Legal Convergence and Endogenous Preferences

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Abstract

We develop a 2-country model of legal dynamics in which each country's social welfare depends generally on its actual law, its culturally ideal one, its technologically efficient one and the actual law of the other country. In our model, countries are better off when all these laws coincide. Moreover, in each country the actual law and the cultural biases of the population respond to the cost of legal diversity, the cost of the divergence between the actual law and both the culturally ideal law and the technological efficiency of regulation. We show that international legal convergence is possible without any coordination between countries. This happens when either there are no efficient legal rules, or when the technologically efficient rule is unique across countries. In that case, legal uniformity is realized in the long run. When there are country specific technologically efficient legal rules, we show that legal convergence is not possible in the long run.

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1 Introduction

Increasing economic and cultural globalizations raise the economic and moral costs of legal differences between countries and plead for an accelerated process of international unification of laws. Yet, the way nations reach a compromise between the respect of culturally ideal law (*i.e.*, legal preferences), actual national law and foreign laws is influenced by several conflicting factors.

First, according to some scholars there are intrinsic limits to the convergence of legal systems. Countries usually choose legal rules that are consistent with the legal framework

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and the social norms they inherited from the past. Law scholars such as Legrand (1996, 1997) share this viewpoint and conclude that international legal unification will not happen spontaneously. Some law-and-economics scholars develop a similar idea via the notion of legal origins (Glaeser and Shleifer, 2002, La Porta *et al.* 2008): countries are tied to the legal system they have chosen in the past (however it was chosen) and this national legal origin constrains further legal evolutions. Legal rules can change over time, but only at the margin, remaining globally within the national legal framework. At an international level, therefore, decentralized policies can only lead to a durable legal status quo. Some authors also advance that countries should indeed converge toward country specific optima, which would result in legal divergence (Balas *et al.*, 2009, Guerriero, 2013). Achieving legal uniformity would thus require international coordination. This legal uniformity, however, might not be the best legal distance between countries.

On the contrary, several scholars in international relations (*e.g.*, Pollack, 2007) and international public law (*e.g.*, Goodman and Jinks, 2004, 2005, 2008, 2013) argue that a key factor of legal convergence is the change in the perception of the world that policy makers experience either by frequent interactions at an international level (*i.e.*, international socialization) and/or by feeling an obligation to comply with a dominant culture (*i.e.*, *acculturation*). In this view, international interactions generate a convergence of legal preferences which in turn leads to legal convergence (*institutional isomorphism* in the terms of Goodman and Jinks, 2005, 2013).[1]

Another factor of legal convergence builds on the idea that legal rules could drift toward efficiency over time. According to Merryman *et al.* (1994), there is often a unique optimal solution to a legal problem. Therefore countries could converge spontaneously toward the same rule when they face the same legal problem, even without any interactions among them. This process is called "Natural convergence" in the comparative law literature. Complications can arise, however, when a legal problem has multiple solutions. It is a well known topic in comparative law that different legal rules can solve the same problem. In such cases functional convergence occurs but formal laws diverge.

Levmore (1986) was one of the first to analyze legal convergence from a Law-and-Economics perspective. According to him, if there is no efficient rule, divergence is likely to arise. Our paper suggests the opposite. In constrast to Levmore, who analyzes legal systems or traditions that do not necessarily engage in interactions, we focus on cases where countries with different legal systems interact. Even if there is no clear efficient rule, legal diversity is costly for these countries. This is the main engine of legal uniformity (which is realized through gradual changes in legal preferences and actual legal systems).

While the study of legal dynamics is a very active topic in the law-and-economics literature, it currently lacks an analytical framework to understand the *joint* evolution of national

[1] In this paper, we focus on the evolution of public rules of law, but the same kind of process can be used to understand the common evolution of some private practices. Concerning corporate governance, for instance, the rise of interactions between private actors at an international level has led to a broad consensus on general principles for firms' objectives and management methods.

preferences and national legal rules, and the conditions leading to legal convergence. To fill this gap, we develop a 2-country model of legal dynamics in which each country's social welfare depends generally on its actual law, its culturally ideal one, its technologically efficient one and the actual law of the other country. In this model, countries are better off when all these laws coincide. Moreover, in each country the actual law and the cultural biases of the population respond to three kinds of costs. The first kind is the cost of legal diversity. The second kind is the cost of the divergence between the actual law and the culturally ideal law. The third and last kind is the the cost of the divergence between the actual law and the technological efficiency of regulation. In order to highlight the relative importance of each factor, we follow an incremental approach: we start with a simple model of legal dynamics with fixed legal preferences, and then progressively introduce endogenous changes in legal preferences (namely endogenous culturally ideal laws), a unique technologically efficient legal rule, and finally several technologically efficient legal rules.

We show that endogenous legal preferences can be a strong vector of legal convergence. When there is no technologically efficient law, endogenous changes in legal preferences alone can be sufficient to achieve *de facto* legal uniformity in the long run. In this way, we show that legal uniformity does not always require international legal coordination. When there is a unique technologically efficient legal rule, we show that national legal systems as well as national legal preferences converge to this level. By contrast, when there are different technologically efficient legal rules, legal divergence occurs in the long run. In such cases, relying on decentralized policies is not sufficient to achieve legal convergence.

The paper unfolds as follows. In section 2, we present our model of legal dynamics and we consider the benchmark case where legal preferences are fixed. In section 3, we analyze legal dynamics when there are endogenous changes in legal preferences. In section 4, we study the interplay between these changes in preferences and the natural convergence hypothesis (we consider in turn the case where there is a unique technologically efficient legal rule and the case where there are multiple national technologically efficient legal rules). We conclude in the last section. The appendix provides the proofs of all the results.

2 A Simple Model of Legal Preferences

2.1 General assumptions

We develop a game theoretical model of legal convergence whose players are two countries and introduce incrementally the different elements that play an important role in legal dynamics[2]. Initially, we assume that the only strategic choices for the players are their laws. Next, we suppose that the strategic choices also comprise their culturally ideal laws (*i.e.*, their legal preferences). Specifically, we consider that both the actual laws and legal preferences can be

[2] Dari-Mattiacci *et al.* (2011) also propose a model of legal dynamics, but its aims are quite different from what we propose to study in this paper (the authors propose an explanation of the existence of cycles in the evolution of legal systems).

associated to points of the real line. We may interpret a point of the real line as being the value of an aggregate index of legal rules concerning a specific issue of the legal system. The construction of aggregate indexes of legal rules is a current practice in the empirical law-and-economics literature. For instance, the Leximetrics database comprises several quantitative indexes of legal rules about corporate governance, creditor law or labour law (see Armour *et al.*, 2009, and Siems, 2011).

More formally, we denote respectively by x_t^i and π_t^i the actual law and the legal preferences of country i ($i = 1, 2$) at date t ($t \in \mathbb{N}$). We assume that there is initially no legal uniformity, *i.e.*, $x_0^1 \neq x_0^2$ and $\pi_0^1 \neq \pi_0^2$. All these numbers are always common knowledge.

At each date, both countries decide simultaneously the magnitude of change of their laws and, when possible, their legal preferences. The decision horizon of a country is one period ahead, as if there were non-overlapping generations of law makers.[3]

Our aim is to study how laws and legal preferences evolve over time. We propose two definitions concerning the issue of legal dynamics:

Definition 1 (Legal Convergence)**.** *There is legal convergence at date t if the legal distance at this date decreases with respect to the previous period: $|x_t^i - x_t^j| < |x_{t-1}^i - x_{t-1}^j|$ ($i \neq j$).*

Definition 2 (Legal Uniformity)**.** *There is legal uniformity at date t when the legal distance is nil, i.e., $|x_t^i - x_t^j| = 0$ ($i \neq j$).*

The preferences of law makers are described by a utility function which is the sum of two continuously differentiable sub-functions, all defined in $\mathbb{R}^2$: $U^i(x^i, x^j) + V^i(x^i, \pi^i)$, $i, j = 1, 2$, and $i \neq j$. The first sub-function U^i describes the cost of divergence of country i 's law from the other country j's law (see Carbonara and Parisi, 2007 or Crettez *et al.*, 2013).[4] We assume that $U^i(x^i, x^j)$ reaches its maximum value if country i's law x^i is equal to country j's law x^j. We also assume that this function is increasing with respect to x^i if $x^i < x^j$ and decreasing if $x^i > x^j$. That is, this function increases where there is legal convergence and decreases otherwise. The second sub-function, $V^i(x^i, \pi^i)$, $i = 1, 2$, describes the cost of the divergence of country i's laws from country i's own legal preferences. We assume that this function reaches its maximum when $x^i = \pi^i$. We also assume that this function is increasing (resp. decreasing) with respect to x^i (resp. with respect to π^i) when $x^i < \pi^i$ and decreasing (resp. increasing) with respect to x^i (resp. with respect to π^i) when $x^i > \pi^i$.[5] It is easy

[3] This assumption could reflect the fact that decision-makers are constantly aware of recurring events which shorten their decision horizon (*e.g.* , elections). It considerably simplifies the analysis.

[4] The costs of legal diversity are well documented when it has an economic content. For example, Rodrik (2004) argues that the diversity of national institutional and legal arrangements is the most important source of transaction costs in international exchanges. According to this author, these costs broadly represent nearly 35% in ad-valorem terms.

[5] For example the sub-function $V^i(x^i, \pi^i)$ models the type of costs assumed by Legrand (1996, 1997) who argues that the foundations of legal systems build on the history, culture and collective preferences of each nation-state. Putting in place a law that would be in dissonance with national preferences would create internal tensions. These tensions are often observed in the European Union when harmonized legal rules, like the Bolkenstein Directive, are at variance with some national legal preferences.

to see that the quadratic function $-(x-y)^2$ satisfies all the above properties, as would all increasing functions of the type $f(x-y)$. Finally we assume that both functions U^i and V^i are continuously differentiable. This implies that the functions V^i , $i=1,2$, satisfy the property:

$$\text{sign}(V_2^i(x,y)) = -\text{sign}(V_1^i(x,y))$$

where V_k^i is the partial derivative of V^i with respect to its k-th argument.

2.2 The Nash equilibrium

For the remaining part of this section, we will assume that legal preferences are constant across time (*i.e.*, $\pi_t^i = \pi^i$, $i=1,2$). We will relax this assumption in the next section.

At each date t, given the value π^i of its legal preferences and the choice made simultaneously by the other country, the law-maker of country i ($i=1,2$, $i \neq j$) chooses the value of its law x_t^i in order to maximize the following utility function:

$$U^i(x_t^i, x_t^j) + V^i(x_t^i, \pi^i). \tag{1}$$

In a Nash equilibrium, the first-order conditions are as follows:

$$U_1^1(\overline{x}_t^1, \overline{x}_t^2) + V_1^1(\overline{x}_t^1, \pi^1) = 0 \tag{2}$$

$$U_1^2(\overline{x}_t^2, \overline{x}_t^1) + V_1^2(\overline{x}_t^2, \pi^2) = 0. \tag{3}$$

Each of these conditions reflects a trade-off between decreasing the cost of legal distance and decreasing the cost of the divergence between the actual law and the culturally ideal law.

The next Proposition establishes that legal uniformity is never satisfied in equilibrium when legal preferences are fixed.

Proposition 1. *When legal preferences are fixed and are initially different ($\pi^1 < \pi^2$), legal convergence stops at a given date and legal uniformity never occurs. Moreover, national laws are closer to each other than national legal preferences:*

$$\pi^1 < \overline{x}^1 < \overline{x}^2 < \pi^2. \tag{4}$$

The distance between laws is smaller than the distance between legal preferences. Moreover, once the laws $\overline{x}^1$ and $\overline{x}^2$ are chosen by national law-makers (that is, at date 0), there is no more convergence because a steady-state is achieved. As a consequence, we observe a strong legal origins effect (Deakin *et al.*, 2007): the distance between laws remains constant over time and legal uniformity never happens. These results arise because there is a trade-off between reducing the legal distance and increasing the gap between the actual laws and the national preferences. Legal uniformity does not occur because it is costly for countries to depart from their legal preferences, even though it is also costly to have laws different from those of other countries. The equilibrium values of the laws are then different from the culturally ideal ones. Once the equilibrium is realized at date 0, there is no reason to depart from it later on. The conditions under which countries make their choice are indeed always the same in each successive period.

