KÄHLER PACKINGS AND SESHADRI CONSTANTS ON PROJECTIVE COMPLEX SURFACES

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Abstract. In analogy to the relation between symplectic packings and symplectic blow ups we show that multiple point Seshadri constants on projective complex surfaces can be calculated as the supremum of radii of multiple Kähler ball embeddings. We exemplify this connection on toric surfaces, also discussing how toric moment maps reflect the packing.

0. Introduction

Symplectic Topology, searching for global properties of symplectic manifolds, developed rather recently out of Hamiltonian Mechanics which was mainly concerned with the automorphisms of symplectic manifolds, or symplectomorphisms, as the flow of Hamiltonian vector fields (see the in-depth treatise of McDuff and Salamon, [MS95]). One of its most striking successes is the analysis and solution of several symplectic packing problems: How large can symplectic balls of a given number be when disjointly embedded in a given symplectic manifold? These questions are especially attractive as they exhibit the fundamental nature of symplectic structures: local flexibility vs (sometimes) global rigidity. In particular, symplectic packing is not so rigid as Euclidean (that is, distance-and-angle preserving) packing, leading to questions like the Kepler conjecture on ball packings (and its solution by Hales [Hal05, Hal12]), but may be less flexible than just volume-preserving packing. So some symplectic packing problems reveal obstructions to packings without gaps, whereas other packings are possible without gaps.

One of the most prominent class of such symplectic packing problems asks how large symplectic balls of a given number can be when disjointly embedded in the complex projective plane $\mathbb{CP}^2$. In more details, consider balls $B_0(r) \subset \mathbb{R}^4$ of radius r centered in 0 together with the symplectic form ω_{std} obtained by restricting the standard symplectic form on $\mathbb{R}^4$. If $\coprod_{q=1}^k B_0(r_q)$ denotes the disjoint union of k of these balls, with possibly different radii, and ω_{FS} denotes a Fubini-Study Kähler form on $\mathbb{CP}^2$, then a symplectic packing of $\mathbb{CP}^2$ with k symplectic balls is defined as a symplectic embedding

$$\iota : \coprod_{q=1}^{k} B_0(r_q) \hookrightarrow \mathbb{CP}^2,$$

that is, ι is a smooth embedding such that $\iota^*\omega_{\mathrm{FS}}|_{B_0(r_q)} = \omega_{\mathrm{std}}$. The symplectic packing problem asks for conditions on the radii r_q for which such a symplectic packing exist, and also how it can be explicitely constructed.

McDuff and Polterovich [MP94] connected this problem first to symplectic blow ups and then to Algebraic Geometry: They showed that a symplectic packing with balls

of radii r_q is only possible if on $\sigma : X \to \mathbb{CP}^2$, the blow up of $\mathbb{CP}^2$ (considered as a complex manifold) in k points $x_1, \ldots, x_k$, there exists a symplectic form representing the cohomology class of $\sigma^* l - \pi \sum_{q=1}^k r_q^2 e_q$, where l and the e_q are Poincaré dual to a line $L \subset \mathbb{CP}^2$ and the exceptional divisors $E_q = \sigma^{-1}(x_q)$. Then they proved that as long as $k \leq 8$ the only obstructions to the existence of such a symplectic form are the same as for the existence of a Kähler form in this cohomology class, namely (-1)-curves on X. Thus, these symplectic packing problems are related to the algebraic-geometric theory of del Pezzo surfaces already extensively studied in the 19th century (see [Man74] for a survey and results).
Next, Biran [Bir97] was able to prove that (-1)-curves remain the only obstacles for symplectic packings with $k \geq 9$ balls, and as a consequence he showed the symplectic analogue of a celebrated algebraic-geometric conjecture named after Nagata, who came across it when solving Hilbert's Fourteenth Problem [Nag59].

Conjecture 0.1 (Nagata)**.** *With notations as above and $k \geq 9$, there is a Kähler form representing the cohomology class $\sigma^* l - \epsilon \cdot \sum_{q=1}^k e_q$ for all $\epsilon < \frac{1}{\sqrt{k}}$ if the blown-up points $x_1, \ldots, x_k$ are chosen sufficiently general.*

This is the Kähler version of a purely algebraic-geometric statement:

Conjecture 0.2 (Nagata, algebraic-geometric version)**.** *Let $C = \{F = 0\}$ be an irreducible algebraic curve in $\mathbb{CP}^2$, given by an irreducible homogeneous polynomial $F = F(X, Y, Z)$ of degree d in three homogeneous variables X, Y, Z and with multiplicity m_q in the point x_q (that is, m_q is the lowest degree of a non-vanishing term in the Taylor series expansion of F around x_q). If the points $x_1, \ldots, x_k \in \mathbb{CP}^2$ are chosen sufficiently general then*

$$\sqrt{k} d \geq \sum_{q=1}^k m_q.$$

Choosing points on $\mathbb{CP}^2$ sufficiently general means that the points should not lie on a union of (not explicitly determined) algebraic curves on $\mathbb{CP}^2$ given by homogeneous polynomial equations. Excluding such "special" positions of the points (e.g. more than 2 points on a complex line) indeed decreases the Seshadri constant. This restriction is not necessary for the centers of symplectically packed balls, due to the greater flexibility of symplectic forms in contrast to Kähler forms.
The equivalence of these two conjectures follows from the fact that Kähler forms representing an integral cohomology class are curvature forms of hermitian metrics on an ample line bundle (that is Kodaira's embedding theorem [Wel80, Thm.III.4.1, III.4.6]) and that intersection numbers of ample cohomology classes with algebraic curves are always positive (that is the easy half of Nakai-Moishezon's Ampleness Criterion [Laz04, Thm.1.2.23]). The Nakai-Moishezon Criterion is applied on the strict transform $\overline{C}$ on X, that is the preimage of C under the blow-up map σ without the exceptional divisors E_q (intersected m_q times by $\overline{C}$).
More on Biran's proof and on what the symplectic methods tell us for the algebraic situation (in particular, why they cannot be used so easily for Nagata's Conjecture) can be found in Biran's lucid survey [Bir01].
The aim of this note is to show that the algebraic conjecture of Nagata is equivalent to a more restricted packing problem, namely a *Kähler packing problem.*

We work in a more general setting: V is assumed to be a 2-dimensional projective complex manifold (surface for short), $x_1, \ldots, x_k \in V$ distinct points on V and L an ample line bundle on V. Sometimes we interpret L also as a divisor on V. Let $\sigma : \widetilde{V} \to V$ be the blow up of the k points $x_1, \ldots, x_k$, with exceptional divisors $E_q = \sigma^{-1}(x_q)$.

