Submitted to Conference Proceedings,
Nankai Satellite Meeting of Statphys 19.

UNITARY QUANTUM GROUPS AND MUTANT KNOTS

H.R.MORTON

Department of Pure Mathematics, University of Liverpool, PO BOX 147, Liverpool L69 3BX, England

Questions of interdependence of quantum group invariants of knots, particularly those arising from the unitary quantum groups $SU(N)_q$, are discussed, with reference to the behaviour of these invariants on mutant pairs of knots.

1 Introduction

This paper is concerned primarily with knot invariants. A fuller account of much of the material about mutants can be found in my article with Cromwell[6], while a wider introduction to the general techniques used here is contained in my expository article[5].

A knot K is taken to mean a simple closed curve in Euclidean space $\mathbf{R}^3$, possibly compactified as the unit sphere S^3. An *invariant* of K is some number or function which, typically, can be calculated from a diagram of K, but which only depends on the curve K up to physical manipulation in space as if it were a closed piece of rope. The term 'link' is used instead of 'knot' when more than one curve is involved. In what follows I shall be mostly concerned with invariants of 'framed' knots or links, where a choice of parallel curve to each component is assumed to be preserved by the manipulations. This can be looked on as dealing with ribbon, rather than rope, and keeping a tally of the twist in each ribbon. (Many recent invariants are sensitive to the twist to some extent, and it has become conventional that a knot diagram should be regarded as specifying the choice of parallel curve which lies alongside each curve in the diagram).

My own interest is in knots and links and in the related topology of 3-dimensional manifolds. In this context a knot invariant may be used to try to tell whether two different diagrams represent the same space curve, up to appropriate manipulations, and more generally to try to reflect some other geometric properties of the knot or of manifolds which can be constructed from it. In recent years a great range of new knot invariants has been discovered, using a wide variety of constructions. At this meeting the most familiar method must be that of vertex models, where the Yang-Baxter equation plays a key role in establishing invariance under topological manipulations of the related knot. The possible uses of such models were realised initially by Jones[2,3] and Turaev[9]. These models themselves were closely connected with the development of quantum groups. Work in this area by Jimbo and others was quickly seen, notably by Turaev and Reshetikhin[7], to lead to a whole range of invariants of knots.

From the knot theory viewpoint it is important to try to establish the extent to which some invariants depend or do not depend on others. It is known, for example, that certain invariants, such as Witten's link invariants, are sections of others, given by suitably specialising variables in an invariant with more parameters. Typically these are quantum group invariants, which in turn can be derived, in the case of the unitary quantum groups, from the 2-variable Homfly polynomial of the link itself or of other geometrically related links. There are thus quite frequently a number of very different ways of finding out equivalent information about the knot or link, and so the question of deciding whether a given invariant depends, theoretically, or even quite explicitly, on others, is increasingly of interest. As an example of a possible independence result, if invariants $I_1, \ldots, I_r$ satisfy $I_1(K) = I_1(K'), \ldots, I_r(K) = I_r(K')$ for all knots K, K' in some subclass of knots, while $J(K) \neq J(K')$ for some choice of K, K' in this same subclass, then clearly the invariant J does not depend in any way on $I_1, \ldots, I_r$.

2 Mutant knots

It is not, in general, easy to find knots with a given value for an invariant I_1, or pairs of knots on which a sequence of invariants takes the same value, and indeed many algebraic features of invariants are not well documented in relation to geometric properties of the knot. The class of *mutant knots*, which I shall now introduce, does however provide a relatively large selection of pairs of knots K, K' for which certain invariants are guaranteed to be the same. It then follows that an invariant which does distinguish some pair of mutants cannot depend in any way on these first invariants.

Here is a picture of the simplest and best-known mutant pair of knots, due to Conway and Kinoshita-Teresaka in about 1960.

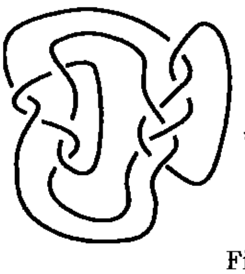
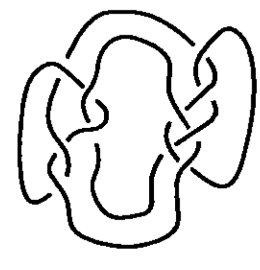
$C =$ , $KT =$ .

Figure 1

The term *mutant* was coined by Conway, and refers to the following general construction.