Proposition 1 above embodies Legrand's viewpoint (1996, 1997), according to which legal convergence should not occur. Countries should have a legal system that remains close to their culturally ideal one. Guerriero (2013) provide theoretical foundations and empirical evidence about this idea that different cultural preferences lead to different legal systems.He shows that after independence 156 transplants changed once and for all their law-making institution to implement their culturally dependent optimal rule.

3 Legal Dynamics with Endogenous Legal Preferences

An important assumption in the model presented in the previous section is that legal preferences do not evolve over time. We relax this assumption in this section. We now assume that each utility function of the law-makers is the sum of three sub-functions. The first two subfunctions are the functions U^i and V^i used in the preceding functions. The third and new subfunction is a continously differentiable function $W^i(\pi_t^i, \pi_{t-1}^i)$. This function describes the cost borne by country i from changing its preferences. We assume that this function reaches its maximum when preferences do not change, *i.e.*, when $\pi_t^i = \pi_{t-1}^i$. We also assume that W^i is increasing with respect to π_t^i when $\pi_t^i < \pi_{t-1}^i$ and decreasing with respect to π_t^i when $\pi_t^i > \pi_{t-1}^i$. Before analyzing the model with endogenous legal preferences, we now justify the relevance of this new assumption.

3.1 Endogenous legal preferences

We can first justify the assumption of endogenous legal preferences using the notion of international socialization which appears in the international relations literature. Pollack (2007) reviews how policy and law makers' preferences change by interacting and socializing at an international level, inducing changes in their vision of the world. Persuasion, for instance, is often used by representatives or non governmental organizations to change their foreign colleagues' preferences.

A second rationale for endogenous legal preferences refers to what Goodman and Jinks (2008, 2013) call *acculturation*: "acculturation... [is] the general process by which actors adopt the beliefs and behavioural patterns of the surrounding culture. This complex social process is driven, at bottom, by identification with a reference group which generates varying degrees of cognitive and social pressures to conform with the behavioural expectations of the wider culture."[6] Acculturation works even if there are no physical international interactions. For example, important foreign constitutional and legislative provisions can effectively cause domestic internalization. Foreign practices are thus examined by a national court not for the legal reasoning of a foreign court, but as evidence of a norm that is widely accepted by other states. The U.S. supreme court's reasoning seems to witness such a process of acculturation (Pollack, 2008).

[6]See Goodman and Jinks (2004, 2013) for more details on the microeconomic process of acculturation.

By displaying norms that are interiorized by states and their representatives, international institutions are an important vector of acculturation. In European Union studies, for instance, a growing number of scholars argue that international institutions, and notably EU ones, shape preferences and identities of individuals and member states in Europe.

Goodman and Jinks (2005, 2013) argue that acculturation is specifically important to understand legal convergence (*i.e.*, what they called *institutional isomorphism*). According to them, legal convergence is primarily the result of acculturation, driven by the desire of individuals to fit in with a reference group of modern states[7].

The assumption of endogenous preferences is also often used in the economic literature (see *e.g.*, Kuran and Sandholm 2008, or Alesina and Reich, 2013). While international socialization and acculturation are essentially sociological notions, they have many features in common with the concept of endogenous preferences as it is developed in economics. In this paper, we try to conciliate the notions of international socialization and acculturation with economics and rational choice theory. Nevertheless, the way we model the evolution of preferences, as resulting from agents' choices, differs from standard assumptions of the economic literature where the evolution process of preferences is exogenous, and relies on evolutionary arguments (Bowles, 1998, Bisin and Verdier, 2001, Kuran and Sandholm, 2008).

Assuming that legal preferences or norms are endogenous in law-and-economics (and institutional economics) as we do is certainly not new (see, *e.g.*, Huck, 1998, Kotsadam and Jakobsson, 2011). Levmore (1986), in his analysis of functional convergence from a Law-and-Economics viewpoint, gives some credit to the possibility of endogenous legal preferences: "The discussion that follows, ..., most readily supports a position that combines an emphasis on law as "a device shaped by the members of society in response to internal conditions in the search for ways and means to translate their basic social postulates into action" with a view that changes (of these postulates) are stimulated when a culture's rules are not consonant with its survival" (Levmore, *ibid*, page 248).

Of course, it could also be argued that norms are constant or changing infrequently. Some evidence, for instance, shows that cultural biases are indeed long-lasting traits (Guiso *et al.*, 2008) which only change due to dramatic investment and consumption needs (Boranbay and Guerriero, 2012) or historical shocks (Nunn and Wantchekon, 2011). In this connection, Boranbay and Guerriero (2012) study a model where a group of agents "costly instills into its members a psychological gain from cooperating, for instance, by attracting a monastic order". They find that the evidence coming from medieval Europe suggests that cultural preferences can evolve in the short and medium run provided that the economic incentives of which they are an expression are sufficiently strong.[8]

[7]Goodman and Jinks (2013) also point to material inducement and persuasion as other means to achieve legal convergence.

[8]Boranbay and Guerriero (2012) is in line with the key insight of evolutionary psychology (Barkow, Cosmides, and Tooby, 1992): group-specific cultural values result from a process that selects, via natural selection or cross-punishment, norms maximizing the fitness of the group's members.

3.2 The Nash equilibrium

Law-makers simultaneously choose their new legal preferences and their legal systems in order to maximize the following utility function:

$$U^1(x_t^1, x_t^2) + V^1(x_t^1, \pi_t^1) + W^1(\pi_t^1, \pi_{t-1}^1), \quad i \neq j. \tag{5}$$

The result of countries' interactions at date t is given by the Nash equilibrium $(\overline{x}_t^1, \overline{\pi}_t^1, \overline{x}_t^2, \overline{\pi}_t^2)$[9]. The following Proposition differentiates between the ideal and actual legal systems and states that the legal distance between countries always decreases over time.

Proposition 2. *Assume that legal preferences are initially different, i.e.* $\pi_{-1}^1 < \pi_{-1}^2$. *Let a sequence of Nash equilibria be given. Then both laws and legal preferences tend to be more alike across time. Moreover, at each date t, the legal distance is smaller than the distance between legal preferences:*

$$\overline{\pi}_{t-1}^1 < \overline{\pi}_t^1 < \overline{x}_t^1 < \overline{x}_t^2 < \overline{\pi}_t^2 < \overline{\pi}_{t-1}^2. \tag{6}$$

The sequence of legal systems $(\overline{x}_t^1, \overline{x}_t^2)$ *and legal preferences* $(\overline{\pi}_t^1, \overline{\pi}_t^2)_t$ *converges respectively to the same limit. Legal uniformity is realized in the long-run.*

This result captures the idea that socialization and acculturation make legal systems and legal preferences more alike accross time (*institutional isomorphism* in Goodman and Jenkins terms).

To intuitively grasp this result, we first observe that in a Nash equilibrium the following optimality conditions are satisfied at all dates t:

$$U_1^1(\overline{x}_t^1, \overline{x}_t^2) + V_1^1(\overline{x}_t^1, \overline{\pi}_t^1) = 0 \tag{7}$$
$$V_2^1(\overline{x}_t^1, \overline{\pi}_t^1) + W_1^1(\overline{\pi}_t^1, \overline{\pi}_{t-1}^1) = 0 \tag{8}$$
$$U_1^2(\overline{x}_t^2, \overline{x}_t^1) + V_1^2(\overline{x}_t^2, \overline{\pi}_t^2) = 0 \tag{9}$$
$$V_2^2(\overline{x}_t^2, \overline{\pi}_t^2) + W_1^2(\overline{\pi}_t^2, \overline{\pi}_{t-1}^2) = 0. \tag{10}$$

The first equality means that the marginal gain of a decrease in legal diversity for country 1 ($U_1^1(\overline{x}_t^1, \overline{x}_t^2)$) is compensated by the marginal cost of the distance between its actual law and its legal preferences ($V_1^1(\overline{x}_t^1, \overline{\pi}_t^1)$). The second equality means that the marginal gain of a decrease in the distance between country 1's laws and its legal preferences is compensated by the marginal cost of changing these preferences. The other equations can be interpreted in a similar way.

To understand the above Proposition let us assume that $\overline{\pi}_{t-1}^1 \leq \overline{x}_t^2 \leq \overline{\pi}_t^2 \leq \overline{\pi}_{t-1}^2$ and let us check that it is in the interest of country 1 to choose $\overline{x}_t^1$ and $\overline{\pi}_t^1$ so as to verify: $\overline{\pi}_{t-1}^1 \leq \overline{\pi}_t^1 \leq \overline{x}_t^1 \leq \overline{x}_t^2 \leq \overline{\pi}_t^2 \leq \overline{\pi}_{t-1}^2$. Indeed, under this ranking, we see that the marginal gain $U_1^1(\overline{x}_t^1, \overline{x}_t^2)$ of making laws closer is non-negative (because U^i). This gain is compensated

[9] While we use the same notations for the equililbrium values of the legal systems as in Proposition , these equilibrium values obtained with endogenous legal preferences are a priori different from the equilibrium values obtained with fixed legal preferences.

by the marginal loss ($V_1^1(\overline{x}_t^1, \overline{\pi}_t^1) \leq 0$) borne by country 1 in making its law more distant from its legal preferences. Moreover, the marginal gain in closing the gap between legal preferences and country 1's law ($V_2^1(\overline{x}_t^1, \overline{\pi}_t^1) \geq 0$) is compensated by the marginal cost of changing its legal preferences ($W_1^1(\overline{\pi}_t^1, \overline{\pi}_{t-1}^1) \leq 0$).

From the first-order conditions, one can deduce that *instant legal uniformity* ($\overline{x}_t^1 = \overline{x}_t^2$) is never a Nash equilibrium. Indeed, given our assumption on U^i, this would imply that $U_1^1(\overline{x}_t^1, \overline{x}_t^2) = 0$ and thus $V_1^1(\overline{x}_t^1, \overline{\pi}_t^1) = 0$. But the later condition only holds when $\overline{x}_t^1 = \overline{\pi}_t^1$. Moreover, from the second equation and our assumption on the function W^i, we have $V_2^1(\overline{x}_t^1, \overline{\pi}_t^1) = 0$, so that $W_1^1(\overline{\pi}_t^1, \overline{\pi}_{t-1}^1) = 0$, and thus $\overline{\pi}_t^1 = \overline{\pi}_{t-1}^1$. The same reasoning applies to country 2, so that we finally have: $\overline{\pi}_{t-1}^1 = \overline{\pi}_{t-1}^2$, which is impossible by assumption.

Using the same argument we can show that the status quo is impossible, and that the only possible outcome is one satisfying the statement of the Proposition.

Assuming that all functions U^i, V^i and W^i, $i = 1, 2$, are *continuously* differentiable enables us to show that complete legal unification occurs in the long run.[10]

Proposition 2 shows that the cost of legal diversity as well as the cost of the distance between actual laws and legal preferences within countries are key factors for legal convergence in the long run. Differences in legal preferences prevent laws from converging instantly and from eliminating the cost of legal diversity. This difference is a key reason why countries in this model are continuously induced to adapt their legal preferences.

We conclude from this Proposition that even in the absence of legal cooperation, legal convergence and legal uniformity can occur in the long run, via a step-by-step process of adjustment of both legal rules and legal preferences. As long as countries can choose their desired mix of changes in actual laws and preferences, it is optimal for them to always change their laws and their legal preferences. This leads to a continuous process of legal convergence, which ends in the long run in legal uniformity (under a mild technical assumption). This contrast strongly with the Proposition 1 and in the terms of Deakin *et al.* (2007), when legal preferences can change over time, we should assist to a "weak legal origin effect". Legal rules can be different in the short run if they are initially different, but they tends to become identical in the long run, because countries also gradually converge in terms of their culturally ideal laws. International convergence of values is then necessary to have international convergence of legal rules.

