Numerical studies of vibration of a four-span continuous plate with rails excited by a moving car with experimental validation

Jing Yanga, Huajiang Ouyang\*a, Dan Stancioiub

aSchool of Engineering, University of Liverpool, Liverpool L69 3GH, UK
Email: H.Ouyang@liverpool.ac.uk

b*Mechanical Engineering and Materials Research Centre, Liverpool John Moores University, Liverpool L3 3AF, UK*

**Abstract**

The vibration of a four-span continuous plate with two rails on top and four extra elastic supports excited by a moving model car is studied through numerical simulations and experiments. Modal testing is carried out to identify Young’s moduli of the plate material and the rail material. Shell elements and beam elements are adopted for the plate and the rails of their Finite Element (FE) model respectively. An offset is required to connect the rails and the plate in the FE model and the offset ratio of the shell element is updated to bring the numerical frequencies of the structure (plate with rails) closest to its experimental frequencies. Modal Superposition (MS) method with numerical modes of the structure and an iterative method are combined to predict the vibration of the structure subjected to the moving car. The displacements of four points of the plate are measured during the crossing of the car and compared with predicted results. The two sets of results agree well, which validates the model of the system. Parametric analysis is then made using the validated system model.

**Keywords** Moving load; Experiment; Parameter identification; Multi-span plate; Dynamic response.

# Introduction

Moving load problems are a kind of dynamic problems in which the load on a subject changes its location in time at a high enough speed. Examples are many, like a train crossing a bridge, a bullet being shot out from a gun, a working crane and so on. A brief overview of moving load problems can be found in Ouyang’s tutorial1. Au *et al*.2 reviewed vehicle-bridge interaction analyses according to bridge types. The interaction between moving vehicles and infrastructures is an important problem needed to be solved accurately for developing safe, economic and environmentally-friendly railways or highways and improving riding comfort for passengers3, 4. One main difficulty in solving this kind of problems, taking the railway for example, is that the contact force between a wheel and a rail varies with time and is dependent on the deflection of the rail at the contact interface as well as the displacement of the wheel, which leads to a coupled problem, implying that it is quite difficult to solve the dynamic contact problem between the rail and the wheel. Kalker5 gave a good review about contact theories and their applications. In this paper, the assumption of rigid dynamic point contact without separation is used to solve the dynamic response of a plate structure subjected to a moving car and it is found good enough to reach accepted accuracy.

As for the dynamic response of big structures, the number of structural modes needed for computation is much smaller than the number of Degrees of Freedom (DOFs) of the structure’s Finite Element (FE) model, thus the Modal Superposition (MS) method is generally thought to be highly efficient in this case, compared with the direct method6. For this self-excited vibration problem, since the traveling speed of the moving vehicle is fairly low, the frequencies involved in the response are not high. It is found that a small number of modes are good enough. Bowe and Mullarkey7 derived the final forms of these two methods for a beam subjected to a moving mass and compared the results by these two methods with the results in the literature. Beam theories and plate theories are commonly used to model simple bridges or rails8, 9, but they are not good enough to model complicated bridges with complex internal structures and boundary conditions. Generally, the analytical modes of complicated structures are not available. In this case, the Finite Element method must be adopted to capture their numerical modes. A good way of using the MS method is that the real modes of complicated structures are approximated by their numerical counterparts in the MS method. Baeza and Ouyang10 studied the vibration of a truss structure excited by a moving oscillator by using the MS method with numerical modes of the truss structure obtained by the FE method. Xia and Zhang11 combined the MS technique with the FE method to simulate the dynamic interaction between the China-Star high-speed train and a bridge with 24 m-span prestressed concrete box girders on the Qin-Shen Special Passenger Railway in China. Li *et al*.12 established FE models of different types of vehicles and bridges to extract numerical modes of the vehicles and bridges and wheel jump is allowed at the contact interface.

To solve two separate sets of equations of the bridge and the vehicles in time domain respectively, there are generally two methods available: the coupled method13-16 and the iterative method17-21. The coupled method combines the equation of motion of the bridge and that of the vehicles together. Then the coefficient matrices of the equation of motion of the whole system are actually time-varying and the equation can be solved by a number of numerical integration methods. On the other hand, the iterative method solves the two sets of equations separately, imposing displacement compatibility and force equilibrium at the contact interface through an iteration algorithm. A numerical integration method is still needed during the iterative process but updating coefficient matrices of the equations at every time step is avoided. Zhang and Xia21 compared time-step iteration (TSI) method with inter-system iteration (ISI) method through vehicle-bridge interaction systems.

