LTH-1000

Spectral flow as a map between N=(2,0)-models

P. Athanasopoulos*[1], A.E. Faraggi*[2] and D. Gepner†[3]

* Department of Mathematical Sciences,
University of Liverpool,
Liverpool L69 7ZL, United Kingdom

† Department of Particle Physics,
Weizmann Institute, Rehovot 76100, Israel

Abstract

The space of (2,0) models is of particular interest among all heterotic-string models because it includes the models with the minimal $SO(10)$ unification structure, which is well motivated by the Standard Model of particle physics data. The fermionic $\mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic-string models revealed the existence of a new symmetry in the space of string configurations under the exchange of spinors and vectors of the $SO(10)$ GUT group, dubbed spinor-vector duality. In this paper we generalize this idea to arbitrary internal rational Conformal Field Theories (RCFTs). We explain how the spectral flow operator normally acting within a general $(2,2)$ theory can be used as a map between $(2,0)$ models. We describe the details, give an example and propose more simple currents that can be used in a similar way.

[1] E-mail address: panos@liv.ac.uk
[2] E-mail address: alon.faraggi@liv.ac.uk
[3] E-mail address: doron.gepner@weizmann.ac.il

1 Introduction

String theory provides a detailed framework to explore the unification of the gauge and gravitational interactions [1]. The construction of phenomenological models that can make contact with the real world has been of great interest and understanding their underlying structure can be especially elucidating. However, the vastness of the *a priori* possible vacuum solutions impedes progress towards the construction of a standard string model. In this respect, the study of various relationships between different models can be very fruitful. In particular, it may have far reaching implications for the interpretation of the landscape of string vacua.

In the heterotic constructions the left and right moving sectors are treated asymmetrically. Of particular interest are the so called $(2,0)$ models[4] because it is known that $N = 1$ spacetime supersymmetry requires (at least) $(2,0)$ world-sheet supersymmetry and because they can accommodate $SO(10)$ unification. The problem is that the space of these models is huge. For example, even though the number of $(2,2)$ Gepner models [2] is quite tractable and they have been studied in detail [3,4], the number of $(2,0)$ models that arise is much greater [4]. For this reason it would be very useful to discover relations in the space of such models.

In this paper we will make a small step in this direction by getting inspiration from a new kind of duality that comes under the name spinor-vector duality and was observed in $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold models [5–10]. It is a duality of the massless spectra of two such models under the exchange of vectorial and spinorial representations of the $SO(10)$ GUT gauge group.

These models turn out to be related through the spectral flow operator and the underlying CFT structure of the spinor-vector duality for $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifolds was discussed in [10]. Even though the form of the duality as expressed in these references is restricted to $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifolds, the important idea that the spectral flow map can be used to relate different $(2,0)$ models is much more general. It is the purpose of this paper to explain the details of this mapping and the exact relationship between the mapped models.

The outline of this paper is as follows: In section 2 we discuss the spinor-vector duality in the fermionic $\mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic-string orbifolds. Understanding how the duality operates in the free fermionic constructions hints at how similar dualities may work in the case of interacting CFTs. In sections 3 and 4 we review the definition of the spectral flow and the simple current formalism which will allow us to construct $(2,0)$ models from a generic $(2,2)$ model. In section 5 we explain how the spectral flow induces a map between different $(2,0)$ models and in section 6 we analyze the consequences of this idea. Section 7 provides an example of how to use the derived results. We conclude with a brief discussion and possible generalizations in sections 8 and 9.

2 The spinor-vector duality case

In this section we outline the spinor-vector duality in the case of the fermionic $\mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic-string orbifolds. The discussion will provide the guide for exploring similar symmetries in models with an interacting internal CFT. The presentation here will be qualitative and further technical details are given in the references.

In the free fermionic formulations of the compactified string [11] all the internal degrees of freedom are represented in terms of free world-sheet fermions. Therefore, in this formulation the internal compactified dimensions are represented in terms of an internal CFT with vanishing interactions. Additionally, the well known relations between two dimensional fermions and bosons entail that the free fermionic formulation is equivalent to a free bosonic formulation, *i.e.* to toroidal orbifolds.

[4] Our convention here is that the left-moving sector is supersymmetric and the right-moving is bosonic.

A string vacuum in the free fermionic formulation is defined in terms of boundary condition basis vectors and the Generalized Gliozzi-Scherk-Olive (GGSO) projection coefficients of the one-loop partition function [11]. The gauge symmetry is generated by spacetime vector bosons that arise from the untwisted as well as the twisted sectors. The spacetime vector bosons arising in the twisted sectors enhance the untwisted gauge group factors under which they are charged. Specific enhancements depend on the states that remain in the physical spectrum after application of the GGSO projections. Similarly, the matter states in the free fermionic models are obtained from the untwisted and twisted sectors. The spinor-vector duality in the free fermionic vacua operates on the matter states in the twisted sectors.

The free fermionic vacua correspond to $\mathbb{Z}_2$ and $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifolds at enhanced symmetry points in the moduli space [12]. In this section we review the spinor-vector duality in $\mathbb{Z}_2$ orbifolds. By doing this we recap the ingredients that are needed for the generalization to interacting internal CFTs. The simplest realization of the spinor-vector duality is in the case of a single $\mathbb{Z}_2$ orbifold acting on the $E_8\times E_8$ heterotic-string compactified on a generic six torus. Taking for simplicity the internal torus as a product of six circles with radii R_i, the partition function (omitting the contribution from the spacetime bosons) reads

$$Z_+ = (V_8 - S_8)\left(\sum_{m,n}\Lambda_{m,n}\right)^{\otimes 6}\left(\overline{O}_{16}+\overline{S}_{16}\right)\left(\overline{O}_{16}+\overline{S}_{16}\right)\,, \tag{1}$$

where as usual, for each circle,

$$p^i_{\rm L,R} = \frac{m_i}{R_i} \pm \frac{n_i R_i}{\alpha'}\,, \qquad \text{and} \qquad \Lambda_{m,n} = \frac{q^{\frac{\alpha'}{4}p_{\rm L}^2}\,\bar q^{\frac{\alpha'}{4}p_{\rm R}^2}}{|\eta|^2}\,, \tag{2}$$

and we have written Z_+ in terms of level-one SO$(2n)$ characters (see for instance [13])

$$\begin{aligned} O_{2n} &= \frac{1}{2}\left(\frac{\theta_3^n}{\eta^n}+\frac{\theta_4^n}{\eta^n}\right)\,, &\qquad V_{2n} &= \frac{1}{2}\left(\frac{\theta_3^n}{\eta^n}-\frac{\theta_4^n}{\eta^n}\right)\,,\\ S_{2n} &= \frac{1}{2}\left(\frac{\theta_2^n}{\eta^n}+i^{-n}\frac{\theta_1^n}{\eta^n}\right)\,, &\qquad C_{2n} &= \frac{1}{2}\left(\frac{\theta_2^n}{\eta^n}-i^{-n}\frac{\theta_1^n}{\eta^n}\right)\,. \end{aligned}$$

