Control theory for classes of nonlinear systems with application in insurance premium-reserve model

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by

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Abstract

This thesis mainly deal with control problems for nonlinear dynamic systems with the application in actuarial science. Research in this field concerning with different classes of nonlinear systems is motivated by theoretical use and possible application. Although it has fruitful literature, there still remains many open problems worthy of thoughtful study.

As an extension of the previous literature in linear time-varying systems, some conventional results of the linear time-varying system can be validated in the commutative class of nonlinear time-varying systems. Next the delay-range-dependent observer design methodology has been developed for the one-sided Lipschitz nonlinear system. Especially delay-range-dependent conditions are formulated and deduced for the system incorporating features of time-varying delays in states and output as well as output nonlinear dynamics with delay-range and delay-derivative bound. Further, we extend the design methodology for controller and observer, through a unified linear matrix inequalities approach, to the one-sided Lipschitz nonlinear time-varying system. The corresponding results for the one-sided Lipschitz nonlinear discrete-time stochastic system are refined and applied in the premium-reserve P-R) modelling in the context of the actuarial science. Thereby the robust H_∞ controller is designed for the premium-reserve system in order to stabilize the accumulated reserve process.

In each chapter, sufficient conditions presenting in tractable way are derived to solve the proposed sub-problems. Several numerical examples are given to illustrate the applicability of the theoretical findings.

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This thesis is finished during the research in Department of Mathematical Sciences in University of Liverpool. It is a great pleasure to have the opportunity to study here.

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Chapter 1

Introduction

1.1 Research problem and Motivation

In recent years various controllability problems for different types of semi-linear and nonlinear dynamical systems have been considered in many publications and monographs. This is clearly related to the wide variety of theoretical results and possible applications.

The problem of controllability and observability for continuous-time and discrete-time linear dynamical systems has been extensively investigated in many papers (see e.g. [42, 43, 44]). This is not true for the nonlinear dynamical systems especially with different types of delays in control and state variables in whatever deterministic or stochastic settings. Similarly, numerous papers concern the design of controller and observer for continuous-time or discrete-time nonlinear dynamical systems. It should be pointed out, that in the proofs of obtained results for nonlinear and semi-linear dynamical systems linearization methods and generalization of open mapping theorem [45, 49] are extensively used.

In the thesis we are motivated to explore the validation of well-known properties of the linear time-varying systems in some special case of nonlinear dynamical systems. Moreover, we extend the observer synthesis methodology for classes of nonlinear system subject to time-varying delays in both states and outputs. The derivation is mathematically simpler in the sense that abstract linear algebra is avoided and only elementary matrix manipulations and linear independence are essential. Further, the results represented in linear matrix inequalities would be tractable and applicable for utilitarian purpose. In terms of realistic application, that control theory in systems starts to play its role in finance and insurance rather than limited in such traditional areas as industrial and chemical process control, reactor control and aerospace engineering. This thesis will also present stabilisation methods for the nonlinear premium-reserve model in the context of actuarial science.

1.2 Main objectives and contributions

In Chapter 3, we assess the fundamental properties of one commutative class of nonlinear time-varying systems. By simplifying the procedure in [19], our work generalise the results proposed by [89] for linear systems and derive feedback stability and stabilisation criteria for this special class of nonlinear time-varying systems .

In Chapter 4, we extend the delay-range-dependent approach for observer design of one-sided Lipschitz nonlinear system incorporating different time-varying state delays, output measurement vector delays in states and output nonlinearities. To further reduce computational complexities in the design of the delay-range-dependent observer, we introduce new algorithm to solve the design problem in Linear-Matrix-Inequality (LMI) form.

In Chapter 5, we present main design conditions for LMI-based dynamical observer and controller strategies for the one-sided Lipschitz nonlinear time-varying system in a unified framework.

In Chapter 6, we study robust stability, stabilization analysis and H_∞ controller design for the quadratic bounded time-varying nonlinear discrete-time stochastic system. Moreover, application of nonlinear stochastic discrete-time control in a non-life insurance problem is discussed in Chapter 7, which extend the research result of this classic problem in non-life insurance.

1.3 Structure of thesis

Chapter 2 firstly introduces the reader some basic notions from the control theory. The relevant work is reviewed with previous results listed as a theoretical basis for this thesis. The history of applying control theory in actuarial literature is also examined to provide the context for the generation of the premium-reserve models in this thesis.

Chapter 3 considers a commutative class of nonlinear time-varying systems. In the form of the pseudo-linear representation analysis, we obtain sufficient conditions for the global controllability and observability through assessing the simple algebraic criteria of rank condition. This condition also allows for the Kalman canonical decomposition. And the general analysis for the global asymptotic stability is explored, which leads to some sufficient conditions for nonlinear state-feedback controller and observer design. Some numerical examples are used to reflect and validate the effectiveness of the theorems, especially compared with the previous outcomes.

In Chapter 4, a novel delay-range-dependent technique is explored for one-sided Lipschitz nonlinear observer design along with time-varying delays in state and output measurement vector. A Lyapunov-Karasovskii (LK) functional is employed whose derivative is estimated by incorporating Jensens inequality to derive stability conditions for observer design using delay-range-dependent scheme. Matrix inequality based stability criteria is established, by exploiting one-sided Lipschitz condition, Schur complement and congruence transformation, guaranteeing asymptotic convergence of state estimation error to the origin. But the consequence of the methodology leads to solve the optimization problem which enhances the solution to the computationally complexed problem. Thus a new algorithm involving the generalised inverse is introduced to convert the delay-range-dependent observer design to the existence of linear matrix inequalities.

