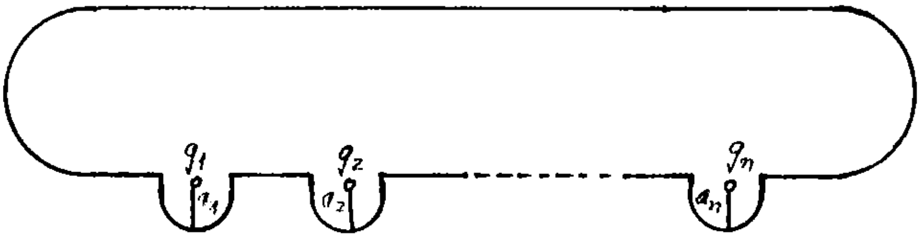
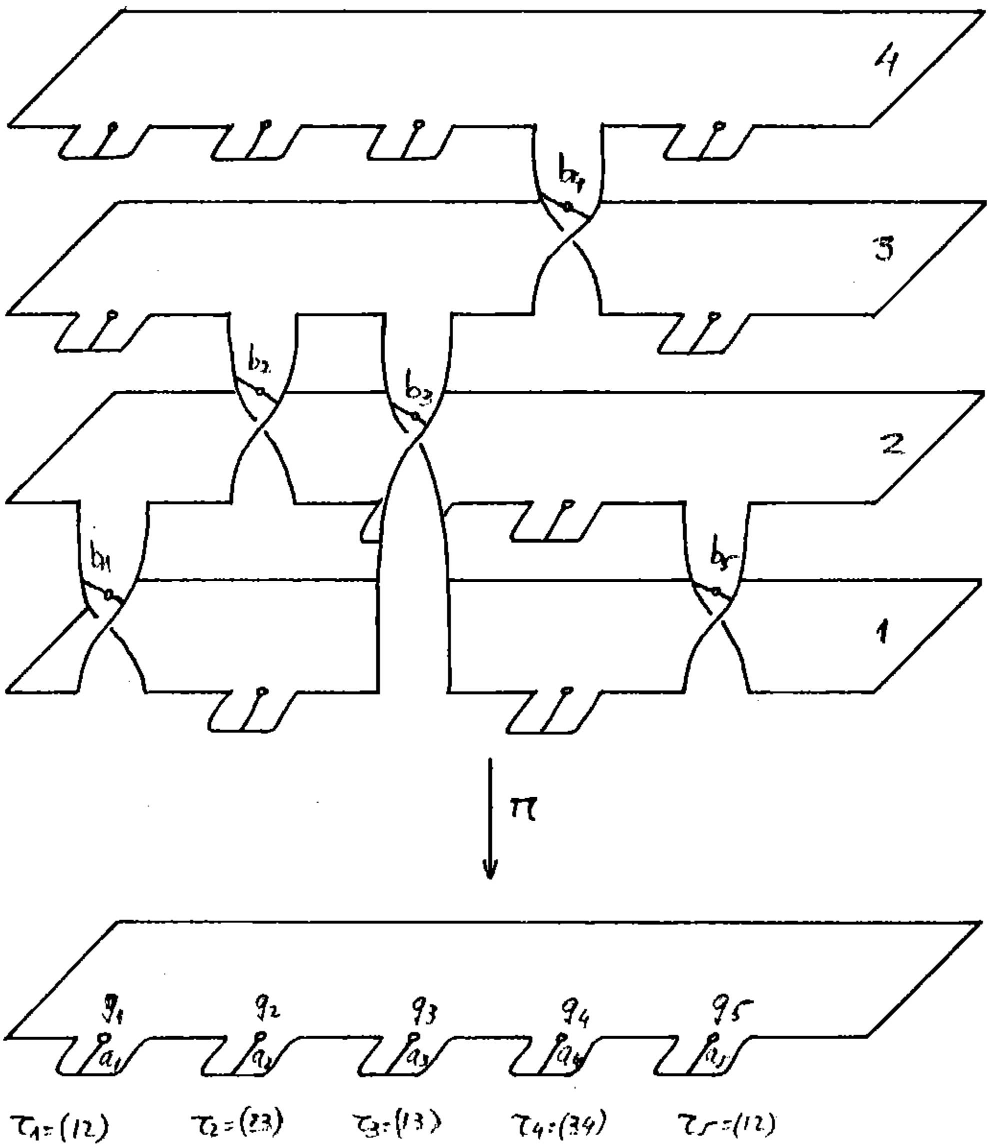

g_1
g_2
g_n
a_1
a_2
a_n

2.1


4
3
2
1
b_1
b_2
b_3
b_4
b_5
π
g_1
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g_3
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g_5
a_1
a_2
a_3
a_4
a_5
$\tau_1=(12)$
$\tau_2=(23)$
$\tau_3=(13)$
$\tau_4=(34)$
$\tau_5=(12)$

2.2

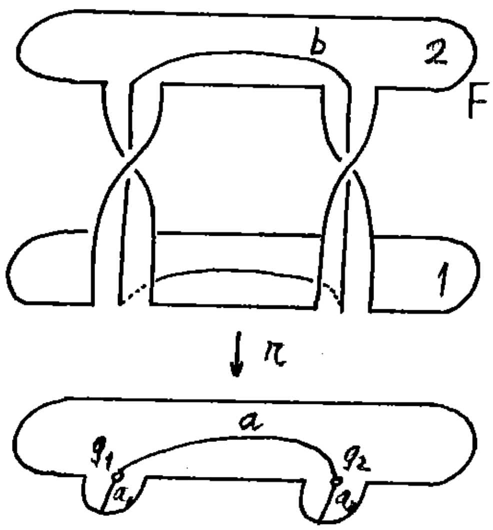
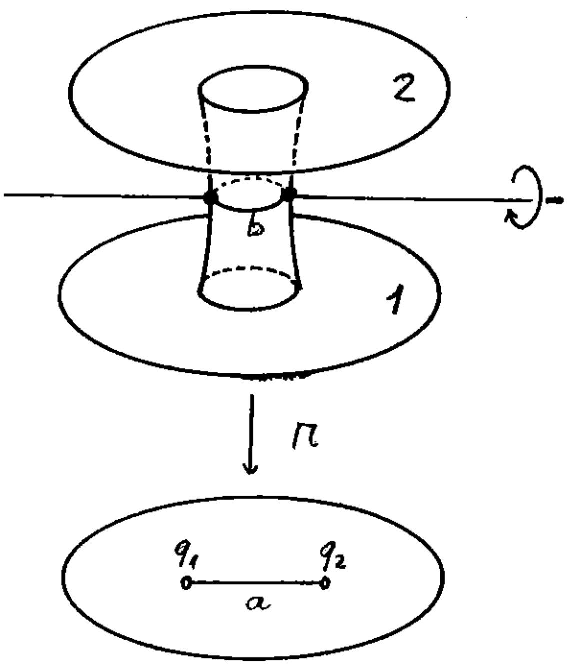


