Active Assignment of Eigenvalues and Eigen-sensitivities for Robust Stabilization of Friction-induced Vibration

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## Abstract

As friction couples tangential and lateral degrees-of-freedom of a structure at contact interfaces, the resulting asymmetric dynamic system is prone to dynamic instability. Using state-feedback control, such a frictional asymmetric system can be stabilized through assigning the system desirable eigenvalues; but uncertainties in system parameters can cause the assigned eigenvalues to deviate from the desired locations and thus stability may be lost.

This study presents a robust stabilization method that assigns both desirable eigenvalues and their sensitivities and thus render assigned eigenvalues stable and insensitive to perturbations in uncertain contact parameters (the friction coefficient, contact damping, and contact stiffness). Our method utilizes receptances of the corresponding symmetric part of the asymmetric system. The optimal control input location is first determined by minimizing the Frobenius norm of the normalized eigen-sensitivity matrix. The normalized eigen-sensitivities indicate that the friction coefficient and contact stiffness intrinsically have similar crucial effects on the stability of the system. To demonstrate the application of the proposed control method, the eigen-sensitivities with respect to only the friction coefficient are assigned. A constrained over-determined least-squares problem is solved to assign both required eigenvalues and eigen-sensitivities. Numerical examples validate the effectiveness of the proposed robust control scheme by Monte Carlo simulations.

*Keywords:* Asymmetric system, friction-induced vibration (FIV), receptance, eigenvalue assignment, eigen-sensitivity assignment, Monte Carlo simulation.

## 1 Introduction

Eigenvalues of a system representing dynamic characteristics of the system are invaluable and their assignment is widely studied in numerous engineering fields, such as control theory [1], vibration suppression [2], electric circuits and finite element model updating [3]. For a vibrating system, its eigenstructure, i.e., eigenvalues and eigenvectors, determines its dynamic response. Therefore, eigenstructure assignment has been extensively investigated for vibration suppression using passive structural modifications and active control [2], [4].

For general systems with mass, damping and stiffness matrices that are symmetric, the main objective of eigenvalue assignment is to assign appropriate natural frequencies or anti-resonances. Mottershead and Ram [2] first introduced a receptance-based inverse method for passive and active pole assignment, in which only a few measured receptances were needed and no knowledge of the mass, damping and stiffness matrices was required. Mottershead and his colleagues proposed receptance-based state-feedback and output-feedback schemes to actively assign appropriate poles/zeros to symmetric systems [4]-[6]. For symmetric systems, Datta et al. [7] were the first to study the quadratic partial eigenvalue assignment problem (PEVAP). A multi-input state-feedback control algorithm was proposed for quadratic PEVAP in [8] and the time-delay problem in PEVAP was investigated in [9]. Tehrani et al. [10] applied a receptance method-based PEVAP to a real lightweight glass-fibre beam. Based on the hybrid method for single-input vibratory systems put forward by Ram et al. [11], Bai et al. [12] developed a multi-step hybrid method for PEVAP of multi-input vibratory systems. In addition, eigenvalue sensitivities were successfully assigned based on receptance measurements in [13].

However, if a system includes certain nonconservative internal forces, the symmetry of its damping and stiffness matrices may be lost. Such a nonconservative system with asymmetric system matrices is likely to have eigenvalues with positive real parts, and thus is prone to instability. Friction, as a kind of nonconservative force, can induce unstable self-excited vibration. In real systems with friction, large amplitude vibration appears and usually noise is generated simultaneously [14], such as, wheel-rail noise [15], machine tool chatter [16] and vehicle brake squeal [17]. To stabilize an unstable asymmetric system, Ouyang et al. assigned eigenvalues of the system to the left half of the complex plane using both structural modifications [18] and active state-feedback control [19]. Singh and Ouyang [20] considered a constant time-delay in the feedback control loop for assigning stable eigenvalues to second-order damped asymmetric systems. Tehrani and Ouyang [21] proposed a partial pole assignment method for asymmetric systems.