We next apply the orbifold projections

$$\begin{aligned} \mathbb{Z}_2 \;:\; g &= (-1)^{F_1+F_2}\delta\,, \\ \mathbb{Z}_2' \;:\; g' &= (x_4,x_5,x_6,x_7,x_8,x_9) \longrightarrow (-x_4,-x_5,-x_6,-x_7,+x_8,+x_9)\,. \end{aligned} \tag{3}$$

F_1 and F_2 in (3) flip the sign in the spinorial representations of $SO(16)_1$ and $SO(16)_2$, generated by $\xi_1 = \{\bar\psi^{1,\cdots,5},\bar\eta^{1,2,3}\}$ and $\xi_2 = \{\bar\phi^{1,\cdots,8}\}$ respectively, and δ shifts the compact X^9 coordinate by half of its period, *i.e.*

$$\delta \;:\; X^9 \to X^9 + \pi R^9 \qquad \Rightarrow \qquad \Lambda_{m,n} \;\to\; (-1)^m \Lambda_{m,n}. \tag{4}$$

The $\mathbb{Z}_2$ projection in (3) breaks the $E_8\times E_8$ gauge group to $SO(16)\times SO(16)$ and preserves $N=4$ spacetime supersymmetry. The additional $\mathbb{Z}_2'$ projection twists the compactified coordinates and preserves only $N=2$ spacetime supersymmetry. Its generator g' reverts the sign of four internal coordinates X^i, $i=4,5,6,7$ and simultaneously breaks one $SO(16)$ to $SO(12)\times SO(4)$.

The action of the $\mathbb{Z}_2\times\mathbb{Z}_2'$ projections on Z_+ is implemented by taking

$$Z_- = \frac{(1+g)}{2}\frac{(1+g')}{2}Z_+\,. \tag{5}$$

The ten-dimensional SO(8) little group is broken to $SO(4)\times SO(4)$. At the same time, the first $SO(16)$ gauge group factor is broken into $SO(12)\times SO(4)$. As a result, the one-loop partition function can be written in terms of the spacetime characters,

$$\begin{aligned} Q_0 &= V_4O_4 - S_4S_4, \qquad & Q_V = V_4O_4 - C_4C_4, \\ Q_S &= O_4C_4 - S_4O_4, \qquad & Q_C = V_4S_4 - C_4V_4. \end{aligned}$$

There are two independent orbits in the partition function and hence one discrete torsion. The full partition function is given by

$$Z_- = Z_{untwisted} + Z_g + Z_{g'} + Z_{gg'}. \tag{6}$$

It consists of the untwisted sector and the three twisted sectors g, g' and gg'. The untwisted sector gives rise to the vector bosons that generate the four dimensional gauge group, whereas the sectors g and gg' give rise to massive states. To note the spinor-vector duality it is sufficient to focus on the states arising from the twisted sector g'. Summation over the GGSO projections in this sector produces the following terms in the partition function:

$$\begin{aligned} Z_{g'} \quad = & \\ & \frac{1}{2}\left(\left|\frac{2\eta}{\theta_4}\right|^4 + \left|\frac{2\eta}{\theta_3}\right|^4\right)\Lambda_{p,q}\left[P_+^{01}\Lambda_{m,n}\left(Q_S\left(\overline{V}_{12}\overline{C}_4\overline{O}_{16} + \overline{S}_{12}\overline{O}_4\overline{S}_{16}\right)\right.\right. \\ & \qquad + \qquad\qquad Q_C\left.\left(\overline{O}_{12}\overline{S}_4\overline{O}_{16} + \overline{C}_{12}\overline{V}_4\overline{S}_{16}\right)\right) \\ & \qquad + \quad P_-^{01}\Lambda_{m,n}\left(Q_S\left(\overline{S}_{12}\overline{O}_4\overline{O}_{16} + \overline{V}_{12}\overline{C}_4\overline{S}_{16}\right)\right. \\ & \qquad + \qquad\qquad Q_C\left.\left.\left(\overline{O}_{12}\overline{S}_4\overline{S}_{16} + \overline{C}_{12}\overline{V}_4\overline{O}_{16}\right)\right)\right] \\ & +\frac{1}{2}\left(\left|\frac{2\eta}{\theta_4}\right|^4 - \left|\frac{2\eta}{\theta_3}\right|^4\right)\Lambda_{p,q}\left[P_+^{01}\Lambda_{m,n}\left(Q_S\left(\overline{O}_{12}\overline{S}_4\overline{O}_{16} + \overline{C}_{12}\overline{V}_4\overline{S}_{16}\right)\right.\right. \\ & \qquad + \qquad\qquad Q_C\left.\left(\overline{V}_{12}\overline{C}_4\overline{O}_{16} + \overline{S}_{12}\overline{O}_4\overline{S}_{16}\right)\right) \\ & \qquad + \quad P_-^{01}\Lambda_{m,n}\left(Q_S\left(\overline{O}_{12}\overline{S}_4\overline{S}_{16} + \overline{C}_{12}\overline{V}_4\overline{O}_{16}\right)\right. \\ & \qquad + \qquad\qquad Q_C\left.\left.\left(\overline{S}_{12}\overline{O}_4\overline{O}_{16} + \overline{V}_{12}\overline{C}_4\overline{S}_{16}\right)\right)\right], \end{aligned} \tag{7}$$

