Risk-sensitive Control and Its Applications

Thesis

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Contents

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Declarations

This dissertation is a result of my own original work and includes nothing which is the outcome of work done in collaboration.

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Abstract

The generalized risk-sensitive control problem is analized in this thesis. If the cost weighting matrices for the state and the control are allowed to be indefinite, then we call the stochastic risk-sensitive control is indefinite. In this thesis we consider both finite and infinite versions of the indefinite generalised risk-sensitive control problem. The change of measure and the completion of squares methods are used in solving this problem. In the infinite cases for both fully and partially observed systems, we introduce a coefficient function into the cost functional, from which the conditions on the controller could be weakened and the limitations of the method are also reduced compared with other relevant papers on similar topics. A group of optimal controllers is obtained in an explicit closed-form for both finite and infinite time horizons in each case. To illustrate the theory, an example of the application in finance is presented to illustrate the theory.

We introduce a *risk-sensitive* version of the classical H_2/H_∞ robust control method for linear stochastic systems with additive noise. Two criteria of *exponential-quadratic* form are employed instead of the usual quadratic criteria. Under the assumption of linear state-feedback controllers, the solutions are found for both the finite and the infinite horizon formulations.

In this thesis we also investigate the risk-sensitive(RS) problem and linear-quadratic(LQ) problem with delays in the control system for a linear stochastic continuous time state. We use a combination of methods,changing measure and

completion of squares methods to find an explicit solution for each case. We also generalised the linear system by adding a delay term in both state and the cost functional, and present several forms of Riccati differential equations. The last chapter contains some preliminary results that we obtained. Further research need to be done in the future.

Chapter 1

Introduction

1.1 Introduction

In this introductory chapter we present a short history of the problem of risk-sensitive control, the robust control and delay system, describe the main contributions of the thesis, and give a brief synopsis of each of the following chapters.

1.2 The problem of risk-sensitive control and robust control

Let $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), \mathbb{P})$ be a complete probability space. Define $(W(t), t \geq 0)$ is a d-dimensional standard Brownian motion. We assume that $\mathcal{F}(t)$ is the augmentation of $\sigma\{W(s)|0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$.
Considering the linear stochastic control system:

$$\begin{cases} dx(t) = [Ax(t) + Bu(t)]dt + Cdw_1(t), \\ \\ dy(t) = Hx(t)dt + D^{\frac{1}{2}}dw_2(t), \\ \\ x(0) = x_0, \\ \\ y(0) = 0, \end{cases} \tag{1.2.1}$$

We assume that $A(\cdot) \in L^\infty(0, T; \mathbb{R}^{n\times n})$, $B(\cdot) \in L^\infty(0, T; \mathbb{R}^{n\times m})$, $C(\cdot) \in L^\infty(0, T; \mathbb{R}^{n\times d})$, $H(\cdot) \in L^\infty(0, T; \mathbb{R}^{q\times n})$, $D(\cdot) \in L^\infty(0, T; \mathbb{R}^{q\times q})$, and $D(t) > 0, \forall t \in [0, T]$, where $L^\infty(0, T; \mathbb{R}^{n\times n})$ is the set of all $\mathbb{R}^{n\times n}$-valued uniformly bounded functions. We further assume that the $\mathcal{F}(t)$-adapted control process $u(\cdot)$ is such that (1.2.1) has a unique strong solution.

We are given $W(t) = [w_1'(t), w_2'(t)]'$, where, $w_1(t)$ and $w_2(t)$ are orderd p and q. And we assume that x_0 and $W(t)$ are independent, and x_0 is a Gaussian random variable with mean μ_0 and variance σ_0.

The *risk-sensitive* control problem is to minimize:

$$\bar{J}(u(\cdot)) := \gamma\mathbb{E}\left\{\exp\left[\frac{\gamma}{2}x'(T)Sx(T) + \frac{\gamma}{2}\int_0^T\Big[x'(t)Qx(t) + u'(t)Ru(t)\Big]dt\right]\right\}, \quad (1.2.2)$$

where $\gamma \in \mathbb{R}, \gamma \neq 0$, is given. And $Q(\cdot) \in L^\infty(0, T; \mathbb{R}^{n\times n})$, $R(\cdot) \in L^\infty(0, T; \mathbb{R}^{m\times m})$, are given matrices with $Q(t) \geq 0, R(t) > 0, \forall t \in [0, T]$.

Finding an optimal $u(\cdot)$ to minimize (1.2.2) when (1.2.1) holds was first introduced by Jacobson in [34]. He found the complete solution to the fully observed system problem, i.e. $H = 0$, and $D = 0$ in (1.2.1). The optimal controller $u(\cdot)$ is found to be in a linear state-feedback form, which is very similar to the linear-quadratic control problem solution (see [80]). But in a risk-sensitive case, optimal $u(t)$ depends on C, which means the controller is related to the intensity of noise.

Whittle [78](see also [79]) solved the discrete case of the general partially observed risk-sensitive control. And the continuous case was first completely settled by Bensoussan and Van Schuppen [5] in 1985. They transferred the partial observation case into the full observation case, and made the problem easily solvable. This method is the most general way to solve all the different partial observation cases in a control problem.

In two recent papers Date and Gashi [17], Date and Gashi [16], a generalization of the cost functional has been introduced as follows:

$$\begin{aligned} J(u(\cdot)) \quad &= \quad \gamma\mathbb{E}\left\{\exp\left[\frac{\gamma}{2}x'(T)Sx(T) + \frac{\gamma}{2}\int_0^T [x'(t)Qx(t) + u'(t)Ru(t)]dt\right.\right. \\ &\qquad \left.\left. + \frac{\gamma}{2}\int_0^T [x'(t)Q_1 + u'(t)R_1]dW(t)\right]\right\}, \end{aligned}$$

for some constant vectors Q_1 and R_1 of proper dimensions. The novelty here is that a *noise* dependent penalty has been introduced. An important feature of this *generalised* risk-sensitive control problem is that it admits an explicit closed-form solution in terms of a certain Riccati type equation.

With more than forty years of development, several related topics concerning the risk-sensitive control problem have been analyzed. Considering the indefinite stochastic control, the linear-quadratic control problem was first found in Chen, Li and Zhou [14].

However, the stochastic Riccati equation in [14] may not be capable handling certain indefinite problems. A lot of research has been undertaken in the indefinite linear-quadratic control area, such as [55], [62], [81], [46], and [54], [86] for the discrete-time case. However, anindefinite risk-sensitive control has not been previously considered for the risk-sensitive control problem, particularly for the partially observed risk-sensitive control problem.

Let us turn our attention to the robust control problem. The H_2/H_∞ problem has been a popular research topic in recent years.

Consider the linear time-varying system to be:

$$\begin{cases} dx(t) = [A(t)x(t) + B_2(t)u(t) + B_1(t)v(t)]dt \\ z(t) = \begin{bmatrix} C(t)x(t) \\ D(t)u(t) \end{bmatrix} \\ x(0) = x_0, \end{cases} \tag{1.2.3}$$

the entries of $A(t)$, $B_1(t)$, $B_2(t)$, $C(t)$, and $D(t)$ are continuous functions of time, and $D'(t)D(t) = I$.

The cost function is given as:

$$J_1(u,v) = \int_0^T z'(t)z(t)dt, \tag{1.2.4}$$

and

$$J_2(u,v) = \int_0^T (\gamma^2 v'(t)v(t) - z'(t)z(t))dt.$$

(1.2.5)

The earliest method used to solve this problem was entropy minimization (see [28], [50], [58], [6], [20] and [85]), until Limebeer, Anderson and Hendel (see [52]) first completely solved the mixed H_2/H_∞ problem by treating the H_2 and H_∞ criteria separately and using a two-player Nash game. And the solutions are found to be state-feedback. The idea of solving the H_2/H_∞ problem is to use two performances to reflect an H_∞ constraint and an H_2 optimality requirement. This research was the first to generate the H_2, and H_∞ as two special cases of the two-player linear-quadratic problem, from which a link between H_2, H_∞ and mixed H_2/H_∞ theories was established in [51] and Sweriduk's research [71].

After the research of Limebeer, Anderson and Hendel, the H_2/H_∞ problem has been widely developed to various fields. However, few results had been obtained until Chen and Zhang [12] who were the first to generate the stochastic mixed state-dependent H_2/H_∞ control problem with *Itô*'s differential systems.

It has been shown that for both finite and infinite horizons, the stochastic H_2/H_∞ control problem is closely related to a pair of coupled DREs(finite) or AREs(infinite). The H_2/H_∞ problem has be developed a lot, but it still seems there is considerable scope when the cost functions are given in exponential form.

The last review comes about the delay system, which is also named dead-time. The linear-quadratic systems with delays has been studied from different viewpoints for many years, with some important papers, such as Ichikawa ([32], [33]), Kwong and Willsky ([41], [42]), R. H. Kwong ([44], [40], [43]), Koivo and Lee, etc. ([36]), adding to the fundamental theories of the delay problem.

Most of the authors in the early period of research on this topic, solved the delay problem using state-space techniques with different approaches. For example, in Ichikawa's paper, a finite number of pure delays occured in a family of evolution equations; Kwong and Willsky focused on a less general control operator with differential delay equations. However the state-space technique can be used only for a special delay structure that contains a finite number of delays and is not generally applied.

Some results have been obtained in the delayed control problem over the last

few decades. For example, a very general case of the linear-quadratic delay problem was solved in the paper of Chen and Wu [13] as follows:
consider the following LQ system:

$$\begin{cases} dx(t) = [Ax(t) + A_1x(t-\delta) + M_tu(t) + M_t^1u(t-\delta)]dt \\ \qquad\qquad + [Cx(t) + C_1x(t-\delta) + D_tu(t) + D_t^1u(t-\delta]dw(t) \\ \ x(t) = \phi(t), \quad u(t) = \eta(t), \quad t \in [-\delta, 0]. \end{cases} \tag{1.2.6}$$

where, $\phi, \eta(t) \in C[-\delta, 0]^n$ is deterministic functions, satisfying $\int_{-\delta}^0 \alpha^2(s)ds < +\infty, \alpha = \phi, \eta$. Giving the cost functional as:

$$J(u(\cdot)) = \frac{1}{2}\mathbb{E}\left[\int_0^T x'(t)Qx(t) + u'(t)Ru(t)dt + x(T)Sx(T)\right]. \tag{1.2.7}$$

They encountered the backward stochastic differential equations:

$$\begin{cases} -dY(t) = f(t, Y(t), Z(t), Y(t+\delta(t)), Z(t+\zeta(t)))dt - Z(t)dB(t), t \in [0, T], \\ \quad Y(t) = \xi(t), \quad Z(t) = \eta(t), \quad t \in [T, T+K], \end{cases} \tag{1.2.8}$$

where $\delta(\cdot)$ and $\zeta(t)$ are $\mathbb{R}^+$-valued functions defined on [0,T]. (See Peng and Yang [60]) for the optimization problem. Using the backward stochastic differential equation method, this paper gives the feedback regulator in terms of the conditional expectation of future information as follows:

$$\begin{aligned} u(t) \quad &= \quad -R^{-1}\Bigg[M_t^\tau y(t) + D_t^\tau z(t) \\ &\qquad + \mathbb{E}^{\mathcal{F}_t}\left((M_{t+\delta}^1)^\tau y(t+\tau) + \left(D_{t+\delta}^1\right)^\tau z(t+\delta)\right)\Bigg], \quad t \in [0, T]. \end{aligned}$$

This paper solved a very general case of the LQ delay problem, however, the result they found about the optimal controller is not explicit and difficult to be applied in practice.

The research on delay systems has developed rapidly in recent years, since the stochastic delay differential equations could be applied to a lot of fields, including finance, engineering and physics. The robust control time-delay problem and H_∞

control problem were solved in Tadmor ([72]- [76]), Zhou [84], Uchida [77], Nagpal [59]. There have been achieved alot if results on the study of time delay systems in various research fields. Some important results in linear-quadratic delay systems have been achieved in recent years, but only a few studies looked at risk-sensitive control. The only paper we could find where authors attempted to solve the exponential cost functional is by Yoneyama [82]. Yoneyama used a change of measure technique to solve the risk-sensitive control problem with delay system of partial observation . However, the delay term in the state system only occurred on the controller $u(t)$ and can not be generally applied. From the results of Yoneyama, we can see that the solutions of the delay system are quite complicated. When a delay state system is combined with a cost functional which involves an exponential expression, it is even more difficult to find an explicit solution. In the respective chapter in this thesis, we are going to be dealing with this delayed-risk-sensitive control problem as a challenge.

1.3 The main contributions

The main contributions of this thesis are as follows:

- The solution for the indefinite optimal control problems of minimizing the generalized form of Jacobson's [34] and Bensoussan's [5] cost functional. We solved both finite and infinite of the full and partial observation cases. Notably for the infinite case, by introducing a coefficient function into the cost functional, we could solve the problems under weaker conditions. The solutions are found using a combination of the change of measure and the completion of squares methods. It is worth mentioning that, the indefinite case analysis has never been done before even for the finite risk-sensitive control.

- The H_2/H_∞ problem has been extensively developed, but it still seems to be a blank when the cost functions are given in exponential form. This thesis focuses on the combination of the stochastic H_2/H_∞ control problem with the risk-sensitive criteria. It is shown that, the H_2/H_∞ control has a pair of controllers which depend on the corresponding Riccati equations solution.

- In this thesis we analyzed the delay system in two different ways: combined with linear-quadratic control problem; and focusing on the exponential criteria: risk-sensitive control. In the linear-quadratic case, we found an explicit solution for the problem rather than a sweeping form of expectation which could not be directly applied. Compared with existed delay models of the risk-sensitive control, we generalised the state system by adding the delay terms in both observable state variables and in the controller.

The following is a short introduction to the chapters.

Chapter 2

In this chapter we review some basic results of stochastic control theory to which we refer later in the thesis. In addition we show the methods from some important papers, that have been used to solve the risk-sensitive control, indefinite control, H_2/H_∞ control and delay systems.

Chapter 3

In this chapter, we analized the *indefinite* generalised risk-sensitive control problem with a fully observed state system in both finite and infinite versions. In each case, the solutions are found in an explicit closed-form. The change of measure and the completion of squares methods are used for this purpose.At the end of the chapter, we give an example that illustrate the theory developed, applied to finance.

Chapter 4

This chapter is concerned with the *indefinite* generalized risk-sensitive control problem with partially observed system. By introducing two matrices and changing of measure, we transformed the partial observation to a classic full observation problem. Using a similar method, as described in the previous chapter, a group of optimal controllers is obtained for both finite and infinite time horizons. In the infinite case we introduce a new coefficient function to the original cost function, whereby reducing the assumptions and conditions as compared with other relevant papers on similar topics.

Chapter 5

In this chapter we discuss robust control, which is a combination of H_2/H_∞ control and risk-sensitive control of stochastic linear systems. Two criteria of exponential-quadratic form are employed instead of the usual quadratic criteria. Under the assumption of linear state-feedback controllers, the solutions are found for both the finite and the infinite time horizon formulations.

Chapter 6

In this chapter we investigate the risk-sensitive problem and the linear-quadratic problem with delay in control system, for a stochastic linear continuous time state. We then generalise the LQ system by adding a delay term in both the state and the cost functionals.

Chapter 7

In the final chapter we summarize the main contributions of the thesis and dentify some interesting open questions for future research.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter we review some basic results of stochastic optimal control theory. These include risk-sensitive control, partially observable stochastic systems, generalized risk-sensitive control, indefinite linear-quadratic control, Robust control, stochastic delay system and Riccati equations. Where applicable we hightlight any new contributions we make with this thesis within the context of the well-known theory.

2.2 Risk-sensitive control

The optimal risk-sensitive control problem was introduced by Jacobson in [34]. He fould the complete solution of the full observation case of the problem in continuous time. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $(w(t), t \geq 0)$ is defined as a $(p+q)$-dimensional standard Brownian motion.
Considering the linear stochastic control system:

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t)]dt + C(t)dw(t), \\ x(0) = x_0, \end{cases} \tag{2.2.1}$$

where $A(t), B(t), C(t)$ are time dependent variables. The risk-sensitive control problem is the optimal control problem to minimise:

$$\bar{J}(u(\cdot)) \equiv \gamma \mathbb{E}\left\{ \exp\left[\frac{\gamma}{2} x'(T)Sx(T) + \frac{\gamma}{2} \int_0^T \Big[x'(t)Qx(t) + u'(t)Ru(t) \Big] dt \right] \right\}, \tag{2.2.2}$$

where $Q(t) \geq 0$, $R(t) > 0$ and $S \geq 0$, $\forall t \in [0, T]$. The optimal control of finding optimal $u(\cdot)$ to minimize (2.2.2) when (2.2.1) holds is found to be in the following form:

$$u^*(x,t) = -R^{-1}B'Px(t), \quad t \in [0,T],$$

where P is the solution of the following Riccati equation:

$$\begin{cases} -\dot{P}(t) = Q(t) + P(t)A(t) + A'(t)P(t) \\ \qquad\qquad - P(t)(B(t)R^{-1}(t)B'(t) - \gamma C(t)C'(t))P(t), \\ P(T) = S. \end{cases}$$

The discrete-time version of the problem has also been done by Jacobson. The results of risk-sensitive control are related to C which is vey different from the LQ problem

2.3 Partially observed risk-sensitive control

For partial observation, Bensoussan and Van Schuppen ([5]) provide the first complete solution to this problem. Given the linear stochastic control system:

$$\begin{cases} dx(t) = [F(t)x(t) + B(t)u(t)]dt + G(t)dw, \\ dy(t) = Hxdt + R^{\frac{1}{2}}db, \\ x(0) = x_0, \\ y(0) = 0, \end{cases} \tag{2.3.1}$$

where $x(t)$ is the state of the system with dimension n, $u(t)$ is the control process with dimensiona m, w and b are standard Wiener processes, $F(\cdot)$, $B(\cdot)$, $G(\cdot)$, $H(\cdot)$, $R(\cdot)$ are given matrices, and $R(t) > 0, \forall t \in [0,T]$. The cost functional is:

$$\bar{J}(u(\cdot)) \equiv \gamma \mathbb{E}\left\{\exp\left[\frac{\gamma}{2}x'(T)Mx(T) + \frac{\gamma}{2}\int_0^T \left[x'(t)Qx(t) + u'(t)Nu(t)\right]dt\right]\right\}, \tag{2.3.2}$$

where $\gamma \neq 0$, is given. $M(\cdot)$, $Q(\cdot)$ and $N(\cdot)$ are given matrices with $M(t) \geq 0$, $Q(t) \geq 0$, $N(t) > 0$, $\forall t \in [0,T]$. Finding optimal $u(\cdot)$ to minimize (2.3.2) when (2.3.1) holds.

Assumption 1. If there exists a $\nu \in (0,\infty)$ such that

$$\mathbb{E}\left[\exp\left(\nu \int_0^T \| u \|^2 \, dt\right)\right] < \infty. \tag{2.3.3}$$

We define the preliminary set of admissible controls as follows:

$$\mathbf{U}_1 = \{u \in \mathcal{L}_y(0,T;R^m) \mid \exists \nu \in (0,\infty)\} \tag{2.3.4}$$

such that (2.3.3) equation holds.

(Notation: for any $t \in T$, $\quad v_t = f(y(\cdot))$ depends only on y before time t.)

Assumption 2.

$$\mathbf{U}_2 = \{u \in \mathbf{U}_1 \mid J(u(\cdot)) < \infty)\} \tag{2.3.5}$$

it is called the class of admissible controls.

Definition 1. For any $u \in \mathbf{U}_2$ let us introduce the following variables:
$r : \Omega \times T \to R^n$

$$dr = [F - PH'R^{-1}H + \gamma PQ]rdt + Budt + PH'R^{-1}dy, \quad r_0 = \gamma. \tag{2.3.6}$$

P is the solution of a filter type Riccati differential equation

$$\dot{P} - FP - PF' + P(H'R^{-1}H - \gamma Q)P - GG' = O, P(0) = P_0, \tag{2.3.7}$$

$\pi^u : \Omega \times R^n \times T \to R$

$$\begin{aligned}\pi^u(x,t) \quad = \quad & \exp\Bigg(\frac{1}{2}(x-r)'P^{-1}(t)(x-r) \\ & + \int_0^T R^{-1}Hrdy - \frac{1}{2}\int_0^T r'H'R^{-1}Hrdt \\ & + \frac{\gamma}{2}\int_0^T (r'Qr + u'Nu)dt + \frac{\gamma}{2}\int_0^T tr(PQ)dt(2\pi)^{\frac{n}{2}} \mid P(t) \mid^{\frac{1}{2}}\Bigg).\end{aligned} \tag{2.3.8}$$

$$K(u(\cdot)) = \mathbb{E}\left[\gamma \int \exp\left(\frac{\gamma}{2} x' M x \pi^u(x,t) dx\right)\right]. \tag{2.3.9}$$

$\Sigma : T \to R^{n\times n}$

$$\dot{\Sigma} - \Sigma GG'\Sigma + F\Sigma + \Sigma F' - \gamma Q + H'R^{-1} = 0, \quad \Sigma(t) = -\gamma M. \tag{2.3.10}$$

Assumption 3.

(1) $H'R^{-1}H - \gamma Q \geq 0$;

(2) $P(t) \geq c_1 I$ for some $c_1 \in (0, \infty)$ and for all t;

(3) $P^{-1}(t) + \Sigma(t) > 0$ for all t.

Theorem 2.3.1. *Assume that Assumption 3 holds and the Riccati equation (2.3.10) has a symmetric bounded solution. For any control u in the class of admissible controls $\boldsymbol{U}_2$ one has the equality:*

$$J(u(\cdot)) = K(u(\cdot)) \tag{2.3.11}$$

where $J(u(\cdot))$ is defined by (2.3.2) and $K(u(\cdot))$ by (2.3.9).

Minimizing $K(u(\cdot))$ by the state equation of dr is a fully observed stochastic control problem. Using this method, the partial observation problem is easily solved.

Theorem 2.3.2. *Consisting the stochastic state system (2.3.1) and the cost functional (2.3.2), the optimal control is given by:*

$$u = -N^{-1}(t)B'(t)S(t)r, \tag{2.3.12}$$

under certain definite conditions. Here S is the solution of a control type Riccati differential equation:

$$\begin{cases} \dot{S} + S(F + \gamma PQ) + (F' + \gamma QP)S + Q - S(BN^{-1}B' - \gamma PH'R^{-1}HP)S = 0, \\ \\ S(T) = \frac{1}{2}[(I + \gamma MP(T))^{-1}M + M(I + \gamma P(T)M)^{-1}]. \end{cases}$$

2.4 Generalized risk-sensitive control

In Date and Gashi's research, they analized the optimal control problems for both of the full and partial state observations, and generalised the risk-sensitive cost

functional. The problem is as follows:
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, given a Brownian motion term $W(t) = [w_1'(t), w_2'(t)]$, with order p and q. Considering the same state system as Bensoussan and Van Schuppen:

$$\begin{cases} dx(t) = [Ax(t) + Bu(t)]dt + Cdw_1(t), \\ \\ dy(t) = Hx(t)dt + F^{\frac{1}{2}}dw_2(t), \\ \\ x(0) = x_0, \\ \\ y(0) = 0, \end{cases} \tag{2.4.1}$$

Introduce noise dependent penalties, a generalization of this cost functional becomes:

$$\begin{aligned} J(u(\cdot)) &= \gamma\mathbb{E}\left\{\exp\left[\frac{\gamma}{2}x'(T)Sx(T) + \frac{\gamma}{2}\int_0^T [x'(t)Qx(t) + u'(t)Ru(t)]dt\right.\right. \\ &+ \left.\left.\frac{\gamma}{2}\int_0^T [x'(t)Q_1 + u'(t)R_1]dW(t)\right]\right\}, \end{aligned}$$

for some constant vectors Q_1 and R_1 of proper dimensions.

Assumption 4. Coefficients R and R_1 are such that

$$R + \frac{\gamma}{4}R_1R_1' > 0 \tag{2.4.2}$$

Definition 2. Introduce the following matrices:

$$\begin{aligned} \bar{A} &\equiv A + \frac{\gamma}{2}CQ_1' - \left(B + \frac{\gamma}{2}CR_1\right)\left[R + \frac{\gamma}{4}R_1R_1'\right]^{-1}\frac{\gamma}{4}R_1Q_1', \\ \bar{B} &\equiv B + \frac{\gamma}{2}CR_1', \\ \bar{R} &\equiv \left[R + \frac{\gamma}{4}R_1R_1'\right]^{-1}. \end{aligned}$$

Assumption 5. For the full observation case, they used two methods: completion of squares and changing of measure to find the solution (see [17]). There is a unique solution to the following Riccati equation:

$$\begin{cases} Q + \dot{P} + PA + A'P + \dfrac{\gamma}{4}\Big(2PC + Q_1\Big)\Big(2PC + Q_1\Big)' - \Bigg[\gamma PB \\ \\ + \dfrac{\gamma^2}{4}\Big(2PC + Q_1\Big)R_1'\Bigg]\Big(R + \dfrac{\gamma}{4}R_1R_1'\Big)^{-1}\Bigg[\gamma PB + \dfrac{\gamma^2}{4}\Big(2PC + Q_1\Big)R_1'\Bigg]' = 0, \\ \\ P(T) = S. \end{cases}$$

Theorem 2.4.1. *Let the Assumptions 4 and 5 hold. There then exists a unique solution to the problem. And the optimal control and the corresponding optimal cost are:*

$$\begin{aligned} u^* = & \quad -\bar{R}^{-1}\left(\bar{B}'P + \frac{\gamma}{4}R_1Q_1'\right)x(t), \\ J^* = & \quad \gamma\mathbb{E}\left[\frac{\gamma}{2}p(0) + x(0)P(0)x(0)\right]. \end{aligned}$$

respectively. Here, $p(t)$ *is the solution to the following ordinary differential equation:*

$$\begin{cases} \dot{p}(t) + tr[C'P(t)C] = 0, \\ \\ p(T) = 0. \end{cases}$$

For the partially observed case, they applied Bensoussan and Van Schuppen's results and transferred the partial case to full case. The following are the results for the generalized partial observed risk-sensitive control:

Assumption 6. There must be at least one differentiable symmetric solution $G : [0, T] \to \mathcal{R}^{n\times n}$ to the equation:

$$2GB + \frac{\gamma}{2}(Q_1 + \widehat{G})R_1' = 0, \quad a.e.t \in [0, T], \tag{2.4.3}$$

where $\widehat{G} \equiv [2GC, 0_{n\times p}]$.

Definition 3. Introduce the following matrices:

$$\widehat{S} \quad \equiv \quad S - G(T),$$

$$\widehat{Q} \equiv Q + \dot{G} + GA + A'G + \frac{\gamma}{4}(Q_1 + \widehat{G})(Q_1 + \widehat{G})',$$

$$\widehat{R} \equiv R + \frac{\gamma}{4} R_1 R_1'.$$

Assumption 7. The matrices R and R_1 are such that $\widehat{R} > 0$.

In the following matrix partitions, matrices M1 and N1 are of dimension $q \times n$ and $q \times m$, respectively:

$$\frac{\gamma}{2}(Q_1' + \widehat{G}') = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}, \qquad \frac{\gamma}{2} R_1' = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}.$$

Definition 4. Introduce the following variables:

$$\widehat{A} \equiv A + CM_1,$$

$$\widehat{B} \equiv B + CN_1,$$

$$\widehat{H} \equiv H + F^{\frac{1}{2}} M_2.$$

and introduce a variable $r(t)$ satisfy:

$$\begin{cases} dr(t) = \Big[\Big(\widehat{A} - \widehat{P}\widehat{H}'\widehat{F}^{-1}\widehat{H} + \gamma\widehat{P}\widehat{Q}\Big)\widehat{r}(t) + \Big(\widehat{B} - \widehat{P}\widehat{H}\widehat{F}^{\frac{1}{2}}N_2\Big)u(t)\Big]dt \\ \qquad\qquad + \widehat{P}\widehat{H}'\widehat{F}^{-1}dy(t), \\ \widehat{r}(0) = \mu(0). \end{cases}$$

Assumption 8. The following Riccati equations have unique global solutions respectively:

$$\begin{cases} \dot{\widehat{P}} - \widehat{A}\widehat{P} - \widehat{P}\widehat{A}' + \widehat{P}(\widehat{H}'\widehat{F}^{-1}\widehat{H} - \gamma\widehat{Q})\widehat{P} - CC' = 0, \\ \widehat{P}(0) = P_0 \end{cases}$$

$$\begin{cases} \dot{\widehat{U}}+(\widehat{A}'+\gamma\widehat{Q}\widehat{P})\widehat{U}+\widehat{U}(\widehat{A}+\gamma\widehat{P}\widehat{Q})-\widehat{U}(\widehat{B}\widehat{R}^{-1}\widehat{B}'-\gamma\widehat{P}\widehat{H}'\widehat{F}^{-1}\widehat{H}\widehat{P})\widehat{U}+\widehat{Q}=0, \\ \\ \widehat{U}(T)=\dfrac{1}{2}\Big[(I-\gamma\widehat{S}\widehat{P}(T))^{-1}\widehat{S}+\widehat{S}(I-\gamma\widehat{P}(T)\widehat{S})^{-1}\Big]. \end{cases}$$

Theorem 2.4.2. *Let the Assumptions 6, 7 and 8 hold. There exist a unique solution to the partial observation problem as follows:*

$$u^*(t)=\widehat{R}^{-1}\widehat{B}'\widehat{U}r(t), \tag{2.4.4}$$

the corresponding cost function is:

$$J^*=\gamma\exp\left[\frac{\gamma}{2}\mu_0'\widehat{U}(0)\mu_0+\frac{\gamma}{2}\int_0^T tr(\widehat{P}\widehat{Q}+\widehat{U}\widehat{P}\widehat{H}'\widehat{F}^{-1}\widehat{H}\widehat{P})dt\right]|[I+\gamma\widehat{S}\widehat{P}(T)]|^{-\frac{1}{2}}. \tag{2.4.5}$$

The novelty here is that a noise dependent penalty has been introduced. An important feature of this generalised risk-sensitive control problem is that it admits an explicit closed-form solution in terms of a certain Riccati type equation. It has been proved by the theorem of Bensoussan and Van Schuppen that both the full and partial state observation problems can be transformed into standard risk-sensitive control problems, the problem can then be solved easily.

