Optimizing Replenishment Policy in an Integrated Supply Chain with Controllable Lead Time and Backorders-Lost Sales Mixture

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Abstract
This paper aims to optimize the inventory replenishment policy in an integrated supply chain consisting of a single supplier and a single buyer. The system under consideration has the features such as backorders-lost sales mixture, controllable lead time, stochastic demand, and stockout costs. The underlying problem has not been studied in the literature. We present a novel approach to formulate the optimization problem, which is able to satisfy the constraint on the number of admissible stockouts per time unit. To solve the optimization problem, we propose two algorithms: an exact algorithm and a heuristic algorithm. These two algorithms are developed based on some analytical properties that we established by analysing the cost function in relation to the decision variables. The heuristic algorithm employs an approximation technique based on an ad-hoc Taylor series expansion. Extensive numerical experiments are provided to demonstrate the effectiveness of the proposed algorithms.
Keywords: Integrated supply chain; replenishment policy; stochastic demand; controllable lead time; backorders; lost sales; joint economic lot size

1. Introduction

Supply chain management involves a range of decision-making tasks including planning and management of sourcing, procurement, conversion, and logistics. Accomplishing these tasks in isolation may lead to conflicts among supply chain parties (Zhao *et al*., 2016). Moreover, the players in a supply chain (such as vendors, retailers, distributors, *etc*.) may belong to different corporate entities and be more prone to minimize their own cost rather than that of the supply chain as a whole. However, this single-sided optimal strategy is not suitable for today's competitive environment (Jha and Shanker, 2013). Increasingly intensive competition has forced companies to seek stronger collaborative relations with their suppliers and/or customers (Yi and Sarker, 2014). Facing fierce market competition, companies that belong to the same supply chain have recognized the significance of synchronizing and coordinating their policies with the objective of reducing operation costs and improving performance (Jia *et al*., 2016).

Nowadays, business companies have realized that a better management of inventories across the supply chain can be achieved through cooperation rather than acting independently This collaborative perspective within the supply chain has led researchers to develop models that consider coordinated inventory replenishment decisions between buyer(s) and vendor(s). The integrated inventory model takes the view of the supply chain's total cost/profit to determine the optimal production-delivery schedule in the context of a centralized system and information sharing. The optimization of replenishment decisions in integrated inventory models can be traced back to the definition of the so-called Joint Economic Lot Size (JELS) problem (Goyal 1977).

After Goyal (1977), an impressive number of studies has emerged in the context of JELS models. The most recent review of JELS models is owed to Glock (2012), which covers the relevant papers published before 2012. In the following, the relevant papers published after 2012 will be reviewed briefly. Das *et al*. (2013) presented an integrated model for a constant deteriorating item, in which shortages are not allowed, but a delay in payment is offered to retailer by supplier. The procurement cost linearly depends on the credit period and the process cost is a linear cost of the quantity purchased by retailer. Diabat (2014) addressed the issue of maximizing the profit of a supply chain under Vendor-Managed Inventory (VMI) given a nonlinear and non-convex objective function. Sadeghi *et al*. (2014a) extended the model of Zavanella and Zanoni (2009) to include fuzzy demand, transportation cost, and several additional constraints. Chang (2014) developed an improved method to optimize a two-echelon supply chain for a deteriorating item where the deterioration rate is constant or follows a probability distribution. Rad *et al*. (2014) studied a vendor-buyer integrated supply chain with imperfect production and shortages, where demand is sensitive to the selling price. Lin and Lin (2014) presented a model involving defective items and quantity discounts, in which the objective is to optimize pricing and ordering strategies. Sadeghi *et al*. (2014b) developed a bi-objective VMI model with one vendor and several retailers, in which different machines work in series to produce a single item. They assumed that the demand is deterministic and known, whereas the budget, required storage space, replenishment frequencies, and average inventory are constrained. Jauhari *et al*. (2014) investigated a single vendor-single buyer supply chain under deterministic demand, taking into account unequal-sized shipments, defective items and carbon emission cost. Braglia *et al*. (2014a; 2014b) first presented a new approach to evaluate physical space occupation costs, and then gave a new cost formulation for an integrated production-inventory model under VMI with consignment stock. Sadeghi and Niaki (2015) proposed a bi-objective VMI model for a supply chain with a single vendor and multiple retailers in which the demand is fuzzy and the vendor faces two constraints: number of orders and available budget. Sadeghi (2015) considered a VMI model for a two-echelon supply chain where several retailers' inventories are replenished with different rates by a single vendor, taking into account a capacity constraint on the vendor's warehouse. Nagaraju *et al*. (2015) studied inventory replenishment decisions for centralized and decentralized supply chain with price dependent demand. Giri *et al*. (2015) developed a single vendor-single buyer model where the capacity of the vendor's warehouse is limited. The model assumes that, whenever the vendor's on-hand inventory level reaches the maximum attainable capacity, she/he follows a modified consignment stock policy until the production lot is completed. Glock and Kim (2015) studied a supply chain model that considers both the downstream flow of materials from the manufacturer to the buyer and the circulation of returnable transport items (RTIs) between two actors. Braglia *et al*. (2016) proposed a novel approach to manage safety stock in a supply chain under VMI with consignment stock where the lead time is assumed to be controllable and costs are evaluated using the present value criterion. Lin (2016) formulated a model that considers partial backlogging and stochastic lead time. The lead time variability is supposed to be controllable through a capital investment. Hariga *et al*. (2016) examined a supply chain where a single vendor supplies a single buyer with a finished product packed in RTIs. The RTI return time is assumed to be stochastic and, in case the return of empty RTIs is delayed, the vendor can rent RTIs from a nearby service provider. Finally, Jauhari *et al*. (2016) studied a three-echelon supply chain under deterministic demand, where the production process is imperfect and inspections include errors.

Real inventory systems are typically characterized by demand uncertainty. When facing stochastic demand, an important issue is concerned with controlling the replenishment lead time (Glock, 2012). In fact, a longer lead time exposes the company to a higher risk of running out of stock; while a shorter lead time may lead to smaller safety stock, better customer service level, reduced stockout losses, and lower expected total costs. Hence, controlling lead time may permit to achieve lower investments in inventory, better product quality, less scrap, reduced storage space requirements, increased productivity, and improved competitive position of the company (Glock, 2012; Hariga, 2000). There exist several approaches to incorporating lead time control: lead time is assumed to be made of several independent components that can be shortened by paying a crashing cost (Chang *et al*., 2006; Jha and Shanker, 2009; Lin *et al*., 2011; Lin, 2013; Panda *et al*., 2014); it can be represented as a linear function of the production rate and the order quantity (Glock, 2012; Song *et al*., 2013; Abdelsalam and Elassal, 2014); it can be considered to be an independent decision variable with a crashing cost expressed as a power or linear function (Chandra and Grabis, 2008) or as an exponential function (Moon *et al*., 2014).

In the context of stochastic demand, another important issue is the assumption of backorders or lost sales to represent customers' purchase behaviour when facing stockouts. Earlier literature often assumed either backorders or lost sales. Recent stochastic inventory models have considered the situation of backorders-lost sales mixture (Chang and Lo, 2009; Sicilia *et al*., 2012; Wang and Tang, 2014; Castellano, 2016). The backorders-lost sales mixture is a more general phenomenon that represents the scenarios in which some customers may be willing to wait until demand is satisfied (such demands are backordered); while others are impatient (such demands are lost). In fact, the case of backorders or lost sales can be treated as a special scenario of backorders-lost sales mixture. However, optimizing safety stock, replenishment policy and lead-time length in supply chains with the backorders-lost sales mixture has been understudied.