The robustness of eigenvalue assignment for both symmetric and asymmetric systems has attracted much attention. For symmetric systems, the well-known study of Kautsky et al. [22] measured robustness using the condition number of the eigenvector matrix of the first-order system. The linear matrix inequality (LMI) approach was used to study robust pole placement for the first-order [23] and second-order [24] linear systems subject to system uncertainties or structural failures. Tehrani et al. [25] presented a sequential multi-input state-feedback method that minimized the norm of the eigen-sensitivities with respect to measurement errors in receptances. Recently, a modified Schur method was proposed for robust eigenvalue assignment in state-feedback control [26]. Araújo et al. [27] assessed eigen-sensitivity to parameter variation in second-order linear systems under state feedback and state derivative feedback. For asymmetric systems with friction, the robustness is of vital importance. Since a large class of friction-induced vibration (FIV) problems is sensitive to variability in friction, material properties, geometry of sliding surfaces, normal load etc.[28], it is difficult to be suppressed [29]. Uncertainties in FIV can arise from different sources, for example, from friction laws themselves [30] or from surface roughness of friction materials [31]. It is preferable to identify the most influential parameters on FIV. For example, Nechak et al. [32] ordered parameters in terms of their impacts and only considered influential parameters to form a Kriging model for robust and optimal design of brakes against dynamic instability.

This research focuses on developing a robust stabilization method for a class of FIV problem when the contact parameters are uncertain. The FIV is manifested as a typical asymmetric, linear time-invariant dynamic system. The uncertain contact parameters are assumed as independent normally distributed random variables. The proposed robust method takes advantage of the receptance method that requires no detailed information of the mass, damping and stiffness matrices. Eigen-sensitivities with respect to uncertain contact parameters are derived using a perturbation method. By minimizing the Frobenius norm of the normalized sensitivity matrix formed by eigen-sensitivities, the optimal control input position is determined which results in the least scatter in assigned eigenvalues. Through examining the real parts of the normalized eigen-sensitivities, the friction coefficient and the contact stiffness are found to intrinsically have similar crucial effects on the stability of the system. To ensure the assigned eigenvalues to be stable and insensitive to the uncertain contact parameters, a constrained over-determined least-squares problem is proposed. By solving this optimisation problem, both required eigenvalues and their normalized eigen-sensitivities with respect to critical uncertain parameters are assigned. Finally, Monte Carlo simulations are conducted to verify that the proposed approach is effective and robust.

## 2 Receptance-based robust state-feedback control

### 2.1 Basic theory

The dynamic equation of a general asymmetric vibrating system with single-input state feedback *u* in Laplace space is formulated as

 

and the single-input *u* formed with displacement and velocity feedbacks is given as

 

In Eq. , **M**, **C** and  are the mass, damping and stiffness matrices, respectively.  and  are respectively the Laplace transforms of the displacement vector and the external force vector. , indicating the positions where the control input  is received, consists of element 1 or 0; For general structural systems,  is positive-define,  and are semi positive-define; Specifically,  and  are the asymmetric matrices induced by contact damping and stiffness at frictional contact interfaces.  are the contact parameters, namely friction coefficient, contact damping, and contact stiffness, respectively. In total, there are  sets of such asymmetric terms. Matrix  indicates the global location of the *k*th frictional degree-of-freedom (DOF). In Eq. , are the displacement and velocity feedback control gain vectors, respectively.

The receptance of a system represents the relationship between force input  and displacement output  Hence, the receptance matrices for the *symmetric* part of the general asymmetric system and for the general *asymmetric* system itself are defined as

 

 

Pre-multiplying matrix  on both sides of Eq. yields

 

and a further manipulation of the above equation results in

 

where  is the receptance matrix of the *closed-loop asymmetric* system and it is calculated using the following equation.

 

where  .

Therefore, the characteristic equation of the closed-loop asymmetric system is

 

With further derivation, the characteristic equation given in Eq. is expressed as the following equation by taking the advantage of the matrix determinant lemma [33].

 

The above equation is the fundamental equation to assign eigenvalues for asymmetric systems. Eigenvalue and eigen-sensitivity assignments are implemented based on this equation later in this paper.

### 2.2 Eigenvalue assignment

Based on Eq. , eigenvalues of the asymmetric system can be assigned utilizing its open-loop receptance  In practice, it is difficult to measure  accurately and its corresponding counterpart  defined in Eq. is measured and utilized instead [6]. The relationship between  and  is

 

The above equation is further simplified using a basic theorem concerning inverse matrices as

 

Since friction-related degrees-of-freedom (DOFs) are very few in comparison with the total number of DOFs of a system, matrix  has only a small number of non-zero elements corresponding to the friction-related DOFs. As a result, matrix  consists of mostly the identity matrix  and thus is quite easy to be inverted.