Definition 0.3. *The multi-point Seshadri constant $\epsilon(L; x_1, \ldots, x_k)$ is defined as*

$$\sup\{\epsilon > 0 : \text{A multiple of } \sigma^* L - \epsilon \sum_{q=1}^{k} E_q \text{ is an ample divisor}\}.$$

Nagata's Conjecture predicts that $\epsilon(L; x_1, \ldots, x_k) = \frac{1}{\sqrt{k}}$ on $V = \mathbb{CP}^2$, where $L \subset \mathbb{CP}^2$ is a line and $x_1, \ldots, x_k \in \mathbb{CP}^2$ are points in sufficiently general position. Seshadri constants were intensively investigated in Algebraic Geometry during the last years, as a measure of local positivity (see e.g. [Laz04, Ch.5]).
Kähler packings can be defined on arbitrary Kähler manifolds:

Definition 0.4. *Let (V, ω) be a n-dimensional Kähler manifold with Kähler form ω. Then a holomorphic embedding*

$$\phi = \coprod_{q=1}^{k} \phi_q : \coprod_{q=1}^{k} B_0(r_q) \to V$$

is called a Kähler embedding of k disjoint complex balls in $\mathbb{C}^n$ centered in 0, of radius r_q, if $\phi_q^*(\omega) = \omega_{\text{std}}$ is the standard Kähler form on $\mathbb{C}^n$ restricted to $B_0(r_q)$.

Returning to the setting where V is a projective surface and L an ample line bundle there are many Kähler forms on V representing the first Chern class $c_1(L)$ of L.

Definition 0.5. *The Kähler packing constant of V and L with k balls is given by*

$$r_K(V, L, x_1, \ldots, x_k) := \sup \left\{ r > 0 : \begin{array}{l} \exists \text{ Kähler form } \omega \in c_1(L) \text{ and Kähler packing} \\ \coprod_{q=1}^{k} \phi_q : \coprod_{q=1}^{k} (B_0(r), \omega_{\text{std}}) \to (V, \omega) \\ \text{with } \phi_q(0) = x_q \end{array} \right\}.$$

Section 1 is devoted to the proof of the following equality:

Theorem 0.6. *In the setting above, $\epsilon(L; x_1, \ldots, x_k) = r_K(V, L, x_1, \ldots, x_k)$.*

It is possible to change the definition of Kähler packings to holomorphic embeddings

$$\coprod_{i=1}^{k} (B_0(R), \frac{\epsilon}{\pi} \cdot \omega_{\text{FS}}) \to (V, \omega)$$

for R arbitrarily large and ω_{FS} the Fubini-Study metric restricted to $B_0(r) \subset \mathbb{C}^2 \subset \mathbb{CP}^2$, without changing the Kähler packing constant. This follows from Lem. 1.2. Thus, Thm. 1.5. in [WN15a] is equivalent to Thm. 0.6 if only one point x_1 is considered, i.e. $k = 1$. Witt Nyström does not study the case of more than one point, but he generalizes Thm. 0.6 for one point to projective complex manifolds of arbitrary dimension, using the obvious generalisations of the notions of Seshadri constants, Kähler packings and Kähler packing constants on a higher-dimensional projective complex manifold with ample line bundle L. See also [WN15b] for related results. For the status of Thm. 0.6 for higher-dimensional projective complex manifolds and arbitrarily many points see Rem. 1.1.

In Section 2 we discuss Kähler packings on a smooth projective complex toric surface V when the blown-up points $x_1, \dots, x_k$ are fixed points of the torus action on V. In this situation it is easy to calculate the multi-point Seshadri constant $\epsilon(L; x_1, \dots, x_k)$, generalizing the case $k = 1$ discussed in [DR99], [BDRH$^+$09], see Cor. 2.3. It is also possible to approximate Kähler packings by Fubini-Study balls using Kähler forms induced by global sections of large enough multiples of L stable under the torus action, see Thm. 2.5. Actually, choosing global sections carefully the same idea works for general surfaces V. The additional tool needed to prove Thm. 0.6 is the symplectic blow-up and blow-down procedure developed by McDuff and Polterovich [MP94, §5], to glue in flat resp. Fubini-Study balls.

Finally, we show in Thm. 2.5 that the toric symplectic moment maps induced by those sections pulled back to the embedded balls approximate the Fubini-Study moment map on a Fubini-Study ball. This gives an interpretation to the change from the toric moment polytope of the line bundle L on V to the toric moment polytope of $\pi^* L - \epsilon \sum_q E_q$ on the blow-up of V (see Prop. 2.2 for a precise statement and Ex. 2.4 for an illustration): The cut-off triangles of the moment polytope are the shadows of the embedded balls under the moment map.

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1. Proof of Theorem 0.6

The inequality $\epsilon(L; x_1, \dots, x_k) \geq r_K(V, L, x_1, \dots, x_k)$ follows immediately by using symplectic blow up constructions on Kähler manifolds as described in [MP94, §5.3, in particular §5.3.A] (see also [Laz04, Lem.5.3.17]).

The idea to prove the other inequality of Thm. 0.6 is to construct Kähler forms on the blow-up of V from global sections of $L^{\otimes m}$ vanishing to higher and higher order in the points $x_1, \dots, x_k$. If the sections are carefully chosen the vanishing is homogeneous in all directions, and the Kähler forms get sufficiently flat around the exceptional divisors E_q over x_q, so that one can glue in a standard Kähler ball of a radius arbitrarily close to $\sqrt{\frac{\epsilon}{\pi}}$. The main technical tool for the gluing procedure is the symplectic blow down described by McDuff and Polterovich [MP94, §5.4].

In more details, recall that the standard Kähler form ω_0 on $\mathbb{C}^2$ is given in affine holomorphic coordinates (x, y) by $\frac{i}{2}(dx \wedge d\overline{x} + dy \wedge d\overline{y})$, whereas the Fubini-Study Kähler form τ_0 on $\mathbb{CP}^1$ is given in homogeneous coordinates $[S : T]$ by $\frac{i}{2\pi}\partial\overline{\partial}\log(S\overline{S} + T\overline{T})$. Note that the latter $(1,1)$-form is well-defined on $\mathbb{CP}^1$ because $S\overline{S} + T\overline{T}$ is homogeneous in S and T, and that it represents $c_1(\mathcal{O}_{\mathbb{CP}^1}(P))$ for any point $P \in \mathbb{CP}^1$. More generally, if $s_0, \dots, s_N$ are sections of a line bundle L on a complex manifold X defining an embedding

$$X \hookrightarrow \mathbb{CP}^N,\ x \mapsto [S_0 : \cdots : S_N] = [s_0(x) : \cdots : s_N(x)]$$

(for example, if the s_i span $H^0(X, L)$ and L is very ample) then we can use this embedding to construct a Kähler form on X, by restricting the Fubini-Study form $\frac{i}{2\pi}\partial\overline{\partial}\log(\sum_{k=0}^N S_k\overline{S}_k)$ in homogeneous coordinates $[S_0 : \cdots : S_N]$ on $\mathbb{CP}^N$ to X. We say that this restricted Kähler form on X is *induced* by the sections $s_0, \dots, s_N$. The Kähler form can also be seen as the curvature form of the hermitian metric h induced by the sections on L, defining the length of the vector $\xi(x)$ for each section

ξ of L and point x on X by

$$\| \xi \|_h^2 := \frac{\xi(x)\overline{\xi(x)}}{\sum_{k=0}^N s_k(x)\overline{s_k(x)}}.$$

