# Delay-Variation-Dependent Stability of Delayed Discrete-Time Systems

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***Abstract*—This note is concerned with the stability analysis of linear discrete-time system with a time-varying delay. A generalized free-weighting-matrix (GFWM) approach is proposed to estimate summation terms in the forward difference of Lyapunov functional, and theoretical study shows that the GFWM approach encompasses several frequently used estimation approaches as special cases. Moreover, an augmented Lyapunov functional with a delay-product type term is constructed to take into account delay changing information. As a result, the proposed GFWM approach, together with the augmented Lyapunov functional, leads to a less conservative delay-variation-dependent stability criterion. Finally, numerical examples are given to illustrate the advantages of the proposed criterion.**

***Index Terms*—Discrete-time system, time-varying delay, stability, generalized free-weighting-matrix approach**

## I. Introduction

Since discrete-time systems have a strong background in engineering applications and time-varying delays appearing in the system may lead to instability, delay-dependent stability analysis of discrete-time systems with a time-varying delay has received extensive attention [1]–[20]. The objective is to derive an effective stability criterion for determining the admissible maximal delay bound (AMDB), under which the stability of system is ensured. One popular method is in the framework of Lyapunov stability theory and linear matrix inequality (LMI) [16]. Although the criteria derived cannot provide the actual AMDB due to the conservatism arising during the construction of the Lyapunov functionals and the estimation of their forward differences [17], they are easily extended to control design problems and uncertain systems [21]. Therefore, many researches have been done in this framework. In order to find the AMDB more accurate, one important issue is to reduce the conservatism of stability criteria. For this purpose, many scholars are devoted to construct appropriate Lyapunov functionals and to develop effective methods to estimate their forward differences.

For the construction of Lyapunov functionals, those with simple form have been widely employed to investigate the stability and/or stabilization problems [1]–[9]. Furthermore, augmented-based and delay-partition-based functionals have been constructed to improve the criteria [10]–[18]. However, there is room for further investigating the construction of the Lyapunov functionals. As reported in literature [22] and [23], for continuous-time systems, an integral term with time-varying delay in its lower limit is included in the Lyapunov functional, in which the derivative of the time-varying delay brings the delay changing information into the criteria. As a result, the obtained criteria are less conservative than those without considering such information. Unfortunately, for the discrete-time systems, the summation term, similar to the aforementioned integral term, cannot provide the delay changing information, since this information is eliminated by the unavoidable enlargement [3], [5]–[7] (see Section II.A for more details). To the best knowledge of authors, no criterion with the delay changing information has been reported so far. It is expected that the stability criteria of discrete-time systems could be improved when this information, if available, is introduced.

During the estimation of the forward difference of functionals, the main difficulty lies in handling the introduced summation term in the form of $\sum_{s=\beta}^{\alpha-1}\Delta x^T(s)R\Delta x(s)$ (see eq. (8) for the detailed definition), which is the most important factor related to the conservatism [15]–[17]. The techniques frequently used for this task can be briefly classified into two categories, the free-weighting matrix (FWM) based method and the inequality-bounding based method. The former estimates the summation term by adding zero-value terms, including He's FWM approach [8] and Kim's zero-value equality approach [9]. For the latter, the summation term is estimated based on summation inequalities, including the widely used Jensen-based inequality (JBI) [1]–[5], [15] and the recently proposed Wirtinger-based inequality (WBI) [16], [17]. It has been shown that the criteria derived by the FWM approach and the JBI are equivalent [3], [5], while those obtained by the WBI are proved to be less conservative than those by the JBI [16], [17]. The results in [16] show that both the WBI and the Kim's zero-value equality approach still lead to conservative criteria. That is to say, the above four estimation approaches still bring the conservatism in a certain extent. Therefore, the development of a less conservative estimation approach is still a significant research direction.

This note investigates the delay-variation-dependent stability of discrete-time systems with a time-varying delay. Firstly, a novel augmented Lyapunov functional is constructed by introducing a delay-product type term, whose derivative brings the delay changing information into the stability criterion. Secondly, a generalized FWM (GFWM) approach is proposed to estimate the summation term appearing in the forward difference of the Lyapunov functional, and it is proved that the GFWM approach encompasses the aforementioned four frequently used estimation approaches. Moreover, a less conservative stability criterion is obtained based on those two techniques. Finally, the advantage of the proposed criterion is illustrated through two classical examples from the literature.

## II. Problem formulation

Consider a discrete-time system with time-varying delay as

$$\begin{cases} x(k+1)=Ax(k)+A_dx(k-d(k)) \\ x(k)=\phi(k), \quad k=-h_2,-h_2+1,\cdots,0 \end{cases} \tag{1}$$

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where $x(k)=[x_1(k),x_2(k),\cdots,x_n(k)]^T$ is the system state; $A$ and $A_d$ are the system matrices; $\phi(k)$ is the initial condition; and $d(k)$ is the time-varying delay satisfying

$$h_1 \le d(k) \le h_2 \tag{2}$$
$$\mu_1 \le \Delta d(k) = d(k+1)-d(k) \le \mu_2 \tag{3}$$

where nonnegative integers $h_1$ and $h_2$ are the delay bounds, and integers $\mu_1$ and $\mu_2$ are the delay variation bounds.

This note aims to analyze the stability of system (1) and to derive less conservative stability criteria by considering the following two problems.