Fig 2.3



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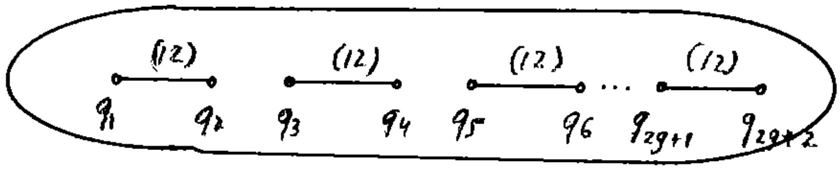

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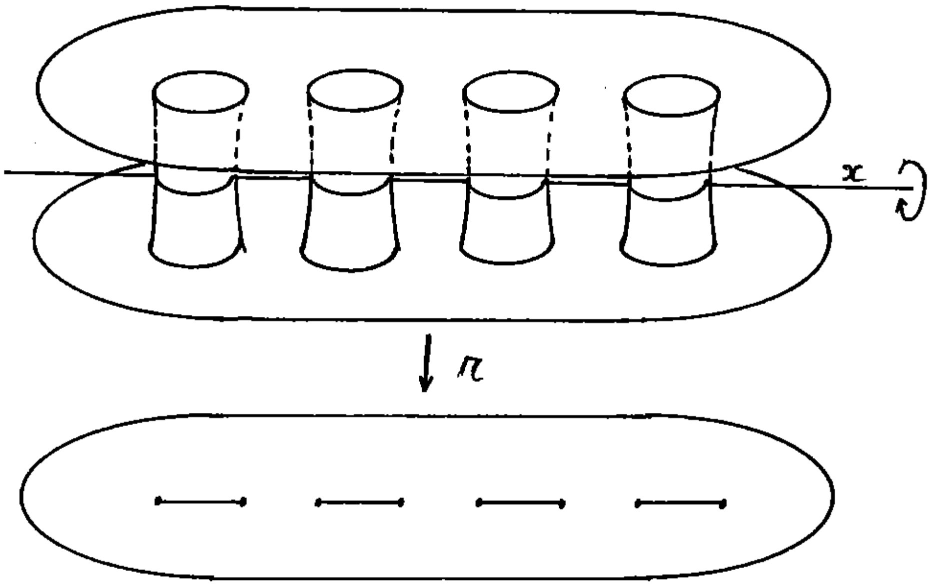
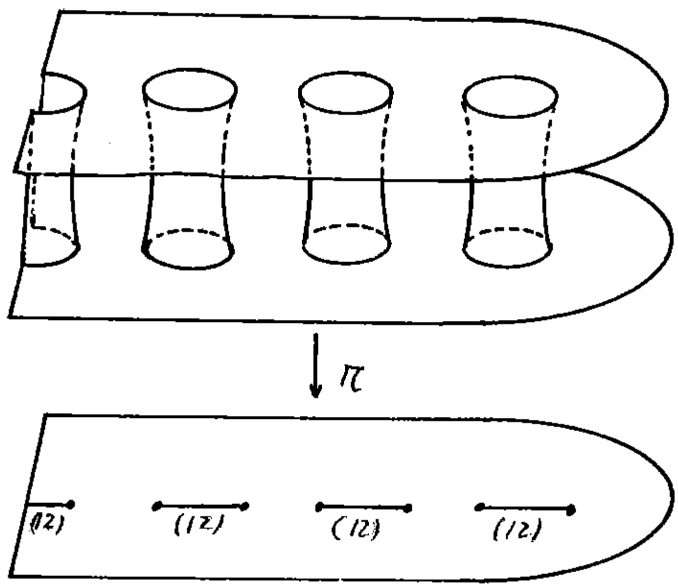


Fig 2-6



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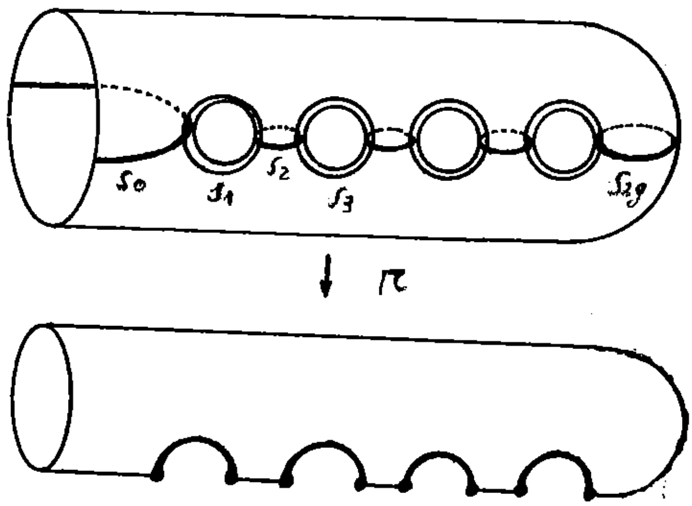
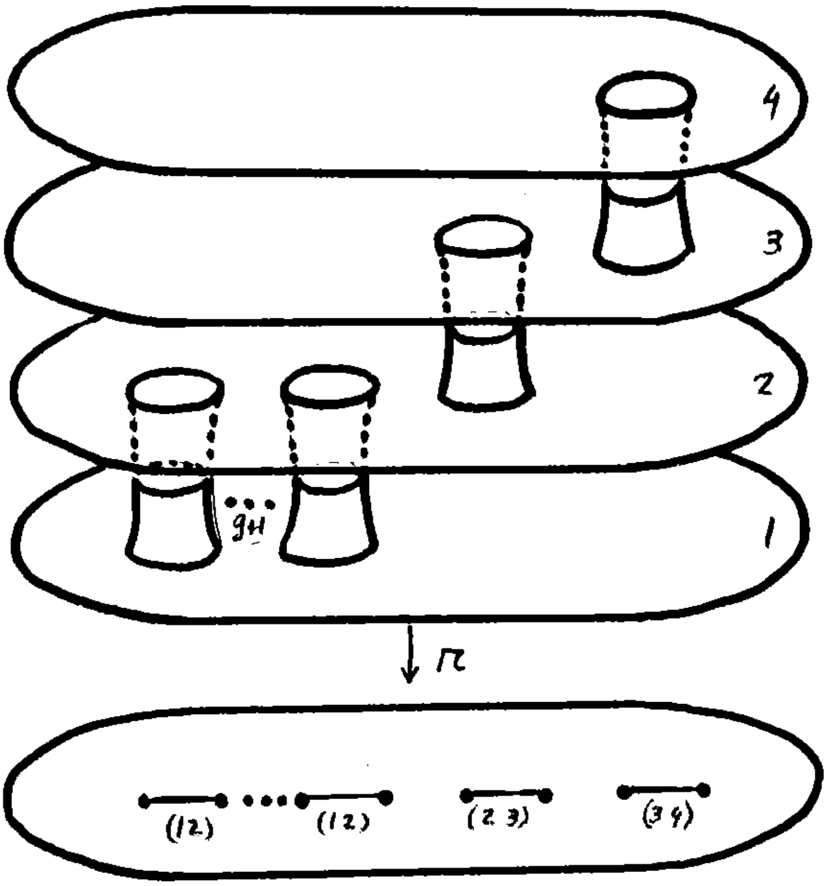