For simplicity, this paper supposes that there is only one frictional contact point in the system and thus only one asymmetric damping and only one stiffness exist at the *k*th DOF. Hence, using the well-known Sharman-Morrison formula [34] and noticing , Eq. is simplified as

 

in which . Column vector  indicates friction related DOFs.  and  have only one element being 1 at respectively the *i*th tangential DOF and the *j*th normal DOF at the contact interface with all the other terms being 0.

Submitting Eq. into Eq. and denoting the result as , one gets

 

Using the above equation, appropriate state-feedback control can actively assign required eigenvalues to the closed-loop system. The corresponding control gain  to assign required eigenvalues is solved using the following equation.

 

where the *i*th (*i*=1,…, *n*) row of  and  are

 

 

in which



and  is the required eigenvalue of the closed-loop system.

### 2.3 Eigen-sensitivity assignment

#### 2.3.1 Derivation of eigen-sensitivity

Equation shows that each eigenvalue of the closed-loop system is a root of this equation. When perturbations are not considered, closed-loop eigenvalues can be assigned exactly to required ones () using the control gain  obtained from Eq. . However, in practical cases, perturbations in system parameters and disturbances always exist due to uncertainty, variability, and changeable operation conditions. In the case of FIV, significant parameters are related to the contact interface, namely, the contact damping , the contact stiffness  and the friction coefficient . These contact parameters are believed to possess a large degree of uncertainty. This is different from [25] in which perturbations appear in measured system receptances. When a small perturbation appears in any of the three uncertain contact parameters ( or ), the assigned closed-loop eigenvalue deviates from the required one by . Thus, based on Eq. , when only the perturbation in friction coefficient is considered, the following equation is derived by taking the total differential of ,

 

Moreover, the perturbed equation of motion still satisfies Eq. as

 

The above equation is derived based on the first-order Taylor expansion. This linear approximation is valid for only small deviations from the nominal values of the parameters.

Comparing Eq. with Eq. , it is obvious that

 

The perturbation idea was described in the robust pole assignment of symmetric systems [25].

Consequently, the sensitivity of the required closed-loop eigenvalue  with respect to the friction coefficient is given as

 

Based on Eq. , the following equations are derived,

 

 

where  is the element of  at the *i*th row and *j*th column, which involves only one non-zero element of  in Eq. . Furthermore,

 

 

 

Consequently, the sensitivity of the closed-loop eigenvalue  with respect to a small perturbation in friction coefficient  is obtained by substituting Eqs. - into Eq. as

 

where .

Similarly, the sensitivities of the closed-loop eigenvalue  with respect to a small change in contact damping  and contact stiffness  are respectively derived as

 

 

where 

#### 2.3.2 Normalized sensitivity matrix

 To evaluate all the sensitivities of the closed-loop eigenvalues with respect to the uncertain contact parameters, a sensitivity matrix is defined as

 

In addition, for a system with *n* degrees-of-freedom, it has *n* pairs of complex conjugate eigenvalues and the sensitivities of each pair of eigenvalues with respect to any uncertain parameter are also a complex conjugate pair. Thus, only a half of the complex conjugate eigen-sensitivities with respect to the uncertain contact parameters is included in the sensitivity matrix . In this matrix,  are the sensitivities of the closed-loop eigenvalue  with respect to the friction coefficient , the contact damping  and the contact stiffness  respectively.

Furthermore, in order to exclude the effect of physical scale, normalized eigen-sensitivities with respect to the contact parameters are defined in the following formula

 

where  are the normalized sensitivities of the required eigenvalue  with respect to the three uncertain contact parameters. Additionally,  are the nominal values of .

Correspondingly, the normalized eigen-sensitivity matrix is defined as

 

#### 2.3.3 Eigen-sensitivity assignment

The eigen-sensitivities with respect to the uncertain contact parameters indicate the degree of deviations of eigenvalues from their nominal values. Therefore, assigning the normalized eigen-sensitivities to small values would make the required eigenvalues insensitive to perturbations of the uncertain contact parameters and remain close to their nominal values. Using Eqs. - and Eq. , the control gain vector  to assign particular normalized eigen-sensitivities is determined from the following equation

 

where  and  are the *i*th (*i*=1,…, *n*) row of  and  respectively.

 

 

in which  given below are the coefficients for assigning the normalized sensitivities of closed-loop eigenvalues with respect to or . Besides,  refers to nominal values of  or .





