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On the Chromatic Number of Non-Sparse Random Intersection Graphs

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Abstract An intersection graph of n vertices assumes that each vertex is equipped with a subset of a global label set. Two vertices share an edge when their label sets intersect. Random intersection graphs (RIGs) (as defined in [11,12]) consider label sets formed by the following experiment: each vertex, independently and uniformly, examines all the labels (m in total) one by one. Each examination is independent and the vertex succeeds to put the label in her set with probability p. Such graphs can capture interactions in networks due to sharing of resources among nodes.

In this paper, we discuss various structural and algorithmic results concerning random intersection graphs and we focus on the computational problem of properly coloring random instances of the binomial random intersection graphs model. For the latter, we consider a range of parameters m, p for which RIGs differ substantially from Erdős-Rényi random graphs and for which greedy approaches fail.

Keywords Random intersection graphs · Proper coloring · Martingales · Probabilistic method

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1 Introduction

We study a simple, yet general family of models, namely *Random Intersection Graphs (RIGs)*. In such models there is a universe $\mathcal{M}$ of *labels* and each one of n vertices selects a random subset of $\mathcal{M}$. Two vertices are connected if and only if their corresponding subsets of labels intersect.

Random intersection graphs may model several real-life applications quite accurately. In fact, there are practical situations where each communication agent (e.g. a wireless node) gets access only to some ports (statistically) out of a possible set of communication ports. When another agent also selects a communication port, then a communication link is implicitly established and this gives rise to communication graphs that look like random intersection graphs. RIG modeling is useful in the efficient blind selection of few encryption keys for secure communications over radio channels ([9]), as well as in k-Secret sharing between swarm mobile devices (see [10]). Furthermore, random intersection graphs are relevant to and capture quite nicely social networking. Indeed, a social network is a structure made of nodes tied by one or more specific types of interdependency, such as values, visions, financial exchange, friends, conflicts, web links etc. Other applications may include oblivious resource sharing in a distributed setting, interactions of mobile agents traversing the web, social networking etc. Even epidemiological phenomena (like spread of disease between individuals with common characteristics in a population) tend to be more accurately captured by this "proximity-sensitive" family of random graphs.

In this paper, we focus on the problem of properly coloring random instances of random intersection graphs. A *proper coloring* of a graph $G = (V, E)$ is an assignment of colors to all vertices in V in such a way that no two adjacent vertices have the same color. A *k-coloring* of G is a coloring that uses k colors. The minimum number of colors that can be used to properly color G is the *(vertex) chromatic number* of G and is denoted by $\chi(G)$. Finding the chromatic number of a graph is a fundamental problem in computer science, with various applications related to collision avoidance and message inhibition methods [1], range assignment problems in directional antennas' optimization [2], coordination aspects of MAC access in sensor networks [3] and more.

Deciding whether a given graph admits a k-coloring for a given $k \geq 3$ is well known to be NP-complete. In particular, it is NP-hard to compute the chromatic number [4]. The best known approximation algorithm computes a coloring of size at most within a factor $O\left(\frac{n(\log\log n)^2}{(\log n)^3}\right)$ of the chromatic number [5]. Furthermore, for any constant $\epsilon > 0$, it is NP-hard to approximate the chromatic number within a factor $n^{1-\epsilon}$ [6].

The intractability of the vertex coloring problem for arbitrary graphs lead researchers to the study of the problem on random instances of various random graphs models. Colouring Erdős - Rényi random graphs (where edges appear independently) was considered in [7] and also [8]. As it seems to be implied by these two works, randomness sometimes allows for smaller chromatic number

than maximum degree whp. For $G_{n,\hat{p}}$, it is shown that whp $\chi(G_{n,\hat{p}}) \sim \frac{d}{\log d}$, where d is the mean degree. We have to point out here that both [7] and [8] prove that there exists a colouring of $G_{n,\hat{p}}$ using around $\frac{d}{\log d}$, but their proof does not lead to polynomial time algorithms.

2 Formal Definition of the Model

Random intersection graphs (also referred to as binomial, or uniform random intersection graphs) were introduced by M. Karoński, E.R. Sheinerman and K.B. Singer-Cohen [11] and K.B. Singer-Cohen [12]. The formal definition of the model is given below:

Definition 1 ((Uniform/Binomial) Random Intersection Graph - $\mathcal{G}_{n,m,p}$ [11,12]) Consider a universe $\mathcal{M} = \{1, 2, \ldots, m\}$ of elements and a set of n vertices V. Assign independently to each vertex $v \in V$ a subset S_v of $\mathcal{M}$, choosing each element $i \in \mathcal{M}$ independently with probability p and draw an edge between two vertices $v \neq u$ if and only if $S_v \cap S_u \neq \emptyset$. The resulting graph is an instance $G_{n,m,p}$ of the random intersection graphs model.

We will say that a property holds in $\mathcal{G}_{n,m,p}$ *with high probability (whp)* if the probability that a random instance of the $\mathcal{G}_{n,m,p}$ model has the property is at least $1 - o(1)$.

In this model we also denote by L_i the set of vertices that have chosen label $i \in M$. Given $G_{n,m,p}$, we will refer to $\{L_i, i \in \mathcal{M}\}$ as its *label representation*. It is often convenient to view the label representation as a bipartite graph with vertex set $V \cup \mathcal{M}$ and edge set $\{(v,i) : i \in S_v\} = \{(v,i) : v \in L_i\}$. We refer to this graph as the *bipartite random graph $B_{n,m,p}$ associated to $G_{n,m,p}$*. Notice that the associated bipartite graph is uniquely defined by the label representation.