To predict accurately dynamic responses of a real system, it is of great significance to determine the actual geometric and material parameters of the systems, to build an appropriate model for the system and to validate the predicted results by experiments. Zhai *et al.*15 established a three-dimensional vehicle-track model and validated it with full-scale field experiments. Arvidsson and Karoumi22 reviewed models and experiments in the field of train-bridge interaction and discussed key model parameters. Liu *et al*.19 measured ambient vibration and high-speed train-induced vibration of a composite railway bridge with seven spans to validate their train-bridge interaction model. The properties, natural frequency and the modal damping ratio of a 25-meter concrete girder specimen were obtained by full scale tests to estimate the dynamic performance of a prestressed concrete girder railway bridge under the passage of a moving train23.

The dynamic response of a single span beam subjected to moving vehicles has drawn much research attention, but relatively smaller amount of research work has been done for the interaction between multi-span beams and moving vehicles8, 9, 24-28 and the experimental study of the vibration of multi-span beams excited by moving vehicles is even less29, 30. Chan and Ashebo30 identified the moving loads acting on a three-span continuous beam through measured bending moments of the beam. Stancioiu *et al*.29 experimentally investigated the vibration of a four-span plate subjected to one or two moving masses under different masses and speeds and compared numerical results with experimental results.

The work in this paper is a further development of the work by Stancioiu *et al*.29. Based on the rig reported by Stancioiu *et al*., two rails and four actuators are attached on top of the plate and on the bottom surface of the plate respectively, and the moving masses are now replaced by a moving car as shown in Fig. 4. The four actuators are applied to impose four time-varying forces at four locations of the plate structure (one location at each span individually) based on an active control algorithm according to the feedback signals which are the displacements measured by four laser displacement transducers, for the purpose of controlling the vibration of the plate structure. However, vibration control is the second phase work after this paper. In this paper, the four actuators do not produce any forces. However, their presence even without actuation presents a small amount of (passive) stiffness that should be accounted for in order to make an accurate model. The moving car is the only excitation source, but the properties of the whole structure are changed locally where actuators are attached, so the stiffness of the actuators should be considered in the model of the whole structure. The model by Stancioiu *et al*. is two dimensional. It is necessary to develop a 3-dimensional model to capture accurate modes of the more complicated structure, including its torsional modes, and to examine the influence of four contact points between the moving model car and the rails on the dynamic response of the car-plate system.

This paper first presents the details of the MS method and an iterative method based on the model of an oscillator moving on a simply supported beam. Numerical modes of the beam are obtained by the FE method and applied in the MS method. This approach can be extended to be a general way of solving moving load problems, which is to use ABAQUS to obtain numerical modes of a structure and to implement MS method using the numerical modes of the structure and iterative method in MATLAB. This approach is verified firstly by the numerical results of the moving oscillator model of Nguyen and Kim18. Then the vibration of a four-span continuous plate with two rails on top and four extra elastic supports (provided by four actuators) excited by a moving model car is studied by simulations and experiments. Experimental results and numerical ones are found to agree very well. Then an in-depth parametric analysis is done using the experimentally validated numerical model.

# 2. MS method by numerical modes and iterative method

*2.1. Moving oscillator model*

The moving oscillator-on-beam model13 is adopted here to demonstrate the combined MS method and the iterative method, as shown in Fig. 1. The beam’s equation of the transverse motion and the equation of the mass in the vertical direction can be written as

|  |  |  |
| --- | --- | --- |
|  | $$ρA\frac{∂^{2}w(x,t)}{∂t^{2}}+EI\frac{∂^{4}w(x,t)}{∂x^{4}}=k\_{v}(u-w(x,t)) ⋅δ(x-vt)$$ | (1) |
|  | $$-m\_{v}g-k\_{v}\left(u(t)-w\left(vt,t\right)\right)=m\_{v}\frac{d^{2}u(t)}{dt^{2}}$$ | (2) |

where $ρ$ and $E$ are the density of the beam and Young’s modulus of the beam respectively; $A$ and $I$ are the area and second moment of area of beam’s cross section; $k\_{v}$ is the stiffness of the spring between the mass and the beam; $m\_{v}$ is the mass of the oscillator; $w$ and $u$ are the transverse deflection of the beam and the vertical displacement of the mass respectively; *v* is the travelling speed of the moving oscillator.



