Efficient Near-Optimal Procedures for Some Inventory Models with Backorders-Lost Sales Mixture and Controllable Lead Time, under Continuous or Periodic Review

Braglia, M., Castellano, D.
Dipartimento di Scienze Giuridiche ed Economiche,
Università Telematica Pegaso, Piazza Trieste e Trento, 48 - 80132 Napoli, Italy.
Tel.: +39 081 7682629; E-mail: davide.castellano@for.unipi.it

Song, D.P.
School of Management, University of Liverpool
Room 128, Chatham Street, Liverpool, L69 7ZH, United Kingdom
Email: Dongping.song@liverpool.ac.uk

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Abstract
This paper considers a number of inventory models with backorders-lost sales mixture, stockout costs, and controllable lead time. The lead time is a linear function of the lot size and includes a constant term that is made of several components. These lot-size-independent components are assumed to be controllable. Both single- and double-echelon inventory systems, under periodic or continuous review, are considered. To authors' knowledge, these models have never been previously studied in literature. The purpose of this paper is to analyse and optimize these novel inventory models. The optimization is carried out by means of heuristics that work on an *ad hoc* approximation of the cost functions. Contrarily to standard optimization methods that use an iterative method, the proposed algorithms exploit closed-form expressions. This peculiarity makes the optimization procedure simpler and more readily applicable in practice than standard approaches. Finally, numerical experiments investigate the efficiency of the proposed heuristics.
Keywords: supply chain; inventory; logistics; lead time; stochastic; heuristics; optimization; joint economic lot size; stockout

1 Introduction

Nowadays, both academicians and practitioners widely recognize that lead time is a critical issue in the area of inventory management. Its reduction (or, rather, its control) is one of the main challenges of the just-in-time (JIT) philosophy (Kim and Benton, 1995). As JIT states, a reduced lead time allows to achieve lower investment in inventory, better product quality, less scrap, reduced storage space requirements, increased productivity, and improved competitive position of the company (Schonberger, 1982; Tersine and Hummingbird, 1995).

The benefits of controlling lead time are particularly relevant when demand is stochastic. In fact, a longer lead time exposes the company to a higher risk of running out of stock. On the contrary, a shorter one may lead to smaller safety stock, improved customer service level, reduced stockout loss, and lower expected total costs (Glock, 2012; Rong and Maiti, 2010).

Liao and Shyu (1991) observed that lead time is made of several independent components (e.g., setup time, process time, queuing time, etc.) and then assumed that these components can be shortened by paying a crashing cost. This piecewise linear-decreasing lead-time crashing cost has been adopted by many successive researchers (e.g., Ouyang et al. (1996); Lin (2009); Panda et al. (2014)). Other models treat lead time as an independent decision variable with a crashing cost expressed as a power or linear function (Chandra and Grabis, 2008) or as an exponential function (Moon et al., 2014).

One major drawback of these formulations is that the lead time is assumed to be independent of the lot size. In fact, since lead time is often made of several components (Liao and Shyu, 1991), some of these components are undoubtedly dependent on the lot size (e.g., the time strictly required to manufacture a production lot is evidently a function of its size). An early investigation about the relationship between lot size and lead time was given by Karmarkar (1987), who asserted that the processing time per batch is a linear function of lot size. This observation has successively been recognized by Kim and Benton (1995), who appear to be the first to introduce a linear relationship between lead time and lot size in the (r,Q) model. This linear function between lead time and lot size has been endorsed by many successive researchers to model single-echelon (Hariga, 1999; Hariga, 2000) or multi-echelon inventory systems (Ben-Daya and Hariga, 2004; Hsiao, 2008a; Glock, 2012; Song et al., 2013; Abdelsalam and Elassal, 2014).

A further aspect that should be considered in a stochastic inventory model is backorders-lost sales mixture (Ouyang et al., 1996; Hsiao, 2008a; Sicilia et al., 2012; Wang and Tang, 2014; Castellano, 2016). This feature is generally adopted to model the different purchasing behaviours of customers when facing stockouts. In fact, some customer may wait until demand is satisfied (such demands are backordered); while others not (such demands are lost).

Although the need for even more generalized models is evident, it is important to highlight that their optimization may be difficult from a practical point of view. Normally, their solution can only be obtained by means of an iterative (or numerical) procedure. In other words, an algorithm is needed to approach a system of complex equations, whose solution may not be immediate to reach in practice. This aspect may thus limit the practical applicability of the model itself (Platt et al., 1997; Eynan and Kropp, 2007; Braglia et al., 2016b).

It should be noted that, in the normal practice of inventory operations, frequent recalculations of the optimal policy over thousands of items are likely to occur. Hence, inventory models have to be solved efficiently in order to be applicable in real-world contexts (Platt et al., 1997). Approximated solution procedures are thus useful in this sense; that is, they are valuable tools to enhance the applicability of complex and generalized models (Braglia et al., 2016a; 2016e).

This paper considers a number of inventory models with backorders-lost sales mixture, stockout costs, and controllable lead time. The lead time is supposed to be a linear function of the lot size and includes a constant term (which can be referred to, e.g., the setup and transportation time) that is made of several components. These lot-size-independent components are assumed to be controllable according to a piecewise linear-decreasing crashing cost. Both single- and double-echelon inventory systems, under periodic or continuous review, are considered. To authors' knowledge, inventory systems with these characteristics have never been investigated in literature previously. The objectives of this paper are twofold: (*i*) to develop and analyse inventory models with the above features; and (*ii*) to propose efficient solution methods that aim to foster their practical application.

The proposed solution procedures are based on approximating part of the cost function according to an *ad hoc* second-order Taylor series expansion. A similar technique has been successfully adopted in previous researches (see, e.g., Eynan and Kropp (2007); and Braglia et al. (2016c; 2016d; 2016f)). Contrarily to standard optimization methods that consist of an iterative procedure to solve the first-order conditions of optimality, this approach exploits closed-form expressions. This peculiarity makes the optimization procedure simpler and more readily applicable. Moreover, the use of closed-form expressions allows to reduce the computational time required by the optimization process. These features favour the practical implementation of any inventory model. The performance of the proposed solution procedures is finally evaluated by means of extensive numerical experiments.

The rest of the paper is organized as follows. Section 2 gives a review of some relevant, recent papers. Section 3 introduces notation and assumptions. Continuous-review and periodic-review inventory models are treated in Section 4 and Section 5, respectively. Section 6 deals with numerical experiments. Finally, conclusions are discussed in Section 7.

2 Literature review

In this paper, single- and double-echelon inventory systems, under periodic or continuous review, are considered. These models include three main features: (*i*) replenishment lead time that is made of two major components, i.e., a lot-size dependent one, and a lot-size-independent controllable one; (*ii*) backorders-lost sales mixture; and (*iii*) stockout costs. The inventory literature is vast, and existing models that embody some of the above characteristics are many. A literature review of some relevant, recent papers will serve the purpose to identify the research gap. Since this paper focuses on single-item models, the literature review will concentrate on this type of inventory systems. The literature review is divided into two sections: inventory systems with lead time independent of lot size, and inventory systems with lead time dependent of lot size. Each section is further divided into two sub-sections: single-echelon inventory systems, and multi-echelon integrated production-inventory systems.

2.1 Inventory systems with lead time independent of lot size

2.1.1 Single-echelon inventory systems

Among recent papers that study single-echelon inventory systems, it is possible to cite the following contributions. Sarkar and Sarkar (2013a) modelled an inventory system with time varying deterioration rate and stock-dependent demand. This model was then extended to consider time-varying backlogging rate (Sarkar and Sarkar, 2013b). Alkhedher et al. (2013) studied a model in which the production process is imperfect and the lead time is not controllable. They considered two cases: the first one is for a predetermined service level case; and the other case is when the service level is a decision variable. Sarkar and Moon (2014) considered the optimization of a continuous review model with setup cost, process quality, and lead time as decision variables. Moon et al. (2014) studied a continuous-review inventory model under distribution-free procedure, with a service level constraint and controllable lead time. Sarkar et al. (2014) faced the problem of optimizing a continuous review inventory system with controllable lead time, defective items, and delay in payments, considering a lead-time demand that is a mixture of Gaussian distribution. Sarkar and Mahapatra (2015) investigated a periodic review inventory model with fuzzy demand, under the assumption that lead time and lost-sale rate are controllable. Shin et al. (2015) studied a continuous review inventory model considering controllable lead time, service level constraint, and transportation discounts. Sarkar et al. (2015a) approached the problem of optimizing a continuous review inventory system with quality improvement and setup cost reduction under a service level constraint. In a further extension of the standard (r,Q) policy, Sarkar et al. (2015b) took into account backorder price discount, process quality improvement, and controllable lead time. Braglia et al. (2016a) developed approximated minimum-cost solutions in closed form to the (S-1,S) inventory policy with complete backordering. In their model, the demand is stochastic and assumed to be Gaussian, and the lead time is fixed. Braglia et al. (2016e) carried out a similar analysis with regard to the (r,Q) policy. They additionally proposed a new cost formulation in which the service level is put in functional dependence with the order quantity.

2.1.2 Multi-echelon integrated production-inventory systems

Among recent papers that consider multi-echelon inventory systems, it is possible to cite the following contributions. Hoque and Goyal (2006) and Hoque (2007; 2009) studied an integrated single vendor-single buyer inventory model with general batch size. Sarkar (2013) developed an integrated production-inventory model with a single supplier and a single buyer under deterministic demand, in which items deteriorate with a rate that is a random variable. Sarkar and Majumder (2013) analysed an integrated vendor-buyer supply chain where setup cost and lead time are controllable. Yi and Sarker (2014) investigated an integrated single vendor-single buyer supply chain under a consignment stock policy. Panda et al. (2014) proposed a two-warehouse fuzzy-stochastic mixture inventory model involving controllable lead time with fully backlogged shortages. Braglia et al. (2016b) considered a single-vendor, single-buyer integrated supply chain with stochastic demand and controllable lead time. Stockout costs are not included. Giri and Roy (2016) studied a single-manufacturer, single-buyer supply chain in two conditions: centralized and decentralized management. A price-dependent, stochastic demand is considered and the lead time is controllable. Jindal and Solanki (2016) investigated a single-vendor, single-buyer supply chain model with quality improvement, backorder price discount, controllable lead time, and mixture of backorders and lost sales. Sarkar (2016) studied a single-vendor, single-buyer supply chain under deterministic and constant demand, with variable backorders, inspection costs, and quantity discounts.