This paper investigates an integrated supply chain taking into account controllable lead time, backorders-lost sales mixture, and stockout costs. The objective is to optimize the replenishment policy and the length of lead time by minimizing the expected total cost per time unit. In the literature, the optimization of the replenishment policy (that implies safety stock and expected shortage) is often tackled by treating the order (or shipment) quantity and the service level as mutually independent decision variables, or by imposing a service level constraint (Jha and Shanker, 2009; Lin *et al*., 2011; Shahpouri *et al*., 2013; Moon *et al*., 2014). This paper proposes a different approach. In particular, we introduce a parameter about the number of admissible stockouts per time unit to express the service level (and thus the safety factor, too) as a function of the order quantity. This allows us to optimize safety stock and expected shortage considering the constraint given by this parameter. Our approach has a couple of unique features that complement the traditional approaches. First, the introduced parameter has physical meaning, representing the number of admissible stockouts per time unit. Thus, the obtained optimal replenishment policy would satisfy the constraint on the number of admissible stockouts per time unit. This is useful in practice. For example, the buyer may impose it as a tangible performance indicator concerning the inventory replenishment policy. This may be particularly true in some circumstances, for example, when the cost coefficients are not accurately defined. Second, as it will be evident in later sections, a further benefit is the reduction of the number of decision variables to be considered in the optimization problem. Note that the amount of calculation effort may increase quickly with the number of items; this advantage may not be negligible when the optimization procedure is applied in practice.

The papers that are most closely related to the present paper are Braglia *et al*. (2014b; 2016). However, their models did not include two fundamental aspects: backorders-lost sales mixture and stockout costs. That is, they mainly focused on the optimization of safety stock, thus neglecting the optimization of expected shortage. To authors' knowledge, the joint optimization of safety stock and expected shortage, taking into account the relationship between service level and order quantity given by the parameter about the number of admissible stockouts per time unit, has never been considered in literature. Hence, we extend previous researches to a different and, under many aspects, more general context, In fact, the problem under consideration is more challenging and harder than that in Braglia *et al*. (2014b; 2016). To solve our optimization problem, we will first formulate an exact algorithm, and then propose an efficient heuristic algorithm based on an approximation technique. The main idea of the approximation technique is to replace part of the cost function with an *ad-hoc* Taylor series expansion so that the approximated cost function becomes more tractable. The heuristic algorithm has computational advantages over the exact algorithm. Numerical experiments will illustrate the effectiveness of the proposed algorithms.

To solve the optimization problem, we present an exact algorithm and an efficient approximated procedure. The heuristic algorithm is given with the additional purpose to foster the application in practice of the proposed model. The approximation technique consists in replacing part of the cost function with an *ad-hoc* Taylor series expansion. Numerical experiments investigate the performance achieved by the proposed algorithms and the sensitivity of the model with respect to some fundamental parameters.

The rest of the paper is organized as follows. In Section 2, we give notation and assumptions. In Section 3, we define the problem and formulate the model. In Section 4, we present the solution methods. Numerical experiments are given in Section 5. Finally, conclusions are drawn in Section 6.

2. Notation and assumptions

The following list gives the main notation adopted in the paper:

Decision variables:

q	Order (or shipment) quantity (quantity unit).
n	Number of shipments per production batch.
L	Length of replenishment lead time (time units).
z	Safety factor.
r	Reorder point (quantity unit), which is an equivalent decision variable to z.

Parameters:

P	Production rate (quantity unit/time unit).
S	Setup cost (money/setup).
h_V	Unit stockholding cost rate for the vendor (money/quantity unit/time unit).
D	Demand rate at the buyer (quantity unit/time unit).
σ	Standard deviation of the demand rate at the buyer (quantity unit/time unit).
A	Ordering cost (money/order).
K	Fixed transportation cost (money/shipment).
h_B	Unit stockholding cost rate for the buyer (money/quantity unit/time unit).
β	Fraction of shortage that is lost.
π_1	Fixed penalty cost per unit shortage (money/quantity unit).
π_0	Marginal profit per unit (money/quantity unit).
k	Number of admissible stockouts per time unit.

Random variables:

X	Lead-time demand.

Functions and operators:

f	Standard normal probability density function (p.d.f.).
F	Standard normal cumulative distribution function (c.d.f.).
G	Standard normal loss function.
$E\left[\cdot\right]$	Mathematical expectation.
$\Pr\left\{\cdot\right\}$	Probability function.
x^+	Maximum between 0 and x.
$\mathbf{1}_{\mathcal{A}}\left(\cdot\right)$	Indicator function on the set $\mathcal{A}$.
$\operatorname{sgn}\left(\cdot\right)$	Sign function.

$\text{erf}(\cdot)$ Error function (Abramowitz and Stegun, 1972).

Sets:

$\mathbb{N}$ Natural numbers.

Our model is based on the following main assumptions:

- One vendor supplies a single item to one buyer.
- The buyer orders a lot of size *q*. The vendor manufactures *nq* units with a finite production rate *P* (with $P > D$) at one setup, and ships in quantity *q* to the buyer over *n* times. For each shipment (of size *q*), the buyer pays a fixed transportation cost *K*. The vendor incurs a setup cost *S* for each production run of size *nq*. The buyer bears an ordering cost A/n for each order of size *q*.
- The buyer adopts a continuous-review (*r*,*q*) inventory replenishment policy. An order of size *q* is placed when the on-hand inventory level drops to the reorder point *r*.
- Shortages are allowed and partially backordered with ratio $1-\beta$. The fraction of shortage with ratio β is lost.
- The reorder point is given by $r = DL + z\sigma\sqrt{L}$, where the first addendum is the expected lead-time demand and the second one is the safety stock.
- The lead time demand *X* is a Gaussian random variable with mean *DL* and standard deviation $\sigma\sqrt{L}$. The expected shortage per cycle is, therefore, given by $E\left[\left(X-r\right)^{+}\right] = \sigma\sqrt{L}G(z)$ (see, *e.g*., Ouyang *et al*. (2004)).
- The time horizon is infinite.

We further assume that the replenishment lead time consists of *M* deterministic and mutually independent components (Chang *et al*., 2006; Jha and Shanker, 2009; Lin *et al*., 2011; Lin, 2013; Panda *et al*., 2014). The generic *m*th component has a minimum duration b_m, a normal duration s_m and a crashing cost per time unit c_m. We order the lead time components in a way such that $c_1 \le c_2 \le ... \le c_M$. Components are crashed one at a time starting with that of least unit crashing cost. If L_m is the length of lead time with components $1, 2, ..., m$ crashed to their minimum durations, then we can write

$$L_m = L_0 - (s_1 - b_1) - (s_2 - b_2) - ... - (s_m - b_m),$$

where $L_0 \equiv \sum_m s_m$.

The lead-time crashing cost $R(L)$ is, therefore, given by

$$R(L) = \sum_{m=1}^{M} 1_{\{L: L_m \le L \le L_{m-1}\}}(L) R_m(L),\ L \in [L_M, L_0], \tag{1}$$

where

$$R_m(L) = c_m(L_{m-1} - L) + c_1(s_1 - b_1) + c_2(s_2 - b_2) + ... + c_{m-1}(s_{m-1} - b_{m-1}).$$

We can note that $R(L)$ is a piecewise-linear, decreasing function in the interval $[L_M, L_0]$. It is also continuous and convex in $[L_M, L_0]$.