We note that by selecting the label set of each vertex using a different distribution, we get random intersection graphs models whose statistical behavior can vary considerably from that of $G_{n,m,p}$. Two of these models that have also been considered in the literature are the following: (a) In the **General Random Intersection Graphs Model** $G_{n,m,\mathbf{p}}$ [13], where $\mathbf{p} = [p_1, p_2, \ldots, p_m]$, the label set S_v of a vertex v is formed by choosing independently each label i with probability p_i. (b) In the **Regular Random Intersection Graphs Model** $G_{n,m,\lambda}$ [14], where $\lambda \in \mathbb{N}$, the label set of a vertex is chosen independently, uniformly at random for the set of all subsets of $\mathcal{M}$ of cardinality λ.

It is worth mentioning that the edges in $G_{n,m,p}$ are not independent. In particular, there is a strictly positive dependence between the existence of two edges that share an endpoint (i.e. $\Pr(\exists\{u,v\}|\exists\{u,w\}) > \Pr(\exists\{u,v\})$). This dependence is stronger the smaller the number of labels $\mathcal{M}$ includes, while it seems to fade away as the number of labels increases. In fact, by using a coupling technique, the authors in [15] prove the equivalence (measured in terms of total variation distance) of uniform random intersection graphs and

Erdős-Rényi random graphs, when $m = n^\alpha, \alpha > 6$. This bound on the number of labels was improved in [16], by showing equivalence of sharp threshold functions among the two models for $\alpha \geq 3$. These results show that random intersection graphs are quite general and that known techniques for random graphs can be used in the analysis of uniform random intersection graphs with a large number of labels.

The similarity between uniform random intersection graphs and Erdős-Rényi random graphs vanishes as the number of labels m decreases below the number of vertices n (i.e. $m = n^\alpha$, for $\alpha \leq 1$). This dichotomy was initially pointed out in [12], through the investigation of connectivity of $G_{n,m,p}$. In particular, it was proved that the connectivity threshold for $\alpha > 1$ is $\sqrt{\frac{\ln n}{nm}}$, but it is $\frac{\ln n}{m}$ (i.e. quite larger) for $\alpha \leq 1$. Therefore, the mean number of edges just above connectivity is approximately $\frac{1}{2} n \ln n$ in the first case (which is equal to the mean number of edges just above the connectivity threshold for Erdős-Rényi random graphs), but it is larger by at least a factor of $\ln n$ in the second case. Other dichotomy results of similar flavor were pointed out in the investigation of the (unconditioned) vertex degree distribution by D. Stark [17], through the analysis of a suitable generating function, and in the investigation of the distribution of the number of isolated vertices by Y. Shang [18].

3 Colouring Non-sparse Random Intersection Graphs

In [19] the authors propose algorithms that whp probability color sparse instances of $G_{n,m,p}$. In particular, for $m = n^\alpha, \alpha > 0$ and $p = o\left(\sqrt{\frac{1}{nm}}\right)$ they show that $G_{n,m,p}$ can be colored optimally. Also, in the case where $m = n^\alpha, \alpha < 1$ and $p = o\left(\frac{1}{m \ln n}\right)$ they show that $\chi(G_{n,m,p}) \sim np$ whp. To do this, they prove that $G_{n,m,p}$ is chordal whp (or equivalently, the associated bipartite graph does not contain cycles) and so a perfect elimination scheme can be used to find a coloring in polynomial time.

In this section, we present in greater detail the results of [20] for coloring non-sparse random instances of $\mathcal{G}_{n,m,p}$. The range of values we consider here is different than the one needed for the algorithms in [19] to work. We study coloring $G_{n,m,p}$ for the case $m = n^\alpha, \alpha \in (0, 1)$, where random intersection graphs differ the most from Erdős-Rényi random graphs, and in particular for the range $mp \leq (1 - \alpha) \ln n$, as well as the denser range $mp \geq \ln^2 n$. We have to note also, that the proof techniques used in [19] cannot be used in the range we consider, since the properties that they examine do not hold in our case.

3.1 Colouring Almost all Vertices

We are going to consider the case where $m = n^\alpha$, for $\alpha \in (0, 1)$ some fixed constant. As mentioned earlier, the area $mp = o\left(\frac{1}{\ln n}\right)$ gives almost surely

instances in which the label graph (i.e. the dual graph where the labels in $\mathcal{M}$ play the role of vertices and the vertices in V play the role of labels) is quite sparse and so $G_{n,m,p}$ can be coloured optimally using $\max_{l\in\mathcal{M}} |L_l|$ colours (see [19]). We will here consider the denser area $mp = \Omega\left(\frac{1}{\ln n}\right)$. In this range of values, it is easy to see that the values of $|L_l|$ are concentrated around np. We were able to prove that even for values of the parameters m, p that give quite denser graphs, we can still use np colours to properly colour most of the graph.[1] Our proof technique is inspired by analogous ideas of Frieze in [21] (see also [8]). The proof of Theorem 1 uses the following Lemma, which first appeared in [13].

Lemma 1 ([13]) *Let $G_{n,m,p}$ be a random instance of the random intersection graphs model. Then the conditional probability that a set of k vertices is an independent set, given that s of them are already an independent set is equal to $\left((1-p)^{k-s} + (k-s)p(1-p)^{k-s-1}\left(1 - \frac{sp}{1+(s-1)p}\right)\right)^m$, where $0 \le s \le k$.*

Proof For a vertex set V', let $X_{V'}$ be the indicator random variable that V' is an independent set, that is

$$X_{V'} = \begin{cases} 1 \text{ if } V' \text{ is an independent set} \\ 0 \text{ otherwise.} \end{cases}$$

Let also V_1', V_2' be two sets of k vertices with s vertices in common. Therefore, we need to find the conditional probability $P\{X_{V_1'} = 1 | X_{V_2'} = 1\}$, i.e. the probability that V_1' is an independent set given that V_2' is an independent set. The main technical tool of the proof is a vertex contraction technique; in particular, we merge several vertices into one *supervertex* and study its probabilistic behaviour.