2.2 Inventory systems with lead time dependent of lot size

2.2.1 Single-echelon inventory systems

Since, to authors’ knowledge, single-echelon inventory models that include a lot size-dependent lead time are relatively limited, it may be preferable to extend this review to older papers, going back of many years. Hariga (1999) revisited the model of Kim and Benton (1995) who investigated a (r,Q) policy with stochastic demand. Shortages are fully backordered and stockout costs are considered. Controllable lead time components are not included. Hariga (2000) extended his previous work (Hariga, 1999) to consider setup cost reduction.

2.2.2 Multi-echelon integrated production-inventory systems

Similarly to the single-echelon case, multi-echelon models that include a lot size-dependent lead time are relatively limited. To give a significant overview about them, it may be preferable to take into account works dated back to many years ago. Ben-Daya and Hariga (2004) appear to be the first researchers to adopt a lot size-dependent lead time into an integrated inventory model. They considered a single-vendor, single-buyer supply chain with normally distributed lead-time demand. Shortages are fully backordered and stockout costs are included. However, controllable lead time components are neglected. The model of Ben-Daya and Hariga (2004) was successively improved by Hsiao (2008b) and Glock (2009). The first author modified the model to consider two reorder points and service levels. The second author introduced unequal-sized batch shipments. Glock (2012) investigated a single-vendor, single-buyer integrated supply chain in which the demand is stochastic, but stockout costs are not included. The lead time includes controllable components. Song et al. (2013) proposed a single-retailer, single-manufacturer inventory model that uses distribution-free procedure. Shortages are fully backordered and production rate is a decision variable. Controllable lead time components are not included. Glock and Ries (2013) analysed a multiple-supplier, single-buyer supply chain with normally distributed lead-time demand, in which shortages are fully backordered. Controllable lead time components are not included. Abdelsalam and Elassal (2014) studied a multi-retailer, single manufacturer and single supplier supply chain. In their model, the demand is stochastic, but stockout costs are not included. Inventory is managed according to a periodic review policy and lead time is not controllable.

Based on the above literature review, it can be seen that there is a research gap. That is, for single- and double-echelon inventory systems, under periodic or continuous review, there is a lack of research to investigate the optimization of inventory control policies by considering the following three main features: (*i*) replenishment lead time consisting of two major components, i.e., a lot-size dependent one, and a lot-size-independent controllable one; (*ii*) backorders-lost sales mixture; and (*iii*) stockout costs. The aim of this paper is to fill this gap.

3 Introductory aspects to the models

The developed models include backorders-lost sales mixture, stockout costs, and controllable lead time. The lead time is a linear function of the lot size and comprises a constant term (e.g., the setup and transportation time) that is made of several components. These lot-size-independent components can be controlled according to a piecewise linear-decreasing crashing cost. Both single- and double-echelon inventory systems, under periodic or continuous review, are considered. For each inventory system, the objective is to determine the replenishment policy and the length of setup and transportation time that minimize the long-run expected total cost per time unit. The contribution of this paper is to develop these novel inventory models and to optimize them by means of efficient and practical heuristic procedures.

The following notation and assumptions are considered in the mathematical formulation.

3.1 Notation

Decision variables:

T Review period or inventory cycle time (time units). Periodic-review case.

Q Order or shipment quantity (quantity units). Continuous-review case.

z Safety factor.

n Number of shipments. Double-echelon system.

s Setup and transportation time (time units).

R Target inventory level (quantity units). An equivalent decision variable to z in the periodic-review case.

r Reorder point (quantity units). An equivalent decision variable to z in the continuous-review case.

Parameters:

D Average demand rate (quantity/time unit).

σ	Standard deviation of demand rate (quantity/time unit).
P	Production rate (quantity/time unit).
β	Fraction of shortages that is lost ($0 \le \beta \le 1$).
A	Ordering cost per order (money/order). Single-echelon system.
A_B	Ordering cost per order (money/order). Double-echelon system.
A_V	Setup cost per production batch (money/setup). Double-echelon system.
h	Inventory holding cost (money/quantity unit/time unit). Single-echelon system.
h_B	Inventory holding cost at buyer (money/quantity unit/time unit). Double-echelon system.
h_V	Inventory holding cost at vendor (money/quantity unit/time unit). Double-echelon system.
K	Transportation cost per shipment (money/shipment). Double-echelon system.
π_0	Marginal profit per unit (money/quantity unit).
π_1	Fixed penalty cost per unit shortage (money/quantity unit).

Random variables:

X	Lead-time demand. Continuous-review case.
Y	Demand during the protection interval. Periodic-review case.

Functions:

$f(\cdot)$	Standard normal probability density function (p.d.f.).
$F(\cdot)$	Standard normal cumulative distribution function (c.d.f.).
$G(\cdot)$	Standard normal loss function.
$\mathbf{1}_{\mathcal{A}}(\cdot)$	Indicator function on the set $\mathcal{A}$.
$\lfloor x \rfloor$	Greatest integer smaller than or equal to *x*.
$\lceil x \rceil$	Smallest integer greater than or equal to *x*.
$\lVert \cdot \rVert$	Euclidean norm.

Sets:

$\mathbb{R}$	Real numbers.
$\mathbb{N}$	Natural numbers.

3.2 Assumptions

In this paper, it is supposed that X (the lead-time demand) and Y (the demand during protection interval) are Gaussian random variables. This hypothesis is motivated by the following two observations:

(1) According to Silver et al. (1998), the so-called Gaussian approximation is reasonable in several practical cases, e.g. for fast-moving items with large lead-time demands and small coefficient of variation, or to model forecast demand.

(2) The Gaussian approximation is helpful in the derivation of some formulas used to develop the models presented in this work.

It is possible to note that the assumption about Gaussian approximation is widely adopted in literature. The reader can be referred, for example, to Ouyang et al. (2007); Zhang et al. (2010); Ho et al. (2011); Guchhait et al. (2012); Mizuyama (2013); Alkhedher et al. (2013); Jindal and Solanki (2016); and Giri and Roy (2016)).

Assumptions for the continuous review case:

- The lead time $L(Q,s)$ is given by $L(Q,s)=QP^{-1}+s$. The first addendum gives the production time per batch *Q* (c.f. explanations in Section 1).
- The lead-time demand *X* is a Gaussian random variable with mean $DL(Q,s)$ and standard deviation $\sigma\sqrt{L(Q,s)}$ (see, e.g., Ben-Daya and Hariga (2004)).
- An order is placed whenever the inventory level falls to *r*, which is expressed as the sum between the expected demand during the lead time and the safety stock, i.e., $r=DL(Q,s)+z\sigma\sqrt{L(Q,s)}$ (see, e.g., Ben-Daya and Hariga (2004)).

Assumptions for the periodic review case:

- The lead time $L(T,s)$ is given by $L(T,s)=TDP^{-1}+s$. The first addendum gives the production time per batch *Q*, given that $Q=TD$ (c.f. explanations in Section 1).
- Inventory is reviewed every *T* time units. A sufficient quantity is ordered up to the target level *R*. The ordered quantity arrives after $L(T,s)$ time units (see, e.g., Abdelsalam and Elassal (2014)).
- The target inventory level *R* is given by $R=D\big(T+L(T,s)\big)+z\sigma\sqrt{T+L(T,s)}$, where $D\big(T+L(T,s)\big)$ is the expected demand during the protection interval and $z\sigma\sqrt{T+L(T,s)}$ is the safety stock (see, e.g., Abdelsalam and Elassal (2014)).

Assumptions concerning the continuous review, double-echelon system (see, e.g., Braglia et al. (2016b)):

- One vendor supplies a single item to one buyer;
- The buyer orders lots of size *Q*. The vendor manufactures nQ with a finite production rate *P* (with $P>D$) at one setup, and ships in quantity *Q* to the buyer over *n* times. The vendor incurs a setup cost A_V for each production run of size nQ. The buyer incurs an ordering cost A_B/n for each order of size *Q*. For each shipment (of lot *Q*), the buyer faces a transportation cost *K*.

The main hypotheses concerning the periodic-review, double-echelon system are similar to those characterizing the continuous-review, double-echelon system; it is only needed to replace the quantity *Q* with TD (see, e.g., Lin (2010)).

With regard to each system and policy, shortages are allowed and partially backordered with ratio $1-\beta$. The fraction of shortages with ratio β is lost. Moreover, the expected total cost per time unit is evaluated over an infinite time horizon. This assumption is widely adopted in literature (Ouyang et al., 1996; Hsiao, 2008a; Sicilia et al., 2012; Wang and Tang, 2014; Castellano, 2016).

With similar arguments to Glock (2012), the setup and transportation time is characterized by a piecewise linear-decreasing crashing cost. According to this formulation, the setup and transportation time is made of *m* mutually independent components, each one having a minimum duration a_j, a normal duration b_j, and a crashing cost per time unit c_j, with $c_1\le c_2\le\ldots\le c_m$. The components of setup and transportation time are crashed one at a time starting with the component of least c_j, and so on. If s_j is the length of setup and transportation time with components $1,2,\ldots,j$ crashed to their minimum duration, then we have $s_j=s_0-\sum_{l=1}^{j}(b_l-a_l)$, where $s_0=\sum_{l=1}^{m}b_l$. The setup and transportation time crashing cost is therefore given by

$$U(s)=\sum_{j=1}^{m}\mathbf{1}_{\{s:s_j\le s<s_{j-1}\}}(s)U_j(s),\text{ with } s\in[s_m,s_0], \tag{1}$$

where $U_j(s)=c_j(s_{j-1}-s)+\sum_{l=1}^{j-1}c_l(b_l-a_l)$. We can note that $U(s)$ is a piecewise-linear, decreasing function in the interval $[s_m,s_0]$. It is also continuous and convex in $[s_m,s_0]$.

In Sections 4 and 5, the mathematical model of the inventory systems analysed in this paper is given, along with the specifically developed optimization procedure. Section 4 and Section 5 deal with single-echelon systems and double-echelon systems, respectively.