Consequently, using the above formulas, the control gain **y**s for assigning required normalized sensitivities of eigenvalues with respect to the uncertain contact parameters can be determined.

### 2.4 Robust state-feedback control

Once a system has uncertain contact parameters, assigned eigenvalues using state-feedback control may deviate from required ones. To ensure the robustness of eigenvalue assignment for asymmetric systems, two considerations need to be satisfied: first, the closed-loop system should remain stable under any combination of the uncertain parameters; second, the eigenvalues of the closed-loop system should stay as close as possible to the required locations.

#### 2.4.1 Robust control input position

Once the system and required eigenvalues are specified, control gain vector  for assigning required eigenvalues is a function of control input vector **b**. Therefore, the normalized closed-loop sensitivity matrix defined in Eq. is also a function of this vector **b**. To make the assigned eigenvalues stable and remain close to the required ones with uncertain contact parameters, the robust control input vector **b** should be determined. Hence, the robust vector **b**rob, which minimizes the Frobenius norm of the normalized sensitivity matrix  with all the assigned eigenvalues sitting on the left half of the complex plane, is determined by solving the minimization problem formulated as

 

In this minimization, the genetic algorithm is applied with nonlinear constraints. The fitness function to be minimized is the Frobenius norm of normalized closed-loop eigen-sensitivity matrix . The first nonlinear constraint is that all the assigned eigenvalues are stable within required stable margins. In this constraint, coefficient  determines the stability margins;  denotes the real part of a complex value;  is the deviation of the real part of the *i*th assigned eigenvalue to its corresponding required one due to perturbations. Moreover,  is calculated using the following formula

 

where represents the absolute value;are deviations of the uncertain contact parameters. Furthermore, the second constraint is that all elements of vector **b** can only be 0 or 1. In Eq. , **B** is the set containing all those possible vector **b**’s.

#### 2.4.2 Robust eigenvalue and eigen-sensitivity assignment

To assign the required eigenvalues and to minimize the deviations of placed eigenvalues from the required ones caused by the uncertain contact parameters, control gain vector **y** which assigns both the required closed-loop eigenvalues and required normalized eigen-sensitivities, is derived as

 

where   and  are defined in Eqs. - for assigning required eigenvalues.  and  are defined in Eqs. - for assigning normalized sensitivities with respect to the uncertain contact parameters. To make assigned eigenvalues insensitive to the uncertain contact parameters, the normalized closed-loop eigen-sensitivities  need to be assigned to small values.

For assigning all the required eigenvalues and all the normalized closed-loop eigen-sensitivities,  has more rows than **y** does. Consequently, Eq. is an over-determined problem and no **y** satisfies this equation exactly. Thereupon, to assign both required eigenvalues and eigen-sensitivities with the requirement that ensures all the assigned eigenvalues to be stable and close to the original required ones, the over-determined problem is modified as the least-squares problem given below

 

where  means the Euclidean norm of a vector;  are weighting coefficients for eigenvalue and eigen-sensitivity assignments respectively. are respectively the assigned and required eigenvalues. The first nonlinear constraint ensures all the assigned eigenvalues to be stable. The second nonlinear constraint uses the coefficient  to force the real parts of the assigned eigenvalues to be close to the required ones.

## 3 Implementation of proposed control method

### 3.1 Friction-induced vibration problem

Lumped-mass models with low DOFs, which provide fundamental understanding of structures, are always used to study FIV [35] and [36]. This paper uses a simplistic lumped-mass model proposed by Nobari et al. [37] for representing a FIV. This model is evolved from a model given in [19] through adding contact damping *c*c at the slider-belt interface. As depicted in Fig. 1, *m*1 has a DOF in the *x* direction; *m*3 has a DOF in the *y* direction; *m*2 moves in both the *x* and *y* directions. An oblique linear spring *k*3 couples vibrations between the *x* and *y* directions was found to trigger instability [38]. *n*1 and *f*1 acting at the slider-belt interface are respectively the normal force and the resultant friction force. *n*1 is a high compressive preload to maintain contact during vibration and *f*1 is governed by Coulomb’s law. The belt moves at a constant speed and the static and kinetic friction coefficients are assumed to be the same deliberately to avoid stick-slip motion.