2.5 Indefinite control

In 2001, Rami, Moore and Zhou [63] identified the generalised(differential) Riccati equation. They derived the generalized form of the optimal controls via the Riccati equation solutions.

Let us state some useful known results on the Moore-Penrose pseudoinverse and matrix equations (see [61]):

Lemma 2.5.1. *(a) If $M\in\mathbb{R}^{m\times n}$, then there exists a unique $M^\dagger\in\mathbb{R}^{m\times n}$, called the Moore-Penrose pseudoinverse, such that*

$$(i)\qquad MM^\dagger M=M,\quad M^\dagger MM^\dagger=M^\dagger,$$

$$(ii)\qquad (MM^\dagger)'=MM^\dagger,\quad (M^\dagger M)'=M^\dagger M,$$

$$(iii) \qquad \textit{if } M \textit{ is symmetric, then } MM^\dagger = M^\dagger M.$$

(b) If L, M, and N are given matrices, then the matrix equation

$$LXM = N, \tag{2.5.1}$$

has a solution X if and only if

$$LL^\dagger NM^\dagger M = N.$$

In this case, any solution to (2.5.1) can be represented as

$$X = L^\dagger NM^\dagger + S - L^\dagger LSMM^\dagger,$$

for some S of proper dimensions.

Recalling the linear-quadratic control problem of [63], consider the linear stochastic control system:

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t)]dt + [C(t)x(t) + D(t)u(t)]dW(t), \\ \\ x(s) = y. \end{cases} \tag{2.5.2}$$

where $(s, y) \in [0, T) \times \mathbf{R}^n$ are the initial time and initial state. The Brownian motion is one-dimensional. The cost function is given as

$$J(u(\cdot)) = \mathbb{E}\left\{\int_s^T [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt + x'(T)Hx(T)\right\}. \tag{2.5.3}$$

The optimal control problem is minimizing the cost functional $J(s, y; u(\cdot))$, for a given $(s, y) \in [0, T) \times \mathbf{R}^n$, over all $u(\cdot) \in U_{ad}$.

Rami et.al. introduce the generalized (differential) Riccati equation(GRE)as follows:

$$\begin{cases} \dot{P} + PA + A'P - (PB + C'PD)(R + D'PD)^{\dagger}(B'P + D'PC) + Q = 0 \\ (R + D'PD)(R + D'PD)^{\dagger}(B'P + D'PC) - (B'P + D'PC) = 0, \\ (R + D'PD) \geq 0, \qquad a.e.t \in [0, T] \\ P(T) = H. \end{cases} \tag{2.5.4}$$

Theorem 2.5.2. *The set of all the optimal controls is as follows:*

$$\begin{aligned} u(Y, z)(t) \quad = \quad & -\Big\{[R + D'PD]^{\dagger}[B'P + D'PC] + Y(t) \\ & -[R + D'PD]^{\dagger}[R + D'PD]Y(t)\Big\} x(t) \\ & +z(t) - [R + D'PD]^{\dagger}[R + D'PD]z(t). \end{aligned} \tag{2.5.5}$$

where $Y(\cdot) \in L^2_{\mathcal{F}}(s, T; R^{n_u \times n})$ and $z(\cdot) \in L^2_{\mathcal{F}}(s, T; R^{n_u})$. The optimal value is:

$$V(s, y) \equiv \inf_{u(\cdot) \in U_{ad}} J(s, y; u(\cdot)) = y'P(s)y. \tag{2.5.6}$$

They also proved that if (2.5.4) admits a solution, it must be in the linear feedback control form, with $Y(t) \equiv 0$ and $z(t) \equiv 0$ in (2.5.5), which means, the solvability of the GRE is sufficient and necessary for the existence of the optimal control for the linear-quadratic problem.

Theorem 2.5.3. *Assume there exists $P(\cdot)$ such that the following equations hold*

$$\begin{cases} \dot{P} + PA + A'P - (PB + C'PD)(R + D'PD)^{\dagger}(B'P + D'PC) + Q = 0 \\ (R + D'PD)(R + D'PD)^{\dagger}(B'P + D'PC) - (B'P + D'PC) = 0, \quad a.e.t \in [s, T] \\ P(T) = H, \end{cases}$$

then P must satisfy

$$(R + D'PD) \geq 0, \qquad a.e.t \in [0, T].$$

The detailed proof is given in [63].

2.6 Robust control

(1) Definition of generalized H_2/H_∞ control:

A generalised H_2/H_∞ problem, sometimes called "mixed H_2/H_∞ problem", is defined as "finding a controller that minimizes an upper bound in the worst case overshoot of a controlled output in response to arbitrary but bounded energy exogenous inputs, subject to an inequality constraint on the H_∞ norm of another closed loop transfer function" by Rotea [64]. A typical control problem that combines H_2 and H_∞-design objectives is described in Figure 2.1:

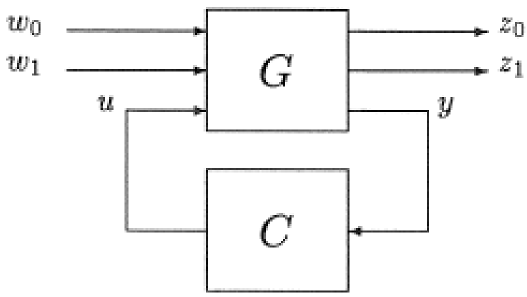

Figure 2.1: A mixed H_2/H_∞ configuration

Here G is the plant and C denotes the controller. w_0 and w_1 are exogenous input vectors, and z_0, z_1 denote the output vectors. u is the input and y is the measured output. G is a state-feedback model defined as follows:

$$G = \begin{cases} \dot{x} = Ax + B_1 w + B_2 u, \\ \\ z_i = Cx + Du, \qquad (i = 0, 1) \\ \\ y = x. \end{cases}$$

All matrices in the above equation are real, with proper dimensions.
Given γ, we wish to minimize the following functions:

$$J = \sup_{w_1 \in \mathcal{P}} \{\|z\|_p^2 - \gamma^2 \|w_1\|_p^2\}$$

here $\|\cdot\|_p$ is the power semi-norm (see[20]).

$$\|z\|_p^2 = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} z'(t)z(t)dt. \tag{2.6.1}$$

(2) The Necessary and Sufficient Conditions For the Existence of Linear Controls:

Theorem 2.6.1. *Given the system:*

$$\begin{cases} dx(t) = [A(t)x(t) + B_1(t)w(t) + B_2(t)u(t)]dt, \\ \\ z(t) = \begin{bmatrix} C(t)x(t) \\ D(t)u(t) \end{bmatrix}, \\ \\ D'D = I, \end{cases} \tag{2.6.2}$$

there exist Nash equilibrium strategies

$$u^*(t,x) \in \Omega$$

$$w^*(t,x) \in \Omega$$

such that

$$J_1(u^*, w^*) \leq J_1(u^*, w) \forall w(t) \in \Omega$$

$$J_2(u^*, w^*) \leq J_1(u, w^*) \forall u(t) \in \Omega$$

where and

$$J_1(u,w) = \int_0^T (\gamma^2 w'(t)w(t) - z'(t)z(t))dt. \tag{2.6.3}$$

and

$$J_2(u,w) = \int_0^T z'(t)z(t)dt, \tag{2.6.4}$$

if and only if the coupled Riccati differential equations

$$\begin{cases} -\dot{P}_1(t) = A'P_1(t) + P_1'(t)A - C'C \\ \qquad\qquad - \begin{bmatrix} P_1(t) & P_2(t) \end{bmatrix} \begin{bmatrix} \gamma^{-2}B_1B_1' & B_2B_2' \\ B_2B_2' & B_2B_2' \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix}, \\ P_1(T) = 0. \end{cases} \tag{2.6.5}$$

$$\begin{cases} -\dot{P}_2(t) = A'P_2(t) + P_2'(t)A + C'C \\ \qquad\qquad - \begin{bmatrix} P_1(t) & P_2(t) \end{bmatrix} \begin{bmatrix} 0 & \gamma^{-2}B_1B_1' \\ \gamma^{-2}B_1B_1' & B_2B_2' \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix}, \\ P_2(T) = 0. \end{cases} \tag{2.6.6}$$

have solutions $P_l(t) \leq 0$ *and* $P_2(t) \leq 0$ *on* $[0, T]$

The detailed proof is given in [52].

2.7 Riccati equation

In solving optimal control problems we encounter the Riccati differential and algebraic equations. In this section, we present some results that show the existence and uniqueness of solutions to above equations. One version of the Riccati differential equation which appears in the derivation of the solution to the stochastic linear-quadratic regulator with state and control dependent noise (see e. g. Wonham [80]), is as follows:

$$\begin{cases} \dot{P}(t) + A(t)'P(t) + P(t)A(t) + \Pi[t, P(t)] \\ \qquad - P(t)B(t)N^{-1}(t)B'(t)P(t) + C'(t)C(t) = 0, \quad t_0 \leq t \leq T, \\ P(T) = P_T \geq 0. \end{cases} \tag{2.7.1}$$

Here $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{p\times n}$, $N(t) \in \mathbb{R}^{m\times m}$, $N(t) > 0$ are given continuous functions of time. $P(t) \in \mathbb{R}^{n\times n}$, and Π is a positive linear map into itself of symmetric $n \times n$ matrices. The algebraic form of (2.7.1), which appears in the derivation of the solution to the infinite horizon, also see e. g. Wonham [80], is given as

$$A'P - PA + \Pi(P) - PBR^{-1}BP + C'C = 0. \tag{2.7.2}$$

Here A, B,C, R are all constants. The following are the main results that show the existence and uniqueness of the solution to the Riccati equation:

Theorem 2.7.1. *Matrix $P(t)$ has the following properties:*
(i)$P(t)$ is absolutely continuous on $[t_0, T]$ and satisfies (2.7.1) almost everywhere.
(ii)$P(t) \geq 0, \quad t_0 \leq t \leq T$, and $P(t)$ is the unique solution of (2.7.1).

Theorem 2.7.2. *If*

$$\inf_K \left| \int_0^\infty e^{t(A-BK)'} \Pi(I) e^{t(A-BK)} dt \right| < 1.$$

then (2.7.2) has at least one solution.

The proof is given in Wonham [80]. The relationship between the solution of the Riccati equation and the optimal control problem can be found in Anderson and Moore's book [1].

2.8 Summary

In this chapter we have presented some fundamental theorems and important results of stochastic optimal control, and delay systems, to which we refer later in the thesis. We have also highlighted the results that will be used in the later chapters.

Chapter 3

Indefinite risk-sensitive control with fully observed system

3.1 Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), \mathbb{P})$ be a complete probability space on which a d-dimensional standard Brownian motion $(W(t), t \geq 0)$ is defined. We assume that $\mathcal{F}(t)$ is the augmentation of $\sigma\{W(s)|0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. Consider the linear stochastic control system:

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t)]dt + C(t)dW(t), \\ \\ x(0) = x_0 \in \mathbb{R}^n, \quad \text{is given.} \end{cases} \tag{3.1.1}$$

We assume that $A(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, $B(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m})$, $C(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times d})$, where $L^\infty(0,T;\mathbb{R}^{n\times n})$ denotes the set of all $\mathbb{R}^{n\times n}$-valued uniformly bounded functions. We further assume that the $\mathcal{F}(t)$-adapted control process $u(\cdot)$ is such that (3.1.1) has a unique strong solution.

The *exponential-quadratic* criterion is defined as:

$$I(u(\cdot)) := \gamma\mathbb{E}\left\{\exp\left[\frac{\gamma}{2}\int_0^T [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt + \frac{\gamma}{2}x'(T)Sx(T)\right]\right\}, \tag{3.1.2}$$

where $\gamma \in \mathbb{R}$. The weighting matrices in (3.1.2) satisfy the following *definiteness* properties:

$$\begin{cases} 0 \leq Q(\cdot) \in L^{\infty}(0,T;\mathbb{R}^{n\times n}), \\ 0 < R(\cdot) \in L^{\infty}(0,T;\mathbb{R}^{m\times m}), \\ 0 \leq S \in \mathbb{R}^{n\times n}. \end{cases} \tag{3.1.3}$$

The *risk-sensitive* control problem is defined as:

$$\begin{cases} \min_{u(\cdot)\in\mathcal{A}} I(u(\cdot)), \\ \text{s.t. } (3.1.1), \end{cases}$$

for some suitably defined *admissible* set $\mathcal{A}$. This problem was introduced and solved by Jacobson [34]. The solution turns out to be unique and of a linear state-feedback form, and with a great similarity with the linear-quadratic (LQ) regulator of deterministic control [1]. A feature unique to the risk-sensitive optimal control law is that it depends on the noise intensity $C(\cdot)$, which is *not* the case with the LQ control of (3.1.1).

After this pioneering work, the partial observation problem was considered by [38], [39], [68], [69], and complete solution obtained in [5]. The discrete-time partial observation problem was solved by Whittle in [78] (see also [79]). For infinite horizon criterion in a Markovian setting, the reader can consult [4], [10], [11]. An important relation with robust controllers was found in [26], [27], whereas the risk-sensitive maximum principle was studied in [48], [49], [31], [66]. A more general version of linear exponential quadratic control, where $x(t)$ evolves in an infinite dimensional Hilbert space is discussed in [22]. The optimal investment problem is particularly suitable for the application of risk-sensitive control; see for example [8], [18], [29].

In two recent papers [16], [17], Date and Gashi generalised the cost functional (3.1.2) by introducing *noise dependent* penalties on the state and control variables. They consider the cost functional

$$J(u(\cdot)) \quad := \quad \gamma\mathbb{E}\left\{\exp\left[\frac{\gamma}{2}\int_0^T \Big[x'(t)Q(t)x(t) + u'(t)R(t)u(t)\Big]dt + \frac{\gamma}{2}x'(T)Sx(T)\right.\right.$$

$$+ \quad \frac{\gamma}{2}\int_0^T \Big[x'(t)Q_1(t) + u'(t)R_1(t)\Big]dW(t)\Big]\Big\}, \tag{3.1.4}$$

and the optimal control problem:

$$\begin{cases} \min\limits_{u(\cdot)\in\mathcal{A}} J(u(\cdot)), \\ \\ \text{s.t. } (3.1.1), \end{cases} \tag{3.1.5}$$

where $Q_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times p})$ and $R_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{m\times p})$. The motivations for considering this kind of a criterion are: it preserves the explicit closed form solvability as a linear state-feedback control law; it appears naturally when dealing with the cost functional that has a cross product term between the state $x(t)$ and the control $u(t)$ under the assumption of partial observation; and it also has an application in optimal investment. One of the assumptions of [16], [17], is that

$$R(t) + \frac{\gamma}{4}R_1(t)R_1'(t) > 0, \quad a.e. \quad t \in [0,T]. \tag{3.1.6}$$

This in particular means that it is no longer necessary for $R(t) > 0$ as in (3.1.3), and it can also be *indefinite*. The cost functionals with indefinite cost matrices also appear in stochastic linear-quadratic control with *multiplicative* noise and its application to optimal investment with a mean-variance criterion (see, for example, [63], [62], [14], [15], [46], [47], [45]).

In this chapter, we weaken condition (3.1.6) even further by assuming

$$R(t) + \frac{\gamma}{4}R_1(t)R_1'(t) \geq 0, \quad a.e. \quad t \in [0,T]. \tag{3.1.7}$$

A consequence of this is that we no longer have a *unique* and *linear* optimal control law, but rather a *parametrised* family of *affine* state feedback laws. This feature is shared with the indefinite LQ control of [63]. Moreover, the derivation of the solution is more involved as compared to [16], [17], which is reflected by less explicit assumptions. In section 3.2, we find all solutions to the optimal control problem (3.1.5) under assumption (3.1.7). As an application, we find the solution to an optimal investment problem with a stochastic interest rate in section 3.4.

As our second contribution, we introduce an *infinite horizon* cost functional:

$$\begin{aligned} J_\infty(u(\cdot)) : \quad &= \lim_{T\to\infty} \frac{\gamma}{f(T)} \log \mathbb{E}\left\{ \exp\left[\frac{\gamma}{2}\int_0^T \Big[x'(t)Qx(t) + u'(t)Ru(t)\Big]dt \right.\right. \\ &\left.\left. + \frac{\gamma}{2}\int_0^T \Big[x'(t)Q_1 + u'(t)R_1\Big]dW(t)\right]\right\}, \end{aligned} \tag{3.1.8}$$

where $f(T)$ is some given positive function. This is clearly the infinite horizon version of (3.1.4) of an *average* type. A new feature of this criterion is the function $f(T)$, which is not necessarily equal to T. This enables the solution of the corresponding optimal control problem under weaker assumptions with regards to the stability of the system. In section 3.3, we find all solutions to the optimal control problem with criterion $J_\infty(u(\cdot))$. We emphasize that in [16], [17] only the *finite* horizon risk-sensitive control problems are considered.

3.2 Finite horizon

Here we are interested in finding *all* solutions to the risk-sensitive control problem (3.1.5), under some weaker assumptions as compared to [16], [17]. As already mentioned, the following is one of our main assumptions (throughout this section we suppress the argument t where appropriate for notational simplicity).

The following *Riccati* differential equation appears naturally in the proof of Theorem 3.2.1:

$$\begin{cases} \dot{P} + PA + A'P + \frac{\gamma}{4}(2PC+Q_1)(2PC+Q_1)' + Q \\ -\Big[PB + \frac{\gamma}{4}(2PC+Q_1)R_1'\Big]\Big(R + \frac{\gamma}{4}R_1R_1'\Big)^\dagger\Big[PB + \frac{\gamma}{4}(2PC+Q_1)R_1'\Big]' = 0, \\ \Big(R + \frac{\gamma}{4}R_1R_1'\Big)\Big(R + \frac{\gamma}{4}R_1R_1'\Big)^\dagger\Big[PB + \frac{\gamma}{4}\Big(2PC+Q_1\Big)R_1'\Big]' \\ \qquad -\Big[PB + \frac{\gamma}{4}\Big(2PC+Q_1\Big)R_1'\Big]' = 0, \\ P(T) = S. \end{cases} \tag{3.2.1}$$

The Riccati differential equation (3.2.1) has a unique solution, which has been proved in [1], and we also give an example in Appendix of the thesis.

We now focus in defining the appropriate *admissible* set of controls $\mathcal{A}$. Let $\mathcal{U}$ denote the set of all $\mathcal{F}(t)$-adapted processes $u(t)$ such that the state equation (3.1.1) has a unique strong solution. For each $u(\cdot) \in \mathcal{U}$ we define:

$$\begin{aligned}
\theta'_u(t) &:= -\frac{\gamma}{2}[2x'(t)PC + x'(t)Q_1 + u'(t)R_1], \\
Z_u(t) &:= \exp\left[-\int_0^t \theta'_u(\tau)dW(\tau) - \frac{1}{2}\int_0^t \theta'_u(\tau)\theta_u(\tau)d\tau\right], \\
Z_u &:= Z_u(T), \\
\widetilde{\mathbb{P}}_u(\alpha) &:= \int_\alpha Z_u(\omega)d\widetilde{\mathbb{P}}(\omega), \qquad \forall \alpha \in \mathcal{F}.
\end{aligned}$$

In order to ensure that $\widetilde{\mathbb{P}}_u$ is a probability measure, we assume that $\theta_u(t)$ satisfies the *Novikov condition*, i.e. for some positive β the following holds:

$$\mathbb{E}\left[e^{(\beta/2)\int_0^T \theta'_u(\tau)\theta_u(\tau)d\tau}\right] < \infty. \tag{3.2.2}$$

We can now define the admissible set of controls as:

$$\mathcal{A} := \{u(\cdot) \in \mathcal{U} \quad \text{such that (3.2.2) holds}\}.$$

As it will become clear from the proof of Theorem 3.2.1, for any $u(\cdot) \in \mathcal{A}$ we have $J(u(\cdot)) < \infty$. The assumption of (3.2.2) appears to be stronger than the assumption of the finiteness of $J(u(\cdot))$, but it is required by our method of solution. Note that for any $u(\cdot) \in \mathcal{A}$ the probability measures $\widetilde{\mathbb{P}}_u$ and $\mathbb{P}$ are *equivalent*, which in particular means that if X is an $\mathcal{F}(T)$-measurable random variable, then:

$$\mathbb{E}[ZX] = \widetilde{\mathbb{E}}_u[X], \tag{3.2.3}$$

Here $\widetilde{\mathbb{E}}_u$ denotes the expectation under $\widetilde{\mathbb{P}}_u$.

Let $Y(\cdot)$ be an $\mathbb{R}^{m\times n}$-valued $\mathcal{F}(t)$-adapted process, $z(\cdot)$ an $\mathbb{R}^m$-valued $\mathcal{F}(t)$-adapted process, and define:

$$K_Y(t) \quad := \quad -(R + \frac{\gamma}{4}R_1R'_1)^\dagger[PB + \frac{\gamma}{4}(2PC + Q_1)R'_1]'$$

$$+ \quad Y(t) - (R + \frac{\gamma}{4}R_1R_1')^{\dagger}(R + \frac{\gamma}{4}R_1R_1')Y(t),$$

$$K_z(t) \quad := \quad z(t) - (R + \frac{\gamma}{4}R_1R_1')^{\dagger}(R + \frac{\gamma}{4}R_1R_1')z(t).$$

We confine the processes $Y(\cdot)$ and $z(\cdot)$ to the following set

$$\mathcal{K} := \{(Y(\cdot), z(\cdot)) : \quad K_Y(\cdot)x(\cdot) + K_z(\cdot) \in \mathcal{A}\}.$$

In other words, the processes $Y(\cdot)$ and $z(\cdot)$ must be such that the control $u_K(t) := K_Y(t)x(t) + K_z(t)$ is admissible.

Theorem 3.2.1. *All solutions to (3.1.5) are given by:*

$$u^*(t) = K_Y(t)x(t) + K_z(t), \tag{3.2.4}$$

with $(Y(\cdot), z(\cdot)) \in \mathcal{K}$. *The optimal cost is:*

$$J^* := J(u^*(\cdot)) = \gamma \exp\left[\frac{\gamma}{2}x'(0)P(0)x(0) + \frac{\gamma}{2}\int_0^T tr(C'PC)dt\right]. \tag{3.2.5}$$

Proof. The proof is a combination of a certain completion of squares and change of measure methods, and the approach of [63]. The differential of the quadratic form $x'(t)P(t)x(t)$ is:

$$d[x'(t)Px(t)] = \{x'(t)\dot{P}x(t) + 2x'(t)P[Ax(t) + Bu(t)] + tr(C'PC)\Big\}dt + 2x'(t)PCdW(t).$$

Integrating both sides from 0 and T, and rearranging the resulting expression, gives:

$$\begin{aligned} 0 \quad &= \quad -x'(T)Sx(T) + x'(0)P(0)x(0) + \int_0^T 2x'(t)PCdW \\ &+ \int_0^T \{x'(t)\dot{P}x(t) + 2x'(t)P[Ax(t) + Bu(t)] + tr(C'PC)\}dt. \end{aligned}$$

The cost functional $J(u(\cdot))$ can now be written as:

$$\begin{aligned} J(u(\cdot)) \quad = \quad & \gamma\mathbb{E}\Bigg\{\exp\left[\frac{\gamma}{2}x'(0)P(0)x(0) + \frac{\gamma}{2}\int_0^T [x'(t)(Q + \dot{P})x(t) + u'(t)Ru(t)]dt\right. \\ & + \frac{\gamma}{2}\int_0^T \{2x'(t)P[Ax(t) + Bu(t)] + tr(C'PC)\}dt \end{aligned}$$

$$+\ \frac{\gamma}{2}\int_0^T [2x'(t)PC + x'(t)Q_1 + u'(t)R_1]dW(t)\Big]\Big\}$$

Due to (3.2.3), for any $u(\cdot) \in \mathcal{A}$, the above expression becomes:

$$\begin{aligned}
J(u(\cdot)) \ &= \ \gamma\widetilde{\mathbb{E}}_u\Big\{\exp\Big[\frac{\gamma}{2}x'(0)P(0)x(0) + \frac{\gamma}{2}\int_0^T tr(C'PC)dt \\
&+\frac{\gamma}{2}\int_0^T x'(t)[Q + \dot{P} + PA + A'P + \frac{\gamma}{4}(2PC+Q_1)(2PC+Q_1)']x(t)dt \\
&\frac{\gamma}{2}\int_0^T \{u'(t)(R+\frac{\gamma}{4}R_1R_1')u(t) + 2x'(t)[PB + \frac{\gamma}{4}(2PC+Q_1)R_1']u(t)\}dt\Big]\Big\}
\end{aligned}$$

For any $(Y(\cdot), z(\cdot)) \in \mathcal{K}$, let us introduce the processes:

$$\begin{aligned}
L_1(t) \ &:= \ Y(t) - (R+\frac{\gamma}{4}R_1R_1')^{\dagger}(R+\frac{\gamma}{4}R_1R_1')Y(t), \\
L_2(t) \ &:= \ z(t) - (R+\frac{\gamma}{4}R_1R_1')^{\dagger}(R+\frac{\gamma}{4}R_1R_1')z(t),
\end{aligned}$$

which, due to the properties of the Moore-Penrose pseudoinverse (see Lemma 2.5.1 (a)), have the property

$$(R+\frac{\gamma}{4}R_1R_1')L_i(t) = (R+\frac{\gamma}{4}R_1R_1')^{\dagger}L_i(t) = 0, \qquad i = 1, 2. \tag{3.2.6}$$

Using this property, as well as other properties of the Moore-Penrose pseudoinverse given in Lemma 2.5.1 (a), we can write $J(u(\cdot))$ as:

$$\begin{aligned}
J(u(\cdot)) \ &= \ \gamma\widetilde{\mathbb{E}}_u\Big\{\exp\Big[\frac{\gamma}{2}x'(0)P(0)x(0) + \frac{\gamma}{2}\int_0^T tr(C'PC)dt \\
&+\frac{\gamma}{2}\int_0^T x'(t)\{\dot{P} + Q + PA + A'P + \frac{\gamma}{4}(2PC+Q_1)(2PC+Q_1)' \\
&-[PB + \frac{\gamma}{4}(2PC+Q_1)R_1']'(R+\frac{\gamma}{4}R_1R_1')^{\dagger}[PB+\frac{\gamma}{4}(2PC+Q_1)R_1']\}x(t)dt \\
&+\frac{\gamma}{2}\int_0^T [u(t) + ((R+\frac{\gamma}{4}R_1R_1')^{\dagger}[PB+\frac{\gamma}{4}(2PC+Q_1)R_1'] + L_1)x(t) + L_2]' \\
&\times(R+\frac{\gamma}{4}R_1R_1')[u(t) + ((R+\frac{\gamma}{4}R_1R_1')^{\dagger}[PB+\frac{\gamma}{4}(2PC+Q_1)R_1'] + L_1)x(t) + L_2]'dt\Big]\Big\}.
\end{aligned}$$

Due to the Riccati differential equation 3.2.1, the term containing $x(t)$ is zero. This simplifies the cost functional into:

$$\begin{aligned} J(u(\cdot)) &= \gamma\widetilde{\mathbb{E}}_u\left\{\exp\left[\frac{\gamma}{2}x'(0)P(0)x(0)+\frac{\gamma}{2}\int_0^T tr(C'PC)dt\right.\right. \\ &+\frac{\gamma}{2}\int_0^T [u(t)+((R+\frac{\gamma}{4}R_1R_1')^\dagger[PB+\frac{\gamma}{4}(2PC+Q_1)R_1']+L_1)x(t)+L_2]' \\ &\left.\left.\times(R+\frac{\gamma}{4}R_1R_1')[u(t)+((R+\frac{\gamma}{4}R_1R_1')^\dagger[PB+\frac{\gamma}{4}(2PC+Q_1)R_1']+L_1)x(t)+L_2]'dt\right]\right\}. \end{aligned} \tag{3.2.7}$$

Since $(R+\frac{\gamma}{4}R_1R_1')\geq 0$, for all $u(\cdot)\in\mathcal{A}$, the following inequality holds:

$$J(u(\cdot))\geq\gamma\exp\left[\frac{\gamma}{2}x'(0)P(0)x(0)+\frac{\gamma}{2}\int_0^T tr(C'PC)dt\right].$$

This lower bound is achieved if:

$$u(t)=-((R+\frac{\gamma}{4}R_1R_1')^\dagger[PB+\frac{\gamma}{4}(2PC+Q_1)R_1']+L_1)x(t)+L_2,$$

which becomes (3.2.4) after substituting the expressions for $L_1(t)$ and $L_2(t)$.