3. Problem definition and model formulation

We focus on a single vendor-single buyer integrated supply chain. It is worth noting that such inventory system represents a milestone in the JELS literature and is still of interest to researchers. The problem under consideration is to optimize the decision variables including the order quantity, the number of shipments per production batch, the length of replenishment lead time, and the safety factor by minimizing the total system cost incurred to both supplier and buyer. The controllable lead time is modelled by a set of independent components that can be shortened by paying a crashing cost. The backorders-lost sales mixture is modelled by a parameter to represent the fraction of shortage that is lost. More importantly, we have introduced a parameter (*k*) to denote the number of admissible stockouts per time unit, which will be treated as a constraint to the optimization problem.

3.1. Standard formulation

In this subsection, we will first derive the standard formulation of the expected total cost per time unit given the assumptions stated in Section 2. After that, we will discuss the need of a new approach in order to take into account the number of admissible stockouts per time unit.

In Figure 1, it is possible to observe how the vendor's inventory and the buyer's inventory vary over time for a sample case with $n = 4$. First, let us derive the costs associated with the buyer. The cycle length is given by q/D. The expected cost per time unit relevant to each shipment is $(A/n + K)D/q$. The expected backorders per cycle and the expected lost sales per cycle are given by $(1-\beta)E\left[(X-r)^+\right]$ and $\beta E\left[(X-r)^+\right]$, respectively. The stockout cost per time unit is therefore given by $\bar{\pi}\sigma\sqrt{L}G(z)D/q$, where $\bar{\pi} \equiv \pi_1 + \beta\pi_0$, which represents the sum of the penalty cost for shortage and the loss of profit due to partial lost sales.. The expected net inventory levels just before receiving an order and at the beginning of the cycle are given by $r - DL + \beta E\left[(X-r)^+\right]$ and $q + r - DL + \beta E\left[(X-r)^+\right]$ respectively. Hence, the expected stockholding cost per time unit is $h_B\left(q/2 + z\sigma\sqrt{L} + \beta\sigma\sqrt{L}G(z)\right)$. The last cost component relevant to the buyer is the lead-time crashing cost per time unit, which is given by $R(L)D/q$.

Second, let us consider the costs that pertain to the vendor. The cycle length for the vendor is nq/D. The expected setup cost per time unit is $\frac{SD}{nq}$. As observed, *e.g.*, by Ouyang *et al.* (2004), the average inventory of the vendor is $q/2\left[n-1-(n-2)D/P\right]$. Therefore, the expected stockholding cost per time unit is $h_V q/2\left[n-1-(n-2)D/P\right]$.

FIGURE 1 HERE

In summary, the long-run expected total cost per time unit for the considered system, in the standard formulation, is:

$$\begin{aligned}\bar{C}(q,z,n,L) \;=\; & (A+S+nK)\frac{D}{nq} + h_B\left(\frac{q}{2} + z\sigma\sqrt{L} + \beta\sigma\sqrt{L}G(z)\right) \\ & + \frac{D}{q}\bar{\pi}\sigma\sqrt{L}G(z) + \frac{D}{q}R(L) + h_V\frac{q}{2}\left[n-1-(n-2)\frac{D}{P}\right].\end{aligned} \tag{2}$$

Given this formulation, the safety stock and the expected shortage are typically optimized considering *q* and *z* as mutually independent decision variables. Additionally, a service level constraint could also be imposed (Jha and Shanker, 2009; Lin *et al.*, 2011; Shahpouri *et al.*, 2013; Moon *et al.*, 2014). However, the above formulation and solution does not ensure the number of stockouts per unit time, which is a more tangible performance indicator. Different from the above traditional approach, we introduce an explicit parameter *k* about the number of admissible stockouts per time unit that converts the service level (and thus the safety factor as well) into a function depending on the order quantity. Such treatment permits us to optimize safety stock and expected shortage, and hence the inventory replenishment policy, taking into account the constraint imposed by *k*. It is worth noting that a further relevant benefit of such treatment is the reduction of the number of decision variables to be considered in the optimization problem. In practical terms, when the amount of calculations needed grows with the number of items, this advantage may not be negligible.

It should be pointed out that in practice the parameter k is helpful to operational managers as it is an important tangible performance indicator. On the other hand, by pre-specifying an appropriate level of k, our approach is able to design the corresponding optimal inventory replenishment policy.

In the next section, we will reformulate Eq. (2) to consider the parameter k so as to put the service level (denoted as p), and consequently service factor z, in functional dependence with the order quantity q.

3.2. New formulation

The new formulation is developed based upon the fundamental relationship between service level p and order quantity q. This expression links p and q by means of the parameter k about the number of admissible stockouts per time unit.

In accordance to Braglia *et al*. (2014b; 2016), the service level p can be expressed as follows:

$$p = p(q) = 1 - \frac{k}{\frac{D}{nq}n} = 1 - \frac{k}{\frac{D}{q}} = 1 - \frac{q}{\xi}, \tag{3}$$

where $\xi \equiv D/k$. Therefore, given k (and D), p is fully determined for any value q. It is possible to assume that $\xi > 1$, as the demand rate is typically larger than the number of admissible stockouts per time unit. Note that the notation $p = p(q)$ would emphasize the functional dependence between service level and order quantity.

It is therefore evident that the introduction of k has allowed us to establish a functional relationship between service level and order quantity. As observed in the previous section, this is a practically important aspect that has been rarely considered in the inventory management literature.

Since $0 < p < 1$, we can write $0 < 1 - q/\xi < 1$; that is, the relation $0 < q < \xi$ must hold. We recall that, in general, the quantity $\alpha \equiv 1 - p$ is the stockout probability. In our model, α represents the accepted stockout risk per ordering cycle.

Let F^{-1} be the quantile function of the standard normal distribution. Under our assumptions, the safety factor z and the service level p are linked through the equation $z = F^{-1}(p)$. This relation can be rewritten as follows by taking into account Eq. (3):

$$z = z(q) = F^{-1}\left(1 - \frac{q}{\xi}\right). \tag{4}$$

If we now insert Eq. (4) into Eq. (2), we get

$$\begin{aligned} C(q,n,L) \quad = \quad & (A + S + nK)\frac{D}{nq} + h_B\left(\frac{q}{2} + z(q)\sigma\sqrt{L} + \beta\sigma\sqrt{L}G(z(q))\right) \\ & + \frac{D}{q}\bar{\pi}\sigma\sqrt{L}G(z(q)) + \frac{D}{q}R(L) + h_V\frac{q}{2}\left[n - 1 - (n-2)\frac{D}{P}\right]. \end{aligned} \tag{5}$$

We can note that, differently from the standard formulations, the safety stock (*i.e.*, $z(q)\sigma\sqrt{L}$) and the expected shortage at the end of the cycle (*i.e.*, $\sigma\sqrt{L}G(z(q))$) are function of the order quantity q. According to Eq. (4), these quantities are parameterized by the number of admissible stockouts per time unit (i.e. k). Therefore, the optimal order quantity defines the optimal level of safety stock and expected shortage that satisfies the constraint given by k.