In Chapter 5, the same LMI-based design conditions for full-order and reduced-order observer are proved for the general one-sided Lipschitz nonlinear time-varying system. In a unified framework, observer-based stabilisation strategies are also worked out. Thus far the concerning systems are assumed to be deterministic with continuous time. In Chapter 6 we would transform the relevant results in discrete time situation subject to stochastic process. Especially, the results obtained in Chapter 6 can be applied in an insurance model in Chapter 7. Firstly, we derive easily testing criteria for stochastic stability and stochastic stabilizability are obtained via non-strict linear matrix inequalities (LMIs). Then a robust H_∞ state feedback controller is designed such that the concerned system not only is internally stochastically stabilizable but also satisfies robust H_∞ performance. Moreover, the previous results of the nonlinearly perturbed discrete stochastic system are generalized to the system with state, control, and external disturbance dependent noise simultaneously.

In Chapter 7, the control problem of Premium-Reserve (P-R) model in non-life insurance is extend. We first describe some basic assumptions and define a new premium rating rule and accumulated reserves process in nonlinear stochastic framework. Then, the control problem of the nonlinear P-R model is investigated under two types of nonlinear assumption: Lipschitz-type nonlinear condition and quadratic bounded nonlinear condition respectively. The result could provide us the solution of robust H_∞ controller for nonlinear stochastic discrete time P-R system.

Chapter 8 is the last chapter, which provides the concluding remarks for the contribution of this thesis as well as some feasible further research directions on the topics in this thesis.

1.4 Notation

Throughout this thesis, the symbol $*$ is used to denote the transposed elements in the symmetric positions of a matrix. The matrices are assumed to have compatible dimensions. The superscript T stands for the matrix transposition. tr(M) stands for the trace of matrix M. diag$\{\cdots\}$ stands for a block-diagonal matrix. For a symmetric matrix $P > 0$ (< 0) means P is positive (negative) definite. I represents identity matrix and 0 denotes zero matrix. $\mathbb{R}^m$ denotes the m dimensional Euclidean space. $\mathbb{N}$ is the set of natural numbers. $\mathbb{E}(\cdot)$ denotes the expectation operator.

Chapter 2

Literature Review and Preliminary Results

2.1 Pseudo-linear Representation methodology

Around 1990s, Banks in [9] and [10] formally introduced the pseudo-linear representation for a class of nonlinear systems, known as the form of state-dependent coefficient (SDC):

$$\dot{x}(t) = A(x)x(t), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{2.1.1}$$

where $A : \mathbb{R}^n \to \mathbb{R}^{n^2}$ is assumed continuously differentiable. This pseudo-linear representation is able to express a general class of nonlinear systems, for example,

$$\dot{x}(t) = f(x(t)), \quad f(0) = 0, \tag{2.1.2}$$

The introduction of the type of equation (2.1.1) is intended to conjecture that the fundamental properties of the nonlinear system can be derived from its corresponding linear representation, as pointed by Kalman [39]. Later in [19], some sufficient conditions for the controllability of nonlinear time-varying system with control:

$$\dot{x}(t) = A(t, x)x(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{2.1.3}$$

are obtained through fixed point theorem, under which the calculation of the Grammian matrix is required and much effort has been taken to ease the computation. It is of interest to study the nonlinear time-varying systems in the same manner as that of a linear time-varying systems. The stability criterion of the free system has also been explored, contrast to the linear system, the negative eigenvalues of $A(x)$ is not sufficient of the stability of system (2.1.1) in Banks & Mhana [9] . They also assume the upper triangularity of $A(x)$ in addition to the hypothesis on situation of its eigenvalues, which implies the solvable Lie algebra generated by the range of $A(x)$ by the use of Lie algebra theorem. But a simple counter example ([86] and [53]) has been proposed with finite escape time. The reason of the conflict may lie in the existence of the global solution to the system which is not guaranteed by the claim in [9]. Langson & Alleyne [53] noticed the problem and tried to fix it by imposing the exponential boundedness on $A(x)$ i.e. $\forall x \in \mathbb{R}^n, \quad ||\exp[A(x(t))t]|| \leq M$, for some real $M > 0$. This endeavour which is not trivial, however, turns out to be not right by a counter example in [60] as the solution to system (2.1.1) may not be expected to be like $x^*(t) = \exp(A(x)t)x_0$ without any restriction on the system coefficient matrix. In other words, the further boundedness condition cannot generally contribute to the conclusion on stability.

2.2 Observer design methodology for Lipschitz and one-sided Lipschitz nonlinear systems

The topic on control and state estimation of nonlinear systems satisfying a Lipschitz condition has been studied for almost four decades, resulting in abundant amount of literature. Especially for the observer synthesis problem on Lipschitz nonlinear system, it is often accomplished by using pseudo-linear techniques which is based on the Lipschitz continuity assumption providing a norm-based form of a nonlinear inequality substituted into the observer error dynamics and the observer error dynamics turning out in a numerically tractable format that is determined by a linear term. For example, in [85] and [76], the authors have obtained sufficient conditions to ensure asymptotic stability of the observer error dynamics. The same conditions in [76] can assure the existence of a reduce-order observer which has been shown in [106]. The proposed design method above is dependent on the solution of a Riccati equation. While the linear matrix inequality (LMI) technique could be seen in [102] and [57] for Lipschitz discrete-time systems and Lipschitz descriptor systems.

Both approaches intend to choose the proper output injection term in the observer dynamics so that the linear part of the observer error dominates the nonlinear terms.

Generally, in the process of utilising the Lipschitz property, this would generate results in significant degree of conservativeness. On the other hand, the nature of Lipschitz continuity condition also hampers the practical use of related results since the Lipschitz is usually region-based and the Lipschitz condition is seldom satisfied in the global sense [21].