4 Single-echelon systems

4.1 Periodic-review case

Under the assumptions stated in Section 3.2, the expected total cost per time unit for the periodic-review, single-echelon inventory system is given by

$$\begin{aligned}C(T,z,s)=&\frac{A}{T}+h\left[\frac{DT}{2}+z\sigma\sqrt{T+L(T,s)}+\beta\sigma\sqrt{T+L(T,s)}G(z)\right]+\\&+\sigma\frac{\bar{\pi}}{T}\sqrt{T+L(T,s)}G(z)+\frac{U(s)}{T},\end{aligned} \tag{2}$$

where $\bar{\pi}\equiv\pi_1+\beta\pi_0$. This cost function can readily be derived following similar arguments to, e.g., Moon and Gallego (1994) or Annadurai and Uthayakumar (2010). In Eq. (2), the first term is the ordering cost; the second term is the inventory holding cost; the third term is the sum of shortage cost and lost marginal profit due to lost sales; and the last term is the setup and transportation time crashing cost. The problem of minimizing Eq. (2) can be expressed as follows:

$$\begin{aligned}\text{P1}\quad \min\quad & C(T,z,s)\\ \text{s.t.}\quad & T>0,\\ & z\in\mathbb{R},\\ & s\in[s_m,s_0].\end{aligned}$$

It is not difficult to verify the following lemma, whose proof can therefore be omitted:

Lemma 1. *For $s\in[s_j,s_{j-1}]$ and (T,z) fixed, $C(T,z,s)$ is concave in s. For $s\in[s_j,s_{j-1}]$ fixed, $C(T,z,s)$ is convex in (T,z).*

According to Lemma 1, it is possible to give the following proposition:

Proposition 1. $\min_{(T,z,s)}C(T,z,s)=\min\left\{\min_{(T,z)}C(T,z,s_j)\,|\,j=0,1,...,m\right\}$.

Proof. It is only needed to show that the optimal solution (T^*,z^*,s^*) to the optimization problem $\min C(T,z,s)$ satisfying s^* taking one of the values in $\{s_0,s_1,...,s_m\}$. Suppose $s_j\le s^*\le s_{j-1}$. Note that $C(T^*,z^*,s)$ is strictly concave in s from Lemma 1. Clearly, $\arg\min C(T^*,z^*,s)$ must take either s_j or s_{j-1}. This competes the proof.

□

To minimize $C(T,z,s)$ in (T,z), for $s\in[s_j,s_{j-1}]$ fixed, it is possible to proceed according to the first-order conditions. The first-order condition in *z* gives:

$$z(T)=F^{-1}\left(1-\frac{h}{h\beta+\frac{\bar{\pi}}{T}}\right)=F^{-1}\left(1-\alpha(T)\right), \tag{3}$$

where $F^{-1}(\cdot)$ is the quantile function of the standard normal distribution, and $\alpha(T)\equiv h\left[h\beta+\bar{\pi}/T\right]^{-1}$. It is worth noting that $\alpha(T)<1$ for values of T such that $hT(1-\beta)<\bar{\pi}$.

With some algebraic manipulations, recalling that $G(z)=f(z)-z\left(1-F(z)\right)$ (Ouyang et al., 1996), and using Eq. (3), Eq. (2) can be rewritten as follows:

$$C(T,s)=\frac{A+U(s)}{T}+h\frac{DT}{2}+\sigma f\left(z(T)\right)\sqrt{T+L(T,s)}\left[h\beta+\frac{\bar{\pi}}{T}\right]. \tag{4}$$

Although $C(T,s)$ (i.e., Eq. (4)) is much simpler than $C(T,z,s)$ (i.e., Eq. (2)), it is relatively difficult to obtain the optimal values of *T* and *z*. This because the first-order conditions (in *T* and *z*) do not have a closed-form solution (note that the equation $\partial C(T,s)/\partial T=0$ has not closed-form solution in *T*). Their solution can only be achieved by means of an iterative (or numerical) procedure that solves a system of two equations in two unknowns (Hariga, 2000; Ben-Daya and Hariga, 2004; Glock, 2012).

One way to circumvent this is to use a Taylor series expansion to approximate part of the cost function (Eynan and Kropp, 2007; Braglia et al., 2016b). This technique will be adopted to develop efficient approximated procedures to optimize the inventory models under consideration.

With regard to Eq. (4), the term $f\left(z(T)\right)\sqrt{T+L(T,s)}$ will be replaced with its second-order Taylor series expansion in *T* in a neighbourhood of $\bar{T}=\sqrt{2\left(A+U(s)\right)(Dh)^{-1}}$, which is the optimal *T* in deterministic conditions, for fixed $s\in\left[s_j,s_{j-1}\right]$. In a neighbourhood of $\bar{T}$, it is thus possible to write

$$f\left(z(T)\right)\sqrt{T+L(T,s)}\approx p_0+p_1\left(T-\bar{T}\right)+\frac{1}{2}p_2\left(T-\bar{T}\right)^2, \tag{5}$$

where:

$$p_0\equiv f\left(z(\bar{T})\right)\sqrt{\bar{T}+L(\bar{T},s)}, \tag{6}$$

$$p_1\equiv z(\bar{T})\sqrt{\left(\bar{T}+L(\bar{T},s)\right)}\left[\frac{d}{dT}\alpha(T)\right]_{T=\bar{T}}+\left(1+\frac{D}{P}\right)\frac{f\left(z(\bar{T})\right)}{2\sqrt{\bar{T}+L(\bar{T},s)}}, \tag{7}$$

$$\begin{aligned}p_2\equiv{}&\frac{z(\bar{T})}{\sqrt{\bar{T}+L(\bar{T},s)}}\left[\frac{d}{dT}\alpha(T)\right]_{T=\bar{T}}\left[1+\frac{D}{P}\right]-\frac{f\left(z(\bar{T})\right)}{4\left(\bar{T}+L(\bar{T},s)\right)^{\frac{3}{2}}}\left[1+\frac{D}{P}\right]^2\\&-\frac{\sqrt{\bar{T}+L(\bar{T},s)}}{f\left(z(\bar{T})\right)}\left[\left[\frac{d}{dT}\alpha(T)\right]_{T=\bar{T}}\right]^2+z(\bar{T})\sqrt{\bar{T}+L(\bar{T},s)}\left[\frac{d^2}{dT^2}\alpha(T)\right]_{T=\bar{T}},\end{aligned} \tag{8}$$

and

$$\frac{d}{dT}\alpha(T)=\frac{h\bar{\pi}}{\left(\bar{\pi}+\beta hT\right)^2},$$
$$\frac{d^2}{dT^2}\alpha(T)=-\frac{2h^2\beta\bar{\pi}}{\left(\bar{\pi}+\beta hT\right)^3}.$$
According to Eq. (5), it is possible to obtain

$$C(T,s)\approx\hat{C}(T,s)=\frac{u}{T}+vT+wT^2+y, \text{ in a neighbourhood of } \bar{T}, \tag{9}$$

where

$$u\equiv A+U(s)+\sigma\bar{\pi}\left(p_0-p_1\bar{T}+\frac{1}{2}p_2\bar{T}^2\right),$$
$$v\equiv\frac{hD}{2}+\sigma h\beta\left(p_1-p_2\bar{T}\right)+\sigma\bar{\pi}\frac{p_2}{2},$$
$$w\equiv\frac{1}{2}\sigma h\beta p_2,$$
$$y\equiv\sigma h\beta\left(p_0-p_1\bar{T}+\frac{1}{2}p_2\bar{T}^2\right)+\sigma\bar{\pi}\left(p_1-p_2\bar{T}\right).$$

Clearly, $\hat{C}(T,s)$ is only an approximation for $C(T,s)$. However, it can be used to determine the cost with good accuracy in a reasonably wide range around $\bar{T}$, as demonstrated in Section 5.

It is possible to note that $\hat{C}(T,s)$ is structured as the total cost function in deterministic conditions plus a constant and a quadratic term with respect to *T*. Such type of cost function is strictly convex in *T* and admits a unique minimum ${}^{*}T$ that coincides with its (unique) stationary point. To find ${}^{*}T$, it is therefore needed to solve the equation $\partial\hat{C}(T,s)/\partial T=0$, which is equivalent to $N(T,s)=0$ where $N(T,s)=2wT^3+vT^2-u$. For the sake of brevity, the explicit expression of the required root of $N(T,s)$ (i.e., ${}^{*}T$) is not given; however, it can easily be obtained according to the procedure proposed by Nickalls (1993).

From Proposition 1, to solve problem P1 it is possible to consider the optimization of $C(T,z,s_j)$ for $j=0,1,...,m$. In conclusion, the procedure proposed to find a near-optimal solution $\left(T^*,z^*,s^*\right)$ to problem P1 and the corresponding cost C^* can be summarized as follows:

Algorithm 1.

Step 1. *Let* $\mathcal{C}\equiv\varnothing$. *For each* s_j, *with* $j=0,...,m$, *do Steps 1.1-1.3.*

Step 1.1. *Set* $s\leftarrow s_j$ *and calculate* ${}^{*}T$ *by minimizing* $\hat{C}(T,s)$ *in T.*

Step 1.2. *Calculate* ${}^{*}z$ *replacing T with* ${}^{*}T$ *in Eq. (3).*

Step 1.3. *Set* $\mathcal{C}\leftarrow\mathcal{C}\cup\left\{C\left({}^{*}T,{}^{*}z,s\right)\right\}$.

Step 2. *Set* $\left(T^*,z^*,s^*\right)\leftarrow\arg\min\mathcal{C}$ *and* $C^*\leftarrow\min\mathcal{C}$.