Fig. 1. Moving oscillator model

Substituting $w\left(x,t\right)=\sum\_{i=1}^{n}φ\_{i}\left(x\right)q\_{i}\left(t\right)$ into Eq. (1), multiplying the resultant equation with $φ\_{i}(x)$ and integrating it over the whole beam length, one can derive

|  |  |  |
| --- | --- | --- |
|  | $$M\ddot{q}+Kq=k\_{v}φ\left(u\left(t\right)-φ^{T}q\right)$$ | (3) |

Eq. (2) can be rewritten as

|  |  |  |
| --- | --- | --- |
|  | $$m\_{v}\ddot{u}(t)+k\_{v}u\left(t\right)=-m\_{v}g+k\_{v}φ^{T}q$$ | (4) |

where $φ=\left[φ\_{1}\left(vt\right), φ\_{2}\left(vt\right),…, φ\_{n}(vt)\right]^{T}$ is mass-normalised mode vector of the beam (without the moving oscillator) and $q=\left[q\_{1},q\_{2},…,q\_{n}\right]^{T}$ is the corresponding modal coordinate vector; $M\_{ij}=δ\_{ij}$, $K\_{ij}=ω\_{i}^{2}δ\_{ij}$, $ω\_{i}^{2}=\frac{EI}{ρA}\left(\frac{iπ}{l}\right)^{4}$. Note that $φ$ in Eq. (3) and Eq. (4) is numerical mode vector of the beam, which approximates its analytical mode. The FE model of the beam is built in ABAQUS and its numerical modes can be obtained by modal analysis and exported into MATLAB.

An iterative method is used to solve the two sets of equations, avoiding updating mass and stiffness matrices in the equation of motion of the whole system. It is performed in every time step in combination with the Newmark integration method and implemented in MATLAB.

From time $t\_{0}$ to time $t\_{0}+∆t$, the iterative scheme works as follows31:

Step 1. calculate $\_{b}=M(a\_{1}+a\_{3}+a\_{4})$ and $\_{v}=m\_{v}(a\_{1}+a\_{3}+a\_{4})$ at time $t\_{0}$, where $a\_{1}=\frac{1}{α∆t^{2}}, a\_{3}=\frac{1}{α∆t}, a\_{4}=\frac{1}{2α}-1 $ and $α$ is a Newmark integration parameter;

Step 2. assume the initial $q$ at time $t\_{0}+∆t$ as $^{0}=$;

Step 3. calculate $^{0}=a\_{2}\left(^{0}-\right)-a\_{5}-a\_{6}$ and $^{0}=a\_{1}\left(^{0}-\right)-a\_{3}-a\_{4}$, where $a\_{2}=\frac{β}{2∆t}, a\_{5}=\frac{β}{α}-1, a\_{6}=\left(\frac{β}{2α}-1\right)∆t$, $β$ is a Newmark integration parameter;

Step 4. calculate $\_{v}^{0}=-m\_{v}g+k\_{v}^{T}^{0}$;

Step 5. the equation of motion of the mass after applying Newmark integration can be obtained as $\left(k\_{v}+a\_{1}m\_{v}\right)^{0}=\_{v}^{0}+\_{v}$, so $^{0}=\frac{\_{v}^{0}+\_{v}}{k\_{v}+a\_{1}m\_{v}}$ ;

Step 6. calculate $^{0}=a\_{2}\left(^{0}-\right)-a\_{5}-a\_{6}$ and $^{0}=a\_{1}\left(^{0}-\right)-a\_{3}-a\_{4}$ if the velocity and acceleration of the mass are needed;

Step 7. calculate $\_{b}^{0}=k\_{v}\left(^{0}-^{T}^{0}\right)$;

Step 8. the equation of motion of the beam after applying Newmark integration scheme becomes $\left(K+a\_{1}M\right)^{'}=\_{b}^{0}+\_{b}$, so $^{'}=\left(K+a\_{1}M\right)^{-1}⋅(\_{b}^{0}+\_{b})$ where $^{'}$ is the new value of $q$ at time $t\_{0}+∆t$;

Step 9. check if the following convergence criterion is satisfied

|  |  |  |
| --- | --- | --- |
|  | $$\frac{Norm\left(^{'}-^{k}\right)}{Norm(^{'}-)}\leq ϵ$$ | (5) |

where $k$ is the iterative number which is 0 here and $ϵ$ (tolerance) is suggested to be between $1.0×10^{-5} $ and $1.0×10^{-8}$ by Yang and Fonder31.

If Eq.(5) is satisfied, the iterative scheme from time $t\_{0}$ to $t\_{0}+∆t$ ends and the next time step starts from step 1 again, otherwise $^{0}$ is updated by $^{1}=^{0}+η(^{'}-^{0})$ , where $η$ is a relaxation coefficient between 0 and 1. The iterative scheme ends when the oscillator leaves the beam. Although the moving oscillator model is used here to demonstrate MS method and iterative method, they can be applied to other more complicated models.

*2.2. Numerical verification*

The properties of the moving oscillator model are the same as those in the reference18: the length of the beam $L=30 m$, $A=7.73 m^{2}$, $I=7.84 m^{4}$, $E=2.825×10^{10} N/m^{2}$, $ρ=5400 kg/m^{3}$, $m\_{v}=4.255×10^{4} kg$, $k\_{v}=1.06×10^{6} N/m$. The critical speed (defined as the speed of a moving constant force at which the structure is excited into resonance at its first mode) of the beam is calculated as $v\_{cr}=\frac{Lω\_{1}}{π}=868.384 km/h$. The time step $∆t$ is constant to be $1×10^{-3}$ s.