### *A. About the delay changing information*

The existing results for system (1) only consider the delay constraint (2) but not take into account the delay variation constraint (3). The delay changing information have been considered to reduce the conservatism for continuous-time system [23]. The stability criterion of system (1) may be improved if similar information is introduced. Whether or not such information is included in the criteria is dependent on the Lyapunov functional.

For continuous-time systems with a delay satisfying $d_1 \le d(t) \le d_2$, Lyapunov functionals usually contain an integral term, $\int_{t-d(t)}^{t-d_1} x^T(s)Qx(s)ds,\ Q>0$, which leads to the term $\dot{d}(t)x^T(t-d(t))Qx(t-d(t))$ in the functional derivative. The $\dot{d}(t)$ here is multiplied by a quadratic term, and using the delay variation constraint to handle this term would introduce the delay changing information into the criteria. However the corresponding summation term for system (1) commonly used is [1], [3], [5]–[7]

$$V_q(x_k) = \sum_{s=k-d(k)}^{k-1} x^T(s)Qx(s) \tag{4}$$

where $x_k$ is the sequence of state defined as $x_k(s)=x(k-s),\ \forall s=-h_2,-h_2+1,\cdots,0$, and $Q$ is a positive definite matrix. Its forward difference is estimated as [1], [3], [5]–[7]

$$\Delta V_q(x_k) = x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) + \sum_{s=k-d(k)-\Delta d(k)+1}^{k-d(k)} x^T(s)Qx(s) \tag{5}$$
$$\le x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) + \sum_{s=k-h_2+1}^{k-h_1} x^T(s)Qx(s) \tag{6}$$

The delay changing $\Delta d(k)$ in (5) is commonly estimated by using $\Delta d(k)+d(k)=d(k+1)$ and the delay constraint (2), instead of directly using the delay variation constraint (3), such that the positive summation term in (6) can be eliminated in the subsequent procedure (refer to [3], [5]–[7] for more details). Thus the delay changing information is entirely disappeared in (6) and it is not included by the criteria.

Therefore, *a new form of Lypapunov functional should be developed to take into account delay variation constraint (3)*. This is the first problem to be investigated.

### *B. About the estimation approach for summation term*

In order to obtain delay-dependent stability criteria, the Lyapunov functionals should contain the following term [17]:

$$V_r(x_k) = \sum_{j=\alpha_1}^{\alpha_2-1} \sum_{s=k+j}^{k-1} \Delta x^T(s)R\Delta x(s) \tag{7}$$

where $\alpha_2=0, \alpha_1=-h_1$ (or $\alpha_2=-h_1, \alpha_1=-h_2$), $\Delta x(k) = x(k+1)-x(k)$, and $R$ is a positive definite matrix. Computing its forward difference leads to

$$\Delta V_r(x_k) = (\alpha_2-\alpha_1)\Delta x^T(k)R\Delta x(k) - \sum_{s=k+\alpha_1}^{k+\alpha_2-1} \Delta x^T(s)R\Delta x(s)$$

As mentioned in [17], how to estimate the summation term in $\Delta V_r(x_k)$ for finding its upper bound is the most important issue during the deriving of the criterion. For simplifying description, let

$$\varphi(\alpha,\beta) = -\sum_{s=\beta}^{\alpha-1} \Delta x^T(s)R\Delta x(s) = -\sum_{s=k+\alpha_1}^{k+\alpha_2-1} \Delta x^T(s)R\Delta x(s) \tag{8}$$

So far, four approaches have been developed and commonly used to estimate the summation term $\varphi(\alpha,\beta)$ and the basic ideas of them are briefly given as follows:

*(1) He's FWM approach [8]:* the following Newton-Leibniz formula based zero-value term is introduced:

$$\varphi_1 = 2f^T(k)M\Big[x(\alpha)-x(\beta)-\sum_{s=\beta}^{\alpha-1}\Delta x(s)\Big] = 0 \tag{9}$$

where $f(k)$ is a suitable vector and $M$ is any matrix. Then $\varphi(\alpha,\beta)$ is estimated as

$$\begin{aligned}\varphi(\alpha,\beta) &= -\sum_{s=\beta}^{\alpha-1} \Delta x^T(s)R\Delta x(s) + \varphi_1\\ &= 2f^T(k)M(x(\alpha)-x(\beta)) + (\alpha-\beta)f^T(k)Xf(k)\\ &\quad - \sum_{s=\beta}^{\alpha-1} \begin{bmatrix} f(k)\\ \Delta x(s)\end{bmatrix}^T \begin{bmatrix} X & M\\ M^T & R\end{bmatrix} \begin{bmatrix} f(k)\\ \Delta x(s)\end{bmatrix}\end{aligned} \tag{10}$$

where $X$ is a symmetrical matrix.

*(2) Kim's zero-value equality approach [9]:* the following zero-value term is introduced

$$\begin{aligned}\varphi_2 = &-\sum_{s=\beta}^{\alpha-1} \Delta x^T(s)T\Delta x(s) - 2\sum_{s=\beta}^{\alpha-1} x^T(s)T\Delta x(s)\\ &+ x^T(\alpha)Tx(\alpha) - x^T(\beta)Tx(\beta) = 0\end{aligned} \tag{11}$$

where $T$ is a symmetric matrix. Then $\varphi(\alpha,\beta)$ is estimated as

$$\begin{aligned}\varphi(\alpha,\beta) &= -\sum_{s=\beta}^{\alpha-1} \Delta x^T(s)R\Delta x(s) + \varphi_2\\ &= x^T(\alpha)Tx(\alpha) - x^T(\beta)Tx(\beta)\\ &\quad - \sum_{s=\beta}^{\alpha-1} \begin{bmatrix} x(s)\\ \Delta x(s)\end{bmatrix}^T \begin{bmatrix} 0 & T\\ T^T & T+R\end{bmatrix} \begin{bmatrix} x(s)\\ \Delta x(s)\end{bmatrix}\end{aligned} \tag{12}$$