4.2 Continuous review policy

Under the assumptions stated in Section 3.2, the expected total cost per time unit for the continuous-review, single-echelon inventory system is given by

$$C(Q,z,s)=\frac{AD}{Q}+h\left[\frac{Q}{2}+z\sigma\sqrt{L(Q,s)}+\beta\sigma\sqrt{L(Q,s)}G(z)\right]+ \\ +\sigma\bar{\pi}\frac{D}{Q}\sqrt{L(Q,s)}G(z)+\frac{D}{Q}U(s). \tag{10}$$

This cost function can readily be derived following similar arguments to, e.g., Ouyang et al. (1996). The first term is the ordering cost; the second term is the inventory holding cost; the third term is the sum of shortage cost and lost marginal profit due to lost sales; and the last term is the setup and transportation time crashing cost. The problem of minimizing Eq. (10) can be expressed as follows:

$$\begin{array}{lll} \text{P2} & \min & C(Q,z,s) \\ & \text{s.t.} & Q\in\mathbb{N}, \\ & & z\in\mathbb{R}, \\ & & s\in[s_m,s_0]. \end{array}$$

If the integrality constraint on Q is relaxed, it is possible to deduce the following lemma, whose proof can be omitted:

Lemma 2. *For $s\in[s_j,s_{j-1}]$ and (Q,z) fixed, $C(Q,z,s)$ is concave in s. For $s\in[s_j,s_{j-1}]$ fixed, $C(Q,z,s)$ is convex in (Q,z).*

The previous lemma leads to the following proposition:

Proposition 2. $\min\limits_{(Q,z,s)} C(Q,z,s)=\min\left\{\min\limits_{(Q,z)} C(Q,z,s_j)\mid j=0,1,\ldots,m\right\}$.

Proof. Similar to that of Proposition 1.

□

With fixed $s\in[s_j,s_{j-1}]$, to minimize $C(Q,z,s)$ in (Q,z) it is required to satisfy the first-order conditions. The first-order condition in *z* gives:

$$z(Q)=F^{-1}\left(1-\frac{h}{h\beta+\frac{\bar{\pi}D}{Q}}\right)=F^{-1}(1-\alpha(Q)). \tag{11}$$

Note that $\alpha(Q)<1$ for values of Q such that $hQ(1-\beta)<D\bar{\pi}$. With some algebraic manipulation and using Eq. (11), Eq. (10) becomes

$$C(Q,s)=\frac{D}{Q}(A+U(s))+h\frac{Q}{2}+\sigma f(z(Q))\sqrt{L(Q)}\left[h\beta+\bar{\pi}\frac{D}{Q}\right]. \tag{12}$$

Since the first-order condition in Q imposed to Eq. (12) has not closed-form solution, the approximation approach described in Section 4.1 can be applied here as well. That is, the term $f(z(Q))\sqrt{L(Q)}$ is approximated with its second-order Taylor series expansion in Q in a neighbourhood of $\bar{Q}\equiv\sqrt{2(A+U(s))D/h}$, which is the optimal Q in deterministic conditions, for fixed $s\in[s_j,s_{j-1}]$. This permits to obtain

$$f(z(Q))\sqrt{L(Q)}\approx p_0+p_1(Q-\bar{Q})+\frac{1}{2}p_2(Q-\bar{Q})^2, \text{ in a neighbourhood of } \bar{Q}, \tag{13}$$

where

$$p_0 \equiv f\left(z\left(\bar{Q}\right)\right)\sqrt{L\left(\bar{Q},s\right)}, \tag{14}$$

$$p_1 \equiv z\left(\bar{Q}\right)\sqrt{L\left(\bar{Q},s\right)}\left[\frac{d}{dQ}\alpha\left(Q\right)\right]_{Q=\bar{Q}} + \frac{f\left(z\left(\bar{Q}\right)\right)}{2P\sqrt{L\left(\bar{Q},s\right)}}, \tag{15}$$

$$\begin{aligned} p_2 \equiv & \frac{z\left(\bar{Q}\right)}{P\sqrt{L\left(\bar{Q},s\right)}}\left[\frac{d}{dQ}\alpha\left(Q\right)\right]_{Q=\bar{Q}} + \\ & -\frac{f\left(z\left(\bar{Q}\right)\right)}{4P^2\left[L\left(\bar{Q},s\right)\right]^{\frac{3}{2}}} - \frac{\sqrt{L\left(\bar{Q},s\right)}}{f\left(z\left(\bar{Q}\right)\right)}\left[\left[\frac{d}{dQ}\alpha\left(Q\right)\right]_{Q=\bar{Q}}\right]^2 + \\ & + z\left(\bar{Q}\right)\sqrt{L\left(\bar{Q},s\right)}\left[\frac{d^2}{dQ^2}\alpha\left(Q\right)\right]_{Q=\bar{Q}}, \end{aligned} \tag{16}$$

with

$$\frac{d}{dQ}\alpha\left(Q\right) = \frac{h\bar{\pi}D}{\left(\bar{\pi}D + \beta hQ\right)^2},$$

$$\frac{d^2}{dQ^2}\alpha\left(Q\right) = -\frac{2h^2\beta\bar{\pi}D}{\left(\bar{\pi}D + \beta hQ\right)^3}.$$

According to Eq. (14), it is possible to write:

$$C\left(Q,s\right) \approx \hat{C}\left(Q,s\right) = \frac{u}{Q} + vQ + wQ^2 + y, \text{ in a neighbourhood of } \bar{Q}, \tag{17}$$

where

$$u \equiv \left(A + U\left(s\right)\right)D + \sigma D\bar{\pi}\left(p_0 - p_1\bar{Q} + \frac{p_2}{2}\bar{Q}^2\right), \tag{18}$$

$$v \equiv \frac{h}{2} + \sigma h\beta\left(p_1 - p_2\bar{Q}\right) + \sigma\bar{\pi}\frac{p_2}{2}D, \tag{19}$$

$$w \equiv \frac{1}{2}\sigma h\beta p_2, \tag{20}$$

$$y \equiv \sigma h\beta\left(p_0 - p_1\bar{Q} + \frac{1}{2}p_2\bar{Q}^2\right) + \sigma\bar{\pi}D\left(p_1 - p_2\bar{Q}\right). \tag{21}$$

It is possible to note that $\hat{C}\left(Q,s\right)$ is structurally identical to $\hat{C}\left(T,s\right)$ (see Eq. (9)). Therefore, the (real-valued) optimum $\hat{Q}$ of $\hat{C}\left(Q,s\right)$ in *Q*, for fixed $s \in \left[s_j, s_{j-1}\right]$, can be found similarly. That is, by imposing the first-order condition a cubic equation is obtained, which can be solved with the procedure given by Nickalls (1993).

From Proposition 2, problem P2 can be solved focusing on the optimization of $C\left(Q,z,s_j\right)$ for $j = 0,1,...,m$. Ultimately, a near-optimal solution $\left(Q^*, z^*, s^*\right)$ to problem P2 and the corresponding cost C^* can be found according to the following algorithm:

Algorithm 2.

Step 1. Let $\mathcal{C} \equiv \varnothing$. For each s_j, with $j = 0,...,m$, do Steps 1.1-1.4.

Step 1.1. Set $s \leftarrow s_j$ and calculate $\hat{Q}$ by minimizing $\hat{C}\left(Q,s\right)$ in Q.

Step 1.2. *If* $C\left(\left\lfloor \hat{Q} \right\rfloor, s\right) \le C\left(\left\lceil \hat{Q} \right\rceil, s\right)$, *then set* ${}^{*}Q \leftarrow \left\lfloor \hat{Q} \right\rfloor$, *otherwise set* ${}^{*}Q \leftarrow \left\lceil \hat{Q} \right\rceil$.

Step 1.3. *Calculate* ${}^{*}z$ *replacing* Q *with* ${}^{*}Q$ *in Eq. (11).*

Step 1.4. *Set* $\mathcal{C} \leftarrow \mathcal{C} \cup \left\{C\left({}^{*}Q, {}^{*}z, s\right)\right\}$.

Step 2. *Set* $\left(Q^{*}, z^{*}, s^{*}\right) \leftarrow \arg\min \mathcal{C}$ *and* $C^{*} \leftarrow \min \mathcal{C}$.

5 Double-echelon systems

5.1 Periodic review policy

Under the assumptions given in Section 3.2, the expected joint total cost per time unit for the periodic-review, double-echelon inventory system can be derived following similar arguments to, e.g., Lin (2010). The expected total cost per time unit for the buyer and for the vendor is

$$C_B(T,n,z,s) = \left(\frac{A_B}{n} + K\right)\frac{1}{T} + h_B\left(\frac{DT}{2} + z\sigma\sqrt{T + L(T,s)} + \beta\sigma\sqrt{T + L(T,s)}G(z)\right) + \\ + \frac{\bar{\pi}}{T}\sigma\sqrt{T + L(T,s)}G(z) + \frac{U(s)}{T},$$

and

$$C_V(T,n) = \frac{A_V}{nT} + h_V\frac{DT}{2}\left(n - 1 - \frac{D}{P}(n-2)\right),$$

respectively. Hence, the expected joint total cost per time unit is given by:

$$\begin{aligned} C(T,n,z,s) &= C_B(T,n,z,s) + C_V(T,n) \\ &= \left(\frac{A_B + A_V}{n} + K\right)\frac{1}{T} + \\ &\quad + h_B\left(\frac{DT}{2} + z\sigma\sqrt{T + L(T,s)} + \beta\sigma\sqrt{T + L(T,s)}G(z)\right) + \\ &\quad + \frac{\bar{\pi}}{T}\sigma\sqrt{T + L(T,s)}G(z) + \frac{U(s)}{T} + h_V\frac{DT}{2}\left(n - 1 - \frac{D}{P}(n-2)\right). \end{aligned} \tag{22}$$

The problem of minimizing Eq. (22) can be expressed as follows:

$$\begin{aligned} \text{P3} \quad \min \quad & C(T,n,z,s) \\ \text{s.t.} \quad & T > 0 \\ & n \in \mathbb{N}, \\ & z \in \mathbb{R}, \\ & s \in [s_m, s_0]. \end{aligned}$$

If the integrality constraint on n is relaxed, it is not difficult to verify the following lemma, whose proof can therefore be omitted:

Lemma 3. *For* $s \in \left[s_j, s_{j-1}\right]$ *and* (T,n,z) *fixed,* $C(T,n,z,s)$ *is concave in s. For* $s \in \left[s_j, s_{j-1}\right]$ *fixed,* $C(T,n,z,s)$ *is convex in* (T,n,z).

The following proposition can be deduced from the previous lemma:

Proposition 3. $\min_{(T,n,z,s)} C(T,n,z,s) = \min\left\{\min_{(T,n,z)} C\left(T,n,z,s_j\right) \mid j = 0,1,\ldots,m\right\}$.

