Cost-Sharing in Generalized Selfish Routing

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Abstract. We study a generalization of atomic selfish routing games where each player may control multiple flows which she routes seeking to minimize their aggregate cost. Such games emerge in various settings, such as traffic routing in road networks by competing ride-sharing applications or packet routing in communication networks by competing service providers who seek to optimize the quality of service of their customers. We study the existence of pure Nash equilibria in the induced games and we exhibit a separation from the single-commodity per player model by proving that the Shapley value is the only cost-sharing method that guarantees it. We also prove that the price of anarchy and price of stability is no larger than in the single-commodity model for general cost-sharing methods and general classes of convex cost functions. We close by giving results on the existence of pure Nash equilibria of a splittable variant of our model.

1 Introduction

Congestion games are a well-studied abstraction of a large collection of applications which includes several network routing games. Rosenthal proposed the model [26, 27] and in the past 15 years, starting with [31], there has been a large body of work in the area (e.g., [2, 4, 6, 7, 8, 10, 11, 13, 17, 18, 28]). Network applications have been one of the main motivations behind the success of the model and *selfish routing* is the paradigmatic example in the study of existence and inefficiency of equilibrium solutions. A selfish routing game is played on a directed graph $G = (V, E)$. Each player i in the game is characterized by a start vertex s_i, a destination vertex t_i, and a flow size w_i. Player i must select an s_i-t_i path that minimizes the sum of the edge costs along the path. The edge costs are increasing functions of the total flow on them and there is a predefined cost-sharing method that dictates how edge costs are distributed among each edge's users. The main assumption is that players reach a Nash equilibrium and the system performance is typically measured by comparing the worst or best Nash equilibrium to the optimal solution in terms of total cost. These metrics are termed the *price of anarchy* (POA) and *price of stability* (POS), respectively [3, 23]. Existence of a pure Nash equilibrium (PNE) and POA/POS performance properties are very well understood for general cost-sharing methods in the selfish routing model [12, 14, 21, 22, 30].

In this paper, we study a generalization of the selfish routing game, which we term *selfish routing with multi-commodity players*. In this generalization, each player may control more than one flow in the network. Similar settings have been studied before in the context of scheduling games [1], in the context of integer splittable routing games [27, 33], and for a special case of our model (where each commodity has the same flow size) in [10]. More specifically, player i is described by a set of commodities Q_i. Each commodity q has a starting vertex s_q, a destination vertex t_q and a flow size w_q. Each player i must pick how to route the flows in Q_i, each on a single path. Applications of our model include routing in road networks where ride-sharing platforms operate and routing in communication networks where connections are operated by service providers. Consider the example of ride-sharing platforms. The game is played on the road network and there is a continuous flow of rides using either platform between each pair of nodes in the graph. The route that each car follows is dictated centrally by the platform that seeks to optimize the aggregate travel time of its flows. In the packet routing application, network connections are routed by competing service providers. Each service provider wishes to optimize the quality of service of their clients and hence routes connections seeking to minimize their aggregate costs.

As a concrete example, consider a network with two nodes s, t, and two parallel edges e_1, e_2, from s to t. The joint cost of each edge is given as $C(x) = x^2$, with x the total flow on the edge. The game has two players. Player 1 wishes to route a flow of size 1 from s to t, while player 2 wishes to route two flows, each of size 1, from s to t. Suppose the cost-sharing method dictates that each commodity traveling through an edge pays an equal share of the joint cost. Player 2 has three options: route both commodities on the same edge that player 1 is using, route both commodities on the other edge, or route the two commodities on different edges. The corresponding costs for player 2 would be 6, 4, and 3, which establishes the latter option as the best response.

1.1 Our Results

In this work, we search for cost-sharing methods that guarantee the existence of pure Nash equilibria in multi-commodity selfish routing games. We also focus on the inefficiency of equilibria and we conduct a comprehensive study of the POA/POS, i.e., the ratio of the total cost in the worst/best Nash equilibrium over the optimal total cost. Our results hold for general cost functions and cost-sharing methods and they also extend to general congestion games.