Fig. 1. An example problem of FIV

In this model, only *x*2 and *y*2 are the friction-related DOFs, thus the friction-induced damping and stiffness matrices are expressed as , where matrix **E** is given below. The mass, damping and stiffness matrices are defined corresponding to displacement vector  as





In the above matrices, , The uncertain contact parameters are assumed to follow Gaussian distributions. Their nominal values and standard deviations are

The open-loop and required closed-loop eigenvalues are listed in Table 1. It is evident that the open-loop system is unstable because it has a pair of eigenvalues with positive real parts.

Table 1. Open-loop and required closed-loop eigenvalues

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *i*th pair of eigenvalues | 1th | 2nd | 3rd | 4th |
| Open-loop **s** | 0.0058.946i | -0.06512.134i | -0.52716.823i | -0.21219.732i |
| Required closed-loop   | -0.29i | -0.412i | -0.5517i | -0.2520i |

Based on Eq. , the characteristic equation of the model shown in Fig. 1 is obtained after further manipulation as follows

 

in which  are vectors indicating the friction-related DOFs.

### 3.2 Robust stabilization

#### 3.2.1 Robust control input position

The model shown in Fig. 1 has four pairs of complex eigenvalues, thus, for this model the normalized sensitivity matrix **S**N defined in Eq. has four rows and three columns. For the model investigated, the control input vector **b** has four elements that are either 1 or 0. Thus, except for the one with all zeros, vector **b** has 15 possible varieties and they are listed in Table 2.

Table 2. Possible values for vector **b**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |

Using Eq. , the robust vector **b**rob is obtained as  which leads to the minimum Frobenius norm of the normalized eigen-sensitivity matrix **S**N. The corresponding control gain vector to assign the required eigenvalues  is obtained as ={-6.727, -12.481, 7.461, 8.102, -0.051, 1.767, 0.342, 0.858}T.

To check this result, the Frobenius norms of the normalized sensitivity matrix under different vector **b**’s are listed in Table 3. In this table, the norm is set as Inf if the first nonlinear constraint in Eq. is not satisfied. It can be seen that the minimum of ||**S**N(**b**)||F is indeed achieved by **b**5.

Table 3. Frobenius norms of normalized sensitivity matrices

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **b***i* | 1 | 2 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| ||**S**N(**b***i*)||F | Inf | 0.159 | Inf | 0.177 | 0.152 | Inf | 0.161 | 0.160 | 0.192 | 0.157 | Inf | 0.182 | Inf | 0.193 | Inf |

#### 3.2.2 Robust eigenvalue and eigen-sensitivity assignment

To minimize the Frobenius norm of normalized sensitivity matrix and assign eigenvalues close to the original required eigenvalues, the constrained over-determined least squares problem in Eq. is proposed to find appropriate closed-loop eigenvalues and corresponding control gain vector.

##### A. Critical uncertain parameter

It is clear that once all of the terms of the normalized sensitivity matrix are made small, deviations of assigned eigenvalues from the required ones due to perturbations in the contact parameters become small. Nevertheless, it is difficult and unnecessary to assign all the eigen-sensitivities with respect to all the contact parameters to small values. Consequently, it is preferable to minimize the eigen-sensitivities only with respect to the crucial contact parameters. To determine the most influential uncertain contact parameter on closed-loop eigenvalues, the normalized sensitivities of the open-loop eigenvalues and the required closed-loop eigenvalues are calculated and plotted. The normalized sensitivities of the open-loop eigenvalues are calculated using Eqs. - and Eq. by setting the control gain vectors as zero vectors and using the open-loop eigenvalue instead of the closed-loop eigenvalues. These two sets of normalized eigen-sensitivities are displayed in the following figure.

|  |  |
| --- | --- |
|  |  |
| a. Normalized open-loop eigen-sensitivities | b. Normalized closed-loop eigen-sensitivities |

Fig. 2. Normalized sensitivities of open-loop and closed-loop eigenvalues (closed-loop system is formed with control gain vector $y\_{λ}$ and robust control input vector **b**rob); $S\_{Nop}\left(μ\right), S\_{Nop}\left(c\_{c}\right), S\_{Nop}(k\_{c})$ stand for the normalized eigen-sensitivities of open-loop eigenvalues with respect to friction coefficient, contact damping and contact stiffness; $S\_{Ncl}\left(μ\right), S\_{Ncl}\left(c\_{c}\right), S\_{Ncl}(k\_{c})$ stand for the normalized eigen-sensitivities of closed-loop eigenvalues with respect to the three uncertain contact parameters.