Proof. Similar to that of Proposition 1.

□

According to the above properties, (n,s) is now kept fixed, with $s\in[s_j,s_{j-1}]$, and the problem of minimizing $C(T,n,z,s)$ in (T,z) is considered. The procedure proposed is described below.

The first-order condition in *z* gives:

$$z(T)=F^{-1}\left(1-\frac{h_B}{h_B\beta+\dfrac{\bar{\pi}}{T}}\right)=F^{-1}\left(1-\alpha(T)\right), \tag{23}$$

which is identical to Eq. (3) with h_B in place of *h*. Consequently, Eq. (22) can be rewritten as follows:

$$\begin{aligned}C(T,n,s)=&\left(\frac{A_B+A_V}{n}+K+U(s)\right)\frac{1}{T}+\\&+h_B\frac{DT}{2}+\sigma f\left(z(T)\right)\sqrt{T+L(T,s)}\left[h_B\beta+\frac{\bar{\pi}}{T}\right]+\\&+h_V\frac{DT}{2}\left(n-1-\frac{D}{P}(n-2)\right).\end{aligned} \tag{24}$$

Since the first-order condition in *T* for Eq. (24) has not closed-form solution, the same approximation technique as that in the previous sections can be adopted. Hence, the term $f\left(z(T)\right)\sqrt{T+L(T,s)}$ is approximated with its second-order Taylor series expansion in *T* in a neighbourhood of $\bar{T}=\sqrt{\left[(A_B+A_V)/n+K+U(s)\right]/H(n)}$, where $H(n)\equiv D/2\left[h_B+h_V\left(n-1-D/P(n-2)\right)\right]$. It is possible to note that $\bar{T}$ is the optimal *T* in deterministic conditions, for fixed (n,s) with $s\in[s_j,s_{j-1}]$. Consequently, with reference to a neighbourhood of $\bar{T}$, it is possible to write

$$f\left(z(T)\right)\sqrt{T+L(T,s)}\approx p_0+p_1\left(T-\bar{T}\right)+\frac{1}{2}p_2\left(T-\bar{T}\right)^2, \tag{25}$$

where p_0, p_1, and p_2 are given by Eqs. (6)-(8), with h_B in place of *h*. According to Eq. (25), $C(T,n,s)$ can be approximated as follows:

$$C(T,n,s)\approx\hat{C}(T,n,s)=\frac{u}{T}+vT+wT^2+y\text{, in a neighbourhood of }\bar{T}, \tag{26}$$

where

$$u\equiv\frac{A_B+A_V}{n}+K+U(s)+\sigma\bar{\pi}\left(p_0-p_1\bar{T}+\frac{1}{2}p_2\bar{T}^2\right), \tag{27}$$

$$v\equiv\sigma h_B\beta\left(p_1-p_2\bar{T}\right)+\sigma\bar{\pi}\frac{p_2}{2}+h_B\frac{D}{2}+h_V\frac{D}{2}\left(n-1-\frac{D}{P}(n-2)\right), \tag{28}$$

$$w\equiv\frac{1}{2}\sigma h_B\beta p_2, \tag{29}$$

$$y\equiv\sigma h_B\beta\left(p_0-p_1\bar{T}+\frac{1}{2}p_2\bar{T}^2\right)+\sigma\bar{\pi}\left(p_1-p_2\bar{T}\right). \tag{30}$$

For fixed (n,s) with $s \in [s_j, s_{j-1}]$, the minimum ${}^{*}T$ of $\hat{C}(T,n,s)$ in T can be found solving the first-order condition. This leads to a cubic equation that can be again approached with the procedure proposed by Nickalls (1993).

From Proposition 3, problem P3 can be solved considering the optimization of $C(T,n,z,s_j)$, for $j = 0,1,...,m$, only. In conclusion, a near-optimal solution (T^*, n^*, z^*, s^*) to problem P3 and the corresponding cost C^* can be found according to the following algorithm:

Algorithm 3.

Step 1. Let $\mathcal{C} \equiv \varnothing$. *For each* s_j, *with* $j = 0,...,m$, *do Steps 1.1-1.6.*

Step 1.1. *Set* $s \leftarrow s_j$, $n = 1$ *and* ${}^{*}C = +\infty$.

Step 1.2. *Calculate* ${}^{*}T$ *by minimizing* $\hat{C}(T,n,s)$ *in T.*

Step 1.3. *If* $C({}^{*}T,n,s) \le {}^{*}C$, *then set* ${}^{*}C \leftarrow C({}^{*}T,n,s)$, $n \leftarrow n+1$ *and go to Step 1.2, otherwise set* ${}^{*}n \leftarrow \max\{1, n-1\}$ *and go to Step 1.4.*

Step 1.4. *Calculate* ${}^{*}T$ *by minimizing* $\hat{C}(T, {}^{*}n, s)$ *in T.*

Step 1.5. *Calculate* ${}^{*}z$ *replacing T with* ${}^{*}T$ *in Eq. (23).*

Step 1.6. *Set* $\mathcal{C} \leftarrow \mathcal{C} \cup \{C({}^{*}T, {}^{*}n, {}^{*}z, s)\}$.

Step 2. *Set* $(T^*, n^*, z^*, s^*) \leftarrow \arg\min \mathcal{C}$ *and* $C^* \leftarrow \min \mathcal{C}$.

5.2 Continuous review policy

Under the assumptions given in Section 3.2, the expected joint total cost per time unit for the continuous-review, double-echelon inventory system can be obtained following similar arguments to, e.g., Lin (2009). The expected total cost per time unit for the buyer and for the vendor is

$$C_B(Q,n,z,s) = \left(\frac{A_B}{n} + K\right)\frac{D}{Q} + h_B\left(\frac{Q}{2} + z\sigma\sqrt{L(Q,s)} + \beta\sigma\sqrt{L(Q,s)}G(z)\right) + \\ +\sigma\pi\frac{D}{Q}\sqrt{L(Q,s)}G(z) + U(s)\frac{D}{Q},$$

and

$$C_V(Q,n) = \frac{A_V}{n}\frac{D}{Q} + h_V\frac{Q}{2}\left(n - 1 - \frac{D}{P}(n-2)\right),$$

respectively. Hence, the expected joint total cost per time unit is:

$$\begin{aligned} C(Q,n,z,s) &= C_B(Q,n,z,s) + C_V(Q,n) \\ &= \left(\frac{A_B + A_V}{n} + K\right)\frac{D}{Q} + h_B\left(\frac{Q}{2} + z\sigma\sqrt{L(Q,s)} + \beta\sigma\sqrt{L(Q,s)}G(z)\right) + \\ &\quad +\sigma\pi\frac{D}{Q}\sqrt{L(Q,s)}G(z) + U(s)\frac{D}{Q} + h_V\frac{Q}{2}\left(n - 1 - \frac{D}{P}(n-2)\right). \end{aligned} \tag{31}$$

The problem of minimizing Eq. (31) can be formalized as follows:

$$\text{P4} \quad \min \quad C(Q,n,z,s)$$

$$\text{s.t.}\quad Q \in \mathbb{N}$$
$$n \in \mathbb{N},$$
$$z \in \mathbb{R},$$
$$s \in [s_m, s_0].$$

If the integrality constraint on Q and n is relaxed, it is relatively easy to prove the following properties:

Lemma 4. *For* $s \in [s_j, s_{j-1}]$ *and* (Q,n,z) *fixed,* $C(Q,n,z,s)$ *is concave in s. For* $s \in [s_j, s_{j-1}]$ *fixed,* $C(Q,n,z,s)$ *is convex in* (Q,n,z).

The previous lemma leads to the following proposition:

Proposition 4. $\min\limits_{(Q,n,z,s)} C(Q,n,z,s) = \min\left\{\min\limits_{(Q,n,z)} C(Q,n,z,s_j) \mid j = 0,1,...,m\right\}.$

Proof. Similar to that of Proposition 1.

□

Given the above properties, it is possible to consider the minimization of $C(Q,n,z,s)$, for fixed (n,s) with $s \in [s_j, s_{j-1}]$. The first-order condition in *z* gives:

$$z(Q) = F^{-1}\left(1 - \frac{h_B}{h_B\beta + \frac{D}{Q}\bar{\pi}}\right) = F^{-1}\left(1-\alpha(Q)\right). \tag{32}$$

Note that Eq. (32) is identical to Eq. (11) with h_B in place of *h*. According to Eq. (32) and with some algebraic manipulations, Eq. (31) can be rewritten as follows:

$$\begin{aligned} C(Q,n,s) = &\left(\frac{A_B + A_V}{n} + K + U(s)\right)\frac{D}{Q} + \\ &+ h_B\frac{Q}{2} + \sigma f\left(z(Q)\right)\sqrt{L(Q,s)}\left[h_B\beta + \frac{D}{Q}\bar{\pi}\right] + \\ &+ h_V\frac{Q}{2}\left(n - 1 - \frac{D}{P}(n-2)\right). \end{aligned} \tag{33}$$

Since the first-order condition in Q imposed to Eq. (33) has not closed-form solution, it is possible to repeat the approximation approach used in the previous models identically. That is, the term $f\left(z(Q)\right)\sqrt{L(Q,s)}$ can be approximated with its second-order Taylor series expansion in Q in a neighbourhood of $\bar{Q} \equiv \sqrt{D\left[(A_B + A_V)/n + K + U(s)\right]/H(n)}$, where $H(n) \equiv 1/2\left[h_B + h_V\left(n - 1 - D/P(n-2)\right)\right]$. Note that $\bar{Q}$ is the optimal Q in deterministic conditions, for fixed (n,s) with $s \in [s_j, s_{j-1}]$. With reference to a neighbourhood of $\bar{Q}$, it is possible to write:

$$f\left(z(Q)\right)\sqrt{L(Q,s)} \approx p_0 + p_1\left(Q - \bar{Q}\right) + \frac{1}{2}p_2\left(Q - \bar{Q}\right)^2, \tag{34}$$

where p_0, p_1, and p_2 are given by Eqs. (14)-(16), with h_B instead of *h*. According to Eq. (34), the following approximation can be achieved:

$$C(Q,n,s) \approx \hat{C}(Q,n,s) = \frac{u}{Q} + vQ + wQ^2 + y\text{, in a neighbourhood of } \bar{Q}, \tag{35}$$

where

$$u \equiv \left(\frac{A_B + A_V}{n} + K + U(s) \right) D + \sigma \bar{\pi} D \left(p_0 - p_1 \bar{Q} + \frac{p_2}{2} \bar{Q}^2 \right), \tag{36}$$

$$v \equiv \sigma h_B \beta \left(p_1 - p_2 \bar{Q} \right) + \sigma \bar{\pi} \frac{p_2}{2} D + \frac{h_B}{2} + \frac{h_V}{2} \left(n - 1 - \frac{D}{P} (n-2) \right), \tag{37}$$

$$w \equiv \frac{1}{2} \sigma h_B \beta p_2, \tag{38}$$

$$y \equiv \sigma h_B \beta \left(p_0 - p_1 \bar{Q} + \frac{1}{2} p_2 \bar{Q}^2 \right) + \sigma \bar{\pi} D \left(p_1 - p_2 \bar{Q} \right). \tag{39}$$

For fixed (n,s), with $s \in [s_j, s_{j-1}]$, the (real-valued) minimum $\hat{Q}$ of $\hat{C}(Q,n,s)$ in Q can be found solving the first-order condition. This leads to a cubic equation that can be solved according to the procedure given by Nickalls (1993).