Let (x, y) be local complex coordinates around $x_q \in V$, and denote by $S := \frac{x}{y}, T := \frac{y}{x}$ the induced homogeneous coordinates on the exceptional divisor $E_q \cong \mathbb{CP}^1$. If $U_q(\delta)$ denotes a ball centered in x_q of sufficiently small radius δ, measured according to the coordinates x, y, then the tubular neighborhood $\sigma^{-1}(U_q(\delta)) \subset \widetilde{V}$ of E_q is projected to $E_q \cong \mathbb{CP}^1$ by a holomorphic map p_q collapsing the lines in $U_q(\delta)$ through $(0,0)$. Furthermore $\sigma^{-1}(U_q(\delta))$ is covered by two charts with coordinates (x, t) resp. (s, y), with transition maps given by $y = xt$ and $s = 1/t$. Note that the exceptional divisor E_q intersects these charts as the vanishing locus of x resp. y.
Now assume that $\epsilon \in \mathbb{Q}$. Then for every integer $n > 0$ such that $n\epsilon$ is also an integer, the line bundle $\widetilde{L}_n := \sigma^*(L^{\otimes n}) \otimes \mathcal{O}_{\widetilde{V}}(-n\epsilon \cdot \sum_{q=1}^k E_q)$ is ample. On $U_q(\delta)$ the line bundle $L^{\otimes n}$ is trivial, hence we can define a hermitian metric h_0 on $L^{\otimes n}_{|U_q(\delta)}$ by

$$\| \xi \|_{h_0}^2 := \frac{\xi(x)\overline{\xi(x)}}{e^{x\overline{x}+y\overline{y}}}$$

with everywhere positive curvature form $\frac{1}{\pi}\omega_0 = \frac{i}{2\pi}(dx \wedge d\overline{x} + dy \wedge d\overline{y})$. If $\sigma^* h_0$ denotes the pulled back metric on $\sigma^*(L^{\otimes n})$ its curvature form $\sigma^*\omega_0$ is everywhere semipositive on $U_q(\delta)$ and positive away from E_q.
The sections of $\mathcal{O}_{\sigma^{-1}(U_q(\delta))}(-n\epsilon \cdot \sum_{q=1}^k E_q)$ given in the coordinates (x, t) of one of the charts covering $\sigma^{-1}(U_q(\delta))$ by

$$\sqrt{\binom{n\epsilon}{j}} t^j, \ j = 0, \ldots, n\epsilon,$$

define a hermitian metric h_q on $\mathcal{O}_{\sigma^{-1}(U_q(\delta))}(-n\epsilon \cdot \sum_{q=1}^k E_q)$. Note that the coefficient of t^j allows to rewrite the metric induced by these sections on $\mathcal{O}_{\sigma^{-1}(U_q(\delta))}(-n\epsilon \cdot \sum_{q=1}^k E_q)$ as a power of the metric induced by the sections 1 and t on $\mathcal{O}_{\sigma^{-1}(U_q(\delta))}(-\sum_{q=1}^k E_q)$. Since the curvature form of h_q is positive on E_q the tensor product $\sigma^* h_0 \otimes h_q$ is a hermitian metric $h_{0,q}$ on $\widetilde{L}_{n|\sigma^{-1}(U_q(\delta))}$ with everywhere positive curvature form

$$\omega_{0,q} := \frac{1}{\pi}\sigma^*\omega_0 + n\epsilon \cdot p_q^*\tau_0.$$

Step 1. For $n \gg 0$ we can find sections $s_0, \ldots, s_N$ spanning $H^0(\widetilde{V}, \widetilde{L}_n)$ such that the induced hermitian metric $\widetilde{h}$ on $\widetilde{L}_n$ has a positive curvature form $\widetilde{\omega}$ satisfying

$$\widetilde{\omega}_{|E_q} = n\epsilon \cdot \tau_0 \text{ and } \widetilde{\omega}(P) = \frac{1}{\pi}(\sigma^*\omega_0)(P) + n\epsilon \cdot (p_q^*\tau_0)(P),$$

for all $q = 1, \ldots, k$ and all points $P \in E_q$: For $n \gg 0$ the line bundle $\widetilde{L}_n$ is sufficiently ample such that the restriction maps

$$\begin{aligned} H^0(\widetilde{V}, \widetilde{L}_n \otimes \mathcal{O}_{\widetilde{V}}(-2\sum_{r=1}^{k} E_r)) &\rightarrow \bigoplus_{r=1}^{k} H^0(E_r, \widetilde{L}_{n|E_r} \otimes \mathcal{O}_{E_r}(-2E_r)) = \\ &= \bigoplus_{r=1}^{k} H^0(E_r, \mathcal{O}_{E_r}(-(n\epsilon+2)E_r)), \end{aligned}$$

$$H^0(\widetilde{V}, \widetilde{L}_n \otimes \mathcal{O}_{\widetilde{V}}(-\sum_{r=1}^{k} E_r)) \rightarrow \bigoplus_{r=1}^{k} H^0(E_r, \mathcal{O}_{E_r}(-(n\epsilon+1)E_r)),$$

$$H^0(\widetilde{V}, \widetilde{L}_n \otimes \mathcal{O}_{\widetilde{V}}(-\sum_{r\neq q} E_r)) \rightarrow \bigoplus_{r\neq q} H^0(E_r, \mathcal{O}_{E_r}(-(n\epsilon+1)E_r)) \oplus H^0(E_q, \mathcal{O}_{E_q}(-n\epsilon E_q))$$

are surjective for each $q = 1, \ldots, k$, by Serre Vanishing [Laz04, Thm.1.2.6]. For each $q = 1, \ldots, k$ we can thus find

- sections in $H^0(\widetilde{V}, \widetilde{L}_n)$ restricting to $S^{n\epsilon+2}, S^{n\epsilon+1}T, \ldots, T^{n\epsilon+2}$ on E_q (when divided by the square of the defining function of E_q) and vanishing to order ≥ 2 on all exceptional divisors $E_r \neq E_q$,
- sections in $H^0(\widetilde{V}, \widetilde{L}_n)$ restricting to (scalar multiples of) $S^{n\epsilon+1}, S^{n\epsilon}T, \ldots, T^{n\epsilon+1}$ on E_q (when divided by the defining function of E_q) and vanishing to order ≥ 2 on all exceptional divisors $E_r \neq E_q$, and
- sections in $H^0(\widetilde{V}, \widetilde{L}_n)$ restricting on E_q to (scalar multiples of) $S^{n\epsilon}, S^{n\epsilon-1}T, \ldots, T^{n\epsilon}$, a basis of $H^0(E_q, \mathcal{O}_{E_q}(-n\epsilon E_q))$ and vanishing to order ≥ 2 on all exceptional divisors $E_r \neq E_q$.