Fig 28



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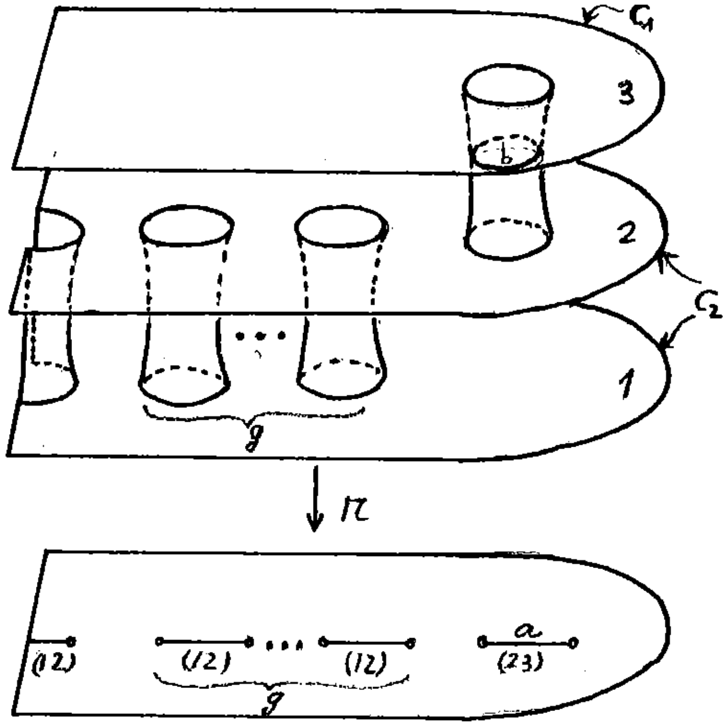
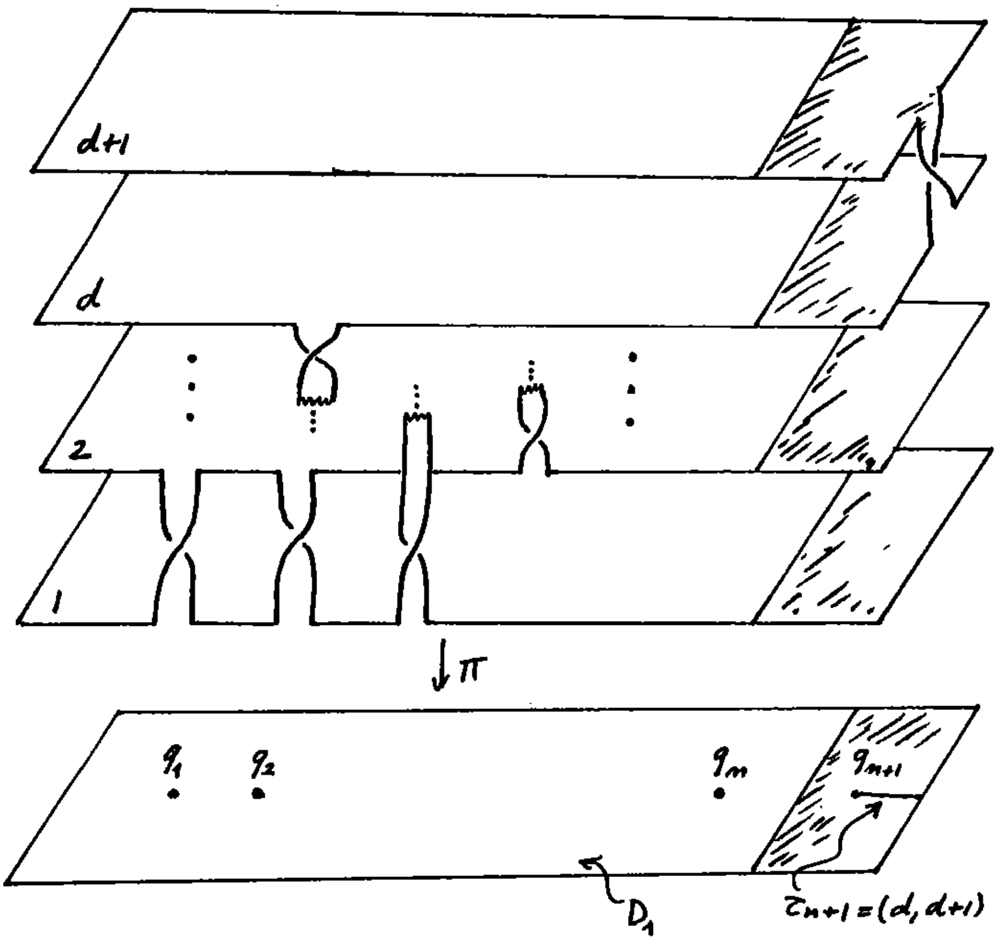


Fig 3.2



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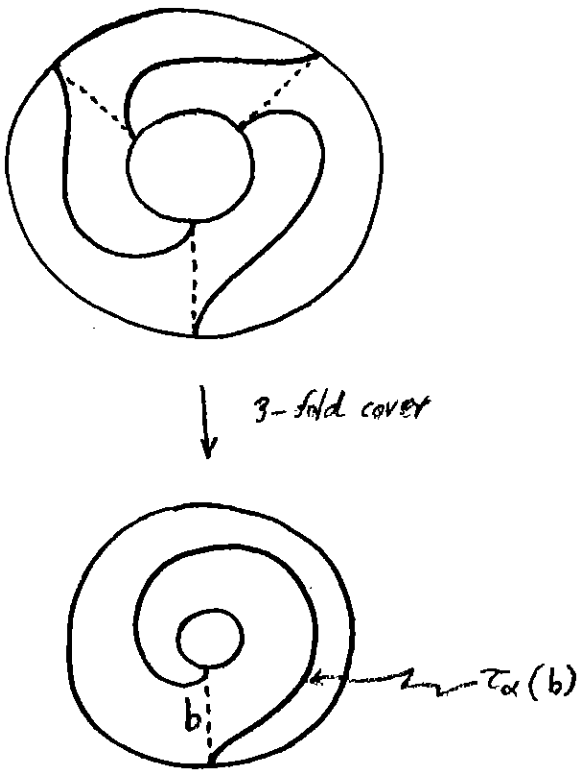