The above figure demonstrates that the normalized eigen-sensitivities with respect to friction coefficient and contact stiffness denoted respectively using asterisks and circles always have similar greater absolute values of the real parts of normalized eigen-sensitivities. This means that friction coefficient and contact stiffness have similar critical effects on the real parts of the eigenvalues, thus have similar critical effects on the stability of the system. Consequently, these two contact parameters are identified as the critical uncertain contact parameters. Fig. 2 also illustrates that although the real parts of the closed-loop eigenvalues are slightly less sensitive to the uncertain contact parameters than those of the open-loop eigenvalues, the imaginary parts of the closed-loop eigenvalues are more sensitive than those of the open-loop eigenvalues. Therefore, much attention should be paid to the uncertainties in the contact parameters when the control input is introduced to the system.

Furthermore, the difference between the normalized eigen-sensitivity with respect to friction coefficient  and contact stiffness  is derived based on Eqs. , and as

 

It is obvious from Eq. that for a certain deterministic system **a** is a function of the required eigenvalue  and the control input position vector **b**. Therefore, once those two variables are specified, vector **a** is constant. Furthermore, for the system studied the second and the fourth terms of **a** are always zero and the first and the third terms of **a** are near to the origin of the complex plane. Moreover, since the friction coefficient and the contact damping are smaller than 1, the difference between  and  is quite small. This demonstrates that the normalized closed-loop eigen-sensitivities with respect to these two uncertain parameters are intrinsically similar to each other. Hence, the normalized eigen-sensitivities with respect to either of these two uncertain contact parameters can be assigned to demonstrate the application and the effectiveness of the proposed method. Consequently, later only the normalized eigen-sensitivities with respect to the friction coefficient are assigned.

##### B. Results of robust eigenvalue and eigen-sensitivity assignment

In Eq. , is set and this means more effort is paid to assign eigen-sensitivities in this study. In this equation,  is set as 0.8 to make the assigned eigenvalues close to the original required ones. The solution of Eq. is = {-6.438, -29.290, 44.954, -14.679, 0.195, 1.454, 0.094, 0.797}T.The closed-loop eigenvalues placed using this control gain vector **y** with no uncertainties involved are = {-0.1139.748i, -0.39611.628i, -0.48517.326i, -0.25219.971i}T.

The normalized sensitivities of the closed-loop eigenvalues  and  are plotted in Fig. 3. It is obvious that the closed-loop system formed with control gain **y** that assigns both eigenvalues and eigen-sensitivities has the smallest normalized sensitivities, in comparison with the normalized sensitivities of the open-loop and closed-loop system that assigns only the eigenvalues. The deviations of eigenvalues plotted in Fig. 4 are calculated using Eq. for the open-loop system, the closed-loop system that assigns only the eigenvalues, and the closed-loop system that assigns both eigenvalues and eigen-sensitivities. This figure shows that the robust stabilization method that assigns both eigenvalues and eigen-sensitivities results in the smallest deviations of eigenvalues. Thus, this control method assigning both eigenvalues and eigen-sensitivities is verified to be robust.

|  |  |
| --- | --- |
|  |  |
| Fig. 3. Normalized eigen-sensitivity of the open-loop and closed-loop systems; $S\_{N}\left(op\right), S\_{N}\left(p\right), S\_{N}(pso)$ stand for the normalized eigen-sensitivities of open-loop system, closed-loop system whose eigenvalues are assigned, and closed-loop system whose eigenvalue and eigen-sensitivities are assigned. | Fig. 4 Deviations of real parts of eigenvalues; $δR\left(op\right)$, $δR\left(p\right)$, $δR\left(pso\right)$ represent deviations of real parts of open-loop system, closed-loop system whose eigenvalues are assigned, and closed-loop system whose eigenvalue and eigen-sensitivities are assigned. |

## 4 Monte Carlo Simulations

In the last section, the robust input vector **b**rob and the robust control gain vector **y** for robustly assigning both eigenvalues and eigen-sensitivities are obtained. In this section, Monte Carlo simulations are implemented to confirm that the proposed method is less vulnerable to uncertainties in the contact parameters. In such simulations, 10 thousand sets of the uncertain parameters are generated. The corresponding eigenvalues of the open-loop and closed-loop systems are plotted. Since conjugate eigenvalue pairs are symmetrical with respect to the real axis of the complex plane, only eigenvalues with positive imaginary part are illustrated in the following figures.