Suppose that a knot K can be decomposed into two oriented 2-tangles F and G as shown in figure 2.

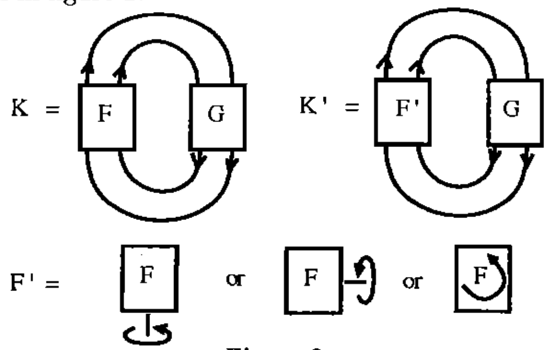


Figure 2

A new knot K' can be formed by replacing the tangle F with the tangle F' given by rotating F through π in one of three ways, reversing its string orientations if necessary. Any of these three knots K' is called a *mutant* of K. It is clear from figure 1 that the knots C and KT are mutants.

It was shown by Conway that mutants K, K' always share the same Alexander polynomial. Lickorish[4] generalised this by proving that mutants must also have identical Homfly polynomials, and hence the same Jones polynomial. Indeed, as I have suggested above, there is a range of other invariants which also agree on mutants. I shall give a brief account of the construction of quantum group invariants, and show how some of these invariants are inevitably 'blind' to mutants, while others can be seen by explicit calculation to distinguish the knots C and KT.

3 Quantum invariants

Reshetikhin and Turaev[7] have described in detail how a finite-dimensional module V_λ over a suitable quantum group can be used to construct an invariant $J_K(V_\lambda)$ of a framed knot K which is a power series in the quantum group parameter h. It can usually be expressed easily in terms of $q = e^h$ or $s = e^{h/2}$.

For the present purposes the important features of a quantum group $\mathcal{G}$ are that it is an algebra with coefficients $\mathbf{C}[[h]]$ which admits tensor products and duals of finite-dimensional modules. In the most frequently used examples each module is completely reducible as a direct sum, as noted by Rosso[8].

The construction of invariants is based on 'colouring' tangles by $\mathcal{G}$-modules, and representing these by module homomorphisms. This construction extends to determine an invariant of framed oriented links when 'coloured' by a choice of module for each component. The invariants are multilinear under direct sums of modules, while a knot K coloured with a tensor product $V \otimes W$ of two modules has the same invariant as the link $K^{(2)}$ made up of two parallel copies of K when coloured by V and W respectively on the two components.

As an illustration of the general construction, imagine that an oriented tangle T has been given and that a choice of module has been made for each string in T, as shown in figure 3.

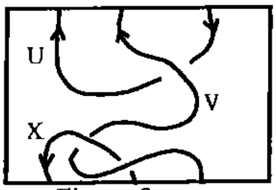
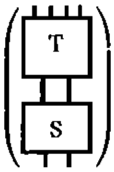
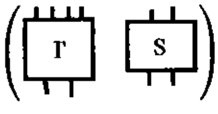
$T =$ 

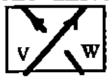
Figure 3

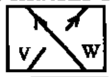
Associate modules $W_{\pm}$ to the top and bottom of T, by taking the tensor product of the modules colouring the strings which leave the bottom, in their order from left to right, as W_- and of those which arrive at the top as W_+, using the dual module where the string orientation is in the opposite sense. Where there are no strings at the top or the bottom, the trivial module $\mathbf{C}[[h]]$ is used. In the example shown we have $W_+ = U \otimes V \otimes U^*$ and $W_- = X^* \otimes X \otimes V$.

The aim is then to represent T by a homomorphism $J_T : W_- \to W_+$, using composite or tensor product of homomorphisms where a tangle has been built up from consistently coloured pieces as shown below.

J  $= J_T \circ J_S$

J  $= J_T \otimes J_S$

Reshetikhin and Turaev show how to use the universal R-matrix for $\mathcal{G}$ to represent the elementary coloured tangles 

and 

and give consistent representations for the local maxima and minima, 

and 

, with any choice of colouring and orientation. The composition laws then determine J_T in terms of these elementary pieces. The essential part of their work is the following result, which is mainly a consequence of properties of the universal R-matrix.