*(3) The JBI approach [4]:* $\varphi(\alpha,\beta)$ is directly estimated using the JBI, i.e.,

$$\varphi(\alpha,\beta) \le \frac{-1}{\alpha-\beta}\sum_{s=\beta}^{\alpha-1} \Delta x^T(s)R\sum_{s=\beta}^{\alpha-1}\Delta x(s) = \frac{-\vartheta_1^T R\vartheta_1}{\alpha-\beta} \tag{13}$$

where $\vartheta_1 = x(\alpha)-x(\beta)$.

*(4) The WBIs [16], [17]:* $\varphi(\alpha,\beta)$ is directly estimated using different WBIs, including

- Type I: Lemma 2 in [17]

$$\varphi(\alpha,\beta) \le -\frac{1}{\alpha-\beta}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix}^T \begin{bmatrix} R & 0\\ 0 & 3\Big(\frac{\alpha-\beta+1}{\alpha-\beta-1}\Big)R\end{bmatrix}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix} \tag{14}$$

- Type II: Lemma 3 in [16]

$$\varphi(\alpha,\beta) \le -\frac{1}{\alpha-\beta}\begin{bmatrix}\vartheta_1\\ \vartheta_3\end{bmatrix}^T\begin{bmatrix}R & 0\\ 0 & 3R\end{bmatrix}\begin{bmatrix}\vartheta_1\\ \vartheta_3\end{bmatrix} \tag{15}$$

- Type III: Corollary 3 in [17]

$$\varphi(\alpha,\beta) \le -\frac{1}{\alpha-\beta}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix}^T\begin{bmatrix}R & 0\\ 0 & 3R\end{bmatrix}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix} \tag{16}$$

where

$$\vartheta_2 = x(\alpha)+x(\beta)-\frac{2}{(\alpha-\beta+1)}\sum_{s=\beta}^{\alpha}x(s)$$
$$\vartheta_3 = \frac{1}{\alpha-\beta}\Big[(\alpha-\beta-1)x(\alpha)+(\alpha-\beta+1)x(\beta)-2\sum_{s=\beta}^{\alpha-1}x(s)\Big]$$

The criteria obtained by the He's FWM approach and the JBI approach have the same conservatism [3]. Although the WBI approach, as well the Kim's zero-value equality approach, can be used to derive less conservative criteria [14], [17], the conservatism still exists in those criteria [16]. That is, four aforementioned approaches still bring the conservatism in a certain extent. Therefore, *the investigation of a less conservative estimation approach is a significant issue*. This is the second problem to be investigated.

## III. New techniques for the above two problems

This section develops a delay-product type term and a GFWM approach to solve the aforementioned problems.

### A. A delay-product type term for the first problem

For the first problem, the following delay-product type term is introduced into the Lyapunov functional:

$$V_1(x_k) = d(k)\xi_1^T(k)P_1\xi_1(k) \tag{17}$$

where $\xi_1(k) = \big[x^T(k), \sum_{s=k-h_1}^{k-1}x^T(s)\big]^T$, and $P_1$ is a symmetric matrix. Then its forward difference can be obtained as

$$\begin{aligned}\Delta V_1(x_k) =& \xi_1^T(k+1)\big[\Delta d(k)P_1\big]\xi_1^T(k+1)\\ &+d(k)\big[\xi_1^T(k+1)P_1\xi_1(k+1)-\xi_1^T(k)P_1\xi_1(k)\big]\end{aligned} \tag{18}$$

The delay variation $\Delta d(k)$ in (18) is multiplied with a quadratic term, and it can be directly handled by using the delay variation constraint (3), instead of the delay bounds constraint like the literature does. Thus the delay changing information can be included in the criterion. That is, the delay-product type term (17) succeeds in solving the first problem.

### B. A GFWM approach for the second problem

This part develops a GFWM approach after proposing two new zero-value equalities, and then proves that the GFWM approach encompasses the approaches mentioned previously.

*(1) Two new zero-value equalities:* The zero-value equalities (9) and (11) in the FWM-based methods play an important role in reducing conservatism. Two new zero-value equalities are introduced here. Firstly, the following equation is true

$$\sum_{s=\beta}^{\alpha-1}x(s) = \sum_{s=\beta}^{\alpha}x(s) - x(\alpha)$$

Thus, the following new zero-value equality is obtained:

$$\varphi_3 = 2f^T(k)L\Big[(\alpha-\beta+1)\sigma(\alpha,\beta)-x(\alpha)-\sum_{s=\beta}^{\alpha-1}x(s)\Big] \tag{19}$$

where $L$ is any matrix and $\sigma(\alpha,\beta)=\sum_{s=\beta}^{\alpha}\frac{x(s)}{\alpha-\beta+1}$.