Hence the implicit idea behind them are then motivated to look for a less restrictive system satisfying the one-side Lipschitz condition which encompasses the Lipschitz continuity condition as a special case. It's first introduced in [32] and [33], where sufficient conditions are gained for asymptotical stability of the error dynamics within one-sided Lipschitz systems. Following this result, the observer design problem has been analysed by several other researchers such as reduced- order observers for such systems given in [93] and a design scheme in terms of Riccati inequalitys shown in [97]. More recently, proposed by [1], the linear matrix inequality (LMI) conditions which can be converted into LMI provide a useful analysis tool and address the fundamental design problem as well. Improved results based on both the Riccati equation and the LMI approaches can be found in [100] and [101]. And the corresponding problem in the discrete-time version has been carried out in [96]. Time delays, varying in an interval and appearing in state, input, and output variables as well as in state derivatives, are frequently encountered in engineering and physical systems. Recently, the delay-dependent observer-design techniques have been developed for one-sided Lipschitz systems in [26] and [15], which can be effectively reformulated for monitoring and control of complex forms of engineering systems.

2.3 Control Theory in Insurance

In spite of its popularity in many other areas, control theory has not been intensively implemented in actuarial science until recent decades. In Non-Life insurance area, the first application of control theory involved in actuarial publications could probably date back to the famous papers by [20] and [13]. They propose for the classical risk theory problem a control action based on a pre-defined level of the surplus (accumulated) reserve, see Figure 2.1. In their papers, both suggest a premium refund action whenever the surplus exceeds a certain limiting level. Under this arrangement, the premium for

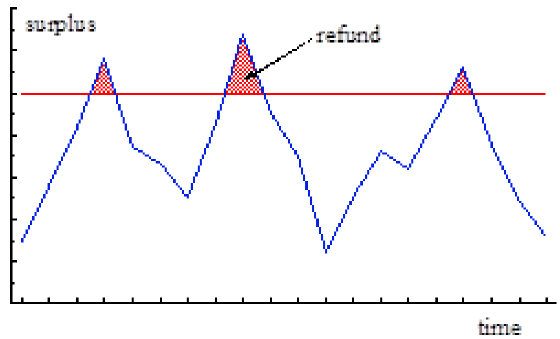

Figure 2.1: De Finetti's approach to control of surplus.

the t^{th} year P_t is determined by the following equation:

$$P_{t+1} = (1+\theta)\mathbb{E}[claims] + \mathbf{1}_{(R_\Pi - R_t)},$$

where $1 > \theta > 0$ is the loading factor; $\mathbb{E}\left[claims\right]$ is the expected claims of current year, R_t is the reserve value at the end of the t^{th} time period, "$R_\Pi \geq 1$" is the pre-defined limiting (barrier) level of reserve and

$$\mathbf{1}_{(R_\Pi - R_t)} = \begin{cases} R_\Pi - R_t, \text{when} \quad R_\Pi - R_t < 0 \\ 0, \text{ when} \quad R_\Pi - R_t > 0. \end{cases}$$

After De Finetti several control theoretical articles have appeared in actuarial publications. The models in these articles have employed both deterministic and stochastic techniques, however, most of them have been linear. Some actuarial works along this line include [78, 11, 7, 8, 58, 59, 73, 74, 75, 87, 107, 108] and [30]. Most of these focus on studying the properties of a given control rule, though some also explore optimal solutions.

Among these works, [7, 8, 58, 59] have tried successfully to implement control theory for solving this interesting actuarial problem. They propose a smooth control action for the determination of the premium which is applied periodically and accordingly to the available information of the surplus process.

Thus, according to their research work, the proposed premium equation has finally received the following form:

$$P_{t+1} = (1+\theta)\mathbb{E}[claims] - \varepsilon R_{t-1}. \tag{2.3.1}$$

Moreover, Balzer and Benjamin [7] also discuss the effect of the delay on the stability of the system and the optimal choice for the feedback factor ε when using equation (2.3.1) with the surplus value with 1 year time delay.

Balzer and Benjamin [8] study further with 4 year time delay. In that paper, a full extension of this kind of investigation is achieved by considering the delay factor as a free parameter τ and by calculating the respective general conditions of stability and optimality for the feedback factor ε. Their result show the linear system becomes unstable when integer time-delay τ is great than 4. So, the premium equation (2.3.1) becomes,

$$P_{t+1} = (1+\theta)\mathbb{E}[claims] - \varepsilon R_{t-\tau}. \tag{2.3.2}$$

Vandebroek and Dhaene [87] prove that the premium equation (2.3.2) is the optimal linear feedback controller for the premium pricing in the case that we require to minimize the probability of ruin along with a smooth pattern for the development of the premiums and reserves. For solving this problem, they use dynamic programming techniques.

Zimbidis and Haberman [108] consider a modelling structure with a discrete-time equation to describe the development of the accumulated reserve process for an insurance system.

Their approach says that the development of the accumulated reserve R_t, at the end of each year, assuming also an accumulation factor $1 + r$ and $r > 0$ which is the respective rate of the investment return of the surplus reserve, is given by

$$R_{t+1} = (1+r)R_t + e(\hat{C}_{t+1} - \varepsilon R_{t-\tau}) - C_{t+1}, \tag{2.3.3}$$

where e is the parameter for the administration expenses and the desired profit margin, which can be expressed as $(1-e)$ of the respective premium.

In their paper, the classical *Root-Locus* (see [81]) method is used for the investigation of the stability of the system and an appropriate feedback factor ε is calculated using a specific premium decision function. Due to the limitation of their method, the analysis of the stability of a P-R process was based on time-invariant parameters and constant delay factors without considering any type of uncertainty.

Recently, [65] - [67] and [94] introduce time-varying delays and uncertainties in their P-R systems under different frameworks. In [66], they propose a P-R system model like system (2.3.4) for different dependent insurance products in a insurance company. This model considers a negative feedback mechanism for the accumulated surplus, it invests the surplus in short-term risk-free assets, and it assumes the accumulated reserves follow a linear stochastic, discrete-time framework considering also a set of different norm-bounded parameter uncertainties ΔJ_t, ΔE_t, ΔZ_t involved in the model.

$$\begin{cases} \underline{R}_{t+1} = \{[J + \Delta J_t] - e[Z + \Delta Z_t]K\}\underline{R}_t(1 + v(t)) - e[E + \Delta E_t]\underline{R}_{t-\tau_t}(1 + v(t)) + \underline{w}_{t+1}, \\ \underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0], \end{cases} \tag{2.3.4}$$

In their papers, the stability of the discrete-time P-R systems with norm-bounded parameter uncertainties and time-varying delay are investigated in a deterministic [65], Markovian regime switching [94] and stochastic framework [66], respectively. They propose H_∞ criteria to be used for the determination of the premium control rule. Most of these papers focus on studying the properties of a given control rule, though some of them also explore feasible solutions to a specific problem employing different optimality criteria. These papers [65]-[67] and [94] are based on discrete time linear approach.