where we defined $P_\pm^{01}$ as

$$P_\pm^{01} = \frac{1 \pm \epsilon_1(-1)^m}{2}. \tag{8}$$

The spinor-vector duality transformation is transparent in the partition function (7). Massless states arise from the untwisted sector and the g'-twisted sector. The internal winding modes in the g and gg'-twisted sectors are shifted by 1/2. The states in these two sectors are therefore massive. The untwisted sector gives rise to spacetime vector bosons that generate the $SO(12)\times SO(4)\times SO(16)$ gauge symmetry and to scalar multiplets that transform in the bi-vector representation of $SO(12)\times SO(4)$. Examining the g'-twisted sector reveals how the spinor-vector duality operates. Massless states arise for vanishing internal momentum and winding modes, *i.e.* $m=n=0$. Depending on the choice of the discrete torsion $\epsilon_1=\pm1$, vanishing lattice modes will therefore arise from $P_+^{01}\Lambda_{m,n}$ or $P_-^{01}\Lambda_{m,n}$, *i.e.*

$$\begin{aligned} \epsilon_1 = +1 \quad &\Rightarrow \quad P_+^{01}\Lambda_{m,n} = \Lambda_{2m,n} \quad \text{and} \quad P_-^{01}\Lambda_{m,n} = \Lambda_{2m+1,n}\ , \\ \epsilon_1 = -1 \quad &\Rightarrow \quad P_-^{01}\Lambda_{m,n} = \Lambda_{2m,n} \quad \text{and} \quad P_+^{01}\Lambda_{m,n} = \Lambda_{2m+1,n}\ . \end{aligned}$$

It follows from the q-expansion of the θ functions that in the case with $\epsilon_1=+1$ the zero lattice modes attach to $Q_S\overline{V}_{12}\overline{C}_4\overline{O}_{16}$, which produces two massless $N=2$ hypermultiplets in the

12 vector representation of $SO(12)$, whereas in the case with $\epsilon_1 = -1$ the zero lattice modes attach to $Q_S\overline{S}_{12}\overline{O}_4\overline{O}_{16}$, which produces a massless $N = 2$ hypermultiplet in the **32** spinorial representation. It is further noted from (7) that in the case with $\epsilon_1 = +1$ the term $Q_S\overline{O}_{12}\overline{S}_4\overline{O}_{16}$ gives rise to eight additional states from the first excited modes of the twisted lattice. Hence, the total number of degrees of freedom $32 = 12 \cdot 2 + 4 \cdot 2$ is preserved under the duality map.

The realization of the spinor-vector duality in this model provides a simple example where its origins can be explored and generalized to cases with interacting world-sheet CFTs. In the toroidal case, since all the data of the compactification is encoded in the toroidal background parameters and the orbifold action on them, it is anticipated that the spinor-vector duality is realizable in terms of a continuous or discrete map between two sets of background parameters. Indeed, in ref. [10] it was shown that the spinor-vector duality map is realized in terms of a continuous interpolation between two Wilson lines. The continuous interpolation, rather than a discrete transformation, is particular to the cases that preserve $N = 2$ spacetime supersymmetry, *i.e.* when a single $\mathbb{Z}_2$ twist is acting on the internal torus. In this case the moduli fields that enable the continuous interpolation exist in the spectrum and are not projected. In the compactifications with $N = 1$ spacetime supersymmetry, these moduli fields are projected out. Therefore, in the $N = 1$ cases the spinor-vector duality map is discrete.

The spinor-vector duality can be regarded as a direct consequence of the breaking of the world-sheet supersymmetry on the bosonic side of the heterotic-string from $N = 2$ to $N = 0$, *i.e.* from $(2, 2)$ world-sheet supersymmetry to $(2, 0)$. In the $(2, 2)$ case the gauge symmetry is enhanced to E_6 (or E_7). In this case the spinor and vector representations of $SO(10)\times U(1)$ (or $SO(12)\times SU(2)$) are embedded in the single **27** (or **56**) representation of E_6 (or E_7). The breaking of the $(2, 2)$ world-sheet supersymmetry to $(2, 0)$ results in the reduction of the enhanced gauge symmetry, by projecting out the spinorial components of the adjoint representation in its decomposition under the corresponding $SO(2n)$ subgroup. At the same time the matter multiplets are split into the spinorial and vectorial components. The GSO projections may retain either the spinorial or the vectorial representation in the massless spectrum. The spinor-vector duality is then induced by the spectral flow operator. The generalization to interacting internal CFTs can therefore proceed along the following lines. We can start with a generic compactification with $(2, 2)$ world-sheet supersymmetry, and subsequently break the $N = 2$ world-sheet supersymmetry on the bosonic side to $N = 0$. There ought to be choices of the breaking that result in different models that are related by the spectral flow operator.

We can illustrate the spinor-vector duality in terms of a spectral flow operator by considering the boundary condition basis vectors [7] in eq. (9):

$$\begin{aligned}
v_1 = S &= \{\psi^\mu, \chi^{1,\dots,6}\},\\
v_{1+i} = e_i &= \{y^i, \omega^i | \bar{y}^i, \bar{\omega}^i\}, \ i = 1, \dots, 6,\\
v_8 = z_1 &= \{\bar{\phi}^{1,\dots,4}\},\\
v_9 = z_2 &= \{\bar{\phi}^{5,\dots,8}\},\\
v_{10} = z_3 &= \{\bar{\psi}^{1,\dots,4}\},\\
v_{11} = z_0 &= \{\bar{\eta}^{0,1,2,3}\},\\
v_{12} = b_1 &= \{\chi^{34}, \chi^{56}, y^{34}, y^{56} | \bar{y}^{34}, \bar{y}^{56}, \bar{\eta}^0, \bar{\eta}^1\},
\end{aligned} \tag{9}$$

where the vector $\mathbf{1} = \{\psi^\mu,\ \chi^{1,\dots,6}, y^{1,\dots,6}, \omega^{1,\dots,6} |\ \bar{y}^{1,\dots,6}, \bar{\omega}^{1,\dots,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,\dots,5}, \bar{\phi}^{1,\dots,8}\}$ is obtained as the linear combination $\mathbf{1} = S + \sum_i e_i + z_0 + z_1 + z_2 + z_3$. In (9) we used the usual notation of the free fermionic formalism [11]. The gauge group generated by vector bosons arising in the 0-sector is $SO(8) \times SO(8) \times SO(8) \times SO(8)$. The gauge symmetry may be enhanced by vector bosons arising from nine additional purely anti-holomorphic sets given by:

$$G = \{ \quad z_0, z_1, z_2, z_3, \quad z_0+z_1, z_0+z_2, z_0+z_3, z_1+z_2, z_1+z_3, z_2+z_3 \ \}. \tag{10}$$

The basis vector b_1 reduces the $N=4 \to N=2$ spacetime supersymmetry and the untwisted gauge symmetry to $SO(8)\times SO(4)\times SO(4)\times SO(8)\times SO(8)$. Additionally, it gives rise to the twisted sector, which produces matter states charged under the four dimensional gauge group. The sixteen sectors $B_1^{pqrs} = b_1 + pe_3 + qe_4 + re_5 + se_6$, with $p,q,r,s \in \{0,1\}$, correspond to the sixteen fixed points of the non-freely acting $\mathbb{Z}_2$ orbifold.