Uniting a basis of $H^0(\widetilde{V}, \widetilde{L}_n \otimes \mathcal{O}_{\widetilde{V}}(-2\sum_{r=1}^{k} E_r)) \subset H^0(\widetilde{V}, \widetilde{L}_n)$ with suitable linear combinations of the sections above we obtain a set of sections $s_0, \ldots, s_N$ spanning $H^0(\widetilde{V}, \widetilde{L}_n)$ which can be subdivided in three disjoint parts for each $q = 1, \ldots, k$: In terms of the (x, t) coordinates in one of the charts around E_q the sections are either of the form

$$\sqrt{\binom{n\epsilon}{j}} t^j + x^3 \cdot f_j(x,t),\ j = 0, \ldots, n\epsilon,\ \text{or}$$

$$x \cdot (\sqrt{\binom{n\epsilon+1}{l}} t^l + x \cdot g_l(x,t)),\ l = 0, \ldots, n\epsilon+1,\ \text{or}\ x^2 \cdot h(x,t),$$

where the f_j, g_l and h are regular functions in x, t and there exists exactly one section of the respective form for each j and each l. By multiplying with $s^{n\epsilon}$ and using $t \cdot s = 1$ and $x = sy$ we obtain expressions for the sections in the (s, y)-coordinates of the other chart around E_q, and these expressions in s, y are completely similar to those in x, t.

Let $\widetilde{h}$ denote the hermitian metric on $\widetilde{L}_n$ and $\widetilde{\omega}$ the Kähler form on $\widetilde{V}$ induced by the sections $\sigma_0, \ldots, \sigma_N$. Using the coordinates (x, t) around E_q (the calculations are completely analogous when using the coordinates (s, y) of the other chart around E_q) the Taylor series expansion of log shows that

$$\begin{aligned}\widetilde{\omega}(0,0) &= \left(\frac{i}{2\pi}\partial\overline{\partial}\log(\sum_{j=0}^{N}\sigma_j(x,t)\overline{\sigma_j(x,t)})\right)(0,0) = \frac{1}{\pi}\left(dx\wedge d\overline{x} + n\epsilon dt\wedge d\overline{t}\right) = \\ &= \frac{1}{\pi}\sigma^*\omega_0(0,0) + n\epsilon p_2^*\tau_0(0,0),\end{aligned}$$

because $F := \sum_{j=0}^{N}\sigma_j(x,t)\overline{\sigma_j(x,t)}$ is a power series in $x, \overline{x}, t, \overline{t}$, and the only terms of order ≤ 2 in F are $1, x\overline{x}, n\epsilon t\overline{t}$. Similarly in other points $P = (0, t_0) \in E_q$: Choose a unitary matrix $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in U(2)$ such that $t_0 = \frac{c}{a}$, and rewrite F in terms of the coordinates (x', t') given by

$$x = x'(a+bt'),\ t = \frac{c+dt'}{a+bt'}.$$

Then $F\cdot|a+bt'|^{2n\epsilon}$ is a power series in $x', \overline{x'}, t', \overline{t'}$, and as before the only terms of order ≤ 2 are $1, x'\overline{x'}, n\epsilon t'\overline{t'}$ because $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in U(2)$ implies that

$$(1+t\overline{t})\cdot|a+bt'|^2 = |a+bt'|^2 + |c+dt'|^2 = 1 + t'\overline{t'}.$$

Since $\partial\overline{\partial}\log|a+bt'|^2 = 0$ we conclude once again that

$$\widetilde{\omega}(P) = \frac{1}{\pi}(dx'\wedge d\overline{x'} + n\epsilon dt'\wedge d\overline{t'}) = \frac{1}{\pi}\sigma^*\omega_0(P) + n\epsilon p_2^*\tau_0(P).$$

Finally, the statement on $\widetilde{\omega}_{|E_q}$ follows because $F_{|x=0} = (1+t\overline{t})^{n\epsilon}$.
Step 2. On the tubular neighborhoods $\sigma^{-1}(U_q(\delta))$ we glue the metrics $\widetilde{h}$ on $\widetilde{L}_n$ and $h_{0,q}$ on $\widetilde{L}_{n|\sigma^{-1}(U_q(\delta))}$: To this purpose we use a partition of unity $(\overline{\rho}_1, \overline{\rho}_2)$ subordinate to the open cover $(\widetilde{V} - \sigma^{-1}(U_q(\delta/2)), \sigma^{-1}(U_q(\delta)))$ of $\widetilde{V}$. We construct $\overline{\rho}_1, \overline{\rho}_2$ from a partition of unity (ρ_1, ρ_2) subordinate to the open cover $(\mathbb{R} - (-\delta^2/4, \delta^2/4), (-\delta^2, \delta^2))$ of $\mathbb{R}$, by setting

$$\overline{\rho}_i(x,t) := \rho_i(|\sigma(x,t)|^2) = \rho_i(|\sigma(s,y)|^2), i = 1,2.$$

Note that we can choose ρ_i such that the first-order partial derivatives of $\overline{\rho}_i$ are bounded by a constant multiple of $1/\delta$ and the second-order partial derivatives of $\overline{\rho}_i$ by a constant multiple of $1/\delta^2$.
Let $s_0, \ldots, s_N$ be the global sections of $\widetilde{L}_n$ constructed in Step 1. Then in coordinates (x,t) around E_q the metric $\widetilde{h}$ induced by these sections is given by

$$\widetilde{h}(s(x,t)) = |s(x,t)|/(\sum_{j=0}^{N}|s_j(x,t)|^2)^{\frac{1}{2}} = |s(x,t)|\cdot e^{-\frac{1}{2}\phi_1(x,t)},$$

with $\phi_1(x,t) = \log(\sum_{j=0}^{N}|s_j(x,t)|^2)$, for each section s of $\widetilde{L}_n$. Similarly,

$$h_{0,q}(s(x,t)) = |s(x,t)|\cdot e^{-\frac{1}{2}\phi_2(x,t)}$$

with $\phi_2 = n\epsilon\log(1+t\overline{t}) + \frac{1}{\pi}x\overline{x}(1+t\overline{t})$. Then the glued metric $\overline{h}$ can be constructed as

$$\overline{h}(s(x,t)) = |s(x,t)|e^{-\frac{1}{2}(\overline{\rho}_1\phi_1 + \overline{\rho}_2\phi_2)} = |s(x,t)|e^{-\frac{1}{2}(\phi_1 + \overline{\rho}_2(\phi_2-\phi_1))}.$$

Its curvature is

$$\frac{i}{2\pi}\left[\partial\overline{\partial}\phi_1 + \partial\overline{\partial}(\overline{\rho}_2(\phi_2-\phi_1))\right] = \frac{i}{2\pi}\left[\partial\overline{\partial}\phi_1 + \partial(\overline{\rho}_2\cdot\overline{\partial}(\phi_2-\phi_1) + \overline{\partial}\overline{\rho}_2\cdot(\phi_2-\phi_1))\right] =$$

$$= \frac{i}{2\pi}\left[\partial\overline{\partial}\phi_1 + \partial\overline{\rho}_2\cdot\overline{\partial}(\phi_2-\phi_1) + \overline{\rho}_2\cdot\partial\overline{\partial}(\phi_2-\phi_1) + \partial\overline{\partial}\overline{\rho}_2\cdot(\phi_2-\phi_1) + \overline{\partial}\overline{\rho}_2\cdot\partial(\phi_2-\phi_1)\right].$$