2.4 Concepts of controllability and observability

A systematic study of control theory was started at the beginning of sixties in the last century. Controllability (or stabilizability) and observability (or detectability) are basic concepts describing the qualitative properties of dynamical control systems and are of paramount importance in the mathematical control theory; see e.g., [35, 36, 37, 38, 39]. They appear as necessary and sometimes as sufficient conditions for the existence of a solution to most control problems.

It is worthwhile mentioning that in the literature there are many different definitions of controllability and observability, which strongly depend on a class of dynamical control systems and on the other hand on the form of admissible controls. So different are the exposition of the subject and the derivation of criteria and proofs. But the definitions are various in approach rather than content as they are motivated by the intention to serve different purposes by developing the concepts of controllability and observability.

The theory of controllability is mainly based on the description in the form of state space for both time-invariant and time-varying linear control systems. To the end, we focus on state-space models of dynamical systems, which provide a robust and universal method for studying controllability of various classes of nonlinear systems.

Definition 2.1. [51, 52] A system is defined to be completely state controllable at time t_0, if for any $t_0 \geq 0$ each initial state $x(t_0)$ in the controllability domain $D \subset \mathbb{R}^n$ can be transferred to any final state $x(t_f)$ in a finite time $t_f > t_0$ in D under some control $u(t)$. If D is the whole state space $\mathbb{R}^n$, the controllability's said to be global. If D is not the whole state space $\mathbb{R}^n$, thus we have the local controllability.

The word "completely" emphasizes that the choice of the initial and final states in D is arbitrary. The dependence on a particular interval $[t_0, t_f]$ can be eliminated by introducing the following definition.

Definition 2.2. [51, 52] A system is defined to be totally state controllable in the controllability domain $D \subset \mathbb{R}^n$, if it's completely state controllable in D on every interval $[t_0, t_f], t_f > t_0$ under some control $u(t)$. If D is the whole state space $\mathbb{R}^n$, the controllability's said to be global. If D is not the whole state space $\mathbb{R}^n$, thus we have the local controllability.

From the definitions above, the problem of controllability is to show the existence of a control function, which steers the solution of the system from its initial state to final state, where the initial and final states may vary over the entire space.

The complete controllability is generally a necessary condition for the existence of a solution to a control problem where t_f is undefined. While the total controllability was found to be a necessary and sufficient condition for the uniqueness of the solution to certain optimal control problems [51].

The concept of "observability" could be quickly dismissed, as is done in [35, 36, 37], by defining it as the dual (in an abstract algebraic sense) of state-controllability. The results obtained for controllability then carry over to observability by a "dualizing" procedure.

Definition 2.3. [51, 52] An unforced system is said to be completely observable on $[t_0, t_f]$, if for given given t_0 and t_f every state $x(t_0)$ in the domain D can be determined from the knowledge of $y(t)$ on $[t_0, t_f]$. If the above is true for every t_0 and some finite $t_f > t_0$, the system is said simply to be completely observable. If the above is true for every t_0 and every $t_f > t_0$, the system is said simply to be totally observable.

Essentially, the controlled system is completely controllable if every desired transition of the system's state can be effected in finite time by some unconstrained control inputs. A system is completely observable if every transition of the plants state eventually affects some of the plants outputs. Mathematically speaking, these concepts are a matter of linear independence of certain scalar or vector time functions.

There are various important relationships between controllability, stability and stabilizability of linear both finite-dimensional and infinite-dimensional control systems. Controllability is also strongly related to the theory of realization and so called minimal realization and canonical forms for linear time-invariant control systems such as the Kalmam canonical form, the Jordan canonical form or the Luenberger canonical form.

2.5 Lipschitz and one-sided Lipschitz nonlinear systems.

2.5.1 System description and observer design

In this thesis, we consider the class of nonlinear systems described by the following set equations:

$$\dot{x} = Ax + f(x,u) \quad y = Cx, \tag{2.5.1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$ are the state vector, control input vector and the output vector. The linear constant matrices of the dynamic system are represented by A, C of appropriate dimensions and the pair (A, C) is assumed to be observable.

Definition 2.4. [1] A nonlinear function $f(x,u)$ is said to be Lipschitz in a region D enclosing the origin if there exists a scalar $l \in \mathbb{R}$ such that the relation

$$\|f(x,u) - f(\overline{x},u)\| \leq l\|x - \overline{x}\|, \tag{2.5.2}$$

holds $\forall x, \overline{x} \in D$, where l is the Lipschitz constant.

Definition 2.5. [1] A nonlinear function $f(x,u)$ is said to be one-sided Lipschitz in a region D enclosing the origin if there exists a scalar $\rho \in \mathbb{R}$ such that the relation

$$\langle f(x,u) - f(\overline{x},u), x - \overline{x} \rangle \leq \rho \|x - \overline{x}\|^2, \tag{2.5.3}$$

holds $\forall x, \overline{x} \in D$, where ρ is the one-sided Lipschitz constant.

Definition 2.6. [1] A nonlinear function $f(x,u)$ is said to satisfy the quadratic inner-boundedness condition in a defined region D, if there exist scalars $\beta, \alpha \in \mathbb{R}$, such that

$$(f(x,u) - f(\overline{x},u))^T (f(x,u) - f(\overline{x},u)) \leq \beta \|x - \overline{x}\|^2 - \alpha \langle x - \overline{x}, f(x,u) - f(\overline{x},u) \rangle \tag{2.5.4}$$

is satisfied for all $x, \overline{x} \in D$.