Towards this goal, fix an element i of $\mathcal{M} = \{1, 2, \ldots, m\}$ and consider two (super)vertices v_1, v_2 of $G(n, m, p)$ that choose each label i independently with probability $p^{(1)}$ and $p^{(2)}$ respectively (the exact values of those probabilities will be determined below in the special case that supervertices consist of independent sets). Let also S_{v_1}, S_{v_2} denote the sets of elements of $\mathcal{M}$ assigned to v_1 and v_2 respectively. Then,

$$\begin{aligned} P\{i \in S_{v_1} | \nexists(v_1, v_2)\} &= P\{i \in S_{v_1}, i \notin S_{v_2} | \nexists(v_1, v_2)\} \\ &= \frac{P\{i \in S_{v_1}, i \notin S_{v_2}, \nexists(v_1, v_2)\}}{P\{\nexists(v_1, v_2)\}} = \frac{p^{(1)}(1-p^{(2)})}{1-p^{(1)}p^{(2)}} \quad (1) \end{aligned}$$

where (v_1, v_2) is an edge between v_1 and v_2. From this we get:

- Conditional on the fact that (v_1, v_2) does not exist, the probabilistic behaviour of vertex v_1 is identical to that of a single vertex that chooses element i of $\mathcal{M}$ independently with probability $\frac{p^{(1)}(1-p^{(2)})}{1-p^{(1)}p^{(2)}}$.

[1] Note however, that this does not mean that the chromatic number is close to np, since the part that is not coloured could be a clique in the worst case.

– Conditional on the fact that (v_1, v_2) does not exist, the probabilistic behaviour of v_1 and v_2 considered as a unit is identical to that of a single vertex that chooses element i of $\mathcal{M}$ independently with probability

$$P\{i \in S_{v_1} \cup S_{v_2} | \nexists(v_1, v_2)\} = P\{i \in S_{v_1} | \nexists(v_1, v_2)\} + P\{i \in S_{v_2} | \nexists(v_1, v_2)\} = \frac{p^{(1)} + p^{(2)} - 2p^{(1)}p^{(2)}}{1 - p^{(1)}p^{(2)}} \quad (2)$$

where i is a fixed element of $\mathcal{M}$. The first of the above equations follows from the observation that if there is no edge between v_1 and v_2 then the sets S_{v_1} and S_{v_2} are disjoint, meaning that element i cannot belong to both of them. The second equation follows from symmetry.

Consider now merging one by one the vertices of $G(n, m, p)$ into one supervertex. Let w_j denote a supervertex of j simple vertices that form an independent set. Note that the probabilistic behaviour of w_j is not affected by the way the merging is done. If w_{j_1}, w_{j_2} are two supervertices representing two disjoint sets of simple vertices, we say that an edge (w_{j_1}, w_{j_2}) exists iff any edge connecting a simple vertex in w_{j_1} and a simple vertex in w_{j_2} exists. Thus, the event $\{\nexists(w_{j_1}, w_{j_2})\}$ is equivalent to the event {the vertices in w_{j_1} together with those in w_{j_2} form an independent set}.

Using equation (2) we can show that $P\{i \in S_{w_2}\} = \frac{2p_i}{1+p}$, $P\{i \in S_{w_3}\} = \frac{3p}{1+2p}$, and by induction

$$P\{i \in S_{w_j}\} = \frac{jp}{1 + (j-1)p} \quad (3)$$

where i is a fixed element of $\mathcal{M}$ and S_{w_j} is the union of all the sets of elements of $\mathcal{M}$ assigned to each simple vertex in w_j, i.e., $S_{w_j} = \bigcup_{v \in w_j} S_v$, where v is a simple vertex and S_v is the set of elements of $\mathcal{M}$ assigned to v. Because of the definition of w_j, the subsets S_v in the above union are disjoint.

Thus, let V_1' be any set of k (simple) vertices and let V_2' be an independent set of k vertices that has s vertices in common with V_1'. Since there is no edge between any vertices in V_2', we can treat the $k-s$ vertices of V_2' not belonging to V_1' and the s vertices belonging to both V_1' and V_2' as two seperate supervertices w_{k-s} and w_s respectively that do not communicate by an edge. Hence, by equations (1), (2) and (3), the probabilistic behaviour of w_s is identical to that of a single vertex w_s' that chooses each element of $\mathcal{M}$ independently with probability $p^{(w_s')}$, given by

$$p^{(w_s')} = \frac{p^{(w_s)}(1 - p^{(w_{k-s})})}{1 - p^{(w_s)}p^{(w_{k-s})}} = \frac{sp}{1 + (k-1)p}. \quad (4)$$

Let now V'' be a set of $k - s$ simple vertices and a vertex identical to w_s'. Then, for a fixed element i of $\mathcal{M}$, each of the $k - s$ simple vertices chooses i independently with probability p, while the supervertex w_s' chooses i independently with probability $p^{(w_s')}$. Therefore, the probability that V_1' is an independent set, given that V_2' is an independent set is the probability

that there is no edge between the simple vertices in $V_1' \backslash V_2'$ and the vertex w_s'. In particular, this is equal to the probability that these vertices have not selected any elements of $\mathcal{M}$ in common. But this is exactly equal to $\left((1-p)^{k-s} + (k-s)p(1-p)^{k-s-1}\left(1 - \frac{sp}{1+(s-1)p}\right)\right)^m$ as stated in the Lemma.

We are now ready to present our theorem on coloring almost all vertices of $G_{n,m,p}$.