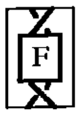
Theorem 1 (Reshetikhin, Turaev) *Alteration of a coloured tangle T by regular isotopy (string manipulation, respecting the choice of framing on each string) leaves the homomorphism J_T unchanged.*

In particular, a diagram of a knot K, coloured by a module V, determines a module homomorphism $\mathbf{C}[[h]] \to \mathbf{C}[[h]]$ which must be multiplication by some scalar. By theorem 1, this scalar, $J_K(V) \in \mathbf{C}[[h]]$, depends only on the framed knot K, and not on the diagram chosen to picture it. For a link, we can make a choice of one module per component; it is sufficient to deal with irreducible modules V, as the invariants behave multilinearly under direct sums of modules.

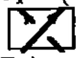
Returning now to the special case of mutants, suppose that K and K' have been coloured by a module V_λ. The coloured tangle F determines a map

$$J_F : V_\lambda \otimes V_\lambda \to V_\lambda \otimes V_\lambda .$$

If we know that $J_{F'} = J_F$ then $J_{K'}(V_\lambda) = J_K(V_\lambda)$, because the remainder of the diagram is the same for K' as for K. Suppose now that $F' = \Delta \circ F \circ \Delta^{-1}$, given pictorially by

$F' =$ 

as in the case of the Conway and Kinoshita-Teresaka pair.

Then $J_{F'} = R_\lambda \circ J_F \circ (R_\lambda)^{-1}$, where $R_\lambda : V_\lambda \otimes V_\lambda \to V_\lambda \otimes V_\lambda$ represents the elementary tangle  coloured by V_λ.

We now give a sufficient condition for equality of mutant invariants in the case above; the result is also true for the two other types of mutant, further details can be found in Morton and Cromwell[6].

Firstly a quick algebraic observation.

Suppose that a module W decomposes into irreducible summands as $W \cong V_1 \oplus \cdots \oplus V_r$. Suppose further that no two of the summands $V_1, \ldots, V_r$ are isomorphic. Then any endomorphism $\alpha : W \to W$ maps each V_i to itself by scalar multiplication with some α_i, by Schur's lemma, and consequently any two endomorphisms $\alpha, \beta : W \to W$ commute.

Suppose now that mutants K, K' are coloured by a module V_λ such that $V_\lambda \otimes V_\lambda$ has *no* repeated summands. Then the homomorphisms R_λ and J_F commute, by the observation above with $W = V_\lambda \otimes V_\lambda$. It follows that $J_{F'} = R_\lambda \circ J_F \circ R_\lambda^{-1} = J_F$ and hence that $J_K(V_\lambda) = J_{K'}(V_\lambda)$.

Examples of this behaviour occur among the invariants given by the unitary quantum groups $SU(N)_q$. The irreducible modules over $SU(N)_q$ can be indexed by Young diagrams λ, just as in the classical case of the Lie algebra of $SU(N)$, and satisfy some relations depending on N, again as in the classical case. The simplest module is the 'fundamental module', $V_{\square}$, of dimension N, associated with the Young diagram $\square$. Decomposition of tensor products of modules in $SU(N)_q$ into irreducibles is determined by the Littlewood-Richardson combinatorial rules on the Young diagrams. These can be used to test any product $V_\lambda \otimes V_\lambda$ for repeated summands. A quick application of the rules shows that when λ consists of a single row or column then $V_\lambda \otimes V_\lambda$ has no repeated summands, and so the resulting invariants agree on mutants.

The following table gives a summary of the known coincidences and differences of unitary invariants on mutants K, K'. Columns in the table are indexed by Young diagrams λ with increasing numbers of cells j, while rows are indexed by N. The entries in the table refer to the difference $J_K(V_\lambda) - J_{K'}(V_\lambda)$ for the $SU(N)_q$ invariant with irreducible module V_λ whose Young diagram is λ. An entry 0 in the table means that the invariant listed agrees on *all* mutant pairs K, K'. Most, but not all, of these entries follow from the condition above. An entry ? means that evidence is unavailable, or inconclusive, while an entry × means that the difference is known to be non-zero for some pair of mutants.

$$J_K(V_\lambda) - J_{K'}(V_\lambda)$$

j	1	2		3			4				
N \ λ	(1)	(1,1)	(2)	(1,1,1)	(2,1)	(3)	(1,1,1,1)	(2,1,1)	(2,2)	(3,1)	(4)
2	0	0	0	0	0	0	0	→	→	→	→
3	0	↓	↓	0	0	0	0	0	0	×	0
4	↓	↓	↓	0	×	0	0	0	0	?	0
5	↓	↓	↓	↓	×	↓	↓	?	↓	?	↓

Figure 4

All entries in the row $N = 2$ are known to be 0, as are all entries in the columns with one or two cells in the Young diagram λ.