The one-sided Lipschitz and quadratic inner-boundedness conditions extrapolate the definitive Lipschitz theory to a more ecumenical category of nonlinear systems and have inbuilt advantages in observer synthesis. For a given function $f(x,u)$ satisfying one-sided Lipschitz conditions in Definitions (2.5)-(2.6), whereas the reverse is not true (see details in [1], [100], [101]). Further, the one-sided Lipschitz constant ρ and the quadratic inner-boundedness parameter β can be any real numbers, unlike the Lipschitz constant, which needs to be always positive.

Consider the following classical Luenberge observer:

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + L(y - C\hat{x}) \tag{2.5.5}$$

where $\hat{x}$ represents the estimate of x and L is to be designed so that the estimation error $e = x - \hat{x}$ asymptotically converges towards zero. The dynamic of the estimation error can be described by

$$\dot{e} = (A - LC)e + \Delta f \tag{2.5.6}$$

where $\Delta f = f(x) - f(\hat{x})$.

Next, we present some known results for this class of nonlinear systems. To clarify the comparisons that we will provide in this thesis, we consider only the methods that use the observer form in equation (2.5.5) and the quadratic Lyapunov function

$$V(e) = e^T P e, \quad with \quad P = P^T > 0, \tag{2.5.7}$$

based on which the following section will disclose the existing results for stability and stabilization of the Lipschitz nonlinear system.

2.5.2 State of the art on existing methods for the Lipschitz nonlinear system

Standard LMI approach

Theorem 2.1. *[70, 76] The estimation error is asymptotically stable if there exist matrices $P = P^T > 0$ and R of adequate dimensions so that the following LMI condition holds:*

$$\begin{bmatrix} A^T P + PA - R^T C - C^T R + I_n & P \\ P & -\frac{1}{\gamma_f^2} I_n \end{bmatrix} < 0. \tag{2.5.8}$$

Then the gain stabilizing the estimation error will be given by $L = P^{-1}R^T$. For more details on this approach, we refer the reader to [76] where other previous results related to this approach have been discussed, namely the pioneering work of [71] and [85].

Riccati equation based approach

Theorem 2.2. *[70, 71] The estimation error is asymptotically stable if there exist scalars $\epsilon > 0$, $\beta \in \mathbb{R}$ and a matrix $P = P^T > 0$ of adequate dimension so that the following Riccati equation holds:*

$$A^T P + PA + \epsilon\gamma_f^2 I_n + \frac{1}{\epsilon} PP - \beta^2 C^T C < 0. \tag{2.5.9}$$

Then the gain stabilizing the estimation error can be chosen as:

$$L = \frac{\beta^2}{2} P^{-1} C^T. \tag{2.5.10}$$

S-procedure lemma based approach

The approach we recall here is based on the use of the well-known S-procedure lemma. This technique was firstly developed in [14] and has been highlighted recently in [70]. Before stating the synthesis condition, we first recall the S-procedure lemma.

Lemma 2.1. *[3, 14, 68, 70, 71, 85]. Let $V_0(\zeta)$ and $V_1(\zeta)$ be two arbitrary quadratic* *forms over* $\mathbb{R}$. *Then* $V_0(\zeta)$ *is a consequence of* $V_1(\zeta) < 0$ *if and only if there exists* $\tau > 0$ *such that*

$$V_0(\zeta) \leq \tau V_1(\zeta), \quad \forall \zeta \in \mathbb{R} - \{0\}$$

Based on the Lemma above, the authors in [14] and [70] gave the following theorem.

Theorem 2.3. *[14, 68, 70, 71, 85] The estimation error is asymptotically stable if there exist a scalar* $\tau > 0$ *and matrices* $P = P^T > 0$ *and* R *of adequate dimensions so that the following LMI condition holds:*

$$\begin{bmatrix} A^T P + PA - R^T C - C^T R + \tau \gamma_f^2 I_n & P \\ P & -\tau I_n \end{bmatrix} < 0. \tag{2.5.11}$$

The gain stabilizing the estimation error is given by: $L = P^{-1} R^T$.

Comparison and discussion in [70] has shown the observer design technique in Theorem 2.3 with less conservatism. Notice that LMIs may be obtained easily by using the Lipschitz property, Youngs relation and Schur complement lemma.

2.5.3 Linear Matrix Inequality techniques in stability analysis of delay systems

In this section, LMI techniques in deriving delay dependent stability conditions will be reviewed.

The time delay is often a source of the generation of oscillation and a source of instability of control systems [50]. Therefore, the problem of stability analysis and control of time-delay systems has attracted much attention during the past years, which is of both practical and theoretical importance.

Many results have been reported using a variety of approaches and techniques. However, much of the focus has been laid on the use of the Lyapunov-Krasovskii theory to derive sufficient stability conditions in the form of linear matrix inequalities.

In the literature, various approaches have been proposed to obtain delay-dependent stability conditions, among which the linear matrix inequality (LMI) approach is the most popular and has played an important role due to the fact that LMIs can be cast into a convex optimisation problem which can be handled efficiently by resorting to recently developed numerical algorithms for solving LMIs [14]. Another reason that makes LMI conditions appealing is their frequent readiness to solve the corresponding synthesis problems once the stability (or other performances) conditions have been established, especially when state feedback is employed.