We now focus in proving that *any* admissible optimal control must be of the form (3.2.4). Let $u(\cdot)\in\mathcal{A}$ be any *optimal* control. From (3.2.8) it follows that it is necessary to have

$$(R+\frac{\gamma}{4}R_1R_1')^{\frac{1}{2}}[u(t)+((R+\frac{\gamma}{4}R_1R_1')^\dagger(PB+\frac{\gamma}{4}(2PC+Q_1)R_1')'+L_1)x(t)+L_2]=0,$$

which after multiplication from the right by $(R+\gamma R_1R_1'/4)^{\frac{1}{2}}$ becomes

$$(R+\frac{\gamma}{4}R_1R_1')[u(t)+((R+\frac{\gamma}{4}R_1R_1')^\dagger(PB+\frac{\gamma}{4}(2PC+Q_1)R_1')'+L_1)x(t)+L_2]=0.$$

Due to (3.2.6), this equation can be written as

$$(R+\frac{\gamma}{4}R_1R_1')u(t)+(R+\frac{\gamma}{4}R_1R_1')(R+\frac{\gamma}{4}R_1R_1')^\dagger(PB+\frac{\gamma}{4}(2PC+Q_1)R_1')'x(t)=0. \tag{3.2.8}$$

This is an equation of the type (2.5.1) with $u(t)$ as the unknown. If we define

$$L := (R + \frac{\gamma}{4}R_1R_1'),$$
$$M := 1,$$
$$N := -(R + \frac{\gamma}{4}R_1R_1')(R + \frac{\gamma}{4}R_1R_1')^\dagger(PB + \frac{\gamma}{4}(2PC + Q_1)R_1')'x(t)$$

then, due to the solution of the Riccati differential equation (see the second equation in (3.2.1)), we know that the condition

$$\begin{aligned} & -(R + \frac{\gamma}{4}R_1R_1')(R + \frac{\gamma}{4}R_1R_1')^\dagger(R + \frac{\gamma}{4}R_1R_1')(R + \frac{\gamma}{4}R_1R_1')^\dagger(PB + \frac{\gamma}{4}(2PC + Q_1)R_1')'x(t) \\ = \; & -(R + \frac{\gamma}{4}R_1R_1')(R + \frac{\gamma}{4}R_1R_1')^\dagger(PB + \frac{\gamma}{4}(2PC + Q_1)R_1')'x(t), \end{aligned}$$

is satisfied. From Lemma 2.5.1 (b) we know that this is a necessary and sufficient condition for the equation (3.2.8) to have a solution. Therefore, there exists a process $S(t)$ such that the solution to (3.2.8) is

$$\begin{aligned} u(t) \;\; = \;\; & -(R + \frac{\gamma}{4}R_1R_1')^\dagger(PB + \frac{\gamma}{4}(2PC + Q_1)R_1')'x(t) + S(t) \\ & -(R + \frac{\gamma}{4}R_1R_1')(R + \frac{\gamma}{4}R_1R_1')^\dagger S(t), \end{aligned}$$

which corresponds to (3.2.4) with $Y(t) = 0$ and $z(t) = S(t)$. Therefore, we have proved that any optimal control must be of the form (3.2.4). $\square$

3.3 Infinite horizon

Here we consider the infinite horizon optimal control problem:

$$\begin{cases} \min\limits_{u(\cdot)\in\mathcal{A}_\infty} J_\infty(u(\cdot)), \\ \\ \text{s.t. } (3.1.1), \end{cases} \tag{3.3.1}$$

where $\mathcal{A}_\infty$ is a suitable admissible set of controls to be defined below. The solution to this problem proceeds in a similar way as to the finite horizon, but we require more assumptions, in particular with regards to the *stability* of the system.

Assume the matrices A, B, C, Q, R, Q_1, R_1, are constant and

$$R + \frac{\gamma}{4} R_1 R_1' \geq 0.$$

The following Riccati *algebraic* equation appears naturally in the proof of Theorem 3.3.1:

$$\begin{cases} PA + A'P + \frac{\gamma}{4}(2PC + Q_1)(2PC + Q_1)' + Q \\ - \left[PB + \frac{\gamma}{4}(2PC + Q_1)R_1'\right]\left(R + \frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left[PB + \frac{\gamma}{4}(2PC + Q_1)R_1'\right]' = 0, \\ \left(R + \frac{\gamma}{4}R_1R_1'\right)\left(R + \frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left[PB + \frac{\gamma}{4}\Big(2PC + Q_1\Big)R_1'\right]' \\ \qquad - \left[PB + \frac{\gamma}{4}\Big(2PC + Q_1\Big)R_1'\right]' = 0, \end{cases} \tag{3.3.2}$$

The Riccati algebraic equation (3.3.2) has a solution, which has also been proved in the book of Anderson and Moore [1].

Let us define that the given function $f : (0, \infty) \to \mathbb{R}$ is such that:

$$\lim_{T\to\infty} \frac{\gamma^2[tr(C'PC)T + x'(0)Px(0)]}{2f(T)} = H, \tag{3.3.3}$$

for some $H \in \mathbb{R}$.

Let $\mathcal{U}$ denote the set of all $\mathcal{F}(t)$-adapted processes $u(t)$ such that the state equation (3.1.1) has a unique strong solution. For each $u(\cdot) \in \mathcal{U}$ we define:

$$\begin{aligned} \theta_u'(t) &:= -\frac{\gamma}{2}[2x'(t)PC + x'(t)Q_1 + u'(t)R_1], \\ Z_u(t) &:= \exp\left[-\int_0^t \theta_u'(\tau)dW(\tau) - \frac{1}{2}\int_0^t \theta_u'(\tau)\theta_u(\tau)d\tau\right], \\ Z_u &:= Z_u(T), \\ \widetilde{\mathbb{P}}_u(\alpha) &:= \int_\alpha Z_u(\omega)d\widetilde{\mathbb{P}}(\omega), \qquad \forall \alpha \in \mathcal{F}. \end{aligned}$$

In order to ensure that $\widetilde{\mathbb{P}}_u$ is a probability measure, we assume that $\theta_u(t)$ satisfies the

Novikov condition, i.e. for some positive β and all $T \in (0, \infty)$ the following holds:

$$\mathbb{E}\left[e^{\beta \int_0^T \theta_u'(\tau)\theta_u(\tau)d\tau/2}\right] < \infty. \tag{3.3.4}$$

Different from the finite horizon, here we further require that the controls satisfy the following stability condition:

$$\lim_{T\to\infty} \frac{\gamma}{f(T)} \log \widetilde{\mathbb{E}}_u \left[e^{-\gamma x'(T)Px(T)/2}\right] = G, \tag{3.3.5}$$

for some $G \in \mathbb{R}$.

The admissible set of controls can now be defined as:

$$\mathcal{A}_\infty := \{u(\cdot) \in \mathcal{U} \quad \text{such that (3.2.2) and (3.3.5) hold}\}.$$

Let $Y(\cdot)$ be an $\mathbb{R}^{m\times n}$-valued $\mathcal{F}(t)$-adapted process, $z(\cdot)$ an $\mathbb{R}^m$-valued $\mathcal{F}(t)$-adapted process, and define:

$$\begin{aligned} K_Y^\infty(t) &:= -(R + \frac{\gamma}{4}R_1R_1')^\dagger[PB + \frac{\gamma}{4}(2PC + Q_1)R_1']' \\ &+ Y(t) - (R + \frac{\gamma}{4}R_1R_1')^\dagger(R + \frac{\gamma}{4}R_1R_1')Y(t), \\ K_z^\infty(t) &:= z(t) - (R + \frac{\gamma}{4}R_1R_1')^\dagger(R + \frac{\gamma}{4}R_1R_1')z(t). \end{aligned}$$

We confine the processes $Y(\cdot)$ and $z(\cdot)$ to the following set

$$\mathcal{K}^\infty := \{(Y(\cdot), z(\cdot)) : \quad K_Y^\infty(\cdot)x(\cdot) + K_z^\infty(\cdot) \in \mathcal{A}_\infty\}.$$

Theorem 3.3.1. *All solutions to (3.3.1) are given by:*

$$u_\infty^*(t) = K_Y^\infty(t)x(t) + K_z^\infty(t), \tag{3.3.6}$$

with $(Y(\cdot), z(\cdot)) \in \mathcal{K}^\infty$. *The optimal cost is:*

$$J_\infty^* := J_\infty(u_\infty^*(\cdot)) = H + G.$$

Proof. We proceed similarly to the proof of Theorem 3.2.1. The differential of the quadratic form $x'(t)Px(t)$ is:

$$d[x'(t)Px(t)] = \{2x'(t)P[Ax(t) + Bu(t)] + tr(C'PC)\Big\}dt + 2x'(t)PCdW(t).$$

Integrating both sides from 0 and T, and rearranging the resulting expression, gives:

$$\begin{aligned} 0 &= -x'(T)Px(T) + x'(0)Px(0) + \int_0^T 2x'(t)PCdW \\ &+ \int_0^T \{x'(t)\dot{P}x(t) + 2x'(t)P[Ax(t) + Bu(t)] + tr(C'PC)\}dt. \end{aligned}$$

The cost functional $J_\infty(u(\cdot))$ can now be written as:

$$\begin{aligned} J_\infty(u(\cdot)) &= \lim_{T\to\infty} \frac{\gamma}{f(T)} \log \mathbb{E}\Big\{\exp\Big[\frac{\gamma}{2}x'(0)Px(0) - \frac{\gamma}{2}x'(T)Px(t) \\ &+\frac{\gamma}{2}\int_0^T [x'(t)Qx(t) + u'(t)Ru(t)]dt \\ &+\frac{\gamma}{2}\int_0^T \{2x'(t)P[Ax(t) + Bu(t)] + tr(C'PC)\}dt \\ &+ \frac{\gamma}{2}\int_0^T [2x'(t)PC + x'(t)Q_1 + u'(t)R_1]dW(t)\Big]\Big\} \end{aligned}$$

For any $u(\cdot) \in \mathcal{A}_\infty$, the above expression becomes:

$$\begin{aligned} J_\infty(u(\cdot)) &= \lim_{T\to\infty} \frac{\gamma}{f(T)} \log \widetilde{\mathbb{E}}_u \Big\{\exp\Big[\frac{\gamma}{2}x'(0)Px(0) - \frac{\gamma}{2}x'(T)Px(t) + \frac{\gamma}{2}tr(C'PC)T \\ &+\frac{\gamma}{2}\int_0^T x'(t)[Q + PA + A'P + \frac{\gamma}{4}(2PC + Q_1)(2PC + Q_1)']x(t)dt \\ &\frac{\gamma}{2}\int_0^T \{u'(t)(R + \frac{\gamma}{4}R_1R_1')u(t) + 2x'(t)[PB + \frac{\gamma}{4}(2PC + Q_1)R_1']u(t)\}dt\Big]\Big\} \end{aligned}$$

For any $(Y(\cdot), z(\cdot)) \in \mathcal{K}_\infty$, let us introduce the processes:

$$\begin{aligned} L_1^\infty(t) &:= Y(t) - (R + \frac{\gamma}{4}R_1R_1')^\dagger(R + \frac{\gamma}{4}R_1R_1')Y(t), \\ L_2^\infty(t) &:= z(t) - (R + \frac{\gamma}{4}R_1R_1')^\dagger(R + \frac{\gamma}{4}R_1R_1')z(t), \end{aligned}$$

which, due to the properties of the Moore-Penrose pseudoinverse (see Lemma 2.5.1

(a)), have the property

$$(R + \frac{\gamma}{4}R_1R_1')L_i^{\infty}(t) = (R + \frac{\gamma}{4}R_1R_1')^{\dagger}L_i^{\infty}(t) = 0, \qquad i = 1, 2. \tag{3.3.7}$$

We can write $J_{\infty}(u(\cdot))$ now as:

$$\begin{aligned}
J_{\infty}(u(\cdot)) &= \lim_{T\to\infty} \frac{\gamma}{f(T)} \widetilde{\mathbb{E}}_u \Big\{ \exp\Big[\frac{\gamma}{2}x'(0)Px(0) - \frac{\gamma}{2}x'(T)Px(t) + \frac{\gamma}{2}tr(C'PC)T \\
&\quad + \frac{\gamma}{2}\int_0^T x'(t)\{Q + PA + A'P + \frac{\gamma}{4}(2PC + Q_1)(2PC + Q_1)' \\
&\quad - [PB + \frac{\gamma}{4}(2PC + Q_1)R_1']'(R + \frac{\gamma}{4}R_1R_1')^{\dagger}[PB + \frac{\gamma}{4}(2PC + Q_1)R_1']\}x(t)dt \\
&\quad + \frac{\gamma}{2}\int_0^T [u(t) + ((R + \frac{\gamma}{4}R_1R_1')^{\dagger}[PB + \frac{\gamma}{4}(2PC + Q_1)R_1'] + L_1^{\infty})x(t) + L_2^{\infty}]' \\
&\quad \times(R + \frac{\gamma}{4}R_1R_1')[u(t) + ((R + \frac{\gamma}{4}R_1R_1')^{\dagger}[PB + \frac{\gamma}{4}(2PC + Q_1)R_1'] + L_1^{\infty})x(t) + L_2^{\infty}]'dt\Big]\Big\}.
\end{aligned}$$

This simplifies further to:

$$\begin{aligned}
J_{\infty}(u(\cdot)) &= \lim_{T\to\infty} \frac{\gamma}{f(T)} \log \widetilde{\mathbb{E}}_u \Big\{ \exp\Big[\frac{\gamma}{2}x'(0)Px(0) - \frac{\gamma}{2}x'(T)Px(t) + \frac{\gamma}{2}tr(C'PC)T \\
&\quad + \frac{\gamma}{2}\int_0^T [u(t) + ((R + \frac{\gamma}{4}R_1R_1')^{\dagger}[PB + \frac{\gamma}{4}(2PC + Q_1)R_1'] + L_1^{\infty})x(t) + L_2^{\infty}]' \\
&\quad \times(R + \frac{\gamma}{4}R_1R_1')[u(t) + ((R + \frac{\gamma}{4}R_1R_1')^{\dagger}[PB + \frac{\gamma}{4}(2PC + Q_1)R_1'] + L_1^{\infty})x(t) + L_2^{\infty}]'dt\Big]\Big\}.
\end{aligned}$$

For all $u(\cdot) \in \mathcal{A}_{\infty}$, the following inequality holds:

$$\begin{aligned}
J_{\infty}(u(\cdot)) &\geq \lim_{T\to\infty} \frac{\gamma}{f(T)} \log \widetilde{\mathbb{E}}_u \exp\Big[\frac{\gamma}{2}x'(0)P(0)x(0) - \frac{\gamma}{2}x'(T)Px(t) + \frac{\gamma}{2}tr(C'PC)T\Big] \\
&= \lim_{T\to\infty} \frac{\gamma^2[x'(0)P(0)x(0) + tr(C'PC)T]}{2f(T)} + \lim_{T\to\infty} \frac{\gamma}{f(T)} \log \widetilde{\mathbb{E}}_u \Big[e^{-\gamma x'(T)Px(t)/2}\Big] \\
&= H + G.
\end{aligned}$$

This lower bound is achieved if $u(t) = u_{\infty}^*(t)$.

The remaining part of the proof in showing that all optimal controls have the form (3.3.6) proceeds as in the proof of Theorem 3.2.1. □

We now focus on deriving some conditions under which the stability requirement (3.3.5) holds under the optimal control $u^*_\infty(\cdot)$. We assume that the processes $Y(\cdot)$ and $z(\cdot)$ have the following special structure:

$$Y(t) = K_0,, \quad z(t) = K_1 x(t) + K_2,$$

for some constant matrices K_0, K_1, and K_2. The optimal control $u^*_\infty(\cdot)$ can now be written as

$$u^*_\infty(t) = -K_3 x(t) + K_4,$$

where

$$\begin{aligned} K_3 = \quad & - \quad \Big\{\Big[R + \frac{\gamma}{4}R_1R_1'\Big]^\dagger \Big[PB + \frac{\gamma}{4}\Big(2PC + Q_1\Big)R_1'\Big]' K_0 \\ & - \quad \Big[R + \frac{\gamma}{4}R_1R_1'\Big]^\dagger \Big[R + \frac{\gamma}{4}R_1R_1'\Big] K_0 \\ & - \quad K_1 + \Big[R + \frac{\gamma}{4}R_1R_1'\Big]^\dagger \Big[R + \frac{\gamma}{4}R_1R_1'\Big] K_1 \Big\}, \\ K_4 = \quad & K_2 \quad - \Big(R + \frac{\gamma}{4}R_1R_1'\Big)^\dagger \Big(R + \frac{\gamma}{4}R_1R_1'\Big) K_2. \end{aligned}$$

By the Girsanov theorem, the process

$$\widetilde{W}(t) = W(t) - \int_0^t \frac{\gamma}{2}[2\big(x'(t)PC\big)' + x'(t)Q_1 + u'(t)R_1]dt, \quad t \geq 0,$$

is a Brownian motion under the measure $\widetilde{\mathbb{P}}_u$. Substituting $u^*_\infty(\cdot)$ into the state equation (3.1.1) gives:

$$\begin{aligned} dx(t) \quad &= \quad \Big[\big(A + \gamma CC'P + \gamma CQ_1\big) - \big(B + \frac{\gamma}{2}CR_1'\big)K_3\Big]x(t)dt + K_4\Big(B + \frac{\gamma}{2}CR_1'\Big)dt + Cd\widetilde{W}(t) \\ &= \quad [\bar{A}x(t) + \bar{K}]dt + Cd\widetilde{W}(t), \end{aligned}$$

where $\bar{A} = \big(A + \gamma CC'P + \gamma CQ_1\big) - \big(B + \frac{\gamma}{2}CR_1'\big)K_3$ and $\bar{K} = K_4\Big(B + \frac{\gamma}{2}CR_1'\Big)$.

3.4 An example of the application on optimal investment

In this section we illustrate the application of our results to the problem of *optimal investment* in a market with a *stochastic* interest rate (see, for example, [21], [37], [35], [56], [65], [9], [57], for a background on the optimal investment problem). Let us consider a market of a *bank account* with price $S_0(t)$, and l stocks with prices $S_i(t)$, $i = 1, ..., l$, that satisfy the following equations:

$$\begin{cases} dS_0(t) = S_0(t)r(t)dt, \\ \\ dS_i(t) = S_i(t)[\mu_i(t)dt + \sigma_i'(t)dW(t)], \quad i = 1, ...l, \\ \\ S_i(0) > 0, \quad i = 0, 1, ..., l, \qquad \text{are given.} \end{cases} \tag{3.4.1}$$

Here $r(t)$ is the interest rate, $\mu_i(t)$ is the appreciation rate, whereas the d-dimensional vector process $\sigma_i(t)$ is the volatility of the stock. All these coefficient processes must be such that the equations (3.4.1) have unique strong solutions.

In this market we consider an investor with an initial wealth of y_0. Let $n_i(t)$ denote the number of shares of asset $S_i(t)$ held by the investor at time t. The value of his portfolio is $y(t) := \sum_{i=0}^{l} n_i(t)S_i(t)$. This portfolio is called *self-financing* if (see, for example, [37]):

$$dy(t) = \sum_{i=0}^{l} n_i(t)dS_i(t).$$

After substituting the differentials of $S_i(t)$ into this equation, and defining $u_i(t) := n_i(t)S_i(t)$, we obtain:

$$dy(t) = [r(t)y(t)dt + B'(t)u(t)]dt + u'(t)\sigma(t)dW(t), \tag{3.4.2}$$

where $u(t) := [u_1(t), ..., u_l(t)]'$, $\sigma'(t) := [\sigma_1'(t), ..., \sigma_l'(t)]$, and $B(t) := [\mu_1(t) - r(t), ..., \mu_l(t) - r(t)]'$. Thus, the portfolio (3.4.2) is a stochastic *control system* with state $y(t)$ and control $u(t)$.

We are interested in the following optimal investment problem with *power utility*:

$$\begin{cases} \max\limits_{u(\cdot)\in\mathcal{A}_p} \mathbb{E}[y^\lambda(T)], \\ \\ \text{s.t. } (3.4.2) \text{ and } y(t) > 0, \ \forall\, t \geq 0 \ a.s., \end{cases},$$

for some $\lambda \in (0,1)$ and a suitable admissible set of controls $\mathcal{A}_p$. Consider the following n-dimensional *factor* process:

$$\begin{cases} dx(t) = A(t)x(t)dt + C(t)dW(t), \\ \\ x(0) \in \mathbb{R}^n. \end{cases}$$

We define the interest rate as:

$$r(t) = x'(t)Q(t)x(t).$$

Thus, the interest rate follows a quadratic-affine term-structure model (QATSM) (see, for example, [16]). Similarly to [8], [7], we assume that for some function $L(t)$ it holds that:

$$B(t) = L(t)x(t).$$

In this formulation, the optimal investment problem appears to be different from our risk-sensitive control problem considered in the previous sections. However, we now show that it can be reformulated as an *example* of the risk-sensitive control problem. We confine ourselves to control processes $u(t)$ that ensure $y(t) > 0$ for $t \geq 0$. In this case the differential of $\log y(t)$ is:

$$\begin{aligned} d\log y(t) &= \frac{1}{y(t)}[r(t)y(t)dt + B'(t)u(t)]dt - \frac{1}{2}\frac{1}{y^2(t)}u'(t)\sigma(t)\sigma'(t)u(t)dt \\ &\quad + \frac{1}{y(t)}u'(t)\sigma(t)dW(t). \end{aligned}$$

By defining the new control process as $v(t) = \frac{1}{y(t)}u(t)$, we have:

$$d\log y(t) = [r(t) + B'(t)v(t) - v'(t)\sigma(t)\sigma'(t)v(t)/2]dt + v'(t)\sigma(t)dW(t).$$

In integral form this can be written as:

$$y(T) = y_0 \exp\left[\int_0^T [r(t) + B'(t)v(t) - v'(t)\sigma(t)\sigma'(t)v(t)/2]dt + \int_0^T v'(t)\sigma(t)dW(t)\right].$$

The optimal investment problem 3.4.3 can now be stated as:

$$\begin{cases} \max\limits_{v(\cdot)\in\mathcal{A}_v} \mathbb{E}\exp\left[\lambda\int_0^T (x'Qx + x'L'v - v'\sigma\sigma'v/2)dt + \lambda\int_0^T v'\sigma dW(t)\right], \\ \\ s.t. \quad dx(t) = A(t)x(t) + C(t)dW(t). \end{cases}$$

Here, for simplicity, we have taken $y_0 = 1$ since it does not effect the form optimal control.

Let $\sigma(t)$ be deterministic and $\sigma(t)\sigma'(t) > 0$. By the completion of squares, we have:

$$\begin{aligned} \frac{1}{2}v'\sigma\sigma'v - x'L'B'v &= \frac{1}{2}[v - (\sigma\sigma')^{-1}Lx]'\sigma\sigma'[v - (\sigma\sigma')^{-1}Lx] - \frac{1}{2}x'L'(\sigma\sigma')^{-1}Lx \\ &= \frac{1}{2}k'\sigma\sigma'k - \frac{1}{2}x'L'(\sigma\sigma')^{-1}Lx, \end{aligned}$$

where $k(t) := v(t) - [\sigma(t)\sigma'(t)]^{-1}L(t)x(t)$. Our optimal investment problem (3.4.3) can now be written as:

$$\begin{cases} \min\limits_{k(\cdot)\in\mathcal{A}_k} -\mathbb{E}\exp\left[-\frac{1}{2}\int_0^T (x'\widetilde{Q}x + k'\widetilde{R}k)dt - \frac{1}{2}\int_0^T (x'\widetilde{Q}_1 + k'\widetilde{R}_1)dW(t)\right], \\ \\ dx(t) = Ax(t)dt + CdW(t), \end{cases}$$

where

$$\widetilde{Q} := -\lambda Q - \lambda L'(\sigma\sigma')^{-1}BL, \quad \widetilde{R} = \lambda\sigma\sigma', \quad \widetilde{Q}_1' = -2\lambda\sigma'(\sigma\sigma')^{-1}L, \quad \widetilde{R}_1' = -2\lambda\sigma.$$

If $\mathcal{A}_k$ is defined as $\mathcal{A}$ of section 3.2, then this an example of the risk-sensitive control problem (3.1.5), and thus Theorem 3.2.1 can be applied. Provided that the assumptions of section 3.2 are satisfied, and since $\widetilde{R} - \widetilde{R}_1\widetilde{R}_1'/4 = \lambda\sigma\sigma'(1-\lambda) > 0$, from Theorem 3.2.1 it follows that the unique solution to (3.4.3) is:

$$k^*(t) = \frac{1}{4\lambda(1-\lambda)}(\sigma\sigma')^{-1}\widetilde{R}_1(2C'P + \widetilde{Q}_1)x(t),$$

where $P(t)$ is the solution to the Riccati differential equation

$$\begin{cases} \dot{P} + PA + A'P + \widetilde{Q} - \dfrac{1}{4}(2PC + \widetilde{Q}_1)(2PC + \widetilde{Q}_1)' \\ \qquad - \frac{1}{16\lambda(1-\lambda)}(2PC + \widetilde{Q}_1)\widetilde{R}_1'(\sigma\sigma')^{-1}(2PC + \widetilde{Q}_1)\widetilde{R}_1'\Big)' = 0 \\ P(T) = 0. \end{cases}$$

The corresponding optimal investment strategy $u^*(\cdot)$ is:

$$u^*(t) = v^*(t)y(t) = [k^* + (\sigma\sigma')^{-1}Lx]y(t).$$

Since this is a linear function of $y(t)$, the requirement of $y(t) > 0$ is satisfied.

3.5 Summary

We have considered a general case of an *indefinite* risk-sensitive control problem for stochastic systems with additive noise. This situation appears when we use a generalised risk-sensitive cost functional. We find all solutions to this problem by the completion of squares and the change of measure methods. Both the finite and infinite horizon cases are considered. The optimal investment problem in a market with a stochastic interest rate appears as a special case of our results.

Chapter 4

Indefinite risk-sensitive control with partially observed system

4.1 Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), \mathbb{P})$ be a complete probability space on which a $p+q$-dimensional standard Brownian motion $(W(t), t \geq 0)$ is defined. We assume that $\mathcal{F}(t)$ is the augmentation of $\sigma\{W(s)|0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. Consider the linear stochastic control system:

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t)]dt + C(t)dw_1(t), \\ \\ dy(t) = H(t)x(t)dt + D^{\frac{1}{2}}(t)dw_2(t), \\ \\ x(0) = x_0 \in \mathbb{R}^n, \quad y(0) = 0 \quad \text{is given.} \end{cases} \tag{4.1.1}$$

We assume that $A(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, $B(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m})$, $C(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times p})$, $H(\cdot) \in L^\infty(0,T;\mathbb{R}^{q\times n})$, $D(\cdot) \in L^\infty(0,T;\mathbb{R}^{q\times q})$ and $D > 0, \quad \forall t \in [0,T]$ where $L^\infty(0,T;\mathbb{R}^{n\times n})$ denotes the set of all $\mathbb{R}^{n\times n}$-valued uniformly bounded functions. We are given $W(t) = [w_1'(t), w_2'(t)]'$, i.e. $w_1(t)$ and $w_2(t)$ are components of $W(t)$ of order p and q. And we assume that x_0 and $W(t)$ are independent, and x_0 is a Gaussian random variable with mean μ_0 and variance σ_0. We further assume that the $\mathcal{F}(t)$-adapted control process $u(\cdot)$ is such that (4.1.1) has a unique strong solution.

Given the cost functional

$$
\begin{aligned}
J(u(\cdot)) \quad := \quad & \gamma \mathbb{E}\left\{ \exp\left[\frac{\gamma}{2} \int_0^T \left[x'(t)Q(t)x(t) + u'(t)R(t)u(t) \right] dt + \frac{\gamma}{2} x'(T)Sx(T) \right.\right. \\
& \left.\left. + \frac{\gamma}{2} \int_0^T \left[x'(t)Q_1(t) + u'(t)R_1(t) \right] dW(t) \right] \right\}, \qquad (4.1.2)
\end{aligned}
$$

and the optimal control problem is to find some controller $u(\cdot)$ which satisfies the following:

$$
\begin{cases}
\min\limits_{u(\cdot)\in\mathcal{A}} J(u(\cdot)), \\
\\
\text{s.t. } (4.1.1),
\end{cases} \qquad (4.1.3)
$$

for some suitably defined *admissible* set $\mathcal{A}$. The weighting matrices in (4.1.2) satisfy the following *definiteness* properties:

$$
\begin{cases}
0 \le Q(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n}), \\
0 < R(\cdot) \in L^\infty(0,T;\mathbb{R}^{m\times m}), \\
0 \le S \in \mathbb{R}^{n\times n}, \\
Q_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times p}), \\
R_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{m\times p}).
\end{cases} \qquad (4.1.4)
$$

With fifty years development, several cases about risk-sensitive control problem have been analized. The partial observation problem was considered by [38], [39], [68], [69], and complete solution obtained in [5].

In the papers of Date and Gashi [16], [17], they solved the generalization case by introducing noise dependent penalties on the state and control variables.

In this chapter, we weaken condition of the indefiniteness even further by assuming

$$
R(t) + \frac{\gamma}{4} R_1(t)R_1'(t) \ge 0, \quad a.e. \quad t \in [0,T]. \qquad (4.1.5)
$$

As a comtribution, we no longer have a unique and linear optimal control law, but rather a parametrised family of state feedback laws. Moreover, the derivation of the solution is more involved as compared to [16], [17], which is reflected by less explicit assumptions. In section 4.2, we find all solutions to the optimal control problem (4.1.3) under assumption (4.1.5).

Our second contribution is that we introduce an *infinite horizon* cost functional:

$$\begin{aligned} J_\infty(u(\cdot)) &:= \lim_{T\to\infty} \frac{\gamma}{f(T)} \log \mathbb{E}\left\{ \exp\left[\frac{\gamma}{2}\int_0^T \left[x'(t)Qx(t) + u'(t)Ru(t) \right] dt \right.\right. \\ &\quad \left.\left. + \frac{\gamma}{2}\int_0^T \left[x'(t)Q_1 + u'(t)R_1 \right] dW(t) \right] \right\}, \end{aligned}$$

where $f(T)$ is some given function. This is clearly the infinite horizon version of (4.1.2) of an *average* type. A new feature of this criterion is the function $f(T)$, which is not necessarily equal to T. This enables the solution of the corresponding optimal control problem under *weaker* assumptions, in particular with regards to the *stability* of the system, a feature important in applications. In section 4.3, we find all solutions to the optimal control problem with criterion $J_\infty(u(\cdot))$, and apply such results to the optimal investment problem, where the relevance of the function $f(T)$ is illustrated. We emphasize that in [16], [17] only the *finite* horizon risk-sensitive control problems are considered.