Theorem 1 *When $m = n^\alpha, \alpha < 1$ and $mp \leq \beta \ln n$, for any constant $\beta < 1-\alpha$. Then a random instance of the random intersection graphs model $\mathcal{G}_{n,m,p}$ contains a subset of at least $n - o(n)$ vertices that can be coloured using np colours, with probability at least $1 - e^{-n^{0.99}}$.*

Proof In what follows, we will denote by $G_{n,m,p}$ an instance of the random intersection graphs model $\mathcal{G}_{n,m,p}$. We also denote by $B_{n,m,p}$ the bipartite graph associated to $G_{n,m,p}$. We prove a slightly stronger property than what the lemma requires.

Assume an arbitrary ordering of the vertices $v_1, v_2, \ldots, v_n$. For $i = 1, 2, \ldots, n$, let B_i be the subgraph of $B_{n,m,p}$ induced by $\cup_{j=1}^{i} v_j \bigcup \mathcal{M}$. We denote by H_i the intersection graph whose bipartite graph has vertex set $V \bigcup \mathcal{M}$ and edge set that is exactly as B_i between $\cup_{j=1}^{i} v_j$ and $\mathcal{M}$, whereas every other edge (i.e. the ones between $\cup_{j=i}^{n} v_j$ and $\mathcal{M}$) appears independently with probability p.

Set $x = np$. Let X denote the size of the largest x-colourable subset of vertices in $G_{n,m,p}$ and let X_i denote the expectation of the largest x-colourable subset in H_i. Notice that X_i is a random variable depending on the overlap between $G_{n,m,p}$ and H_i. Obviously, $X = X_n$ and setting $X_0 = E[X]$, we have $|X_i - X_{i+1}| \leq 1$, for all $i = 1, 2, \ldots, n$. It is straightforward to verify that the sequence $X_0, X_1, \ldots, X_n$ is a Doob Martingale (see also Chapter 9 of [22]). Hence, by applying Azuma's inequality, we have that

$$\Pr(|X - E[X]| \geq t) \leq 2e^{-\frac{t^2}{2n}}.$$

Set now $k_0 = \frac{(1-\epsilon^2)n}{x}$, where ϵ is a positive constant that is arbitrarily close to 0. For $t = \epsilon\frac{k_0 x}{1+\epsilon} = \epsilon(1-\epsilon)n$, Azuma's inequality becomes

$$\Pr\left(|X - E[X]| \geq \epsilon(1-\epsilon)n\right) \leq 2e^{-\frac{\epsilon^2 n}{3}}. \tag{5}$$

Let now Y denote the number of x-colourable subsets of $(1+\epsilon)\frac{xk_0}{1+\epsilon}$ vertices in $G_{n,m,p}$, that can be split in exactly x independent sets (i.e. chromatic classes) of size exactly k_0. We can now verify that, proving that $\Pr(Y > 0)$ is greater or equal to the right hand side of inequality (5), i.e. $2e^{-\frac{\epsilon^2 n}{3}}$, then we will have proven that (a) $E[X] \geq \frac{xk_0}{1+\epsilon}$ and (b) that the values of X are concentrated around something greater than $\frac{xk_0}{1+\epsilon}$ with high probability. More specifically, (a) comes from the observation that the event $\{Y > 0\}$ implies the event $\{X \geq xk_0\}$, hence $\Pr(Y > 0) \leq \Pr(X \geq xk_0) = \Pr\left(X - \frac{xk_0}{1+\epsilon} \geq \frac{\epsilon xk_0}{1+\epsilon}\right) =$

$\Pr\left(X - \frac{xk_0}{1+\epsilon} \geq \epsilon(1-\epsilon)n\right)$. If now $E[X]$ was strictly less than $\frac{xk_0}{1+\epsilon}$, then this would mean that $\Pr(Y>0) < \Pr\left(X - E[X] \geq \epsilon(1-\epsilon)n\right)$ which by (5) is less than $2e^{-\frac{\epsilon^2 n}{3}}$. Hence, proving that $\Pr(Y>0) \geq 2e^{-\frac{\epsilon^2 n}{3}}$ could only mean that $E[X] \geq \frac{xk_0}{1+\epsilon}$. Part (b) then follows as well.

The remarks (a) and (b) described above are sufficient to prove the theorem, since ϵ can be as small as possible. Since Y is a nonnegative random variable that takes only integral values, in order to bound $\Pr(Y>0)$, we will use the well known inequality (see also exercise 1 of Chapter 4 in [23])

$$\Pr(Y>0) \geq \frac{E^2[Y]}{E[Y^2]}.$$

Since every colour class considered in Y must have exactly k_0 vertices and obviously different colour classes must not overlap, we get that

$$E[Y] = \prod_{i=1}^{x} \binom{n-(i-1)k_0}{k_0} \left((1-p)^{k_0} + k_0 p(1-p)^{k_0-1}\right)^m$$

where the term $\left((1-p)^{k_0} + k_0 p(1-p)^{k_0-1}\right)^m \stackrel{def}{=} p_1$ is the probability that a colour class is indeed an independent set, that is no two vertices in it have a common label. Similarly, we have that

$$E[Y^2] \leq E[Y] \sum_{k_1,\ldots,k_x \leq k_0} \prod_{i=1}^{x} \binom{k_0}{k_i}\binom{n-ik_0}{k_0-k_i} p_2$$

where p_2 is the conditional probability that a colour class of k_0 vertices is an independent set, given that k_i of them are already an independent set. By Lemma 1, we have that

$$p_2 \stackrel{def}{=} \left((1-p)^{k_0-k_i} + (k_0-k_i)p(1-p)^{k_0-k_i-1}\left(1 - \frac{k_i p}{1+(k_i-1)p}\right)\right)^m.$$