Regarding the existence of pure Nash equilibria, we show that applying the *Shapley value* per edge, with the weight of a player on an edge being the sum of the commodity sizes she places on the edge, results in a potential game and, hence, guarantees the existence of a pure Nash equilibrium. On the contrary, we show that weighted Shapley values may induce games such that no pure Nash equilibrium exists, which exhibits a separation from the single-commodity case, where each player controls only one commodity. Given that the class of weighted Shapley values are the unique cost-sharing methods that guarantee pure Nash equilibria in the single-commodity player model [15], our results suggest that the Shapley value is essentially the unique anonymous cost-sharing method that guarantees pure Nash equilibria in the multi-commodity player model.

With respect to the inefficiency of equilibria, we prove upper bounds on the POA that match the ones from the single-commodity per player model. Our bounds work for general (convex) cost functions and for general cost sharing methods satisfying the following natural assumptions [12], which we briefly discuss afterwards and explain in more detail in Section 1.3:

1. Every cost function in the game is continuous, increasing and convex.
2. Cost-sharing is consistent when player sets generate costs in the same way.
3. For convex resource cost functions, the cost share of a player on a resource is a convex function of her flow on the resource.

Assumption 1 is standard in congestion-type settings. For example, linear cost functions have obvious applications in many network models, as do queueing delay functions, while higher degree polynomials (such as quartic) have been proposed as realistic models of road traffic [32]. With assumption 2, the cost sharing method only charges players according to how they contribute in the total cost and there is no other way of discrimination between them. Assumption 3 asks that the curvature of the cost shares is consistent, i.e., given assumption 1, that the share of a player on a resource is a convex function of her weight (otherwise, we would get that the cost share of the player increases in a slower than convex way but the total cost of the constant weight of players increases in a convex way, which we view as unfair).

The POS is an interesting concept and it is very well motivated in cases where the players can be started in an initial configuration or where a trusted mediator can suggest a solution to the players. This suggests that the POS is especially interesting in cases where a pure Nash equilibrium exists. Therefore, on the POS side, we focus on the Shapley value, the only cost-sharing method that guarantees existence of a pure Nash equilibrium in our setting. We prove that the POS is equal to the POS of the single-commodity case for general classes of cost functions.

Finally, we study an extension to the *splittable* model, where players may split their commodities across different paths. In particular, we study the existence of pure Nash equilibria in that setting and mention interesting open problems.

1.2 Related Work

Previous works in [20, 10, 27, 33] study settings that share similarities to multi-commodity routing games. In [27], Rosenthal studies weighted routing games where each player may split her integer flow size among different subflows of integer size. Focusing on the proportional cost-sharing method (that charges each player a cost proportional to her flow on an edge), he proves that there exist such games with no PNE. In [33], the authors identify special cases where PNE exist in Rosenthal's model. Our approach differs from the work in [27] and [33]. We study general multi-commodity players and not only players who control unit flows with the same start vertex. In [20], it is shown that there exist games where merging atomic players into a coalition (similarly into a multi-commodity player) may degrade the quality of the induced PNE when proportional sharing is used.

In a small contrast, we focus on worst-case metrics and show that the POA and POS of multi-commodity player games is no worse than in the single-commodity case, for general cost-sharing methods. Finally, in [10], the authors focus on coalitions of atomic players in routing games (equivalent to multi-commodity players) and mostly on the objective of minimizing the maximum cost. For the sum of costs objective, which we consider in this paper, they prove that the game always admits a pure Nash equilibrium under proportional cost-sharing and quadratic edge cost functions. We provide more comprehensive results with respect to the existence of pure Nash equilibria for general methods and general classes of cost functions.

On the cost-sharing side, the authors in [15] characterize the class of (generalized) weighted Shapley values as the only methods to guarantee existence of a PNE when each player controls one commodity. We exhibit a separation from this result by showing that weighted Shapley values do not guarantee pure Nash equilibria existence in the multi-commodity extension. With respect to the POA and POS of cost-sharing in routing games, [12] provides general tight bounds, which, in this work, we generalize to the multi-commodity player model.

1.3 Preliminaries

In this section, we present the notation and preliminaries for our model in terms of a general *congestion game with multi-commodity players*. In such a game, there is a set Q of k commodities which are partitioned into $n \leq k$ non-empty and disjoint subsets $Q_1, Q_2, \ldots Q_n$. Each set of commodities Q_i, for $i = 1, 2, \ldots, n$, is controlled by an independent player. Denote $N = \{1, 2, \ldots, n\}$ the set of players. The players in N share access to a set of resources E. Each resource $e \in E$ has a flow-dependent cost function $C_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. As stated in assumption 1 (section 1.1), we assume the cost functions of the game are drawn from a given set $\mathcal{C}$ of allowable cost functions, such that every $C \in \mathcal{C}$ must be continuous, increasing and convex. We also make the mild technical assumption that the set $\mathcal{C}$ is closed under (i) scaling and (ii) dilation, meaning that if $C(x) \in \mathcal{C}$, then (i) $C(a \cdot x) \in \mathcal{C}$ and also (ii) $a \cdot C(x) \in \mathcal{C}$, for every positive a.