3.3 Evidence about legal convergence

In this section, we have shown that legal convergence can occur spontaneously when preferences change and legal diversity is costly. This result is illustrated by the findings of Goodman and Jinks (2013). They notice that nation-states have adopted similar institutional forms and policies across a range of issue-areas, such as education, environmental protection, scientific research, and state administration (see Goodman and Jinks, 2013, chapter

[10]The requirement that the functions U^i, V^i and W^i be continuously differentiable ensures that the first-order conditions are satisfied at the steady-state of the dynamics.

4). State convergence with respect to women's rights is an example of legal convergence in these issue-areas. As Goodman and Jinks (2004) put it:

> After an initial stage of early adopters, the number of states providing women the right to vote increased steeply and included most states before the rate of adoption tapered off. Additionally, an important finding indicates a "contagion" effect: once the norm was institutionalized, a strong predictor for whether an individual state would enact women's suffrage was whether other states in its region had done so in the past five years. The overall findings suggest that, compared with local conditions such as the strength of domestic women's rights groups, countries apparently are affected much less strongly by internal factors and much more strongly by shifts in the international logic of political citizenship.

Some recent examples of international waves of change have occurred in European countries, for example on the possibility to abort freely or to contract same-sex marriage. International references to foreign laws were constantly made during the legislative procedures of change. Abortion is now legal in nearly all European countries. Countries with catholic roots such as Spain and Portugal have changed their laws later but in the long run, culture and religion have not been barriers to legal convergence.[11]

4 Legal Dynamics with Endogenous Legal Preferences and the Natural Convergence Assumption

4.1 Natural convergence

Discussions about legal dynamics in the law-and-economics literature often focus on the idea that some legal rules are economically more efficient than others. Levmore (1986) introduces an explanation of variety and uniformity in legal systems. Considering that many legal rules serve to channel behavior, the author argues that "we should find more uniformity across legal systems when theory tells us that a rule matters. For example, since it is easy to predict the deterioration of the social and economic fabric of any society if there are no deterrents to theft, we should expect to find thieves liable at least for what they have taken, and probably more. For many of the same reasons, we should expect negligent behavior to be discouraged as well". Such examples refer to functional convergence.

A corresponding view in the comparative law literature is that if countries face identical problems and if there is only one optimal solution for a given legal problem, countries will converge spontaneously toward this optimal solution in the long run (Merryman *et al.*, 1994). This view is summarized by Zweigert and Kötz (1998) as follows: "Different legal systems give the same or very similar solutions, even as to detail, to the same problems of life,

[11] Of course, theses correlations are not a *proof* of the effects of international socialization on legal convergence. As mentioned by a referee, there exists a plethora of possible unobserved determinants of legal changes and preferences evolution that could account for the observed correlations.

despite the great differences in their historical development, conceptual structure, and style of operations". In corporate law for instance, Hansmann and Kraakman (2001) argue that a broad consensus has emerged about conceptual elements such as the necessity of managing the firm in the interest of shareholders. This consensus has led to similar changes in legal rules over the world. Competitive pressures have also induced convergence towards efficient law in the long run. Roe (2000, 2001) even argues that competitive forces tend to erase the differences between the economic points of view of political parties (*e.g.*, left parties have became less hostile to shareholders over time).

We now take into account in the model the cost of not choosing an efficient legal rule. We assume that the preferences of law-makers are represented by the sum of the three sub-functions used in the preceding function and the (continuously differentiable) sub-function $Z^i(x_t^i, x^i)$, where x^i is the efficient rule for country i. With these preferences, Z^i is maximized when $x_t^i = x^i$; it is increasing with respect to x_t^i when $x_t^i < x^i$, and decreasing when $x_t^i > x^i$.

For some legal scholars, the determination of the optimal legal rule is country or system specific. For example, a civil law system can be as efficient as a common law (see, *e.g.*, Garoupa and Ligüerre, 2011, Guerriero, 2013). In terms of our model, this means that we can either assume that there is only one efficient legal rule on a given issue, or assume that there are multiple country dependent efficient legal rules. In the former case we have formally: $x^1 = x^2 = x^*$. In the latter case, $x^1 \neq x^2$. We study legal dynamics in each of these cases in turn.

4.2 The case with a unique optimal legal rule

We assume that at each date t the law-makers maximize the following objective which summarizes the elements aforementioned:

$$U^i(x_t^i, x_t^j) + V^i(x_t^i, \pi_t^i) + W^i(\pi_t^i, \pi_{t-1}^i) + Z^i(x_t^i, x), \quad i \neq j. \tag{11}$$

4.2.1 The Nash equilibrium

In a Nash equilibrium the first-order conditions are as follows:

$$U_1^1(\overline{x}_t^1, \overline{x}_t^2) + V_1^1(\overline{x}_t^1, \overline{\pi}_t^1) + Z_1^1(\overline{x}_t^1, x) = 0 \tag{12}$$

$$V_2^1(\overline{x}_t^1, \overline{\pi}_t^1) + W_1^1(\overline{\pi}_t^1, \overline{\pi}_{t-1}^1) = 0 \tag{13}$$

$$U_1^2(\overline{x}_t^2, \overline{x}_t^1) + V_1^2(\overline{x}_t^2, \overline{\pi}_t^2) + Z_1^2(\overline{x}_t^2, x) = 0 \tag{14}$$

$$V_2^2(\overline{x}_t^2, \overline{\pi}_t^2) + W_1^2(\overline{\pi}_t^2, \overline{\pi}_{t-1}^2) = 0. \tag{15}$$

Only the first and third conditions differ from those of the preceding section. In addition to the cost of legal distance and the cost of the difference between the actual law and legal preferences, the choice of law now takes into account the cost of legal inefficiency (*i.e.*, the cost of choosing a law different from the efficient one). For instance, decreasing further the legal distance with country j brings about an increase in the utility of country i which is compensated by the negative effects of an increase in the inefficiency of its law and/or an increase in the distance with its legal preferences.

When there is a unique optimal legal rule, we can establish the following proposition.

Proposition 3. *Let* $(x_t^1, x_t^2, \pi_t^1, \pi_t^2)_t$ *be a sequence of Nash equilibria. Then the laws and legal preferences of both countries converge and legal uniformity is achieved in the long run. Moreover, the limit law is the unique efficient one:* $\lim_{t\to+\infty} x_t^i = \lim_{t\to+\infty} \pi_t^i = x$, $i = 1, 2$.

Both countries finally adopt the efficient rule. Proposition 3 is similar to Proposition 2 in the sense that endogenous preferences allow for legal uniformity in the long run. But here, the long run value of the legal rule is the (unique) efficient one. It always pays to change a country's legal preferences so as to make them closer to the efficient legal rules.

The property that legal uniformity occurs in the long run depends strongly on the assumption that there is a unique optimal legal rule. When there are multiple optimal legal rules we will show that this result does not hold anymore.

4.3 The case with country specific optimal legal rules

We now analyze legal dynamics when the efficient legal rule is country dependent (we denote by x^i the efficient legal rule of country i, $i = 1, 2$, $x^1 \neq x^2$)). As stated by Deakin *et al.* (2007), "legal rules are endogenized by local economic conditions and political context. This is a theme common both to modern comparative legal doctrine of functional analysis and to the varieties-of-capitalism approach. The same effect might be achieved in one system by a rule of law and in another by self-regulatory instruments of soft law". For example, an efficient labour law can be country specific because the weight given to job protection is not the same everywhere.

We now assume that at each date t the law-makers maximize the following objective:

$$U^i(x_t^i, x_t^j) + V^i(x_t^i, \pi_t^i) + W^i(\pi_t^i, \pi_{t-1}^i) + Z^i(x_t^i, x^i), \quad i \neq j. \tag{16}$$

4.3.1 The Nash equilibrium

In a Nash equilibrium the first-order conditions are as follows:

$$U_1^1(\overline{x}_t^1, \overline{x}_t^2) + V_1^1(\overline{x}_t^1, \overline{\pi}_t^1) + Z_1^1(\overline{x}_t^1, x^1) = 0 \tag{17}$$

$$V_2^1(\overline{x}_t^1, \overline{\pi}_t^1) + W_1^1(\overline{\pi}_t^1, \overline{\pi}_{t-1}^1) = 0 \tag{18}$$

$$U_1^2(\overline{x}_t^2, \overline{x}_t^1) + V_1^2(\overline{x}_t^2, \overline{\pi}_t^2) + Z_1^2(\overline{x}_t^2, x^2) = 0 \tag{19}$$

$$V_2^2(\overline{x}_t^2, \overline{\pi}_t^2) + W_1^2(\overline{\pi}_t^2, \overline{\pi}_{t-1}^2) = 0. \tag{20}$$

These conditions and their interpretations are similar to those obtained for the case where there is a unique efficient legal rule.

4.3.2 Steady state and legal dynamics

Using the equilibrium first-order conditions, we obtain the next result concerning the steady state of the dynamics.

Proposition 4. *In a steady state, we have: $x^1 < \overline{\pi}^1 = \overline{x}^1 < \overline{x}^2 = \overline{\pi}^2 < x^2$. In a steady-state, the actual laws coincide with legal preferences. Moreover, in each country the actual law differs from the corresponding optimal legal rules.*

The above result shows that there is no legal uniformity in a steady state. Moreover, in each country the actual law coincide with its legal preferences. These preferences, however, do not coincide with the efficient legal rules. The distance between the laws remains positive but smaller than the difference between the country's efficient legal rules. There is thus a compromise between the efficiency loss and the cost of legal distance.

The general study of the dynamics, the convergence and stability properties of the steady state is quite complex and it is not easy to obtain general results. The next Lemma, however, illustrates a case where the legal dynamics remain in the interval $[x^1, x^2]$

Lemma 1. *Assume that π^i_{t-1} belongs to $[x^1, x^2]$, $i = 1, 2$. Then, x^i_s and π^i_s belong to $[x^1, x^2]$, for all date $s \geq t$, $i = 1, 2$.*

It is difficult to obtain more convergence results.[12] Yet, we can prove more results with quadratic preferences.

4.3.3 Global stability of the steady-state in the quadratic case

We will now use the following quadratic specification of the objective functions:

$$\mathcal{U}^1 = -\frac{\alpha_1}{2}(x^1_t - x^2_t)^2 - \frac{\beta_1}{2}(x^1_t - \pi^1_t)^2 - \frac{\gamma_1}{2}(\pi^1_t - \pi^1_{t-1})^2 - \frac{\theta_1}{2}(x^1_t - x^1)^2 \tag{21}$$

$$\mathcal{U}^2 = -\frac{\alpha_2}{2}(x^2_t - x^1_t)^2 - \frac{\beta_1}{2}(x^2_t - \pi^2_t)^2 - \frac{\gamma_1}{2}(\pi^2_t - \pi^2_{t-1})^2 - \frac{\theta_2}{2}(x^2_t - x^2)^2. \tag{22}$$

With these functions the first-order necessary conditions are as follows:

$$-\alpha_1(x^1_t - x^2_t) - \beta_1(x^1_t - \pi^1_t) - \theta_1(x^1_t - x^1) = 0 \tag{23}$$

$$\beta_1(x^1_t - \pi^1_t) - \gamma_1(\pi^1_t - \pi^1_{t-1}) = 0 \tag{24}$$

$$-\alpha_2(x^2_t - x^1_t) - \beta_2(x^2 - \pi^2_t) - \theta_2(x^2_t - x^2) = 0 \tag{25}$$

$$\beta_2(x^2_t - \pi^2_t) - \gamma_2(\pi^2_t - \pi^2_{t-1}) = 0. \tag{26}$$

We can check that the steady-state is given by the following expressions:

$$x^i_\infty = \pi^i_\infty = \frac{(\alpha_j + \theta_j)\theta_i x^i + \alpha_i \theta_j x^j}{\alpha_i \theta_j + \theta_i(\alpha_j + \theta_j)}, \; i \neq j, \; i = 1, 2 \; j = 1, 2. \tag{27}$$

Concerning the dynamics, we can establish the following proposition:

Proposition 5. *In the quadratic case, the steady-state is globally stable.*

[12] We can prove, however, that under the assumptions of Lemma 1 there are an infinite number of times for which the sequence $(x^i_t, \pi^i_t)_t$ $(i = 1, 2)$ is arbitrarily close to the steady state studied above.