For specific choices of the GGSO projection coefficients the gauge group is enhanced. The vector bosons arising from the sector z_3 may enhance the $SO(8)\times SO(4)\times SO(4)$ symmetry to $SO(12)\times SO(4)$, which may be enhanced further to $E_7\times SU(2)$. In the case of E_7 both z_3 and z_0 are generators of the E_7 gauge group. In the case of $SO(12)$ the matter representations are obtained from the following sectors: the two sectors B_1^{pqrs} and $B_1^{pqrs}+z_3$ give the vectorial **12** representation and the two sectors $B_1^{pqrs}+z_0$ and $B_1^{pqrs}+z_3+z_0$ the spinorial **32**.

For appropriate choices of the GGSO phases either the spinorial or the vectorial representations from a given sector are retained in the spectrum. If both the spinorial and the vectorial states are retained in a given sector, the $SO(12)\times SU(2)$ symmetry is necessarily enhanced to E_7. We note therefore that it is precisely the basis vector z_0 that acts as the spectral flow operator. For an appropriate choice of the phases it acts as a generator of E_7, whereas when the E_7 symmetry is broken to $SO(12)\times SU(2)$, coupled with appropriate mapping of the GGSO projections, the spinor-vector duality map is induced. Examining the basis vectors in (9) we see that z_0 is precisely the mirror of the basis vector S, which is the spacetime supersymmetry generator on the fermionic side of the heterotic-string. Hence, S is an operator of the left-moving $N=2$ world-sheet supersymmetry, whereas z_0 is an operator of the world-sheet supersymmetry on the bosonic side.

An important feature of the $\mathbb{Z}_2\times\mathbb{Z}_2'$ models is that the spectral flow operator is of order two, *i.e.* the sector $2z_0$ is identified with the untwisted sector. This leads to two different models, as explained above, related via the spinor-vector duality. In this paper we generalize these ideas to arbitrary internal RCFTs. For these, the spectral flow operator will generically be of order greater than two leading naturally to a bigger family of models. In the following sections we explain how these models are related in the most general case.

3 The spectral flow

To handle the most general case in what follows, we will be slightly changing our notation from the one used in the previous section and in the free fermionic language. Our starting point here is generic (2,2) heterotic models with an internal CFT with c=9. The standard examples of interacting constructions are the Gepner models [2] in which the internal CFT is a product of minimal models, but all our arguments are completely general. A general state in such a model is of the form:

$$\Phi_L \otimes \Phi_R \tag{11}$$

and the right-moving part which we wish to focus on is of the form

$$\Phi_R = (w)(h,Q)(p), \tag{12}$$

where w is an $SO(10)$ weight (o,v,s,c) and p an E_8 weight. The appearance of the $SO(10)$ and E_8 weights is because of the bosonic string map which is used to construct a modular invariant heterotic-string theory from a type II theory. It replaces the $\widehat{so}(2)_1$ Kac-Moody algebra with an $\widehat{so}(10)_1 \times (\widehat{e_8})_1$ one [1].

The mass formula is

$$\begin{aligned} \frac{\alpha' M_R^2}{2} &= h_{\rm TOT} - \frac{c}{24} \\ &= \frac{w^2}{2} + h + \frac{p^2}{2} + N_R - 1 \ , \end{aligned} \tag{13}$$

where we have used the fact that $c = 24$ for the bosonic string and we have also included the contribution N_R from the oscillators corresponding to the spacetime bosons.

By definition a CFT is said to have $N = 2$ world-sheet supersymmetry if it includes four fields:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \ , \tag{14}$$

$$G^{\pm}(z) = \sum_{n\in\mathbb{Z}} G^{\pm}_{n\pm a} z^{-n-\frac{3}{2}\mp a} \ , \tag{15}$$

$$J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1} \ , \tag{16}$$

that satisfy the algebra [1]:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \ , \\ \left[L_m, G^{\pm}_{n\pm a}\right] &= (\frac{m}{2} - n \mp a)G^{\pm}_{m+n\pm a} \ , \\ [L_m, J_n] &= -nJ_{m+n} \ , \\ [J_m, J_n] &= \frac{c}{3} m\delta_{m+n,0} \ , \\ \left[J_m, G^{\pm}_{n\pm a}\right] &= \pm G^{\pm}_{m+n\pm a} \ , \\ \{G^+_{m+a}, G^-_{n-a}\} &= 2L_{m+n} + (m-n+2a)J_{m+n} + \frac{c}{3}\Big((m+a)^2 - \frac{1}{4}\Big)\delta_{m+n,0} \ , \\ \{G^+_{m+a}, G^+_{n+a}\} &= \{G^-_{m-a}, G^-_{n-a}\} = 0 \ , \end{aligned} \tag{17}$$

where a is a real parameter that describes how the fermionic superpartners $G^{\pm}$ of T transform:

$$G^{\pm}(e^{2\pi i} z) = -e^{\mp 2\pi i a} G^{\pm}(z). \tag{18}$$

The algebras for a and $a+1$ are isomorphic. $a \in \mathbb{Z}$ corresponds to the R sector and $a \in \mathbb{Z} + \frac{1}{2}$ corresponds to the NS sector. A state is completely described by the eigenvalues h (called the conformal dimension) and Q (called the $U(1)$ charge) of the operators L_0 and J_0 that form the Cartan subalgebra:

$$|\phi\rangle = |h, Q\rangle. \tag{19}$$

We also note that the algebra is invariant under the following transformation which is known as the *spectral flow*:

$$\begin{aligned} L^{\eta}_n &= L_n + \eta J_n + \frac{c}{6}\eta^2 \delta_{n,0} \ , \\ G^{\eta\pm}_{n\pm a} &= G^{\eta\pm}_{n\pm(a+\eta)} \ , \\ J^{\eta}_n &= J_n + \frac{c}{3}\eta\delta_{n,0}. \end{aligned} \tag{20}$$