The optimization of safety stock and expected shortage, and hence of the inventory replenishment policy, under the constraint given by k can be achieved by solving the following problem:

$$
\begin{aligned}
(\mathrm{P}) \quad & \min \quad C(q,n,L) \\
& \text{s.t.} \quad q \in \left]0,\xi\right[, \\
& \qquad\; n \in \mathbb{N}, \\
& \qquad\; L \in [L_M, L_0].
\end{aligned}
$$

4. Solution methods for problem (P)

4.1. Exact optimization algorithm

The cost function in Eq. (5) is new and there is no existing algorithm available to solve it. We will propose two algorithms to solve it. Firstly, we start with an analyse of the relationships between the cost function $C(q, n, L)$ and the decision variables. The purpose is to obtain some interesting properties that could be used to design the exact optimization algorithm for problem (P). For fixed (q,n) and $L \in [L_m, L_{m-1}]$, we have

$$\frac{\partial^2}{\partial L^2} C(q,n,L) = -\frac{\sigma}{4}\frac{D}{qL^{\frac{3}{2}}}\left[h_B q z(q) + G(z(q))(\bar{\pi} D + q\beta)\right] < 0,$$

which means that $C(q,n,L)$ is strictly concave in L. Hence, the minimum in L lies on one of the endpoints of $[L_m, L_{m-1}]$; that is, we can write

$$\min_{(q,n,L)} C(q,n,L) = \min\left\{\min_{(q,n)} C(q,n,L_m) \mid m = 0,1,\ldots,M\right\}.$$

If we relax the integrality constraint on n, we can note that

$$\frac{\partial^2}{\partial n^2} C(q,n,L) = \frac{2D(A+S)}{qn^3} > 0.$$

Therefore, $C(q,n,L)$ is strictly convex in n, for fixed (q,L).

Now, let us consider a fixed (n,L). We can observe that $z(q) \to +\infty$ as $q \to 0^+$. In fact, $p(q) \to 1$ as $q \to 0^+$, and $F(z) \to 1$ asymptotically, *i.e.*, as $z \to +\infty$. With a similar argument, we have that $z(q) \to -\infty$ as $q \to \xi^-$. Consequently, it follows that $G(z(q)) \to +\infty$ as $q \to \xi^-$ and that $G(z(q)) \to 0$ as $q \to 0^+$.

Given the above limit properties, it is therefore possible to show that $C(q,n,L) \to \infty \cdot \operatorname{sgn}(k\bar{\pi} - h_B(1-\beta))$ as $q \to \xi^-$ and that $C(q,n,L) \to +\infty$ as $q \to 0^+$. Moreover, since the shortage cost is typically greater than the stockholding cost over the inventory cycle of the buyer, one can reasonably assume that $k\bar{\pi} - h_B(1-\beta) > 0$. Hence, we can write that $C(q,n,L) \to +\infty$ as q tends to 0^+ or ξ^-. If we further consider that $C(q,n,L)$ is continuous in q for $q \in \left]0,\xi\right[$, then the minimum of $C(q,n,L)$ in q over $\left]0,\xi\right[$ necessarily lies on a stationary point.

It is not possible to analytically determine the number of stationary points that $C(q,n,L)$ admits in q over $\left]0,\xi\right[$. This is based on two facts: (*i*) the First-Order Condition of optimality in q cannot be solved in closed form, and (*ii*) to prove the convexity of $C(q,n,L)$ in q is hard. Therefore, the problem concerned with the optimization of $C(q,n,L)$ in q over $\left]0,\xi\right[$, for fixed (n,L), can only be solved by adopting an iterative/numerical technique or a meta-heuristic algorithm. We would, however, observe that extensive numerical experiments have shown that $C(q,n,L)$ is convex in q

over $]0,\xi[$, for fixed (n,L), under a wide range of parameter values. This suggests that $C(q,n,L)$ admits a unique stationary point in q over $]0,\xi[$.

In conclusion, if we let C^* denote the minimum cost and (q^*,n^*,L^*) denote the optimal solution, we propose the following exact algorithm to solve problem (P):

Algorithm 1.

Step 1. Set $C^* = +\infty$.

Step 2. For each $m = 0,1,\ldots,M$ *, do Steps 2.1-2.4.*

Step 2.1. Set $n=1$ *and* ${}^*C = +\infty$.

Step 2.2. Find $\hat{q} \leftarrow \arg\min_{q\in]0,\xi[} C(q,n,L_m)$.

Step 2.3. If ${}^*C \ge C(\hat{q},n,L_m)$*, then set* ${}^*C \leftarrow C(\hat{q},n,L_m)$, ${}^*q \leftarrow \hat{q}$, $n \leftarrow n+1$ *and go to Step 2.2; otherwise, set* ${}^*n \leftarrow n-1$ *and go to Step 2.4.*

Step 2.4. If $C^* \ge C({}^*q,{}^*n,L_m)$*, set* $C^* \leftarrow C({}^*q,{}^*n,L_m)$, $q^* \leftarrow {}^*q$, $n^* \leftarrow {}^*n$ *and* $L^* \leftarrow L_m$.

The stockout probability, the safety factor and the safety stock corresponding to the optimum solution are respectively given by $\alpha^* \equiv \frac{q^*}{\xi}$, $z^* \equiv F^{-1}(1-\alpha^*)$ and $U^* \equiv z^*\sigma\sqrt{L}$. Note that the proposed exact algorithm in Algorithm 1 makes use of the analytical properties of the cost function that were established earlier on. Moreover, Algorithm 1 is able to find the optimal solution to problem (P). An alternative method is to use meta-heuristic algorithms to optimize (q, n, L) simultaneously. However, applying meta-heuristic algorithms directly to optimize multi-dimensional problems is more time-consuming and cannot guarantee the optimality of the (Pasandideh *et al.*, 2011; Shahvari *et al.*, 2012; Shahvari and Logendran, 2016; 2017).

In Algorithm 1, the one-dimensional minimization problem at Step 2.2 is solved using an iterative/numerical technique or a meta-heuristic algorithm. This aspect may incur difficulties in applying the model in practice (Platt *et al.*, 1997; Eynan and Kropp, 2007; Castellano, 2016). For example, there may be technical difficulties strictly concerned with the implementation of the particular method chosen to solve the sub-problem at Step 2.2, and/or the issues related to the required computational effort, which may become significant as the number of items to be managed grows. For that reason, the development of an efficient approximated solution procedure to approach problem (P), which is able to find a “good” solution at the cost of a reasonable computational effort, is therefore encouraged. We will propose a heuristic optimization algorithm in the next section.

4.2. Heuristic optimization algorithm

The heuristic solution method is based on the idea of replacing $z(q)$ and $G(z(q))$, in Eq. (5), with the respective second-order Taylor series expansion in q centred in

$$\bar{q} \equiv \sqrt{\frac{2J(n,L)}{h_B + 2H(n)}},$$

where $J(n,L) \equiv [n-1-(n-2)D/P]h_V/2$ and $H(n) \equiv D[(A+S)1/n + K + R(L)]$. It is possible to note that $\bar{q}$ is the q-value that minimizes the expected total cost in deterministic conditions, for a given (n,L). A similar approximation method has been used in literature (Eynan and Kropp, 2007; Braglia *et al.*, 2016; Castellano, 2016).

We would informally observe that the minimum-cost solution in stochastic conditions lies in a neighbourhood of the minimum-cost solution in deterministic conditions, for the same inventory system and with same parameter values. The “distance” between the two solutions clearly increases as the degree of randomness of the system (evaluated, for example, by $Cv \equiv \sigma/D$) grows.

Therefore, following an intuitive argument, the smaller *Cv*, the better the approximation of the optimum provided by the heuristic approach. In the numerical study section, details about the error will be provided.