This result means that whatever the initial values of the laws and legal preferences in a given country, their values converge to the same number, which is country dependent, and which lies between x^1 and x^2. Legal uniformity does not occur in the long run (the legal distance remains constant).

4.4 Evidence of "natural convergence"

Many scholars have documented a great wave of convergence of legal rules which have an important economic content (see, *e.g.*, Zweigert and Kötz, 1998, who use a comparative law approach). The main active domains concern corporate law and stock market law, both of which have transnational implications. At the same time, all rules in the domain of corporate law and stock market law are not equivalent from an economic efficiency viewpoint. In continental Europe and Japan, changes in securities law and (to a lesser extent) in corporate law have increased the sensitivity of managers to the interests of external (minority) shareholders and there is evidence of some convergence of legal rules (Hansmann and Kraakman, 2001, Lele and Siems, 2006, Perraudin *et al.*, 2013). Siems (2008) provides a detailed comparative law analysis of the process of legal convergence in shareholder law. While he proposes nuanced views about this process, Siems highlights some trends of converging changes in national legal cultures (Civil Law and Common Law, Western and Asian law) and some trends of convergence in shareholder law. According to him, this diminishing distance between legal views and legal rules of different countries is driven partly by the internationalization of the economy and partly by the pressures of specific groups (companies, shareholders, foreign countries and international organizations) to converge. These different trends support the results of this section.

5 Conclusion

In this paper we have shown that legal convergence can occur in the long run when legal preferences are endogenous. To reduce the cost of legal diversity, countries can adapt their legal systems *and* their preferences. A mix of theses changes is optimal. It is indeed doubtful that adapting to legal diversity can be carried out only by changing legal systems. Making concessions on preferences is also necessary. The concessions reduce the cost borne out of the difference between laws and legal norms, which arise when countries change their legal systems to reduce legal diversity. If the legal preferences of a country can evolve over time, laws and preferences should gradually converge toward the legal systems and preferences of other nations, while at the same time these other nations follow a similar pattern.

Our conclusion with regard to the possibility of achieving legal convergence differs for example from the view of Legrand (1997), to whom laws depend essentially on the preferences of nations. If preferences are fixed and different across countries, legal convergence will not take place, and can even be undesirable.

Our model implies that the speed of legal convergence is different according to different laws, depending negatively on the magnitude of the costs of change, and positively on the importance of the process of acculturation/socialization. The speed of convergence could be lower when the costs of change are high and when international interactions are infrequent, or do not induce a process of international socialization. The ongoing process of globalization and internationalization of cultural and moral values[13] should then increase legal convergence.

When legal problems have solutions that can be ranked depending on their efficiency and when the optimal solution is unique, unification occurs in the long run because all countries tend to choose the optimal rule. Legal preferences change in the way that this optimal rule is the preferred rule in the long run.

An optimal solution, however, is not necessarily unique. For instance, choosing between the death penalty or a life sentence, allowing same-sex marriages or not, depend fundamentally on the cultural and moral values of people, of norms within a given country, rather than discussions on the intrinsic efficiency of the rule. In such cases, theories of legal evolution that build on natural convergence can not explain the process of legal evolution. Convergence cannot occur by this means. Convergence, is still is often observed for these kinds of rules. We can explain this fact by the effects of acculturation/socialization. From this viewpoint, international socialization can explain legal convergence even when there is no legal rule intrinsically preferable to others.

A potential limit of our analysis is that in our model law-makers do not take into account the future and the past. This assumption is made to simplify the study of the dynamics.[14] The fact that we do not consider the past, *i.e.*, the cost of legal change, overestimates the speed of legal convergence because we eliminate one reason which would explain the difficulty of changing a legal system. Neglecting the past is a strong assumption. For instance, in common law jurisdictions, an appellate judge fixes in each period a rule taking into account the cost of justifying the distance of such law from the precedent set by a previous appellate judge (Gennaioli and Shleifer, 2007).[15]We neglect this phenomenon, *i.e.*, stare decisis.

In a different but related paper (see Crettez *et al.*, 2013) we have addressed the issue of the effect of the length of the decision horizon in a model where the law-makers bear the cost of changing the laws (*i.e.*, where the past matters for the law-makers). We have found that a long decision horizon does not prevent legal convergence. There is no reason to think that this property would not hold here too. Moreover, while we overestimate the speed of legal convergence by not considering the past, we also underestimate the speed of convergence by not considering the future. If the decision-horizon of the law makers were long, they would take into account the impact of their present decisions on the future costs of legal distance and the future costs of changing legal preferences. As a result, legal convergence would probably be faster. To sum up, a change in the decision horizon of policymakers should

[13]For reference about this process, see Goodman and Kinks (2013) for the globalisation of values, or Cowen (2004) for the globalization of culture.

[14]Otherwise we would have two additional state-variables, one for each country, and it would not be easy to study the game, even in the quadratic case.

[15]We thank an anonymous referee for this remark.

change the speed of convergence but not the process of convergence itself. Our proposition, while based on strong assumptions, should remain true when policymakers take into account the future or the past.

In our model also, legal convergence – be it in the short or the long run – always occurs in a non-cooperative way. Legal cooperation can speed up or slow down and stop legal convergence (legal convergence may not necessarily be optimal). An interesting topic for further research would be to investigate whether and when legal competition is preferable to legal cooperation in a setting where preferences are endogenous.

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Appendix

Proof of Proposition 1. Assume that $\overline{x}^1 \geq \overline{x}^2$. Given our assumptions on functions U^i and V^i this implies $U_1^1(\overline{x}^1, \overline{x}^2) \leq 0$, and then $V^1(\overline{x}^1, \pi^1) \leq 0$. Thus $\overline{x}^1 \geq \pi^1$. Similarly, we have $U^2(\overline{x}^2, \overline{x}^1) \geq 0$, and then $V^2(\overline{x}^2, \pi^2) \leq 0$. This later inequation implies that $x^2 \geq \pi^2$. Therefore, we have $\pi^2 \leq \overline{x}^2 \leq \overline{x}^1 \leq \pi^1$, which is a contradiction. □

Proof of Proposition 2. By definition, a Nash equilibrium at date t satisfies the following optimality conditions:

$$U_1^1(\overline{x}_t^1, \overline{x}_t^2) + V_1^1(\overline{x}_t^1, \overline{\pi}_t^1) = 0, \tag{28}$$

$$V_2^1(\overline{x}_t^1, \overline{\pi}_t^1) + W_1^1(\overline{\pi}_t^1, \overline{\pi}_{t-1}^1) = 0, \tag{29}$$

$$U_1^2(\overline{x}_t^2, \overline{x}_t^1) + V_1^2(\overline{x}_t^2, \overline{\pi}_t^2) = 0, \tag{30}$$

$$V_2^2(\overline{x}_t^2, \overline{\pi}_t^2) + W_1^2(\overline{\pi}_t^2, \overline{\pi}_{t-1}^2) = 0. \tag{31}$$

Let us first prove that $\overline{x}_t^1 \leq \overline{x}_t^2$. If not, $\overline{x}_t^1 > \overline{x}_t^2$. Then, $U_1^1(\overline{x}_t^1, \overline{x}_t^2) < 0$ and from (28), it follows that: $V_1^1(\overline{x}_t^1, \overline{\pi}_t^1) > 0$. This implies that $x_t^1 < \overline{\pi}_t^1$. Using a similar argument with equation (30), one may show that: $\overline{x}_t^2 > \overline{\pi}_t^2$.

Using our assumptions on the functions V^i in (29) and (31), we have: $\text{sign}\big(V_1^1(\overline{x}_t^1, \overline{\pi}_t^1)\big) = -\text{sign}\big(V_2^1(\overline{x}_t^1, \overline{\pi}_t^1)\big)$ and $\text{sign}\big(V_1^2(\overline{x}_t^2, \overline{\pi}_t^2)\big) = -\text{sign}\big(V_2^2(\overline{x}_t^2, \overline{\pi}_t^2)\big)$.

Since, $V_1^1(\overline{x}_t^1, \overline{\pi}_t^1) > 0$, it follows that $W_1^1(\overline{\pi}_t^1, \overline{\pi}_{t-1}^1) > 0$. This implies that: $\overline{\pi}_t^1 < \overline{\pi}_{t-1}^1$. Similarly, we may prove that: $\overline{\pi}_t^2 > \overline{\pi}_{t-1}^2$. We have thus proven that:

$$\overline{\pi}_{t-1}^2 < \overline{\pi}_t^2 < \overline{x}_t^2 < \overline{x}_t^1 < \overline{\pi}_t^1 < \overline{\pi}_{t-1}^1, \tag{32}$$

which is a contradiction. Then, we must have: $\overline{x}_t^1 \leq \overline{x}_t^2$. It then follows that $U_1^1(\overline{x}_t^1, \overline{x}_t^2) \geq 0$ and from (28), $V_1^1(\overline{x}_t^1, \overline{\pi}_t^1) \leq 0 \Rightarrow \overline{x}_t^1 \geq \overline{\pi}_t^1$. As $\text{sign}\big(V_1^1(\overline{x}_t^1, \overline{\pi}_t^1)\big) = -\text{sign}\big(V_2^1(\overline{x}_t^1, \overline{\pi}_t^1)\big)$, one has $V_2^1(\overline{x}_t^1, \overline{\pi}_t^1) \geq 0$ and from (29), one has $W_1^1(\overline{\pi}_t^1, \overline{\pi}_{t-1}^1) \leq 0$. This implies $\overline{\pi}_t^1 \geq \overline{\pi}_{t-1}^1$.

A similar reasoning leads to: $\overline{x}_t^2 \leq \overline{\pi}_t^2$ and $\overline{\pi}_t^2 \leq \overline{\pi}_{t-1}^2$. We have thus proven:

$$\overline{\pi}_{t-1}^1 \leq \overline{\pi}_t^1 \leq \overline{x}_t^1 \leq \overline{x}_t^2 \leq \overline{\pi}_t^2 \leq \overline{\pi}_{t-1}^2. \tag{33}$$

Now if $\overline{x}_t^1 = \overline{x}_t^2$, one can see, by using the same kind of argument as above, that $\overline{\pi}_{t-1}^1 = \overline{\pi}_{t-1}^2$, which is a contradiction. Then, since $U_1^i(x_t^i, x_t^j)(-1)^{1+i} > 0$, $i \neq j$, one can easily see that $\overline{\pi}_{t-1}^2 < \overline{\pi}_t^2 < \overline{x}_t^2 < \overline{x}_t^1 < \overline{\pi}_t^1 < \overline{\pi}_{t-1}^1$. The fact that this property holds at each date t follows by induction.

Let us now prove that legal uniformity is realized in the long run. The sequence $(\overline{\pi}_t^1)_t$ is non-decreasing and upper-bounded. It thus converges to a limit $\overline{\pi}^1$. The sequence $(\overline{\pi}_t^2)_t$ is non-increasing and lower-bounded. It thus also converges to a limit $\overline{\pi}^2$. Now consider the sequence $(\overline{x}^1)_t$ (which lies in the set $[\overline{\pi}_{-1}^1, \overline{\pi}_{-1}^2]$). Let $\overline{\pi}^1$ be a limit of any converging subsequence. Using equation (29) and by passing to the limit, we obtain:

$$V_2^1(\overline{x}^1, \overline{\pi}^1) + W_1^1(\overline{\pi}^1, \overline{\pi}^1) = 0. \tag{34}$$

We know that $W_1^1(\overline{\pi}^1, \overline{\pi}^1) = 0$. Thus $V_2^1(\overline{x}^1, \overline{\pi}^1) = 0$. It follows that $V_1^1(\overline{x}^1, \overline{\pi}^1) = 0$. This implies that: $\overline{x}^1 = \overline{\pi}^1$. Since this is true for all limits of converging subsequences of $(\overline{x}_t^1)_t$ this implies that this sequence converges to $\overline{\pi}^1$. By the same argument, we can show that the sequence $(\overline{x}_t^2)_t$ converges to $\overline{\pi}^2$. Finally, using equation (28) and passing to the limit, we obtain that:

$$U_1^1(\overline{x}^1 \overline{x}^2) + V_1^1(\overline{x}^1, \overline{\pi}^1) = 0 \tag{35}$$

Since $\overline{x}^1 = \overline{\pi}^1$ this equation reduces to: $U_1^1(\overline{x}^1 \overline{x}^2) = 0$. This implies that $\overline{x}^1 = \overline{x}^2$. The proof is complete. □

Proof of Proposition 3. The proof goes through a series of lemmas. We will first consider the cases where $\max_i \overline{\pi}_{t-1}^i \leq x$. Then, we will address the case where $\min_i \overline{\pi}_{t-1}^i < x < \max_i \overline{\pi}_{t-1}^i$. Before that, we will prove the next lemma.