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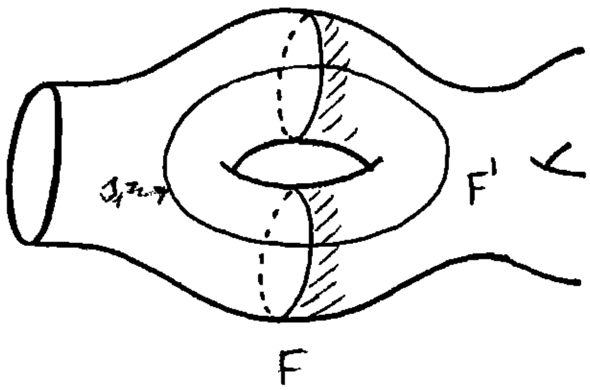

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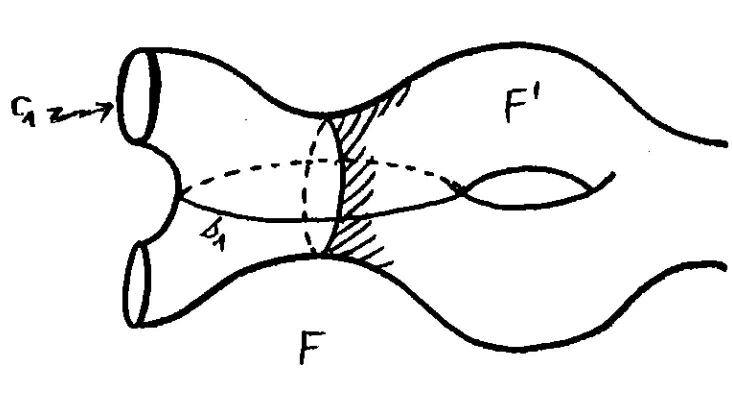

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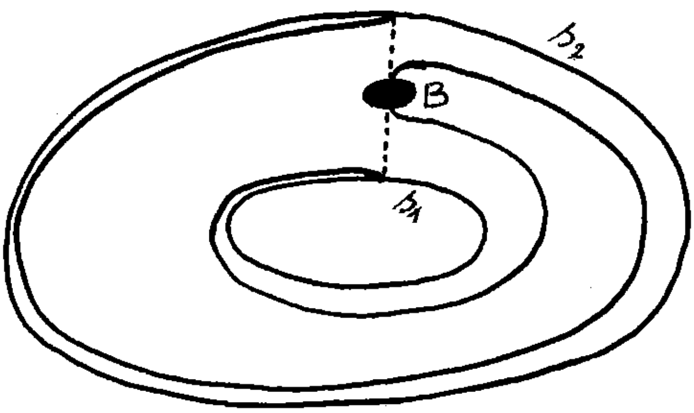

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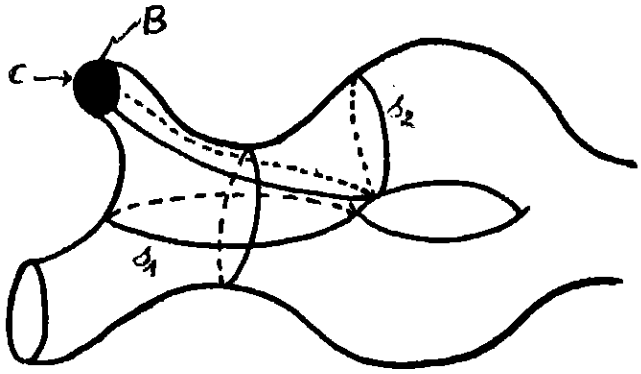

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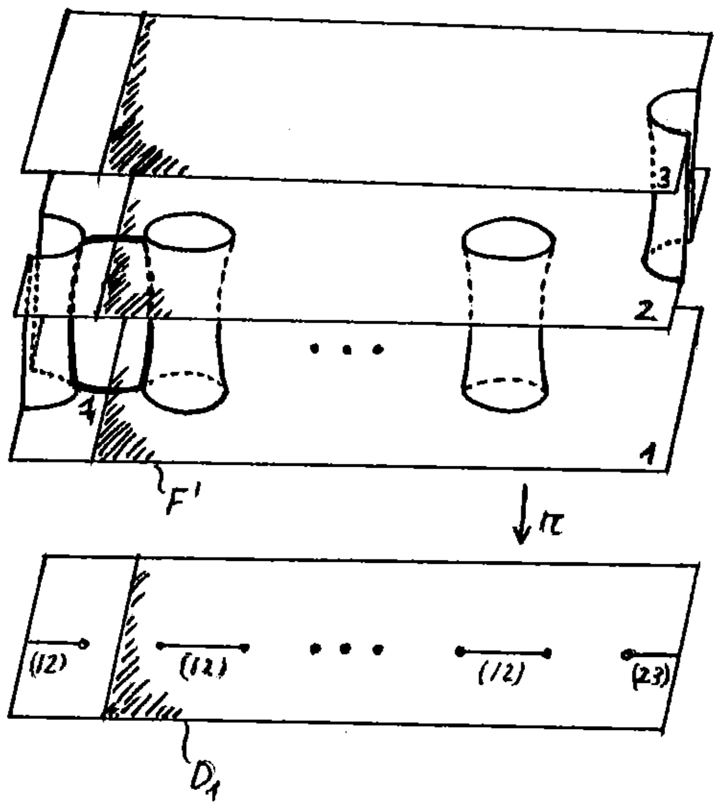