For simplicity, we will review the LMI techniques in deriving stability results for the single-delay case. However, the LMI techniques presented in the following can be extended to the multiple-delay case in a straightforward manner. In this section, two classes of time-delay systems will be considered, which are,

$$\Sigma_1 = \begin{cases} \dot{x}(t) = Ax(t) + A_1 x(t-h) \\ x(t) = \phi(t), \quad \forall t \in [-h, 0] \end{cases} \tag{2.5.12}$$

and

$$\Sigma_2 = \begin{cases} \dot{x}(t) = Ax(t) + A_1 x(t-h(t)) \\ x(t) = \phi(t), \quad \forall t \in [-\bar{h}, 0] \end{cases} \tag{2.5.13}$$

where $x(t) \in \mathbb{R}^n$ is the state; $\phi(t)$ is the continuous initial condition. The scalar $h > 0$ is the constant delay of system (Σ_1), while $h(t)$ is the time-varying delay of system (Σ_2), which is assumed to be continuous and satisfies

$$0 < h(t) \leq \bar{h}. \tag{2.5.14}$$

In both the time-delay systems (Σ_1) and (Σ_2), A and A_1 are known real constant matrices. It is noted that stability results on (Σ_1) obtained by the method of Lyapunov-Krasovskii functional can be easily extended to systems with differentiable time-varying delays. It will be developed later in chapter 4.

At the end of this section, we list some helpful lemma as follows.

Lemma 2.2. ***Schur complement lemma:*** *For a given matrix* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$ *with* A_{11} *and* A_{12} *are symmetric, then the following conditions are equivalent:*

1. $A < 0$
2. $A_{11} < 0, A_{22} - A_{12}^T A_{11}^{-1} A_{12} < 0$
3. $A_{22} < 0, A_{11} - A_{12} A_{22}^{-1} A_{12}^T < 0$

Lemma 2.3. ***Young's inequality:*** *$\forall a, b \in \mathbb{R}^n$ and $\epsilon > 0$, $\forall R > 0$*

$$2a^T b \leq \epsilon a^T R a + \frac{1}{\epsilon} b^T R^{-1} b$$

Lemma 2.4. ***Jensen's inequality:*** *For any constant matrix $M \in \mathbb{R}^{m\times m}$, $M = M^T > 0$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \to \mathbb{R}^m$ such that the integrations concerned are well defined, then*

$$\gamma \int_0^{\gamma} \omega^T(\beta) M \omega(\beta) d\beta \geq \left(\int_0^{\gamma} \omega(\beta) d\beta \right)^T M \left(\int_0^{\gamma} \omega(\beta) d\beta \right) \tag{2.5.15}$$

2.5.4 Delay-dependent stability conditions

Newton-Leibniz formula

By using the Newton-Leibniz formula and noting system (2.5.12), we have

$$\begin{aligned} x(t-h) &= x(t) - \int_{t-h}^{t} \dot{x}(\alpha) d\alpha \\ &= x(t) - \int_{t-h}^{t} [Ax(\alpha) + A_1 x(\alpha - h)] d\alpha. \end{aligned} \tag{2.5.16}$$

This together with system (2.5.12) gives

$$\dot{x}(t) = (A + A_1)x(t) - A_1 \int_{t-h}^{t} [Ax(\alpha) + A_1 x(\alpha - h)] d\alpha. \tag{2.5.17}$$

Note that the asymptotic stability of the time-delay system in equation (2.5.17) implies that of the system in (Σ_1). For this reason, we now turn to study the stability of system in equation (2.5.17). To this end, we choose a Lyapunov-Krasovskii functional candidate as follows:

$$V(t, x_t) = x(t)^T P^{-1} x(t) + \int_{-h}^{0} \int_{t+\theta}^{t} x(\alpha)^T A_1^T Q_1^{-1} A_1 x(\alpha) d\alpha d\theta + \int_{-h}^{0} \int_{t-h+\theta}^{t} x(\alpha)^T A_1^T Q_2^{-1} A_1 x(\alpha) d\alpha d\theta, \tag{2.5.18}$$

where $P > 0$, $Q_1 > 0$ and $Q_2 > 0$. Then, the stability condition for (2.5.17) is obtained in the following theorem.

Theorem 2.4. *[16] The time delay system in eq(2.5.17) is asymptotically stable for any delay h satisfying $0 < h \leq \bar{h}$ if there exist matrices $P > 0$, $Q_1 > 0$ and $Q_2 > 0$*

such that

$$\begin{bmatrix} \Omega & \bar{h}PA^T & \bar{h}PA_1^T \\ \bar{h}AP & -Q_1 & 0 \\ \bar{h}A_1P & 0 & -Q_2 \end{bmatrix} < 0, \tag{2.5.19}$$

where $\Omega = (A + A_1)P + P(A + A_1)^T + A_1(Q_1 + Q_2)A_1^T$.

Chapter 3

Controllability and stabilization of a commutative class of nonlinear time-varying systems

3.1 Introduction

In this chapter, we consider the nonlinear time-varying system (2.1.3), let $A(t,x)$ be written as

$$A(t,x) = \sum_{i=1}^{m} a_i(t,x(t))A_i x(t), \tag{3.1.1}$$

where A_i's are assumed to be mutually commutative i.e. A_i's satisfy the following conditions:

$$A_i A_j = A_j A_i, \quad \forall i,j = 1,2,\ldots,m. \tag{3.1.2}$$

Its worth mentioning that, as a special case, of the corresponding linear system have been studied in Wu [89], Zhu [105], Leiva & Zambrano [55] and Date & Gashi [18].

$$\begin{cases} \dot{x}(t) = A(t)x(t) + Bu(t) = \sum_{i=1}^{m} a_i(t)A_i x(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{3.1.3}$$

$A_i \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times l}$ and A_i are commutative matrices of each other. Particularly, sufficient conditions for controllability and stability of the system 3.1.3 have been proposed in Wu [89].

In order to gain more insight into the controllability, the stability and the observability problems of the nonlinear system, we thereby attempt to resolve in (2.1.3) by the means of 1) the boundedness on the $A(x)$ to ensure the existence of the global solution to the pesudo-linear dynamics by transforming into a fixed-point problem for constructing a proper mapping on an invariant subset 2) the commutativity on matrix $A(x)$ ensures the state transition matrix expressed in a explicit and closed form.