Combining the above, we conclude that

$$\begin{aligned}\frac{1}{\Pr(Y>0)} \leq \frac{E[Y^2]}{E^2[Y]} &\leq \sum_{k_1,\ldots,k_x \leq k_0} \prod_{i=1}^{x} \frac{\frac{k_0!}{k_i!(k_0-k_i)!}\frac{(n-ik_0)!}{(k_0-k_i)!(n-(i+1)k_0+k_i)!}}{\frac{(n-(i-1)k_0)!}{k_0!(n-ik_0)!}} \frac{p_2}{p_1} \\ &\leq \sum_{k_1,\ldots,k_x \leq k_0} \prod_{i=1}^{x} \frac{\left(\frac{k_0!}{(k_0-k_i)!}\right)^2}{k_i!(n-ik_0)^{k_i}} \frac{p_2}{p_1} \qquad (6)\end{aligned}$$

The fraction $\frac{p_2}{p_1}$ can be bounded in a quite straightforward manner as follows

$$\begin{aligned}\sqrt[m]{\frac{p_2}{p_1}} &\leq \frac{(1-p)^{k_0-k_i} + (k_0-k_i)p(1-p)^{k_0-k_i-1}}{(1-p)^{k_0} + k_0 p(1-p)^{k_0-1}} \\ &= \frac{1-p+(k_0-k_i)p}{1-p+k_0 p}(1-p)^{-k_i} = \left(1 - \frac{k_i p}{1-p+k_0 p}\right)(1-p)^{-k_i} \\ &\leq e^{-\frac{k_i p}{1-p+k_0 p} + k_i p} = e^{\frac{k_0 k_i p^2 - k_i p^2}{1-p+k_0 p}} \leq e^{k_0 k_i p^2}\end{aligned}$$

where the last inequality follows since $k_0 \to \infty$ for $mp = O(\ln n)$ and $m = n^\alpha, \alpha < 1$.

For $i = 1, \ldots, x$, let $A_i \stackrel{def}{=} \frac{\left(\frac{k_0!}{(k_0-k_i)!}\right)^2}{k_i!(n-ik_0)^{k_i}} \frac{p_2}{p_1}$, so that $\frac{E[Y^2]}{E^2[Y]} \leq \sum_{k_1,\ldots,k_x \leq k_0} \prod_{i=1}^{x} A_i$. When $k_i = 0$, then trivially $A_i = 1$. On the other hand, when $1 \leq k_i \leq k_0$, using the inequalities $\frac{k_0!}{(k_0-k_i)!} \leq k_0^{k_i}, k_i! \geq \left(\frac{k_i}{e}\right)^{k_i}$ and the fact that $xk_0 = (1-\epsilon^2)n$, we can see that

$$A_i \leq \frac{k_0^{2k_i}}{k_i^{k_i}(n)^{k_i}} e^{mk_0k_ip^2} = e^{2k_i \ln k_0 - k_i \ln k_i - k_i \ln n + mk_0k_ip^2 + O(k_i \ln \ln n)} \qquad (7)$$

We now distinguish two cases.

(a) $1 \leq k_i \leq \frac{k_0}{\ln^2 n}$. Then $A_i \leq e^{2k_i \ln n + mk_0k_ip^2} \leq e^{k_i(2\ln n + mp)} = e^{O\left(\frac{k_0}{\ln n}\right)}$, since $mp = O(\ln n)$.

(b) $\frac{k_0}{\ln^2 n} < k_i \leq k_0$. Then $A_i \leq e^{\alpha k_i \ln n - k_i \ln n + mk_0k_ip^2 + O(k_i \ln \ln n)} \leq e^{(\alpha - 1 + \beta)k_i \ln n + O(k_i \ln \ln n)} = o(1)$, since $\beta < 1 - \alpha$. We should also mention that the $O(\cdot)$ part of the expontent is different than the $O(\cdot)$ part of the exponent in (7).

The crucial observation now is that, for all values of k_i, $A_i^x \leq e^{O\left(\frac{n}{\ln n}\right)}$. As a final note, the total number of terms in the sum $\sum_{k_1,\ldots,k_x \leq k_0}$ is $(k_0+1)^x = e^{x \ln (k_0+1)} \leq e^{n^{1-\alpha} \ln^2 n}$.

By (6), we then have that

$$\Pr(Y > 0) \geq e^{-n^{1-\alpha} \ln^2 n - O\left(\frac{n}{\ln n}\right)} \geq 2e^{-\frac{\epsilon^2 n}{3}}$$

which concludes the proof.

It is worth noting here that the proof of Theorem 1 can also be used similarly to prove that $\Theta(np)$ colours are enough to colour $n - o(n)$ vertices even in the case where $mp = \beta \ln n$, for any constant $\beta > 0$. However, finding the exact constant multiplying np is technically more difficult.

3.2 A Polynomial Time Algorithm for the Case $mp \geq \ln^2 n$

In the following algorithm every vertex chooses i.u.a.r (independently, uniformly at random) a preference in colours, denoted by $shade(\cdot)$ and every label l chooses a preference in the colours of the vertices in L_l, denoted by $c_l(\cdot)$.

Algorithm CliqueColour:
Input: An instance $G_{n,m,p}$ of $\mathcal{G}_{n,m,p}$ and its associated bipartite $B_{n,m,p}$.
Output: A proper colouring $G_{n,m,p}$.

1. for every $v \in V$ choose a colour denoted by $shade(v)$ independently, uniformly at random among those in $\mathcal{C}$;

2. for every $l \in \mathcal{M}$ choose a colouring of the vertices in L_l such that for every colour in $\{c \in \mathcal{C} : \exists v \in L_l \text{ with } shade(v) = c\}$ there is exactly one vertex in the set $\{u \in L_l : shade(u) = c\}$ having $c_l(u) = c$ while the rest remain uncoloured;
3. set $U = \emptyset$ and $C = \emptyset$;
4. **for** $l = 1$ **to** m **do** {
5. colour every vertex in $L_l \backslash U \cup C$ according to $c_l(\cdot)$ iff there is no collision with the colour of a vertex in $L_l \cap C$;
6. include every vertex in L_l coloured that way in C and the rest in U; }
7. let $\mathcal{H}$ denote the (intersection) subgraph of $G_{n,m,p}$ induced by the vertices in U;
8. give a proper colouring of $\mathcal{H}$ using a new set of colours $\mathcal{C}'$;
9. **output** a colouring of $G_{n,m,p}$ using $|\mathcal{C} \cup \mathcal{C}'|$ colours;

The following result concerns the correctness of Algorithm CliqueColor.