The dynamic responses of the moving oscillator system given in Fig. 2 are very close to the analytic results in Fig. 9 by Nguyen and Kim18, which indicates that the approach presented in this paper works well. Please note that only the first mode is used in the computation for comparison, as such is the case for the ‘analytic’ results18.

 (a) (b) 

(c) (d) 

Fig. 2. Comparison of vertical displacements of bridge mid-span and accelerations of car body / sprung mass at speeds of 50 km/h (a, b) and 360 km/h (c, d) (‘Analytic’ refers to the ‘Analytic’ curve in Fig. 918)

# 3. Theoretical development and experimental study of a moving-car-on-bridge model

*3.1. Experimental setup*

A four-span continuous plate with four grounded elastic springs (representing four actuators) and a moving car on top is shown in Fig. 3. The whole length of the plate is 3.6 m with four equal spans. Front and rear ramps are located before and after the plate respectively. The displacements of four points of the plate are measured by four laser displacement transducers. Figure 4 shows the first two spans and the first span in more detail. The actuators are attached to the plate via thin rods functioning as elastic springs and they will be used in vibration control to be reported in a separate paper. Their average stiffness is measured to be around 8548 N/m. The mass of each thin rod is 50 g. A 4.335 kg model car can travel along the two rails glued on top of the plate. The width of the plate is 101.67 mm and the thickness of the plate is 3.16 mm. The two rails are the same and the width and height of each rail are 6.85 mm and 8.52 mm respectively.



Fig. 3. Lateral view of the whole experimental setup (unit: cm)

 

Fig. 4. Pictures of experimental setup: (a) first two spans, (b) first span

It should be pointed that vibration experiments of a 4-span continuous structure of this degree of complexity subjected to a moving car have not been reported in the literature.

*3.2. Car model*

The model car is treated as a rigid body which consists of two DOFs (pitch $θ$ and heave $z$) as shown in Fig. 5. The centre of gravity of the car is assumed to be at its geometric centre. The parameters of the car model are: $m\_{v}=4.335 kg$ , moment of inertial around the y axis $I\_{v}=0.012045 kg m^{2}$, its wheel span $s=0.126$ m and the car body length $L\_{c}=0.208$ m.

 

Fig. 5. Idealization of the moving car: (a) elevation; (b) cross-sectional view at the front axle

There are three stages representing different car locations on the infrastructure: stage 1 - the whole car at front ramp and its front wheel at starting point of the plate; stage 2 - the whole car on the plate and its rear wheel at starting point of the plate; stage 3 - the whole car on the plate and its front wheel at end point of the plate. The dynamic response of the system during the time period of the car moving from stage 1 to stage 3 is simulated and compared with experimental measurement. As the ramps are quite short and they are not connected with the plate, they can be taken to be rigid and without movement.

The equations of motion of the moving rigid body in the vertical plane during stage 1 to stage 3 can be written as

$m\_{v}\ddot{z\left(t\right)}=-m\_{v}g-k\_{1}\left[z\left(t\right)-θ\left(t\right)\frac{s}{2}-w\left(x\_{1},y\_{1},t\right)\right]-k\_{2}\left[z\left(t\right)-θ\left(t\right)\frac{s}{2}-w\left(x\_{2},y\_{2},t\right)\right]-k\_{3}\left[z\left(t\right)+θ\left(t\right)\frac{s}{2}-w\left(x\_{3},y\_{3},t\right)\right]-k\_{4}\left[z\left(t\right)+θ\left(t\right)\frac{s}{2}-w\left(x\_{4},y\_{4},t\right)\right]$ (6)

$I\_{v}\ddot{θ(t)}=k\_{1}\left[z\left(t\right)-θ\left(t\right)\frac{s}{2}-w\left(x\_{1},y\_{1},t\right)\right]\frac{s}{2}+k\_{2}\left[z\left(t\right)-θ\left(t\right)\frac{s}{2}-w\left(x\_{2},y\_{2},t\right)\right]\frac{s}{2}-k\_{3}\left[z\left(t\right)+θ\left(t\right)\frac{s}{2}-w\left(x\_{3},y\_{3},t\right)\right]\frac{s}{2}-k\_{4}\left[z\left(t\right)+θ\left(t\right)\frac{s}{2}-w\left(x\_{4},y\_{4},t\right)\right]\frac{s}{2}$ (7)

where $w\left(x\_{i},y\_{i},t\right) \left( i=1, 2, 3, 4\right)$ is the deflection of the rail at the contact point with $i$th spring; $k\_{i}$ is the stiffness of the $i$th spring and *z*(*t*) is the displacement of the gravity centre of the car body.