Strategies. Each commodity $q \in Q$ has a set of possible strategies $\mathcal{P}^q \subseteq 2^E$. Associated with each commodity q is a weight w_q, which has to be allocated to a strategy in $\mathcal{P}^q$. For a player i, a strategy $P_i = (P_q)_{q \in Q_i}$ defines the strategy for each commodity q player i controls. An outcome $P = (P_1, P_2, \ldots, P_n)$ is a tuple of strategies of the n players.

Load. For an outcome P, the flow $f_e^i(P)$ of a player i on resource e equals the sum of the weights of all her commodities using e, i.e., $f_e^i(P) = \sum_{q \in Q_i, e \in P_q} w_q$. The total flow on a resource e is given as $f_e(P) = \sum_{i \in N} f_e^i(P)$. We use $x_e(P)$ for the set of players who assign positive flow on resource e on an outcome P.

Cost shares. The cost sharing method of the game determines how the flow-dependent joint cost of a resource $C_e(f_e(P))$ is divided among its users. Given an outcome P, we write $\chi_{ie}(P)$ for the cost of player i on resource e, such that $\sum_{i \in N} \chi_{ie}(P) = C_e(f_e(P))$. The cost of a player i, $X_i(P)$, is the sum of her costs on each resource, $X_i(P) = \sum_{e \in E} \chi_{ie}(P)$. For any $T \subseteq N$, let $f_e^T(P)$ be the vector of the flows that each player in T assigns to e. Then the cost share of player i can also be defined as a function of the player's identity, the resource's cost function and the vector of flows assigned to e, i.e., $\chi_{ie}(P) = \xi(i, f_e^N(P), C_e)$.

In this paragraph we explain in more detail assumption 2 and 3 from Section 1.1, which are needed for our general POA results in Section 3. Assumption 2 states that the cost-sharing method only charges players based on how they contribute to the joint cost. More specifically, assume we scale the joint cost on a resource by a positive factor β, i.e., $C'_e(f_e(P)) = \beta \cdot C_e(f_e(P))$. Given that the same players use this resource, the new cost shares of the players would be a scaled by factor β version of their initial cost shares, i.e., $\xi'(i, f_e^N(P), C_e) = \beta \cdot \xi(i, f_e^N(P), C_e)$. This is given by scaling and replication arguments. Last, we make the fairness-related assumption 3 which states that the cost share of a player on a resource is a convex function of her flow.

We now define a specific class of cost-sharing methods, which is important in our analysis.

Weighted Shapley values. The *weighted Shapley value* defines how the cost $C_e(\cdot)$ of resource e is distributed among the players using it. Given an ordering π of N, let $F_e^{<i,\pi}(P)$ be the sum of flows of the players preceding i in π. Then the marginal cost increase caused by player i is $C_e(F_e^{<i,\pi}(P) + f_e^i(P)) - C_e(F_e^{<i,\pi}(P))$. For a given distribution Π over orderings, the cost share of player i on resource e is $E_{\pi \sim \Pi}[C_e(F_e^{<i,\pi}(P) + f_e^i(P)) - C_e(F_e^{<i,\pi}(P))]$. For the weighted Shapley value, the distribution over orderings is given by a sampling parameter $\lambda_e^i(P)$ for each player i. The last player in the ordering is picked proportional to the sampling parameters $\lambda_e^i(P)$. Then this process is repeated iteratively for the remaining players.

As in [12], we study a parameterized class of weighted Shapley values defined by a parameter γ. For this class, $\lambda_e^i(P) = f_e^i(P)^\gamma$ for all players i and resources e. For $\gamma = 0$, this reduces to the (standard) *Shapley value*, where we have a uniform distribution over orderings.