From the mutual commutativity of constant matrices A_i's and the bounded scalar function $a_i(t, x)$, the computation of the state transition matrix of a nonlinear time-variant system can be done in the same way as that of a linear time-variant system and yield more explicit information on eigenvalues of $A(t, x)$. With the aid of pseudo-linear dynamics by resolving a fixed-point problem, we would provide sufficient conditions for the globally complete controllability of the system through simple algebraic rank criteria. It helps to avoid falling into difficult calculation of the determinant of the controllability Grammian matrix based on procedure from [19]. Furthermore, as the Kalman canonical decomposition can be derived, we have gained feedback stability criterion and stabilisation criterion of controller design for the system (2.1.3).

3.2 Controllability

Consider the nonlinear time-varying system with control:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{m} a_i(t, x(t)) A_i x(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{3.2.1}$$

where $A_i \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times l}$ and A_i are commutative matrices of each other , $i = 1, ..., m$. The state x is an n-vector and the control input u is an m-vector. The coefficient functions $a_i : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}, i = 1, ..., m$ are piecewise continuous functions of t and continuous functions of x.

The corresponding pseudo-linear form of system (3.2.1) would be:

$$\begin{cases} \dot{x}_z(t) = \sum_{i=1}^{m} a_i(t; z) A_i x_z(t) + Bu(t), \\ x_z(t_0) = x_0, \end{cases} \tag{3.2.2}$$

for a specified function $z(\cdot) \in C_nT$, denoting the Banach space of continuous $\mathbb{R}^n$-valued functions on $T = [t_0, t_f]$. For each fixed $z(\cdot) \in C_nT$, system (3.2.2) is linear. The complete solution is given by

$$x_z(t) = \phi(t, t_0; z)x_0 + \int_{t_0}^{t} \phi(t, s; z)Bu(s)ds \tag{3.2.3}$$

Lemma 3.1. *The state transition matrix of system (3.2.3) is denoted as:*

$$\phi(t, t_0; z) = \exp\left[\sum_{i=1}^{m} \int_{t_0}^{t} a_i(s; z)dsA_i\right]$$

which can be represented in the form as follows:

$$\phi(t, t_0; z) = \sum_{k=0}^{n^m-1} \sum_{k_1=0}^{n-1} \cdots \sum_{k_m=0}^{n-1} g_k(t, t_0; z)(A_1^{k_1} \cdots A_m^{k_m}) \tag{3.2.4}$$

where $g_k(t, t_0; z)$ are scalar functions.

Proof. Similarly in [72], by the virtue of the Cayley-Hamilton theorem and community of matrices A_i's,

$$\begin{aligned}
\phi(t, t_0; z) &= \exp\left[\int_{t_0}^{t} a_1(s; z)dsA_1\right] \cdots \exp\left[\int_{t_0}^{t} a_1(s; z)dsA_1\right] \\
&= \left[\sum_{k_1=0}^{n-1} g_{k_1}(t, t_0; z)A_1^{k_1}\right] \cdots \left[\sum_{k_m=0}^{n-1} g_{k_m}(t, t_0; z)A_m^{k_m}\right] \\
&= \sum_{k=0}^{n^m-1} \sum_{k_1=0}^{n-1} \cdots \sum_{k_m=0}^{n-1} g_k(t, t_0; z)(A_1^{k_1} \cdots A_m^{k_m})
\end{aligned}$$

The following theorem gives conditions under which the nonlinear system is global controllable.

Theorem 3.1. *The system (3.2.1) is globally completely (totally) controllable at t_f, if the conditions below are satisfied:*

(a) The integrator $|\int_{t_0}^{t} a_i(s, x(s))ds| \leq M$ all $x(\cdot) \in C_nT$, $t \in T$, $i = 1, 2...m$, here M is positive real constant.

(b) The coefficient functions $g_k(t, t_0; z)$ in (3.2.4) is assumed to be linearly independent from each other for all t_0 and for some (all) finite $t_f > t_0$.

(c) The collection of vectors

$$A_1^{k_1} A_2^{k_2} ... A_m^{k_m} b_i \qquad k_1, ..., k_m = 0, 1, ..., n-1$$

spans n dimensions. The b_i are m columns of the matrix B.

Proof. Based on the similar procedure by Davison & Kunze in [19] , we then design the controller:

$$u(t_0, t, t_f; z) = B^T \phi^T(t_0, t; z) G^{-1}(t_0, t_f; z) \left\{ \phi^{-1}(t_f, t_0; z) x_f - x_0 \right\}, \tag{3.2.5}$$

with which the system can be steered from x_0 to the pre-set final state x_f. And the controllability Gramian matrix is denoted by

$$G(t_0, t; z) = \int_{t_0}^{t} \phi(t_0, s; z) B B^T \phi^T(t_0, s; z) ds,$$

which is positive definite since the system is completely controllable if and only if the rows of the matrix $\phi(t, t_0; z)B$ for $t \in T$ are linearly independent functions [52]. The elements of the inverse of Gramian matrix $G^{-1}(t_0, t; z)$ are denoted as $g_{ij}(t_0, t_f; z)$.