From Proposition 4, problem P4 can be solved taking into account the optimization of $C(Q,n,z,s_j)$ for $j = 0,1,...,m$. Ultimately, a near-optimal solution (Q^*, n^*, z^*, s^*) to problem P4 and the corresponding cost C^* can be found with the following algorithm:

Algorithm 4.

Step 1. *Let* $\mathcal{C} \equiv \varnothing$. *For each* s_j, *with* $j = 0,...,m$, *do Steps 1.1-1.8.*

Step 1.1. *Set* $s \leftarrow s_j$, $n = 1$ *and* ${}^*C = +\infty$.

Step 1.2. *Calculate* $\hat{Q}$ *minimizing* $\hat{C}(Q,n,s)$ *in Q.*

Step 1.3. *If* $C(\lfloor \hat{Q} \rfloor, n, s) \le C(\lceil \hat{Q} \rceil, n, s)$, *then set* ${}^*Q \leftarrow \lfloor \hat{Q} \rfloor$, *otherwise set* ${}^*Q \leftarrow \lceil \hat{Q} \rceil$.

Step 1.4. *If* $C({}^*Q, n, s) \le {}^*C$, *then set* ${}^*C \leftarrow C({}^*Q, n, s)$, $n \leftarrow n+1$ *and go to Step 1.2, otherwise set* ${}^*n \leftarrow \max\{1, n-1\}$ *and go to Step 1.5.*

Step 1.5. *Calculate* $\hat{Q}$ *minimizing* $\hat{C}(Q, {}^*n, s)$ *in Q.*

Step 1.6. *If* $C(\lfloor \hat{Q} \rfloor, {}^*n, s) \le C(\lceil \hat{Q} \rceil, {}^*n, s)$, *then set* ${}^*Q \leftarrow \lfloor \hat{Q} \rfloor$, *otherwise set* ${}^*Q \leftarrow \lceil \hat{Q} \rceil$.

Step 1.7. *Calculate* ${}^*z$ *replacing Q with* ${}^*Q$ *in Eq. (32).*

Step 1.8. *Set* $\mathcal{C} \leftarrow \mathcal{C} \cup \{ C({}^*Q, {}^*n, {}^*z, s) \}$.

Step 2. *Set* $(T^*, n^*, z^*, s^*) \leftarrow \arg\min \mathcal{C}$ *and* $C^* \leftarrow \min \mathcal{C}$.

6 Numerical study

This section presents numerical experiments carried out to test the performance of the proposed solution procedures in terms of achieved error and required computational effort. For what concerns the error analysis, an approach based on the design of experiments (DOE) is followed. In this way, both the magnitude of the error and the influence of parameters on the error can be assessed. For each parameter, two disjoint intervals of possible values are considered. That is, it is assumed that each parameter can take values within two different levels: “low” (labelled with “1”) or “high” (labelled with “2”).

For each combination of parameter levels, 30 trials have been done. In each trial, parameter values are randomly drawn within the corresponding intervals. With regard to a single trial, the error is evaluated in terms of absolute percentage error (APE). The mean absolute percentage error (MAPE) is then taken as output (i.e., as performance indicator) for that particular combination of parameter levels. With reference to a given inventory model, the APE corresponding to the generic *k*th trial, i.e., APE_k, is defined as follows:

$$\mathrm{APE}_k \equiv \left|\frac{C\left(\mathbf{X}^*\right)-C\left(\hat{\mathbf{X}}^*\right)}{C\left(\mathbf{X}^*\right)}\right| \times 100,$$

where:

- $\mathbf{X}^*$ is the solution obtained with a genetic algorithm (GA) performed within MATLAB® R2013b;
- $\hat{\mathbf{X}}^*$ is the solution obtained with the proposed method;
- $C(\cdot)$ is the (true) cost function.

The MAPE is calculated as follows:

$$\mathrm{MAPE} \equiv \frac{1}{M}\sum_{k=1}^{M}\mathrm{APE}_k,$$

where *M* is the total number of trials for each levels combination.

It should be observed that:

- GA has been adopted to obtain the solution taken into reference to calculate the APE because it is simple to use and is extensively recognized as a valuable optimization tool (Chaudhry and Luo, 2005; Sivanandam and Deepa, 2008). Since the objective is not to achieve a fine tuning of the GA, default parameter values specified in MATLAB® have been adopted. It is important to point out that it has been verified that the solution found by GA is not worse than that obtained by the proposed optimization approaches and by the iterative procedure typically implemented in literature.
- GA has been used to optimize the “reduced” cost functions, i.e., the cost functions rewritten taking into consideration the first-order condition in *z*. Moreover, GA has operated considering Propositions 1-4. That is, GA has been repeated for each s_j, for $j = 0,1,...,m$, keeping (in each repetition) fixed $s = s_j$. In this regard, note that if GA approaches the considered optimization problems without consideration about the results given by Proposition 1-4, it could be slower and therefore less efficient.

In experiments, the time unit is expressed in years. The setup and transportation time are assumed to be made of three components, whose durations are reported in Table 1. Table 2 shows the intervals associated with parameter levels. Values in Tables 1 and 2 have been taken from literature (Braglia et al., 2016b). It is possible to observe that Table 2 includes the coefficient of variation of demand *Cv*, i.e., $Cv \equiv \sigma/D$, instead of the standard deviation σ. Moreover, it is assumed that $Cv \leq 0.3$. In fact, the normal approximation to the demand during lead time (continuous review policy) or during protection interval (periodic review policy) is appropriate for small values of *Cv*. In other words, it is necessary that the probability of achieving negative values be negligible (Zipkin, 2000).

TABLE 1 HERE

TABLE 2 HERE

For the single-echelon inventory system, Figure 1 and Figure 2 show the mean absolute percentage error (MAPE) for the periodic review and the continuous review cases, respectively. A clarification about how to read these pictures deserves to be made. Each cell in these figures represents the typical output of an interaction plot, between two factors, in a DOE analysis (there exists interaction when the effect of one factor depends on the level of the other factor). In the present context, a factor is a model parameter, and the response variable is MAPE. This type of plot investigates the presence of interaction effect, on the response variable, between two factors. Parallel lines in an interaction plot indicate no interaction. Moreover, the greater the difference in slope between the lines, the higher the degree of interaction.

From Figures 1 and 2, it is evident that the efficiency achieved by the proposed approximation approach is high: the maximum MAPE is about 1% in the periodic review case and 0.15% in the continuous review case. For what concerns the sensitivity of the error with respect to parameters, it is possible to note that:

- Cv, β, and D affect the error with positive direction (i.e., the error increases as they grow).
- A, h, and P affect the error with negative direction (i.e., the error decreases as they grow).
- The effect of the other parameters is negligible.

FIGURE 1 HERE

FIGURE 2 HERE

With regard to the double-echelon inventory system, Figure 3 and Figure 4 show the results for the periodic review and continuous review cases, respectively. Again, the efficiency of the proposed approximation approach is very good: the maximum MAPE is about 0.6% in the periodic review case and 0.5% in the continuous review case.

Concerning the effect of parameters on the error in the periodic review case, it is possible to observe that:

- Cv, β, D, and K provide a not negligible effect with positive direction.
- P, A_B, and A_V have limited impact with negative direction.
- The other parameters have a practically negligible influence.

With regard to the continuous review case, the following observations can be made:

- K, P, and h_V have a not negligible impact with positive direction.
- A_V, h_B, D, β, and Cv affect the error with negative direction.
- The other parameters have a practically negligible influence.

FIGURE 3 HERE

FIGURE 4 HERE

To evaluate the computational efficiency of the proposed solution methods, a comparison with the standard iterative approach and with the previously introduced GA has been carried out. The comparison has been made in terms of time needed to solve 2000 randomly generated problems.

Although the time difference on a single problem is in the order of a few seconds (on average), the discrepancy of performances over several problems may become significant. It should also be noted that, in practice, a retailer may manage thousands of items, and the relevant control variables (e.g., the order quantity and the reorder point in the continuous review policy) are required to be recalculated frequently. It is therefore practically useful to evaluate the computational efficiency in terms of time needed to solve a relatively large set of (randomly generated) problems.

The setup and transportation time are supposed to be made of three components (see Table 1). Parameters take values within the intervals shown in Table 2. Tests have been made on a PC with an Intel® Core™ i7 processor at 2.4GHz and 16GB of RAM.

Results are presented in Table 3. The minimum percentage of computational time reduction achieved by the proposed optimization approach is about 98.5% and 76.5% with respect to GA and to the solution method with iterative procedure, respectively. These results are both relevant to the double-echelon inventory system under continuous review. With respect to GA, the maximum percentage of computational time reduction is about 99.8%, obtained in the single-echelon inventory system under both periodic and continuous review policies. With respect to the solution method with iterative procedure, the maximum percentage of computational time reduction is approximately 83.9%, achieved in the single-echelon inventory system under periodic review policy.