With reference to a neighbourhood of $\bar{q}$ and recalling that $G(z) = f(z) - z(1-F(z))$ (see, *e.g.*, Ouyang *et al*. (2004)), we can write

$$z(q) \approx d_1 + d_2(q-\bar{q}) + \frac{1}{2}d_3(q-\bar{q})^2, \tag{6}$$

$$G(z(q)) \approx d_4 + d_5(q-\bar{q}) + \frac{1}{2}d_6(q-\bar{q})^2, \tag{7}$$

where

$$d_1 \equiv z(\bar{q}),$$

$$d_2 \equiv -\frac{1}{\xi f(z(\bar{q}))},$$

$$d_3 \equiv \frac{z(\bar{q})}{\left[\xi f(z(\bar{q}))\right]^2},$$

$$d_4 \equiv G(z(\bar{q})),$$

$$d_5 \equiv \frac{1-F(z(\bar{q}))}{\xi f(z(\bar{q}))},$$

$$d_6 \equiv \frac{G(z(\bar{q}))}{\left[\xi f(z(\bar{q}))\right]^2}.$$

With some algebraic manipulations, Eqs. (6) and (7) can conveniently be rewritten as follows:

$$z(q) \approx g_1 + g_2 q + g_3 q^2, \tag{8}$$

$$G(z(q)) \approx g_4 + g_5 q + g_6 q^2, \tag{9}$$

where

$$g_1 \equiv d_1 - d_2\bar{q} + \frac{1}{2}d_3\bar{q}^2,$$

$$g_2 \equiv d_2 - d_3\bar{q},$$

$$g_3 \equiv \frac{1}{2}d_3,$$

$$g_4 \equiv d_4 - d_5\bar{q} + \frac{1}{2}d_6\bar{q}^2,$$

$$g_5 \equiv d_5 - d_6\bar{q},$$

$$g_6 \equiv \frac{1}{2}d_6.$$

According to Eqs. (8) and (9), $C(q,n,L)$ can, therefore, be approximated in a neighbourhood of $\bar{q}$ by the following function:

$$\hat{C}(q,n,L) = \frac{u}{q} + vq + wq^2 + y,$$

whose coefficients are given by

$$u \equiv J(n,L) + g_4\sigma\bar{\pi}D\sqrt{L},$$

$v \equiv H(n)+\frac{h_B}{2}+g_6\sigma D\bar{\pi}\sqrt{L}+h_B\sigma(g_2+\beta g_5)\sqrt{L}$,

$w \equiv h_B\sigma(g_3+\beta g_6)\sqrt{L}$,

$y \equiv g_5\sigma\bar{\pi}D\sqrt{L}+h_B\sigma(g_1+\beta g_4)\sqrt{L}$.

For fixed (n,L), $\hat{C}(q,n,L)$ corresponds to the deterministic EOQ cost structure plus a constant and a quadratic term in *q*. Imposing the First-Order Condition of optimality in *q*, we have

$$\frac{\partial}{\partial q}\hat{C}(q,n,L)=0 \Leftrightarrow N(q,n,L)=0,$$

where

$$N(q,n,L)=2wq^3+vq^2-u,$$

which is a cubic equation in *q* (for fixed (n,L)).

Let us assume that the coefficient of $N(q,n,L)$ are positive. It is possible to note that $N(q,n,L)$ always (*i.e.*, for any configuration of its roots) admits a unique positive real root. Let us denote such root by $\hat{q}$. The explicit expression of $\hat{q}$ can easily be found according to the procedure proposed by Nickalls (1993).

In conclusion, the following heuristic algorithm can be adopted to approach problem (P) so as to find a near-optimal solution $(\tilde{q},\tilde{n},\tilde{L})$ and the corresponding cost $\tilde{C}$:

Algorithm 2.

Step 1. Set $\tilde{C}=+\infty$.

Step 2. For each $m=0,1,...,M$ *, do Steps 2.1-2.4.*

Step 2.1. Set $n=1$ *and* ${}^*\tilde{C}=+\infty$.

Step 2.2. Find $\hat{q}$ *as the unique positive solution to* $N(q,n,L)=0$.

Step 2.3. If ${}^*\tilde{C}\ge C(\hat{q},n,L_m)$*, then set* ${}^*\tilde{C}\leftarrow C(\hat{q},n,L_m)$, ${}^*\tilde{q}\leftarrow\hat{q}$, $n\leftarrow n+1$ *and go to Step 2.2; otherwise, set* ${}^*\tilde{n}\leftarrow n-1$ *and go to Step 2.4.*

Step 2.4. If $\tilde{C}\ge C({}^*\tilde{q},{}^*\tilde{n},L_m)$*, set* $\tilde{C}\leftarrow C({}^*\tilde{q},{}^*\tilde{n},L_m)$, $\tilde{q}\leftarrow{}^*\tilde{q}$, $\tilde{n}\leftarrow{}^*\tilde{n}$ *and* $\tilde{L}\leftarrow L_m$.

The stockout probability, the safety factor and the safety stock relevant to the solution $(\tilde{q},\tilde{n},\tilde{L})$ are given by $\tilde{\alpha}\equiv\tilde{q}/\xi$, $\tilde{z}\equiv F^{-1}(1-\tilde{\alpha})$ and $\tilde{U}\equiv\tilde{z}\sigma\sqrt{L}$, respectively.

5. Numerical study

5.1. Performance analysis

In this section, we first evaluate the error achieved by the heuristic algorithm. Then, the exact algorithm is compared with the heuristic algorithm in terms of computational time required to solve a batch of randomly generated problems.

With regard to the error analysis, we adopt the Design of Experiments approach. In this way, we are able to assess both the magnitude of the error that is achieved and the influence of parameters on the error. For each parameter, we consider two disjoint intervals of values. That is, we assume that a parameter can take values belonging to two different levels: “low” (labelled “1”) or “high” (labelled “2”). For each combination of parameter levels, 30 trials have been carried out. In a given trial, parameter values are randomly drawn within the interval relevant to the level that corresponds to each parameter.

The magnitude of the error that pertains to a certain combination of parameter levels is evaluated by means of the Mean Absolute Percentage Error (MAPE) calculated over all 30 trials:

$$\text{MAPE} \equiv \frac{1}{30}\sum_{i=1}^{30}\text{APE}_i ,$$

where APE_i is the Absolute Percentage Error in the *i*th trial:

$$\text{APE}_i \equiv \frac{\left|C_i^* - \tilde{C}_i\right|}{C_i^*}\times 100 .$$

In the expression of APE_i, C_i^* is the cost of the optimum solution obtained by Algorithm 1, while $\tilde{C}_i$ is the cost of the heuristic solution obtained by Algorithm 2.

The minimization problem at Step 2.2 in Algorithm 1 has been approached exploiting a Simulated Annealing (SA) algorithm taken from Optimization Toolbox™ in MATLAB® R2015b. Parameters have been kept to default values. SA is selected due to its proved effectiveness and robustness to solve minimization problems (Suman and Kumar, 2006).

In our experiments, we take the year as time unit. The lead time is assumed to be made of three components, whose duration and unit crashing cost are given in Table 1. Table 2 shows the intervals associated with the parameter levels.

TABLE 1 HERE

TABLE 2 HERE

It is worth noting that Table 2 shows the coefficient of variation *Cv* of the demand rate instead of its standard deviation *σ*, which can, therefore, be readily obtained as $\sigma = Cv \cdot D$.

We would remind the reader that the normal approximation to the lead-time demand is appropriate for small values of *Cv*. That is, the approximation is more accurate for increasingly smaller values of $\Pr\{X \le 0\}$ (Zipkin, 2000).