It should be noticed that the deflection of the ramp is zero due to its rigid body and no-movement assumption. The car is given a push at the start and allowed to travel along the rails freely. The time instants when it passes 4 midpoints are recorded by 4 electromagnetic sensors. It is found that its travelling speed can be taken to be constant.

*3.3. Parameter identification for structural model*

To know the area and moment of inertia of the rail section, a clear picture of the rail section was taken and its profile was then detected by using image processing and saved in a CAD drawing. Figure 6 shows the picture of the rail section and the CAD drawing of its profile. The two figures look similar but not the same, which could be caused by the error of edge detecting for the picture. The area and second moment of area of the rail section are calculated from its CAD drawing to be around $A\_{r}=28.0 mm^{2}$ and $I\_{r}=207.0 mm^{4}$ respectively. The cross section of the plate is rectangular. The area and second moment of area of the plate are calculated to be $A\_{p}=bh=321.3 mm^{2}$ and $I\_{p}=\frac{bh^{3}}{12}=267.3 mm^{4}$ respectively, where *b* is the width of the plate and *h* is the height of the plate. The density of the rail and the plate are identified to be around 8356.5 $kg/m^{3}$ and 7699.8 $kg/m^{3}$ respectively.

The Young’s modulus of a Euler-Bernoulli beam can be identified from measured natural frequencies

 $f\_{k}=\frac{β\_{k}^{2}}{2πl^{2}}\sqrt{\frac{EI}{ρA}}$ (8)

where $f\_{k}$ is the *k*th frequency of the beam and can be obtained by modal testing; $β\_{k}$ is the *k*th wave number of the beam and can be calculated by using beam’s theory according to its boundary conditions ; $l$ is the length of one span of the beam. $l$, $ρ$, *A* and *I* can be identified directly or indirectly, so *E* can be calculated inversely from this frequency formula. Modal testing is carried out on a plate specimen and a rail specimen separately to obtain their natural frequencies. The Young’s moduli of the rails and the plate are calculated to be around 86.6 GPa and 183.4 GPa respectively from Eq. (8).

 

Fig. 6. The picture of rail section and its CAD drawing

*3.4. FE model of the structure*

A 3D FE model of the plate with supports and rails is built in ABAQUS. 480 shell elements (S4R) are used for the plate and 160 beam elements (B31) for each rail which are tied to the plate. Four spring elements and four mass elements are tied to the plate too, modelling the elastic springs and the mass of the thin rods. For the boundary conditions of the plate, only the rotational degrees-of-freedom are allowed at the five pin supports. Modal analysis is done in ABAQUS to predict the frequencies of the structure which are compared with the experimental frequencies obtained by modal testing. An offset ratio for the shell elements is needed to represent the connection between the rails to the plate since the neutral axis of the rails and the neutral plane of the plate is at a short distance apart. As the structural FE model is the approximation of the actual structure, this value of the offset may not be the distance between the rails’ neutral axis and the plate’s neutral plane and it should be determined through model updating, which turns out to be 1.87 and brings the numerical frequencies of the structure close to their experimental counterparts.

Table 1 shows the differences between the first eight experimental frequencies and the corresponding numerical frequencies. It can be seen from the table that the differences are less than 5.0%, which indicates that the FE model of the structure is good enough for dynamic analysis. The first eight mode shapes of the structure obtained from ABAQUS are very similar to the mode shapes of the four-span continuous beam. The first torsional mode of the structure occurs as its 9th mode.

Table 1. Comparison between experimental and numerical frequencies of the structure

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Mode | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| Experimental | 21.242 | 23.381 | 28.632 | 36.374 | 68.541 | 73.564 | 83.080 | 93.560 |
| Numerical | 20.376 | 22.582 | 28.180 | 34.990 | 65.417 | 71.036 | 82.704 | 95.278 |
| Difference | -4.1% | -3.4% | -1.6% | -3.8% | -4.6% | -3.4% | -0.5% | 1.8% |

The equation of motion of the FE model of the structure can be expressed as

|  |  |  |
| --- | --- | --- |
|  | $$M\ddot{X}(t)+KX\left(t\right)=F(t)$$ | (9) |

where $M$and $K$are mass matrix and stiffness matrix of the structure model respectively; $F(t)$ is the force vector which changes with the movement of the car. The concentrated forces $F\_{i}$ acting on the structure at the $i$th wheel location can be expressed as

|  |  |  |
| --- | --- | --- |
|  | $$F\_{i}=k\_{i}\left(z-w\left(x\_{i},y\_{i}\right)-\frac{s}{2}θ\right), i=1, 2 $$ | (10) |
|  | $$F\_{i}=k\_{i}\left(z-w\left(x\_{i}, y\_{i}\right)+\frac{s}{2}θ\right), i=3, 4$$ | (11) |

What should be noticed is that the contact force at the $i$th wheel is $-F\_{i}$ as compression force is defined as positive force in contact mechanics.