The Taylor series expansion of log and the properties of the sections s_j discussed in Step 1 show that $\phi_2 - \phi_1$ expands to a power series in x, t only containing terms of order ≥ 3. Hence the remarks on the partial derivatives of $\overline{\rho}_1$ and $\overline{\rho}_2$ imply that around $(x,t) = (0,0)$ all summands but the first converge everywhere on $\sigma^{-1}(U_q(\delta)) - \sigma^{-1}(U_q(\delta/2))$ to 0 when δ tends to 0. Since $\frac{i}{2\pi}\partial\overline{\partial}\phi_1$ is strictly positive being the curvature of $\widetilde{h}$, it follows that $\overline{h}$ is a positive metric on $\widetilde{L}^n$ for n sufficiently large and δ sufficiently small. Calling $\overline{\omega}$ the Kähler form obtained as the curvature of $\overline{h}$ we have that

$$\overline{\omega}_{|\sigma^{-1}(U_q(\delta/2))} = \omega_{0,q}.$$

Step 3. We glue in standard Kähler balls of radius $\sqrt{\frac{\epsilon}{\pi}}$ to $(\widetilde{V}, \frac{1}{n}\overline{\omega})$ replacing the exceptional divisors E_q: Let $\mathcal{L}(r)$ denote the preimage of the ball $B(r)$ centered in $0 \in \mathbb{C}^2$ under the standard blow-up σ of $\mathbb{C}^2$ in 0 and let $\rho(\delta,\epsilon)$ denote the Kähler form

$$\rho(\delta,\epsilon) := \delta\cdot\sigma^*\omega_0 + \epsilon\cdot p_2^*\tau_0$$

on $\mathcal{L}(r)$, for $\delta, \epsilon > 0$. The construction of $\overline{\omega}$ implies that an appropriate rescaling of the (x,y)-coordinates around x_q without changing the homogeneous coordinates S, T on E_q yields holomorphic embeddings

$$\phi_q : \mathcal{L}(1+\epsilon_q) \hookrightarrow \widetilde{V}$$

such that $\phi_q^*(\frac{1}{n}\overline{\omega}) = \rho(\delta_q, \epsilon)$, for some $\epsilon_q, \delta_q > 0$. The symplectic blow-down construction in [MP94, §5.4, in particular §5.4.A] shows that there exist a Kähler form on V representing $c_1(L)$ and a Kähler embedding of k standard balls of radii $\sqrt{\frac{\epsilon}{\pi}}$ into V, wrt this Kähler form. □

Remark 1.1. The equality of Thm. 0.6 should hold in arbitrary dimensions, using the obvious generalisations of the notions of Seshadri constants, Kähler packings and Kähler packing constants: The proof of Thm. 0.6 above should generalize to higher dimensions. However, especially calculating the estimates needed for the glueing process will become rather tedious, so to stay on the safe side Thm. 0.6 is stated only for surfaces – an interesting enough case, in view of Nagata's Conjecture. The author does not know whether the arguments of Witt Nyström [WN15a, WN15b] can be adapted to the case of several points.

We finally show that a Fubini-Study ball can be glued into a flat Kähler ball and vice versa, by suitably modifying the symplectomorphism between flat Kähler balls and Fubini-Study balls given by

$$\phi : (B_0(1), \omega_{\mathrm{std}}) \to (\mathbb{C}^2, \omega_{\mathrm{FS}}), \quad (z_1, z_2) \mapsto \frac{1}{(1-\sum_{i=1}^2 |z_i|^2)^{\frac{1}{2}}}\cdot(z_1,z_2)$$

(see [MS95, Ex.7.14]) and its inverse:

Lemma 1.2. *For all $R, \epsilon, \lambda > 0$ there is a Kähler form $\tau = \tau(R,\epsilon,\lambda)$ on $\mathbb{C}^2$ such that*

$$\tau_{|B_0(R)} = \lambda^2\cdot\frac{1}{R^2+1}\omega_{\mathrm{std}} \text{ and } \tau_{|\mathbb{C}^2 - B_0(R+\epsilon)} = \lambda^2\omega_{\mathrm{FS}}.$$

For all $0 < \epsilon < \frac{1}{2}$, $\lambda > 0$ *there is a Kähler form* σ *on* $B_0(1)$ *such that*

$$\sigma_{|B_0(1-2\epsilon)} = \lambda^2 \cdot \frac{1}{4\epsilon(1-\epsilon)}\omega_{\mathrm{FS}} \text{ and } \sigma_{|B_0(1)-B_0(1-\epsilon)} = \lambda^2\omega_{\mathrm{std}}.$$

Proof. We obtain τ as the pull back of $\lambda^2\omega_{\mathrm{FS}}$ via the monotone embedding given in polar coordinates $u \in S^3, r \in (0,\infty)$ on $\mathbb{C}^n - \{0\}$ by

$$\begin{array}{ccccccl} (u,r) & \mapsto & (u, r/R) & \mapsto & (u, r(R^2-r^2)^{-\frac{1}{2}}) & & \text{on } B_0(R) - \{0\}, \\ z & \mapsto & z & & & & \text{on } \mathbb{C}^n - B_0(R+\epsilon) \end{array}$$

and smoothened on $B_0(R+\epsilon) - B_0(R)$. Then τ is a Kähler form by [MP94, §5.1] and satisfies the requested properties.

Similarly, we obtain σ as the pull back of $\lambda^2\omega_{\mathrm{std}}$ via the monotone embedding given by

$$\begin{array}{ccccccl} (u,r) & \mapsto & \left(u, \frac{r}{2(\epsilon(1-\epsilon))^{\frac{1}{2}}}\right) & \mapsto & \left(u, \frac{r}{(r^2+2(\epsilon(1-\epsilon))^{\frac{1}{2}})^{\frac{1}{2}}}\right) & & \text{on } B_0(1-2\epsilon) - \{0\}, \\ z & \mapsto & z & & & & \text{on } B_0(1) - B_0(1-\epsilon) \end{array}$$

and smoothened on $B_0(1-\epsilon) - B_0(1-2\epsilon)$. □

Note that rescaling yields symplectomorphisms

$$(B_0(R), \lambda^2 \cdot \frac{1}{R^2+1}\omega_{\mathrm{std}}) \to (B_0(\frac{R}{(R^2+1)^{\frac{1}{2}}}), \lambda^2\omega_{\mathrm{std}})$$

and

$$(B_0(1-2\epsilon), \lambda^2 \cdot \frac{1}{4\epsilon(1-\epsilon)}\omega_{\mathrm{FS}}) \to (B_0(\frac{1-2\epsilon}{2(\epsilon(1-\epsilon))^{\frac{1}{2}}}), \lambda^2\omega_{\mathrm{FS}}).$$

Thus Lem. 1.2 implies that we can change the definition of a Kähler packing with balls as indicated after Def. 0.5 without changing the Kähler packing constant. Note that globally on V, the Chern class of the Kähler form is not changed by gluing in the balls because the new Kähler form is constructed as a pullback via a map homotopic to the identity.

2. Kähler packings on toric surfaces

We start with fixing notations and recall some facts on toric varieties following [Ful93]: $N \cong \mathbb{Z}^n$ denotes a lattice of rank n, $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ the dual lattice of N, Δ a fan of rational strongly convex polyhedral cones $\sigma \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $X(\Delta)$ the n-dimensional toric variety associated to Δ, together with the natural action of the torus $T_N \cong (\mathbb{C}^*)^n$ on $X(\Delta)$.