Theorem 2 (Correctness) *Given an instance $G_{n,m,p}$ of the random intersection graphs model, algorithm CliqueColour always finds a proper colouring.*

Proof For the sake of contradiction, suppose that in the colouring proposed by the algorithm there are two vertices v_1 and v_2 that are connected and have been assigned to the same colour c. This of course means that these two vertices have at least one label in common. Since the sets $\mathcal{C}$ and $\mathcal{C}'$ are disjoint and the colouring of $\mathcal{H}$ provided at step 8 of the algorithm is proper, the only way that such a collision would arise is if both v_1 and v_2 belong to C. This means that both were coloured by the first pass of the algorithm and also $shade(v_1) = shade(v_2) = c$. Let l be the smallest indexed label in $|S_{v_1} \cap S_{v_2}|$. It is easy to see then that we come to a contradiction, as label l and step 5 will guarantee that at least one of the two vertices lies in U.

The following theorem concerns the efficiency of algorithm CliqueColour, provided that $mp \geq \ln^2 n$ and $p = o\left(\frac{1}{\sqrt{m}}\right)$. Notice that for p larger than $\frac{1}{\sqrt{m}}$, every instance of the random intersection graphs model $\mathcal{G}_{n,m,p}$, with $m = n^\alpha, \alpha < 1$, is complete whp.

Theorem 3 (Efficiency) *Algorithm CliqueColour succeeds in finding a proper $\Theta\left(\frac{nmp^2}{\ln n}\right)$-colouring of $G_{n,m,p}$ in polynomial time whp, provided that $mp \geq \ln^2 n, p = o\left(\frac{1}{\sqrt{m}}\right)$ and $m = n^\alpha, \alpha < 1$.*

Proof For $s \in \mathcal{C}$, let Z_c denote the number of vertices $v \in V$ such that $shade(v) = c$. Z_c is a binomial random variable, so by Chernoff bounds we can see that, for any positive constant β_1 that can be arbitrarily small

$$\Pr\left(\left|Z_c - \frac{n}{|\mathcal{C}|}\right| \geq \frac{\beta_1 n}{|\mathcal{C}|}\right) \leq 2e^{-\frac{\beta^2 n}{3|\mathcal{C}|}}.$$

For $|\mathcal{C}| = \Theta\left(\frac{mnp^2}{\ln n}\right)$ and $p = o\left(\frac{1}{\sqrt{m}}\right)$, we can then use Boole's inequality to see that there is no $c \in \mathcal{C}$ such that $\left|Z_c - \frac{n}{|\mathcal{C}|}\right| \geq \frac{\beta_1 n}{|\mathcal{C}|}$, with probability $1-o(1)$, i.e. almost surely.

Using the same type of arguments, we can also verify that for arbitrarily small positive constants β_2 and β_3, we have that $\Pr(\exists v \in V : ||S_v| - mp| \geq \beta_2 mp) = o(1)$ and $\Pr(\exists l \in \mathcal{M} : ||L_l| - np| \geq \beta_3 np) = o(1)$, for all $mp = \omega(\ln n)$ and $m = n^\alpha, \alpha < 1$.

We will now prove that the maximum degree of the graph $\mathcal{H}$ is small enough to allow a proper colouring of $\mathcal{H}$ using $\mathcal{C}' = \Theta\left(\frac{nmp^2}{\ln n}\right)$ colours. For a label $l \in \mathcal{M}$ let Y_l denote the number of vertices $v \in L_l$ such that $c_l(v) \neq shade(v)$. In order for a label l not to be able to assign colour $shade(v)$ to $v \in L_l$, it should be the case that it has assigned colour $shade(v)$ to another vertex $u \in L_l$ with $shade(u) = shade(v)$. Hence, the only way to have a collision is when two or more vertices with the same shade have all chosen label l. Notice also that in order to have $Y_l \geq k$, the number of different shades appearing among the vertices that have chosen label l should be at most $|L_l| - k$. This means that $\Pr(Y_l \geq k) \leq \binom{|L_l|}{k}\left(\frac{|L_l|-k}{|\mathcal{C}|}\right)^k$. Given the concentration bound for $|L_l|$, we have that

$$\Pr(\exists l : Y_l \geq k) \leq m\binom{(1+\beta_3)np}{k}\left(\frac{(1+\beta_3)np-k}{|\mathcal{C}|}\right)^k + o(1) \leq m\left(\frac{3np}{k}\right)^k\left(\frac{2np}{|\mathcal{C}|}\right)^k + o(1).$$

By now setting $k = \frac{np}{\ln n}$ and for $|\mathcal{C}| \geq 18\frac{mnp^2}{\ln n}$ we then have that, with probability $1 - o(1)$, there is no label $l \in \mathcal{M}$ such that $Y_l \geq \frac{np}{\ln n}$.