$X(t)$ is the nodal displacement vector of the structure which can be expressed as

|  |  |  |
| --- | --- | --- |
|  | $$X\left(t\right)=Φq(t)$$ | (12) |

where $Φ$ is the mass-normalised modal matrix (numerical modes) of the structure model and $q(t)$ is the vector of the modal coordinates.

Substituting Eq. (12) into Eq. (9) and multiplying $Φ^{T}$on both sides of Eq. (9) lead to

|  |  |  |
| --- | --- | --- |
|  | $$\ddot{q}(t)+diag[ω\_{i}^{2}]q\left(t\right)=Φ^{T}F(t)$$ | (13) |

 where $ω\_{i}$ is the *i*th natural frequency of the structure and it is found that using 8 or moremodes does not make a big difference on dynamic response of the system, so 8 modes are used. The torsional modes of the structure do not produce much influence on the structural dynamics.

 *3.5. Experimental verification of the system model*

 The structural model and the car model are combined and solved by the aforementioned iterative method in MATLAB. It is found from simulations that adopting $1×10^{5}$ N/m or bigger values for the stiffness of springs $k\_{1}$, $k\_{2}$, $k\_{3}$ and $k\_{4}$ does not make much difference, so the stiffness of springs $k\_{1}$, $k\_{2}$, $k\_{3}$ and $k\_{4}$ is taken to be $1×10^{5}$ N/m for modelling the rigid contact between the wheels and rails. The average velocity of the car is measured to be around 1.05 m/s. The constant time step length is 0.0005 s. The dynamic responses of the plate at four measured points are predicted by the system model and compared with experimental measurements obtained by laser displacement transducers.

Figure 7 gives the displacements measured by four laser transducers and the deflections of the four points on the plate predicted by the system model respectively. It can be seen from the figure that the downward plate deflections from simulation and experiment are almost the same in magnitude, but numerical results of the upward deflections are slightly bigger than the experimental ones. In addition, all the numerical responses lag slightly behind the experimental ones. The difference between the two sets of results could be caused by the inevitable errors in the identified parameters and the assumption of constant car velocity used in the system model. Note that the car’s travelling speed decreases slightly due to friction as it moves forward. Generally, the numerical results agree well with experimental ones, which indicates that the system model is good enough to predict the structural dynamic response.



Fig. 7. Comparison between displacements measured by laser transducers and numerical displacements of the plate at measurement points

# Numerical analysis

*4.1. Moving rigid-body model and moving mass model*

To see the difference between the results by treating the car as a 2-DOF rigid body and the results by treating it as a moving (point) mass, a comparison is made below. The mass is the same as the car and the stiffness of the contact spring for the moving mass model is taken as the total stiffness between the car body and the plate: $k\_{v}=4×10^{5}$ N/m. The deflections of the plate at the four measured points from the two models are shown in Fig. 8. It can be seen that there is little difference between the results from the two models: the time histories are almost the same, but the magnitude obtained by the moving mass model is slightly bigger. However, if a torsional mode of the plate is excited, a moving rigid-body model needs to be adopted to take account of the influence of this torsional mode.



Fig. 8. Comparison between numerical displacements of the plate at measurement points by moving rigid-body model and those by moving mass model

The displacement of the centre of the car body and the displacement of the mass are shown in Fig. 9. It can be found in the figure that the displacement of the car body is basically the same as, but slightly smaller than the displacement of the mass.



Fig. 9. Comparison between displacement of rigid-body centre and displacement of moving mass

*4.2. The influence of car speed*

Figure 10 shows the influence of car speed on the dynamic response of the four mid-span points (not the same as the four measured points) of the structure. The critical speed of the structure is $v\_{cr}=38.236$ m/s.

The reference static deflections are obtained in ABAQUS by placing the car at the mid-spans of the plate so that the four wheels present four concentrated forces. Dynamic Amplification Factor (DAF) is defined as the ratio between the maximum dynamic response of the mid-span point of one span and its static deflection with the car on the mid-span of this span. It can be seen from the figure that the maximum dynamic responses of the mid-span points basically increase with the ratio of the car speed to the critical speed in the range of 0.015 to 0.15. A small local peak occurs at the ratio of 0.075. The DAFs of the mid-span points in spans 2-4 fluctuate very much when the speed ratio is greater than 0.135. This is probably because that the influence of vibration of front spans on rear spans becomes strong under high car speed. It is found that negative contact forces always occur when car speed ratio is greater than 0.2. However, the contact force can only be greater or equal to zero (loss of contact) in reality, so the dynamic responses with car speed ratio greater than 0.2 are not presented in this paper. Fryba3 showed that the maximum dynamic response of a simply supported beam under a moving constant force increases firstly and then decreases with the speed and the peak response happens at the speed ratio in the range of 0.5 to 0.7 for different damping of the beam. What should be paid attention to is that the car speed that leads to loss of contact changes with car mass as shown in Table 2. For a heavier car, a lower car speed for loss of contact is found.