This also implies the existence of a *spectral flow operator* U_η that acts on states in the following way:

$$U_\eta|h,Q\rangle = |h_\eta,Q_\eta\rangle = \big|h-\eta Q+\frac{\eta^2 c}{6},Q-\frac{c\eta}{3}\big\rangle. \tag{21}$$

Of particular interest are the states

$$\big|\frac{3}{8},\pm\frac{3}{2}\big\rangle_{\rm R} = U_{\mp\frac{1}{2}}|0,0\rangle_{\rm NS}\ , \tag{22}$$

because they can be combined with the s and c weight vectors of $SO(10)$ with the smallest possible length to give massless states. Indeed, such vectors are of the form

$$w = (\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}) \tag{23}$$

and have $w^2=\frac{5}{4}$. An even number of minus signs corresponds to s and an odd number of minus signs to c. We then note from (13) that whenever the internal CFT has $N=2$ world-sheet supersymmetry the states

$$\pm\,\beta_0 = (\pm c)(\frac{3}{8},\pm\frac{3}{2})(0). \tag{24}$$

will be part of the massless spectrum. These states describe gauge bosons in the $\mathbf{16}$ and $\overline{\mathbf{16}}$ of $SO(10)$ and, in conjunction with the $U(1)$ symmetry of the $N=2$ algebra, they extend $SO(10)$ to E_6. This proves our previous claim that the $N=2$ superconformal algebra on the bosonic sector is associated with E_6 gauge symmetry. The states in (24) are an extension of the spectral flow operator of the internal CFT. We call these states the spectral flow operator as well.

4 The simple current formalism

Since we already started from a (2,2) model, there will be a modular invariant partition function (MIPF) describing it. It will be of the form

$$Z[\tau,\bar\tau] = \sum_{i,j}\chi_i(\tau)M_{ij}\chi_j(\bar\tau), \tag{25}$$

where χ_i are the characters of the chiral algebra and M_{ij} a modular invariant. For our examples, we take this to be the partition function of the usual Gepner models, *i.e.* after the projections of the universal simple currents β_0 and β_i have been applied to ensure spacetime supersymmetry [2]. Nevertheless, the approach is very general and valid whenever the simple current method can be used to construct modular invariants. This includes any RCFT and potentially some non-rational CFTs in which the chosen simple current defines a finite orbit as well. To avoid this complication we restrict ourselves to RCFTs through this paper.

As explained in the introduction we are not interested in the $(2,2)$ models *per se* but rather in the $(2,0)$ that we get after breaking the E_6 symmetry on the right. A consistent and modular invariant $(2,0)$ model can be derived from a $(2,2)$ model through the simple current construction [4, 14]. This is the same as the beta method for Gepner models and it practically amounts to orbifolding the original $(2,2)$ model. The result is that states not invariant under the action of the simple current are projected out and new states appear in twisted sectors. We will use both notations J and β for a simple current[5] and we will focus on simple currents that break E_6 on the right to $SO(10)$. The MIPF for the resulting model is then

$$Z[\tau,\bar\tau] = \sum\chi_i(\tau)M_{ik}M_{kj}(J)\chi_j(\bar\tau), \tag{26}$$

[5]Using multiplicative notation for the action of J and additive notation for the action of β.

where

$$M_{kj}(J) = \frac{1}{N}\sum_{n=1}^{N_J}\delta(\Phi_k, J^n\Phi_j)\delta_{\mathbb{Z}}(Q_J(\Phi_k) + \frac{n}{2}Q_J(J)) \tag{27}$$

is called a simple current modular invariant (SCMI) and N is a normalization constant ensuring that the vacuum only appears once. In practical terms, the above formula means that:

i) Only states whose left part is connected to the right through J will appear in the partition function, *i.e.* states with $\Phi_L = J^n\Phi_R = \Phi_R + n\beta$. This defines the n-th J-twisted sector.

ii) Only states invariant under the projection will appear in the partition function. This is expressed in the constraint $Q_J(\Phi) + \frac{n}{2}Q_J(J) \in \mathbb{Z}$. Q_J is called the monodromy charge and is defined as

$$Q_J(\Phi) = h(\Phi) + h(J) - h(J\Phi) \qquad \text{mod } 1. \tag{28}$$

The easiest way to see that this is the appropriate condition for invariance under the J projection is to note that the monodromy charge is conserved modulo 1 in operator products and thus implies the existence of a phase symmetry $\Phi \to e^{-2\pi i Q_J(\Phi)}\Phi$. This induces a cyclic group of order N_J. N_J is called the order of J and it can also be proven that $Q_J(\Phi)$ is quantized in units of $1/N_J$ [14].

The definition (28) is for any general RCFT. For Gepner models, where $\Phi = (w_\Phi)(\vec{l}_\Phi, \vec{q}_\Phi, \vec{s}_\Phi)(p_\Phi)$ and $J = (w_J)(\vec{l}_J, \vec{q}_J, \vec{s}_J)(p_J)$, it takes the explicit form:

$$Q_J(\Phi) = -w_J \cdot w_\Phi - p_J \cdot p_\Phi + \sum_{i=1}^{r}\left(\frac{-l^i_\Phi l^i_J + q^i_\Phi q^i_J}{2(k_i+2)} - \frac{s^i_\Phi s^i_J}{4}\right). \tag{29}$$

In this form it is easy to see that

$$Q_\beta(\Phi) = Q_\Phi(\beta) \quad \text{and} \quad Q_{\beta_1+\beta_2}(\Phi) = Q_{\beta_1}(\Phi) + Q_{\beta_2}(\Phi), \tag{30}$$

i.e. the monodromy charge is symmetric and linear with respect to its arguments. These properties are true in general [14].