For a label $l \in \mathcal{M}$ now let W_l be the number of vertices $v \in L_l$ such that $shade(v) = c_l(v)$ but they remained uncoloured, hence included in $\mathcal{H}$. In order for a vertex $v \in L_l$ to be counted in W_l, there should exist a label j prior to l (i.e. a label among $1, \ldots l-1$) such that $v \in L_j$ and there is another vertex $u \in L_j$ with $shade(u) = shade(v)$. The probability that this happens is at most $p\left(1-(1-p)^{Z_{shade(v)}}\right)\left(1+(1-p)+(1-p)^2+\cdots\right) = 1-(1-p)^{Z_{shade(v)}}$. The crucial observation now is that, because choices of labels by vertices (of the same shade or not) is done independently and because the vertices counted in W_l have (by definition of the colouring $c_l(\cdot)$ in step 2 of the algorithm) different shades, the inclusion in W_l of any vertex $u \in L_l$ with $shade(u) = c_l(u)$ does not affect the inclusion of another $v \in L_l \backslash \{u\}$ with $shade(v) = c_l(v)$. Hence, taking also into account the concentration bound for $Z_{shade(v)}$ and $|L_l|$, we have that

$$\Pr(\exists l : W_l \geq k') \leq m\binom{(1+\beta_3)np}{k'}\left(1-(1-p)^{(1+\beta_1)\frac{n}{|\mathcal{C}|}}\right)^{k'} + o(1).$$

By now setting $k' = \frac{np}{\ln n}$ and using the relation $(1-x)^y \sim 1-xy$, valid for all x, y such that $xy = o(1)$, we have that when $|\mathcal{C}| \geq 18\frac{mnp^2}{\ln n}$, there is no label l such that $W_l \geq \frac{np}{\ln n}$, with high probability.

We have then proved that the number of vertices in U of the algorithm that have chosen a specific label is with high probability at most $\frac{2np}{\ln n}$. Since, for any vertex v in $G_{n,m,p}$ has $|S_v| \leq (1+\beta_2)mp$, we conclude that the maximum degree in $\mathcal{H}$ satisfies $\max_{v\in\mathcal{H}} degree_{\mathcal{H}}(v) \leq (1+\beta_2)mp\frac{2np}{\ln n}$. It is then evident that we can colour $\mathcal{H}$ greedily, in polynomial time, using $\frac{2.1nmp^2}{\ln n}$ more colours, with high probability. Hence, we can colour $G_{n,m,p}$ in polynomial time, using at most $\frac{20.1nmp^2}{\ln n}$ colours in total.

It is worth noting here that the number of colours used by the algorithm in the case $mp \geq \ln^2 n, p = O\left(\frac{1}{\sqrt[4]{m}}\right)$ and $m = n^\alpha, \alpha < 1$ is of the correct order of magnitude. Indeed, by the concentration of the values of $|S_v|$ around mp for any vertex v with high probability, one can use the results of [24] for the uniform random intersection graphs model $G_{n,m,\lambda}$, with $\lambda \sim mp$ to provide a lower bound on the chromatic number. Indeed, it can be easily verified that the independence number of $G_{n,m,\lambda}$, for $\lambda = mp \geq \ln^2 n$ is at most $\Theta\left(\frac{\ln n}{mp^2}\right)$, which implies that the chromatic number of $G_{n,m,\lambda}$ (and hence of the $G_{n,m,p}$ because of the concentration of the values of $|S_v|$) is at least $\Omega\left(\frac{nmp^2}{\ln n}\right)$.

3.3 Colouring Random Hypergraphs

The model of random intersection graphs $\mathcal{G}_{n,m,p}$ could also be though of as generating random Hypergraphs. The Hypergraphs generated have vertex set V and edge set $\mathcal{M}$. There is a huge amount of literature concerning colouring hypergraphs. However, the question about colouring there seems to be different from the one we answer in this paper. More specifically, a proper colouring of a hypergraph seems to be any assignment of colours to the vertices, so that no monochromatic edge exists. This of course implies that fewer colours than the the chromatic number (studied in this paper) are needed in order to achieve this goal.

We would also like to mention that as far as $\mathcal{G}_{n,m,p}$ is concerned, the problem of finding a colouring such that no label is monochromatic seems to be quite easier when p is not too small.

Theorem 4 *Let $G_{n,m,p}$ be a random instance of the model $\mathcal{G}_{n,m,p}$, for $p = \omega(\frac{\ln m}{n})$ and $m = n^\alpha$, for any fixed $\alpha > 0$. Then with high probability, there is a polynomial time algorithm that finds a k-colouring of the vertices such that no label is monochromatic, for any fixed integer $k \geq 2$.*

Proof By Chernoff bounds and Boole's inequality we can easily show that, for any constant $\epsilon > 0$ that can be arbitrarily small

$$\Pr(\exists l : ||L_l| - np| \geq \epsilon np) \leq 2me^{-\frac{\epsilon^2 np}{3}} \to 0$$

for any $p = \omega(\frac{\ln m}{n})$.

If we were to choose the colour of each vertex independently, uniformly at random among the available colours, then the mean number of monochromatic edges in $G_{n,m,p}$, would be almost surely (given that the above concentration bound holds)

$$E[\#\text{ monochromatic edges}] = \sum_{l \in \mathcal{M}} k^{1-|L_l|} \leq mk^{1-(1-\epsilon)np} < 1.$$

Then, using the method of conditional expectations (see [25,26]) we can derive an algorithm that finds the desired colouring in time $O(mn^{k+2})$. Indeed, since $E[\#\text{ monochromatic edges}] < 1$, there must be a vertex v and a colour c, such that colouring v with c guarantees that $E[\#\text{ monochromatic edges}|colour(v) = c] < 1$. This, combined with the fact that, given any colouring $\mathcal{C}_S$ of any subset S of vertices, we can compute $E[\#\text{ monochromatic edges}|\mathcal{C}_S]$ in time $O(nm)$, leads to the desired algorithm.

4 Other Combinatorial Problems in RIGs

We conclude this paper by briefly mentioning some works related to the design and average case analysis of efficient approximation algorithms on RIGs for various combinatorial problems. Some of these results, as well as the techniques used for the analysis, highlight and take advantage of the intricacies and special structure of random intersection graphs, while others are adapted from the field of Erdős-Rényi random graphs. For further results on various models of random intersection graphs, we refer the reader to the recent review paper [27].