4.2 Finite Horizon

Here we are interested in finding *all* solutions to the risk-sensitive control problem (4.1.3), under some weaker assumptions as compared to [16], [17]. As already mentioned, the following is one of our main assumptions (throughout this section we suppress the argument t where appropriate for notational simplicity).

For simplicity, we define the following matrices:

$$\bar{A} = A + \frac{\gamma}{2}CQ'_{11} - P\left(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)' D^{-1}\left(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right) + \gamma P\left(Q + \frac{\gamma}{4}Q_1Q'_1\right),$$

$$\bar{B} = B + \frac{\gamma}{2}CR'_{11},$$

$$\bar{C} = P\left(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)' D^{-\frac{1}{2}},$$

where

$$Q_1 = \begin{bmatrix} Q_{11} \\ Q_{12} \end{bmatrix} \qquad R_1 = \begin{bmatrix} R_{11} \\ R_{12} \end{bmatrix} \tag{4.2.1}$$

We now focus in defining the appropriate *admissible* set of controls $\mathcal{A}$. Let $\mathcal{U}$ denote the set of all $\mathcal{F}(t)$-adapted processes $u(t)$ such that the state equation (4.1.1) has a unique strong solution. For each $u(\cdot) \in \mathcal{U}$ we define:

$$\begin{aligned}
\theta'_u(t) &:= -\frac{\gamma}{2}\Big[x'(t)Q_1 + u'(t)R_1\Big], \\
Z_u(t) &:= \exp\left[-\int_0^t \theta'_u(\tau)dW(\tau) - \frac{1}{2}\int_0^t \theta'_u(\tau)\theta_u(\tau)d\tau\right], \\
Z_u &:= Z_u(T), \\
\widetilde{\mathbb{P}}_u(\alpha) &:= \int_\alpha Z_u(\omega)d\widetilde{\mathbb{P}}(\omega), \qquad \forall \alpha \in \mathcal{F}.
\end{aligned}$$

In order to ensure that $\widetilde{\mathbb{P}}_u$ is a probability measure, we assume that $\theta_u(t)$ satisfies the *Novikov condition*, i.e. for some positive β the following holds:

$$\mathbb{E}\left[e^{(\beta/2)\int_0^T \theta'_u(\tau)\theta_u(\tau)d\tau}\right] < \infty. \tag{4.2.2}$$

We can now define the admissible set of controls as:

$$\mathcal{A} := \{u(\cdot) \in \mathcal{U} \quad \text{such that (4.2.2) holds}\}.$$

As it will become clear from the proof of Theorem 4.2.1, for any $u(\cdot) \in \mathcal{A}$ we have $J(u(\cdot)) < \infty$. The assumption of (4.2.2) appears to be stronger than the assumption of the finiteness of $J(u(\cdot))$, but it is required by our method of solution. Note that for any $u(\cdot) \in \mathcal{A}$ the probability measures $\widetilde{\mathbb{P}}_u$ and $\mathbb{P}$ are *equivalent*, which in particular means that if X is an $\mathcal{F}_T$-measurable random variable, then:

$$\mathbb{E}[ZX] = \widetilde{\mathbb{E}}_u[X], \tag{4.2.3}$$

Here $\widetilde{\mathbb{E}}_u$ denotes the expectation under $\widetilde{\mathbb{P}}_u$.

We define:

$$d\widetilde{y} = dy - \frac{\gamma}{2} D^{\frac{1}{2}} R'_{12} u(t) dt,$$

and the process $\overline{w}_2 : \Omega \times T \to \mathbb{R}^d$ is a standard Wiener process:

$$\widetilde{w}_2 = \overline{w}_2 - \int_0^T D^{-\frac{1}{2}} \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q'_{12} \right) x(t) dt,$$

$$d\widetilde{y}(t) = D^{\frac{1}{2}} d\overline{w}_2$$

and by Girsanov theorem, we define the change of probability that:

$$\begin{aligned}
\frac{d\overline{P}}{dP} &= \exp\left\{ \int_0^T D^{\frac{1}{2}} \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q'_{12} \right) x(t) d\overline{w}_2 \right. \\
&\quad \left. - \frac{1}{2} \int_0^T x'(t) \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q'_{12} \right)' D^{-1} \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q'_{12} \right) x(t) dt \right\} \\
&= \exp\left\{ \int_0^T D^{-1} \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q'_{12} \right) x(t) d\widetilde{y} \right. \\
&\quad \left. - \frac{1}{2} \int_0^T x'(t) \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q'_{12} \right)' D^{-1} \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q'_{12} \right) x(t) dt \right\}.
\end{aligned} \tag{4.2.4}$$

We introduce a vector $r : \Omega \times T \to \mathcal{R}^n$

$$\begin{aligned}
dr(t) &= \left[A + \frac{\gamma}{2} C Q'_{11} - P \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q'_{12} \right)' D^{-1} \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q'_{12} \right) + \gamma P \left(Q \right. \right. \\
&\quad \left. \left. + \frac{\gamma}{4} Q_1 Q'_1 \right) \right] r(t) dt + \left(B + \frac{\gamma}{2} C R'_{11} \right) u(t) dt + P \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q'_{12} \right)' D^{-1} d\widetilde{y},
\end{aligned} \tag{4.2.5}$$

where P(t) is the solution of the following Riccati differential equation:

$$\begin{cases} \dot{P} - (A + \frac{\gamma}{2} C Q_{11}')P - P(A + \frac{\gamma}{2} C Q_{11}')' + P\Big[(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}')' \\ \qquad \times D^{-\frac{1}{2}}(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}') - \gamma(Q + \frac{\gamma}{4} Q_1 Q_1')\Big]P - CC' = 0, \\ P(0) = P_0, \end{cases} \tag{4.2.6}$$

The following Riccati differential equation appears naturally in the proof of Theorem 4.2.1:

$$\begin{cases} \gamma\left(Q + \frac{\gamma}{4} Q_1 Q_1'\right) - \left(H + \frac{\gamma}{2} R^{\frac{1}{2}} Q_{12}'\right)' R^{-1}\left(H + \frac{\gamma}{2} R^{\frac{1}{2}} Q_{12}'\right) + \dot{M}_1 + M_1 \overline{A} \\ + \overline{A}' M_1 + \Big[I + M_1 P\Big]\left(H + \frac{\gamma}{2} R^{\frac{1}{2}} Q_{12}'\right)' R^{-1}\left(H + \frac{\gamma}{2} R^{\frac{1}{2}} Q_{12}'\right)\Big[I + M_1 P\Big]' \\ - \frac{1}{\gamma^2} M_1 \overline{B}\left(R + \frac{\gamma}{4} R_1 R_1'\right)^{\dagger} \overline{B}' M_1 = 0, \\ \left(R + \frac{\gamma}{4} R_1 R_1'\right)\left(R + \frac{\gamma}{4} \overline{B}' M_1\right) = 0, \\ M_1(T) = S. \end{cases} \tag{4.2.7}$$

The Riccati differential equation (4.2.7) has a unique solution, the proof is given in Anderson and Moore's book [1].

Assumption 9. $Q_1 R_1' = 0$.

Let $Y(\cdot)$ be an $\mathbb{R}^{m\times n}$-valued $\mathcal{F}(t)$-adapted process, $z(\cdot)$ an $\mathbb{R}^m$-valued $\mathcal{F}(t)$-adapted process, and define:

$$\begin{aligned} L_Y(t) &:= -(R + \frac{\gamma}{4} R_1 R_1')^{\dagger} M_1 \overline{B} + Y(t) - (R + \frac{\gamma}{4} R_1 R_1')^{\dagger}(R + \frac{\gamma}{4} R_1 R_1')Y(t), \\ L_z(t) &:= z(t) - (R + \frac{\gamma}{4} R_1 R_1')^{\dagger}(R + \frac{\gamma}{4} R_1 R_1')z(t). \end{aligned}$$

We confine the processes $Y(\cdot)$ and $z(\cdot)$ to the following set

$$\mathcal{L} := \{(Y(\cdot), z(\cdot)) : \quad L_Y(\cdot)x(\cdot) + L_z(\cdot) \in \mathcal{A}\}.$$

In other words, the processes $Y(\cdot)$ and $z(\cdot)$ must be such that the control $u_L(t) := L_Y(t)x(t) + L_z(t)$ is admissible.

Theorem 4.2.1. *All solutions to (4.1.3) are given by:*

$$u^*(t) = L_Y(t)r(t) + L_z(t), \\ , \tag{4.2.8}$$

with $(Y(\cdot), z(\cdot)) \in \mathcal{L}$. *The optimal cost is:*

$$J^* := J(u^*(\cdot)) = \gamma \exp\left[\frac{1}{2}r'(0)M_1(0)r(0) + \frac{1}{2}\int_0^T \gamma tr\left[P\left(Q + \frac{\gamma}{4}Q_1Q_1'\right)\right] + tr\left[\bar{C}'M_1\bar{C}\right]dt\right]. \tag{4.2.9}$$

Proof. The proof is a combination of a certain completion of squares and change of measure methods, and the approach of [63]. Due to (4.2.3), for any $u(\cdot) \in \mathcal{A}$, the cost functional $J(u(\cdot))$ becomes:

$$\begin{aligned} J(u(\cdot)) \quad &= \quad \frac{1}{\gamma}\widetilde{\mathbb{E}}_u\Bigg\{\exp\left[\frac{\gamma}{2}\int_0^T \Big[x'(t)\left(Q + \frac{\gamma}{4}Q_1Q_1'\right)x(t)\right. \\ &\left. +u'(t)\left(R + \frac{\gamma}{4}R_1R_1'\right)u(t) + \frac{\gamma}{2}x'(t)Q_1R_1'u(t)\Big]dt + x'(T)Sx(T)\right]\Bigg\}. \end{aligned} \tag{4.2.10}$$

By the property on $dW(t)$, we can write:

$$d\widetilde{W}(t) = dW(t) - \frac{\gamma}{2}\Big(x'(t)Q_1 + u'(t)R_1\Big)'dt,$$

$$dW(t) \quad = \quad \begin{bmatrix} d\widetilde{w}_1(t) \\ \\ d\widetilde{w}_2(t) \end{bmatrix} + \frac{\gamma}{2}\begin{bmatrix} Q_{11}'x(t) + R_{11}'u(t) \\ \\ Q_{12}'x(t) + R_{12}'u(t) \end{bmatrix}dt.$$

The state equation under the new probability measure $\widetilde{\mathbb{P}}$ and in terms of the new control variable $u(t)$ is:

$$\begin{cases} dx(t) = \Big[\Big(A + \frac{\gamma}{2} C Q_{11}'\Big) x(t) + \Big(B + \frac{\gamma}{2} C R_{11}'\Big) u(t)\Big] dt + C d\widetilde{w}_1(t), \\ \\ dy(t) = \Big[\Big(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}'\Big) x(t) + \frac{\gamma}{2} D^{\frac{1}{2}} R_{12}' u(t)\Big] dt + D^{\frac{1}{2}} d\widetilde{w}_2(t), \\ \\ x(0) = x_0, \quad y(0) = 0. \end{cases} \tag{4.2.11}$$

the equation 4.2.11 can be transferred to:

$$\begin{cases} dx(t) = \Big[\Big(A + \frac{\gamma}{2} C Q_{11}'\Big) x(t) + \Big(B + \frac{\gamma}{2} C R_{11}'\Big) u(t)\Big] dt + C d\widetilde{w}_1(t), \\ \\ d\widetilde{y}(t) = \Big(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}'\Big) x(t) dt + D^{\frac{1}{2}} d\widetilde{w}_2(t), \\ \\ x(0) = x_0, \quad y(0) = 0. \end{cases} \tag{4.2.12}$$

With the assumption 9, and substitute all results above into the equation 4.2.10:

$$\begin{aligned} J(u(\cdot)) \quad &= \quad \frac{1}{\gamma} \widetilde{\mathbb{E}}_u \Big\{ \exp \Big[\int_0^T D^{-1} \Big(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}'\Big) x(t) d\widetilde{y} + x'(T) S x(T) \\ &+ \int_0^T \Big[x'(t) \Big[\frac{\gamma}{2} \Big(Q + \frac{\gamma}{4} Q_1 Q_1'\Big) - \frac{1}{2} \Big(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}'\Big)' D^{-1} \Big(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}'\Big) \Big] x(t) \\ &\quad + \frac{\gamma}{2} u'(t) \Big(R + \frac{\gamma}{4} R_1 R_1'\Big) u(t) \Big] dt \Big] \Big\} \end{aligned} \tag{4.2.13}$$

Introduce a cost functional $K(u(\cdot))$:

$$\begin{aligned} K(u(\cdot)) \quad &= \quad \frac{1}{\gamma} \widetilde{\mathbb{E}}_u \Big\{ \exp \Big[\int_0^T \Big[D^{-\frac{1}{2}} \Big(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}'\Big) r(t) \Big] d\overline{w}_2 + r'(T) S r(T) \\ &- \frac{1}{2} \int_0^T r'(t) \Big[\Big(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}'\Big)' D^{-1} \Big(H + \frac{\gamma}{2} D^{\frac{1}{2} Q_{12}'}\Big) - \gamma \Big(Q + \frac{\gamma}{4} Q_1 Q_1'\Big) \Big] r(t) \\ &- \gamma u'(t) \Big(R + \frac{\gamma}{4} R_1 R_1'\Big) u(t) - \gamma tr \Big[P \Big(Q + \frac{\gamma}{4} Q_1 Q_1'\Big) \Big] dt \Big] \Big\}. \end{aligned} \tag{4.2.14}$$

By Bensoussan and Van Schuppen's [5]theorem, minimization of the cost functional $J(u(\cdot))$ subject to equation for $dx(t)$ after changing of measure is equivalent to the minimization of the cost functional $K(u(\cdot))$ respect to $dr(t)$.

The differential of the quadratic form $r'(t)M_1(t)r(t)$ is:

$$\begin{aligned} d[r'(t)M_1 r(t)] &= r'(t)\dot{M}_1 r(t) + 2r'(t)M_1\Big[\bar{A}r(t) + \bar{B}u(t)\Big] + tr[\bar{C}M_1\bar{C}] \\ &\quad +2r'(t)M_1\bar{C}d\bar{w}_2. \end{aligned}$$

Integrating both sides from 0 and T, and rearranging the resulting expression, gives:

$$\begin{aligned} 0 &= -r'(T)Sr(T) - r'(0)M_1(0)r(0) + \int_0^T 2r'(t)M_1\bar{C}d\bar{w}_2. \\ &\quad + \int_0^T \Big\{ r'(t)\dot{M}_1 r(t) + 2r'(t)M_1\Big[\bar{A}r(t) + \bar{B}u(t)\Big] + tr[\bar{C}M_1\bar{C}]\Big\}dt \end{aligned} \tag{4.2.15}$$

The new cost functional $K(u(\cdot))$ can now be written as:

$$\begin{aligned} K(u(\cdot)) &= \frac{1}{\gamma}\widetilde{\mathbb{E}}_u\Big\{\exp\Big[\frac{1}{2}r'(0)M_1(0)r(0) + \int_0^T \Big\{r'(t)\left[I + M_1 P\right]\Big(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\Big)' D^{-\frac{1}{2}}\Big\} d\bar{w}_2 \\ &\quad + \frac{1}{2}\int_0^T r'(t)\Big\{\gamma\Big(Q + \frac{\gamma}{4}Q_1Q'_1\Big) - \Big(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\Big)' D^{-1}\Big(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\Big) \\ &\qquad + \dot{M}_1 + M_1\bar{A} + \bar{A}'M_1\Big\}r(t)dt \\ &\quad + \frac{1}{2}\int_0^T \gamma\Big\{u'(t)\Big(R + \frac{\gamma}{4}R_1R'_1\Big)u(t) + 2r'(t)M_1\bar{B}u(t)\Big\}dt \\ &\quad + \frac{1}{2}\int_0^T \Big\{\gamma tr\Big[P\Big(Q + \frac{\gamma}{4}Q_1Q'_1\Big)\Big] + tr\Big[\bar{C}'M_1\bar{C}\Big]\Big\}dt\Big]\Big\}. \end{aligned} \tag{4.2.16}$$

Now we are going to deal with the Brownian Motion term $d\bar{w}_2$ in $K(u(\cdot))$. Let $\mathcal{U}$ denote the set of all $\mathcal{F}(t)$-adapted processes $u(t)$ such that the state equation (4.1.1) has a unique strong solution. For each $u(\cdot) \in \mathcal{U}$ we define:

$$\widetilde{\theta}'_u(t) := r'(t)\Big[I + M_1 P\Big]\Big(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\Big)' D^{-\frac{1}{2}},$$

$$
\begin{aligned}
\widetilde{Z}_u(t) &:= \exp\left[-\int_0^t \widetilde{\theta}'_u(\tau)dW(\tau) - \frac{1}{2}\int_0^t \widetilde{\theta}'_u(\tau)\overline{\theta}_u(\tau)d\tau\right], \\
\widetilde{Z}_u &:= \widetilde{Z}_u(T), \\
\widehat{\mathbb{P}}_u(\alpha) &:= \int_\alpha \overline{Z}_u(\omega)d\widetilde{\mathbb{P}}(\omega), \qquad \forall \alpha \in \mathcal{F}.
\end{aligned}
$$

Applying the same method as before, for any $u(\cdot) \in \mathcal{A}$, the cost functional $K(u(\cdot))$ becomes:

$$
\begin{aligned}
K(u(\cdot)) = {} & \frac{1}{\gamma}\widehat{\mathbb{E}}_u\Bigg\{\exp\Bigg[\frac{1}{2}r'(0)M_1(0)r(0) + \frac{1}{2}\int_0^T \left\{\gamma tr\left[P\left(Q + \frac{\gamma}{4}Q_1Q_1'\right)\right] + tr\left[\overline{C}'M_1\overline{C}\right]\right\}dt \\
& + \frac{1}{2}\int_0^T r'(t)\Big\{\gamma\left(Q + \frac{\gamma}{4}Q_1Q_1'\right) - \left(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)'D^{-1}\left(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right) \\
& + \dot{M}_1 + M_1\overline{A} + \overline{A}'M_1 + [I + M_1P]\left(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)' \\
& \times D^{-1}\left(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)[I + M_1P]'\Big\}r(t)dt \\
& + \frac{1}{2}\int_0^T \gamma\left\{u'(t)\left(R + \frac{\gamma}{4}R_1R_1'\right)u(t) + 2r'(t)M_1\overline{B}u(t)\right\}dt\Bigg]\Bigg\}.
\end{aligned} \tag{4.2.17}
$$

For any $(Y(\cdot), z(\cdot)) \in \mathcal{L}$, let us introduce the processes:

$$
\begin{aligned}
L_1(t) &:= Y(t) - (R + \frac{\gamma}{4}R_1R_1')^\dagger(R + \frac{\gamma}{4}R_1R_1')Y(t), \\
L_2(t) &:= z(t) - (R + \frac{\gamma}{4}R_1R_1')^\dagger(R + \frac{\gamma}{4}R_1R_1')z(t),
\end{aligned}
$$

which, due to the properties of the Moore-Penrose pseudoinverse (see Lemma 2.5.1 (a)), have the property

$$
(R + \frac{\gamma}{4}R_1R_1')L_i(t) = (R + \frac{\gamma}{4}R_1R_1')^\dagger L_i(t) = 0, \qquad i = 1, 2. \tag{4.2.18}
$$

Using this property, as well as other properties of the Moore-Penrose pseudoinverse given in Lemma 2.5.1 (a), we can write $K(u(\cdot))$ as:

$$\begin{aligned}
K(u(\cdot)) \;=\; & \gamma\widehat{\mathbb{E}}_u\left\{\exp\left[\frac{1}{2}r'(0)M_1(0)r(0)+\frac{1}{2}\int_0^T\left\{\gamma tr\left[P\left(Q+\frac{\gamma}{4}Q_1Q_1'\right)\right]+tr\left[\bar{C}'M_1\bar{C}\right]\right\}dt\right.\right.\\
&+\frac{1}{2}\int_0^T r'(t)\left\{\gamma\left(Q+\frac{\gamma}{4}Q_1Q_1'\right)-\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)'D^{-1}\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)\right.\\
&\quad+\dot{M}_1+M_1\bar{A}+\bar{A}'M_1+[I+M_1P]\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)'D^{-1}\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)\\
&\quad\left.\times[I+M_1P]'-\frac{1}{\gamma^2}M_1\bar{B}\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\bar{B}'M_1\right\}r(t)dt\\
&+\frac{1}{2}\int_0^T\left[u(t)+\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left(\bar{B}'M_1+L_1(t)\right)r(t)+L_2(t)\right]'\\
&\quad\left.\left.\times\left(R+\frac{\gamma}{4}R_1R_1'\right)\left[u(t)+\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left(\bar{B}'M_1+L_1(t)\right)r(t)+L_2(t)\right]dt\right]\right\}.
\end{aligned}\tag{4.2.19}$$

Due to the Riccati differential equation 4.2.7, the term containing $r(t)$ is zero. This simplifies the cost functional into:

$$\begin{aligned}
K(u(\cdot)) \;=\; & \gamma\widehat{\mathbb{E}}_u\left\{\exp\left[\frac{1}{2}r'(0)M_1(0)r(0)+\frac{1}{2}\int_0^T\left\{\gamma tr\left[P\left(Q+\frac{\gamma}{4}Q_1Q_1'\right)\right]+tr\left[\bar{C}'M_1\bar{C}\right]\right\}dt\right.\right.\\
&+\frac{1}{2}\int_0^T\left[u(t)+\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left(\bar{B}'M_1+L_1(t)\right)r(t)+L_2(t)\right]'\\
&\quad\left.\left.\times\left(R+\frac{\gamma}{4}R_1R_1'\right)\left[u(t)+\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left(\bar{B}'M_1+L_1(t)\right)r(t)+L_2(t)\right]dt\right]\right\}.
\end{aligned}\tag{4.2.20}$$

Since $\left(R+\frac{\gamma}{4}R_1R_1'\right)\geq 0$, for all $u(\cdot)\in\mathcal{A}$, the following inequality holds:

$$K(u(\cdot))\geq\gamma\exp\left[\frac{1}{2}r'(0)M_1(0)r(0)+\frac{1}{2}\int_0^T\gamma tr\left[P\left(Q+\frac{\gamma}{4}Q_1Q_1'\right)\right]+tr\left[\bar{C}'M_1\bar{C}\right]\right].$$

This lower bound is achieved if:

$$u(t)=-\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left(\bar{B}'M_1+L_1(t)\right)r(t)-L_2(t),$$

which becomes (4.2.8) after substituting the expressions for $L_1(t)$ and $L_2(t)$.

We now focus in proving that *any* admissible optimal control must be of the form (4.2.8). Let $u(\cdot) \in \mathcal{A}$ be any *optimal* control. From (4.2.20) it follows that it is necessary to have

$$\left(R + \frac{\gamma}{4}R_1R_1'\right)^{\frac{1}{2}} \left[u(t) + \left(R + \frac{\gamma}{4}R_1R_1'\right)^{\dagger} \left(\bar{B}'M_1 + L_1(t)\right) r(t) + L_2(t)\right] = 0,$$

which after multiplication from the right by $\left(R + \frac{\gamma}{4}R_1R_1'\right)^{\frac{1}{2}}$ becomes

$$\left(R + \frac{\gamma}{4}R_1R_1'\right) \left[u(t) + \left(R + \frac{\gamma}{4}R_1R_1'\right)^{\dagger} \left(\bar{B}'M_1 + L_1(t)\right) r(t) + L_2(t)\right] = 0.$$

Due to (4.2.18), this equation can be written as

$$\left(R+\frac{\gamma}{4}R_1R_1'\right) u(t)+\left(R+\frac{\gamma}{4}R_1R_1'\right) \left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger} \bar{B}'M_1r(t)=0. \tag{4.2.21}$$

This is an equation of the type (2.5.1) with $u(t)$ as the unknown. If we define

$$\begin{aligned} L &:= \left(R + \frac{\gamma}{4}R_1R_1'\right), \\ M &:= 1, \\ N &:= -\left(R + \frac{\gamma}{4}R_1R_1'\right) \left(R + \frac{\gamma}{4}R_1R_1'\right)^{\dagger} \bar{B}'M_1r(t), \end{aligned}$$

then, due to the Riccati differential equation (see the second equation in (4.2.7)), we know that the condition

$$\begin{aligned} & -\left(R + \frac{\gamma}{4}R_1R_1'\right) \left(R + \frac{\gamma}{4}R_1R_1'\right)^{\dagger} \left(R + \frac{\gamma}{4}R_1R_1'\right) \left(R + \frac{\gamma}{4}R_1R_1'\right)^{\dagger} \bar{B}'M_1r(t) \\ = & -\left(R + \frac{\gamma}{4}R_1R_1'\right) \left(R + \frac{\gamma}{4}R_1R_1'\right)^{\dagger} \bar{B}'M_1r(t), \end{aligned}$$

is satisfied. From Lemma 2.5.1 (b) we know that this is a necessary and sufficient condition for the equation (4.2.21) to have a solution. Therefore, there exists a process $S(t)$ such that the solution to (4.2.21) is

$$u^*(t) = -\left(R + \frac{\gamma}{4}R_1R_1'\right)^{\dagger} \bar{B}'M_1(t)r(t) + S(t) - \left(R + \frac{\gamma}{4}R_1R_1'\right) \left(R + \frac{\gamma}{4}R_1R_1'\right)^{\dagger} S(t).$$

which corresponds to (4.2.8) with $Y(t) = 0$ and $z(t) = S(t)$. Therefore, we have proved that any optimal control must be of the form (4.2.8). □

4.3 Infinite horizon

Here we consider the infinite horizon optimal control problem:

$$\begin{cases} \min\limits_{u(\cdot)\in\mathcal{A}_\infty} J_\infty(u(\cdot)), \\ \\ \text{s.t. } 4.1.1 \text{ holds}, \end{cases} \tag{4.3.1}$$

where $\mathcal{A}_\infty$ is a suitable admissible set of controls to be defined below. The solution to this problem proceeds in a similar way as to the finite horizon, but we require more assumptions, in particular with regards to the *stability* of the system.

The matrices $A, B, C, H, D, Q, R, Q_1, R_1$, are constant and

$$\left(R + \frac{\gamma}{4} R_1 R_1'\right) \geqslant 0.$$

For simplicity, we define the following matrices:

$$\begin{aligned} \bar{A} &= A + \frac{\gamma}{2} C Q_{11}' - P\left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}'\right)' D^{-1}\left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}'\right) + \gamma P\left(Q + \frac{\gamma}{4} Q_1 Q_1'\right), \\ \bar{B} &= B + \frac{\gamma}{2} C R_{11}', \\ \bar{C} &= P\left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}'\right)' D^{-\frac{1}{2}}, \end{aligned}$$

where

$$Q_1 = \begin{bmatrix} Q_{11} \\ Q_{12} \end{bmatrix}, \qquad R_1 = \begin{bmatrix} R_{11} \\ R_{12} \end{bmatrix}.$$

We now focus in defining the appropriate admissible set of controls $\mathcal{A}$. Let $\mathcal{U}$ denote the set of all $\mathcal{F}(t)$-adapted processes $u(t)$ such that the state equation (4.1.1) has a unique strong solution. For each $u(\cdot) \in \mathcal{U}$ we define:

$$\begin{aligned} \theta_u'(t) &:= -\frac{\gamma}{2}[2x'(t)PC + x'(t)Q_1 + u'(t)R_1], \\ \\ Z_u(t) &:= \exp\left[-\int_0^t \theta_u'(\tau)dW(\tau) - \frac{1}{2}\int_0^t \theta_u'(\tau)\theta_u(\tau)d\tau\right], \end{aligned}$$

$$\begin{aligned} Z_u &:= Z_u(T), \\ \widetilde{\mathbb{P}}_u(\alpha) &:= \int_\alpha Z_u(\omega) d\widetilde{\mathbb{P}}(\omega), \qquad \forall \alpha \in \mathcal{F}. \end{aligned}$$

In order to ensure that $\widetilde{\mathbb{P}}_u$ is a probability measure, we assume that $\theta_u(t)$ satisfies the *Novikov condition*, i.e. for some positive β the following holds:

$$\mathbb{E}\left[e^{\frac{\beta}{2}\int_0^T \theta'_u(\tau)\theta_u(\tau)d\tau}\right] < \infty. \tag{4.3.2}$$

Different from the finite horizon, here we further require that the controls satisfy the following stability condition:

$$\lim_{T\to\infty} \frac{\gamma}{f(T)} \log \widehat{\mathbb{E}}_u \left[e^{-\frac{\gamma}{2}r'(T)M_2 r(T)}\right] = \pi_2, \tag{4.3.3}$$

for some $\pi_2 \in \mathbb{R}$.

The admissible set of controls can now be written as:

$$\mathcal{A}_\infty := \{u(\cdot) \in \mathcal{U} \quad \text{such that (4.3.2) and (4.3.3) holds}\}.$$

Note that for any $u(\cdot) \in \mathcal{A}$ the probability measures $\widetilde{\mathbb{P}}_u$ and $\mathbb{P}$ are *equivalent*, which in particular means that if X is an $\mathcal{F}_T$-measurable random variable, then:

$$\mathbb{E}[ZX] = \widetilde{\mathbb{E}}_u[X], \tag{4.3.4}$$

Here $\widetilde{\mathbb{E}}_u$ denotes the expectation under $\widetilde{\mathbb{P}}_u$.