Lemma 2. *Let $(\overline{x}_t^1, \overline{x}_t^2, \overline{\pi}_t^1, \overline{\pi}_t^2)$ be a Nash equilibrium at date t. Then we must have either $\overline{x}_t^i \geq \overline{\pi}_t^i \geq \overline{\pi}_{t-1}^i$ or $\overline{x}_t^i \leq \overline{\pi}_t^i \leq \overline{\pi}_{t-1}^i$, $i = 1, 2$.*

Proof. Let us consider the necessary condition for optimality:

$$V_2^i(\overline{x}_t^i, \overline{\pi}_t^i) + W_1^i(\overline{\pi}_t^i, \overline{\pi}_{t-1}^i) = 0. \tag{36}$$

Recall that $\text{sign}(V_2^i(x, y)) = -\text{sign}(V_1^i(x, y))$. If $\overline{x}_t^i \geq \overline{\pi}_t^i$, then $V_1^i(\overline{x}_t^i, \overline{\pi}_t^i) \leq 0$, and then, $V_2^i(\overline{x}_t^i, \overline{\pi}_t^i) \geq 0$. Thus, we have $W_1^i(\overline{\pi}_t^i, \overline{\pi}_{t-1}^i) \leq 0$, so that $\overline{\pi}_t^i \geq \pi_{t-1}^i$. By the same reasoning, we can show that is if $\overline{x}_t^i \leq \overline{\pi}_t^i$, then $\overline{\pi}_t^i \leq \pi_{t-1}^i$. □

- **The case where $\overline{\pi}_{t-1}^1 < \overline{\pi}_{t-1}^2 < x$ (assuming w.l.o.g that $\max_i \overline{\pi}_{t-1}^i = \overline{\pi}_{t-1}^2$)**

Lemma 3. *In a Nash equilibrium, we have : $\overline{x}_t^1 > \overline{\pi}_t^1$.*

Proof. Assume that $\overline{x}_t^1 \leq \overline{\pi}_t^1$. Then we cannot have $\overline{x}_t^1 \leq \overline{x}_t^2$. Indeed, in this case, we have $Z_1^1(\overline{x}_t^1, x) \leq 0$, and so $\overline{x}_t^1 \geq x$. But from Lemma 2 this implies that $\overline{\pi}_{t-1}^1 \geq x$ which is impossible. So we necessarily have $\overline{x}_t^1 > \overline{x}_t^2$. Now assume that $\overline{x}_t^2 \leq \overline{\pi}_t^2$. Then, from Lemma 1, we must have $Z_1^2(\overline{x}_t^2, x) \leq 0$, so that $\overline{x}_t^2 \geq x$. Again, this implies that $\overline{x}_t^1 \geq x$, and thus $\overline{\pi}_{t-1}^1 \geq x$, which is a contradiction. Now consider the case where $\overline{x}_t^2 > \overline{\pi}_t^2$. Since $\overline{x}_t^1 > \overline{x}_t^2$, it follows from Lemma 1 that $\overline{\pi}_{t-1}^1 > \overline{\pi}_{t-1}^2$, which is impossible. □

From the two preceding lemmas, we know that in a Nash equilibrium, we always have $\overline{x}_t^1 > \overline{\pi}_t^1 \geq \overline{\pi}_{t-1}^1$.

Lemma 4. *In a Nash equilibrium, we always have* $\max\{\overline{x}_t^1, \overline{x}_t^2\} \leq x$.

Proof. Assume that $\overline{x}_t^1 \geq \overline{x}_t^2$. As $\overline{x}_t^1 > \overline{\pi}_t^1$, we must have $Z_1^1(\overline{x}_t^1, x) \geq 0$, so that $\overline{x}_t^1 \leq x$ and the result follows. Suppose instead that $\overline{x}_t^1 < \overline{x}_t^2$. If $\overline{x}_t^2 \leq \overline{\pi}_t^2$, then as $Z_1^2(\overline{x}_t^2, x) \geq 0$, we must have $\overline{x}_t^2 \leq x$ and the result follows again. If, on the other hand, we have $\overline{x}_t^2 > \overline{\pi}_t^2$, then since we have $\overline{\pi}_t^2 < \overline{\pi}_{t-1}^2 < x$, the conclusion follows. □

Lemma 5. *In a Nash equilibrium, we have:* $\min\{\overline{x}_t^1, \overline{x}_t^2\} \geq \overline{\pi}_{t-1}^1$.

Proof. If $\overline{x}_t^2 \geq \overline{x}_t^1$, this is immediate (this stems from Lemmas 2 and 3). Assume now that $\overline{x}_t^1 > \overline{x}_t^2$. Assume that $\overline{x}_t^2 \leq \overline{\pi}_t^2$. Then $Z_1^2(\overline{x}_t^2, x) \leq 0$ and thus $\overline{x}_t^2 \geq x$. So $\overline{\pi}_{t-1}^2 \geq x$, which is a contradiction. Then consider the case $\overline{x}_t^2 > \overline{\pi}_t^2$. As $\overline{x}_t^1 > \overline{x}_t^2$ and $\overline{\pi}_{t-1}^2 > \overline{\pi}_{t-1}^1$, the conclusion follows from Lemma 2. ☐

Lemma 6. *In a Nash equilibrium, we have:* $\max\{\overline{\pi}_t^1, \overline{\pi}_t^2\} \leq x$.

Proof. We already know that $\overline{x}_t^1 > \overline{\pi}_t^1$ and $\overline{x}_t^1 \leq x$ (Lemmas 3 and 4). Let us show that $\overline{\pi}_t^2 \leq x$. If $\overline{x}_t^2 \geq \overline{\pi}_t^2$, this is immediate (Lemma 3). Assume now that $\overline{x}_t^2 < \overline{\pi}_t^2$. If $\overline{\pi}_t^2 \geq x$ then $\overline{\pi}_{t-1}^2 \geq x$ (Lemma 2), which is impossible and the result follows. ☐

Lemma 7. *In a Nash equilibrium we have:* $\min\{\overline{\pi}_{t-1}^1, \overline{\pi}_{t-1}^2\} = \overline{\pi}_{t-1}^1 \leq \min\{\overline{\pi}_t^1, \overline{\pi}_t^2\}$.

Proof. If $\overline{\pi}_t^2 \geq \overline{\pi}_t^1$, this is immediate, since $\overline{\pi}_t^1 \geq \overline{\pi}_{t-1}^1$ (Lemmas 2 and 3), and $\overline{\pi}_{t-1}^1 < \overline{\pi}_{t-1}^2$ by assumption. Assume now that $\overline{\pi}_t^2 < \overline{\pi}_t^1$. Suppose that $\overline{\pi}_t^2 < \overline{\pi}_{t-1}^1$. If $\overline{x}_t^2 \geq \overline{\pi}_t^2$, this is impossible (because of Lemma 2 and the assumption that $\overline{\pi}_{t-1}^1 < \overline{\pi}_{t-1}^2$). If $\overline{x}_t^2 < \overline{\pi}_t^2$, then from Lemma 2 we have $\overline{x}_t^2 < \overline{\pi}_t^2 < \overline{\pi}_{t-1}^1$, from Lemma 5 this is impossible (because $\overline{\pi}_{t-1}^1 \leq \overline{x}_t^1$). ☐

Remark. When $\overline{\pi}_{t-1}^2 < \overline{\pi}_{t-1}^1 < x$, we have the same kind of results as above: one only has to substitute $\overline{x}_t^2$, for $\overline{x}_t^1$ and $\overline{\pi}_t^2$ for $\overline{\pi}_t^1$.

• **The case where** $\overline{\pi}_{t-1}^1 = \overline{\pi}_{t-1}^2 < x$.

Lemma 8. *In a Nash equilibrium, we have :* $\overline{x}_t^1 > \overline{\pi}_t^1$.

Proof. Same as the proof of Lemma 3. ☐

Lemma 9. *In a Nash equilibrium, we always have* $\max\{\overline{x}_t^1, \overline{x}_t^2\} \leq x$.

Proof. Same proof as that of Lemma 4 ☐

Lemma 10. *In a Nash equilibrium, we have:* $\min\{\overline{x}_t^1, \overline{x}_t^2\} \geq \overline{\pi}_{t-1}^1$.

Proof. Same proof as that of Lemma 5 ☐

Lemma 11. *In a Nash equilibrium, we have:* $\max\{\overline{\pi}_t^1, \overline{\pi}_t^2\} \leq x$.

Proof. We already know that $\overline{x}_t^1 \geq \overline{\pi}_t^1$ and $\overline{x}_t^1 \leq x$ (lemmas 8 and 9). Let us show that $\overline{\pi}_t^2 \leq x$. If $\overline{x}_t^2 \geq \overline{\pi}_t^2$, this is immediate (Lemma 9). Assume now that $\overline{x}_t^2 < \overline{\pi}_t^2$. If $\overline{\pi}_t^2 \geq x$ then $\overline{\pi}_{t-1}^2 \geq x$ (because of Lemma 2), which is impossible and the result follows. ☐

Remark. The proof is similar to the proof of Lemma 6.

Lemma 12. *In a Nash equilibrium we have:* $\overline{\pi}_{t-1}^1 = \overline{\pi}_{t-1}^2 = \min\{\overline{\pi}_{t-1}^1, \overline{\pi}_{t-1}^2\} \leq \min\{\overline{\pi}_t^1, \overline{\pi}_t^2\}$.

Proof. If $\overline{\pi}_t^2 \geq \overline{\pi}_t^1$, this is immediate, since $\overline{\pi}_t^1 \geq \overline{\pi}_{t-1}^1$, and $\overline{\pi}_{t-1}^1 = \overline{\pi}_{t-1}^2$ by assumption. Assume now that $\overline{\pi}_t^1 \geq \overline{\pi}_t^2$. Assume that $\overline{\pi}_t^2 < \overline{\pi}_{t-1}^1$. If $\overline{x}_t^2 \geq \overline{\pi}_t^2$, this is impossible (because $\overline{\pi}_{t-1}^1 = \overline{\pi}_{t-1}^2$ and $\overline{\pi}_t^2 \geq \overline{\pi}_{t-1}^2$). If $\overline{x}_t^2 < \overline{\pi}_t^2$, then we have $\overline{x}_t^2 < \overline{\pi}_t^2 < \overline{\pi}_{t-1}^1$, but from Lemma 10 this is impossible. ☐

Remark. The proof is similar to that as Lemma 7.

- **The Case where** $\overline{\pi}_{t-1}^1 < \overline{\pi}_{t-1}^2 = x$.

Lemma 13. *In a Nash equilibrium, we have :* $\overline{x}_t^1 > \overline{\pi}_t^1$.

Proof. Same proof as that of Lemma 3. $\square$

Lemma 14. *In a Nash equilibrium, we always have* $\max\{\overline{x}_t^1, \overline{x}_t^2\} \leq x$.

Proof. The proof is similar to that of Lemma 4. Assume that $\overline{x}_t^1 \geq \overline{x}_t^2$. As $\overline{x}_t^1 > \overline{\pi}_t^1$ (Lemma 13), we must have $Z_1^1(\overline{x}_t^1, x) \geq 0$, so that $\overline{x}_t^1 \leq x$ and the conclusion follows. Suppose instead that $\overline{x}_t^1 < \overline{x}_t^2$. If $\overline{x}_t^2 \geq \overline{\pi}_t^2$, then as $Z_1^2(\overline{x}_t^2, x) \geq 0$, we must have $\overline{x}_t^2 \leq x$ and the conclusion follows again. If, on the other hand, we have $\overline{x}_t^2 < \overline{\pi}_t^2$, then since we have $\overline{\pi}_t^2 \leq \overline{\pi}_{t-1}^2 = x$, the proof is complete. $\square$

Lemma 15. *In a Nash equilibrium, we have:* $\min\{\overline{x}_t^1, \overline{x}_t^2\} \geq \overline{\pi}_{t-1}^1$.