Another thing to note is that if J and J' are simple currents then JJ' is a simple current as well. In fact, we can generalize (27) to the case where we orbifold by $J_1, \cdots, J_i, \cdots$ simultaneously. To simplify the notation let $\vec{n}$ label the twisted sectors and define

$$[\vec{n}]k \equiv J_1^{n_1}\cdots J_i^{n_i}\cdots\Phi_k \equiv \Phi_k + \sum_i n_i\beta_i.$$

Then the most general SCMI is [15]:

$$M_{k,[\vec{n}]k} = \frac{1}{N}\prod_i \delta_{\mathbb{Z}}(Q_{J_i}(\Phi_k) + X_{ij}n^j). \tag{31}$$

The matrix X is defined modulo 1 and its elements are quantized as $X_{ij} = \frac{n_{ij}\in\mathbb{Z}}{\gcd(N_i,N_j)}$. It also satisfies $X_{ij} + X_{ji} = Q_{J_i}(J_j)$. This fixes its symmetric part completely. The remaining freedom in choosing the antisymmetric part corresponds to discrete torsion [15].

5 Outline of the idea

We start with a particular simple current J. Any J would do, but for the reasons explained in the introduction the simple currents that we have in mind will break E_6, thus giving a $(2,0)$ model. We call the $(2,0)$ model that is derived this way $\mathcal{M}_0$. We also know that J_0 (β_0) is generically a simple current of every $(2,2)$ model since it is the spectral flow operator that enhances the symmetry to E_6 on the right. This naturally defines a whole family of models $\{\mathcal{M}\}_\alpha$ that are derived through the simple currents J, J_0 and linear combinations of them with and without discrete torsion.

The task of examining how the spectra of these models are related to each other is very fascinating and daunting at the same time. We will not attempt to carry out the analysis in its full generality here. Instead, we will restrict ourselves to the more modest goal of explaining how the mapping induced by the spectral flow J_0 (β_0) works.

6 Mapping induced by the spectral flow

Here we focus on the family of models $\mathcal{M}_0, \cdots, \mathcal{M}_m$ that are derived through the simple currents $J, JJ_0, \cdots, JJ_0^m$ or equivalently $\beta, \beta+\beta_0, \cdots, \beta+m\beta_0$. This family will have N_{β_0} members where N_{β_0} is the order of β_0. Our goal is to study how the massless spectra in these models are related. To that end, we take a closer look at the model $\mathcal{M}_m$.

We start by examining the untwisted sector[6]. Massless states in the original $(2,2)$ model will also belong to the $\mathcal{M}_m$ model if they survive the invariance projections. Note that

$$Q_{\beta+m\beta_0}(\Phi) = Q_\beta(\Phi) + mQ_{\beta_0}(\Phi) = Q_\beta(\Phi) \mod 1, \tag{32}$$

where in the last step we used the fact that $Q_{\beta_0}(\Phi) \in \mathbb{Z}$ because Φ belongs to the original $(2,2)$ model. This proves that $Q_{\beta+m\beta_0}(\Phi) \in \mathbb{Z} \Leftrightarrow Q_\beta(\Phi) \in \mathbb{Z}$ and therefore the untwisted sectors of every model in the $\mathcal{M}$ family are identical.

Let us now consider the twisted sectors. Note that models $\mathcal{M}_{m_1}$ and $\mathcal{M}_{m_2}$ will in general have a different number of twisted sectors since $\beta + m_1\beta_0$ and $\beta + m_2\beta_0$ will be of different order. Let us analyze the n-twisted sector of the $\mathcal{M}_m$ model. A very useful formula can be found by rearranging (28) as

$$\begin{aligned} h(\Phi+\beta) &= h(\Phi) + h(\beta) - Q_\beta(\Phi) , \\ \text{and by induction:} \quad h(\Phi+m\beta) &= h(\Phi) + mh(\beta) - mQ_\beta(\Phi) - \frac{m(m-1)}{2}Q_\beta(\beta) , \end{aligned} \tag{33}$$

where the equations are understood mod 1. Massless states in the n-twisted sector of $\mathcal{M}_m$ are of the form

$$\Phi_L \otimes (\tilde{\Phi}_L + n(\beta + m\beta_0)) , \tag{34}$$

where this time we have written the tilde explicitly to remind us that we have applied the bosonic string map. In the notation of equation (12) this is simply [2]:

$$\tilde{\Phi}_L = \Phi_L + (v)(0,0)(0). \tag{35}$$

The massless condition gives

$$h(\Phi_L) = \frac{1}{2}, \quad h(\tilde{\Phi}_L) = 1 \quad \text{and} \quad h(\tilde{\Phi}_L + n\beta + nm\beta_0) = 1. \tag{36}$$

[6]Here and in what follows untwisted sector means untwisted with respect to the simple current that defines the model, *i.e.* states with $n = 0$ in (34). The states might be twisted with respect to other simple currents that were present in the original (2,2) model but this does not affect our argument.

Furthermore, as explained before and as can be seen from (27), the states must also satisfy the invariance condition

$$Q_{\beta+m\beta_0}(\tilde{\Phi}_L) + \frac{n}{2} Q_{\beta+m\beta_0}(\beta + m\beta_0) \in \mathbb{Z}. \tag{37}$$

Using linearity of the monodromy charge and the fact that $Q_{\beta_0}(\tilde{\Phi}_L) \in \mathbb{Z}$ and $Q_{\beta_0}(\beta_0) \in 2\mathbb{Z}$ because $\tilde{\Phi}_L$ and β_0 belonged to the massless spectrum of the original $(2,2)$ model, the invariance condition becomes

$$Q_\beta(\tilde{\Phi}_L) + \frac{n}{2} Q_\beta(\beta) + mnQ_{\beta_0}(\beta) \in \mathbb{Z}. \tag{38}$$

We can also further manipulate (36) to derive another condition. Bearing in mind that in what follows all the calculations are mod 1, we get:

$$\begin{aligned}
0 = 1 \quad &= \quad h(\tilde{\Phi}_L + n\beta + nm\beta_0) \\
&\overset{(33)}{=} \quad h(\tilde{\Phi}_L + n\beta) + \underbrace{nmh(\beta_0)}_{\in\mathbb{Z}} - \underbrace{nmQ_{\beta_0}(\tilde{\Phi}_L)}_{\in\mathbb{Z}} - n^2 mQ_{\beta_0}(\beta) - \underbrace{\frac{nm(nm-1)}{2}}_{\in\mathbb{Z}} \underbrace{Q_{\beta_0}(\beta_0)}_{\in\mathbb{Z}} \\
&= \quad h(\tilde{\Phi}_L + n\beta) - n^2 mQ_{\beta_0}(\beta) \\
&\overset{(33)}{=} \quad \underbrace{h(\tilde{\Phi}_L)}_{=1=0} + nh(\beta) - nQ_\beta(\tilde{\Phi}_L) - \frac{n(n-1)}{2} Q_\beta(\beta) - n^2 mQ_{\beta_0}(\beta) \\
&\overset{(38)}{=} \quad nh(\beta) + \frac{n}{2} Q_\beta(\beta)
\end{aligned} \tag{39}$$

Or in other words,

$$n\Big(h(\beta) + \frac{1}{2} Q_\beta(\beta)\Big) \in \mathbb{Z}. \tag{40}$$

Equations (38) and (40) are the main results of this section. In general, these conditions are necessary but not sufficient because of the inherent uncertainty in the definition of the monodromy charge which is given mod 1. Nevertheless, the beauty of this general argument is that starting from an arbitrary $(2,0)$ model we get a handle on the massless spectrum in any twisted sector of any model in the family.