If we let Cv_X be the coefficient of variation of the lead-time demand, it is possible to observe that $Cv_X < Cv$, which implies

$$\Pr\{X \le 0\} = \frac{1}{2}\left[1-\text{erf}\left(\frac{\sqrt{2}}{2Cv_X}\right)\right] < \mu(Cv) = \frac{1}{2}\left[1-\text{erf}\left(\frac{\sqrt{2}}{2Cv}\right)\right].$$

In Figure 1, it is possible to see a plot of $\mu(Cv)$ for $0 < Cv \le 0.5$. To assure a small $\Pr\{X \le 0\}$ in our numerical experiments, we have taken $Cv \le 0.4$ ($\mu(0.4) \cong 0.6\%$).

FIGURE 2 HERE

The results of the error analysis are shown in Figure 3. The first evident outcome is that the maximum MAPE is about 0.015%. This result is satisfactory and proves numerically that the heuristic procedure is actually effective. For what concerns the effect of parameters on the error, we can observe that:

- The parameters that greatly affect the error with positive direction (*i.e.*, the higher the parameter value, the greater the error) are *Cv*, *D* and h_B.
- The parameters that greatly affect the error with negative direction (*i.e.*, the higher the parameter value, the smaller the error) are *K*, *β*, h_V and π_1.
- The influence given by the other parameters is practically negligible.

FIGURE 3 HERE

With regard to the analysis about the computational effort required, we compare the performance of the heuristic procedure (Algorithm 2) with that of the exact algorithm (Algorithm 1). The minimization problem at Step 2.2 of Algorithm 1 is again approached with SA.

The comparison has been carried out in terms of computational time needed to solve 1000 randomly generated problems. Parameter values have been randomly drawn within the intervals in Table 2. Although the time difference on a single problem is in the order of a few seconds (on average), the discrepancy among performances over several problems may become significant.

We would further observe that thousands of items may be needed to manage in practice (*e.g.*, consider a large retailer). Hence, the control variables (*i.e.*, the order quantity, the number of shipments and the lead time) necessitate to be determined for all items. Moreover, these decision variables plausibly require to be recalculated frequently as the system parameters may change over time.

Tests have been made on a PC with an Intel® Core™ i7 processor at 2.4GHz with 16GB of RAM. We have observed the following results:

- Algorithm 1 has required 14,124 seconds;
- Algorithm 2 has required 35 seconds.

The advantage in terms of computational time provided by the heuristic algorithm is evident: the percentage of computational time reduction is more than 99%.

In conclusion, our numerical tests have shown that the approximated minimization procedure is highly efficient. Comparing to the exact algorithm, the heuristic algorithm achieves practically negligible error and the computational time is substantially smaller. Therefore, the approximated procedure appears to be promising for a practical application.

5.2. Sensitivity analysis

In this section, we evaluate how the minimum-cost solution $\left(q^*, n^*, L^*\right)$, the minimum cost C^* and other significant quantities (*i.e.*, the stockout probability α^* and the safety stock U^*) react to variations in the values of *Cv* of *k*. The parameter values relevant to the lead-time components are given in Table 1. Table 3 shows the value of those parameters that have been kept fixed. The results of this analysis are shown in Table 4. We can draw the following conclusions about this analysis.

For fixed *Cv*, q^* is increasing for small values of *k*. For $0.1 \le k \le 0.3$, there is a change in the trend; that is, q^* becomes decreasing in *k*. This becomes more noticeable for $0.3 \le k \le 1.1$. For small *Cv*, this fact is not clearly visible due to the order of magnitude adopted in the table. However, it can be observed at the fourth decimal place. The opposite behaviour characterizes n^*. In fact, it decreases (increases) as q^* increases (decreases). This is particularly evident for $Cv \ge 0.4$. With fixed *k*, q^* is increasing as *Cv* increases, while n^* is decreasing in *Cv*. A similar behaviour to q^* characterizes C^*. This first results permit to observe that, for fixed *Cv*, the optimal order quantity becomes smaller as the number of admissible stockouts per time unit increases. That is, if the manager accepts a greater stockout risk, then the optimal order quantity decreases, and consequently the (optimal) number of shipments increases (the trend of q^* and n^* is clearly opposite). On the other hand, for fixed *k*, if the variability (*i.e.*, *Cv*) increases, then the system tends to increase the optimal order quantity and to decrease the number of shipments. This effect is due to the natural behaviour of the system to contrast (because *k* is fixed) the growth of the stockout probability, which evidently tends to increase with the variability of the lead-time demand.

For fixed *Cv*, L^* is constant in *k*, while it is decreasing in *Cv* for fixed *k*. This is an expected result since a higher variability leads the system to shorten the lead time so as to optimize replenishments. In fact, if *k* is fixed, then the system tends to react to the growth in the variability in order to compensate the increase of the stockout probability. That is, if the variability of the lead-time demand (which is an increasing function of L) increases because Cv grows, then the system evidently tends to shorten L.

Let us now consider the behaviour of the stockout probability α^*. It is clearly increasing in k, for fixed Cv. This is not surprising as a greater k means accepting a higher number of stockouts, which therefore leads to a larger stockout probability. The same happens if we let Cv vary and keep k fixed. This is evident as Cv measures the system variability. Hence, if the variability in the system increases, for fixed k, then the stockout probability grows.

With fixed Cv, the safety stock U^* diminishes as k increases. On the contrary, U^* grows with Cv for a given k. In fact, the safety stock operates as a buffer designed to protect the system against randomness in the demand. It is therefore obvious that the higher the variability (*i.e.*, Cv), the larger the buffer (*i.e.*, U^*).

TABLE 3 HERE

TABLE 4 HERE
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6. Conclusions

When facing uncertain demand, two important issues deserve more attention. The first is lead time control, which is an effective mechanism to reduce the variability of demand during the lead time. The second is the way to represent customers' purchase behaviour when facing stockouts, e.g. backorders or lost sales. The aim of this paper was to optimize safety stock and expected shortage in an integrated supply chain consisting of a single vendor and a single buyer. The system under consideration has the following features: backorders-lost sales mixture, controllable lead time, stochastic demand, and stockout costs. The decision variables include order quantity, reorder point, and number of shipments and length of lead time.

Our main contributions include: (i) we investigated the inventory replenishment problem in an integrated supply chain with backorders-lost sales mixture and controllable lead time, which has been understudied in the literature. Traditionally, customers' purchase behaviour is often assumed to be backorders only or lost sales only. However, the backorders-lost sales mixture is a more appropriate representation since it generalizes the case of backorders or lost sales (e.g. they can be included in our model by letting $\beta = 0$ or 1). Hence, our study enriches the literature on inventory management in integrated supply chains. (ii) Contrarily to standard approaches that treat the order quantity and the service level as mutually independent decision variables, we adopted a novel approach to formulate the optimization problem. Specifically, we put the service level and the order quantity in functional dependence by means of the parameter about the number of admissible stockouts per time unit. This parameter was introduced as a constraint in the problem of determining the replenishment policy (*i.e.*, order quantity, reorder point, and number of shipments) and the length of lead time that minimize the long-run expected total cost per time unit. Hence, the optimal policy obtained through our approach defines the best level of safety stock and expected shortage that satisfies the constraint given by the number of admissible stockouts per time unit. (iii) To solve the optimization problem, we present two algorithms: an exact algorithm and a heuristic algorithm. These two algorithms were developed based on some analytical properties that we established by analysing the cost function in relation to the decision variables. The heuristic algorithm employed an approximation technique based on an ad-hoc Taylor series expansion. The first set of numerical experiments demonstrated that the heuristic solution procedure can find near-optimal solutions in a wide range of scenarios, and is highly efficient in terms of required computational effort. Additional tests were performed to investigate the sensitivity of the model with respect to some system parameters.