The entries in the column with $\lambda = (2,1)$ are known to be 0 for $N = 3$ and to be non-zero on the Conway and Kinoshita-Teresaka pair for $N \geq 4$, following calculations of Morton and Cromwell[6].

More recent calculations with Ryder have shown that when $N = 3$ and $\lambda = (3,1)$ the difference is also non-zero on this pair. The difference in this case was found to be

$$(s^{46}-s^{44}+2\,s^{40}-4\,s^{38}+2\,s^{36}+3\,s^{34}-4\,s^{32}+6\,s^{30}-s^{28}-3\,s^{26}+6\,s^{24}-4\,s^{22}+4\,s^{20}+2\,s^{18}-5\,s^{16}+5\,s^{14}-2\,s^{12}-2\,s^{10}+4\,s^{8}-2\,s^{6}+s^{2}-1)$$
$$(s^8-s^{-8})^2(s^7-s^{-7})(s^6-s^{-6})(s^5-s^{-5})(s^4-s^{-4})^2(s^3-s^{-3})^2(s^2-s^{-2})(s-s^{-1})^3,$$
up to a power of $s=e^{h/2}$.

When any of the quantum invariants is written as a power series in h the coefficient of h^d is a rather restricted type of invariant of the knot known as a Vassiliev finite-type invariant. It has been shown by Chmutov et al [1] that for $d \leq 9$ every such invariant will agree on mutant pairs. It is thus of interest to look at the degree of the lowest term in h in any of the non-zero differences in the table. In the column with $\lambda =$  the first non-zero term for Conway/Kinoshita-Teresaka is the term in h^{11}, while the entry reported above for the same pair of knots when $N=3$ and $\lambda =$  can quickly be seen to begin with the term in h^{13}.

One consequence of this non-zero entry with $N=3$ is a certain guarantee that the $SU(3)_q$ invariants of a knot can not all be derived from knowledge of all its $SU(2)_q$ invariants. Similar guarantees of the independence of the $SU(N+1)_q$ invariants from the $SU(N)_q$ invariants are anticipated, but are not so far available for $N>2$.

It is worth noting that these invariants have alternative descriptions in more combinatorial or geometric terms. For example, all the invariants for one Young diagram, as N varies, can be organised as a Laurent polynomial in two variables v, z, and the invariant for row N recovered by the substitution $v=s^N, z=s-s^{-1}$ where $s=e^{h/2}$. The first column, with the fundamental representation, comes in this way from the Homfly polynomial, while the column for each Young diagram with j cells comes from a linear combination of the Homfly polynomials of suitable satellite knots of the original knot, consisting of certain cables with j strings.

Acknowledgments

I would like to thank the Royal Society for a travel grant which enabled me to attend this meeting, and the organisers at Nankai University for their local support.

References

1. S.V. Chmutov, S.V. Duzhin and S.K. Lando, Vassiliev knot invariants, I. Introduction, preprint 1994.
2. V.F.R. Jones, Notes on a talk in Atiyah's seminar, 1986.

3. V.F.R. Jones, On knot invariants related to some statistical mechanical models, *Pacific J. Math* 137 (1989), 311-334.
4. W.B.R. Lickorish, Linear skein theory and link polynomials, *Topology and its Applications,* 27 (1987), 265-274.
5. H.R. Morton, Invariants of links and 3-manifolds from skein theory and from quantum groups. In 'Topics in knot theory', the Proceedings of the NATO Summer Institute in Erzurum 1992, NATO ASI Series C 399, ed. M.Bozhüyük. Kluwer 1993, 107-156.
6. H.R. Morton and P.R. Cromwell, Distinguishing mutants by knot polynomials. Preprint 1994, to appear in J. Knot Theory Ramif.
7. N.Y. Reshetikhin and V.G. Turaev, Ribbon graphs and their invariants derived from quantum groups, *Commun. Math. Phys.* 127 (1990), 1-26.
8. M. Rosso, Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra, *Comm. Math. Phys.* 117 (1988), 581-593.
9. V.G. Turaev, The Yang-Baxter equation and invariants of links, *Invent. Math.* 92 (1988), 527-553.