We define:

$$d\widetilde{y} = dy - \frac{\gamma}{2} D^{\frac{1}{2}} R'_{12} u(t) dt,$$

and the process $\overline{w}_2 : \Omega \times T \to \mathbb{R}^d$ is a standard Wiener process:

$$\widetilde{w}_2 = \overline{w}_2 - \int_0^T D^{-\frac{1}{2}} \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q'_{12}\right) x(t) dt,$$

therefore,

$$d\widetilde{y}(t) = D^{\frac{1}{2}} d\overline{w}_2$$

and by Girsanov theorem, we define the change of probability that:

$$\begin{aligned}\frac{d\overline{P}}{dP} &= \exp\Big\{\int_0^T D^{\frac{1}{2}}\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)x(t)d\overline{w}_2 \\ &\qquad -\frac{1}{2}\int_0^T x'(t)\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)'D^{-1}\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)x(t)dt\Big\} \\ &= \exp\Big\{\int_0^T D\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)x(t)d\widetilde{y} \\ &\qquad -\frac{1}{2}\int_0^T x'(t)\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)'D^{-1}\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)x(t)dt\Big\}\end{aligned} \tag{4.3.5}$$

We introduce a vector $r:\Omega\times T\to\mathbb{R}^n$

$$\begin{aligned}dr(t) &= \Big[A+\frac{\gamma}{2}CQ'_{11}-P\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)'D^{-1}\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)+\gamma P\,(Q \\ &\qquad +\frac{\gamma}{4}Q_1Q'_1\big)\Big]r(t)dt+\left(B+\frac{\gamma}{2}CR'_{11}\right)u(t)dt+P\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)'D^{-1}d\hat{y}.\end{aligned} \tag{4.3.6}$$

P(t) is the solution of the following Riccati differential equation:

$$\begin{cases}\dot{P}-(A+\frac{\gamma}{2}CQ'_{11})P-P(A+\frac{\gamma}{2}CQ'_{11})'+P\Big[(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12})' \\ \qquad \times D^{-\frac{1}{2}}(H+\frac{\gamma}{2}D^{-1}Q'_{12})-\gamma(Q+\frac{\gamma}{4}Q_1Q'_1)\Big]P-CC'=0, \\ P(0)=P_0,\end{cases} \tag{4.3.7}$$

The following Riccati differential equation appears naturally in the proof of

Theorem 4.3.1:

$$\begin{cases} \gamma\left(Q+\frac{\gamma}{4}Q_1Q_1'\right)-\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)'D^{-1}\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)+M_2\overline{A}+\overline{A}'M_2 \\ +[I+M_2P]'\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)'D^{-1}\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)[I+M_2P]' \\ -\frac{1}{\gamma^2}M_2\overline{B}\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\overline{B}'M_2=0, \\ \left(R+\frac{\gamma}{4}R_1R_1'\right)\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left(R_{11}C'M_2+B'M_2\right)=0, \end{cases} \tag{4.3.8}$$

The Riccati differential equation (4.3.8) has a solution, and the proof is in Anderson and Moore's book [1].

Assumption 10. $Q_1R_1'=0$.

Let us define that the given function $f:(0,\infty)\to\mathbb{R}$ is such that:

$$\lim_{T\to\infty}\frac{\gamma\left[r'(0)M_2r(0)+tr[P(Q+\frac{\gamma}{4}Q_1Q_1')]T\right]+tr(\overline{C}'M_2\overline{C})T}{2f(T)}=\pi_1,$$

for some $\pi_1\in\mathbb{R}$.

Let $Y(\cdot)$ be an $\mathbb{R}^{m\times n}$-valued $\mathcal{F}(t)$-adapted process, $z(\cdot)$ an $\mathbb{R}^m$-valued $\mathcal{F}(t)$-adapted process, and define:

$$\begin{aligned} L_Y^{\infty}(t) &:= -\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}M_2\overline{B}+Y(t)-\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left(R+\frac{\gamma}{4}R_1R_1'\right)Y(t), \\ L_z^{\infty}(t) &:= z(t)-\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left(R+\frac{\gamma}{4}R_1R_1'\right)z(t). \end{aligned}$$

We confine the processes $Y(\cdot)$ and $z(\cdot)$ to the following set

$$\mathcal{L}^{\infty}:=\{(Y(\cdot),z(\cdot)):\quad L_Y^{\infty}(\cdot)x(\cdot)+L_z^{\infty}(\cdot)\in\mathcal{A}_{\infty}\}.$$

Theorem 4.3.1. *All solutions to (4.3.1) are given by:*

$$u_{\infty}^*(t)=L_Y^{\infty}(t)r(t)+L_z^{\infty}(t), \tag{4.3.9}$$

with $(Y(\cdot), z(\cdot)) \in \mathcal{K}^\infty$*. The optimal cost is:*

$$J^*_\infty := J_\infty(u^*_\infty(\cdot)) = \pi_1 + \pi_2.$$

Proof. We proceed similarly to the proof of Theorem 4.2.1. For any $u(\cdot) \in \mathcal{A}$, the cost functional $J(u(\cdot))$ can be written as:

$$\begin{aligned} J_\infty(u(\cdot)) &= \lim_{T\to\infty} \frac{1}{f(T)} \gamma \log \widetilde{\mathbb{E}}_u \Bigg\{ \exp\left[\frac{\gamma}{2}\int_0^T \left[x'(t)\left(Q + \frac{\gamma}{4}Q_1Q_1'\right)x(t)\right.\right. \\ &\quad \left.\left. + u'(t)\left(R + \frac{\gamma}{4}R_1R_1'\right)u(t) + \frac{\gamma}{2}x'(t)Q_1R_1'u(t)\right]dt\right]\Bigg\}. \end{aligned} \tag{4.3.10}$$

By the property on $dW(t)$, we can write:

$$d\widetilde{W}(t) = dW(t) - \frac{\gamma}{2}\Big(x'(t)Q_1 + u'(t)R_1\Big)' dt,$$

$$dW(t) = \begin{bmatrix} d\widetilde{w}_1(t) \\ \\ d\widetilde{w}_2(t) \end{bmatrix} + \frac{\gamma}{2}\begin{bmatrix} Q_{11}'x(t) + R_{11}'u(t) \\ \\ Q_{12}'x(t) + R_{12}'u(t) \end{bmatrix}.$$

The state equation under the new probability measure $\widetilde{\mathbb{P}}$ and in terms of the new control variable $u(t)$ is:

$$\begin{cases} dx(t) = \Big[\Big(A + \frac{\gamma}{2}CQ_{11}'\Big)x(t) + \Big(B + \frac{\gamma}{2}CR_{11}'\Big)u(t)\Big]dt + Cd\widetilde{w}_1(t), \\ \\ dy(t) = \Big[\Big(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\Big)x(t) + \frac{\gamma}{2}D^{\frac{1}{2}}R_{12}'u(t)\Big]dt + D^{\frac{1}{2}}d\widetilde{w}_2(t), \\ \\ x(0) = x_0, \quad y(0) = 0. \end{cases} \tag{4.3.11}$$

the equation 4.3.11 can be transferred to:

$$\begin{cases} dx(t) = \Big[\Big(A + \frac{\gamma}{2}CQ'_{11}\Big)x(t) + \Big(B + \frac{\gamma}{2}CR'_{11}\Big)u(t)\Big]dt + Cd\widetilde{w}_1(t), \\ d\widetilde{y}(t) = \Big(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\Big)x(t)dt + D^{\frac{1}{2}}d\widetilde{w}_2(t), \\ x(0) = x_0, \quad y(0) = 0. \end{cases} \tag{4.3.12}$$

With the assumption 10, and substitute all results above into 4.3.10 and by Girsanov theorem, we have:

$$\begin{aligned} J_\infty(u(\cdot)) &= \lim_{T\to\infty}\frac{1}{f(T)}\gamma\log\widetilde{\mathbb{E}}_u\Big\{\exp\Big[\int_0^T D^{-1}\Big(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\Big)x(t)d\widetilde{y} \\ &+ \int_0^T\Big[x'(t)\Big[\frac{\gamma}{2}\Big(Q + \frac{\gamma}{4}Q_1Q'_1\Big) - \frac{1}{2}\Big(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\Big)'D^{-1}\Big(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\Big)\Big]x(t) \\ &+ \frac{\gamma}{2}u'(t)\Big(R + \frac{\gamma}{4}R_1R'_1\Big)u(t)\Big]dt\Big]\Big\}. \end{aligned} \tag{4.3.13}$$

Introduce the following cost functional $K(u(\cdot))$:

$$\begin{aligned} K(u(\cdot)) &= \lim_{T\to\infty}\frac{1}{f(T)}\gamma\log\widetilde{\mathbb{E}}_u\Big\{\exp\Big[\int_0^T\Big[D^{-\frac{1}{2}}\Big(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\Big)r(t)\Big]d\overline{w}_2 \\ &- \frac{1}{2}\int_0^T r'(t)\Big[\Big(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\Big)'D^{-1}\Big(H + \frac{\gamma}{2}D^{\frac{1}{2}Q'_{12}}\Big) - \gamma\Big(Q + \frac{\gamma}{4}Q_1Q'_1\Big)\Big]r(t) \\ &- \gamma u'(t)\Big(R + \frac{\gamma}{4}R_1R'_1\Big)u(t) - \gamma tr\Big[P\Big(Q + \frac{\gamma}{4}Q_1Q'_1\Big)\Big]dt\Big]\Big\}. \end{aligned} \tag{4.3.14}$$

By Bensoussan and Van Schuppen's [5]theorem, minimization of the cost functional $J_\infty(u(\cdot))$ respect to $dx(t)$ is equivalent to the minimization of the cost functional $K(u(\cdot))$ respect to $dr(t)$.

The differential of the quadratic form $r'(t)M_1(t)r(t)$ is:

$$\begin{aligned} d[r'(t)M_2r(t)] &= 2r'(t)M_2\Big[\overline{A}r(t) + \overline{B}u(t)\Big] + tr[\overline{C}M_2\overline{C}] \\ &+ 2r'(t)M_2\overline{C}d\overline{w}_2. \end{aligned}$$

Integrating both sides from 0 and T, and rearranging the resulting expression, gives:

$$\begin{aligned}0 &= -r'(T)M_2r(T) + r'(0)M_2r(0) + \int_0^T 2r'(t)M_2\overline{C}d\overline{w}_2 \\ &\quad + \int_0^T \left\{2r'(t)M_2\left[\overline{A}r(t) + \overline{B}u(t)\right] + tr[\overline{C}M_2\overline{C}]\right\} dt.\end{aligned}$$

The cost functional $K_\infty(u(\cdot))$ can now be written as:

$$\begin{aligned}K_\infty(u(\cdot)) &= \lim_{T\to\infty}\frac{1}{f(T)}\gamma\log\widetilde{\mathbb{E}}_u\left\{\exp\left[\int_0^T\left\{r'(t)\left[I+M_2P\right]\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)'D^{-\frac{1}{2}}\right\}d\overline{w}_2\right.\right. \\ &\quad +\frac{1}{2}\int_0^T r'(t)\{\gamma\left(Q+\frac{\gamma}{4}Q_1Q_1'\right) - \left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)'D^{-1}\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right) \\ &\qquad +M_2\overline{A}+\overline{A}'M_2\Big\}r(t)dt \\ &\quad +\frac{1}{2}\int_0^T\gamma\left\{u'(t)\left(R+\frac{\gamma}{4}R_1R_1'\right)u(t)+2r'(t)M_2\overline{B}u(t)\right\}dt \\ &\quad +\frac{1}{2}\int_0^T\left\{\gamma tr\left[P\left(Q+\frac{\gamma}{4}Q_1Q_1'\right)\right]+tr\left[\overline{C}'M_2\overline{C}\right]\right\}dt \\ &\quad \left.\left. -\frac{1}{2}r'(T)M_2r(T)+\frac{1}{2}r'(0)M_2r(0)\right]\right\}.\end{aligned} \tag{4.3.15}$$

Now we are going to deal with the Brownian Motion term $d\overline{w}_2$ in $K(u(\cdot))$. Let $\mathcal{U}$ denote the set of all $\mathcal{F}(t)$-adapted processes $u(t)$ such that the state equation (4.1.1) has a unique strong solution. For each $u(\cdot)\in\mathcal{U}$ we define:

$$\begin{aligned}\widetilde{\theta}'_u(t) &:= r'(t)\Big[I+M_2P\Big]\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)'D^{-\frac{1}{2}}, \\ \widetilde{Z}_u(t) &:= \exp\left[-\int_0^t\widetilde{\theta}'_u(\tau)dW(\tau)-\frac{1}{2}\int_0^t\widetilde{\theta}'_u(\tau)\overline{\theta}_u(\tau)d\tau\right], \\ \widetilde{Z}_u &:= \widetilde{Z}_u(T), \\ \widehat{\mathbb{P}}_u(\alpha) &:= \int_\alpha \overline{Z}_u(\omega)d\widetilde{\mathbb{P}}(\omega), \qquad \forall\alpha\in\mathcal{F}.\end{aligned}$$

Applying the same method as before, for any $u(\cdot) \in \mathcal{A}_\infty$, the above expression becomes:

$$\begin{aligned}
K_\infty(u(\cdot)) \quad = \quad & \lim_{T\to\infty} \frac{1}{f(T)} \gamma \log \widehat{\mathbb{E}} \Bigg\{ \exp \Bigg[\frac{1}{2} r'(0) M_2 r(0) - \frac{1}{2} r'(T) M_2 r(T) \\
& + \frac{1}{2} \int_0^T \left\{ \gamma tr \left[P \left(Q + \frac{\gamma}{4} Q_1 Q_1' \right) \right] + tr \left[\overline{C}' M_2 \overline{C} \right] \right\} dt \\
& + \frac{1}{2} \int_0^T r'(t) \Bigg\{ \gamma \left(Q + \frac{\gamma}{4} Q_1 Q_1' \right) - \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}' \right)' D^{-1} \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}' \right) \\
& \qquad + M_2 \overline{A} + \overline{A}' M_2 + [I + M_2 P] \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}' \right)' \\
& \qquad \times D^{-1} \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}' \right) [I + M_2 P]' \Bigg\} r(t) dt \\
& + \frac{1}{2} \int_0^T \gamma \left\{ u'(t) \left(R + \frac{\gamma}{4} R_1 R_1' \right) u(t) + 2 r'(t) M_2 \bar{B} u(t) \right\} dt \Bigg] \Bigg\}. \qquad (4.3.16)
\end{aligned}$$

For any $(Y(\cdot), z(\cdot)) \in \mathcal{L}_\infty$, let us introduce the processes:

$$\begin{aligned}
L_1^\infty(t) \quad &:= \quad Y(t) - \left(R + \frac{\gamma}{4} R_1 R_1' \right)^+ \left(R + \frac{\gamma}{4} R_1 R_1' \right) Y(t), \\
L_2^\infty(t) \quad &:= \quad z(t) - \left(R + \frac{\gamma}{4} R_1 R_1' \right)^+ \left(R + \frac{\gamma}{4} R_1 R_1' \right) z(t).
\end{aligned}$$

which, due to the properties of the Moore-Penrose pseudoinverse (see Lemma 2.5.1 (a)), have the property

$$\left(R + \frac{\gamma}{4} R_1 R_1' \right) L_i^\infty(t) = \left(R + \frac{\gamma}{4} R_1 R_1' \right)^\dagger L_i^\infty(t) = 0, \qquad i = 1, 2. \qquad (4.3.17)$$

Using this property, as well as other properties of the Moore-Penrose pseudoinverse given in Lemma 2.5.1 (a), we can write $K(u(\cdot))$ as:

$$\begin{aligned}
K_\infty(u(\cdot)) \quad = \quad & \lim_{T\to\infty} \frac{1}{f(T)} \gamma \log \widehat{\mathbb{E}}_u \Bigg\{ \exp \Bigg[\frac{1}{2} r'(0) M_2 r(0) - \frac{1}{2} r'(T) M_2 r(T) \\
& + \frac{1}{2} \int_0^T \left\{ \gamma tr \left[P \left(Q + \frac{\gamma}{4} Q_1 Q_1' \right) \right] + tr \left[\overline{C}' M_2 \overline{C} \right] \right\} dt \\
& + \frac{1}{2} \int_0^T r'(t) \Bigg\{ \gamma \left(Q + \frac{\gamma}{4} Q_1 Q_1' \right) - \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}' \right)' D^{-1} \left(H + \frac{\gamma}{2} D^{\frac{1}{2}} Q_{12}' \right)
\end{aligned}$$

$$
\begin{aligned}
&+M_2\overline{A}+\overline{A}'M_2+[I+M_2P]\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)'D^{-1}\left(H+\frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)\\
&\times[I+M_2P]'-\frac{1}{\gamma^2}M_2\overline{B}\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\overline{B}'M_2\Big\}r(t)dt\\
&+\frac{1}{2}\int_0^T\left[u(t)+\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left(\overline{B}'M_2+L_1\right)r(t)+L_2\right]'\\
&\times\left(R+\frac{\gamma}{4}R_1R_1'\right)\left[u(t)+\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left(\overline{B}'M_2+L_1\right)r(t)+L_2\right]dt\Big]\Big\}.
\end{aligned}
\tag{4.3.18}
$$

Due to the Riccati differential equation, the term containing $r(t)$ is zero. This simplifies the cost functional into:

$$
\begin{aligned}
K_\infty(u(\cdot)) &= \lim_{T\to\infty}\frac{1}{f(T)}\gamma\log\widehat{\mathbb{E}}_u\Big\{\exp\Big[\frac{1}{2}r'(0)M_2r(0)-\frac{1}{2}r'(T)M_2r(T)\\
&+\frac{1}{2}\int_0^T\left\{\gamma tr\left[P\left(Q+\frac{\gamma}{4}Q_1Q_1'\right)\right]+tr\left[\overline{C}'M_2\overline{C}\right]\right\}dt\\
&+\frac{1}{2}\int_0^T\left[u(t)+\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left(\overline{B}'M_2+L_1\right)r(t)+L_2\right]'\\
&\times\left(R+\frac{\gamma}{4}R_1R_1'\right)\left[u(t)+\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left(\overline{B}'M_2+L_1\right)r(t)+L_2\right]dt\Big]\Big\}.
\end{aligned}
\tag{4.3.19}
$$

Sinace $\left(R+\frac{\gamma}{4}R_1R_1'\right)\geq 0$, for all $u(\cdot)\in\mathcal{A}$, the following inequality holds:

$$
\begin{aligned}
K(u(\cdot)) &\geq \lim_{T\to\infty}\frac{1}{f(T)}\gamma\log\widehat{\mathbb{E}}_u\Big\{\exp\Big[\frac{1}{2}r'(0)M_2r(0)-\frac{1}{2}r'(T)M_2r(T)\\
&+\frac{1}{2}\int_0^T\left\{\gamma tr\left[P\left(Q+\frac{\gamma}{4}Q_1Q_1'\right)\right]+tr\left[\overline{C}'M_2\overline{C}\right]\right\}\\
&= \lim_{T\to\infty}\frac{\gamma\left[r'(0)M_2r(0)+tr\left[P\left(Q+\frac{\gamma}{4}Q_1Q_1'\right)\right]T\right]+tr\left[\overline{C}'M_2\overline{C}\right]T}{2f(T)}\\
&+\lim_{T\to\infty}\frac{1}{f(T)}\gamma\log\widehat{\mathbb{E}}_u\left[e^{-\frac{\gamma}{2}r'(T)M_2r(T)}\right]\\
&= \pi_1+\pi_2
\end{aligned}
\tag{4.3.20}
$$

This lower bound is achieved if:

$$u(t) = u^*_\infty(t),$$

which becomes (4.3.9) after substituting the expressions for $L_1(t)$ and $L_2(t)$.

The remaining part of the proof in showing that all optimal controls have the form (4.3.9) proceeds as in the proof of Theorem 4.3.1.

We now focus on deriving some conditions under which the stability requirement (4.3.3) holds under the optimal control $u^*_\infty(\cdot)$. We assume that the processes $Y(\cdot)$ and $z(\cdot)$ have the following special structure:

$$\begin{aligned}&Y(t) \text{ is a constant matrix S,}\\ &z(t) \text{ is a state feed-back process, i.e.} z(t) = G_1 r(t) + G_2\end{aligned}$$

for some constant S, G_1 and G_2. The optimal control $u^*_\infty(\cdot)$ can now be written as

$$u^* = G_3 r(t) + G_4,$$

where

$$\begin{aligned}G_3 &= -\left(R + \frac{\gamma}{4}R_1R_1'\right)^\dagger \left[\bar{B}'M_2 + S - \left(R + \frac{\gamma}{4}R_1R_1'\right)^\dagger \left(R + \frac{\gamma}{4}R_1R_1'\right) S\right]\\ &\quad -G_1 + \left(R + \frac{\gamma}{4}R_1R_1'\right)^\dagger \left(R + \frac{\gamma}{4}R_1R_1'\right) G_1,\end{aligned}$$

and

$$G_4 = -G_2 + \left[R + \frac{\gamma}{4}R_1R_1'\right]^\dagger \left[R + \frac{\gamma}{4}R_1R_1'\right] G_2.$$

By Girsanov theorem, the process:

$$d\widetilde{w}_2(t) = d\overline{w}_2(t) - r'(t)\Big[I + M_2P\Big]' \left(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)' D^{-\frac{1}{2}}dt, \tag{4.3.21}$$

is a Brownian motion under the measure $\widehat{\mathbb{P}}_u$. Substituting $u^*(t)$ into $dr(t)$, gives:

$$dr(t) = \left\{\left[\bar{A} + \bar{B}G_3 + \bar{C}D^{\frac{1}{2}}(D^{-\frac{1}{2}})'\left(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q_{12}'\right)(I + M_2P)'\right]r(t) + \bar{B}G_4\right\}dt$$

$$+\bar{C}D^{\frac{1}{2}}d\widetilde{w}_2(t).$$

For simplicity, we define the following matrices:

$$\begin{aligned}
\bar{G}_1 &= \bar{A} + \bar{B}G_3 + \bar{C}D^{\frac{1}{2}}(D^{-\frac{1}{2}})' \left(H + \frac{\gamma}{2}D^{\frac{1}{2}}Q'_{12}\right)(I + M_2P)', \\
\bar{G}_2 &= \bar{B}G_4, \\
\bar{G}_3 &= \bar{C}D^{\frac{1}{2}}.
\end{aligned}$$

Therefore, we can write the equation of $dr(t)$ as:

$$dr(t) = \left[\bar{G}_1 r(t) + \bar{G}_2\right]dt + \bar{G}_3 d\widetilde{w}_2(t). \tag{4.3.22}$$

□

4.4 Summary

We solved the generalized risk-sensitive control problem in an *indefinite* case in this chapter. By using a combination method of the completion of squares and the changing of probability measure, we have obtained explicit solutions to optimal control problems in both finite and infinite horizon. Especially for the infinite case, we introduced a general function into the cost functional, from which weaker conditions are needed for solving similar infinite optimal control problems. It will be interesting to extend these ideas to discrete-time setting, and explore the relation with robust controllers based on H_∞ control theory.

Chapter 5

Robust risk-sensitive control

5.1 Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), \mathbb{P})$ be a complete probability space on which a d-dimensional standard Brownian motion $(W(t), t \geq 0)$ is defined. We assume that $\mathcal{F}(t)$ is the augmentation of $\sigma\{W(s)|0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. Consider the linear stochastic control system:

$$\begin{cases} dx(t) = [A(t)x(t) + B_2(t)u(t) + B_1(t)v(t)]dt + A_1(t)dW(t) \\ \\ z(t) = \begin{bmatrix} C(t)x(t) \\ \\ D(t)u(t) \end{bmatrix}, \quad D'(t)D(t) = I \quad \forall t \geq 0, \\ \\ x(0) = x_0 \in \mathbb{R}^n \quad \text{is given.} \end{cases} \tag{5.1.1}$$

Here $x(\cdot)$ is the state of the system, $u(\cdot)$ is the control process, $v(\cdot)$ is the *disturbance*, and $z(\cdot)$ is the *output* of the system. We assume that:

$$\begin{aligned} A(\cdot) &\in L^\infty(0,T;\mathbb{R}^{n\times n}), \quad B_2(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n_u}), \quad B_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n_v}), \\ \\ A_1(\cdot) &\in L^\infty(0,T;\mathbb{R}^{n\times d}), \quad C(\cdot) \in L^\infty(0,T;\mathbb{R}^{m_c\times n}), \quad D(\cdot) \in L^\infty(0,T;\mathbb{R}^{m_d\times n_u}), \end{aligned}$$

where $L^\infty(0,T;E)$ denotes the set of all E-valued uniformly bounded functions. If $u(\cdot)$ and $v(\cdot)$ are square integrable processes, i.e. if $u(\cdot) \in L_{\mathcal{F}}(0,T;\mathbb{R}^{n_u})$ and $v(\cdot) \in L_{\mathcal{F}}(0,T;\mathbb{R}^{n_v})$, then (5.1.1) has a unique strong solution.

The H_2/H_∞ control problem is well known (see, for example, [28], [50], [58], [6], [20], [85], [53]). For deterministic systems, i.e. when $A_1(t) = 0$ for all $t \geq 0$, this problem is formulated in [52] as follows: given the *two* cost functionals

$$\begin{aligned} I_1(u(\cdot), v(\cdot)) &:= \int_0^T z'(t)z(t)dt, \\ I_2(u(\cdot), v(\cdot)) &:= \int_0^T (\theta^2 v'(t)v(t) - z'(t)z(t))dt, \end{aligned}$$

for some positive θ, find the optimal pair $(u^*(\cdot), v^*(\cdot))$ of the *optimal control* $u^*(\cdot)$ and the *worst case disturbance* $v^*(\cdot)$, that satisfy the *Nash equilibrium*

$$I_1(u^*(\cdot), v^*(\cdot)) \leq I_1(u(\cdot), v^*(\cdot)), \tag{5.1.2}$$

$$I_2(u^*(\cdot), v^*(\cdot)) \leq I_2(u^*(\cdot), v(\cdot)). \tag{5.1.3}$$

The inequality (5.1.2) indicates that $u^*(\cdot)$ minimizes the quadratic cost of the output (i.e. the *output energy*) under the worst-case disturbance, which corresponds to the "H_2" part of the problem. The inequality (5.1.3) ensures that under the worst-case disturbance the effect of the disturbance on the output, as measured by the quadratic cost I_2, is bounded, and this corresponds to the "H_∞" part of the problem. The infinite horizon formulation is very similar, but it requires that the system be stable under the pair $(u^*(\cdot), v^*(\cdot))$, which is not a requirement in the finite horizon formulation.

In [52], the solution to this control problem is obtained under the assumption of linear state feedback form of $u(\cdot)$ and $v(\cdot)$. It was shown that if a certain *coupled* pair of Riccati differential equations has a solution, then a unique explicit solution exists. It was also shown that the pair $(u^*(\cdot), v^*(\cdot))$ of deterministic systems is also the *optimal* pair for the *stochastic* systems with additive noise, i.e. when $A_1(t) \neq 0$ for all $t \geq 0$. This is an expected result considering the fact that both $I_1(u(\cdot), v(\cdot))$ and $I_2(u(\cdot), v(\cdot))$ are quadratic. In particular, in this case the pair $(u^*(\cdot), v^*(\cdot))$ is *independent* of the noise *intensity* $A_1(t)$.

There has been a great interest and progress in various *stochastic* versions of the H_2/H_∞ control problem in the past two decades (see, for example, [12]). The focus in recent years has been in the stochastic systems with *multiplicative* noise. However, in all these papers the criteria have been kept in the *quadratic form*.

In this chapter, we *generalise* the H_2/H_∞ control problem of [52] by using the following *exponential* quadratic criteria:

$$
\begin{aligned}
J_1(u(\cdot), v(\cdot)) &:= \frac{1}{\gamma_1}\mathbb{E}\exp\left[\frac{\gamma_1}{2}\int_0^T z'(t)z(t)dt\right],\\
J_2(u(\cdot), v(\cdot)) &:= \frac{1}{\gamma_2}\mathbb{E}\exp\left[\frac{\gamma_2}{2}\int_0^T (\theta^2 v(t)'v(t) - z'(t)z(t))dt\right],
\end{aligned}
$$

for some positive γ_1 and γ_2. The aim now is to find a pair $(u^*(\cdot), v^*(\cdot))$ such that the following inequalities hold:

$$
J_1(u^*(\cdot), v^*(\cdot)) \le J_1(u(\cdot), v^*(\cdot)), \quad \forall u(\cdot) \in \mathcal{A}_u \tag{5.1.4}
$$

$$
J_2(u^*(\cdot), v^*(\cdot)) \le J_2(u^*(\cdot), v(\cdot)), \quad \forall v(\cdot) \in \mathcal{A}_v. \tag{5.1.5}
$$

for some suitably defined sets $\mathcal{A}_u$ and $\mathcal{A}_v$.

Two motivations for this generalisation are that we still obtain an explicit solution for this control problem, and by using risk-sensitive criteria we further increase the potential for applications of this control method. In what follows, we consider separately the finite and infinite horizon cases. To simplify the notation, we suppress the argument t where appropriate.