Proof. If $\overline{x}_t^2 \geq \overline{x}_t^1$, this is immediate (this stems from Lemmas 2 and 13). Assume now that $\overline{x}_t^1 > \overline{x}_t^2$. Assume that $\overline{x}_t^2 \leq \overline{\pi}_t^2$. Then $Z_1^2(\overline{x}_t^2, x) \leq 0$ and thus $\overline{x}_t^2 \geq x$. So $\overline{\pi}_{t-1}^2 = \overline{\pi}_t^2 = \overline{x}_t^2 = x$. Since $\overline{\pi}_{t-1}^1 < x$, we are done. Then consider the case $\overline{x}_t^2 > \overline{\pi}_t^2$. As $\overline{x}_t^1 > \overline{x}_t^2$ and from Lemma 2, $\overline{\pi}_t^2 \geq \overline{\pi}_{t-1}^2 = x > \overline{\pi}_{t-1}^1$, the conclusion follows. $\square$

Lemma 16. *In a Nash equilibrium, we have:* $\max\{\overline{\pi}_t^1, \overline{\pi}_t^2\} \leq x$.

Proof. We already know that $\overline{x}_t^1 \geq \overline{\pi}_t^1$ and $\overline{x}_t^1 \leq x$ (Lemmas 13 and 14). Let us show that $\overline{\pi}_t^2 \leq x$. If $\overline{x}_t^2 \geq \overline{\pi}_t^2$, as from Lemma 2, $\overline{\pi}_t^2 \geq \overline{\pi}_{t-1}^2 = x$, and from Lemma 14, $\overline{x}_t^2 \leq x$, we have $\overline{x}_t^2 = \overline{\pi}_t^2 = x$. Assume now that $\overline{x}_t^2 < \overline{\pi}_t^2$. From Lemma 2, $\overline{\pi}_t^2 \leq \overline{\pi}_{t-1}^2 = x$. The result then follows easily. $\square$

Lemma 17. *In a Nash equilibrium we have:* $\min\{\overline{\pi}_{t-1}^1, \overline{\pi}_{t-1}^2\} = \overline{\pi}_{t-1}^1 \leq \min\{\overline{\pi}_t^1, \overline{\pi}_t^2\}$.

Proof. If $\overline{\pi}_t^2 \geq \overline{\pi}_t^1$, this is immediate, since $\overline{\pi}_t^1 \geq \overline{\pi}_{t-1}^1$ (Lemma 2 and Lemma 13). Assume now that $\overline{\pi}_t^2 < \overline{\pi}_t^1$. The case $\overline{\pi}_t^2 < \overline{\pi}_{t-1}^1$ is impossible. Indeed, if $\overline{x}_t^2 \geq \overline{\pi}_t^2$, from Lemma 2 $\overline{\pi}_t^2 \geq \overline{\pi}_{t-1}^2$, and then $\overline{\pi}_{t-1}^1 > \overline{\pi}_{t-1}^2$ is a contradiction. If $\overline{x}_t^2 < \overline{\pi}_t^2$, then we have $\overline{x}_t^2 \leq \overline{\pi}_t^2 < \overline{\pi}_{t-1}^1$, but from Lemma 15 this is impossible. $\square$

Remark The previous analysis enables one to handle the case where $\overline{\pi}_{t-1}^2 < \overline{\pi}_{t-1}^1 = x$.

- **The case** $\overline{\pi}_{t-1}^1 = \overline{\pi}_{t-1}^2 = x$

It is clear that in this case, there are legal and preferences unifications: $\overline{x}_t^1 = \overline{x}_t^2 = \overline{\pi}_t^1 = \overline{\pi}_t^2$.

Remarks. From the preceding lemmas, we have:

$$\pi_{-1}^1 = \min\{\pi_{-1}^1, \pi_{-1}^2\} \leq \min\{\overline{\pi}_0^1, \overline{\pi}_0^2\}, \tag{37}$$

$$\max\{\overline{\pi}_0^1, \overline{\pi}_0^2\} \leq x, \tag{38}$$

$$\pi_{-1}^1 = \min\{\pi_{-1}^1, \pi_{-1}^2\} \leq \min\{\overline{x}_0^1, \overline{x}_0^2\}, \tag{39}$$

$$\max\{\overline{x}_0^1, \overline{x}_0^2\} \leq x. \tag{40}$$

Assume that: $\overline{\pi}_0^1 \leq \overline{\pi}_0^2$. Then, from the same lemmas, we will have

$$\overline{\pi}_0^1 \leq \min\{\overline{\pi}_1^1, \overline{\pi}_1^2\}, \tag{41}$$

$$\max\{\overline{\pi}_1^1, \overline{\pi}_1^2\} \leq x, \tag{42}$$

$$\overline{\pi}_0^1 \leq \min\{\overline{x}_1^1, \overline{x}_1^2\}, \tag{43}$$

$$\max\{\overline{x}_1^1, \overline{x}_1^2\} \leq x. \tag{44}$$

If not ($\overline{\pi}_0^2 < \overline{\pi}_0^1$), we will have:

$$\overline{\pi}_0^2 \leq \min\{\overline{\pi}_1^1, \overline{\pi}_1^2\} \tag{45}$$

$$\max\{\overline{\pi}_1^1, \overline{\pi}_1^2\} \leq x \tag{46}$$

$$\overline{\pi}_0^2 \leq \min\{\overline{x}_1^1, \overline{x}_1^2\} \tag{47}$$

$$\max\{\overline{x}_1^1, \overline{x}_1^2\} \leq x. \tag{48}$$

To put it differently, we have:

$$\min\{\overline{\pi}_0^1, \overline{\pi}_0^2\} \leq \min\{\overline{\pi}_1^1, \overline{\pi}_1^2\} \tag{49}$$

$$\max\{\overline{\pi}_1^1, \overline{\pi}_1^2\} \leq x \tag{50}$$

$$\min\{\overline{\pi}_0^1, \overline{\pi}_0^2\} \leq \min\{\overline{x}_1^1, \overline{x}_1^2\} \tag{51}$$

$$\max\{\overline{x}_1^1, \overline{x}_1^2\} \leq x. \tag{52}$$

It is easy to see that these inequations are satisfied for all dates t (and not only at dates 0 and 1), whether $\overline{\pi}_{t-1}^1 \leq \overline{\pi}_{t-1}^2$ or not. This implies that $(\overline{x}_t^1, \overline{x}_t^2, \overline{\pi}_t^1, \overline{\pi}_t^2) \in [\min\{\pi^1_{-1}, \pi^2_{-1}\}, x]^4$ for all t. Moreover, the sequence $(\lambda_t)_t$ where $\lambda_t = \min\{\overline{\pi}_t^1, \overline{\pi}_t^2\}$, is non-decreasing and upper-bounded. It therefore has a limit $\lambda \leq x$. The next Lemma insures that $\lambda = x$, and thus that the sequence of equilibria converges to x (legal uniformity is achieved in the long run).

Convergence in the case $\max\{\pi^1_{-1}, \pi^2_{-1}\} \leq x$

Lemma 18. *Let a sequence of Nash equilibria $(\overline{x}_t^1, \overline{x}_t^2, \overline{\pi}_t^1, \overline{\pi}_t^2)_t$ be given, with* $\max\{\pi^1_{-1}, \pi^2_{-1}\} \leq x$. *Then, this sequence has a limit which satisfies: $\overline{\pi}_\infty^1 = \overline{\pi}_\infty^2 = \overline{x}_\infty^1 = \overline{x}_\infty^2 = x$.*

Proof. From Lemma 2, we know that at each date t, there are only four possible kinds of equilibria:

1. $\overline{\pi}_{t-1}^i \leq \overline{\pi}_t^i \leq \overline{x}_t^i$, $i = 1, 2$.
2. $\overline{\pi}_{t-1}^1 \leq \overline{\pi}_t^1 \leq \overline{x}_t^1$ and $\overline{x}_t^2 \leq \overline{\pi}_t^2 \leq \overline{\pi}_{t-1}^2$,
3. $\overline{x}_t^1 \leq \overline{\pi}_t^1 \leq \pi_{t-1}^1$ and $\overline{\pi}_{t-1}^2 \leq \overline{\pi}_t^2 \leq \overline{x}_t^2$,
4. $\overline{x}_t^i \leq \overline{\pi}_t^i \leq \overline{\pi}_{t-1}^i$, $i = 1, 2$.

We notice that in the last case, we must have: $\overline{x}_t^i = \overline{\pi}_t^i$, $i = 1, 2$. Indeed, since we know that $V_i^i(\overline{x}_t^i, \overline{\pi}_t^i) \geq 0$ and $Z_1^i(\overline{x}_t^i, x) \geq 0$, $i = 1, 2$. This implies that $U^i(\overline{x}_t^i, \overline{x}_t^j) \leq 0$, $i, j = 1, 2$, $i \neq j$. These last inequations mean that $\overline{x}_t^1 = \overline{x}_t^2$. As $V_i^i(\overline{x}_t^i, \overline{\pi}_t^i) + Z_1^i(\overline{x}_t^i, x) = 0$, the conclusion follows. If $\overline{x}_t^i = \overline{\pi}_t^i = x$ at date t, these equations also hold for all posterior periods and the Lemma follows.

Assume now that the fourth kind of equilibrium never occurs. Assume also that there is an infinite number of these periods in which the first kind of equilibrium occurs. In all of these periods, the sequence $(\overline{\pi}^1_{s-1}, \overline{\pi}^1_s, \overline{x}^1_s, \overline{\pi}^2_{s-1}, \overline{\pi}^2_s, \overline{x}^2_s)$ lies in a compact set. There thus exists a converging subsequence with a limit $(\hat{\pi}^1, \overline{\pi}^1, \overline{x}^1, \hat{\pi}^2, \overline{\pi}^2, \overline{x}^2)$. The coordinates of this limit satisfy by construction: $\hat{\pi}^1 \leq \overline{\pi}^1 \leq \overline{x}^1$ and $\hat{\pi}^2 \leq \overline{\pi}^2 \leq \overline{x}^2$.

Recall now that the sequence $(\lambda_t)_t$ is non-decreasing and upper-bounded (we have defined above $\lambda_t = \min\{\overline{\pi}^1_t, \overline{\pi}^2_t\}$). Let λ be its limit. Thus we have $\min\{\hat{\pi}^1, \hat{\pi}^2\} = \lambda$, and $\min\{\pi^1, \pi^2\} = \lambda$.[16]

Assume that: $\hat{\pi}^1 = \overline{\pi}^1 = \lambda$. Then we have: $W^1_1(\overline{\pi}^1, \hat{\pi}^1) = 0$, and then $V^1_2(\overline{x}^1, \overline{\pi}^1) = 0$. This implies that: $\overline{x}^1 = \overline{\pi}^1 = \lambda$. From $U^1(\overline{x}^1, \overline{x}^2) + V^1_1(\overline{x}^1, \overline{\pi}^1) + Z^1_1(\overline{x}^1, x) = 0$, we have: $U^1(\overline{x}^1, \overline{x}^2) \leq 0$, and thus $\overline{x}^1 \geq \overline{x}^2$. Thus $\lambda \geq \overline{x}^2$. But we also have for all dates s $\lambda_s \leq \min\{\overline{x}^1_s, \overline{x}^2_s\}$. Hence, $\lambda \leq \min\{\overline{x}^1, \overline{x}^2\}$. Therefore $\overline{x}^2 = \lambda$. This then implies that $\overline{x}^i = \overline{\pi}^i = x$, $i = 1, 2$.

The same argument applies if $\hat{\pi}^2 = \overline{\pi}^2 = \lambda$.