7 An example

The fact that these conditions are necessary provides a prime test for where *not* to look for massless states in a particular model. This can be of great importance when performing a computer scan in the space of models, so we give an example below.

Our starting point is the Gepner model $k^r = 2^6$, which is a (2,2) model. In this model the internal CFT is a product of 6 minimal models each of which has central charge $c = \frac{3k}{k+2} = \frac{3}{2}$. All states will be of the form (11) but this time the internal CFT state is completely described by three vectors $\vec{l}, \vec{q}$ and $\vec{s}$ so we will be using the notation $\Phi_R = (w)(\vec{l}, \vec{q}, \vec{s})(p = 0)$ instead. For the sake of the argument let us focus our attention on the massless charged spectrum in this model, which of course will fall into the fundamental (**27**) or anti-fundamental ($\overline{\mathbf{27}}$) representation of E_6. Without loss of generality, we will study states in the **27**, which under the $SO(10)$ group decomposes into $\mathbf{10} + \mathbf{16} + \mathbf{1}$. Let us briefly remind the reader that the right-moving part of such massless states will then be of the form:

- **10**s: $\Phi_R = (v)(\Phi^I)(p = 0)$ with

$$\Phi^I \in \Big\{ \underline{(0,0,0)^4(0,2,2)^2},\ \underline{(0,0,0)^2(1,-1,0)^4},\ \underline{(0,0,0)^3(0,2,2)(1,-1,0)^2} \Big\}\ ,$$

- **16**s: $\Phi_R = (c)(\Phi^{II})(p=0)$ with

$$\Phi^{II} \in \Big\{\underline{(0,-1,-1)^4(0,1,1)^2},\ \underline{(0,-1,-1)^2(1,-2,-1)^4},\ \underline{(0,-1,-1)^3(0,1,1)(1,-2,-1)^2}\Big\}\ ,$$

- **1**s: $\Phi_R = (w=0)(\Phi^{III})(p=0)$ with

$$\Phi^{III} \in \Big\{\underline{(0,-2,-2)^4(0,0,0)^2},\ \underline{(0,-2,-2)^2(1,-3,-2)^4},\ \underline{(0,-2,-2)^3(0,0,0)(1,-3,-2)^2}\Big\}\ ,$$

where underlining means permutations.

In this model β_0 has the usual form

$$\beta_0 = (c)(0,1,1)^6(p=0) \tag{41}$$

and is of order $N_{\beta_0} = 8$. We choose the simple current with which we will orbifold our theory to be

$$\beta = \ (w=0)(2,1,-1)(0,0,0)^5(p=0)\ , \tag{42}$$

which is also of order $N_\beta = 8$ and we note that $Q_\beta(\beta_0) = \frac{3}{8} \notin \mathbb{Z}$. Therefore the gauge bosons extending $SO(10)$ to E_6 are indeed projected out and we end up with a $(2,0)$ model. As explained in the previous section, this process naturally induces a whole family of models $\mathcal{M}_0, \cdots, \mathcal{M}_7$ that arise if we orbifold by $\beta, \cdots, \beta + 7\beta_0$ respectively.

The untwisted sector in all of these models will be the same and it will consist of all the states mentioned above that satisfy the invariance condition

$$Q_\beta(\Phi_R) \in \mathbb{Z} \quad \Leftrightarrow \quad \frac{-2l_1 + q_1 + 2s_1}{8} \in \mathbb{Z}. \tag{43}$$

For the $n-$twisted sector we will use equation (40). $h(\beta)$ can be readily calculated from the known formula for Gepner models [2]:

$$h = \sum_{i=1}^{r}\left(\frac{l_i(l_i+2) - q_i^2}{4(k_i+2)} + \frac{s_i^2}{8}\right) \tag{44}$$

and we find that

$$n\Big(h(\beta) + \frac{1}{2}Q_\beta(\beta)\Big) = n\Big(\frac{9}{16} + \frac{1}{2}(-\frac{5}{8})\Big) = \frac{n}{4} \in \mathbb{Z}. \tag{45}$$

This means that massless states can only arise in the untwisted $n = 0$ sector, which we have already studied, or in the $n = 4$ twisted sector. In the latter sector the right-moving part of the states will be of the form

$$\begin{aligned}
\Phi_R &= \tilde{\Phi}_L + 4(\beta + m\beta_0) \\
&= \tilde{\Phi}_L + 4\beta + 4m\beta_0 \\
&= \tilde{\Phi}_L + (w=0)(0,4,0)(0,0,0)^5(p=0) + m(w=0)(0,4,0)^6(p=0) \\
&= \begin{cases} \Phi_L + (w=0)(0,4,0)(0,0,0)^5(p=0) & \text{if m even} \\ \Phi_L + (w=0)(0,0,0)(0,4,0)^5(p=0) & \text{if m odd} \end{cases}\ ,
\end{aligned} \tag{46}$$

where we have used the properties [2] that for Gepner models q is defined mod $2(k+2)$, s is defined mod 4 and we have also performed the identification $(l,q,s) \equiv (k-l, q+k+2, s+2)$ multiple times. A quick comparison with Φ^I, Φ^{II} and Φ^{III} given above shows that states of the form (46) cannot be massless charged states, so the spectrum consists of the states in the untwisted sector only.

Once more, the power of this method is that it allowed us to check only one twisted sector ($n = 4$) for massless states, as opposed to checking as many as seven of them for each model that we would *a priori* expect in this example.