From the practical perspective, an operational manager may impose the number of admissible stockouts per time unit as a tangible performance indicator concerning the inventory replenishment policy. By pre-specifying an appropriate level of this parameter, our approach can be employed to design the optimal inventory replenishment policy subject to the specified constraint. The proposed heuristic solution procedure is able to produce high quality solutions with small computational effort, which implies that the solution procedure is readily applicable in practice. Moreover, the results obtained from the sensitivity analysis offer useful managerial insights into how the decision variables should response to the changes of key system parameters.

Several future developments are possible. The sensitivity analysis has shown that a minimum cost with respect to the number of admissible stockouts per time unit could exist. Hence, a method that permits to optimize costs with respect to this parameter may be investigated. It should be noted that a major limitation of the considered supply chain model is that it is relatively simple. Therefore, future works may be devoted to extend the proposed approach to more complex inventory systems. In addition, other types of uncertainties, e.g. machine breakdown, unreliable transportation, and defective or deteriorating items, could be incorporated.

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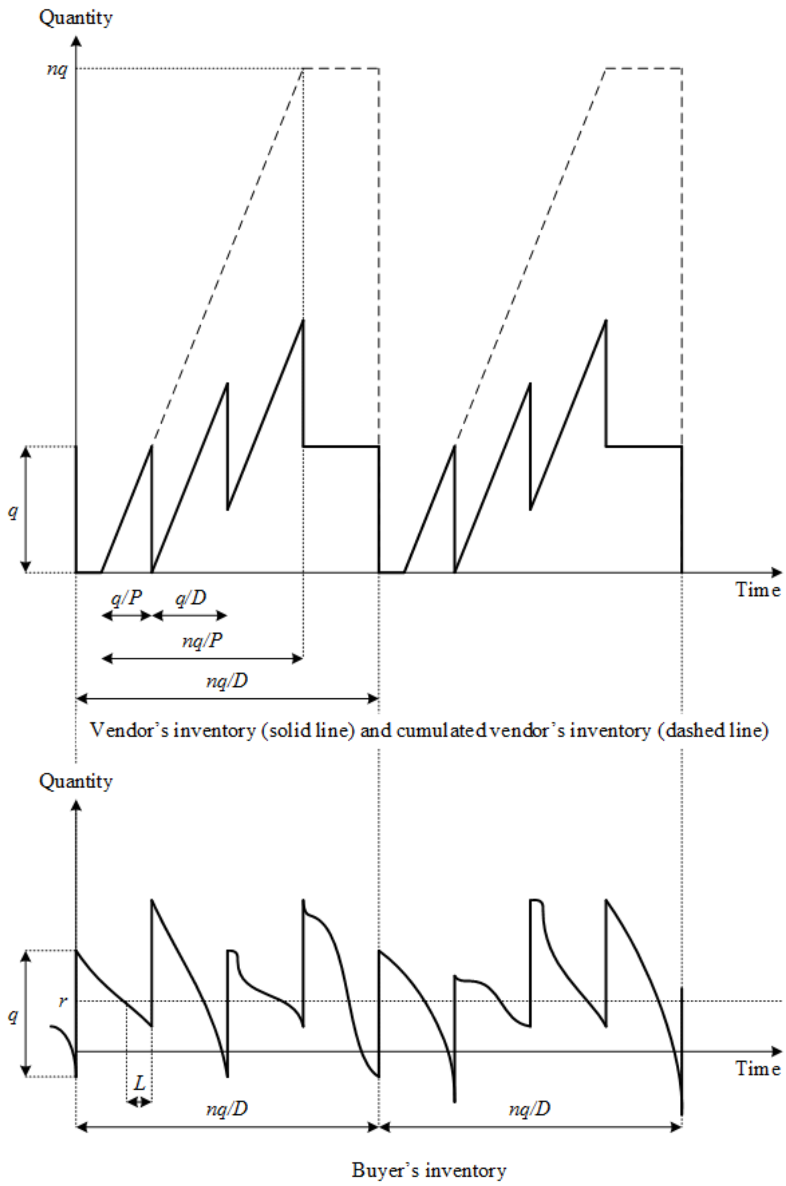

Figure 1. Inventory pattern for the vendor and the buyer.

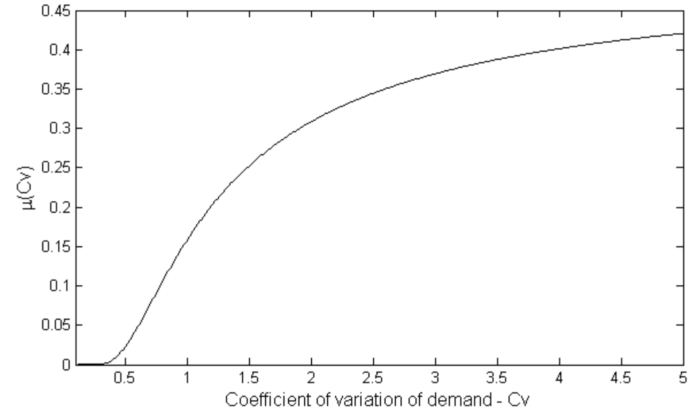

Figure 2. Plot of $\mu(Cv)$ for $0 < Cv \le 5$.

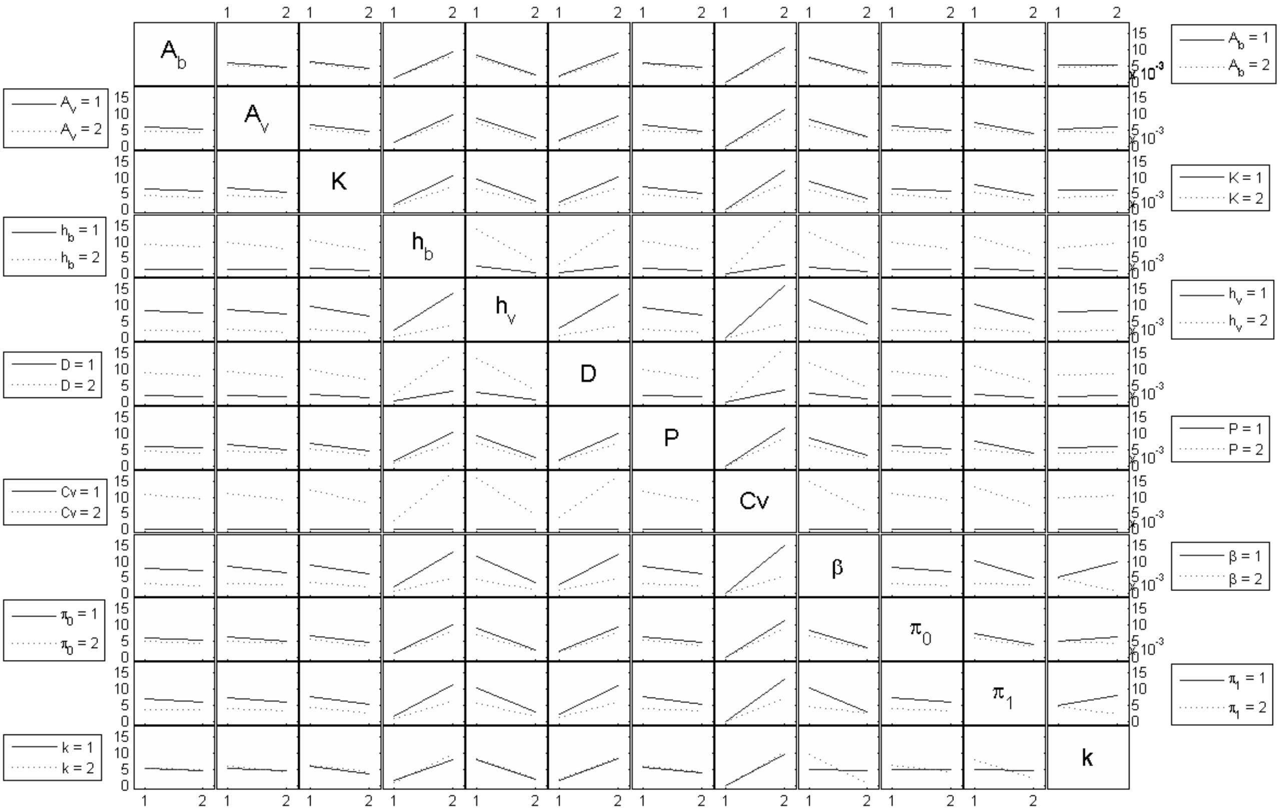

Figure 3. Results of the error analysis.