Pure Nash equilibrium. We now proceed with the definition of our solution concept. The *pure Nash equilibrium* (PNE) condition on an outcome P states that for every player i it must be the case that

$$X_i(P) \leq X_i(P'_i, P_{-i}), \text{ for any other strategy } P'_i. \tag{1}$$

Social cost. The social cost in the game is given by the sum of the player costs, i.e.,

$$SC(P) = \sum_{i \in N} X_i(P) = \sum_{i \in N} \sum_{e \in E} \xi(i, f_e^N(P), C_e) = \sum_{e \in E} C_e(f_e(P)). \tag{2}$$

Price of Anarchy and Price of Stability. Let $\mathcal{Z}$ be the set of outcomes and $\mathcal{Z}^{\mathcal{N}}$ be the set of pure Nash equilibria outcomes of the game. Then the *price* *of anarchy* (POA) and the *price of stability* (POS) are defined as follows,

$$POA = \frac{\max_{P \in \mathcal{Z}^{\mathcal{N}}} SC(P)}{\min_{P \in \mathcal{Z}} SC(P)} \quad \text{and} \quad POS = \frac{\min_{P \in \mathcal{Z}^{\mathcal{N}}} SC(P)}{\min_{P \in \mathcal{Z}} SC(P)}. \tag{3}$$

The POA and POS for a class of games are defined as the largest such ratios among all games in the class.

2 Existence of Pure Nash Equilibria

Our first result proves that applying the (standard) Shapley value (with respect to the player flows $f_e^i(P)$) on each resource, induces a potential game. Recall that, for the Shapley value cost-sharing, we have a uniform distribution over orderings, i.e., we use the definition of weighted Shapley values in Section 1.3 with every sampling parameter equal to 1.

Theorem 1. *Congestion games with multi-commodity players under Shapley cost sharing are exact potential games.*

Proof. Consider any ordering π of the players in N and let $f_e^{\leq i,\pi}(P)$ denote the vector that we get after truncating $f_e^N(P)$ by removing all entries for players that succeed i in π. We prove that the following is a potential function of the game,

$$\Phi(P) = \sum_{e \in E} \sum_{i \in N} \xi(i, f_e^{\leq i,\pi}(P), C_e). \tag{4}$$

Hart and Mas-Colell [19] proved that (4) is independent of the ordering π in which players are considered. Let $P' = (P_i', P_{-i})$. It suffices to show that $\Phi(P) - \Phi(P')$ equals the change in the cost of player i. Focus on a single resource e and let π be one of the orderings that places the flow of player i, $f_e^i(P)$, in the last position. Then, the potential on resource e loses a term equal to

$$\xi(i, f_e^{\leq i,\pi}(P), C_e) = \xi(i, f_e^N(P), C_e)$$

and gains a term equal to

$$\xi(i, f_e^{\leq i,\pi}(P_i', P_{-i}), C_e) = \xi(i, f_e^N(P_i', P_{-i}), C_e),$$

which is precisely the change in the cost of player i on e. Summing over all edges gives the desirable $\Phi(P) - \Phi(P') = X_i(P) - X_i(P')$, which completes the proof. $\square$

One might expect that, similarly to standard congestion games, the same potential function argument would apply under weighted Shapley values as well. However, this is not the case.

Theorem 2. *There is a congestion game with multi-commodity players admitting no PNE for any weighted Shapley value defined by sampling weights of the form $f_e^i(P)^\gamma$ with $\gamma > 0$ or $\gamma < 0$.*

Proof. We prove this theorem by showing two examples admitting no PNE, for $\gamma > 0$ and $\gamma < 0$. Due to page limitations, we restrict to the description of the instances. For the $\gamma > 0$ case: Consider two players, 1 and 2, who compete for two parallel (meaning each commodity must pick exactly one of them) resources e,e' with identical cost functions $C_e(x) = C_{e'}(x) = x^{1+\delta}$ with $\delta > 0$ and $\frac{\gamma}{\delta}$ a large positive number (note that for $\delta = 0$, we have linear cost functions where in this case we have an equilibrium. As soon as we deviate from linearity, we use convexity to construct an example with no equilibrium). Player 1 controls a unit commodity $p \in Q_1$. Player 2 controls two commodities $q, q' \in Q_2$, with $w_{q'} = 1$ and $w_q = k$, for k a very large number. Recall, that the sampling weight of a player i on a resource e is given by $\lambda_e^i = (f_e^i)^\gamma$. This means that smaller weights are favoured when constructing the weighted Shapley ordering. In particular, for $k \to \infty$, if commodities p, q share the same resource, then the probability that q goes last in the Shapley ordering becomes 1 and the cost of commodity q would be $(k+1)^d - 1$.