### 4.1 Open-loop system

The open-loop eigenvalues of the system studied are plotted in Fig. 5. In this figure, the eigenvalues of the deterministic system and uncertain system (with the uncertain contact parameters) are marked using asterisks and circles respectively. This figure demonstrates that the open-loop system has one pair of unstable eigenvalues with positive real parts when the contact parameters are deterministic. On the other hand, when uncertainties in the contact parameters are considered, the stability of the system becomes uncertain. The system is stable only if all the real parts of eigenvalues are negative. Nevertheless, the circles in Fig. 5 indicate that the uncertain open-loop system could be more unstable with greater positive real parts of its eigenvalues. This figure illustrates the significant effect of the uncertain contact parameters on the stability of the system.

### 4.2 Closed-loop system with only eigenvalue assignment

In Fig. 6, circles represent the eigenvalues of 15 different closed-loop systems formed with 15 possible control input vector **b**’s that are listed in Table 2. In these closed-loop systems, the contact parameters are uncertain. The control gain vector ’s used in these closed-loop systems are calculated using Eq. for assigning the required eigenvalues when the contact parameters are deterministic. In addition, the required eigenvalues are denoted as asterisks in Fig. 6. When uncertainties of the contact parameters are present, the closed-loop system can be even more unstable than the corresponding open-loop system (i.e., the closed-loop eigenvalues have larger positive real parts), even if the control gain vectors ’s that stabilize the corresponding deterministic open-loop system are applied. Consequently, one needs to understand the importance of taking uncertainties in the contact parameters into consideration.

|  |  |
| --- | --- |
|  |  |
| Fig. 5. Open-loop eigenvalues with and without uncertain contact parameters; Asterisks represent eigenvalues of deterministic system; circles represent eigenvalues of uncertain systems.  | Fig. 6. Eigenvalues of uncertain closed-loop systems; Circles represent eigenvalues of closed-loop systems formed with different control input vectors; Diamonds represent eigenvalues of closed-loop system formed with the robust control input vector **b**rob. Asterisks represent required eigenvalues. |

After the assigned closed-loop eigenvalues are calculated and shown as circles in Fig. 6, the robust control input vector **b** is determined in two steps based on the statistics of the assigned eigenvalues. First, since the possible optimal vector **b** should at least assign all of the closed-loop eigenvalues to the left half of the complex plane, six eligible control input vectors are selected. The corresponding numbers of unstable placed eigenvalues Nun are listed in Table 4. Additionally, only one member of each pair of complex conjugate eigenvalues is examined in this table.

 Table 4. Number of unstable placed closed-loop eigenvalues with different control input vector ’s

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **b***i* | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| Nun | 122 | 88 | 125 | 12 | 0 | 1256 | 0 | 0 | 0 | 0 | 627 | 302 | 267 | 23 | 0 |

For Fig. 6, the deviation of each placed eigenvalue (denoted as circles) from the required one (denoted as asterisks) is evaluated only for the stable closed-loop systems. The deviation is the distance from the assigned eigenvalues to the required ones. Using Eq. , the mean value  and standard deviation  of the distances from the *i*th required eigenvalue to the *i*th placed eigenvalues using vector **b***j* are calculated to determine the robust vector **b**. For 10 thousands assigned eigenvalues, the sums of these mean values and standard deviations are listed in Table 5.

 

Table 5. Mean values and standard deviations of the deviations of placed eigenvalues from required eigenvalues

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **b**j | **b**5 | **b**7 | **b**8 | **b**9 | **b**10 | **b**15 |
|   | 0.646 | 0.680 | 0.680 | 0.682 | 0.657 | 0.734 |
|  | 0.483 | 0.502 | 0.503 | 0.505 | 0.484 | 0.549 |

Table 5 demonstrates that **b**5 = {0, 0, 1, 1}T simultaneously leads to the minimum mean value and minimum standard deviation. Accordingly, this vector **b** that places all the closed-loop eigenvalues to the stable region and leads to the smallest deviations from the required eigenvalues is confirmed as the robust control input vector **b**rob. The placed eigenvalues using **b**rob are represented using diamonds in Fig. 6, and these placed eigenvalues are closer to the required ones (denoted as asterisks) than the others. Furthermore, this result is identical to the one obtained in the previous section, hence, the proposed robust control method is validated to be able to determine the robust position of control input in one run. Moreover, the proposed strategy is much more efficient when there are millions of DOFs in a system, as there is no need to run the time-consuming Monte Carlo simulation to find the robust control input position. The robust control input vector **b**rob can be obtained using the proposed strategy only in one run.