Secondly, the following lemma is recalled to propose the second zero-value equality:

*Lemma 1:* (Abel's transformation [24]) For two sequences $\{a_i\}_{i=n_1}^{n_2}$ and $\{b_i\}_{i=n_1}^{n_2}$ with $n_2>n_1$, the following holds

$$\sum_{i=n_1}^{n_2}a_ib_i = a_{n_2}B_{n_2}-\sum_{i=n_1}^{n_2-1}B_i(a_{i+1}-a_i);\quad B_i=\sum_{s=n_1}^{i}b_s \tag{20}$$

Two sequences in (20) are defined as:

$$\{a_s\}_{s=n_1}^{n_2} = \{\gamma_1+\gamma_2 s\}_{s=\beta}^{\alpha-1},\ \{b_s\}_{s=n_1}^{n_2} = \{\Delta x_i(s)\}_{s=\beta}^{\alpha-1} \tag{21}$$

Using Lemma 1 yields

$$\begin{aligned}&\sum_{s=\beta}^{\alpha-1}(\gamma_1+\gamma_2 s)\Delta x_i(s)\\ &=(\gamma_1+\gamma_2\alpha)x_i(\alpha)-[\gamma_1+\gamma_2(\beta-1)]x_i(\beta)-\gamma_2\sum_{s=\beta}^{\alpha}x_i(s)\end{aligned} \tag{22}$$

Then letting $\gamma_1+\gamma_2\alpha=1$ and $\gamma_1+\gamma_2(\beta-1)=-1$. Thus, the following can be obtained:

$$\sum_{s=\beta}^{\alpha-1}\chi(s)\Delta x(s) = x(\alpha)+x(\beta)-2\sigma(\alpha,\beta) \tag{23}$$

where $\chi(s)=\gamma_1+\gamma_2 s = \frac{-\alpha-\beta+1}{\alpha-\beta+1}+\frac{2}{\alpha-\beta+1}s$. Thus, for any matrix $N$, the following new zero-value equality is obtained:

$$\varphi_4 = 2f^T(k)N\Big[x(\alpha)+x(\beta)-2\sigma(\alpha,\beta)-\sum_{s=\beta}^{\alpha-1}\chi(s)\Delta x(s)\Big] \tag{24}$$

*(2) The GFWM approach:* The sum of $\varphi_1$, $\varphi_2$, $\varphi_3$, and $\varphi_4$, defined in (9), (11), (19), and (24), respectively, leads to the following zero-value equality:

$$\begin{aligned}\phi_1(M,T,L,N,\alpha,\beta) &= \varphi_1+\varphi_2+\varphi_3+\varphi_4\\ &=\Phi_1(M,T,L,N,\alpha,\beta)\\ &\quad-\sum_{s=\beta}^{\alpha-1}\begin{bmatrix}\eta_1(k,s)\\ \eta_2(s)\end{bmatrix}^T\begin{bmatrix}0 & \Theta\\ \Theta^T & \check{T}\end{bmatrix}\begin{bmatrix}\eta_1(k,s)\\ \eta_2(s)\end{bmatrix}=0\end{aligned} \tag{25}$$

where

$$\begin{aligned}\Phi_1(M,T,L,N,\alpha,\beta) =& x^T(\alpha)Tx(\alpha)-x^T(\beta)Tx(\beta)\\ &+2f^T(k)\Big\{M\,[x(\alpha)-x(\beta)]+L\,[(\alpha-\beta+1)\sigma(\alpha,\beta)-x(\alpha)]\\ &+N\,[x(\alpha)+x(\beta)-2\sigma(\alpha,\beta)]\Big\}\end{aligned} \tag{26}$$

$$\eta_1(k,s)=\begin{bmatrix}f(k)\\ \chi(s)f(k)\end{bmatrix},\quad \eta_2(s)=\begin{bmatrix}x(s)\\ \Delta x(s)\end{bmatrix} \tag{27}$$

$$\Theta=\begin{bmatrix}L & M\\ 0 & N\end{bmatrix},\quad \check{T}=\begin{bmatrix}0 & T\\ T & T\end{bmatrix} \tag{28}$$

And some elementary calculus lead that

$$\sum_{s=\beta}^{\alpha-1}\eta_1^T(k,s)\begin{bmatrix}X & Y\\ Y^T & Z\end{bmatrix}\eta_1(k,s)=\bar{\Phi}_2(X,Z,\alpha,\beta)\le\Phi_2(X,Z,\alpha,\beta)$$

where $X$ and $Z$ are positive definite symmetric matrices, and $Y$ is any matrix, and

$$\bar{\Phi}_2(X,Z,\alpha,\beta)=(\alpha-\beta)f^T(k)\Big[X+\frac{(\alpha-\beta-1)}{3(\alpha-\beta+1)}Z\Big]f(k) \tag{29}$$

$$\Phi_2(X,Z,\alpha,\beta)=(\alpha-\beta)f^T(k)\Big[X+\frac{Z}{3}\Big]f(k) \tag{30}$$

Therefore, using the new zero-value equality (25) to estimate the $\varphi(\alpha,\beta)$ leads to

$$\begin{aligned}
&\varphi(\alpha,\beta)\\
&=-\sum_{s=\beta}^{\alpha-1}\Delta x^T(s)R\Delta x(s)+\phi_1(M,T,L,N,\alpha,\beta)\\
&=\Phi_1(M,T,L,N,\alpha,\beta)+\bar{\Phi}_2(X,Z,\alpha,\beta)-\Phi_3(\bar{X},\Theta,\bar{T}) \quad (31)\\
&\leq \Phi_1(M,T,L,N,\alpha,\beta)+\Phi_2(X,Z,\alpha,\beta)-\Phi_3(\bar{X},\Theta,\bar{T}) \quad (32)
\end{aligned}$$

where

$$\Phi_3(\bar{X},\Theta,\bar{T})=\sum_{s=\beta}^{\alpha-1}\eta_3^T(k,s)\begin{bmatrix}\bar{X} & \Theta\\ \Theta^T & \bar{T}\end{bmatrix}\eta_3(k,s) \tag{33}$$

and $\bar{X}=\begin{bmatrix}X & Y\\ Y^T & Z\end{bmatrix}$, $\bar{T}=\begin{bmatrix}0 & T\\ T & T+R\end{bmatrix}$, $\eta_3(k,s)=\begin{bmatrix}\eta_1(k,s)\\ \eta_2(s)\end{bmatrix}$.