Assume now that: $\hat{\pi}^1 = \lambda$ and $\overline{\pi}^2 = \lambda$. This implies that $W^2_1(\overline{\pi}^2, \hat{\pi}^2) \geq 0$ (since $\overline{\pi}^2 \leq \hat{\pi}^2$). Thus $V^2_2(\overline{x}^2, \overline{\pi}^2) \leq 0$. Therefore, $V^2_1(\overline{x}^2, \overline{\pi}^2) \geq 0$. Since by construction $\overline{\pi}^2 \leq \overline{x}^2$, we have $\overline{x}^2 = \overline{\pi}^2 = \lambda$. But since $U^2(\overline{x}^2, \overline{x}^2) + V^2_1(\overline{x}^2, \overline{\pi}^2) + Z^2_1(\overline{x}^2, x) = 0$, this implies that $U^2(\overline{x}^2, \overline{x}^1) \leq 0$. Thus $\overline{x}^2 \geq \overline{x}^1$. Since $\overline{x}^1 \geq \lambda$, we have $\overline{x}^1 = \overline{\pi}^1 = \lambda$. But these equations mean that $\overline{x}^i = \overline{\pi}^i = x$, $i = 1, 2$.

The case where $\overline{\pi}^1 = \lambda$ and $\hat{\pi}^2 = \lambda$ can be handled as above.

We can then conclude that $\lambda = x$.

Assume now that there is only a finite number of periods in which the first kind of equilibrium, we can construct a converging subsequence with a limit $(\hat{\pi}^1, \overline{\pi}^1, \overline{x}^1, \hat{\pi}^2, \overline{\pi}^2, \overline{x}^2)$ which satisfies: $\hat{\pi}^1 \leq \overline{\pi}^1 \leq \overline{x}^1$ and $\overline{x}^2 \leq \overline{\pi}^2 \leq \hat{\pi}^2$ with $\min\{\hat{\pi}^1, \hat{\pi}^2\} = \lambda$, and $\min\{\overline{\pi}^1, \overline{\pi}^2\} = \lambda$.

Assume that: $\hat{\pi}^1 = \overline{\pi}^1 = \lambda$. From $W^1(\overline{\pi}^1, \hat{\pi}^1) = 0$, we can show that $\overline{x}^1 = \overline{\pi}^1 = \lambda$. Thus $\lambda = \overline{x}^1 \geq \overline{x}^2$. But since we also have $\overline{x}^2 \geq \overline{x}^1$ (from $U^2(\overline{x}^2, \overline{x}^1) + V^2_1(\overline{x}^2, \overline{\pi}^2) + Z^2_1(\overline{x}^2, x) = 0$), we have $\overline{x}^1 = \overline{x}^2 = \lambda$ and then $\lambda = x$.

Assume that $\hat{\pi}^1 = \lambda$ and $\overline{\pi}^2 = \lambda$. We have then $\lambda \geq \overline{x}^2$ and $\lambda \leq \overline{x}^1$. But we also have $\overline{x}^1 \leq \overline{x}^2$, therefore $\overline{x}^1 = \overline{x}^2$. It follows that $\overline{x}^i = \hat{\pi}^i = \overline{\pi}^i = \lambda$, $i = 1, 2$, and then $\lambda = x$.

Assume then that $\hat{\pi}^2 = \lambda$, and $\overline{\pi}^2 = \lambda$. From $W^2(\overline{\pi}^2, \hat{\pi}^2) = 0$, we can show that $\overline{x}^2 = \overline{\pi}^2 = \lambda$. Since $\overline{x}^2 \geq \overline{x}^1$ and $\overline{x}^1 \geq \lambda$, we again have $\overline{x}^i = \hat{\pi}^i = \overline{\pi}^i = \lambda$, $i = 1, 2$, and then $\lambda = x$.

The last case to be considered is where $\hat{\pi}^2 = \lambda$, and $\overline{\pi}^1 = \lambda$. From $W^2(\overline{\pi}^2, \hat{\pi}^2) \geq 0$, we can show that $V^2_2(\overline{x}^2, \overline{\pi}^2) \leq 0$ and then that $V^1_1(\overline{x}^2, \overline{\pi}^2) \geq 0$. Then from $U^2(\overline{x}^2, \overline{x}^1) + V^2_1(\overline{x}^2, \overline{\pi}^2) + Z^2_1(\overline{x}^2, x) = 0$, we have $\overline{x}^2 \geq \overline{x}^1$. This proves that $\overline{x}^1 = \lambda$. Thus, $\overline{x}^i = \hat{\pi}^i = \overline{\pi}^i = \lambda$, $i = 1, 2$, and then $\lambda = x$.

If there is only a finite number of periods in which the second kind of equilibrium occurs, there is then an infinite number of times in which the third kind of equilibrium occurs. In that case, we can prove as in the last case that $\lambda = x$. The proof is complete. □

[16]This is because the subsequences $\min\{\overline{\pi}^1_{s_k}, \overline{\pi}^2_{s_k}\}$, and $\min\{\overline{\pi}^1_{s_k-1}, \overline{\pi}^2_{s_k-1}\}$ both converge to λ.

The preceding conclusion also holds, by symmetry for the case where $x \leq \min\{\pi^1_{-1}, \pi^2_{-1}\}$.

- **The Case where $\pi^1_{-1} < x < \pi^2_{-1}$**

Assume that $\overline{\pi}^1_{t-1} < x < \overline{\pi}^2_{t-1}$ (the same conclusion would apply if $\overline{\pi}^2_{t-1} < x < \overline{\pi}^1_{t-1}$).

Lemma 19. *Let $(\overline{x}^1_t, \overline{x}^2_t, \overline{\pi}^1_t, \overline{\pi}^2_t)$ be a Nash equilibrium.Then we necessarily have: $\overline{\pi}^1_{t-1} \leq \overline{\pi}^1_t < \overline{x}^1_t$ and $\overline{x}^2_t < \overline{\pi}^2_t < \overline{\pi}^2_{t-1}$.*

Proof. It is easy to see that Lemma 2 holds, so that $\overline{x}^1_t > \overline{\pi}^1_t$. Assume that $\overline{\pi}^2_t = \overline{\pi}^2_{t-1}$. Then we must have $V^2_2(\overline{x}^2_t, \overline{\pi}^2_t) = 0$, and then $\overline{x}^2_t = \overline{\pi}^2_t$. But this in turn implies that: $U^2_1(\overline{x}^2_t, \overline{x}^1_t) + Z^2_1(\overline{x}^2_t, x) = 0$. If $\overline{x}^2_t \geq \overline{x}^1_t$, then $\overline{x}^2_t \leq x$ so that $\overline{\pi}^2_{t-1} \leq x$ which is impossible. Assume then that $\overline{x}^2_t \leq \overline{x}^1_t$. So $\overline{x}^2_t \geq x$. Since $\overline{x}^1_t \geq x$ and $\overline{x}^1_t \geq \overline{x}^2_t$, we must have $\overline{x}^1_t \leq \overline{\pi}^1_t$. which is impossible. We have therefore proved that $\overline{\pi}^2_t \neq \overline{\pi}^2_{t-1}$. This proves also that $\overline{x}^2_t \neq \overline{\pi}^2_t$.

Suppose now that $\overline{x}^2_t > \overline{\pi}^2_t > \overline{\pi}^2_{t-1}$. If $\overline{x}^2_t \geq \overline{x}^1_t$, then $Z^2_1(\overline{x}^2_t, x) \geq 0$, which implies that $x \geq \overline{x}^2_t$. This is impossible since this would imply that $\overline{\pi}^2_{t-1} < x$. If $\overline{x}^1_t \geq \overline{x}^2_t$, then $Z^1_1(\overline{x}^1_t, x) \geq 0$, and thus $\overline{x}^1_t \leq x$. Again, we would have $\overline{\pi}^2_{t-1} < x$, which is impossible. □

Lemma 20. *Let a sequence of Nash equilibria $(\overline{x}^1_t, \overline{x}^2_t, \overline{\pi}^1_t, \overline{\pi}^2_t)_t$ be given, with $\pi^1_{-1} < x < \pi^2_{-1}$. Then all the terms of the sequences converge to x: $\lim_{t\to+\infty} \overline{\pi}^1_t = \lim_{t\to+\infty} \overline{x}^1_t = \lim_{t\to+\infty} \overline{x}^2_t = \lim_{t\to+\infty} \pi^2_t = x$.*

Proof. Assume that $\pi^1_{-1} < x < \pi^2_{-1}$. Then Lemma 19 holds. Therefore, if $\overline{\pi}^1_0 \geq x$, necessarily we have $\overline{x}^1_0 \geq x$ and thus $\overline{x}^2_0 \geq x$ (because we must have $\overline{x}^2_t \geq \overline{x}^1_t$) which in turn implies $\overline{\pi}^2_0 \geq x$. If $\overline{\pi}^2_0 \leq x$, then necessarily $\overline{x}^2_0 < x$. Therefore $\overline{x}^2_0 \geq \overline{x}^1_0$ and then $\overline{\pi}^1_0 < x$. The preceding conclusions hold for $\overline{\pi}^1_t \geq x$ and $\overline{\pi}^2_t \leq x$ respectively if $\overline{\pi}^1_{t-1} < x < \overline{\pi}^2_{t-1}$.
Suppose that there is no $s \in \mathbb{N}$ such that $\overline{\pi}^1_s \geq x$ *or* $\overline{\pi}^2_s \leq x$ (so: $\overline{\pi}^1_s < x < \overline{\pi}^2_s$ for all s). By Lemma 19, the sequence $(\overline{\pi}^1_t)_t$ is a non-decreasing sequence bounded above by x, whereas $(\overline{\pi}^2_t)_t$ is a non-increasing sequence bounded below by x. Then we can show (using subsequences if necessary) that $(\overline{\pi}^1_t)_t$ and $(\overline{\pi}^2_t)_t$ as well as the sequences $(\overline{x}^1_t)_t$ and $(\overline{\pi}^2_t)_t$ go to x.
If, on the other hand, there is s such that $\overline{\pi}^1_s \geq x$ *or* $\overline{\pi}^2_s \leq x$ (with $\overline{\pi}^1_{s-1} < x < \overline{\pi}^2_{s-1}$), then we are in a case that has already been considered (both $\overline{\pi}^1_s$ and $\overline{\pi}^2_s$ are either lower or greater than x). Our convergence results for these cases prove the statement of the Lemma. □

Since we have proved the convergence result for all initial conditions, the proof is complete. □

Proof of Proposition 4. In a steady-state, we have $W^i_1(\overline{\pi}, \overline{\pi}) = 0$, $i = 1, 2$. So $V^i_2(\overline{x}^i, \overline{\pi}^i) = 0$, and thus $\overline{x}^i = \overline{\pi}^i$. Necessarily, $\overline{x}^1 < \overline{x}^2$. Otherwise, from the first equation, we would have $U^1_1 \leq 0$, and then $Z^1_1 \geq 0$, which would imply $\overline{x}^1 \leq x^1$. But from the corresponding equation for country 2, we would also have: $Z^2_1 \leq 0$, or $\overline{x}^2 \geq x^2$. So, we would have: $x^2 \leq \overline{x}^2 \leq \overline{x}^1 \leq x^1$, which is a contradiction. Now, $\overline{x}^1 < \overline{x}^2$ implies $U^1_1 > 0$, and then $\overline{x}^1 > x^1$. This also implies $Z^2_1 > 0$ and thus $\overline{x}^2 < x^2$. □

Proof of Lemma 1. Assume that $\overline{x}_t^1 < x^1$. Then suppose $\overline{x}_t^2 \leq \overline{x}_t^1$. Therefore $\overline{x}_t^2 < x^2$. Necessarily, $V_1^2(\overline{x}_t^2, \overline{\pi}_t^2) < 0$. So $\overline{x}_t^2 > \overline{\pi}_t^2$, and thus, $\overline{x}_t^2 > \overline{\pi}_t^2 > \overline{\pi}_{t-1}^2$. But this is a contradiction. So, we must have: $\overline{x}_t^1 \leq \overline{x}_t^2$. This then implies that $V_1^1(\overline{x}_t^1, \overline{\pi}_t^1) < 0$, so $\overline{x}_t^1 > \overline{\pi}_t^1 > \overline{\pi}_{t-1}^1$, which is a contradiction again. As a consequence, we must have $\overline{x}_t^1 \geq x^1$. By symmetry, we have $\overline{x}_t^2 \leq x^2$.