8 Some further generalizations

There are many ways to generalize the above ideas to generate even more relationships in the space of $(2,0)$ models. For example, we are not restricted to using only β_0 but the natural splitting of the states into an $SO(10)$ part, an internal $N = 2$ CFT and an E_8 part suggests that any

$$\beta_{0'} = (w)(\beta_0^{\mathrm{CFT}})(p)$$

would generate its own orbit of $(2,0)$ models. Furthermore, when the internal CFT can be written as a tensor product of $N = 2$ superconformal theories each term comes with a spectral flow operator β_0^i. We can then go one step further and use only some of the β_0^i's instead of the entire β_0^{CFT}.

Finally, as explained earlier, the presence of a simple current J that breaks (2,2) to $(2,0)$ increases the possibilities even further. We can now have any linear combination of J, with any of the β's mentioned above, with or without discrete torsion, and any such simple current will create its own orbit in the space of $(2,0)$ models.

In this paper we have shown explicitly how to use one of these mappings, the spectral flow β_0, to generate an entire family of models and we have derived useful expressions for the analysis of the spectra of these models. We believe that having not just one, but a big selection of such mappings as explained above will prove to be an important tool in the classification of $(2,0)$ models.

9 Conclusions

Heterotic-string vacua with $(2,0)$ world-sheet supersymmetry are particularly interesting from a phenomenological point of view, as they reproduce the $SO(10)$ GUT structure, which is well motivated by the Standard Model data. Ultimately, the confrontation of a string vacuum with low scale experimental data will be achieved by associating it with an effective smooth quantum field theory limit. However, while the moduli spaces of (2,2) heterotic-string compactifications, and consequently their smooth limit, are reasonably well understood, this is not the case for those with $(2,0)$ world-sheet supersymmetry. Indeed, the study of these moduli spaces is an area of intense contemporary research [16].

In this paper we discussed how the spinor-vector duality, which was observed in the framework of heterotic-string compactifications with free world-sheet CFTs, can be extended to those with general RCFTs. The recipe adopted from the free case is the following: We start with a (2,2) compactification and break the world-sheet supersymmetry on the bosonic side. The spectral flow operator, that operates as a symmetry generator of the (2,2) theory, then induces a map between the string vacua of the $(2,0)$ theory. As such, the map induced by the spectral flow operator provides a useful tool to explore the moduli spaces of $(2,0)$ heterotic-string compactifications. The question of interest in this respect is twofold. First, is this description complete? Namely, do all $(2,0)$ heterotic-string compactifications descend from (2,2) theories? Second, what is the imprint of this map in the effective field theory limit? We hope to return to these questions in future publications.

Acknowledgments

A.E.F. thanks the Weizmann Institute and the Theoretical Physics Department at the University of Oxford for hospitality. P.A. acknowledges support from the Hellenic State Scholarships Foundation (IKY). This work was supported in part by the STFC (ST/J000493/1).

References

[1] For comprehensive reviews and references see *e.g.* :
L.E. Ibanez and A.M. Uranga, *String theory and particle physics: an introduction to string phenomenology*, Cambridge University Press, 2012;
R. Blumenhagen, D. Lüst and S. Theisen, *Basic concepts of string theory*, Springer 2013.

[2] D. Gepner, *Phys. Lett.* B 199 (1987) 380;
D. Gepner, *Nucl. Phys.* B 296 (1988) 757.

[3] A. Font, L.E. Ibanez, F. Quevedo and A. Sierra, *Nucl. Phys.* B 337 (1990) 119.

[4] A. Schellekens and S. Yankielowicz, *Nucl. Phys.* B 330 (1990) 103.

[5] A.E. Faraggi, C. Kounnas and J. Rizos, *Nucl. Phys.* B 774 (2007) 208, arXiv:hep-th/0611251.

[6] A.E. Faraggi, C. Kounnas and J. Rizos, *Phys. Lett.* B 648 (2007) 84, arXiv:hep-th/0606144.

[7] A.E. Faraggi, C. Kounnas and J. Rizos, *Nucl. Phys.* B 799 (2007) 19, arXiv:0712.0747 [hep-th].

[8] T. Catelin-Jullien, A.E. Faraggi, C. Kounnas, J. Rizos, *Nucl. Phys.* B 812 (2009) 103, arXiv:0807.4084 [hep-th].

[9] C. Angelantonj, A.E. Faraggi, M. Tsulaia, *JHEP* 1007 (2010) 4, arXiv:1003.5801 [hep-th].

[10] A.E. Faraggi, I. Florakis, T. Mohaupt, M. Tsulaia, *Nucl. Phys.* B 848 (2011) 332, arXiv:1101.4194 [hep-th].

[11] H. Kawai, D.C. Lewellen, and S.H.-H. Tye, *Nucl. Phys.* B 288 (1987) 1;
I. Antoniadis, C. Bachas, and C. Kounnas, *Nucl. Phys.* B 289 (1987) 87;
I. Antoniadis and C. Bachas, *Nucl. Phys.* B 298 (1988) 586.

[12] A.E. Faraggi, *Phys. Lett.* B 326 (1994) 62; arXiv:hep-ph/9311312;
A.E. Faraggi, *Phys. Lett.* B 544 (2002) 207, arXiv:hep-th/0206165;
A.E. Faraggi, S. Forste and C. Timirgaziu, *JHEP* 0608 (2006) 057, arXiv:hep-th/0605117;
R. Donagi and A.E. Faraggi, *Nucl. Phys.* B 694 (2004) 187, arXiv:hep-th/0403272;
R. Donagi and K. Wendland, *J. Geom. Phys.* **59** (2009) 942, arXiv:0809.0330 [hep-th].

[13] C. Angelantonj and A. Sagnotti, *Phys. Rep.* **371** (2002) 1, arXiv:hep-th/0204089.

[14] A. Schellekens and S. Yankielowicz, *Nucl. Phys.* B 327 (1989) 673.

[15] M. Kreuzer and A. Schellekens, *Nucl. Phys.* B 411 (1994) 97, arXiv:hep-th/9306145.

[16] For recent examples see *e.g.* :
L.B. Anderson, J. Gray and E. Sharpe, arXiv:1402.1532 [hep-th];
X. de la Ossa and E.E. Svanes, arXiv:1402.1725 [hep-th];
M. Bertolini, I.V. Melnikov and M.R. Plesser, arXiv:1402.1751 [hep-th].