A toric variety $X(\Delta)$ is covered by the affine toric varieties $U_\sigma := \mathbb{C}[\sigma^\vee \cap M]$, $\sigma \in \Delta$, where $\sigma^\vee = \{u \in M_{\mathbb{R}} : \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\}$ is the dual cone to σ in $M_{\mathbb{R}}$ and $\sigma^\vee \cap M$ is a semigroup in M.

A toric variety $X(\Delta)$ is complete if the support $|\Delta| = \bigcup_{\sigma\in\Delta} \sigma$ covers all of $N_{\mathbb{R}}$.

T_N-invariant morphisms $X(\Delta') \to X(\Delta)$ between two n-dimensional toric varieties correspond to abelian group homomorphisms $\alpha : N \to N$ such that $\alpha_{\mathbb{R}} := \alpha \otimes_{\mathbb{Z}} \mathbb{R} : N_{\mathbb{R}} \to N_{\mathbb{R}}$ maps each cone of Δ' into a cone of Δ.

Cones $\sigma \in \Delta$ of maximal dimension n correspond to T_N-fixed points $x_\sigma \in X(\Delta)$ whereas cones in Δ of lower dimension correspond to higher-dimensional T_N-orbits in $X(\Delta)$. In particular the rays $\tau \in \Delta$ correspond to $(n-1)$-dimensional T_N-orbits whose closures are the irreducible T_N-stable Weil divisors on $X(\Delta)$. For a ray $\tau \in \Delta$

let $v_\tau \in N$ denote the generator of τ in N, and D_τ the Weil divisor corresponding to τ.
A T_N-stable Cartier divisor D on $X(\Delta)$ is defined by elements $u_D(\sigma) \in M$ for each $\sigma \in \Delta$ of maximal dimension n such that $u_D(\sigma) - u_D(\sigma') \in (\sigma \cap \sigma')^\perp$. The corresponding Weil divisor is given as $D = -\sum_{\sigma \supset \tau \in \Delta \text{ ray}} \langle u_D(\sigma), v_\tau \rangle D_\tau$.
$X(\Delta)$ is nonsingular if each cone $\sigma \in \Delta$ is generated by n vectors $v_1, \ldots, v_n \in N$ that are a $\mathbb{Z}$-basis of N. In that case the T_N-stable Weil divisors coincide with the T_N-stable Cartier divisors. Note also that the dual cone $\sigma^\vee \subset M_\mathbb{R}$ and the semigroup $\sigma^\vee \cap M \subset M$ are generated by a $\mathbb{Z}$-basis of M if σ is generated by a $\mathbb{Z}$-basis.

Proposition 2.1 ([Ful93, Sec.2.4])**.** *Let $X(\Delta)$ be a nonsingular toric variety and $\sigma \in \Delta$ a cone of maximal dimension corresponding to the T_N-fixed point x_σ.*
Then the blow up of $X(\Delta)$ in x_σ is given by the morphism $X(\Delta') \to X(\Delta)$ where Δ' is constructed from Δ by subdividing σ into n cones σ_i generated by

$$v_1, \ldots, v_{i-1}, v_1 + \cdots + v_n, v_{i+1}, \ldots, v_n$$

where $v_1, \ldots, v_n \in N$ are spanning σ and also are a $\mathbb{Z}$-basis of the lattice N.
The exceptional divisor on $X(\Delta')$ is T_N-stable and corresponds to the ray τ generated by $v_1 + \cdots + v_n$. □

A T_N-stable Cartier divisor $D = \sum_{\tau \in \Delta \text{ ray}} a_\tau D_\tau$ on $X(\Delta)$ defines a rational convex polyhedron P_D in $M_\mathbb{R}$,

$$P_D = \{u \in M_\mathbb{R} : \langle u, v_\tau \rangle \geq -a_\tau \text{ for all rays } \tau \in \Delta\}.$$

The elements of $P_D \cap M$ correspond to T_N-stable generators of the space of global sections of $\mathcal{O}_{X(\Delta)}(D)$.
D is ample if and only if the elements $u_D(\sigma) \in M$ describing D are exactly the vertices of P_D (see [Ful93, p.70]). If (and only if) such an ample T_N-stable divisor exists on $X(\Delta)$ and $X(\Delta)$ is complete then the toric variety $X(\Delta)$ is projective.
The following two results can be found in [BDRH$^+$09, §4] but we present the proof for the convenience of the reader and because some details are needed later on.

Proposition 2.2 ([BDRH$^+$09, §4])**.** *Let $X(\Delta)$ be an n-dimensional non-singular projective toric variety, $\sigma \in \Delta$ a cone of maximal dimension n with corresponding T_N-fixed point x_σ and $\pi : X(\Delta') \to X(\Delta)$ the blow up of $X(\Delta)$ in x_σ, with exceptional divisor E_σ, as constructed in Prop. 2.1. Let D be an ample T_N-stable Cartier divisor on $X(\Delta)$ with associated polyhedron P_D.*

(a) *Let $v_1, \ldots, v_n \in N$ be the generators of the edges of σ and $w_1, \ldots, w_n \in M$ the generators of the edges of $\sigma^\vee$. If $\sigma' \in \Delta$ is a cone of maximal dimension n intersecting σ in the facet spanned by $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ then the vertices $u_D(\sigma)$ and $u_D(\sigma')$ of P_D differ by a multiple $\epsilon_i w_i$ of w_i, $\epsilon_i > 0$.*
(b) *$D_\epsilon := \pi^* D - \epsilon E_\sigma$ is an ample ($\mathbb{Q}$-)divisor if and only if $\epsilon < \min_{i=1,\ldots,n} \epsilon_i$, and its associated polyhedron P_{D_ϵ} is obtained from P_D by taking away the simplex with vertex $u_D(\sigma)$ and edges ϵw_i starting in $u_D(\sigma)$.*

Proof. Let v_i' span the one edge τ' of σ' that is not an edge of σ. In particular, v_i' and v_i lie on different sides of the hyperplane spanned by $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$. Consequently, $\langle w_i, v_i' \rangle < 0$.