5.2 Finite horizon

We confine ourselves to *linear* state feedback controls and disturbances. The sets of all such controls and disturbances are defined as:

$$
\mathcal{U} := \{u(\cdot): \quad u(t) = K_u(t)x(t) \quad \text{where } K_u(\cdot) \in L^\infty(0, T; \mathbb{R}^{n_u \times n})\},
$$

$$
\mathcal{V} := \{v(\cdot): \quad v(t) = K_v(t)x(t) \quad \text{where } K_v(\cdot) \in L^\infty(0, T; \mathbb{R}^{n_v \times n})\}.
$$

There exists a unique solution pair $(P_1(\cdot), P_2(\cdot))$ to the following coupled Riccati differential equations:

$$\begin{cases} \dot{P}_1 + C'C + P_1A + A'P_1 - P_1(B_2B_2' - \gamma_1A_1A_1')P_1 - \theta^{-2}(P_1B_1B_1'P_2 + P_2B_1B_1'P_1) = 0, \\ \\ \dot{P}_2 - C'C + P_2A + A'P_2 - P_2(\theta^{-2}B_1B_1' - \gamma_2A_1A_1')P_2 - P_1B_2B_2'P_1 - P_2B_2B_2'P_1 - P_1B_2B_2'P_2 = 0, \\ \\ P_1(T) = 0, \quad P_2(T) = 0. \end{cases}$$

The main aim of this section is to show that there exists a unique pair $(u^*(\cdot), v^*(\cdot))$ such that the inequalities (5.1.4) and (5.1.5) hold, and is given by:

$$u^*(t) \;:=\; -B_2'P_1x(t), \tag{5.2.1}$$

$$v^*(t) \;:=\; -\theta^{-2}B_1'P_2x(t). \tag{5.2.2}$$

If the noise $v^*(\cdot)$ is applied to the system (5.1.1), then for any $u(\cdot) \in \mathcal{U}$ we define:

$$\begin{aligned} \theta_u'(t) &:= -\gamma_1 x'(t)P_1(t)A_1, \\ Z_u(t) &:= \exp\left[-\int_0^t \theta_u'(\tau)dW(\tau) - \frac{1}{2}\int_0^t \theta_u'(\tau)\theta_u(\tau)d\tau\right], \\ Z_u &:= Z_u(T), \\ \widetilde{\mathbb{P}}_u &:= \int_\alpha Z_u(\omega)d\mathbb{P}(\omega), \quad \forall \alpha \in \mathcal{F}. \end{aligned}$$

A sufficient condition for $\widetilde{\mathbb{P}}_u$ to be a probability measure is the following Novikov condition:

$$\mathbb{E}\left\{\exp\left[\frac{\beta_u}{2}\int_0^T \theta_u'(t)\theta_u(t)dt\right]\right\} < \infty, \tag{5.2.3}$$

for some $\beta_u > 0$. We can now formulate the admissible set of controls as:

$$\mathcal{A}_u := \{u(\cdot) \in \mathcal{U} \quad \text{that satisfy (5.2.3)}\}.$$

Similarly, if the control $u^*(\cdot)$ is applied to the system (5.1.1), then for any $v(\cdot) \in \mathcal{V}$

we define:

$$\begin{aligned}
\theta_v'(t) &:= -\gamma_2 x'(t) P_2(t) A_1, \\
Z_v(t) &:= \exp\left[-\int_0^t \theta_v'(\tau) dW(\tau) - \frac{1}{2}\int_0^t \theta_v'(\tau)\theta_v(\tau) d\tau \right], \\
Z_v &:= Z_v(T), \\
\widetilde{\mathbb{P}}_v &:= \int_\alpha Z_v(\omega) d\mathbb{P}(\omega), \qquad \forall \alpha \in \mathcal{F}.
\end{aligned}$$

The sufficient Novikov condition for $\widetilde{\mathbb{P}}_v$ to be a probability measure is:

$$\mathbb{E}\left\{ \exp\left[\frac{\beta_v}{2} \int_0^T \theta_v'(t)\theta_v(t) dt \right] \right\} < \infty, \tag{5.2.4}$$

for some $\beta_v > 0$. The set of disturbances that we consider is:

$$\mathcal{A}_v := \{ v(\cdot) \in \mathcal{V} \quad \text{that satisfy (5.2.4)} \}.$$

Assumption 11. $(u^*(\cdot), v^*(\cdot)) \in \mathcal{A}_u \times \mathcal{A}_v$.

Theorem 5.2.1. *There exists a unique pair $(u^*(\cdot), v^*(\cdot))$ that satisfies the inequalities (5.1.4) and (5.1.5), and is given by (5.2.1) and (5.2.2). In this case we have*

$$\begin{aligned}
J_1(u^*(\cdot), v^*(\cdot)) &= \frac{1}{\gamma_1} \exp\left[\frac{\gamma_1}{2} x'(0) P_1(0) x(0) + \frac{\gamma_1}{2} \int_0^T tr(A_1' P_1 A_1) dt \right], \\
J_2(u^*(\cdot), v^*(\cdot)) &= \frac{1}{\gamma_2} \exp\left[\frac{\gamma_2}{2} x'(0) P_2(0) x(0) + \frac{\gamma_2}{2} \int_0^T tr(A_1' P_2 A_1) dt \right].
\end{aligned}$$

Proof. We first consider $J_1(u(\cdot), v^*(\cdot))$ with $u(\cdot) \in \mathcal{A}_u$. Since

$$\begin{aligned}
0 &= x'(0) P_1(0) x(0) \\
&+ \int_0^T [x' \dot{P}_1 x + 2x' P_1 (Ax + B_2 u + B_1 v^*) + tr(A_1' P_1 A_1)] dt + \int_0^T 2x' P_1 A_1 dW,
\end{aligned}$$

we can write $J_1(u(\cdot), v^*(\cdot))$ as:

$$J_1(u(\cdot), v^*(\cdot)) = \frac{1}{\gamma_1}\mathbb{E}\exp\left[\frac{\gamma_1}{2}x'(0)P_1(0)x(0) + \frac{\gamma_1}{2}\int_0^T tr(A_1'P_1A_1)dt\right.$$

$$+\frac{\gamma_1}{2}\int_0^T x'(\dot{P}_1 + C'C + P_1A + A'P_1 - \theta^{-2}P_1B_1B_1'P_2 - \theta^{-2}P_2B_1B_1'P_1)xdt$$

$$\left.+\frac{\gamma_1}{2}\int_0^T (u'u + 2x'P_1B_2u)dt + \frac{\gamma_1}{2}\int_0^T 2x'P_1A_1dW(t)\right].$$

Let $\widetilde{\mathbb{E}}_u$ denote the expectation under the probability measure $\widetilde{\mathbb{P}}_u$. For any $u(\cdot) \in \mathcal{A}_u$ we can write $J_1(u(\cdot), v^*(\cdot))$ as:

$$J_1(u(\cdot), v^*(\cdot)) = \frac{1}{\gamma_1}\widetilde{\mathbb{E}}_u\exp\left[\frac{\gamma_1}{2}x'(0)P_1(0)x(0) + \frac{\gamma_1}{2}\int_0^T tr(A_1'P_1A_1)dt\right.$$

$$+\frac{\gamma_1}{2}\int_0^T x'(\dot{P}_1 + C'C + P_1A + A'P_1 - \theta^{-2}P_1B_1B_1'P_2 - \theta^{-2}P_2B_1B_1'P_1 + \gamma_1P_1A_1A_1'P_1)xdt$$

$$\left.+\frac{\gamma_1}{2}\int_0^T (u'u + 2x'P_1B_2u)dt\right].$$

The completion of squares gives:

$$u'u + 2x'P_1B_2u = (u + B_2'P_1x)'(u + B_2'P_1x) - x'P_1B_2B_2'P_1x.$$

Due to our Riccati equation on $P_1(\cdot)$, for all $u(\cdot) \in \mathcal{A}_u$ we have:

$$\begin{aligned} J_1(u(\cdot), v^*(\cdot)) &= \frac{1}{\gamma_1}\widetilde{\mathbb{E}}_u\exp\left[\frac{\gamma_1}{2}x'(0)P_1(0)x(0) + \frac{\gamma_1}{2}\int_0^T tr(A_1'P_1A_1)dt\right. \\ &+ \left.\frac{\gamma_1}{2}\int_0^T (u + B_2'P_1x)'(u + B_2'P_1x)dt\right], \\ &\geq \frac{1}{\gamma_1}\exp\left[\frac{\gamma_1}{2}x'(0)P_1(0)x(0) + \frac{\gamma_1}{2}\int_0^T tr(A_1'P_1A_1)dt\right], \end{aligned}$$

with equality if and only if $u(t) = -B_2'P_1(t)x(t)$ for *a.e.* $t \in [0, T]$.

We now consider $J_2(u^*(\cdot), v(\cdot))$ for all $v(\cdot) \in \mathcal{A}_v$. Since

$$\begin{aligned} 0 &= x'(0)P_2(0)x(0) \\ &+ \int_0^T [x'\dot{P}_2 x + 2x'P_2(Ax + B_2u^* + B_1v) + tr(A_1'P_2A_1)]dt + \int_0^T 2x'P_2A_1 dW(t), \end{aligned}$$

we can write $J_2(u^*(\cdot), v(\cdot))$ as

$$J_2(u^*(\cdot), v(\cdot)) = \frac{1}{\gamma_2}\mathbb{E}\exp\Bigg[\frac{\gamma_2}{2}x'(0)P_2(0)x(0) + \frac{\gamma_2}{2}\int_0^T tr(A_1'P_2A_1)dt$$

$$+\frac{\gamma_2}{2}\int_0^T x'(\dot{P}_2 + P_2A + A'P_2 - P_2B_2B_2'P_1 - P_1B_2B_2'P_2 - C'C - P_1B_2B_2'P_1)xdt$$

$$+\frac{\gamma_2}{2}\int_0^T (\theta^2 v'v + 2x'P_2B_1v)dt + \frac{\gamma_2}{2}\int_0^T 2x'P_2A_1 dW(t)\Bigg].$$

If we denote by $\widetilde{\mathbb{E}}_v$ the expectation under the probability measure $\widetilde{\mathbb{P}}_v$, then for any $v(\cdot) \in \mathcal{A}_v$ we have:

$$J_2(u^*(\cdot), v(\cdot)) = \frac{1}{\gamma_2}\widetilde{\mathbb{E}}_v\exp\Bigg[\frac{\gamma_2}{2}x'(0)P_2(0)x(0) + \frac{\gamma_2}{2}\int_0^T tr(A_1'P_2A_1)dt$$

$$+\frac{\gamma_2}{2}\int_0^T x'(\dot{P}_2 + P_2A + A'P_2 - P_2B_2B_2'P_1 - P_1B_2B_2'P_2 - C'C - P_1B_2B_2'P_1 + \gamma_2P_2A_1A_1'P_2)xdt$$

$$+\frac{\gamma_2}{2}\int_0^T (\theta^2 v'v + 2x'P_2B_1v)dt\Bigg].$$

The completion of squares gives:

$$\theta^2 v'v + 2x'P_2B_1v = (v + \theta^{-2}B_1'P_2x)'\theta^2(v + \theta^{-2}B_1'P_2x) - \theta^{-2}x'P_2B_1B_1'P_2x(t).$$

Due to our Riccati equation on $P_2(\cdot)$, for any $v(\cdot) \in \mathcal{A}_v$ we have:

$$\begin{aligned} J_2(u^*(\cdot), v(\cdot)) &= \frac{1}{\gamma_2}\widetilde{\mathbb{E}}_v\exp\Bigg[\frac{\gamma_2}{2}x'(0)P_2(0)x(0) + \frac{\gamma_2}{2}\int_0^T tr(A_1'P_2A_1)dt \\ &+ \frac{\gamma_2}{2}\int_0^T (v + \theta^{-2}B_1'P_2x)'\theta^2(v + \theta^{-2}B_1'P_2x)dt\Bigg]. \end{aligned}$$

$$\geq \frac{1}{\gamma_2} \exp\left[\frac{\gamma_2}{2} x'(0)P_2(0)x(0) + \frac{\gamma_2}{2}\int_0^T tr(A_1'P_2A_1)dt\right],$$

with equality if and only if $v(t) = -\theta^{-2}B_1'P_2(t)x(t)$ for *a.e.* $t \in [0,T]$. □

5.3 Infinite horizon

Here we consider the *infinite* horizon version of our problem. The derivation is similar to the finite horizon, but it is more involved due to certain *stability* requirements, which are absent in the finite horizon case. The matrices A, A_1, B_1, B_2, C, D, are constant. We consider the following two functionals, which are the infinite horizon versions of $J_1(u(\cdot),v(\cdot))$ and $J_2(u(\cdot),v(\cdot))$:

$$\begin{aligned}
J_1^\infty(u(\cdot),v(\cdot)) &= \lim_{T\to\infty} \frac{1}{f_1(T)\gamma_1} \ln \mathbb{E} \exp\left[\frac{\gamma_1}{2}\int_0^T z'zdt\right], \\
J_2^\infty(u(\cdot),v(\cdot)) &= \lim_{T\to\infty} \frac{1}{f_2(T)\gamma_2} \ln \mathbb{E} \exp\left[\frac{\gamma_2}{2}\int_0^T (\theta^2 v'v - z'z)dt\right].
\end{aligned}$$

where $f_1(T)$ and $f_2(T)$ are given positive functions. Our aim is to find a pair $(u^*(\cdot),v^*(\cdot))$ such that the following inequalities hold:

$$J_1^\infty(u^*(\cdot),v^*(\cdot)) \leq J_1^\infty(u(\cdot),v^*(\cdot)), \quad \forall u(\cdot) \in \mathcal{A}_u^\infty \tag{5.3.1}$$

$$J_2^\infty(u^*(\cdot),v^*(\cdot)) \leq J_2^\infty(u^*(\cdot),v(\cdot)), \quad \forall v(\cdot) \in \mathcal{A}_v^\infty. \tag{5.3.2}$$

for some suitably defined sets $\mathcal{A}_u^\infty$ and $\mathcal{A}_v^\infty$. We confine ourselves only to linear *constant* state feedback controls and disturbances. The sets of all such controls and disturbances are defined as:

$$\mathcal{U}^\infty := \{u(\cdot): \quad u(t) = K_u x(t) \quad \text{where } K_u \in \mathbb{R}^{n_u\times n}\},$$

$$\mathcal{V}^\infty := \{v(\cdot): \quad v(t) = K_v x(t) \quad \text{where } K_v \in \mathbb{R}^{n_v\times n}\}.$$

There exists a solution pair (P_1,P_2) to the following coupled Riccati algebraic equations:

$$\begin{cases} C'C + P_1A + A'P_1 - P_1(B_2B_2' - \gamma_1 A_1A_1')P_1 - \theta^{-2}(P_1B_1B_1'P_2 + P_2B_1B_1'P_1) = 0, \\ \\ -C'C + P_2A + A'P_2 - P_2(\theta^{-2}B_1B_1' - \gamma_2 A_1A_1')P_2 - P_1B_2B_2'P_1 - P_2B_2B_2'P_1 - P_1B_2B_2'P_2 = 0, \end{cases}$$

The proof can be found in Anderson and Moore's book [1].

Let us define that functions $f_1(T)$ and $f_2(T)$ are such that:

$$\lim_{T\to\infty} \frac{tr(A_1'P_1A_1)T + x'(0)P_1x(0)}{2f_1(T)} = g_1 \in \mathbb{R},$$

$$\lim_{T\to\infty} \frac{tr(A_1'P_2A_1)T + x'(0)P_2x(0)}{2f_2(T)} = g_2 \in \mathbb{R}.$$

We show later in this section that there exists a unique pair $(u^*_\infty(\cdot), v^*_\infty(\cdot))$ that satisfies the inequalities (5.3.1) and (5.3.2), and is given by:

$$u^*_\infty(t) \;:=\; -B_2'P_1x(t), \tag{5.3.3}$$

$$v^*_\infty(t) \;:=\; -\theta^{-2}B_1'P_2x(t). \tag{5.3.4}$$

If the noise $v^*_\infty(\cdot)$ is applied to the system (5.1.1), then for any $u(\cdot) \in \mathcal{U}^\infty$ we define:

$$\begin{aligned} \theta_u'(t) &:= -\gamma_1 x'(t)P_1A_1, \\ Z_u(t) &:= \exp\left[-\int_0^t \theta_u'(\tau)dW(\tau) - \frac{1}{2}\int_0^t \theta_u'(\tau)\theta_u(\tau)d\tau \right], \\ Z_u &:= Z_u(T), \\ \widetilde{\mathbb{P}}_u &:= \int_\alpha Z_u(\omega)d\mathbb{P}(\omega), \quad \forall \alpha \in \mathcal{F}. \end{aligned}$$

A sufficient condition for $\widetilde{\mathbb{P}}_u$ to be a probability measure is the following Novikov condition:

$$\mathbb{E}\left\{ \exp\left[\frac{\beta_u}{2} \int_0^T \theta_u'(t)\theta_u(t)dt \right] \right\} < \infty, \tag{5.3.5}$$

for some $\beta_u > 0$. The controls $u(\cdot)$ are restricted further, so that the following *stability* condition holds:

$$\lim_{T\to\infty} \frac{1}{f_1(T)\gamma_1} \ln \widetilde{\mathbb{E}}_u \exp\left[-\frac{\gamma_1}{2} x(T)' P_1 x(T)\right] = h_1 \in \mathbb{R}. \tag{5.3.6}$$

We can now formulate the admissible set of controls as:

$\mathcal{A}_u^\infty := \{u(\cdot) \in \mathcal{U}$ such that (5.3.5) holds for all $T \in (0,\infty)$, and (5.3.6) holds$\}$.

Similarly, if the control $u_\infty^*(\cdot)$ is applied to the system (5.1.1), then for any $v(\cdot) \in \mathcal{V}^\infty$ we define:

$$\begin{aligned}
\theta_v'(t) &:= -\gamma_2 x'(t) P_2 A_1, \\
Z_v(t) &:= \exp\left[-\int_0^t \theta_v'(\tau) dW(\tau) - \frac{1}{2}\int_0^t \theta_v'(\tau)\theta_v(\tau) d\tau\right], \\
Z_v &:= Z_v(T), \\
\widetilde{\mathbb{P}}_v &:= \int_\alpha Z_v(\omega) d\mathbb{P}(\omega), \qquad \forall \alpha \in \mathcal{F}.
\end{aligned}$$

The sufficient Novikov condition for $\widetilde{\mathbb{P}}_v$ to be a probability measure is:

$$\mathbb{E}\left\{\exp\left[\frac{\beta_v}{2}\int_0^T \theta_v'(t)\theta_v(t) dt\right]\right\} < \infty, \tag{5.3.7}$$

for some $\beta_v > 0$. We further restrict the set of permitted disturbances, so that the following holds:

$$\lim_{T\to\infty} \frac{1}{f_2(T)\gamma_2} \ln \mathbb{E} \exp\left[-\frac{\gamma_2}{2} x(T)' P_2 x(T)\right] = h_2 \in \mathbb{R}. \tag{5.3.8}$$

The set of disturbances that we consider can now be defined as:

$\mathcal{A}_v := \{v(\cdot) \in \mathcal{V}$ such that (5.3.7) holds for all $T \in (0,\infty)$, and (5.3.8) holds$\}$.

Assumption 12. $(u_\infty^*(\cdot), v_\infty^*(\cdot)) \in \mathcal{A}_u^\infty \times \mathcal{A}_v^\infty$.

Theorem 5.3.1. *There exists a unique pair $(u_\infty^*(\cdot), v_\infty^*(\cdot))$ that satisfies the inequalities (5.3.1) and (5.3.2), and is given by (5.3.3) and (5.3.4). In this case we* *have*

$$\begin{aligned} J_1^\infty(u^*(\cdot), v^*(\cdot)) &= g_1 + h_1, \\ J_2^\infty(u^*(\cdot), v^*(\cdot)) &= g_2 + h_2. \end{aligned}$$

Proof. Let us first consider $J_1^\infty(u(\cdot), v_\infty^*(\cdot))$ with $u(\cdot) \in \mathcal{A}_u^\infty$. Since

$$\begin{aligned} 0 &= -x'(T)P_1x(T) + x'(0)P_1x(0) \\ &+ \int_0^T [2x'P_1(Ax + B_2u + B_1v_\infty^*) + tr(A_1'P_1A_1)]dt + \int_0^T 2x'P_1A_1dW, \end{aligned}$$

we can write $J_1^\infty(u(\cdot), v_\infty^*(\cdot))$ as:

$$J_1^\infty(u(\cdot), v_\infty^*(\cdot)) = \lim_{T\to\infty} \frac{1}{f_1(T)\gamma_1} \ln \mathbb{E} \exp \left[\frac{\gamma_1}{2} x'(0)P_1x(0) + \frac{\gamma_1}{2} tr(A_1'P_1A_1)T \right.$$

$$+\frac{\gamma_1}{2} \int_0^T x'(C'C + P_1A + A'P_1 - \theta^{-2}P_1B_1B_1'P_2 - \theta^{-2}P_2B_1B_1'P_1)xdt$$

$$\left. -\frac{\gamma_1}{2} x'(T)P_1x(T) + \frac{\gamma_1}{2} \int_0^T (u'u + 2x'P_1B_2u)dt + \frac{\gamma_1}{2} \int_0^T 2x'P_1A_1dW(t) \right].$$

For any $u(\cdot) \in \mathcal{A}_u^\infty$ we can write $J_1^\infty(u(\cdot), v_\infty^*(\cdot))$ as:

$$J_1^\infty(u(\cdot), v_\infty^*(\cdot)) = \lim_{T\to\infty} \frac{1}{f_1(T)\gamma_1} \ln \widetilde{\mathbb{E}}_u \exp \left[\frac{\gamma_1}{2} x'(0)P_1x(0) + \frac{\gamma_1}{2} tr(A_1'P_1A_1)T \right.$$

$$+\frac{\gamma_1}{2} \int_0^T x'(C'C + P_1A + A'P_1 - \theta^{-2}P_1B_1B_1'P_2 - \theta^{-2}P_2B_1B_1'P_1 + \gamma_1P_1A_1A_1'P_1)xdt$$

$$\left. -\frac{\gamma_1}{2} x'(T)P_1x(T) + \frac{\gamma_1}{2} \int_0^T (u'u + 2x'P_1B_2u)dt \right].$$

The completion of squares gives:

$$u'u + 2x'P_1B_2u = (u + B_2'P_1x)'(u + B_2'P_1x) - x'P_1B_2B_2'P_1x.$$

Due to our Riccati equation on P_1, for all $u(\cdot) \in \mathcal{A}_u^\infty$ we have:

$$\begin{aligned}
J_1^\infty(u(\cdot), v_\infty^*(\cdot)) &= \lim_{T\to\infty} \frac{1}{f_1(T)\gamma_1} \ln \widetilde{\mathbb{E}}_u \exp\left[\frac{\gamma_1}{2} x'(0)P_1x(0) + \frac{\gamma_1}{2} tr(A_1'P_1A_1)T\right. \\
&- \left.\frac{\gamma_1}{2} x'(T)P_1x(T) + \frac{\gamma_1}{2}\int_0^T (u+B_2'P_1x)'(u+B_2'P_1x)dt\right], \\
&\geq g_1 + h_1,
\end{aligned}$$

with equality if and only if $u(t) = -B_2'P_1x(t)$ for $a.e.$ $t \in [0,\infty)$.

We now consider $J_2^\infty(u_\infty^*(\cdot), v(\cdot))$ for all $v(\cdot) \in \mathcal{A}_v^\infty$. Since

$$\begin{aligned}
0 &= -x'(T)P_2x(T) + x'(0)P_2x(0) \\
&+ \int_0^T [2x'P_2(Ax + B_2u_\infty^* + B_1v) + tr(A_1'P_2A_1)]dt + \int_0^T 2x'P_2A_1dW(t),
\end{aligned}$$

we can write $J_2^\infty(u_\infty^*(\cdot), v(\cdot))$ as

$$\begin{aligned}
&J_2^\infty(u_\infty^*(\cdot), v(\cdot)) = \lim_{T\to\infty} \frac{1}{f_2(T)\gamma_2} \ln \mathbb{E} \exp\left[\frac{\gamma_2}{2} x'(0)P_2x(0) + \frac{\gamma_2}{2} tr(A_1'P_2A_1)T\right. \\
&+\frac{\gamma_2}{2}\int_0^T x'(P_2A + A'P_2 - P_2B_2B_2'P_1 - P_1B_2B_2'P_2 - C'C - P_1B_2B_2'P_1)xdt \\
&\left.-\frac{\gamma_2}{2} x'(T)P_2x(T) + \frac{\gamma_2}{2}\int_0^T (\theta^2 v'v + 2x'P_2B_1v)dt + \frac{\gamma_2}{2}\int_0^T 2x'P_2A_1dW(t)\right].
\end{aligned}$$

For any $v(\cdot) \in \mathcal{A}_v^\infty$ we have:

$$\begin{aligned}
&J_2^\infty(u_\infty^*(\cdot), v(\cdot)) = \lim_{T\to\infty} \frac{1}{f_2(T)\gamma_2} \ln \widetilde{\mathbb{E}}_v \exp\left[\frac{\gamma_2}{2} x'(0)P_2x(0) + \frac{\gamma_2}{2} tr(A_1'P_2A_1)T\right. \\
&+\frac{\gamma_2}{2}\int_0^T x'(P_2A + A'P_2 - P_2B_2B_2'P_1 - P_1B_2B_2'P_2 - C'C - P_1B_2B_2'P_1 + \gamma_2P_2A_1A_1'P_2)xdt \\
&\left.-\frac{\gamma_2}{2} x'(T)P_2x(T) + \frac{\gamma_2}{2}\int_0^T (\theta^2 v'v + 2x'P_2B_1v)dt\right].
\end{aligned}$$

The completion of squares gives:

$$\theta^2 v'v + 2x'P_2B_1v = (v + \theta^{-2}B_1'P_2x)'\theta^2(v + \theta^{-2}B_1'P_2x) - \theta^{-2}x'P_2B_1B_1'P_2x(t).$$

Due to our Riccati equation on P_2, for any $v(\cdot) \in \mathcal{A}_v^\infty$ we have:

$$\begin{aligned}
J_2^\infty(u_\infty^*(\cdot), v(\cdot)) &= \lim_{T\to\infty} \frac{1}{f_2(T)\gamma_2} \ln \widetilde{\mathbb{E}}_v \exp\Bigg[\frac{\gamma_2}{2}x'(0)P_2x(0) + \frac{\gamma_2}{2}tr(A_1'P_2A_1)T \\
&- \frac{\gamma_2}{2}x'(T)P_2x(T) + \frac{\gamma_2}{2}\int_0^T (v + \theta^{-2}B_1'P_2x)'\theta^2(v + \theta^{-2}B_1'P_2x)dt\Bigg]. \\
&\geq g_2 + h_2,
\end{aligned}$$

with equality if and only if $v(t) = -\theta^{-2}B_1'P_2(t)x(t)$ for *a.e.* $t \in [0, \infty)$. □

5.4 Summary

In this chapter we introduce a risk-sensitive version of the H_2/H_∞ control method for linear stochastic systems with additive noise. With the assumption of linear state-feedback controllers, both the finite and infinite horizon cases are considered, and explicit solutions in terms of Riccati equations are obtained. It will be interesting to extend these ideas to nonlinear and discrete-time systems.

Chapter 6

Linear-quadratic control and risk-sensitive control problems for stochastic linear control system with delay

6.1 Introduction

Time-delay systems are also called systems with aftereffect or dead-time. The LQ optimal control theory of systems with delays in state and control variables has been studied by several authors from different view-points, and some important papers build up the fundamental theories of the delay problem, such as Ichikawa [32][33], Kwong and Willsky [41][42], R. H. Kwong [44] [40] [43], Koivo and Lee etc. [36].

Most of the authors in the early period of the research for this topic solved the delay problem by the state-space techniques with different approaches. For example, Ichikawa aims at a general theory for a family of evolution equations with a control operator containing a finite number of pure delays; the work of Kwong and Willsky deals with differential delay equations and a less general control operator which does not contain pure delays. However the state-space technique only apply to control operators with a special delay structure which contains a finite number of delays and an integral term on the length of the memory but not applied generally.

In the paper of Chen and Wu [13], consider the following LQ system:

$$\begin{cases} dx(t) = [Ax(t) + A_1x(t-\delta) + M_tu(t) + M_t^1u(t-\delta)]dt \\ \qquad\qquad + [Cx(t) + C_1x(t-\delta) + D_tu(t) + D_t^1u(t-\delta]dw(t) \\ x(t) = \phi(t), u(t) = \eta(t), \quad t \in [-\delta, 0]. \end{cases} \tag{6.1.1}$$

where, ϕ, $\eta(t) \in C[-\delta, 0]^n$ is deterministic functions, satisfying $\int_{-\delta}^0 \alpha^2(s)ds < +\infty$, $\alpha = \phi$. Giving the cost functional as:

$$J(u(\cdot)) = \frac{1}{2}\mathbb{E}\Bigg[\int_0^T x'(t)Qx(t) + u'(t)Ru(t)dt + x(T)Sx(T)\Bigg]. \tag{6.1.2}$$

They encountered the stochastic differential equations:

$$\begin{cases} -dY(t) = f(t, Y(t), Z(t), Y(t+\delta(t)), Z(t+\zeta(t)))dt - Z(t)dB(t), \quad t \in [0, T], \\ Y(t) = \xi(t), Z(t) = \eta(t), \quad t \in [T, T+K], \end{cases}$$

Where Y and z are the solution pair to a specific backward stochastic equation, $\delta(\cdot)$ and $\zeta(t)$ are R^+-valued functions defined on [0,T] (See Peng and Yang [60] in the optimization problem). By the backward stochastic differential equation (FBSDE) method, this paper gives the feedback regulator in terms of the conditional expectation of the future information as follows:

$$\begin{aligned} u(t) &= -R^{-1}\Bigg[M_t^\tau y(t) + D_t^\tau z(t) \\ &\quad + \mathbb{E}^{\mathcal{F}_t}\left((M_{t+\delta}^1)^\tau y(t+\tau) + \left(D_{t+\delta}^1\right)^\tau z(t+\delta)\right)\Bigg], \quad t \in [0, T]. \end{aligned} \tag{6.1.3}$$

This paper solved a very general case of the LQ delay problem, however, the result they found about the optimal controller is not explicit and difficult to be applied in real situation. The advantage of our approach is that our solution is in a feed-back form, which is not the case in Chen and Wu, where the solution is found in a conditional expectation.