The estimation shown in (31) and (32) not only combines two exiting FWM-based zero-value equalities, $\varphi_1$ and $\varphi_2$, but also introduces two new FWM-based zero-value equalities, $\varphi_3$ and $\varphi_4$. Thus, it is named as the GFWM approach.

*(3) The comparison of the GFWM approach with the existing approaches:* Theoretical studies are carried out to show that the GFWM approach encompasses the existing ones.

- The He's FWM approach: letting $T=0$, $L=0$, $N=0$, $Z=0$, and $Y=0$ yields

$$\begin{aligned}
&\Phi_1(M,T,L,N,\alpha,\beta)+\bar{\Phi}_2(X,Z,\alpha,\beta)-\Phi_3(\bar{X},\Theta,\bar{T})\\
&=2f^T(k)M(x(\alpha)-x(\beta))+(\alpha-\beta)f^T(k)Xf(k)\\
&\quad-\sum_{s=\beta}^{\alpha-1}\begin{bmatrix}f(k)\\ \Delta x(s)\end{bmatrix}^T\begin{bmatrix}X & M\\ M^T & R\end{bmatrix}\begin{bmatrix}f(k)\\ \Delta x(s)\end{bmatrix}
\end{aligned}$$

  That is, equality (31) reduces to equality (10). Thus the GFWM approach encompasses the He's FWM approach.
- The Kim's zero-value equality approach: letting $M=0$, $X=0$, $L=0$, $N=0$, $Z=0$, and $Y=0$ yields

$$\begin{aligned}
&\Phi_1(M,T,L,N,\alpha,\beta)+\bar{\Phi}_2(X,Z,\alpha,\beta)-\Phi_3(\bar{X},\Theta,\bar{T})\\
&=x^T(\alpha)Tx(\alpha)-x^T(\beta)Tx(\beta)-\sum_{s=\beta}^{\alpha-1}\begin{bmatrix}x(s)\\ \Delta x(s)\end{bmatrix}^T\bar{T}\begin{bmatrix}x(s)\\ \Delta x(s)\end{bmatrix}
\end{aligned}$$

  That is, equality (31) reduces to equality (12). Thus the GFWM approach encompasses the Kim's zero-value equality approach.
- The JBI approach: letting $f(k)=[x^T(\alpha)\ x^T(\beta)]^T$, $X=MR^{-1}M^T$, $M=\frac{R}{\alpha-\beta}[-I\ I]^T$, $T=0$, $L=0$, $N=0$, $Z=0$, and $Y=0$ and using Schur complement yield

$$\begin{aligned}
&\Phi_1(M,T,L,N,\alpha,\beta)+\bar{\Phi}_2(X,Z,\alpha,\beta)-\Phi_3(\bar{X},\Theta,\bar{T})\\
&=\frac{-\vartheta_1^TR\vartheta_1}{\alpha-\beta}-\sum_{s=\beta}^{\alpha-1}\begin{bmatrix}f(k)\\ \Delta x(s)\end{bmatrix}^T\begin{bmatrix}MR^{-1}M^T & M\\ M^T & R\end{bmatrix}\begin{bmatrix}f(k)\\ \Delta x(s)\end{bmatrix}\\
&\leq-\frac{\vartheta_1^TR\vartheta_1}{\alpha-\beta}
\end{aligned}$$

  That is, equality (31) reduces to inequality (13). Thus, the GFWM approach encompasses the JBI approach. (Note that the summation term in the second line above is non-negative based on the Schur complement.)
- The WBI approach: letting $f(k)=[x^T(\alpha),x^T(\beta),\sigma^T(\alpha,\beta)]^T$, $X=MR^{-1}M^T$, $Y=MR^{-1}N^T$, $Z=NR^{-1}N^T$, $M=\frac{R}{\alpha-\beta}[-I\ I\ 0]^T$, $N=\frac{3(\alpha-\beta+1)R}{(\alpha-\beta)(\alpha-\beta-1)}[-I\ \ -I\ \ 2I]^T$, $L=0$, and $T=0$ considering that $\frac{\alpha-\beta+1}{\alpha-\beta-1}\geq\frac{(\alpha-\beta+1)^2}{(\alpha-\beta)^2}>1$ and $\vartheta_3=\frac{(\alpha-\beta+1)}{(\alpha-\beta)}\vartheta_2$, and using Schur complement yield

