Trefoil knots without tritangent planes

H.R.Morton

Abstract.

A quick proof is given that trefoil knots constructed explicitly as (3,2) curves on a torus by stereographic projection from S^3 have no tritangent planes.

Introduction.

In [F], Freedman asked whether every generic smooth knotted curve in $\mathbf{R}^3$ must have a tritangent plane. Some years ago I was able to give the makings of a counterexample, which was a polygonal knot without polygonal tritangent planes, where a plane is said to be tangent at a vertex of a polygonal curve if it is a local supporting plane to the curve, [M]. I hoped to convert this into a smooth example by a small deformation, but I was never able to show how to make a suitable choice.

Recently Montesinos and Nuño, [MN], have shown that a certain smooth embedding of a trefoil on a torus in $\mathbf{R}^3$ has no tritangent planes, using some detailed numerical estimates in their proof.

In this note I give a similar, but not identical, embedding of a trefoil as a (3,2) curve on a torus, with a quick proof that it has no tritangent plane.

The more familiar view of a trefoil as a (2,3) curve on the torus clearly has two tritangent planes, which can be imagined by setting down a wire model on a table. The table touches the torus along a circle, and this circle is met three times by the curve.

It might initially be expected that any closed curve set down on a plane will come to rest with three points in contact with the plane. A much more common position, however, is that there are only two points in contact, and the curve can rock backwards and forwards, like a rocking chair on two runners. Attempts to rock it far enough so that a third point comes in contact with the plane may well fail when on of the two pieces of curve disappears inside the convex hull. A tritangent plane need not, of course, be a support plane; it is quite easy to construct knots without tritangent support planes.

Construction.

For this example I start with a trefoil knot $T \subset S^3$, embedded parametrically as

$$(a\cos 3\theta, a\sin 3\theta, b\cos 2\theta, b\sin 2\theta),$$

with $a^2+b^2=1, a,b\neq 0$.

Stereographic projection $h : S^3 - \{(0,0,0,1)\} \to \mathbf{R}^3$ is the homeomorphism defined by

$$h(x,y,z,t) = \frac{1}{1-t}(x,y,z).$$

The curves which I will use as examples are then the trefoil knots $h(T) \subset \mathbf{R}^3$, for each choice of (a,b).

Theorem. *The curve $h(T) \subset \mathbf{R}^3$, which is a trefoil knot, has no tritangent plane.*

Proof: Points on $h(T)$ can be written parametrically as

$$\mathbf{r}(\theta) = \frac{1}{1 - b\sin 2\theta}(a\cos 3\theta, a\sin 3\theta, b\cos 2\theta).$$

Points of intersection of $h(T)$ with the plane whose equation is

$$c_1 x_1 + c_2 x_2 + c_3 x_3 = c_4$$

are given by

$$a_1 \cos 3\theta + a_2 \sin 3\theta + a_3 \cos 2\theta + a_4 \sin 2\theta = c_4, \tag{1}$$

where $(a_1, a_2, a_3, a_4) = (ac_1, ac_2, bc_3, bc_4)$. We may find the intersection points by writing (1) in terms of $z = e^{i\theta}$. The equation then becomes

$$\alpha_1 z^3 + \alpha_2 z^{-3} + \alpha_3 z^2 + \alpha_4 z^{-2} = c_4, \tag{2}$$

where $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (a_1 - ia_2, a_1 + ia_2, a_3 - ia_4, a_3 + ia_4)$. This can be rewritten as $P(z) = 0$, where P is a polynomial of degree 6. There are then at most six points of intersection of the plane with the curve $h(T)$, given by roots of P on the unit circle in $\mathbf{C}$. For the plane to be tritangent these roots must occur as three repeated pairs, and the polynomial P then has the form $P = Q^2$ for some cubic polynomial $Q(z)$.

We may assume that $\alpha_1 \neq 0$, otherwise the plane cannot be tritangent. Write P as a monic polynomial, to get

$$P(z) = z^6 + \frac{\alpha_3}{\alpha_1} z^5 - \frac{c_4}{\alpha_1} z^3 + \frac{\alpha_4}{\alpha_1} z + \frac{\alpha_2}{\alpha_1}. \tag{3}$$

Lemma. *If the polynomial*

$$P(z) = z^6 + \beta_1 z^5 + \beta_2 z^3 + \beta_3 z + \beta_4$$

is a perfect square with all its roots on the unit circle then $\beta_1 = \beta_3 = 0$.

Proof: Suppose that $P = Q^2$, with $Q(z) = z^3 + cz^2 + dz + e$. Equating coefficients gives

$$\begin{aligned} \beta_1 &= 2c, \\ \beta_3 &= 2d, \\ 0 &= c^2 + 2d, \\ \text{and } 0 &= d^2 + 2ec. \end{aligned}$$

Then $c^4 + 8ec = 0$, so either $c = 0$, when $\beta_1 = \beta_3 = 0$, or $c^3 = -8e$.

In the second case Q has a root $\frac{1}{2}c$ and the sum of the other two roots is then $-\frac{3}{2}c$. Now $|e| = 1$, since all the roots are on the unit circle, so $|c| = 2$. Then the sum of the remaining two roots will have modulus 3 which is impossible. □

The theorem now follows at once. For if any plane is tritangent then the resulting polynomial P in (3) is a perfect square. By the lemma we have $\alpha_3/\alpha_1 = \alpha_4/\alpha_1 = 0$, so that $a_3 = a_4 = 0$ and thus $c_3 = c_4 = 0$. Then $P(z) = z^6 + \alpha_2/\alpha_1$, which is not a perfect square, and thus did not arise from a tritangent plane. □

Remarks. The image under stereographic projection of the points

$$(a\cos\theta, a\sin\theta, b\cos\varphi, b\sin\varphi)$$

with $a^2 + b^2 = 1$ is a torus in $\mathbf{R}^3$, given by rotating a circle about the x_3-axis. Polar coordinates relative to this axis are given by θ, but φ is not the angle measured from the centre of the cross-sectional circle.

In [MN] the torus on which the trefoil lies is also given by rotating a circle around the x_3-axis, but in place of φ they use the angle from the centre of the rotated circle as a second coordinate in describing the trefoil, to get a parametrisation as

$$((A + B\cos\varphi)\cos\theta, (A + B\cos\varphi)\sin\theta, B\sin\varphi)$$

with $\theta = 3t, \varphi = 2t$. An attempt to analyse this curve on the lines above meets the difficulty that the polynomial corresponding to P which determines the planar intersections has degree 10 and not 6.

References.

[F] Freedman, M.H. *Planes triply tangent to curves with non-vanishing torsion.* Topology, 19 (1980), 1-8.

[M] Morton, H.R. Talk at Southampton University, June 1986.

[MN] Montesinos Amilibia, A. and Nuño Ballesteros, J.J. *A knot without tritangent planes.* Preprint, University of Valencia (1989).

Department of Pure Mathematics
The University
LIVERPOOL L69 3BX
ENGLAND

2 April 1990.