Fig. 10. DAF of mid-span points against car speed ratio (*m*v= 4.335 kg and $L\_{c}$=0.208 m)

Table 2. Speed ratio for loss of contact

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Car mass / structure mass | 0.1 | 0.4 | 0.7 | 1.1 |
| Car speed / critical speed | 0.5 | 0.2 | 0.15 | 0.1 |

*4.3. The influence of span ratio*

The relationship between the dynamic response of the structure and the wheelbase with different mass ratios (car mass / structure mass) is shown in Fig. 11. The mass ratio is changed by changing the car mass only. It is found from the figure that DAF basically does not vary with mass ratio under the same car peed. Increasing wheelbase can make the dynamic response of the structure decrease in the range of span ratios (wheelbase / beam span) between 0 and 0.8, and increase to a steady state afterwards. That is to say that a shorter car can cause a greater structural dynamic response, which is verified by the previous fact that the displacements of the structure obtained from the moving mass model are bigger than that from the moving rigid-body model. It is interesting to find that the dynamic response of the first span reaches a steady state with shorter wheelbase than that of other spans as shown in the figure. The dynamic responses for several wheelbases are shown in Fig. 12. It can be seen that initially when the car span ratio is 0.3 the dynamic response of one mid-span point only has one downward peak, but has two downward peaks for car span ratio of 0.8 or higher and the time difference between the two peaks increases with wheelbase. The two sets of wheel loads are like two separate loads at a certain distance apart, which may cause two peaks in the dynamic response of the structure. When the wheelbase increases, it seems like that the distance between two mass loads increases. In consequence, the time interval between the two peaks becomes bigger.

(a)  (b) 

(c)  (d) 

Fig. 11. Dynamic responses of mid-span points against wheelbase at car speed *v*=1.05 m/s with mass ratios of (a) 0.2, (b) 0.4, (c) 0.6, (d) 0.8





Fig. 12. Dynamic responses of mid-span points for span ratios of (a) 0.3, (b) 0.8, (c) 1.7, (d) 3.0 with mass ratio of 0.4

*4.4. The influence of contact spring stiffness*

The influence of contact spring stiffness on the dynamic response of the structure can be seen in Fig. 13 (*m*v=4.335 kg and *L*c=0.208 m). As can be seen in the figure, the contact spring stiffness does not influence the dynamic response of the structure much under car speed of 1.05 m/s, but the influence becomes stronger at higher car speeds. It is interesting to find that the magnitude of the contact force between the first car wheel and the plate structure does not differ much with contact spring stiffness at *v*=1.05 m/s as shown in Fig. 14. This is because when the car speed is low, the inertial force in the contact force is small and the main contribution to the contact force is the dead-weight of the car which is constant. However, higher frequencies of the contact force occur when the contact spring stiffness increases at the car speed of *v*=1.05 m/s, as found in Fig. 15. This is also true for the car speed of *v*=5.74 m/s shown in Fig. 17. On the other hand, the magnitude of structural response differs significantly with changing contact spring stiffness at *v*=5.74 m/s as shown in Fig. 13 and the magnitude of the contact force basically increases with contact spring stiffness at *v*=5.74 m/s as shown in the Fig. 16. Figure 18 and Figure 19 show that the dynamic response of the car body decreases quite much in magnitude with contact spring stiffness for both car speeds (1.05 m/s and 5.74 m/s). It can be seen from Fig. 20 that the car basically oscillates slightly faster with larger contact spring stiffness when the car moves at 5.74 m/s. However, this is not true when *v*=1.05 m/s. It can be estimated from Fig. 18 that the car almost oscillates at the same frequency with different contact spring stiffness. Please note that the above phenomenon found for the contact force between the first car wheel and the rail is also true for the contact forces between other car wheels and the rails, although they are not presented in this paper.

When the car speed is 1.05 m/s, the basic frequency of the moving car (different from the natural frequency of the car) is related to the car speed and equals to $f\_{v}=\frac{v}{L\_{0}}=\frac{1.05}{0.9}=1.167 Hz$. This frequency and the natural frequency of the car in the vertical bouncing mode presented in Table 3 are the source of the frequencies of the contact force, which can be found in Fig. 15. When the car speed is 5.74 m/s, the basic frequency of the moving car related to the car speed becomes $f\_{v}=\frac{v}{L\_{0}}=6.373$ Hz. This frequency and the natural frequency of the car shown in Table 3 are the source of the frequencies of the contact force and can be found in Fig. 17. The high frequency of the contact force looks like the integer multiples of $f\_{v}$, which seem quite obvious in Fig. 17 (c).