Component - m	Duration (days)		Unit crashing cost (moneys/day)
	Normal - s_m	*Minimum* - b_m	
1	12	7	0.3
2	12	7	2.1
3	10	8	5.3

Table 1. Lead time data.

Parameters	Levels	
	Low ("1")	High ("2")
A_B	[50, 70]	[100, 120]
A_V	[140, 160]	[190, 210]
K	[15, 25]	[35, 45]
h_B	[15, 18]	[25, 28]
h_V	[2, 6]	[10, 14]
D	[700, 1000]	[1400, 1700]
P	[2000, 2300]	[2700, 3000]
Cv[1]	[0.05, 0.15]	[0.3, 0.4]
β	[0.1, 0.2]	[0.8, 0.9]
π_0	[80, 90]	[140, 150]
π_1	[20, 30]	[60, 70]
k	[0.07, 0.10]	[0.7, 1.0]

[1]Coefficient of variation of demand, i.e., $Cv \equiv \sigma/D$.

Table 2. Intervals where parameters take values.

Parameters	Values
A_B	107
A_V	204
K	18
h_B	28
h_V	10
D	797
P	2278
β	0.87
π_0	148
π_1	28

Table 3. Parameter values adopted in the sensitivity analysis.

	Number of admissible stockouts per year - *k*							
Cv	*0.01*	*0.05*	*0.1*	*0.3*	*0.5*	*0.7*	*0.9*	*1.1*
0.001	q^* = 34.34	q^* = 34.34	q^* = 34.34	q^* = 34.34	q^* = 34.34	q^* = 34.33	q^* = 34.32	q^* = 34.32
	n^* = 8	n^* = 8	n^* = 8	n^* = 8	n^* = 8	n^* = 8	n^* = 8	n^* = 8
	L^* = 34	L^* = 34	L^* = 34	L^* = 34	L^* = 34	L^* = 34	L^* = 34	L^* = 34
	C^* = 2665	C^* = 2662	C^* = 2661	C^* = 2661	C^* = 2663	C^* = 2665	C^* = 2668	C^* = 2671
	α^* = 0.04%	α^* = 0.22%	α^* = 0.43%	α^* = 1.29%	α^* = 2.15%	α^* = 3.02%	α^* = 3.88%	α^* = 4.74%
	U^* = 0.81	U^* = 0.69	U^* = 0.64	U^* = 0.54	U^* = 0.49	U^* = 0.46	U^* = 0.43	U^* = 0.41
0.008	q^* = 34.51	q^* = 34.53	q^* = 34.54	q^* = 34.54	q^* = 34.51	q^* = 34.47	q^* = 34.41	q^* = 34.35
	n^* = 8	n^* = 8	n^* = 8	n^* = 8	n^* = 8	n^* = 8	n^* = 8	n^* = 8
	L^* = 34	L^* = 34	L^* = 34	L^* = 34	L^* = 34	L^* = 34	L^* = 34	L^* = 34
	C^* = 2825	C^* = 2802	C^* = 2795	C^* = 2796	C^* = 2809	C^* = 2828	C^* = 2849	C^* = 2873
	α^* = 0.04%	α^* = 0.22%	α^* = 0.43%	α^* = 1.30%	α^* = 2.16%	α^* = 3.03%	α^* = 3.89%	α^* = 4.74%
	U^* = 6.48	U^* = 5.55	U^* = 5.11	U^* = 4.33	U^* = 3.93	U^* = 3.65	U^* = 3.43	U^* = 3.25
0.04	q^* = 39.03	q^* = 39.15	q^* = 39.20	q^* = 39.18	q^* = 39.02	q^* = 38.80	q^* = 35.23	q^* = 34.94
	n^* = 7	n^* = 7	n^* = 7	n^* = 7	n^* = 7	n^* = 7	n^* = 8	n^* = 8
	L^* = 29	L^* = 29	L^* = 29	L^* = 29	L^* = 29	L^* = 29	L^* = 29	L^* = 29
	C^* = 3515	C^* = 3410	C^* = 3376	C^* = 3380	C^* = 3444	C^* = 3532	C^* = 3633	C^* = 3743
	α^* = 0.05%	α^* = 0.25%	α^* = 0.49%	α^* = 1.47%	α^* = 2.45%	α^* = 3.41%	α^* = 3.98%	α^* = 4.82%
	U^* = 29.62	U^* = 25.28	U^* = 23.20	U^* = 19.56	U^* = 17.69	U^* = 16.39	U^* = 15.75	U^* = 14.94
0.1	q^* = 47.45	q^* = 53.54	q^* = 53.68	q^* = 53.59	q^* = 47.40	q^* = 46.48	q^* = 46.02	q^* = 45.14
	n^* = 6	n^* = 5	n^* = 5	n^* = 5	n^* = 6	n^* = 6	n^* = 6	n^* = 6
	L^* = 22	L^* = 22	L^* = 22	L^* = 22	L^* = 22	L^* = 22	L^* = 22	L^* = 22
	C^* = 4665	C^* = 4434	C^* = 4357	C^* = 4367	C^* = 4512	C^* = 4710	C^* = 4940	C^* = 5194
	α^* = 0.06%	α^* = 0.34%	α^* = 0.67%	α^* = 2.02%	α^* = 2.97%	α^* = 4.11%	α^* = 5.20%	α^* = 6.23%
	U^* = 63.42	U^* = 53.04	U^* = 48.35	U^* = 40.12	U^* = 36.88	U^* = 34.01	U^* = 31.82	U^* = 30.05
0.4	q^* = 71.79	q^* = 87.71	q^* = 88.72	q^* = 87.87	q^* = 71.00	q^* = 66.70	q^* = 54.58	q^* = 45.64
	n^* = 4	n^* = 3	n^* = 3	n^* = 3	n^* = 4	n^* = 4	n^* = 5	n^* = 6
	L^* = 22	L^* = 22	L^* = 22	L^* = 22	L^* = 22	L^* = 22	L^* = 22	L^* = 22
	C^* = 9893	C^* = 8931	C^* = 8606	C^* = 8658	C^* = 9290	C^* = 10,145	C^* = 11,111	C^* = 12,143
	α^* = 0.09%	α^* = 0.55%	α^* = 1.11%	α^* = 3.31%	α^* = 4.45%	α^* = 5.86%	α^* = 6.16%	α^* = 6.30%
	U^* = 244.28	U^* = 199.00	U^* = 178.91	U^* = 143.81	U^* = 133.08	U^* = 122.63	U^* = 120.63	U^* = 119.76

Table 4. Results of the sensitivity analysis.