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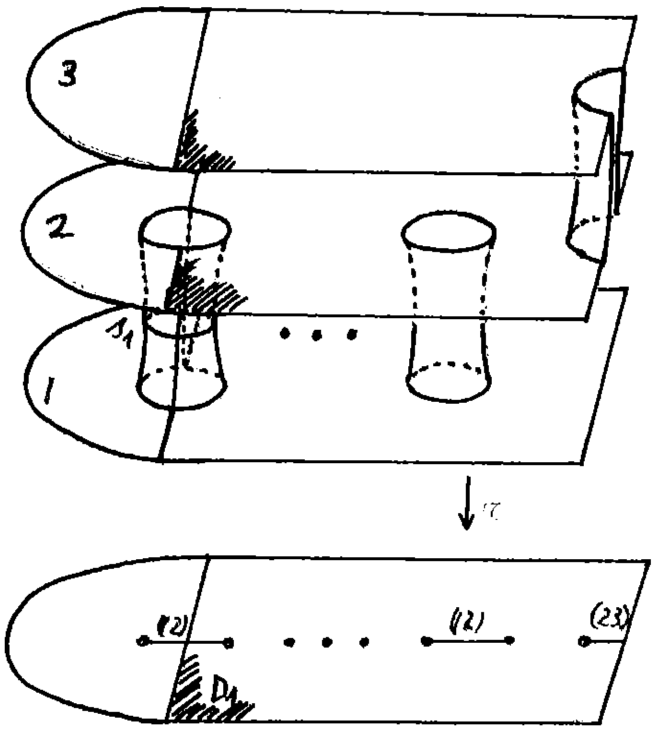
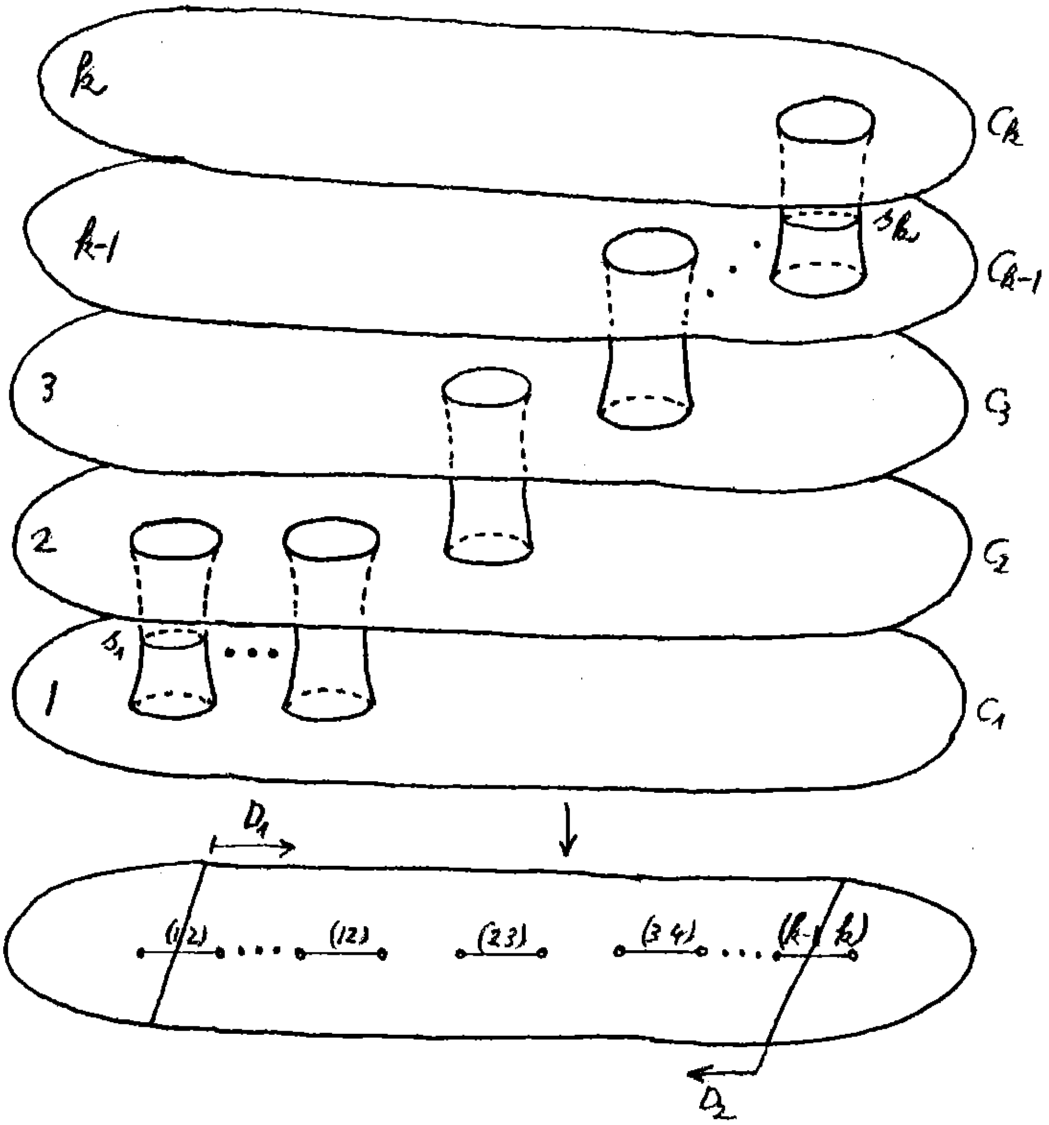


Fig 39



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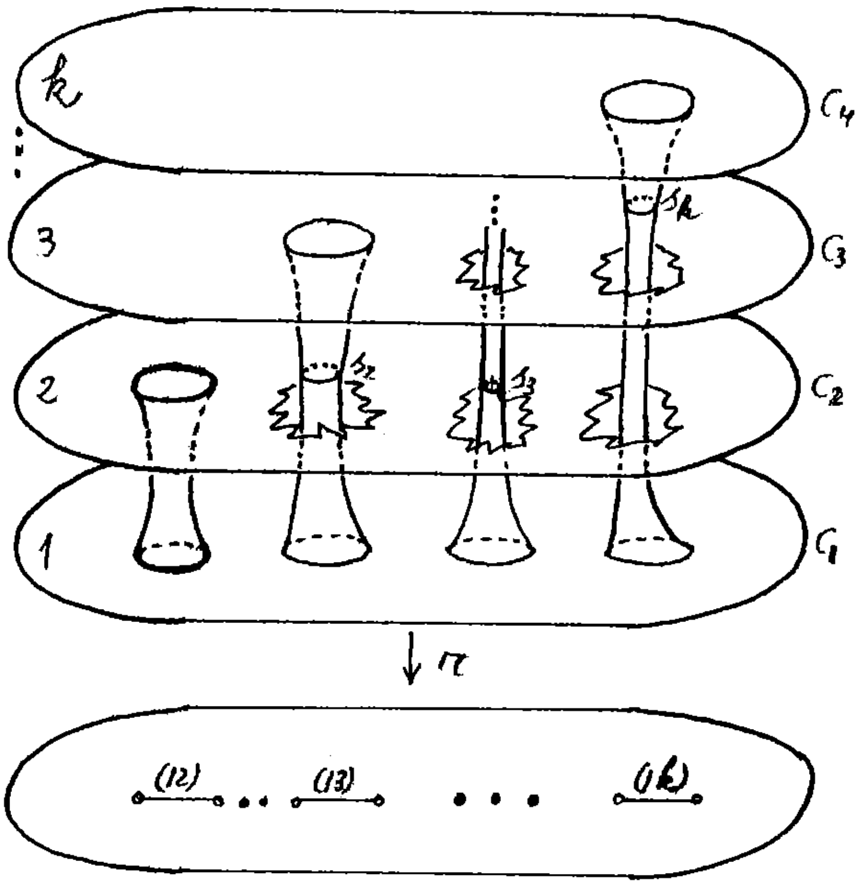
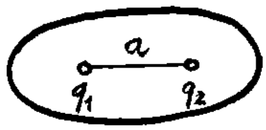