The research on delay system developed rapidly in recent years, since the stochastic delay differential equations could be applied in a lot of fields, such as finance, engineering and physics. The robust control problem and H_∞ control problem for time-delay systems were solved in a lot of papers, such as Tadmor([72]- [76]), Zhou [84], Uchida [77], Nagpal [59].

In the past decades, there are few papers related with risk-sensitive delay control problem have been published. In Speyer and Banavar [2], a differential game approach was employed to solve such problem for continuous time systems, while a dynamic programming was applied [70]. In Zhao and Cui [83], the risk-sensitive estimation problem for systems with constant delay was considered by employing indefinite space approach. In Yoneyama [82], he used a change of measure technique to solve the partially observed risk-sensitive control problem with delay system. However, the delay term in the state system is only occurred on the controller $u(t)$.

In this chapter we analyzed the delay system in two different directions: one is combined with LQ control problem, and the other one is focused on the exponential criteria: generalized risk-sensitive control. The organization of this chapter is as follows: in section 2, we pay attention to the LQ problem. In section 3, we focused on the generalized risk-sensitive control problem with delay system, and a conclusion is given in section 4.

6.2 Linear-quadratic problem with delay system

In this section, we are dealing with cases of the linear-quadratic problem with input delays in a particular time horizon, i.e. $t \in [0, 2\delta]$.

6.2.1 Problem formulation and assumptions

Consider the linear stochastic control system:

$$\begin{cases} dx(t) = [A_1x(t) + A_2x(t-\delta)]dt + [A_3x(t-\delta) + B_1u(t) + B_2u(t-\delta)]dw(t), \quad t \in [0, 2\delta], \\ \\ x(t) = \phi(t), \quad u(t) = \eta(t), \quad t \in [-\delta, 0]. \end{cases} \tag{6.2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $u(t)$ is $\mathcal{F}$-adapted process such that (6.2.1) has a unique solution, A_1, A_2, A_3, B_1, B_2 are given matrices in proper dimensions, δ is a positive real number.

The cost functional is as follows:

$$J(u(\cdot)) \equiv \mathbb{E}\left\{ \int_0^{2\delta} \big[x(t)Q_1x(t) + x(t-\delta)Q_2x(t-\delta) + u(t)'R_1u(t) + u(t-\delta)R_2u(t-\delta)\big]\, dt \right\}, \tag{6.2.2}$$

where Q_1, Q_2, R_1, R_2 are given matrices in proper dimensions.

We wish to solve the problem:

$$\begin{cases} \min\limits_{u(\cdot)\in\mathcal{A}} J(u(\cdot)), \\ \\ \text{s.t. (6.2.1) holds,} \end{cases} \tag{6.2.3}$$

$\mathcal{A}$ is a set where u is the solution to the problem (6.2.3).

For simplicity, let us define the following matrices:

$$x_t = \begin{bmatrix} x(t) \\ \\ x(t-\delta) \end{bmatrix}, \quad u_t = \begin{bmatrix} u(t) \\ \\ u(t-\delta) \end{bmatrix}, \quad w_t = \begin{bmatrix} w(t) \\ \\ w(t-\delta) \end{bmatrix},$$

$$\bar{A}_1 = \begin{bmatrix} A_1 & A_2 \\ \\ 0 & A_1 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 0 \\ \\ A_2\phi(t) \end{bmatrix}, \quad \bar{A}_3 = \begin{bmatrix} 0 & A_3 \\ \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_4 = \begin{bmatrix} 0 \\ \\ A_3\phi(t) \end{bmatrix},$$

$$\bar{B}_1 = \begin{bmatrix} B_1 & B_2 \\ \\ 0 & B_1 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 0 \\ \\ B_2\eta(t) \end{bmatrix}.$$

Consider the following linear equations:

Matrix $P(t) = \begin{bmatrix} P_1(t) & P_2(t) \\ P_2(t) & P_3(t) \end{bmatrix}$ satisfies the following:

$$\begin{cases} \dot{P}(t) + P(t)\bar{A}_1 + \bar{A}_1P(t) + \bar{Q} + \bar{A}_3P(t)\bar{A}_3 = 0, \\ \\ P(2\delta) = 0. \end{cases} \tag{6.2.4}$$

and another matrix $G(t)$ which satisfies:

$$\begin{cases} \dot{G}' + G'(t)\bar{A}_1 + 2\bar{A}_2' P(t) + 2\bar{A}_4' P(t)\bar{A}_3 + 2\bar{B}_2' P\bar{A}_3 = 0, \\ \\ G(2\delta) = 0. \end{cases} \tag{6.2.5}$$

We define the following variables for simplicity:

$$\begin{aligned} \widetilde{R} &= R_1 + B_1' H(t)B_1(t) + R_2 + B_2' P_1(t)B_2 + 2B_1' P_2(t)B_2 + B_1' P_3(t)B_1 \\ &\quad -\Big[B_2' P_1(t)B_1 + B_1' P_2(t)B_1\Big]\Big[R_1 + B_1' P_1(t)B_1\Big]^{-1}\Big[B_1' P_1(t)B_2 + B_1' P_2(t)B_1\Big], \\ K_1 &= \Big[B_2' P_1(t) + B_1' P_2(t)\Big]A_3 - \Big[B_2' P_1(t)B_1 + B_1' P_2(t)B_1\Big]\Big[R_1 + B_1' P_1(t)B_1\Big]^{-1} B_1' P_1(t)A_3, \\ K_2 &= B_1' H(t)A_3\phi(t) + B_1' H(t)B_2\eta(t) + \Big[B_2' P_2(t) + B_1' P_3(t)\Big]A_3\phi(t) \\ &\quad +\Big[B_2' P_2(t) + B_1' P_3(t)\Big]B_2\eta(t) - \Big[B_2' P_1(t)B_1 + B_1' P_2(t)B_1\Big] \\ &\quad \times\Big[R_1 + B_1' P_1(t)B_1\Big]^{-1}\Big[B_1' P_2(t)A_3\phi(t) + B_1' P_2(t)B_2\eta(t)\Big]. \end{aligned}$$

Let introduce a matrix $H(t)$ which satisfies:

$$\begin{cases} Q_1 + \dot{H}(t) + H(t)A_1 + A_1 H(t) - A_3' P_1(t)B_1 \\ \\ \quad \times\Big[R_1 + B_1' P_1(t)B_1\Big]^{-1} B_1' P_1 A_3 - K_1' \widetilde{R}^{-1} K_1 \quad = 0, \\ \\ H(\delta) = P_1(\delta). \end{cases} \tag{6.2.6}$$

and matrix $M(t)$ where:

$$\begin{cases} \dot{M}(t) + \frac{1}{2}\Big[M'(t)A_1 + A_1'M(t)\Big] + 2A_2'H(t)\phi(t) - 2K_2'\widetilde{R}^{-1}K_1 \\ \quad - 2\Big[A_3'P_2(t)B_1\phi(t) + B_2'P_2(t)B_1\eta(t)\Big]\Big[R_1 + B_1'P_1(t)B_1\Big]^{-1} B_1'P_1A_3 = 0, \\ M(\delta) = 2P_2(\delta)x(0) + G_1(\delta). \end{cases} \tag{6.2.7}$$

The following are the main results of this section:

Theorem 6.2.1. *Let the above equations hold, the unique solution to problem 6.2.3 and the corresponding cost are as follow respectively:*

$$u^*(t) = \begin{cases} -\widetilde{R}^{-1}\Big[\frac{1}{2}x_t[A_3P_1B_2 + A_3P_2B_1] + \frac{1}{2}[A_3P_2B_2\phi(t) + A_3P_3B_1\phi(t)] \\ \quad - [x_tA_3P_1B_1 + A_3P_2B_1\phi(t) + B_2P_2B_1\eta(t)] \\ \quad \times [R_1 + B_1P_1B_1]^{-1}[B_2P_1B_1 + B_1P_2B_1]\Big]', \quad \text{if } t \in [0,\delta], \\ -\Big[R_1 + B_1'P_1(t)B_1\Big]^{-1}\Big[[B_1'P_1(t)B_2 + B_1'P_2(t)B_1]u(t-\delta) \\ \quad + B_1'P_1(t)A_3x(t-\delta) + B_1'P_2(t)A_3\phi(t) + B_1'P_2(t)B_2\eta(t)\Big], \quad \text{if } t \in [\delta, 2\delta]. \end{cases} \tag{6.2.8}$$

and

$$\begin{aligned} J(u(\cdot)) \;=\; & \mathbb{E}\int_0^{2\delta}\Big\{\bar{A}_4'P(t)\bar{A}_4 + 2\bar{A}_4'P(t)\bar{B}_2 + \bar{B}_2'P(t)\bar{B}_2 + G'(t)\bar{A}_2\Big\}dt \\ & +\mathbb{E}\int_0^{\delta}\Big\{Q_2(t)\phi^2(t) + R_2(t)\eta^2(t) + A_3'H(t)A_3\phi^2(t) + 2A_3'H(t)B_2\phi(t)\eta(t) \\ & \quad +B_2'H(t)B_2\eta^2(t) + M'(t)A_2\phi(t) + x'(0)H(0)x(0) + M(0)x(0) \end{aligned}$$

$$
\begin{aligned}
&+x'(0)P_3(\delta)x(0)+G_2(\delta)x(0)-K_2'\widetilde{R}^{-1}K_2\Bigg\}dt \\
-\mathbb{E}\int_\delta^{2\delta}&\Bigg\{\Big[A_3'P_2(t)\phi(t)B_1+B_2'P_2(t)\eta(t)B_1\Big]\Big[R_1+B_1'P_1(t)B_1\Big]^{-1} \\
&\times\Big[A_3'P_2(t)\phi(t)B_1+B_2'P_2(t)\eta(t)B_1\Big]'\Bigg\}dt. \qquad (6.2.9)
\end{aligned}
$$

The proof of the Theorem 1 will be given in the following subsection.

6.2.2 Proof of Theorem 6.2.1

From (6.2.1), we can rewrite the state system as

$$
\begin{cases}
dx(t)=[A_1x(t)+A_2x(t-\delta)]dt+[A_3x(t-\delta)+B_1u(t)+B_2u(t-\delta)]dw(t), \quad t\geq 0,\\
dx(t-\delta)=[A_1x(t-\delta)+A_2\phi(t)]dt+[A_3\phi(t)+B_1u(t-\delta)+B_2\eta(t)]dw(t-\delta), \quad t>\delta,\\
x(t)=\phi(t), u(t)=\eta(t), \quad t\in[-\delta,0].
\end{cases} \qquad (6.2.10)
$$

The state system can be written as:

$$
dx_t=[\bar{A}_1x_t+\bar{A}_2]dt+[\bar{A}_3x_t+\bar{A}_4+\bar{B}_1u_t+\bar{B}_2]dw_t, \quad t\geq\delta. \qquad (6.2.11)
$$

The cost functional is as follows:

$$
J(u(\cdot)) \;\equiv\; \mathbb{E}\left\{\int_0^{2\delta}\left[x_t'\bar{Q}x_t+u_t'\bar{R}u_t\right]dt\right\}. \qquad (6.2.12)
$$

where

$$
\bar{Q}=\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2\end{bmatrix}, \bar{R}=\begin{bmatrix} R_1 & 0 \\ 0 & R_2\end{bmatrix}.
$$

The differential of the quadratic form $x'(t)P(t)x(t)$ is:

$$
x_{2\delta}'P(2\delta)x_{2\delta}-x_\delta'P(\delta)x_\delta \;=\; \int_\delta^{2\delta}\Big\{x_t\dot{P}(t)x_t+2x_t'P(t)(\bar{A}_1x_t+\bar{A}_2)
$$

$$+(x_t'\bar{A}_3' + \bar{A}_4' + u_t'\bar{B}_1' + \bar{B}_2')P(t)(\bar{A}_3x_t + \bar{A}_4 + \bar{B}_1u_t + \bar{B}_2)\Big\}dt$$

$$+\int_\delta^{2\delta} 2x_t'P(t)(\bar{A}_3x_t + \bar{A}_4 + \bar{B}_1u_t + \bar{B}_2)dw_t,$$

By the assumption on matrix $G(t)$, it is clear that:

$$-\mathbb{E}\Big[G'(\delta)x_\delta\Big] = \mathbb{E}\int_\delta^{2\delta}\Big[(\dot{G}'(t)+G'(t)\bar{A}_1)x_t+G'(t)\bar{A}_2\Big]dt+\mathbb{E}\int_\delta^{2\delta}G'(t)(\bar{A}_3x_t+\bar{A}_4+\bar{B}_1u_t+\bar{B}_2)dw_t.$$

Therefore, $J(u(\cdot))$ can be written as:

$$\begin{aligned}
J(u(\cdot)) \quad = \quad & \mathbb{E}\int_0^\delta\Big[x'(t)Q_1x(t) + \phi'(t)Q_2\phi(t) + u(t)'R_1u(t) + \eta'(t)R_2\eta(t)\Big]dt \\
& +\mathbb{E}\Big[G'(\delta)x_\delta + x_\delta'P(\delta)x_\delta\Big] \\
& +\mathbb{E}\int_\delta^{2\delta}\Big\{x_t'\Big[\dot{P}(t) + P(t)\bar{A}_1 + \bar{A}_1'P(t) + \bar{Q} + \bar{A}_3'P(t)\bar{A}_3\Big]x_t \\
& \quad +\Big[2\bar{A}_2'P(t) + 2\bar{A}_4'P(t)\bar{A}_3 + 2\bar{B}_2'P\bar{A}_3 + \dot{G}(t) + G'(t)\bar{A}_1\Big]x_t \\
& \quad +u_t'\Big[\bar{R} + \bar{B}_1'P(t)\bar{B}_1\Big]u_t + 2u_t'\Big[\bar{B}_1'P(t)\bar{A}_3x_t + \bar{B}_1'P(t)\bar{A}_4 + \bar{B}_1'P(t)\bar{B}_2\Big] \\
& \quad +\Big[\bar{A}_4'P(t)\bar{A}_4 + 2\bar{A}_4'P(t)\bar{B}_2 + \bar{B}_2'P(t)\bar{B}_2 + G'(t)\bar{A}_2\Big]\Big\}dt. \qquad (6.2.13)
\end{aligned}$$

By the equations of $P(t)$ and $G(t)$, we have all terms related to x_t eliminated. Since the cost function $J(u(\cdot))$ is divided into two parts based on different time horizons, we have to do some analysis before applying the completion of squares method on u_t terms:

$$\begin{aligned}
& u_t'\Big[\bar{R} + \bar{B}_1'P(t)\bar{B}_1\Big]u_t + 2u_t'\Big[\bar{B}_1'P(t)\bar{A}_3x_t + \bar{B}_1'P(t)\bar{A}_4 + \bar{B}_1'P(t)\bar{B}_2\Big] \\
= \quad & u'(t)\Big[R_1 + B_1'P_1(t)B_1\Big]u(t) + 2u'(t)\Big[\big(B_1'P_1(t)B_2 + B_1'P_2(t)B_1\big)u(t-\delta) \\
& +B_1'P_1(t)A_3x(t-\delta) + B_1'P_2(t)A_3\phi(t) + B_1'P_2(t)B_2\eta(t)\Big]
\end{aligned}$$

$$
\begin{aligned}
&+u'(t-\delta)\Big[R_2 + B_2'P_1(t)B_2 + 2B_1'P_2(t)B_2 + B_1'P_3(t)B_1\Big]u(t-\delta)\\
&+2u'(t-\delta)\Big[\big(B_2'P_1(t) + B_1'P_2(t)\big)A_3x(t-\delta) + \big(B_2'P_2(t) + B_1'P_3(t)\big)A_3\phi(t)\\
&\qquad + \big(B_2'P_2(t) + B_1'P_3(t)\big)B_2\eta(t)\Big]. \qquad (6.2.14)
\end{aligned}
$$

Substitute above equation into (6.2.13), and we perform the completion of squares for the terms inside the integral which contains $u(t)$:

$$
\begin{aligned}
&u'(t)\Big[R_1 + B_1'P_1(t)B_1\Big]u(t) + 2u'(t)\Big[\big(B_1'P_1(t)B_2 + B_1'P_2(t)B_1\big)u(t-\delta)\\
&\qquad +B_1'P_1(t)A_3x(t-\delta) + B_1'P_2(t)A_3\phi(t) + B_1'P_2(t)B_2\eta(t)\Big]\\
= \;&\Big\{u(t) + \Big[R_1 + B_1'P_1(t)B_1\Big]^{-1}\Big[\big(B_1'P_1(t)B_2 + B_1'P_2(t)B_1\big)u(t-\delta)\\
&\qquad +B_1'P_1(t)A_3x(t-\delta) + B_1'P_2(t)A_3\phi(t) + B_1'P_2(t)B_2\eta(t)\Big]\Big\}'\Big[R_1 + B_1'P_1(t)B_1\Big]\\
&\times\Big\{u(t) + \Big[R_1 + B_1'P_1(t)B_1\Big]^{-1}\Big[\big(B_1'P_1(t)B_2 + B_1'P_2(t)B_1\big)u(t-\delta)\\
&\qquad +B_1'P_1(t)A_3x(t-\delta) + B_1'P_2(t)A_3\phi(t) + B_1'P_2(t)B_2\eta(t)\Big]\Big\}\\
&-\Big[\big(B_1'P_1(t)B_2 + B_1'P_2(t)B_1\big)u(t-\delta)\\
&\qquad +B_1'P_1(t)A_3x(t-\delta) + B_1'P_2(t)A_3\phi(t) + B_1'P_2(t)B_2\eta(t)\Big]'\Big[R_1 + B_1'P_1(t)B_1\Big]^{-1}\\
&\times\Big[\big(B_1'P_1(t)B_2 + B_1'P_2(t)B_1\big)u(t-\delta)\\
&\qquad +B_1'P_1(t)A_3x(t-\delta) + B_1'P_2(t)A_3\phi(t) + B_1'P_2(t)B_2\eta(t)\Big].
\end{aligned}
$$

Therefore, we obtained the optimal $u(t) = u(t)^*$ when $t \in [\delta, 2\delta]$ as following:

$$
\begin{aligned}
u^*(t) &= -\Big[R_1 + B_1' P_1(t) B_1\Big]^{-1} \Big[(B_1' P_1(t) B_2 + B_1' P_2(t) B_1) u(t-\delta) \\
&\quad + B_1' P_1(t) A_3 x(t-\delta) + B_1' P_2(t) A_3 \phi(t) + B_1' P_2(t) B_2 \eta(t)\Big]. \quad (6.2.15)
\end{aligned}
$$

Now let us turn attention to the time horizon $t \in [0, \delta]$. First we substitute u^* back to $J(u(\cdot))$:

$$
\begin{aligned}
J(u(\cdot)) &= \mathbb{E}\int_0^{\delta} \Big[x'(t) Q_1 x(t) + Q_2 \phi^2(t) + u(t)' R_1 u(t) + R_2 \eta^2(t)\Big] dt \\
&\quad + G_1'(\delta) x(\delta) + G_2'(\delta) x(0) + x'(\delta) P_1(\delta) x(\delta) + 2x'(\delta) P_2(\delta) x(0) + x'(0) P_3(\delta) x(0) \\
&\quad + \mathbb{E}\int_0^{2\delta} \Big[\bar{A}_4' P(t) \bar{A}_4 + 2\bar{A}_4' P(t) \bar{B}_2 + \bar{B}_2' P(t) \bar{B}_2 + G'(t) \bar{A}_2\Big] dt \\
&\quad + \mathbb{E}\int_{\delta}^{2\delta} \Big\{u(t) + \Big[R_1 + B_1' P_1(t) B_1\Big]^{-1} \Big[\Big(B_1' P_1(t) B_2 + B_1' P_2(t) B_1\Big) u(t-\delta) \\
&\quad + B_1' P_1(t) A_3 x(t-\delta) + B_1' P_2(t) A_3 \phi(t) + B_1' P_2(t) B_2 \eta(t)\Big]\Big\}' \Big[R_1 + B_1' P_1(t) B_1\Big] \\
&\quad \times \Big\{u(t) + \Big[R_1 + B_1' P_1(t) B_1\Big]^{-1} \Big[\Big(B_1' P_1(t) B_2 + B_1' P_2(t) B_1\Big) u(t-\delta) \\
&\quad + B_1' P_1(t) A_3 x(t-\delta) + B_1' P_2(t) A_3 \phi(t) + B_1' P_2(t) B_2 \eta(t)\Big]\Big\} dt \\
&\quad + \mathbb{E}\int_0^{\delta} \Big\{u'(t) \Big[R_2 + B_2' P_1(t) B_2 + 2B_1' P_2(t) B_2 + B_1' P_3(t) B_1\Big] u(t) \\
&\quad + 2u'(t) \Big[\Big(B_2' P_1(t) + B_1' P_2(t)\Big) A_3 x(t) + \Big(B_2' P_2(t) + B_1' P_3(t)\Big) A_3 \phi(t) \\
&\quad + \Big(B_2' P_2(t) + B_1' P_3(t)\Big) B_2 \eta(t)\Big] \\
&\quad - \Big[u'(t) \Big(B_2' P_1(t) B_1 + B_1' P_2(t) B_1\Big)
\end{aligned}
$$

$$+x'(t)A_3'P_1(t)B_1 + A_3'P_2(t)B_1\phi(t) + B_2'P_2(t)B_1\eta(t)\Big]\Big[R_1 + B_1'P_1(t)B_1\Big]^{-1}$$

$$\times\Big[\Big(B_1'P_1(t)B_2 + B_1'P_2(t)B_1\Big)u(t)$$

$$+B_1'P_1(t)A_3x(t) + B_1'P_2(t)A_3\phi(t) + B_1'P_2(t)B_2\eta(t)\Big]\Big\}dt. \qquad (6.2.16)$$

Let us take some analysis on the expression above. First, recall the function (6.2.1), where $t \in [0, \delta]$, the equation can be written as:

$$dx(t) = [A_1x(t) + A_2\phi(t)]dt + [A_3\phi(t) + B_1u(t) + B_2\eta(t)]dw(t)$$

The differential of the quadratic form $x'(t)H(t)x(t)$ is:

$$\begin{aligned} dx'(t)H(t)x(t) \quad = \quad & \Big[x'(t)\dot{H}(t)x(t) + 2x'(t)H(t)(A_1x(t) + A_2\phi(t)) \\ & + \left[A_3'\phi(t) + u'(t)B_1' + B_2'\eta(t)\right]H(t)\left[A_3\phi(t) + B_1u(t) + B_2\eta(t)\right]\Big]dt \\ & +2x'(t)H(t)\left[A_3\phi(t) + B_1u(t) + B_2\eta(t)\right]dw(t), \end{aligned}$$

it is clear that:

$$\begin{aligned} & \mathbb{E}\Big[x'(\delta)H(\delta)x(\delta) - x'(0)H(0)x(0)\Big] \\ = \quad & \mathbb{E}\int_0^\delta \Big\{x'(t)\Big[\dot{H}(t) + H(t)A_1 + A_1'H(t)\Big]x(t) + 2x'(t)H(t)A_2\phi(t) \\ & +u'(t)B_1'H(t)B_1u(t) + 2u'(t)\Big[B_1'H(t)A_3\phi(t) + B_1'H(t)B_2\eta(t)\Big] \\ & +A_3'H(t)A_3\phi^2(t) + 2A_3'H(t)B_2\phi(t)\eta(t) + B_2'H(t)B_2\eta^2(t)\Big\}dt. \quad (6.2.17) \end{aligned}$$

The differential of $M(t)x(t)$ is:

$$dM(t)x(t) \quad = \quad \Big[\dot{M}(t)x(t) + M'(t)(A_1x(t) + A_2\phi(t)\Big]dt$$

$$+M'(t)\Big[A_3\phi(t)+B_1u(t)+B_2\eta(t)\Big]dw(t),$$

we have:

$$\mathbb{E}[M(\delta)x(\delta)]-M(0)x(0)=\mathbb{E}\int_0^{\delta}(\dot{M}(t)+M'(t)A_1)x(t)+M'(t)A_2\phi(t)dt.\quad(6.2.18)$$

Substitute equation (6.2.17) and (6.2.18) into (6.2.16), we can get:

$$\begin{aligned}
J(u(\cdot)) &= \mathbb{E}\int_0^{2\delta}\Big[\bar{A}_4'P(t)\bar{A}_4+2\bar{A}_4'P(t)\bar{B}_2+\bar{B}_2'P(t)\bar{B}_2+G'(t)\bar{A}_2\Big]dt\\
&\quad+\mathbb{E}\int_{\delta}^{2\delta}\Bigg\{u(t)+\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}\Big[[B_1'P_1(t)B_2+B_1'P_2(t)B_1]u(t-\delta)\\
&\quad+B_1'P_1(t)A_3x(t-\delta)+B_1'P_2(t)A_3\phi(t)+B_1'P_2(t)B_2\eta(t)\Big]\Bigg\}'[R_1+B_1'P_1(t)B_1]\\
&\quad\times\Bigg\{u(t)+\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}\Big[[B_1'P_1(t)B_2+B_1'P_2(t)B_1]u(t-\delta)\\
&\quad+B_1'P_1(t)A_3x(t-\delta)+B_1'P_2(t)A_3\phi(t)+B_1'P_2(t)B_2\eta(t)\Big]\Bigg\}dt\\
&\quad+\mathbb{E}\int_0^{\delta}\Bigg\{x'(t)\Big[Q_1+\dot{H}(t)+H(t)A_1+A_1H(t)\\
&\qquad-A_3'P_1(t)B_1\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}B_1'P_1(t)A_3\Big]x(t)\\
&\quad+\Big[2A_2'H(t)\phi(t)+\dot{M}(t)+\frac{1}{2}\Big[M'(t)A_1+A_1'M(t)\Big]-2\Big[A_3'P_2(t)B_1\phi(t)\\
&\quad+B_2'P_2(t)B_1\eta(t)\Big]\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}B_1'P_1(t)A_3\Big]x(t)\\
&\quad+u'(t)\Bigg\{R_1+B_1'H(t)B_1(t)+R_2+B_2'P_1(t)B_2+2B_1'P_2(t)B_2+B_1'P_3(t)B_1\\
&\quad-\Big[B_2'P_1(t)B_1+B_1'P_2(t)B_1\Big]\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}\Big[B_1'P_1(t)B_2+B_1'P_2(t)B_1\Big]\Bigg\}u(t)\\
&\quad+2u'(t)\Bigg\{B_1'H(t)A_3\phi(t)+B_1'H(t)B_2\eta(t)+\Big[B_2'P_1(t)+B_1'P_2(t)\Big]A_3x(t)
\end{aligned}$$

$$
\begin{aligned}
&+\Big[B_2'P_2(t)+B_1'P_3(t)\Big]A_3\phi(t)+\Big[B_2'P_2(t)+B_1'P_3(t)\Big]B_2\eta(t)\\
&-\Big[B_2'P_1(t)B_1+B_1'P_2(t)B_1\Big]\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}\\
&\times\Big[B_1'P_1(t)A_3x(t)+B_1'P_2(t)A_3\phi(t)+B_1'P_2(t)B_2\eta(t)\Big]\Big\}\Big\}dt\\
+\mathbb{E}\int_0^\delta&\Big\{Q_2(t)\phi^2(t)+R_2(t)\eta^2(t)+A_3'H(t)A_3\phi^2(t)+2A_3'H(t)B_2\phi(t)\eta(t)\\
&+B_2'H(t)B_2\eta^2(t)+M'(t)A_2\phi(t)+x'(0)H(0)x(0)+M(0)x(0)\\
&+x'(0)P_3(\delta)x(0)+G_2(\delta)x(0)\Big\}dt\\
-\mathbb{E}\int_\delta^{2\delta}&\Big\{\Big[A_3'P_2(t)\phi(t)B_1+B_2'P_2(t)\eta(t)B_1\Big]\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}\\
&\times\Big[A_3'P_2(t)\phi(t)B_1+B_2'P_2(t)\eta(t)B_1\Big]'\Big\}dt.
\end{aligned}
\tag{6.2.19}
$$

We can rewrite the $u(t)$ terms in (6.2.19) as:

$$
\begin{aligned}
&u'(t)\widetilde{R}u(t)+2u'(t)\Big[K_1x(t)+K_2\Big]\\
=\ &\Big\{u'(t)+\widetilde{R}^{-1}\Big[K_1x(t)+K_2\Big]\Big\}'\widetilde{R}\Big\{u'(t)+\widetilde{R}^{-1}\Big[K_1x(t)+K_2\Big]\Big\}\\
&\qquad-\Big[x'(t)K_1'+K_2'\Big]\widetilde{R}^{-1}\Big[K_1x(t)+K_2\Big].
\end{aligned}
$$

The terms including the state x(t), including the last term in the above equation, are:

$$
\begin{aligned}
x'(t)\Big\{&Q_1+\dot{H}(t)+H(t)A_1+A_1H(t)-K_1'\widetilde{R}^{-1}K_1\\
&-A_3'P_1(t)B_1\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}B_1'P_1(t)A_3\Big\}x(t)
\end{aligned}
$$

$$+\Big\{2A_2'H(t)\phi(t)+\dot{M}(t)+\frac{1}{2}\Big(M'(t)A_1+A_1'M(t)\Big)-2K_2'\widetilde{R}^{-1}K_1$$
$$-2\Big[A_3'P_2(t)B_1\phi(t)+B_2'P_2(t)B_1\eta(t)\Big]\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}B_1'P_1(t)A_3\Big\}x(t). \quad (6.2.20)$$