4.1 Independent Sets

The problem of the existence and efficient construction of large independent sets in general random intersection graphs is considered in [13]. Concerning existence, exact formulae are derived for the expectation and variance of the number of independent sets of any size, by using a *vertex contraction technique*. This technique involves the characterization of the statistical behavior of an independent set of any size and highlights an *asymmetry* in the edge appearance rule of random intersection graphs. In particular, it is shown that the probability that any fixed label i is chosen by some vertex in a k-size S with no edges is exactly $\frac{kp_i}{1+(k-1)p_i}$. On the other hand, there is no closed formula for the respective probability when there is at least one edge between the k vertices (or even when the set S is complete)! The special structure of random intersection graphs is also used in the design of efficient algorithms for constructing quite large independent sets in uniform random intersection graphs. By analysis, it is proved that the approximation guarantees of algorithms using the label representation of random intersection graphs are superior to that of well known greedy algorithms for independent sets when applied to instances of $\mathcal{G}_{n,m,p}$.

4.2 Hamilton Cycles

In [28], the authors investigate the existence and efficient construction of *Hamilton cycles* in uniform random intersection graphs. In particular, for the case $m = n^\alpha, \alpha > 1$ the authors first prove a general result that allows one to apply (with the same probability of success) any algorithm that finds a Hamilton cycle with high probability in a $G_{n,M}$ random graph (i.e. a graph chosen equiprobably form the space of all graphs with M edges). The proof is done by using a simple coupling argument. A more complex coupling was given in [29], resulting in a more accurate characterization of the threshold function for Hamiltonicity in $G_{n,m,p}$ for the whole range of values of α. From an algorithmic perspective, the authors in [28] provide an expected polynomial time algorithm for the case where $m = O\left(\sqrt{\frac{n}{\ln n}}\right)$ and p is constant. For the more general case where $m = o\left(\frac{n}{\ln n}\right)$ they propose a *label exposure* greedy algorithm that succeeds in finding a Hamilton cycle in $G_{n,m,p}$ with high probability, even when the probability of label selection is just above the connectivity threshold.

4.3 Maximum Cliques

In [30], the authors consider maximum cliques in the uniform random intersection graphs model $\mathcal{G}_{n,m,p}$. It is proved that, when the number of labels is not too large, we can use the label choices of the vertices to find a maximum clique in polynomial time (in the number of labels m and vertices n of the graph). Most of the analytical work in the paper is devoted in proving the *Single Label Clique Theorem*. Its proof includes a coupling to a graph model where edges appear independently and in which we can bound the size of the maximum clique by well known probabilistic techniques. The theorem states that when the number of labels is less than the number of vertices, any large enough clique in a random instance of $\mathcal{G}_{n,m,p}$ is formed by a single label. This statement may seem obvious when p is small, but it is hard to imagine that it still holds for *all* "interesting" values for p. Indeed, when $p = o\left(\sqrt{\frac{1}{nm}}\right)$, by slightly modifying an argument of [19], one can see that $G_{n,m,p}$ almost surely has no cycle of size $k \geq 3$ whose edges are formed by k distinct labels (alternatively, the intersection graph produced by reversing the roles of labels and vertices is a tree). On the other hand, for larger p a random instance of $\mathcal{G}_{n,m,p}$ is far from perfect[2] and the techniques of [19] do not apply. By using the Single Label Clique Theorem, a tight bound on the clique number of $G_{n,m,p}$ is proved, in the case where $m = n^\alpha, \alpha < 1$. A lower bound in the special case where mp^2 is constant, was given in [12]. We considerably broaden this range of values to also include vanishing values for mp^2 and also provide an asymptotically tight upper bound.

[2] A *perfect graph* is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph. Consequently, the clique number of a perfect graph is equal to its chromatic number.

Finally, as yet another consequence of the Single Label Clique Theorem, the authors in [30] prove that the problem of inferring the complete information of label choices for each vertex from the resulting random intersection graph is *solvable* whp; namely, the maximum likelihood estimation method will provide a unique solution (up to permutations of the labels).[3] In particular, given values m, n and p, such that $m = n^\alpha, 0 < \alpha < 1$, and given a random instance of the $\mathcal{G}_{n,m,p}$ model, the label choices for each vertex are uniquely defined.

4.4 Expansion and Random Walks

The edge expansion and the cover time of uniform random intersection graphs is investigated in [31]. In particular, by using first moment arguments, the authors first prove that $G_{n,m,p}$ is an expander whp when the number of labels is less than the number of vertices, even when p is just above the connectivity threshold (i.e. $p = (1+o(1))\tau_c$, where τ_c is the connectivity threshold). Second, the authors show that random walks on the vertices of random intersection graphs are whp *rapidly mixing* (in particular, the mixing time is logarithmic on n). The proof is based on upper bounding the second eigenvalue of the random walk on $G_{n,m,p}$ through coupling of the original Markov Chain describing the random walk to another Markov Chain on an associated random bipartite graph whose conductance properties are appropriate. Finally, the authors prove that the *cover time* of the random walk on $G_{n,m,p}$, when $m = n^\alpha, \alpha < 1$ and p is at least 5 times the connectivity threshold is $\Theta(n \log n)$, which is optimal up to a constant. The proof is based on a general theorem of Cooper and Frieze [32]; the authors prove that the degree and spectrum requirements of the theorem hold whp in the case of uniform random intersection graphs. The authors also claim that their proof also carries over to the case of smaller values for p, but the technical difficulty for proving the degree requirements of the theorem of [32] increases.

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[3] More precisely, if $\mathcal{B}$ is the set of different label choices that can give rise to a graph G, then the problem of inferring the complete information of label choices from G is *solvable* if there is some $B^* \in \mathcal{B}$ such that $\Pr(B^*|G) > \Pr(B|G)$, for all $\mathcal{B} \ni B \neq B^*$.

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