We switch to the $\gamma < 0$ case: Consider players $i = 1, 2, \ldots, k$, who compete for two parallel resources e_1, e_2 with identical cost functions $C_{e_1}(x) = C_{e_2}(x) = x^3$. Player k controls two commodities $p, q \in Q_k$ with weights $w_p = k$ and $w_q = 1$. Each player $i < k$ controls only one commodity $r_i \in Q_i$ with $w_{r_i} = 1$. The sampling weight of a player i on a resource e is given by $\lambda_e^i = (f_e^i)^\gamma$, for $\gamma < 0$. □

2.1 Alternative Cost-Sharing based on Commodity Weights

One might consider a different way of generalizing weighted Shapley values to multi-commodity congestion games: Apply a weighted Shapley value on the commodity weights by charging a player the sum of the weighted Shapley values of the commodities controlled by her. These cost-sharing methods coincide when all commodities have unit weights, which is equivalent to proportional cost-sharing, i.e., every player pays a cost-share that is proportional to her flow on any given resource. Below we use one such instance with unit commodities to prove that applying a weighted Shapley value method on commodity weights does not guarantee pure Nash equilibrium existence.

Our instance is based on an example in [10], where Fotakis et al. prove that network unweighted congestion games with linear resource cost functions and equal cardinality coalitions do not have the *finite improvement property*, therefore they admit no potential function. Their example translates to a restricted setting of our model where each player controls an equal number of unit commodities. We strengthen their result by proving non-existence of pure Nash equilibria for congestion games with multi-commodity players and cubic resource cost functions (we construct even a network congestion game with no pure Nash equilibrium).

A similar example has already been given by Rosenthal [27]. However, Rosenthal's example uses concave cost functions, which we disallow in our setting. In contrast, our proof uses only convex functions.

Theorem 3. *There is a congestion game with multi-commodity players and cubic cost functions admitting no pure Nash equilibrium under weighted Shapley sharing applied on commodity weights.*

3 The POA and POS of Multi-Commodity Games

In this section we prove that the POA and POS of multi-commodity congestion games are no larger than those of their single-commodity counterparts, for any cost-sharing method and class of cost functions satisfying the natural assumptions in Section 1.3. Due to space limits, the POA proof is omitted. It follows along the lines of the proof for the single-commodity per player model [12].

Theorem 4. *The POA of multi-commodity congestion games under a cost-sharing method ξ and with costs drawn from a given class of increasing and convex cost functions $\mathcal{C}$, such that $\xi, \mathcal{C}$ satisfy assumptions 1, 2, and 3, is equal to the POA of single-commodity congestion games induced by ξ and $\mathcal{C}$.*

Theorem 5. *The POS of Shapley value based multi-commodity congestion games with costs drawn from a given class of increasing and convex cost functions $\mathcal{C}$, is equal to the corresponding POS of the single-commodity case.*

Proof. We begin with the potential function of the game (4) and we prove the following lemma which we use to prove the upper bound on POS. Briefly, the lemma states the following. For any instance with N players and any strategy profile, we can construct a new instance with $N+1$ players by splitting one player in half into two new players. Then this can only reduce the potential value of the game. More precisely, we do this by splitting in half the flow of each commodity controlled by a player i on a resource creating two new commodities, which we assign to the new players i' and i''.

Lemma 1. *Consider an outcome P of the game and assume that on a resource e, we substitute the total flow of a player i with the flows of two other players i',i'' such that $f_e^{i'}(\hat{P}) = f_e^{i''}(\hat{P}) = \frac{f_e^i(P)}{2}$. Then we claim that*

$$\Phi_e(P) \geq \Phi'_e(\hat{P}),$$

where $\Phi'_e(\hat{P})$ is the potential value of resource e after the substitution.