By the equations of $H(t)$ and $M(t)$, the whole expression (6.2.20) will be zero. Therefore, we have the optimal $u(t)=u^*(t)$, where $t\in[0,\delta]$ as follows:

$$\begin{aligned}
u^*(t) &= -\widetilde{R}^{-1}\Big[K_1x(t)+K_2\Big]\\
&= -\widetilde{R}^{-1}\Big\{\Big(\Big[B_2'P_1(t)+B_1'P_2(t)\Big]A_3-\Big[B_2'P_1(t)B_1+B_1'P_2(t)B_1\Big]\\
&\qquad\times\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}B_1'P_1(t)A_3\Big)x(t)\\
&\qquad+B_1'H(t)A_3\phi(t)+B_1'H(t)B_2\eta(t)+\Big[B_2'P_2(t)+B_1'P_3(t)\Big]A_3\phi(t)\\
&\qquad+\Big[B_2'P_2(t)+B_1'P_3(t)\Big]B_2\eta(t)-\Big[B_2'P_1(t)B_1+B_1'P_2(t)B_1\Big]\\
&\qquad\times\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}\Big[B_1'P_2(t)A_3\phi(t)+B_1'P_2(t)B_2\eta(t)\Big]\Big\}
\end{aligned} \quad (6.2.21)$$

Therefore, $J(u(\cdot))$ could be writen as:

$$\begin{aligned}
J(u(\cdot)) &= \mathbb{E}\int_0^{2\delta}\Big\{\bar{A}_4'P(t)\bar{A}_4+2\bar{A}_4'P(t)\bar{B}_2+\bar{B}_2'P(t)\bar{B}_2+G'(t)\bar{A}_2\Big\}dt\\
&\quad+\mathbb{E}\int_\delta^{2\delta}\Big\{u(t)+\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}\Big[\Big(B_1'P_1(t)B_2+B_1'P_2(t)B_1\Big)u(t-\delta)\\
&\quad+B_1'P_1(t)A_3x(t-\delta)+B_1'P_2(t)A_3\phi(t)+B_1'P_2(t)B_2\eta(t)\Big]\Big\}'\Big[R_1+B_1'P_1(t)B_1\Big]\\
&\quad\times\Big\{u(t)+\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}\Big[[B_1'P_1(t)B_2+B_1'P_2(t)B_1]u(t-\delta)\\
&\quad+B_1'P_1(t)A_3x(t-\delta)+B_1'P_2(t)A_3\phi(t)+B_1'P_2(t)B_2\eta(t)\Big]\Big\}dt
\end{aligned}$$

$$+\mathbb{E}\int_0^\delta \Bigg\{u+\widetilde{R}^{-1}\Big[\Big(\Big[B_2'P_1(t)+B_1'P_2(t)\Big]A_3-\Big[B_2'P_1(t)B_1+B_1'P_2(t)B_1\Big]$$

$$\times\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}B_1'P_1(t)A_3\Big)x(t)$$

$$+B_1'H(t)A_3\phi(t)+B_1'H(t)B_2\eta(t)+\Big[B_2'P_2(t)+B_1'P_3(t)\Big]A_3\phi(t)$$

$$+\Big[B_2'P_2(t)+B_1'P_3(t)\Big]B_2\eta(t)-\Big[B_2'P_1(t)B_1+B_1'P_2(t)B_1\Big]$$

$$\times\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}\Big[B_1'P_2(t)A_3\phi(t)+B_1'P_2(t)B_2\eta(t)\Big]\Big]\Bigg\}'\widetilde{R}$$

$$\times\Bigg\{u+\widetilde{R}^{-1}\Big[\Big(\Big[B_2'P_1(t)+B_1'P_2(t)\Big]A_3-\Big[B_2'P_1(t)B_1+B_1'P_2(t)B_1\Big]$$

$$\times\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}B_1'P_1(t)A_3\Big)x(t)$$

$$+B_1'H(t)A_3\phi(t)+B_1'H(t)B_2\eta(t)+\Big[B_2'P_2(t)+B_1'P_3(t)\Big]A_3\phi(t)$$

$$+\Big[B_2'P_2(t)+B_1'P_3(t)\Big]B_2\eta(t)-\Big[B_2'P_1(t)B_1+B_1'P_2(t)B_1\Big]$$

$$\times\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}\Big[B_1'P_2(t)A_3\phi(t)+B_1'P_2(t)B_2\eta(t)\Big]\Big]\Bigg\}dt$$

$$+\mathbb{E}\int_0^\delta \Bigg\{Q_2(t)\phi^2(t)+R_2(t)\eta^2(t)+A_3'H(t)A_3\phi^2(t)+2A_3'H(t)B_2\phi(t)\eta(t)$$

$$+B_2'H(t)B_2\eta^2(t)+M'(t)A_2\phi(t)+x'(0)H(0)x(0)+M(0)x(0)$$

$$+x'(0)P_3(\delta)x(0)+G_2(\delta)x(0)-K_2'\widetilde{R}^{-1}K_2\Bigg\}dt$$

$$-\mathbb{E}\int_\delta^{2\delta}\Bigg\{\Big[A_3'P_2(t)\phi(t)B_1+B_2'P_2(t)\eta(t)B_1\Big]\Big[R_1+B_1'P_1(t)B_1\Big]^{-1}$$

$$\times\left[A_3'P_2(t)\phi(t)B_1+B_2'P_2(t)\eta(t)B_1\right]'\Bigg\}dt. \tag{6.2.22}$$

When we applied the optimal u^* into the equation above, we have:

$$\begin{aligned}
J(u(\cdot)) \quad = \quad & \mathbb{E}\int_0^{2\delta}\left\{\bar{A}_4'P(t)\bar{A}_4+2\bar{A}_4'P(t)\bar{B}_2+\bar{B}_2'P(t)\bar{B}_2+G'(t)\bar{A}_2\right\}dt\\
& +\mathbb{E}\int_0^{\delta}\Bigg\{Q_2(t)\phi^2(t)+R_2(t)\eta^2(t)+A_3'H(t)A_3\phi^2(t)+2A_3'H(t)B_2\phi(t)\eta(t)\\
& \qquad +B_2'H(t)B_2\eta^2(t)+M'(t)A_2\phi(t)+x'(0)H(0)x(0)+M(0)x(0)\\
& \qquad +x'(0)P_3(\delta)x(0)+G_2(\delta)x(0)-K_2'\widetilde{R}^{-1}K_2\Bigg\}dt\\
& -\mathbb{E}\int_\delta^{2\delta}\Bigg\{\left[A_3'P_2(t)\phi(t)B_1+B_2'P_2(t)\eta(t)B_1\right]\left[R_1+B_1'P_1(t)B_1\right]^{-1}\\
& \qquad \times\left[A_3'P_2(t)\phi(t)B_1+B_2'P_2(t)\eta(t)B_1\right]'\Bigg\}dt. \qquad (6.2.23)
\end{aligned}$$

Compared with Chen and Wu [13], the delay terms appear in both state and control in our system, and the cost functional is more general. However, the time horizon in our research is specific and the state is a special case of Chen and Wu. But the method we used to solve the linear-quadratic problem is completely new. And the result we found is in a feed-back form, but in Chen and Wu. it is in a conditional expectation.

6.3 Risk-sensitive control problem with delay system

In this section, we deal with the risk-sensitive control problem with delay system.

6.3.1 Problem formulation and assumption

Consider the linear stochastic control system:

$$\begin{cases} dx(t) = \Big[A_1 x(t) + A_2 \displaystyle\sum_{K_1=1}^{n} x(t-K_1\delta) + B_1 u(t) + B_2 \sum_{K_2=1}^{n} u(t-K_2\delta)\Big]dt \\ \qquad\qquad + \Big[C_1 u(t) + C_2 \displaystyle\sum_{K_3=1}^{n} u(t-K_3\delta) + C_3\Big]dw(t) \\ x(t) = \phi(t), u(t) = \eta(t), \qquad t \in [-\delta, 0] \end{cases} \tag{6.3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state process and $u(t)$ is an $\mathcal{F}$-adapted process. A_1, A_2, B_1, B_2, C_1, C_2, C_3, K_1, K_2, K_3 are constants.

The cost functional is as follows:

$$J(u(\cdot)) : \;=\; \mathbb{E}\Bigg\{ \exp\bigg[\int_0^T [Qx(t) + u'(t)Ru(t)]dt + Q_1 x(T)\bigg]\Bigg\}. \tag{6.3.2}$$

where Q, R and Q_1 are constant matrices with proper dimensions.

We wish to solve the problem:

$$\begin{cases} \displaystyle\min_{u(\cdot)\in\mathcal{A}} J(u(\cdot)), \\ \\ \text{s.t. } 6.3.1 \text{ holds}, \end{cases} \tag{6.3.3}$$

$\mathcal{A}$ is a set where u is the solution to the problem 6.3.3.

Define the following matrices:

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ x(t-\delta) \\ \vdots \\ x(t-n\delta) \end{bmatrix}, \quad \bar{u}(t) = \begin{bmatrix} u(t) \\ u(t-\delta) \\ \vdots \\ u(t-n\delta) \end{bmatrix}, \quad \bar{w}(t) = \begin{bmatrix} w(t) \\ w(t-\delta) \\ \vdots \\ w(t-n\delta) \end{bmatrix},$$

$$\bar{A}_1 = \begin{bmatrix} A_1 & A_2 & A_2 & \dots & A_2 \\ 0 & A_1 & A_2 & \dots & A_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_1 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ A_2\phi(t) \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} B_1 & B_2 & B_2 & \dots & B_2 \\ 0 & B_1 & B_2 & \dots & B_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_1 \end{bmatrix},$$

$$\bar{B}_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ B_2\eta(t) \end{bmatrix}, \quad \bar{C}_1 = \begin{bmatrix} C_1 & C_2 & C_2 & \dots & C_2 \\ 0 & C_1 & C_2 & \dots & C_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & C_1 \end{bmatrix}, \quad \bar{C}_2 = \begin{bmatrix} C_3 \\ C_3 \\ \vdots \\ C_2\eta(t) + C_3 \end{bmatrix}.$$

The following ordinary differential equation has a unique global solution:

$$\begin{cases} Q + \dot{P}(t) + \dfrac{1}{2}P'(t)\bar{A}_1 + \dfrac{1}{2}\bar{A}_1' P(t) = 0, \\ \\ P(T) = Q_1, \end{cases} \tag{6.3.4}$$

Assumption 13.

$$R + \frac{1}{2}\bar{C}_1' P(t)P'(t)\bar{C}_1 > 0. \tag{6.3.5}$$

We now focus in defining the appropriate *admissible* set of controls $\mathcal{A}$. Let $\mathcal{U}$ denote the set of all $\mathcal{F}(t)$-adapted processes $u(t)$ such that the state equation (6.3.1) has a unique strong solution.

Let $\theta'(t)$, $Z(t)$, and Z be defined as:

$$\begin{aligned} \theta'(t) &\equiv P'(t)\left(\bar{C}_1\bar{u}(t) + \bar{C}_2\right), \\ Z(t) &\equiv \exp\left[-\int_0^t \theta'(\tau)d\bar{w}(\tau) - \frac{1}{2}\int_0^t \theta'(\tau)\theta(\tau)d\tau \right], \\ Z &\equiv Z(T). \end{aligned}$$

Let the new probability measure $\widetilde{\mathbb{P}}$ be defined as:

$$\widetilde{\mathbb{P}}(\alpha) \equiv \int_\alpha Z(\omega)d\widetilde{\mathbb{P}}(\omega), \quad \forall \alpha \in \mathcal{F}.$$

By Girsanov theorem, the process

$$\widetilde{w}(t) \equiv \bar{w}(t) - \int_0^t \theta(\tau)d\tau,$$

is a standard Brownian motion. In order to ensure that $\widetilde{\mathbb{P}}_u$ is a probability measure, we assume that $\theta_u(t)$ satisfies the *Novikov condition*, i.e. for some positive β the following holds:

$$\mathbb{E}\left[e^{(\beta/2)\int_0^T \theta_u'(\tau)\theta_u(\tau)d\tau}\right] < \infty. \tag{6.3.6}$$

We can now define the admissible set of controls as:

$$\mathcal{A} := \{u(\cdot) \in \mathcal{U} \quad \text{such that (6.3.6) holds}\}.$$

Since $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent probability measures, for any $\mathcal{F}_t$-measurable random variable X, we have:

$$\mathbb{E}[ZX] = \widetilde{\mathbb{E}}[X],$$

where $\widetilde{\mathbb{E}}(\cdot)$ is the expectation with respect to the probability measure $\widetilde{\mathbb{P}}$.

Theorem 6.3.1. *Let the above assumption above holds, the optimal control and the corresponding cost are respectivly:*

$$\begin{aligned}
\bar{u}^*(t) &= -\frac{1}{2}\left[R + \frac{1}{2}\bar{C}_1'P(t)P'(t)\bar{C}_1\right]^{-1}\left[\bar{B}_1'P(t) + \bar{C}_1P(t)P'(t)\bar{C}_2P\right], \\
J(u^*) &= \exp\left[P'(0)\bar{x}(0) + \int_{n\delta}^T [P'(t)\bar{A}_2 + P'(t)\bar{B}_2 + P(t)\bar{K}_2'\bar{K}_2P(t)]dt\right].
\end{aligned} \tag{6.3.7}$$

The proof is given in the following subsection. The optimal solution is found by a combination of change of measure and completion of squares method.

6.3.2 Proof of Theorem 6.3.1

From the given system $dx(t)$, we can write:

$$\begin{cases} dx(t) = [A_1x(t) + A_2\sum_{K_1=1}^{n} x(t-K_1\delta) + B_1u(t) + B_2\sum_{K_2=1}^{n} u(t-K_2\delta)]dt \\ \quad + [C_1u(t) + C_2\sum_{K_3=1}^{n} u(t-K_3\delta) + C_3]dw(t), \quad t \geq 0, \\ dx(t-\delta) = [A_1x(t-\delta) + A_2\sum_{K_1=2}^{n} x(t-K_1\delta) + B_1u(t-\delta) + B_2\sum_{K_2=2}^{n} u(t-K_2\delta)]dt \\ \quad + [C_1u(t-\delta) + C_2\sum_{K_3=2}^{n} u(t-K_3\delta) + C_3]dw(t-\delta), \quad t > \delta, \\ \vdots \\ dx(t-n\delta) = [A_1x(t-n\delta) + A_2\phi(t) + B_1u(t-n\delta) + B_2\eta(t)]dt \\ \quad + [C_1u(t-n\delta) + C_2\eta(t) + C_3]dw(t-n\delta) \quad t > n\delta, \\ x(t) = \phi(t), u(t) = \eta(t), \qquad t \in [-\delta, 0]. \end{cases} \tag{6.3.8}$$

We could write the system as:

$$d\bar{x}(t) = [\bar{A}_1\bar{x}(t) + \bar{B}_1\bar{u}(t) + \bar{A}_2 + \bar{B}_2]dt + [\bar{C}_1\bar{u}(t) + \bar{C}_2]d\bar{w}(t), t \geq n\delta. \tag{6.3.9}$$

The differential of the term $P'(t)\bar{x}(t)$ is:

$$\begin{aligned} dP'(t)\bar{x}(t) = [\dot{P}'(t)\bar{x}(t) + P'(t)(\bar{A}_1\bar{x}(t) + \bar{A}_2 + \bar{B}_1\bar{u}(t) + \bar{B}_2)]dt \\ +P'(t)(\bar{C}_1\bar{u}(t) + \bar{C}_2)d\bar{w}(t), \end{aligned}$$

it is clear that:

$$\begin{aligned} J(u(\cdot)) \quad = \quad & \mathbb{E}\Bigg\{ \exp\Bigg[\Bigg\{P'(0)\bar{x}(0) - P'(T)\bar{x}(T) + Q_1\bar{x}(T)\Bigg\} \\ & + \int_{n\delta}^{T}\Bigg\{Q\bar{x}(t) + \bar{u}(t)'R\bar{u}(t) + \dot{P}'(t)\bar{x}(t) + P'(t)\Big[\bar{A}_1\bar{x}(t) + \bar{A}_2 + \bar{B}_1\bar{u}(t) + \bar{B}_2\Big]\Bigg\}dt \\ & + \int_{n\delta}^{T}\Bigg\{P'(t)\Big[\bar{C}_1\bar{u}(t) + \bar{C}_2\Big]\Bigg\}d\bar{w}(t)\Bigg]\Bigg\}. \end{aligned} \tag{6.3.10}$$

Under the new probability measure $\widetilde{\mathbb{P}}$, the control problem (6.3.3) could be transformed into the standard risk-sensitive control problem of Jacobson (see, [34]). The cost functional $J(u(\cdot))$ can now be written as:

$$\begin{aligned} J(u(\cdot)) &= \widetilde{\mathbb{E}}\Bigg\{\exp\Bigg[\Bigg[\Bigg\{P'(0)\bar{x}(0)+\Big(P(t)\bar{A}_2+P(t)\bar{B}_2+\frac{1}{2}\bar{C}_2'P(t)P'(t)\bar{C}_2\Big)T\Bigg\} \\ &\quad +\int_{n\delta}^{T}\Bigg\{\Big[Q+\dot{P}(t)+\frac{1}{2}P'(t)\bar{A}_1+\frac{1}{2}\bar{A}_1'P(t)\Big]\bar{x}(t) \\ &\quad +\bar{u}'(t)\Big[(R+\frac{1}{2}\bar{C}_1'P(t)P'(t)\bar{C}_1\Big]\bar{u}(t)+\Big[P'(t)\bar{B}_1+\bar{C}_2'P(t)P'(t)\bar{C}_1'\Big]\bar{u}(t)\Bigg\}dt\Bigg]\Bigg\} \end{aligned} \tag{6.3.11}$$

We now perform the completion of squares for the terms inside the above integrals that contain $\bar{u}(t)$, as follows:

$$\begin{aligned} &\bar{u}'(t)\Big[(R+\frac{1}{2}\bar{C}_1'P(t)P'(t)\bar{C}_1\Big]\bar{u}(t)+\Big[P'(t)\bar{B}_1+\bar{C}_2'P(t)P'(t)\bar{C}_1'\Big]\bar{u}(t) \\ = &\Bigg\{\bar{u}(t)+\frac{1}{2}\Big[R+\frac{1}{2}\bar{C}_1'P(t)P'(t)\bar{C}_1\Big]^{-1}\Big[\bar{B}_1'P(t)+\bar{C}_1P(t)P'(t)\bar{C}_2\Big]\Bigg\}'\Big[R+\frac{1}{2}\bar{C}_1'P(t)P'(t)\bar{C}_1\Big] \\ &\times\Bigg\{\bar{u}(t)+\frac{1}{2}\Big[R+\frac{1}{2}\bar{C}_1'P(t)P'(t)\bar{C}_1\Big]^{-1}\Big[\bar{B}_1'P(t)+\bar{C}_1P(t)P'(t)\bar{C}_2\Big]\Bigg\} \\ &-\frac{1}{4}\Big[\bar{B}_1'P(t)+\bar{C}_1P(t)P'(t)\bar{C}_2\Big]'\Big[R+\frac{1}{2}\bar{C}_1'P(t)P'(t)\bar{C}_1\Big]^{-1}\Big[\bar{B}_1'P(t)+\bar{C}_1P(t)P'(t)\bar{C}_2\Big]. \end{aligned} \tag{6.3.12}$$

The terms including the state $x(t)$ are

$$\Big[Q+\dot{P}(t)+\frac{1}{2}P'(t)\bar{A}_1+\frac{1}{2}\bar{A}_1'P(t)\Big]\bar{x}(t),$$

by the equation od $P(t)$, the above expression is zero. The cost functional $J(u(\cdot))$ can now be written as:

$$\begin{aligned} J(u(\cdot)) &= \widetilde{\mathbb{E}}\Bigg\{\exp\Bigg[\Bigg[\Bigg\{P'(0)\bar{x}(0)+\Big(P(t)\bar{A}_2+P(t)\bar{B}_2+\frac{1}{2}\bar{C}_2'P(t)P'(t)\bar{C}_2-\frac{1}{4}\Big[\bar{B}_1'P(t) \\ &\quad +\bar{C}_1P(t)P'(t)\bar{C}_2\Big]'\Big[R+\frac{1}{2}\bar{C}_1'P(t)P'(t)\bar{C}_1\Big]^{-1}\Big[\bar{B}_1'P(t)+\bar{C}_1P(t)P'(t)\bar{C}_2\Big]\Big)T\Bigg\} \end{aligned}$$

$$+\int_0^T \left\{\bar{u}(t)+\frac{1}{2}\Big[R+\frac{1}{2}\bar{C}_1'P(t)P'(t)\bar{C}_1\Big]^{-1}\Big[\bar{B}_1'P(t)+\bar{C}_1P(t)P'(t)\bar{C}_2\Big]\right\}'$$
$$\times\Big[R+\frac{1}{2}\bar{C}_1'P(t)P'(t)\bar{C}_1\Big]$$
$$\times\left\{\bar{u}(t)+\frac{1}{2}\Big[R+\frac{1}{2}\bar{C}_1'P(t)P'(t)\bar{C}_1\Big]^{-1}\Big[\bar{B}_1'P(t)+\bar{C}_1P(t)P'(t)\bar{C}_2\Big]\right\}dt\Bigg]\Bigg\}. \tag{6.3.13}$$

Since $R+\frac{1}{2}\bar{C}_1'P(t)P'(t)\bar{C}_1>0$, we have that for all $u(\cdot)$ the following inequality holds:

$$J(u(\cdot)) \geq \widetilde{\mathbb{E}}\Bigg\{\exp\Bigg[\Bigg\{P'(0)\bar{x}(0)+\Big(P(t)\bar{A}_2+P(t)\bar{B}_2+\frac{1}{2}\bar{C}_2'P(t)P'(t)\bar{C}_2-\frac{1}{4}\Big[\bar{B}_1'P(t)$$
$$+\bar{C}_1P(t)P'(t)\bar{C}_2\Big]'\Big[R+\frac{1}{2}\bar{C}_1'P(t)P'(t)\bar{C}_1\Big]^{-1}\Big[\bar{B}_1'P(t)+\bar{C}_1P(t)P'(t)\bar{C}_2\Big]\Big)T\Bigg\}, \tag{6.3.14}$$

the lower bound is achieved if and only if:

$$\bar{u}^*(t)=-\frac{1}{2}\Big[R+\frac{1}{2}\bar{C}_1'P(t)P'(t)\bar{C}_1\Big]^{-1}\Big[\bar{B}_1'P(t)+\bar{C}_1P(t)P'(t)\bar{C}_2P\Big]. \tag{6.3.15}$$

Lemma 6.3.2. *If u is zero, and $r(t)=Q\bar{x}(t)$, i.e. the interest rate model with delay, then J^*, the optimal cost, is the price of a zero-coupon bond. Since*

$$P(0,T)=\mathbb{E}\left[\exp\left[\int_0^T r(u)du\right]\right],$$

The optimal cost is:

$$J^*=\exp\left[P'(0)\bar{x}(0)+\int_0^T[P'(t)\bar{A}_2+P'(t)\bar{B}_2+P(t)\bar{K}_2'\bar{K}_2P(t)]dt\right]. \tag{6.3.16}$$

6.4 Conclusion

This chapter analyzes the stochastic linear quadratic control and risk-sensitive control problems with delay systems. We generalize the linear quadratic delay problem and find the explicit solution to each case. Compared with the previous research, the optimal solution we found is in a feed-back form, and it is not the case in Chen and Wu [13] which is found in a conditional expectation. Since this chapter is just a start of some preliminary results, There are some further research could be done in the future, for example, the generalized case for the risk-sensitive control, the extension of time horizon to infinite, and also it could be extend in time-varying process.

Chapter 7

Conclusions

7.1 Introduction

In this chapter, we summarize the main contributions of the thesis and list some interesting open questions for future research.

7.2 Chapter 3

In this chapter, we consider a general case of an indefinite risk-sensitive control problem for fully observed stochastic systems with additive noise. This situation occurs when we use a generalised risk-sensitive cost functional. We find all explicit solutions to this problem using a combination of the completion of squares and the change of measure methods. We consider both the finite and infinite horizon cases. In particular, for the infinite case, we introduce a general function into the cost functional, from which weaker conditions are needed for solving similar infinite optimal control problems. The optimal investment problem in a market with a stochastic interest rate appears as a special case in our results.
These are a few interesting problems that could be look at in the future:

- The conditions for the existence and uniqueness of the solutions to the Riccati equation.
- The discrete case of the indefinite risk-sensitive control problem.
- Applications of risk-sensitive control in engineering and financial mathematics.
- State system with time-varying processes.

7.3 Chapter 4

We solve the generalized risk-sensitive control problem with a partially observed system for an indefinite case in this chapter. This is an extension of chapter 3. By applying Bensoussan and Van Schuppen's theorem [5] of equivalence, we transform the partially observed system to a fully observed problem, and obtain explicit solutions to optimal control problems in both finite and infinite horizons.
It will be interesting to extend these ideas to the follows:

- When we are transforming the partially observed system to a fully observed system, there are several conditions and assumptions made which seems to be strong to be hold. It will be interesting to weaken these conditions and give the explicit solutions to the Riccati equations in those assumptions.
- In this chapter, we find the solution to the state system with constant matrices, it would be desirable to explore the time-varying controls.
- We focus on the continuous risk-sensitive control, and it would be of interest to discover the solutions to a discrete-time risk-sensitive control with partially observed system.
- The application of risk-sensitive control with partial observation to engineering and financial mathematics.

7.4 Chapter 5

In this chapter we introduce a risk-sensitive version of the H_2/H_∞ control method for linear stochastic systems with additive noise. Two criteria of exponential-quadratic form are employed instead of the usual quadratic criteria. Both the finite and infinite horizon cases are considered under the assumption of linear state-feedback controllers, and explicit solutions in terms of Riccati equations are obtained.
It will be interesting to extend these ideas to nonlinear and discrete-time systems.

7.5 Chapter 6

This chapter analyzes the stochastic linear-quadratic control and risk-sensitive control problems with delay systems. We generalise the linear-quadratic delay problem and find the explicit solution to each case. Compared with previous research, the optimal solution we found is in a feed-back form, which is not the case in Chen and Wu [13] who found it to be a conditional expectation.

Since this chapter gives only some preliminary results, further research could be done, for example, the generalised case for the risk-sensitive control, extension of the time horizon to infinite, and alsoextend the research for a time-varying process.

7.6 Summary

In this thesis, the generalised risk-sensitive control problem is studied. Three related topics have been analyzed: the indefinite risk-sensitive control with full observation, the indefinite risk-sensitive control with partial observation and the H_2/H_∞ control with risk-sensitive cost functional. Explicit solutions are found for each problem. In this last chapter, we have pointed out the main contributions of this thesis and introduced come interesting open questions for future research.

Appendix: a special case of the solution for the Riccati equation

In this section, we give an example of the solution of the Riccati equation. Let $Q_1R_1' = 0$. Then the algebraic equation part becomes:

$$\left(R+\frac{\gamma}{4}R_1R_1'\right)\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left[PB+\frac{\gamma}{4}\cdot 2PCR_1'\right]'-\left[PB+\frac{\gamma}{4}\cdot 2PCR_1'\right]'=0, \tag{7.6.1}$$

$$\left[\left(R+\frac{\gamma}{4}R_1R_1'\right)\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left[B+\frac{\gamma}{2}CR_1'\right]-\left[B+\frac{\gamma}{2}CR_1'\right]'\right]P=0. \tag{7.6.2}$$

Thus, if

$$\left(R+\frac{\gamma}{4}R_1R_1'\right)\left(R+\frac{\gamma}{4}R_1R_1'\right)^{\dagger}\left[B+\frac{\gamma}{2}CR_1'\right]-\left[B+\frac{\gamma}{2}CR_1'\right]'=0, \tag{7.6.3}$$

equation 7.6.2 holds.
Under our assumption of $Q_1R_1' = 0$, the differential equation part becomes:

$$\begin{gathered}\dot{P}+PA+A'P+\frac{\gamma}{4}(2PC+Q_1)(2PC+Q_1)'+Q\\ -P\left[B+\frac{\gamma}{2}CR_1'\right]\left(R+\frac{\gamma}{2}R_1R_1'\right)^{\dagger}\left[B+\frac{\gamma}{2}CR_1'\right]'P=0,\end{gathered}$$

which is

$$\begin{gathered}\dot{P}+P(A+\frac{\gamma}{2}CQ_1')+(A+\frac{\gamma}{2}CQ_1')'P+\frac{\gamma}{4}Q_1Q_1'+Q\\ -P\left[\left[B+\frac{\gamma}{2}CR_1'\right]\left(R+\frac{\gamma}{2}R_1R_1'\right)^{\dagger}\left[B+\frac{\gamma}{2}CR_1'\right]'-\gamma CC'\right]P=0.\end{gathered}$$

Thus, this is a Riccati equation for deterministic LQ control, and it has a unique solution if

$$\begin{cases} S \geq 0, \\ \frac{\gamma}{4} Q_1 Q_1' + Q_1 \geq 0, \\ \left[B + \frac{\gamma}{2} C R_1'\right] \left(R + \frac{\gamma}{2} R_1 R_1'\right)^{\dagger} \left[B + \frac{\gamma}{2} C R_1'\right]' - \gamma C C' > 0 \qquad (or = 0). \end{cases} \tag{7.6.4}$$

Example Let

$$\begin{cases} B + \frac{\gamma}{2} C R_1' = 0, \\ \frac{\gamma}{4} Q_1 Q_1' + Q = 0, \\ S = 0. \end{cases}$$

Thus, equation 7.6.3 is satisfied, and the Riccati equation becomes

$$\begin{cases} \dot{P} + P(A + \frac{\gamma}{2} C Q_1') + (A + \frac{\gamma}{2} C Q_1') P + \gamma P C C' P = 0, \\ P(T) = 0 \end{cases} \tag{7.6.5}$$

which has a unique solution $P(t) = 0$.

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