If $D = \sum_{\tau\in\Delta \text{ ray}} a_\tau D_\tau$ then $u_D(\sigma) = -\sum_{i=1}^n a_i w_i$ because $\langle u_D(\sigma), v_i\rangle = -a_i$, and the w_i are a $\mathbb{Z}$-basis of M dual to the $\mathbb{Z}$-basis v_i of N. Since D is ample we must have $\langle u_D(\sigma), v_i'\rangle > -a_{\tau'}$
All these facts imply that there is $\epsilon_i > 0$ such that $\langle u_D(\sigma)+\epsilon_i w_i, v_i'\rangle = -a_{\tau'}$ whereas $\langle u_D(\sigma) + \epsilon_i w_i, v_j\rangle = -a_j$ for $j = 1,\dots,i-1,i+1,\dots,n$. By the same argument as before, we may conclude $u_D(\sigma) + \epsilon_i w_i = u_D(\sigma')$, and (a) is shown.
For (b) note that $w_1 - w_i, \dots, w_i, \dots, w_n - w_i$ generate the semigroup $\sigma_i^\vee \cap M$ as these elements are a dual basis to $v_1, \dots, v_1 + \cdots + v_n, \dots, v_n$. Furthermore,

$$D_\epsilon = \pi^* D - \epsilon E_\sigma = \sum_{\tau\in\Delta \text{ ray}} a_\tau D_\tau + (\sum_{\tau\in\Delta \text{ ray}} a_\tau) E_\sigma - \epsilon E_\sigma.$$

Consequently, σ_i corresponds to the vertex of P_{D_ϵ} given by

$$-\sum_{k=1,k\neq i}^n a_k(w_k - w_i) - (\sum_{k=1}^n a_k) w_i + \epsilon w_i = -\sum_{k=1}^n a_k w_k + \epsilon w_i = u_D(\sigma) + \epsilon w_i.$$

So D_ϵ is ample if $\epsilon < \epsilon_i$ for all $i = 1,\dots,n$, and the polyhedron P_{D_ϵ} replaces the vertex $u_D(\sigma)$ of P_D by the vertices $u_D(\sigma) + \epsilon w_i$, cutting off the simplex as described. □

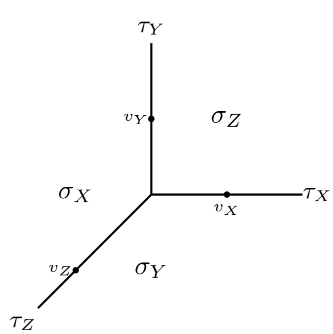
Corollary 2.3. *Let $X(\Delta)$ be an n-dimensional non-singular projective toric variety, let $\pi : X(\Delta') \to X(\Delta)$ be the blow up of $X(\Delta)$ in several T_N-fixed points $x_{\sigma_1}, \dots, x_{\sigma_k}$, with exceptional divisors E_k, and let D be an ample T_N-stable Cartier divisor on $X(\Delta)$. Then $\pi^* D - \epsilon \sum_{l=1}^k E_k$ is an ample ($\mathbb{Q}$)-divisor if, and only if,*

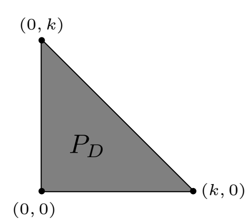
$$\epsilon < \frac{1}{2} \min_{1\le l\le k, 1\le i\le n} \epsilon_i^l,$$

where the $\epsilon_i^l > 0$ are those numbers determined for each edge τ_i of the cones σ_l in Prop. 2.2. □

Example 2.4. Consider the nonsingular toric projective variety $\mathbb{P}^2_{\mathbb{C}}$, on which the torus $T_N \cong (\mathbb{C}^*)^2$ acts as $(s,t)\cdot[X : Y : Z] = [sX : tY : Z]$. The three cones of maximal dimension in the fan Δ describing $\mathbb{P}^2_{\mathbb{C}} = X(\Delta)$ are separated by the rays τ_X, τ_Z and τ_Y spanned by $v_X = (1,0)$, $v_Z = (-1,-1)$ and $v_Y = (0,1)$ in N, respectively. σ_Z, σ_Y and σ_X correspond to the three T_N-fixed points $x_Z = [0 : 0 : 1]$, $x_Y = [0 : 1 : 0]$ and $x_X = [1 : 0 : 0]$ in $\mathbb{P}^2_{\mathbb{C}}$, respectively. The rays τ_X, τ_Z and τ_Y correspond to the T_N-stable divisors $D_X = \{X = 0\}$, $D_Z = \{Z = 0\}$ and $D_Y = \{Y = 0\}$, respectively. These are lines in $\mathbb{P}^2_{\mathbb{C}}$, hence linearly equivalent divisors. For an integer $k > 0$ the moment polytope P_D of the divisor $D := kD_Z$ is

$$P_D = \{(u_1,u_2) \in \mathbb{R}^2 : u_1, u_2 \ge 0, u_1 + u_2 \le k\}.$$



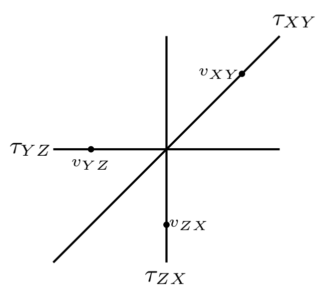


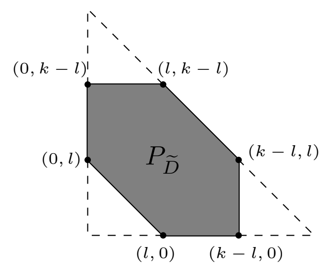
Blowing up the T_N-fixed points x_Z, x_Y and x_X yields a toric variety $\widetilde{X} = X(\widetilde{\Delta})$ with fan $\widetilde{\Delta}$ obtained from Δ by subdividing the cones σ_Z, σ_Y and σ_X with rays spanned by $v_{XY} = (1,1)$, $v_{ZX} = (0,-1)$ and $v_{YZ} = (-1,0)$, respectively. These rays correspond to the exceptional divisors E_Z, E_Y and E_X, respectively, of the blow up morphism $\pi : \widetilde{X} \to \mathbb{P}^2_{\mathbb{C}}$. The rays τ_X, τ_Z and τ_Y correspond to the strict π-transforms of D_X, D_Z and D_Y, respectively. Consequently, for $\widetilde{D} = k\pi^*D_Z - lE_X - lE_Z - lE_Y$ the moment polytope $P_{\widetilde{D}}$ is given as

$$P_{\widetilde{D}} = \{(u_1,u_2) \in \mathbb{R}^2 : 0 \leq u_1 \leq k-l, 0 \leq u_2 \leq k-l, l \leq u_1 + u_2 \leq k\},$$

and $\widetilde{D}$ is ample if and only if $0 < l < \frac{k}{2}$. This implies that

$$\epsilon(D; x_Z, x_Y, x_X) = \frac{1}{2}.$$





We now construct approximations to Kähler packings on toric varieties using T_N-stable global sections of high enough multiples of L. We also investigate how the toric symplectic moment maps induced by these global sections pull back to the embedded balls.

Recall that on an n-dimensional projective toric variety X with very ample divisor D, a toric moment map is given by

$$\mu : X \to \frac{1}{\sum_{u \in P_D \cap M} |x^u|^2} \cdot \sum_{x \in P_D \cap M} |x^u|^2 \cdot u \in \mathbb{R}^n$$

where the $x^u \in H^0(X, \mathcal{O}_X(D))$ are a basis of T_N-stable global sections. Then $\mu(X) = P_D$ [Ful93, Ch.4.2] and μ is a symplectic moment map for the S_N-action on X where $S_N \subset T_N$ is the real torus subgroup of T_N given by points $(z_1, \ldots, z_n)$ with $|z_i| = 1$ [MS95, Ex.5.48]. These properties do not change when we multiply the x^u with arbitrary constants $c_u \in \mathbb{C}^*$. Furthermore,

$$\mu_{\mathrm{std}} : B_0(r) \to \mathbb{R}^n, (z_1, \ldots, z_n) \mapsto (|z_1|^2, \ldots, |z_n|^2)$$

is a symplectic moment map for the standard S_N-action on a ball $B_0(r)$.