TABLE 3 HERE

Observing the outcomes of the analysis done in this section, the reader can note that the optimization procedures appear to reach better solutions for the continuous review policy. The largest error achieved in the periodic review case is nearly 1%, which is a small value, though. Moreover, comparing single- and double-echelon systems, it is evident that the performance of their respective optimization approaches is, on average, similar. That is, in both cases, the average maximum MAPE is about 0.5%. Even in this comparison, the error can be considered substantially negligible.

With regard to the computational requirements of the proposed solution procedures, the percentage of computational reduction is at least equal to 76.5%. This value is obtained with respect to the second fastest solution method, i.e., the iterative algorithm. In particular, the reader can observe that the computational requirements are more limited in the single-echelon case, rather than in the double-echelon case. This evidently depends on the number of steps needed by the algorithms, which are obviously less in the optimization of single-echelon systems. Moreover, approaching continuous review inventory models is a little computationally more onerous than optimizing periodic review inventory models. This can be noticed by observing that, considering the same number of echelons, algorithms for continuous review systems (i.e., Algorithm 2 and Algorithm 4) include more steps than algorithms relevant to periodic review systems (i.e., Algorithm 1 and Algorithm 3). These additional steps are required to evaluate the optimal integer value of lot size.

In conclusion, the real-world application of the developed inventory models seems therefore promising. In fact, the implementation of the proposed heuristics is relatively immediate, e.g., a simple spreadsheet may be used. Moreover, their overall performance has been demonstrated to be satisfactory. Given these peculiarities, the solution procedures provided in the previous sections may therefore be able to foster the adoption in practice of the inventory models presented in this paper.

7 Conclusions

This paper studied some novel inventory models with backorder-lost sales mixture, stockout costs, and controllable lead time. In particular, the lead time was supposed to be made of two main terms. The first one is a linear function of the lot size; the second one is a constant term (which can be referred to, e.g., the setup and transportation time) that includes several controllable components. These controllable components can be shortened according to a piecewise linear-decreasing crashing cost. Both single- and double-echelon inventory systems, under periodic or continuous review policy, were considered.

The problem of optimizing the proposed inventory models was approached by means of specifically developed heuristics. These solution methods work on an approximation of the cost function obtained replacing part of its expression with an *ad hoc* second-order Taylor series expansion. Contrarily to standard methods that consist of an iterative procedure to solve the first-order conditions of optimality, this approximation approach permitted to exploit closed-form formulas in the optimization process.

Numerical experiments were carried out to examine the performance of the proposed optimization procedures. First, the error was assessed in terms of both magnitude and sensitivity on parameters. Then, the required computational effort was investigated observing the time needed to solve a batch of randomly generated problems.

These tests proved that the proposed optimization procedures are highly efficient in terms of both achieved error and required computational effort. It is also possible to note that these procedures can easily be implemented within a simple spreadsheet. Hence, their real-world application may be promising. These peculiarities permit to observe that the provided solution methods may be able to foster the adoption in practice of the inventory models presented in this paper.

Future works could be devoted to extend the proposed models to the case where demand distribution is unspecified and a distribution-free approach needs to be adopted. Moreover, it may be possible to investigate novel practical optimization procedures for different and more complex inventory models.

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	Durations (days)	
Component - j	*Normal -* b_j	*Minimum -* a_j
1	12	7
2	12	7
3	10	8

Table 1. Components of setup and transportation time.

Parameter	Levels	
	Low (1)	*High (2)*
A	[60, 80]	[140, 160]
A_B	[10, 20]	[40, 50]
A_V	[200, 250]	[350, 400]
F	[10, 15]	[35, 40]
h	[5, 10]	[20, 25]
h_B	[12, 15]	[17, 20]
h_V	[2, 5]	[7, 10]
D	[300, 400]	[900, 1000]
P	[1300, 1400]	[1900, 2000]
Cv^1	[0.05, 0.10]	[0.25, 0.30]
β	[0.1, 0.2]	[0.8, 0.9]
π_0	[80, 90]	[140, 150]
π_1	[20, 30]	[60, 70]
c_1	[0.2, 0.4]	[0.8, 1.0]
c_2	[1.8, 2.0]	[3.0, 3.2]
c_3	[4.0, 4.2]	[5.8, 6.0]

[1]Coefficient of variation of demand, i.e., $Cv \equiv \sigma / D$

.

Table 2. Range of values for each parameter.

Optimization approach	Inventory systems			
	Single Echelon		***Double echelon***	
	Periodic review	*Continuous review*	*Periodic review*	*Continuous review*
GA	≅ 3120 seconds	≅ 4110 seconds	≅ 4720 seconds	≅ 5875 seconds
Optimization method with iterative procedure	≅ 31 seconds	≅ 37 seconds	≅ 322 seconds	≅ 361 seconds
Our Optimization method	≅ 5 seconds	≅ 8 seconds	≅ 53 seconds	≅ 85 seconds

Table 3. Computational time required to solve 2000 randomly generated optimization problems.

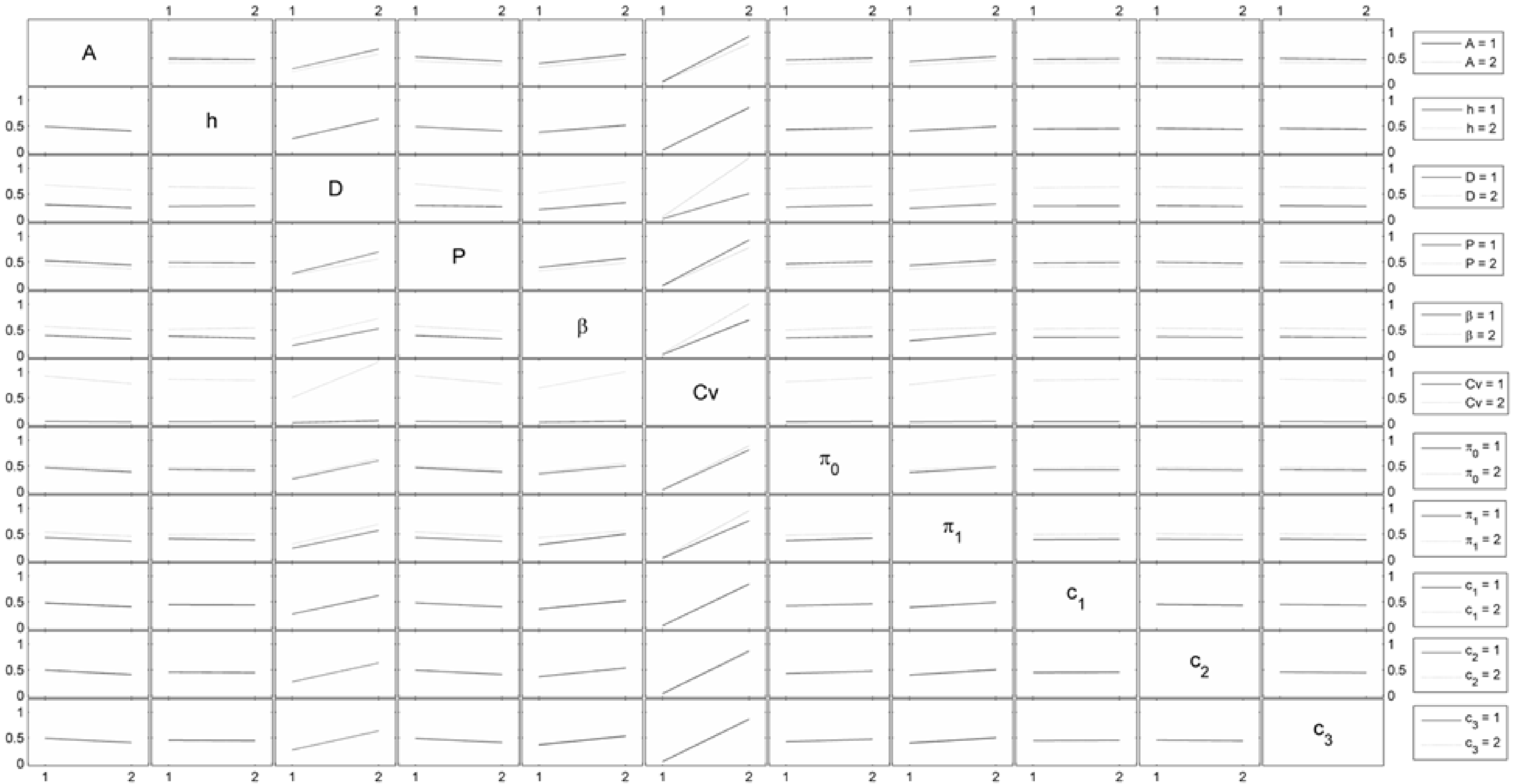

Figure 1. Results of the error analysis. Single-echelon inventory system under periodic review.