Fig. 3.11



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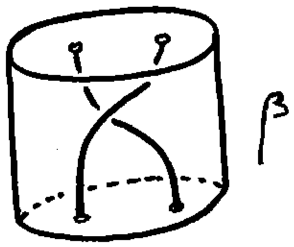

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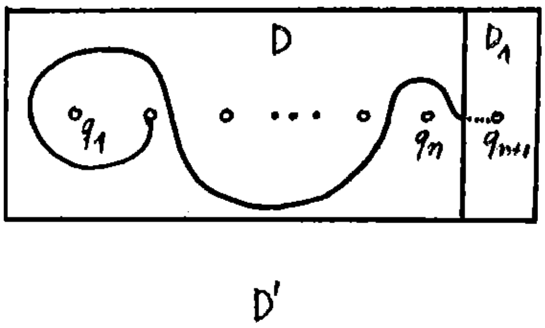

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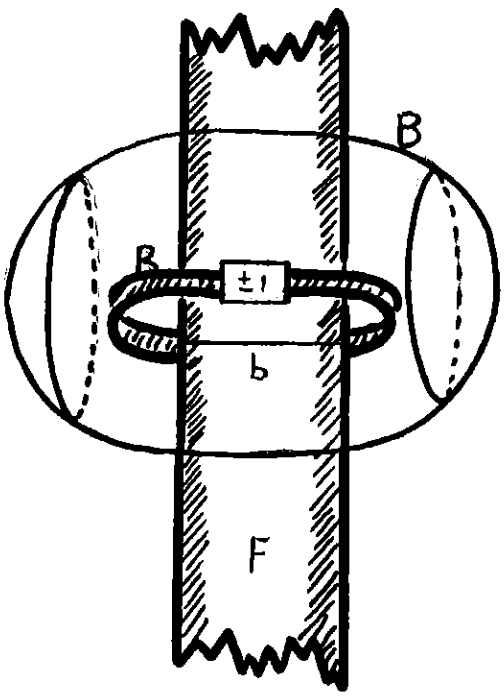

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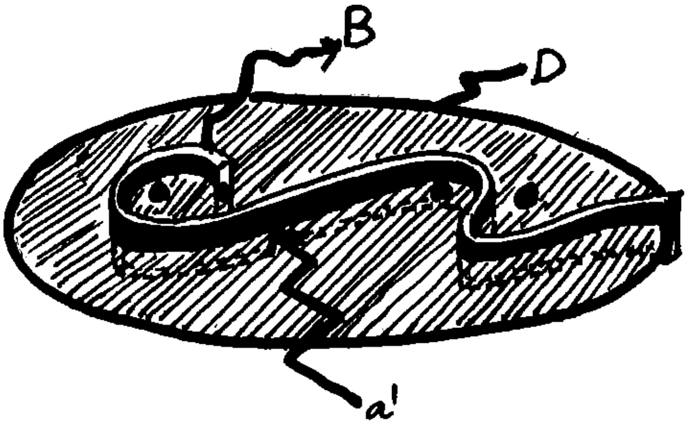

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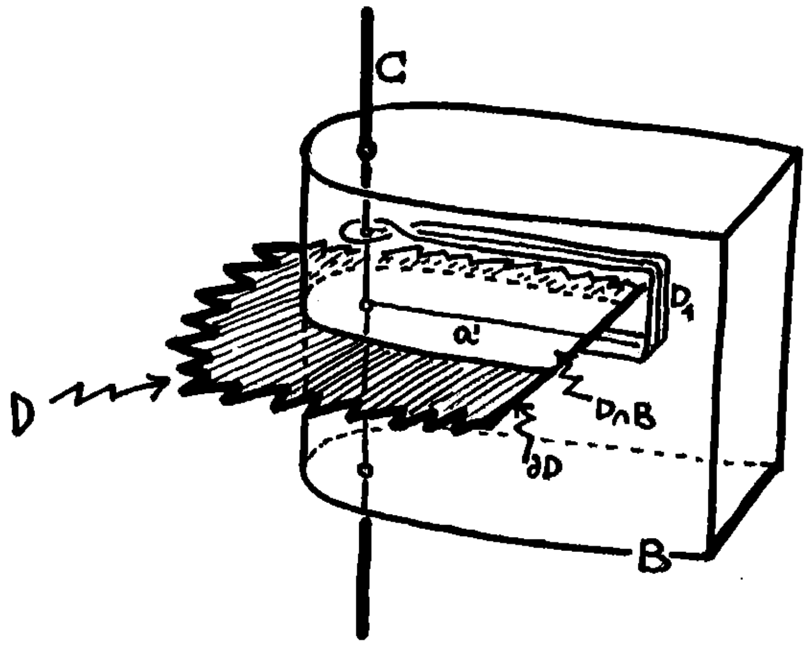

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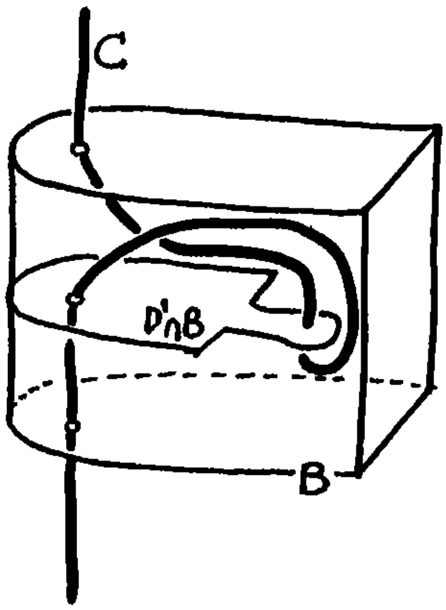