Proof. First, rename the flows such that the substituted one $f_e^i(P)$ to have the highest index. Assign indices i' and i'' to the new ones, with $i < i' < i''$ in ordering π. Then, for any resource e, the new potential value equals to

$$\Phi'_e(\hat{P}) = \sum_{j=1}^{i-1} \xi(j, f_e^{\leq j,\pi}(P), C_e) \; + \; \xi(i', (f_e^{<i,\pi}(P), f_e^{i'}(\hat{P})), C_e) \; + \\ + \; \xi(i'', (f_e^{<i,\pi}(P), f_e^{i'}(\hat{P}), f_e^{i''}(\hat{P})), C_e).$$

Note that the contribution to the potential value of the players before player i is the same as before the substitution. Therefore it is enough to show that

$$\xi(i, f_e^{<i,\pi}(P), C_e) \geq \xi(i', (f_e^{<i,\pi}(P), f_e^{i'}(\hat{P})), C_e) \; + \\ + \; \xi(i'', (f_e^{<i,\pi}(P), f_e^{i'}(\hat{P}), f_e^{i''}(\hat{P})), C_e). \tag{5}$$

To simplify, in what follows call

$$\xi = \xi(i, f_e^N(P), C_e),$$
$$\xi' = \xi(i', (f_e^{<i,\pi}(P), f_e^{i'}(\hat{P})), C_e),$$
$$\xi'' = \xi(i'', (f_e^{<i,\pi}(P), f_e^{i'}(\hat{P}), f_e^{i''}(\hat{P})), C_e).$$

Define as $x_e^i(\pi)$ the set of players preceding player i in π. Then, for every ordering π and permutation τ^i of set $x_e^i(\pi) \cup \{i\}$, define as $F_e^{<i,\pi,\tau^i}(P)$ the sum of players' flows who precede i in both π and τ^i. Let now $|x_e(P)| = r$. By definition of Shapley values, we get

$$\xi = \frac{1}{r!} \sum_{\tau^i} \left(C_e \left(F_e^{<i,\pi,\tau^i}(P) + f_e^i(P) \right) - C_e \left(F_e^{<i,\pi,\tau^i}(P)) \right) \right), \tag{6}$$

$$\xi' = \frac{1}{r!} \sum_{\tau^i} \left(C_e \left(F_e^{<i,\pi,\tau^i}(P) + f_e^{i'}(\hat{P}) \right) - C_e \left(F_e^{<i,\pi,\tau^i}(P) \right) \right). \tag{7}$$

For ξ'', since the position of $f_e^{i'}(\hat{P})$ in the ordering is unspecified, we give an upper bound for this value as follows. For any permutation τ, let $A(\tau)$ be the marginal cost increase caused by $f_e^{i''}(\hat{P})$ when she precedes $f_e^{i'}(\hat{P})$ in π, and $B(\tau)$ when she succeeds. That is

$$A(\tau) = C_e \left(F_e^{<i,\pi,\tau^i}(P) + f_e^{i''}(\hat{P}) \right) - C_e \left(F_e^{<i,\pi,\tau^i}(P) \right),$$
$$B(\tau) = C_e \left(F_e^{<i,\pi,\tau^i}(P) + f_e^{i'}(\hat{P}) + f_e^{i''}(\hat{P}) \right) - C_e \left(F_e^{<i,\pi,\tau^i}(P) + f_e^{i'}(\hat{P}) \right). \tag{8}$$

Let now p equal the probability of $f_e^{i'}(\hat{P})$ preceding $f_e^{i''}(\hat{P})$. Then, the definition of the Shapley value gives

$$\xi'' = (1-p) \cdot \frac{1}{r!} \cdot \sum_{\tau^i} A(\tau) + p \cdot \frac{1}{r!} \cdot \sum_{\tau^i} B(\tau). \tag{9}$$

Due to convexity, $A(\tau) \leq B(\tau)$. Therefore, by substituting $A(\tau)$ with $B(\tau)$ in definition (9), we get the following upper bound for ξ'',

$$\xi'' \leq \frac{1}{r!} \sum_{\tau^i} B(\tau). \tag{10}$$

Towards proving inequality (5), we have

$$\begin{aligned} \xi' + \xi'' \overset{(7),(10)}{\leq} & \frac{1}{r!} \sum_{\tau^i} C_e \left(F_e^{<i,\pi,\tau^i}(P) + f_e^{i''}(\hat{P}) \right) - C_e \left(F_e^{<i,\pi,\tau^i}(P) \right) \\ & + C_e \left(F_e^{<i,\pi,\tau^i}(P) + f_e^{i'}(\hat{P}) + f_e^{i''}(\hat{P}) \right) \\ & - C_e \left(F_e^{<i,\pi,\tau^i}(P) + f_e^{i'}(\hat{P}) \right). \end{aligned}$$