Theorem 2.5. *Let $X(\Delta)$ be a nonsingular projective toric surface and $\pi : X(\Delta') \to X(\Delta)$ the blow-up of T_N-fixed points x_σ corresponding to 2-dimensional cones $\sigma \in \Delta$, with $E_\sigma \subset X(\Delta')$ the exceptional divisor over x_σ.*
Let L be an ample divisor over $X(\Delta)$ and let $0 < \epsilon \in \mathbb{Q}$ such that $L_\epsilon := \pi^*L - \epsilon \cdot \sum_\sigma E_\sigma$ is ample on $X(\Delta')$. Then for $k \gg 0$ with $k\epsilon$ an integer and $\delta > 0$ there exist T_N-stable global sections $s_0^{(\delta)}, \ldots, s_{N_k}^{(\delta)} \in H^0(X(\Delta), L^{\otimes k\epsilon})$ inducing the Kähler form ω_δ on $X(\Delta)$ and the moment map $\mu_\delta : X(\Delta) \to \mathbb{R}^2$, and there exist embeddings $\phi_\sigma^{(\delta)} : B_0(\sqrt{\frac{\epsilon}{\pi}}) \to X(\Delta)$ with $\phi_\sigma^{(\delta)}(0) = x_\sigma$ such that

(1) $\phi_\sigma^{(\delta)*}\omega_\delta$ *tends to* ω_{std} *on* $B_0(\sqrt{\frac{\epsilon}{\pi}})$ *for* $\delta \to 0$, *and*
(2) $\mu_\delta \circ \phi_\sigma^{(\delta)} : B_0(\sqrt{\frac{\epsilon}{\pi}}) \to \mathbb{R}^2$ *tends to the standard moment map* μ_{std} *on* $B_0(\sqrt{\frac{\epsilon}{\pi}})$ *for* $\delta \to 0$.

Both limits hold pointwise, but also in the appropriate $\mathcal{C}^\infty$*-topologies.*

Proof. Let D be a T_N-stable Cartier divisor on $X(\Delta)$ such that $L = \mathcal{O}_{X(\Delta)}(D)$, and choose a sufficiently divisible $k \gg 0$. For a 2-dimensional cone $\sigma \in \Delta$ the two generators $w_1, w_2 \in M$ of the edges of $\sigma^\vee \cap M$ correspond to affine coordinates z_1, z_2 on $U_\sigma \cong \text{Spec } \mathbb{C}[\sigma^\vee \cap M] \cong \mathbb{A}^2_{\mathbb{C}}$ centered in x_σ. Then all the T_N-stable global sections of $L^{\otimes k\epsilon}$ can be written as monomial terms $c_u z^u$, with $c_u \in \mathbb{C}$ and $u \in (P_{k\epsilon D} - u_{k\epsilon D}(\sigma)) \cap M$.
Since $\epsilon < \epsilon(L; x_1, \dots, x_k)$ Cor. 2.3 implies that global sections $z_1^a z_2^b$ with $0 \leq a + b \leq k\epsilon$ do not coincide with global sections of that form but with respect to affine coordinates on $U_{\sigma'}$, σ' another 2-dimensional cone in Δ.
For each 2-dimensional cone $\sigma \in \Delta$ we choose the coefficient $c_{a,b}$ of the monomial $z_1^a z_2^b$, $0 \leq a + b \leq k\epsilon$, to be the square root of the coefficient of the monomial $|z_1|^{2a}|z_2|^{2b}$ in $(\delta^2 + |z_1|^2 + |z_2|^2)^{k\epsilon}$. For all the other T_N-stable global sections we choose the coefficient to be 1.
The T_N-stable global sections provided with these coefficients induce a Kähler form ω_δ whose restriction to U_σ is

$$\omega_{\sigma|U_\sigma} = \frac{1}{k} \cdot \frac{i}{2\pi} \partial\overline{\partial} \log \left((\delta^2 + |z_1|^2 + |z_2|^2)^{k\epsilon} + \text{ terms in } |z_1|^2, |z_2|^2 \text{ of order } > k\epsilon \right)$$

and a moment map whose restriction to U_σ is

$$\mu_{\delta|U_\sigma}(z) = \frac{1}{k} \cdot \frac{1}{\sum_{u \in P_{k\epsilon D} - k\epsilon u_D(\sigma)} |c_u|^2 |z|^{2u}} \sum_{u \in P_{k\epsilon D} - k\epsilon u_D(\sigma)} |c_u|^2 |z|^{2u} \cdot u.$$

For the embedding

$$\phi_{\delta,R} : B_0(R) \to U_\sigma \subset X(\Delta), z \mapsto \delta \cdot z$$

we obtain that

$$\begin{aligned} \phi_{\delta,R}^*\omega_\delta \quad &= \quad \tfrac{1}{k} \cdot \tfrac{i}{2\pi} \partial\overline{\partial} \log \left(\delta^{2k\epsilon}(1 + |z_1|^2 + |z_2|^2)^{k\epsilon} + \text{ terms in of order } > k\epsilon \text{ in } \delta^2 \right) \\ &\xrightarrow{\delta \to 0} \quad \tfrac{1}{k} \cdot \tfrac{i}{\pi} \cdot k\epsilon \cdot \partial\overline{\partial} \log(1 + |z_1|^2 + |z_2|^2) = \epsilon \cdot \omega_{\text{FS}}. \end{aligned}$$

Similarly, $\mu_\delta \circ \phi_{\delta,R}$ tends to

$$z \mapsto \frac{1}{k} \cdot \frac{1}{(1 + |z_1|^2 + |z_2|^2)^{k\epsilon}} \cdot \sum_{|u| \leq k\epsilon} \frac{|c_u|^2}{\delta^{2(k\epsilon - |u|)}} |z|^{2u} \cdot u$$

for $\delta \to 0$, and that is the toric moment map generated by the global sections $\binom{k\epsilon}{u_1+u_2}\binom{u_1+u_2}{u_1} z_1^{u_1} z_2^{u_2}$, $0 \leq u_1 + u_2 \leq k\epsilon$, hence the symplectic moment map with respect to $\epsilon \cdot \omega_{\text{FS}}$.
Rescaling the symplectomorphism $\phi : (B_0(1), \omega_{\text{std}}) \to (\mathbb{C}^n, \omega_{\text{FS}})$ discussed before Lem. 1.2 and noting that ϕ is S_N-invariant we deduce properties (1) and (2). The limit processes on $B_0(\sqrt{\frac{\epsilon}{\pi}})$ obviously hold pointwise, but also work in the appropriate $\mathcal{C}^\infty$-topologies because the involved functions are power series in δ and the real and imaginary parts of the complex coordinates converging in a neighborhood of $B_0(\sqrt{\frac{\epsilon}{\pi}})$. □

Note that the limits of ω_δ and μ_δ on $X(\Delta)$ do not exist, as the embeddings degenerate to maps onto points. Instead one needs the techniques in the proof of Thm. 0.6 to glue in flat resp. Fubini-Study balls.

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