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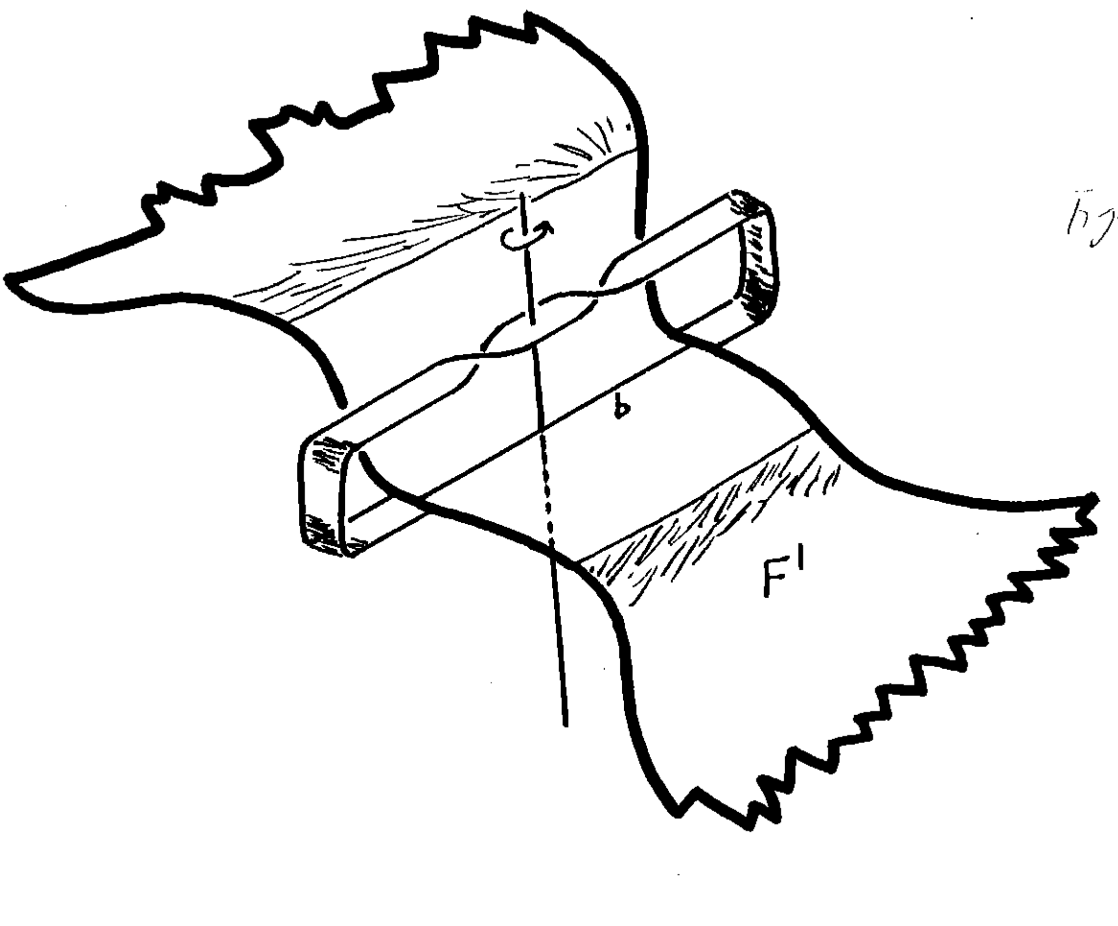