$$\begin{aligned}
&\Phi_1(M,T,L,N,\alpha,\beta)+\bar{\Phi}_2(X,Z,\alpha,\beta)-\Phi_3(\bar{X},\Theta,\bar{T})\\
&=-\frac{1}{\alpha-\beta}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix}^T\begin{bmatrix}R & 0\\ 0 & 3\left(\frac{\alpha-\beta+1}{\alpha-\beta-1}\right)R\end{bmatrix}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix}\\
&\quad-\sum_{s=\beta}^{\alpha-1}\begin{bmatrix}f(k)\\ \chi(s)f(k)\\ \Delta x(s)\end{bmatrix}^T\begin{bmatrix}MR^{-1}M^T & MR^{-1}N^T & M\\ NR^{-1}M^T & NR^{-1}N^T & N\\ M^T & N^T & R\end{bmatrix}\begin{bmatrix}f(k)\\ \chi(s)f(k)\\ \Delta x(s)\end{bmatrix}\\
&\leq-\frac{1}{\alpha-\beta}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix}^T\begin{bmatrix}R & 0\\ 0 & 3\left(\frac{\alpha-\beta+1}{\alpha-\beta-1}\right)R\end{bmatrix}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix} \quad (34)\\
&\leq-\frac{1}{\alpha-\beta}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix}^T\begin{bmatrix}R & 0\\ 0 & 3\frac{(\alpha-\beta+1)^2}{(\alpha-\beta)^2}R\end{bmatrix}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix} \quad (35)\\
&=-\frac{1}{\alpha-\beta}\begin{bmatrix}\vartheta_1\\ \vartheta_3\end{bmatrix}^T\begin{bmatrix}R & 0\\ 0 & 3R\end{bmatrix}\begin{bmatrix}\vartheta_1\\ \vartheta_3\end{bmatrix} \quad (36)\\
&\leq-\frac{1}{\alpha-\beta}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix}^T\begin{bmatrix}R & 0\\ 0 & 3R\end{bmatrix}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix} \quad (37)
\end{aligned}$$

  That is, equality (31) reduces to inequality (14), (15), or (16). Thus, the GFWM approach encompasses three different WBI approaches. (Note that the summation term in the third line is non-negative based on the Schur complement.)

Based on the above discussion, the existing approaches can be considered as special cases of the GFWM approach and can be easily obtained by fixing some free-weighting matrices in the GFWM approach. It means that the GFWM approach is a general form of the existing ones and that it can lead to less conservative results since those free-weighting matrices provide more freedom for checking the feasibility of the obtained LMI-based criteria. Therefore, the proposed GFWM approach is a feasible solution of the second problem.

## IV. Stability of linear discrete-time system with time-varying delay

In this section, the proposed techniques are applied to analyze the stability of a linear discrete-time system with a time-varying delay i.e., (1). The following notations are introduced for simplifying the description of subsequent parts:

$$h_{1d}(k)=d(k)-h_1,\quad h_{2d}(k)=h_2-d(k) \tag{38}$$

$$\xi_2(k)=\left[x^T(k),\ \sum_{s=k-h_1}^{k-1}x^T(s),\ \sum_{s=k-h_2}^{k-h_1-1}x^T(s)\right]^T \tag{39}$$

$$v_1(k)=\sum_{s=k-h_1}^{k}\frac{x(s)}{h_1+1},\ v_2(k)=\sum_{s=k-d(k)}^{k-h_1}\frac{x(s)}{h_{1d}(k)+1} \tag{40}$$

$$v_3(k)=\sum_{s=k-h_2}^{k-d(k)}\frac{x(s)}{h_{2d}(k)+1} \tag{41}$$

$$\begin{aligned}\zeta(k)=\big[&x^T(k),\ x^T(k-h_1),\ x^T(k-d(k))\\ &x^T(k-h_2),\ v_1^T(k),\ v_2^T(k),\ v_3^T(k)\big]^T\end{aligned} \tag{42}$$

For system (1), the proposed GFWM approach, together with a Lyapunov functional containing the delay-product term (17), leads to the following asymptotical stability criterion.

*Theorem 1:* For given scalars $h_i$, $\mu_i$, $i = 1,2$, if there exist symmetric matrices $P_i, Q_i, U_i, T_j, X_j, Z_j, i = 1,2, j = 1,2,3$; and any matrices $L_j, M_j, N_j, Y_j, j = 1,2,3$, such that

$$Q_k > 0,\ U_k > 0,\ \Psi_k = P_2 + h_k \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} > 0,\ k = 1,2 \quad (43)$$

$$\Psi_{2+k} = \begin{bmatrix} \bar{X}_k & \Theta_k \\ \Theta_k^T & \bar{T}_k + \bar{U}_k \end{bmatrix} \geq 0,\ k = 1,2,3 \quad (44)$$

$$\Psi(d(k), \Delta d(k))|_{d(k)=h_i, \Delta d(k)=\mu_j} < 0, i=1,2; j=1,2 \quad (45)$$

where the related notations are given at the top of next page, then system (1) with the time-varying delay satisfying (2) and (3) is asymptotically stable.

*Proof:* the above theorem is proved by following three steps.

*Step 1: Constructing a Lyapunov functional.* Define a Lyapunov functional as

$$\begin{aligned} V(x_k) = & V_1(x_k) + \xi_2^T(k) P_2 \xi_2(k) + \sum_{s=k-h_1}^{k-1} x^T(s) Q_1 x(s) \\ & + \sum_{s=k-h_2}^{k-h_1-1} x^T(s) Q_2 x(s) + \sum_{j=-h_1}^{-1} \sum_{s=k+j}^{k-1} \eta_2^T(s) U_1 \eta_2(s) \\ & + \sum_{j=-h_2}^{-h_1-1} \sum_{s=k+j}^{k-1} \eta_2^T(s) U_2 \eta_2(s) \end{aligned} \quad (46)$$

where $P_i$, $Q_i$, and $U_i, i = 1,2$, are any symmetric matrices, $V_1(x_k)$ is defined in (17), $\xi_2(k)$ is defined in (39), and $\eta_2(s)$ is defined in (27).