Since $f_e^{i'}(\hat{P}) = f_e^{i''}(\hat{P}) = \frac{f_e^i(P)}{2}$, we get

$$\xi' + \xi'' \leq \frac{1}{r!} \sum_{\tau^i} \left(C_e \left(F_e^{<i,\pi,\tau^i}(P) + f_e^i(P) \right) - C_e \left(F_e^{<i,\pi,\tau^i}(P) \right) \right) \overset{(6)}{=} \xi,$$

as desired. This completes Lemma's 1 proof. □

We now continue to the proof for the POS upper bound. By repeatedly applying Lemma 1, we can break the total flow on each resource in flows of infinitesimal size without increasing the value of the potential. This implies that

$$\Phi_e(P) \geq \int_0^{f_e(P)} \frac{C_e(x)}{x} dx. \tag{11}$$

Now call P^* the optimal outcome and $P = \arg\min_{P'} \Phi(P')$ the minimizer of the potential function, which is, by definition, also a PNE. Then

$$\begin{aligned} SC(P^*) &\overset{(4)}{\geq} \Phi(P^*) \overset{\text{Def.}P}{\geq} \Phi(P) \overset{(11)}{\geq} \sum_{e \in E} \int_0^{f_e(P)} \frac{C_e(x)}{x} dx \\ &= \frac{\sum_{e \in E} \int_0^{f_e(P)} \frac{C_e(x)}{x} dx}{\sum_{e \in E} C_e(f_e(P))} \cdot SC(P) \geq \min_{e \in E} \frac{\int_0^{f_e(P)} \frac{C_e(x)}{x} dx}{C_e(f_e(P))} \cdot SC(P). \end{aligned}$$

Rearranging yields the upper bound $POS \leq \max_{C \in \mathcal{C}, x > 0} \frac{C(x)}{\int_0^x \frac{C(x')}{x'} dx'}$, which completes the proof of Theorem 5.

Corollary 1 *For polynomials with non-negative coefficients and degree at most d, the POS of the Shapley value is at most $d+1$, which asymptotically matches the lower bound of [7] for single commodity per player.*

4 Splittable games

We conclude the paper with a discussion of interesting open problems on cost-sharing in the *splittable* version [5, 9, 16, 20, 25, 29] of congestion games with multi-commodity players and with some results. In the splittable version of such games, the weight w_q of a commodity $q \in Q$ can be split among its strategies in $\mathcal{P}^q$; i.e., a fractional strategy of commodity $q \in Q$ is a vector $P_q = (w_{q,P})_{P \in \mathcal{P}^q} \in \mathbb{R}_{\geq 0}^{|\mathcal{P}^q|}$ with $\sum_{P \in \mathcal{P}^q} w_{q,P} = w_q$. For the unsplittable version, vector P_q has only one non-zero and equal to w_q component, which is not necessarily the case for the splittable games. For the single-commodity per player model, it is known that the proportional sharing method, having players paying a cost share proportional to their flows on each resource, guarantees existence of a pure Nash equilibrium. Moreover, the POA of this simple cost-sharing method is well understood [29].

Understanding the POA of other cost-sharing methods both in the single- and multi-commodity models is an interesting open question. Similarly, it is interesting to study questions pertaining to the existence of pure Nash equilibria in such games, which we do next.

A result from Orda et al. [25] implies the existence of pure Nash equilibria in the multi-commodity splittable model, if the cost share of a player on a resource is a convex function of her flow on the resource. The result in [25] is based on the Kakutani Fixed Point theorem. This immediately gives us existence of pure Nash equilibria for the *standard* Shapley cost sharing. We strengthen this result by showing that such games are exact potential games [24] and thus best response dynamics converge to a pure Nash equilibrium. The proof of the following theorems can be found in Appendix.

Theorem 6. *Splittable congestion games with multi-commodity players under Shapley cost sharing are exact potential games.*

As soon as we deviate to the weighted Shapley value method, we prove that they do not guarantee PNE existence. Our proof uses the fact that the cost shares of the players are not necessarily convex anymore in this setting.

Theorem 7. *For parameterised weighted Shapley values with (finite) parameter γ, PNE are not guaranteed to exist for splittable congestion games with multi-commodity players.*

For $\gamma = \infty$, we can even construct a counter example that uses only single-commodity players.

Theorem 8. *For parameterised weighted Shapley values with parameter $\gamma = +\infty$, PNE are not guaranteed to exist even for* single-*commodity players.*

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