Fig 55

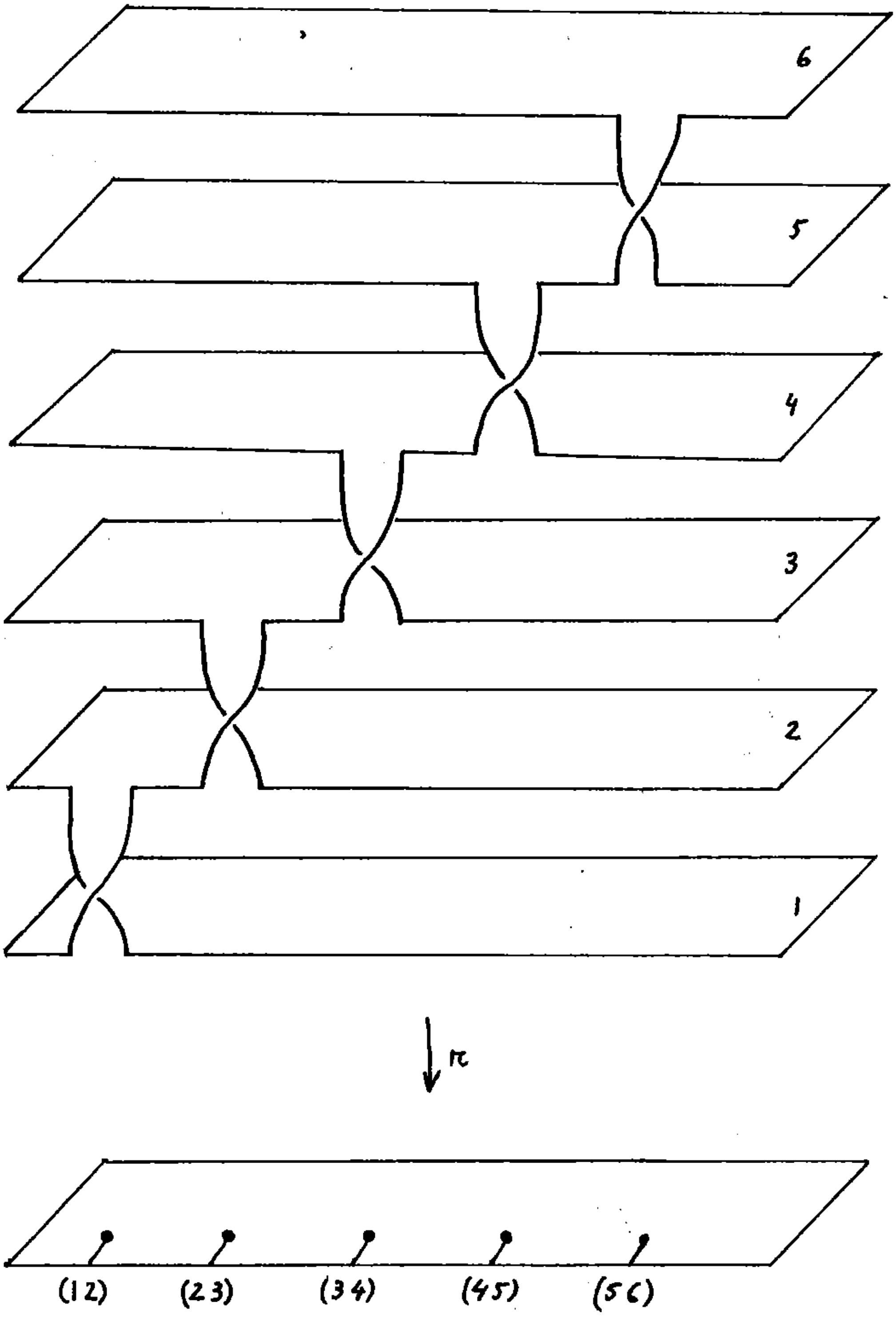

Fig. 5.6

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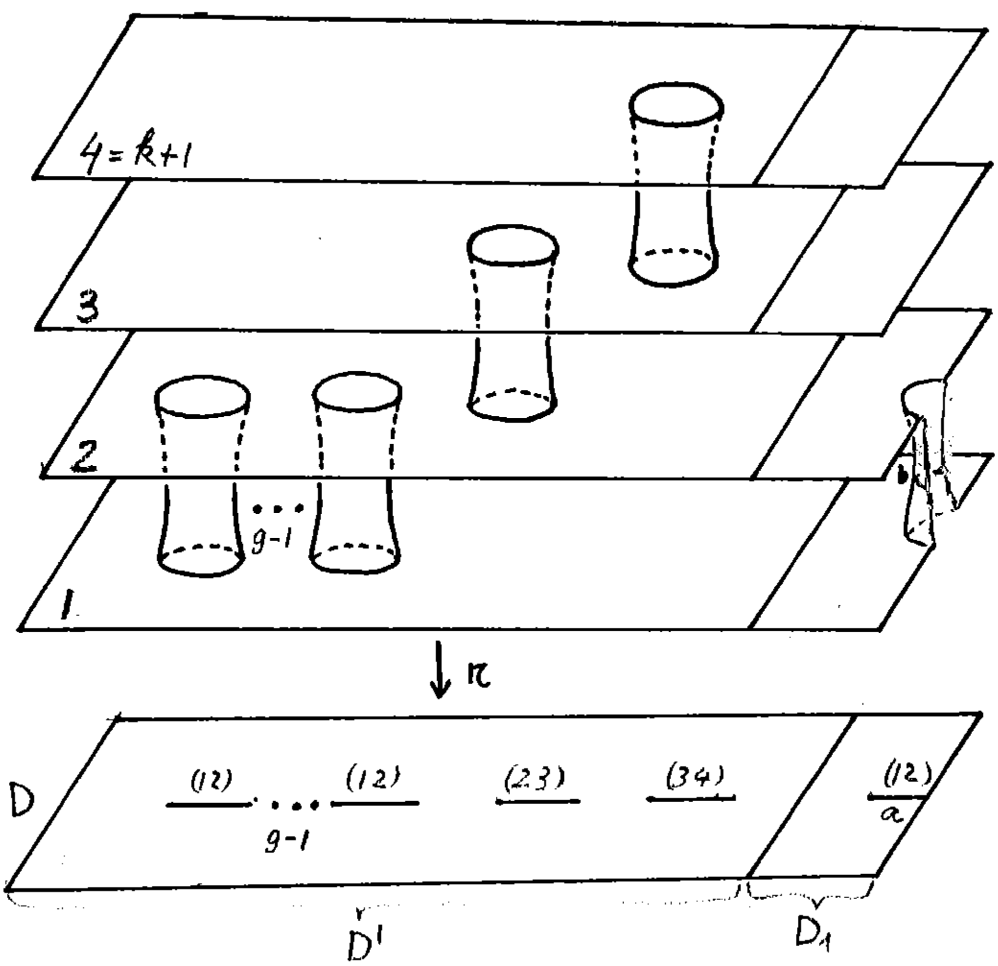

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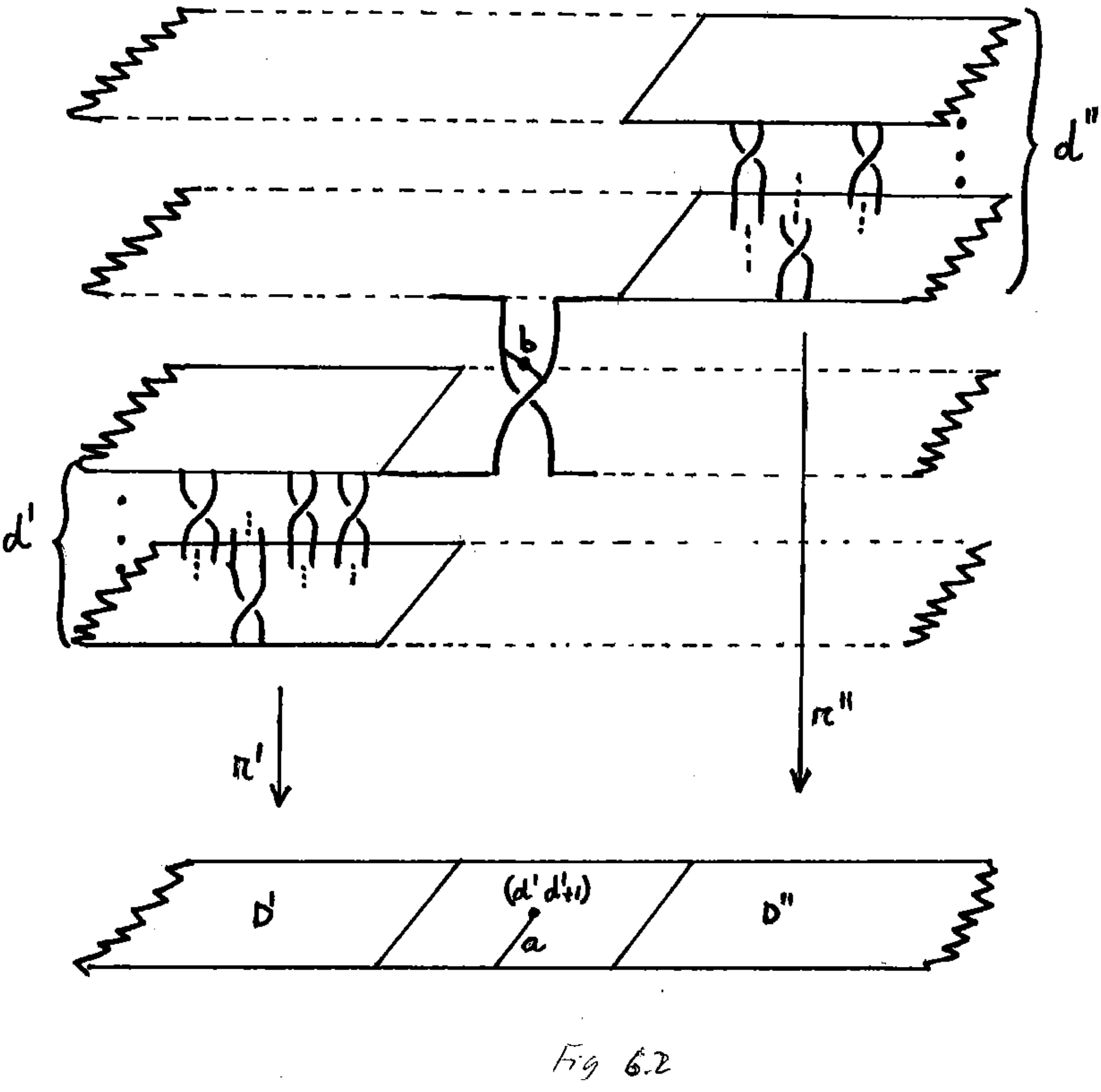

Fig 6.2