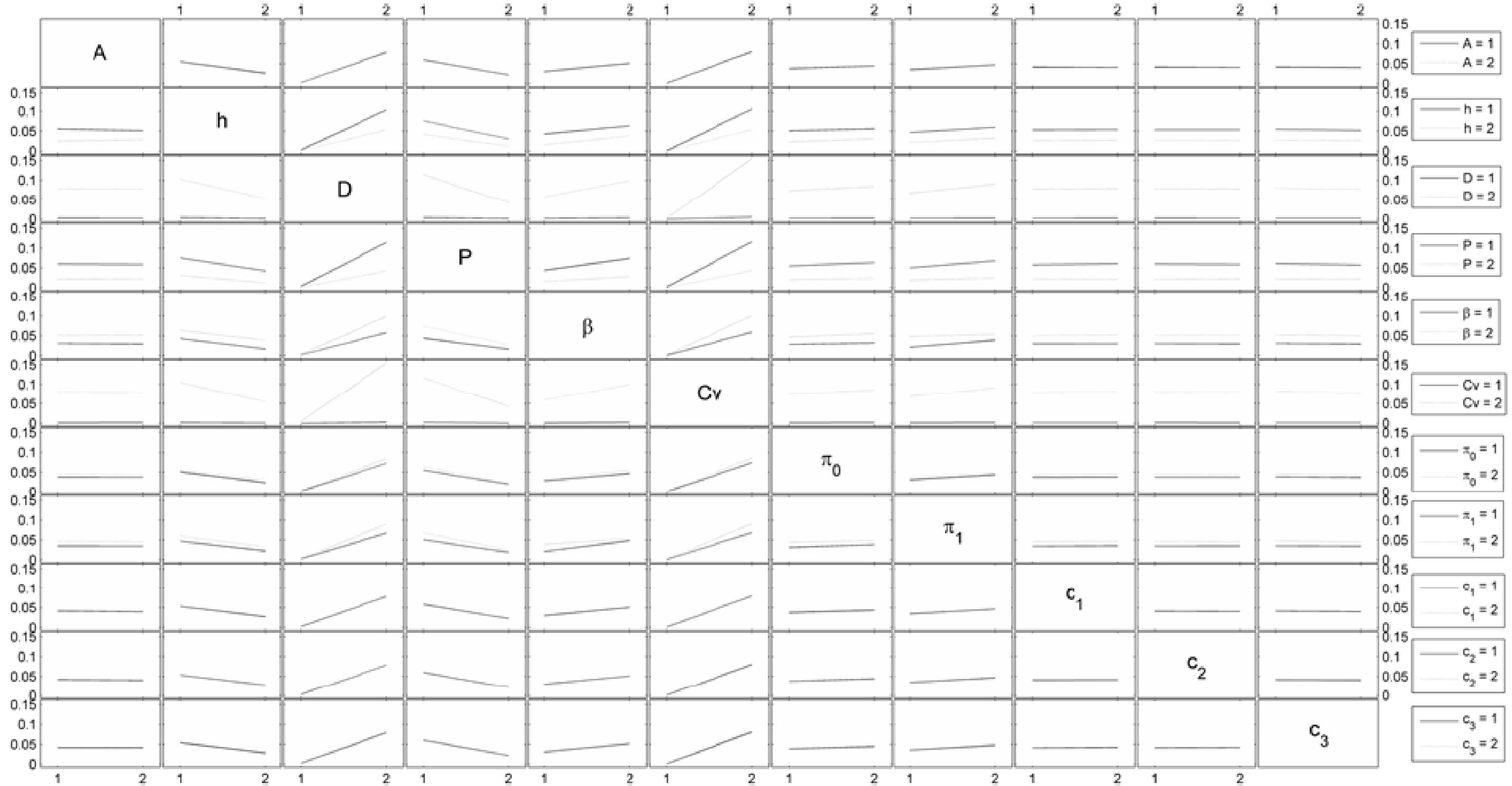

Figure 2. Results of the error analysis. Single-echelon inventory system under continuous review.

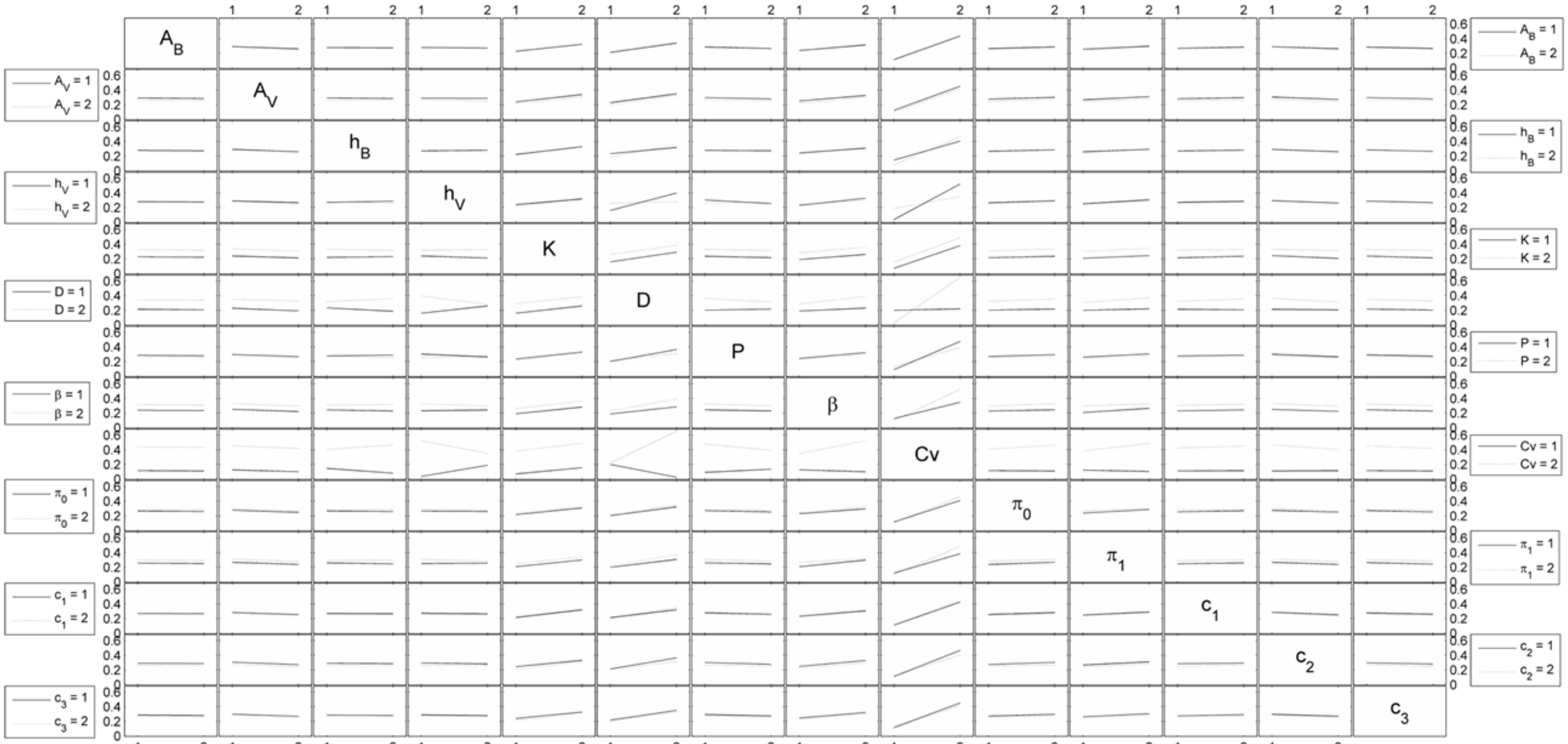

Figure 3. Results of the error analysis. Double-echelon inventory system under periodic review.

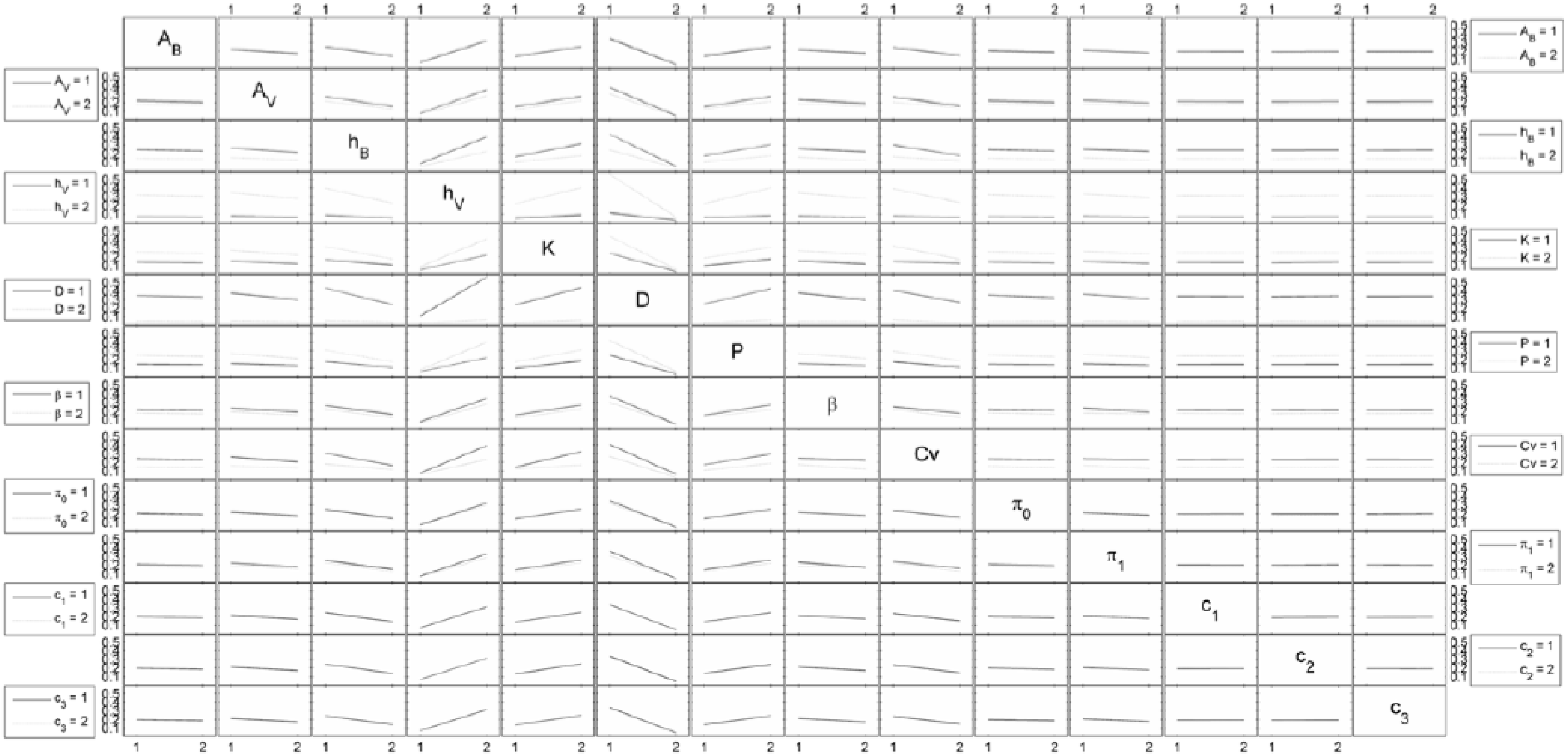

Figure 4. Results of the error analysis. Double-echelon inventory system under continuous review.