*Step 2: Proving condition (43) ensures the positive definite and the radially unbounded of the functional, $V(x_k)$.* Based on the convex combination approach [23], the holding of $\Psi_k > 0, k = 1,2$ leads that the following is true

$$P_2 + d(k) \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} > 0 \quad (47)$$

Thus, LMI (43) ensures that $V(x_k) \geq \epsilon ||x(k)||^2$ for a sufficiently small $\epsilon > 0$, which means the functional (46) is positive definite for all $x(k) \neq 0$ and is also radially unbounded.

*Step 3: Proving conditions (44) and (45) ensure the negative of the forward difference of Lyapunov functional, $\Delta V(x_k)$.* Calculating the forward difference of the functional yields

$$\Delta V(x_k) = \zeta^T(k) \left[\Upsilon_1(d(k), \Delta d(k)) + \Upsilon_2\right] \zeta(k) + \Delta V_b(x_k)$$

where $\zeta(k)$ is defined in (42), $\Upsilon_1(d(k), \Delta d(k))$ and $\Upsilon_2$ are defined in (45), and

$$\begin{aligned} \Delta V_b(x_k) = & - \sum_{s=k-h_1}^{k-1} \eta_2^T(s) U_1 \eta_2(s) - \sum_{s=k-d(k)}^{k-h_1-1} \eta_2^T(s) U_2 \eta_2(s) \\ & - \sum_{s=k-h_2}^{k-d(k)-1} \eta_2^T(s) U_2 \eta_2(s) \end{aligned} \quad (48)$$

The proposed GFWM approach is applied to estimate the summation terms in the above equation. The vector $f(k)$ in (25) can be component of some or all vectors in $\zeta(k)$ and here it is assumed to be the following form:

$$f(k) = \left[x^T(k), x^T(k-d(k)), v_1^T(k), v_2^T(k), v_3^T(k)\right]^T \quad (49)$$

Then using equality (31) to estimate the first term of (48) and using inequality (32) to estimate the second and the third terms in (48) yield

$$\begin{aligned} & \Delta V_b(x_k) \quad (50) \\ & \leq \zeta^T(k) \left[\Upsilon_3 + \Upsilon_4(d(k))\right] \zeta(k) - \sum_{s=k-h_1}^{k-1} \eta_3^T(k,s) \Psi_3 \eta_3(k,s) \\ & - \sum_{s=k-d(k)}^{k-h_1-1} \eta_3^T(k,s) \Psi_4 \eta_3(k,s) - \sum_{s=k-h_2}^{k-d(k)-1} \eta_3^T(k,s) \Psi_5 \eta_3(k,s) \end{aligned}$$

where $\Upsilon_3$ and $\Upsilon_4(d(k))$ are defined in (45), and $\Psi_i, i = 3,4,5$ are defined in (44). Based on (44), $\Delta V(x_k)$ is estimated as

$$\Delta V(x_k) \leq \zeta^T(k) \Psi\big(d(k), \Delta d(k)\big) \zeta(k) \quad (51)$$

Since the $d^2(k)$-dependent terms in $\Pi_{3d(k)}^T P_2 \Pi_{3d(k)}$ and $\Pi_{4d(k)}^T P_2 \Pi_{4d(k)}$ are canceled each other, $\Psi\big(d(k), \Delta d(k)\big)$ is a linear function of $d(k)$ and $\Delta d(k)$, i.e., it can be expressed as $\Omega_1 + d(k)\Omega_2 + \Delta d(k)\Omega_3$ with $\Omega_i, i = 1,2,3$ being $k$-independent matrices combination. By using the convex combination method [23] and following the similar analysis in [25], (45) can guarantee the $\Psi\big(d(k), \Delta d(k)\big) < 0$. Thus, LMIs (44) and (45) ensures the negative definite of the $\Delta V(x_k)$.

Based on Steps 2 and 3, system (1) is asymptotically stable when LMIs (43)-(45) hold. This completes the proof. ■

*Corollary 1:* For a system in which the delay changing information is unavailable (i.e., $\mu_i$ is unknown), the criterion for such case can be directly obtained by setting $P_1 = 0$.

## V. Numerical Examples

Based on two numerical examples, the advantage of the proposed criteria, compared with the existing ones, is verified via the comparison of the calculated AMDBs.

*Example 1:* Consider system (1) with the following parameters [17]:

$$A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix},\ A_d = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix} \quad (52)$$

In this example, constraint (3) is not considered for comparing with the literature. The AMDBs of $h_2$ for various $h_1$ calculated by different criteria are listed in Table I. The results show that the results by the proposed criterion are bigger than (or equal to for some cases) the ones reported in the literature, which indicates the less conservatism of the proposed criterion.

TABLE I
The AMDBs of $h_2$ for different $h_1$ and unknown $\mu_i$ (Example 1)

| Methods | $h_1$: 2 | 6 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| [2]–[8], [13], [15] | $<21$ | $<21$ | $\leq 24$ | $\leq 27$ | $\leq 31$ |
| [18] | 21 | 21 | 24 | 27 | 31 |
| [17] | 21 | 21 | 24 | 28 | 32 |
| [16] | 20 | 21 | 25 | 28 | 32 |
| [14] | 22 | 22 | 25 | 28 | 32 |
| Corollary 1 | 22 | 22 | 26 | 29 | 32 |