Table 3. Natural frequencies (Hz) of the car in the vertical bouncing mode and rotation mode

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Spring stiffness (N/m) | 1.0E+02 | 1.0E+03 | 1.0E+04 | 1.0E+05 |
| Frequency in bouncing | 1.529 | 4.835 | 15.288 | 48.345 |
| Frequency in rotation | 2.648 | 8.374 | 26.479 | 83.734 |

 

 

Fig. 13. The influence of contact stiffness of one spring on DAF at (a) 1st, (b) 2nd, (c) 3rd, (d) 4th mid-span points under different car speeds

 

 

Fig. 14. Contact force at first wheel for contact stiffness of one spring (a) $1.0×10^{2}$, (b) $1.0×10^{3}$, (c) $1.0×10^{4}$, (d) $1.0×10^{5}$ N/m at car speed of 1.05 m/s

 

 

Fig. 15. Single-sided amplitude spectrum of contact force at first wheel for contact stiffness of one spring (a) $1.0×10^{2}$, (b) $1.0×10^{3}$, (c) $1.0×10^{4}$, (d) $1.0×10^{5}$ N/m at car speed of 1.05 m/s





Fig. 16. Contact force at first wheel for contact stiffness of one spring (a) $1.0×10^{2}$, (b) $1.0×10^{3}$, (c) $1.0×10^{4}$, (d) $1.0×10^{5}$ N/m at car speed of 5.74 m/s

 

 

Fig. 17. Single-sided amplitude spectrum of contact force at first wheel for contact stiffness of one spring (a) $1.0×10^{2}$, (b) $1.0×10^{3}$, (c) $1.0×10^{4}$, (d) $1.0×10^{5}$ N/m at car speed of 5.74 m/s

 

 

Fig. 18. Displacement of car body centre for contact stiffness of one spring (a) $1.0×10^{2}$, (b) $1.0×10^{3}$, (c) $1.0×10^{4}$, (d) $1.0×10^{5}$ N/m with car speed of 1.05 m/s





Fig. 19. Displacements of car body centre for contact stiffness of one spring (a) $1.0×10^{2}$, (b) $1.0×10^{3}$, (c) $1.0×10^{4}$, (d) $1.0×10^{5}$ N/m at car speed of 5.74 m/s



 

Fig. 20. Single-sided amplitude spectrum of displacements of car body centre for contact spring of one spring (a) $1.0×10^{2}$, (b) $1.0×10^{3}$, (c) $1.0×10^{4}$, (d) $1.0×10^{5}$ N/m at car speed of 5.74 m/s

# Conclusions

Moving load problems are difficult to analyse and no commercial general-purpose structural analysis software packages can deal with them efficiently. A general and easy way to solve moving load problems is presented in this paper by using ABAQUS and MATLAB. Finite Element (FE) models of complicated structures are built in ABAQUS and their numerical modes are obtained and imported into MATLAB codes where Modal Superposition (MS) and an iterative method are implemented. The moving oscillator model is adopted to verify the proposed approach. The numerical results show that this approach works well with high efficiency.

A four-span continuous plate with two rails on top and four extra elastic supports excited by a moving model car is studied numerically and experimentally. The car is treated as a rigid body with two degrees-of-freedom in contact with the rails via four springs. An FE model of the plate with rails is built in ABAQUS to obtain its numerical modes and the parameters used in the FE model of the structure are identified based on measured frequencies. It is found that the torsional modes of the structure do not have much influence on the structural dynamic response within the speed range studied. The numerical prediction of structural displacements at four measured points of the structure agrees well with the experimental measurements. Further numerical studies reveal some interesting results. The structural response and car displacement from the moving rigid-body model are slightly smaller than those from the moving mass model. Increasing car speed can increase structural dynamic response in a certain speed range and loss of contact between the wheel and rail occurs at a lower car speed for a heavier car. Basically, a longer car decreases structural dynamic response but does not make much influence on the structural response when the wheelbase is 1.5 times of one span of the structure or longer. When the car speed is low, the change of contact spring stiffness does not make much difference in structural response, but at a high car speed, the influence becomes strong. Additionally, at high car speeds larger contact spring stiffness basically leads to bigger contact forces between the car wheels and the rails and slightly quicker oscillation of the car. Regardless of the car speed, the frequencies of the contact forces basically increase but the dynamic response of the moving car body decreases, with contact spring stiffness.

**Acknowledgements**

The first author would like to express his gratitude to the China Scholarship Council and the University of Liverpool who sponsor the first author’s PhD study and research work. The work is partly supported by EPSRC (EP/H022287/1).

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