Now let us prove that $\overline{x}_t^1 \leq x^2$. If we had $\overline{x}_t^1 > x^2$ this would imply $V_1^1(\overline{x}_t^1, \overline{\pi}_t^1) > 0$, and thus $\overline{x}_t^1 < \overline{\pi}_t^1 < \overline{\pi}_{t-1}^1$ which is a contradiction. We can prove likewise that $\overline{x}_t^2 \geq x^1$.

Let us now show that $\overline{\pi}_t^i \in [x^1, x^2]$, i=1,2. Without loss of generality, we only prove the result for $\overline{\pi}_t^1$. If $\overline{\pi}_t^1 \geq \overline{\pi}_{t-1}^1$, then $\overline{x}_t^1 \geq \overline{\pi}_t^1 \geq \overline{\pi}_{t-1}^1 \geq x^1$. On the other hand, if $\overline{\pi}_{t-1}^1 \geq \overline{\pi}_t^1$, then, $\overline{\pi}_{t-1}^1 \geq \overline{\pi}_t^1 \geq \overline{x}_t^1 \geq x^1$. By induction, we can conclude the proof that $(\overline{x}_s^i, \overline{\pi}_s^i) \in [x^1, x^2] \times [x^1, x^2]$, for all $s \geq t$, and $i = 1, 2$. □

Remark. Since $(\overline{x}_s^1, \overline{\pi}_s^1, \overline{x}_s^2, \overline{\pi}_s^2)$ is in $[x^1, x^2]^4$ for $s \geq t-1$, there is a convergence subsequence (as $[x^1, x^2]^4$ is compact). This proves that there are an infinite number of times for which the sequence is arbitrarily close to the steady-state studied above.

Proof of Proposition 5. To analyze the dynamics, we can express $(\overline{x}_t^1, \overline{x}_t^2)$ with respect to $(\overline{\pi}_t^1, \overline{\pi}_t^2)$. We have:

$$\begin{pmatrix} (\alpha_1 + \beta_1 + \theta_1) & -\alpha_1 \\ -\alpha_2 & (\alpha_2 + \beta_2 + \theta_2) \end{pmatrix} \begin{pmatrix} \overline{x}_t^1 \\ \overline{x}_t^2 \end{pmatrix} = \begin{pmatrix} \beta_1 \overline{\pi}_t^1 + \theta_1 x^1 \\ \beta_2 \overline{\pi}_t^2 + \theta_2 x^2 \end{pmatrix} \tag{53}$$

We define:

$$A = \begin{pmatrix} (\alpha_1 + \beta_1 + \theta_1) & -\alpha_1 \\ -\alpha_2 & (\alpha_2 + \beta_2 + \theta_2) \end{pmatrix} \tag{54}$$

We find that:

$$\begin{pmatrix} \overline{x}_t^1 \\ \overline{x}_t^2 \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} (\alpha_2 + \beta_2 + \theta_2) & \alpha_1 \\ \alpha_2 & (\alpha_1 + \beta_1 + \theta_1) \end{pmatrix} \begin{pmatrix} \beta_1 \overline{\pi}_t^1 + \theta_1 x^1 \\ \beta_2 \overline{\pi}_t^2 + \theta_2 x^2 \end{pmatrix} \tag{55}$$

where:

$$\det(A) = (\alpha_1 + \beta_1 + \theta_1)(\alpha_2 + \beta_2 + \theta_2) - \alpha_1 \alpha_2 \tag{56}$$

$$= \alpha_1(\beta_2 + \theta_2) + \alpha_2(\beta_1 + \theta_1) + (\beta_1 + \theta_1)(\beta_2 + \theta_2) > 0 \tag{57}$$

We then deduce that:

$$\overline{x}_t^1 = \frac{1}{\det(A)} \left(\beta_1(\alpha_2 + \beta_2 + \theta_2)\overline{\pi}_t^1 + \theta_1(\alpha_2 + \beta_2 + \theta_2)x^1 + \alpha_1(\beta_2 \overline{\pi}_t^2 + \theta_2 x^2) \right) \tag{58}$$

$$\overline{x}_t^2 = \frac{1}{\det(A)} \left(\beta_2(\alpha_1 + \beta_1 + \theta_1)\overline{\pi}_t^2 + \theta_2(\alpha_1 + \beta_1 + \theta_1)x^2 + \alpha_2(\beta_1 \overline{\pi}_t^1 + \theta_1 x^1) \right) \tag{59}$$

By plugging (58) and (59) in equations (24) and (26) respectively we obtain the the following linear equations system:

$$\frac{\beta_1}{\det(A)} \left[\beta_1(\alpha_2 + \beta_2 + \theta_2)\overline{\pi}_t^1 + \theta_1(\alpha_2 + \beta_2 + \theta_2)x^1 + \alpha_1(\beta_2 \overline{\pi}_t^2 + \theta_2 x^2) \right] - \beta_1 \overline{\pi}_t^1 - \gamma_1 \overline{\pi}_t^1 + \gamma_1 \overline{\pi}_{t-1}^1 = 0 \tag{60}$$

$$\frac{\beta_1}{\det(A)} \left[\beta_2(\alpha_1 + \beta_1 + \theta_1)\overline{\pi}_t^2 + \theta_2(\alpha_1 + \beta_1 + \theta_1)x^2 + \alpha_2(\beta_1 \overline{\pi}_t^1 + \theta_1 x^1) \right] - \beta_2 \overline{\pi}_t^2 - \gamma_2 \overline{\pi}_t^2 + \gamma_2 \overline{\pi}_{t-1}^2 = 0 \tag{61}$$

We can rewrite the above system as follows:

$$\begin{pmatrix} \beta_1^2(\alpha_2+\beta_2+\theta_2-(\beta_1+\gamma_1)\det(A) & \alpha_1\beta_1\beta_2 \\ \alpha_2\beta_1\beta_2 & \beta_2^2(\alpha_1+\beta_1+\theta_1)-(\beta_2+\gamma_2)\det(A) \end{pmatrix} \begin{pmatrix} \overline{\pi}_t^1 \\ \overline{\pi}_t^2 \end{pmatrix} = \begin{pmatrix} -\beta_1\theta_1(\alpha_2+\beta_2+\theta_2)x^1-\gamma_1\overline{\pi}_{t-1}^1\det(A)-\alpha_1\beta_1\theta_2x^2 \\ -\beta_2\alpha_2\theta_1x^1-\beta_2(\alpha_1+\beta_1+\theta_1)\theta_2x^2-\gamma_2\overline{\pi}_{t-1}^2\det(A) \end{pmatrix} \tag{62}$$

We let B be the matrix given by:

$$B = \begin{pmatrix} \beta_1^2(\alpha_2+\beta_2+\theta_2)-(\beta_1+\gamma_1)\det(A) & \alpha_1\beta_1\beta_2 \\ \alpha_2\beta_1\beta_2 & \beta_2^2(\alpha_1+\beta_1+\theta_1)-(\beta_2+\gamma_2)\det(A) \end{pmatrix} \tag{63}$$

We therefore have:

$$\begin{pmatrix} \overline{\pi}_t^1 \\ \overline{\pi}_t^2 \end{pmatrix} = \frac{1}{\det(B)} \begin{pmatrix} \beta_2^2(\alpha_1+\beta_1+\theta_1)-(\beta_2+\gamma_2)\det(A) & -\alpha_1\beta_1\beta_2 \\ -\alpha_2\beta_1\beta_2 & \beta_1^2(\alpha_2+\beta_2+\theta_2)-(\beta_1+\gamma_1)\det(A) \end{pmatrix} \times \begin{pmatrix} -\beta_1\theta_1(\alpha_2+\beta_2+\theta_2)x^1-\beta_1\alpha_1\theta_2x^2-\gamma_1\overline{\pi}_{t-1}^1\det(A) \\ -\beta_2\alpha_2\theta_1x^1-\beta_2(\alpha_1+\beta_1+\theta_1)\theta_2x^2-\gamma_2\overline{\pi}_{t-1}^2\det(A) \end{pmatrix} \tag{64}$$

where:

$$\det(B) = \Big(\beta_1^2(\alpha_2+\beta_2+\theta_2)-(\beta_1+\gamma_1)\det(A)\Big)\Big(\beta_2^2(\alpha_1+\beta_1+\theta_1)-(\beta_2+\gamma_2)\det(A)\Big)-\alpha_1\alpha_2\beta_1^2\beta_2^2 \tag{65}$$

Or:

$$\begin{pmatrix} \overline{\pi}_t^1 \\ \overline{\pi}_t^2 \end{pmatrix} = \begin{pmatrix} -\frac{\big(\beta_2^2(\alpha_1+\beta_1+\theta_1)-(\beta_2+\gamma_2)\det(A)\big)\gamma_1\det(A)}{\det(B)} & \frac{\alpha_2\beta_1\beta_2\gamma_2\det(A)}{\det(B)} \\ \frac{(\alpha_1\beta_1\beta_2)\gamma_1\det(A)}{\det(B)} & -\frac{\big(\beta_1^2(\alpha_2+\beta_2+\theta_2)-(\beta_1+\gamma_1)\det(A)\big)\gamma_2\det(A)}{\det(B)} \end{pmatrix} \begin{pmatrix} \overline{\pi}_{t-1}^1 \\ \overline{\pi}_{t-2}^2 \end{pmatrix}$$
$$+ \begin{pmatrix} [(\beta_2^2(\alpha_1+\beta_1+\theta_1)-(\beta_2+\gamma_2)\det(A))\,(-\beta_1\theta_1(\alpha_2+\beta_2+\theta_2)x^1-\beta_1\alpha_1\theta_2x^2) \\ +\alpha_1\beta_1\beta_2^2\theta_1x^1+\alpha_1\beta_1\beta_2^2(\alpha_1+\beta_1+\theta_1)\theta_2x^2]\,\frac{1}{\det(B)} \\ [(\beta_1\alpha_2\beta_2)(\beta_1\theta_1(\alpha_2+\beta_2+\theta_2)x^1+\alpha_1\beta_1\theta_2x^2) \\ +(\beta_1^2(\alpha_2+\beta_2+\theta_2)-(\beta_1+\gamma_1)\det(A))(-\alpha_2\beta_2\theta_1x^1-\beta_2(\alpha_1+\beta_1+\theta_1)\theta_2x^2)]\,\frac{1}{\det(B)} \end{pmatrix} \tag{66}$$

We let C be defined as follows:

$$C = \begin{pmatrix} -\frac{\big(\beta_2^2(\alpha_1+\beta_1+\theta_1)-(\beta_2+\gamma_2)\det(A)\big)\gamma_1\det(A)}{\det(B)} & \frac{\alpha_2\beta_1\beta_2\gamma_2\det(A)}{\det(B)} \\ \frac{(\alpha_1\beta_1\beta_2)\gamma_1\det(A)}{\det(B)} & -\frac{\big(\beta_1^2(\alpha_2+\beta_2+\theta_2)-(\beta_1+\gamma_1)\det(A)\big)\gamma_2\det(A)}{\det(B)} \end{pmatrix} \tag{67}$$

We want to check that all the modulus of the characteristic roots of C are strictly less than one. Let $P(\lambda)$ be the characteristic polynomial of C, *i.e.*,

$$P(\lambda) = \lambda^2 - T\lambda + D = 0 \tag{68}$$

where D and T denote the determinant and the trace of C respectively. The necessary and sufficient conditions for all the modulus of the characteristic roots to be strictly lower than one are:

$$P(1) = 1 - T + D > 0, \tag{69}$$

$$P(-1) = 1 + T + D > 0, \tag{70}$$

$$|P(0)| = |D| < 1. \tag{71}$$

We can check that:

Lemma 21. *We have* $det(B) > 0$, $T > 0$, $D = \frac{det(A)^2}{det(B)}\gamma_1\gamma_2$.

Cumbersome algebra reveals that:

Lemma 22. *We have:* $|D| < 1$.

Cumbersome algebra also reveals that:

Lemma 23. *We have:* $1 - T + D > 0$.

From the necessary and sufficient conditions for the modulus of the characteristic roots to be strictly lower than one and the previous Lemma, we can conclude that in the quadratic case, the steady-state is globally stable.

□