*Example 2:* Consider system (1) with the following parameters [14]:

$$A = \begin{bmatrix} 1 & 0 & 0.01 & 0 \\ 0 & 1 & 0 & 0.01 \\ -0.009 & 0.009 & 0.9996 & 0.0004 \\ 0.009 & -0.009 & 0.0004 & 0.9996 \end{bmatrix}, A_d = \begin{bmatrix} 0 \\ 0 \\ 0.01 \\ 0 \end{bmatrix} \begin{bmatrix} 0.1284 \\ -0.1380 \\ -0.3049 \\ 0.0522 \end{bmatrix}^T$$

The AMDBs of $h_2$ for different $\mu$ and $h_1$ obtained by different criteria are listed in Table II. The existing criteria and Corollary 1 cannot use the available delay changing information, and 'n/a' in table indicates those cases. The result from Corollary 1 shows that the proposed criterion provides the bigger AMDB

$$\bar{X}_i=\begin{bmatrix}X_i & Y_i\\ Y_i^T & Z_i\end{bmatrix},\Theta_i=\begin{bmatrix}L_i & M_i\\ 0 & N_i\end{bmatrix},\bar{T}_i=\begin{bmatrix}0 & T_i\\ T_i^T & T_i\end{bmatrix},i=1,2,3;\ \bar{U}_1=U_1,\bar{U}_2=\bar{U}_3=U_2$$

$$\Psi(d(k),\Delta d(k))=\Upsilon_1(d(k),\Delta d(k))+\Upsilon_2+\Upsilon_3+\Upsilon_4(d(k))$$

$$\Upsilon_1(d(k),\Delta d(k))=(\Delta d+d(k))\Pi_1^TP_1\Pi_1-d(k)\Pi_2^TP_1\Pi_2+\Pi_{3d(k)}^TP_2\Pi_{3d(k)}-\Pi_{4d(k)}^TP_2\Pi_{4d(k)}$$

$$\Pi_1=\begin{bmatrix}e_1+e_s\\ (h_1+1)e_5-e_2\end{bmatrix},\Pi_2=\begin{bmatrix}e_1\\ (h_1+1)e_5-e_1\end{bmatrix},\Pi_{3d(k)}=\begin{bmatrix}\Pi_1\\ \Pi_{0d(k)}-e_3-e_4\end{bmatrix},\Pi_{4d(k)}=\begin{bmatrix}\Pi_2\\ \Pi_{0d(k)}-e_2-e_3\end{bmatrix},\Pi_{0d(k)}=[h_{1d}(k)+1]e_6+[h_{2d}(k)+1]e_7$$

$$\Upsilon_2=e_1^TQ_1e_1-e_2^T(Q_1-Q_2)e_2-e_4^TQ_2e_4+[e_1^T,\ e_s^T][h_1U_1+(h_2-h_1)U_2][e_1^T,\ e_s^T]^T$$

$$\Upsilon_3=\sum_{i=1}^{3}\left\{e_i^TT_ie_i-e_{i+1}^TT_ie_{i+1}+Sym\Big[e_f^T\Big(M_i(e_i-e_{i+1})+N_i(e_i+e_{i+1}-2e_{i+4})-L_ie_i\Big)\Big]\right\}+Sym\Big[(h_1+1)e_f^TL_1e_5\Big]+h_1e_f^TX_1e_f+\frac{h_1(h_1-1)}{3(h_1+1)}e_f^TZ_1e_f$$

$$\Upsilon_4(d(k))=Sym\left[(h_{1d}(k)+1)e_f^TL_2e_6+(h_{2d}(k)+1)e_f^TL_3e_7\right]+h_{1d}(k)e_f^TX_2e_f+h_{2d}(k)e_f^TX_3e_f+\frac{h_{1d}(k)}{3}e_f^TZ_2e_f+\frac{h_{2d}(k)}{3}e_f^TZ_3e_f$$

$$e_s=[A-I,\ 0_{n\times n},\ A_d,\ 0_{n\times n},\ 0_{n\times n},\ 0_{n\times n},\ 0_{n\times n}];\quad e_i=[0_{n\times(i-1)n},\ I_{n\times n},\ 0_{n\times(7-i)n}],i=1,2,\cdots,7;\quad e_f=[e_1^T,\ e_3^T,\ e_5^T,\ e_6^T,\ e_7^T]^T$$

of $h_2$ than the existing ones do. The result from Theorem 1 shows that the AMDBs decrease as the increasing of the delay changing bound and are all bigger than the one for unknown $\mu$, which shows that the consideration of delay changing information is useful to reduce the conservatism.

TABLE II
MAXIMAL BOUNDS, $h_2$, FOR DIFFERENT $\mu$ AND $h_1$

| Methods | $\mu=-\mu_1=\mu_2$ | | | | Unknown |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 5 | |
| [2], [3], [6], [7], [14] | n/a | n/a | n/a | n/a | $\leq 135$ |
| Corollary 1 ($h_1=1$) | n/a | n/a | n/a | n/a | 138 |
| Theorem 1 ($h_1=1$) | 148 | 143 | 141 | 139 | |
| Theorem 1 ($h_1=10$) | 150 | 146 | 144 | 142 | |

## VI. CONCLUSION

This note has investigated the delay-variation-dependent stability of the discrete-time system with a time-varying delay. Firstly, the Lypapunov functional with a delay-product type term has been developed to introduce the delay changing information into the stability criterion at the first time. Secondly, the GFWM approach has been proposed to estimate summation terms in the forward difference of functional, and the theoretical study has been given to show the GFWM approach encompasses the commonly used approaches and has less conservatism. Finally, those techniques have led to new stability criteria for the delayed linear discrete-time system, and the numerical examples have been used to verify the effectiveness of the proposed criteria.

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