Positivity of knot polynomials on positive links

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Abstract

We answer a question of Jones concerning the positivity of the two-variable (Homfly) knot polynomial P_L when L is a positive link. We show that, in this case, $P_L(v,z) > 0$ for $v \in (0,1)$ and $z > 0$.

In the past five years there has been an explosion of activity in knot theory. It was sparked off by Jones' discovery [3] of a new polynomial invariant for links and the subsequent development of a two-variable generalisation. The two-variable polynomial link invariant was discovered simultaneously by several groups of mathematicians [2,6] working independently, and for an oriented link L it is denoted $P_L(v,z)$.

Recently Jones pointed out to the second author that the functions $P_L(s^n, s^{-1}-s)$ for various n derived from P_L are always positive for $s \in (0,1)$ when L can be presented as the closure of a positive braid. He asked whether this positivity result still holds for the wider class of positive links, that is, for the set of oriented links which possess a diagram in which every crossing is of the type labelled D_+ in figure 1. In this note we show that this is indeed the case.

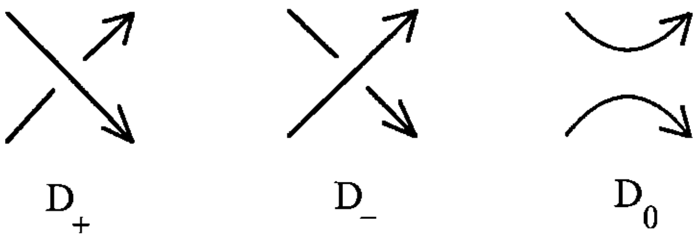

Figure 1

The function $P_L(s^n, s^{-1}-s)$ was originally seen to be positive for $s \in (0,1)$ on the class of positive closed braids by regarding it in the more physically significant context as a link invariant derived from the fundamental representation of the quantum group $SU(n)_q$ with $s^2 = q$. The result for positive braids is also an immediate consequence of the calculation of P_L as formulated in [5].

The case $n = 0$ gives the familiar Alexander polynomial Δ_L, in the form

$$P_L(1, s^{-1}-s) = \Delta_L(s^2),$$

which in turn is closely related to the Conway polynomial ∇_L, where

$$P_L(1,z) = \nabla_L(z).$$

In [1] the first author showed that $\nabla_L(z)$ is positive when L is a positive link and $z > 0$. The same technique also shows the following:

Theorem. *For any positive link L the polynomial $P_L(v,z)$ evaluates to a positive number when $z > 0$ and $v \in (0,1)$.*

Remark. An affirmative answer to Jones' question is an easy corollary, for when $s \in (0,1)$ then $s^n \in (0,1)$ and $s^{-1} - s > 0$.

Proof: The polynomial P_L can be defined recursively by a relation between three oriented link diagrams D_+, D_- and D_0 which are identical except within a small neighbourhood where they differ as shown in figure 1. The recurrence relation

$$v^{-1}P(D_+) - vP(D_-) = zP(D_0)$$

together with the normalising relation

$$P(unknot) = 1.$$

can be used to calculate the polynomial of any oriented link since any diagram can be converted into a diagram of a trivial link by switching cross-overs. This combinatorial approach to the construction of P is detailed in [4] where it is shown that the resulting polynomial depends only on the link and not on the order in which the crossings are analysed.

A particular calculation of P_L for a link L can be recorded by a rooted binary tree in which the vertices are labelled by link diagrams and the edges with monomials in v and z such that

(1) the root vertex is labelled with a diagram of L

(2) each terminal vertex is labelled with a trivial link

(3) each triple (parent, leftchild, rightchild) is of the form

$$(D_+, D_-, D_0) \text{ or } (D_-, D_+, D_0)$$

(4) each edge is labelled as in figure 2.

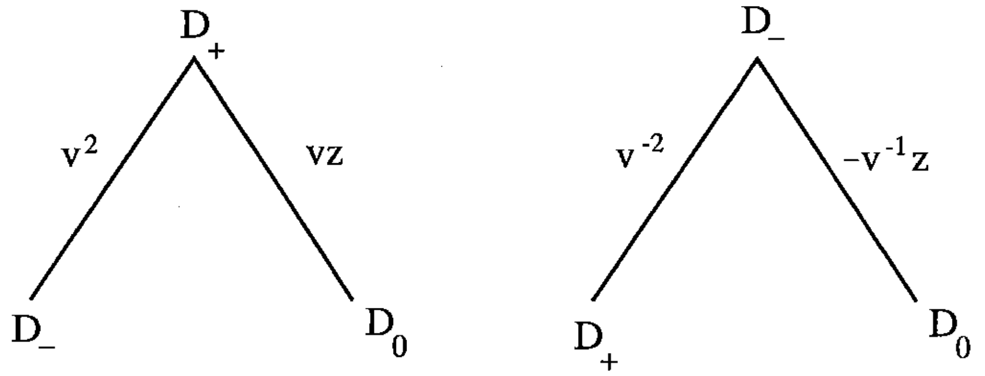

Figure 2

A path in the tree from the root to a terminal vertex represents a sequence of crossing switches and eliminations that converts L to a trivial link. In [1, theorem 2] it was shown that on each such path no crossing needs to be altered more than once.

Let π_i denote the product of the edge labels for the edges on the (unique) path between a terminal vertex T_i and the root, and let μ_i denote the number of components in the trivial link that labels T_i. Then

$$P_L(v, z) = \sum_i \pi_i \delta^{(\mu_i - 1)}$$

where $\delta = \dfrac{v^{-1} - v}{z}$.

Suppose that the diagram of L is positive so that all its crossings are of type D_+. Then it follows from [**1**, theorem 2] that every triple (parent, leftchild, rightchild) in the tree has the form (D_+, D_-, D_0). Hence each application of the recurrence relation is of the form

$$P(D_+) = v^2 P(D_-) + vzP(D_0).$$

Thus all the π_i's are products of positive monomials, and hence are positive. Since we assume $z > 0$, we have $\delta > 0$ precisely when $v \in (0,1)$. So, under the hypotheses of the proposition, P_L is a sum of positive terms, and therefore is positive. $\square$

Example. Let K denote the knot 6_3. Then $\nabla_K(z) = z^4 + z^2 + 1$. However $P_K(\frac{1}{2}, 1) = -\frac{3}{2}$ and hence K is not a positive knot. In fact, 6_3 is the first non-positive knot which has a positive Conway polynomial. (The knot 6_3 is also shown to be non-positive by methods in [**1**].)

References

[**1**] P. R. Cromwell, *Homogeneous links*, J. London Math. Soc (2) 39 (1989) 535-552.

[**2**] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. C. Millett and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. 12 (1985) 239-246.

[**3**] V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc 12 (1985) 103-111.

[**4**] W. B. R. Lickorish and K. C. Millett, *A polynomial invariant for oriented links*, Topology 26 (1987) 107-141.

[**5**] H. R. Morton, *Polynomials from braids*, Contemporary Math. 78 (1988) 575-585.

[**6**] J. H. Przytycki and P. Traczyk, *Invariants of links of Conway type*, Kobe J. Math. 4 (1987) 115-139.

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