### 4.3 Closed-loop system with eigenvalue and eigen-sensitivity assignment

In this subsection, Monte Carlo simulation is implemented to verify that the robustness of the closed-loop system to the uncertain contact parameters is further enhanced by assigning appropriate eigen-sensitivities. As in the previous simulation, the contact parameters are considered uncertain and 10 thousand samples of them are generated. Moreover, the robust control input vector **b**rob is applied to form closed-loop systems in this simulation.

In Fig. 7, eigenvalues of two closed-loop systems formed with control gain vector  and **y** are plotted. The eigenvalues denoted using diamonds are the ones of the closed-loop system formed with the control gain vector . This control gain vector can exactly assign the required eigenvalues  (denoted as squares) for the deterministic system. The closed-loop eigenvalues denoted as diamonds in Fig. 7 are the same ones depicted as diamonds in Fig. 6. Furthermore, the eigenvalues marked as triangles in Fig. 7 are those of the closed-loop system formed with the control gain vector **y**. The control gain vector **y** obtained based on Eq. places the eigenvalues to  (denoted as hexagrams) and assigns corresponding eigen-sensitivities near zero with deterministic contact parameters. In addition, the eigenvalues plotted as circles and asterisks are respectively the ones of uncertain and deterministic open-loop systems.



Fig. 7. Eigenvalues of open-loop and closed-loop systems with uncertain and deterministic contact parameters;are eigenvalues of uncertain open-loop system, uncertain closed-loop system formed with control input vector , and uncertain closed-loop system formed with control input vector **y**; are eigenvalues of corresponding deterministic systems;

This figure demonstrates that due to the uncertainties in the contact parameters the closed-loop eigenvalues cannot be exactly assigned to the required places. Using control gain vector , the assigned eigenvalues (denoted as triangles)get closer to their nominal values (denoted as hexagrams) than those marked as diamonds to their nominal values (denoted as squares) which are eigenvalues of the deterministic systems. Furthermore, all of the closed-loop eigenvalues, denoted using diamonds and triangles in Fig. 7, are on the left half of the complex plane, which shows that the proposed control method can determine appropriate control input to stabilize the unstable open-loop system. In conclusion, this figure illustrates that the robustness of the closed-loop system can be further enhanced through appropriate eigen-sensitivity assignment, even with the robust control input vector **b**rob , which indicates the control input positions that lead to the most insensitive solution to the uncertain contact parameters.

Although the above robust eigen-sensitivity assignment is carried out on a low-DOFs system, it is capable of dealing with large systems when they are represented by a suitable Kriging model [39],[40]. Other ways of model reduction are also possible, for example, through component mode tuning method [31], and proper orthogonal decomposition method [41], [42].

## 5 Conclusions

Asymmetric dynamic systems are likely to possess eigenvalues with positive real parts and are prone to dynamic instability. One way of stabilizing such unstable systems is to assign their eigenvalues with negative real parts. However, due to uncertainties in system parameters, when only eigenvalues of a system are assigned the assigned eigenvalues can deviate from the required ones and thus the stability of the system is not guaranteed. One remedy is to assign eigen-sensitivities at the same time.

The eigen-sensitivities of the closed-loop asymmetric system for friction-induced vibration with respect to the uncertain contact parameters are derived using a perturbation method based on the receptance of the symmetric part of the asymmetric system. The normalized sensitivity matrix is defined using the eigen-sensitivities with respect to the uncertain contact parameters. The presented method offers an effective method to determine the robust control input position vector which minimizes the Frobenius norm of the normalized sensitivity matrix, thus deviations of closed-loop eigenvalues from their required locations due to uncertain contact parameters are minimized. The robustness of stabilization is further enhanced by determining a robust control gain vector that ensures all the closed-loop eigenvalues are stable and close to the required ones. This vector is obtained by solving a constrained over-determined least-squares problem in which both required eigenvalues and eigen-sensitivities are assigned.

In a specific example of friction-induced vibration, the critical uncertain contact parameters are found to be the friction coefficient and contact stiffness for the stability of the asymmetric system. The normalized eigen-sensitivities with respect to only the friction coefficient are assigned to verify the robustness of the proposed control scheme. Monte Carlo simulations further validate that the proposed method is effective.

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