By inserting the controller to this solution to the system (3.2.2):

$$x_z(t) = \phi(t_0, t; z) x_0 + \int_{t_0}^{t} \phi(t, s; z) B u(s) ds. \tag{3.2.6}$$

We now formulate nonlinear operator explicitly:

$$P(z)(t) = \phi(t, t_0; z) \left\{ x_0 + G(t_0, t; z) \cdot G^{-1}(t_0, t_f; z) \left[\phi^{-1}(t_f, t_0; z) x_f - x_0 \right] \right\}. \tag{3.2.7}$$

Define norm of matrix $A \in \mathbb{R}^{n \times n}$ $|A| = \max_j \sum_{i=1}^{n} |A|_{ij}$ and the norm of $z(t)$ in $C_n T$ $||z|| = \max\{\sum_{i=1}^{n} z_i(t) : t \in T\}$

Thus,

$$||P_z(t)|| \le \left\{ C|x_0| + (C-1) \exp\left[mM \sum_i^m |A_i| \right] |x_f| \right\} \exp\left[mM \sum_i^m |A_i| \right] = K$$

where

$$C = 1 + \exp\left[mM\sum_{i}^{m}|A_i|\right]\times\exp\left[mM\sum_{i}^{m}|A_i^T|\right]\times|B|\times|B^T|\times(t_f - t_0)\left[n\max_j\sum_{i=1}^{n}|g_{ij}(t_0,t_f;z(s))|\right],$$

Therefore, the domain set of the operator could be :

$$\Phi = \{z|z \in C_nT; \quad ||z|| \leq K\}$$

where K is the same constant as the upper bound of the operator $P(z)(t)$. Let $\Omega = \{x|x = P(z); z \in \Phi\}$ be its image set. Hence, from the above discussion, the nonlinear operator $P(z)(t)$ mapping is continuous and invariant from the closed convex subset Φ into Ω. Besides the compactness of the image set Ω can be demonstrated due to the Arzela-Ascoli theorem [77]. So we can conclude, in application of the Schauder's theorem applies, the nonlinear operator $P(z)(t)$ is proved to have (at least) a fixed point namely z^* which is the very solution of (3.2.2) and (3.2.3) evaluating at z^*.

On the other hand, there isn't any non-zero vector $P \in \mathbb{R}^n$ so that $P^T\phi(t,t_0;z)B = 0$ for all $t \in T$, otherwise the system (3.2.2) is not completely controllable at t_f.

$$P^T\phi(t,t_0;z)B = P^T\phi(t,t_0;z)B = P^T\sum_{k=0}^{n^m-1}\sum_{k_1=0}^{n-1}\cdots\sum_{k_m=0}^{n-1}g_k(t,z)(A_1^{k_1}...A_m^{k_m})B \quad (3.2.8)$$

Due to the assumption of independence of coefficient functions $g_k(t,z)$, there doesn't exist any non-zero vector $P \in \mathbb{R}^n$ so that $P^T\sum_{k_1=0}^{n-1}\cdots\sum_{k_m=0}^{n-1}(A_1^{k_1}...A_m^{k_m})B = 0$ for all $t \in T$. Consequently, the collection of vectors $A_1^{k_1}A_2^{k_2}...A_m^{k_m}b_i$ will span a n-dimensional subset of $\mathbb{R}^n$, $k_1,...,k_m = 0,1,2,...,n-1$; $i = 1,\ldots,l$. □

Remark 3.1. With the aid of this result, we can infer that the system is controllable immediately if the controllability matrix

$$\bar{C} = [B, A_1B, ...A_1A_2B, ..., (A_1A_2...A_m)B, ..., (A_1A_2...A_m)^{n-1}B]$$

has full column rank.

We have fortune to see that the rank condition for testing the complete controllability from linear theories is still valid in nonlinear analysis. Conversely, the theorem can also be available for linear time-variant systems which is regarded as special cases of nonlinear systems, but limited in the sense of sufficiency.

3.3 Observability

The concept of observability could be defined as the dual of state controllability. The results obtained for controllability would be transferred for the observability by a dualizing procedure.

Consider the unforced nonlinear time-varying system:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{m} a_i(t, x(t)) A_i x(t), \\ y(t) = Cx(t), \end{cases} \tag{3.3.1}$$

Theorem 3.2. *The unforced system (3.3.1) is globally completely (totally) observable at t_f, if the conditions below are satisfied:*

(a)The integrator $|\int_{t_0}^{t} a_i(s, x(s))ds| \leq M$ all $x(\cdot) \in C_n T$, $t \in T$, $i = 1, 2...m$, here M is positive real constant.

(b) The coefficient functions $g_k(t, t_0; z)$ in equation (3.2.4) is assumed to be linearly independent from each other for all t_0 and for some (all) finite $t_f > t_0$.

(c) The collection of vectors

$$[A_1^{k_1} A_2^{k_2} ... A_m^{k_m}]^T c_i \qquad k_1, ..., k_m = 0, 1, ..., n-1$$

spans n dimensions. The c_i are m columns of the matrix C^T.

Proof. The procedure proceeds in the same way as shown in the proof of Theorem 3.1. The solution to the unforced system equations (3.3.1), in terms of the state x and output y, is given by

$$x_z(t) = \phi(t, t_0; z)x_0 \tag{3.3.2}$$

$$y_z(t) = Cx_z(t) = C\phi(t, t_0; z)x_0 \tag{3.3.3}$$

The sufficient and necessary condition for complete observability is the columns of $C\phi(t, t_0; z)$ are linearly independent on $[t_0, t_f]$. Therefore, there isn't any non-zero vector $P \in \mathbb{R}^n$ so that $P^T\phi(t, t_0; z)B = 0$ for all $t \in T$, otherwise there's certain initial state such that $y(t) \equiv 0$ on $[t_0, t_f]$. The condition of linear independence can be expressed by

$$C\phi(t,t_0;z)P = C\phi(t,t_0;z)P = C\sum_{k=0}^{n^m-1}\sum_{k_1=0}^{n-1}\cdots\sum_{k_m=0}^{n-1} g_k(t,z)(A_1^{k_1}...A_m^{k_m})P \qquad (3.3.4)$$

Due to the assumption of independence of coefficient functions $g_k(t,z)$, there doesn't exist any non-zero vector $P \in \mathbb{R}^n$ so that $C\sum_{k_1=0}^{n-1}\cdots\sum_{k_m=0}^{n-1}(A_1^{k_1}...A_m^{k_m})P = 0$ for all $t \in T$. Consequently, the collection of vectors $A_1^{k_1}A_2^{k_2}...A_m^{k_m}c_i$ will span a n-dimensional subset of R^n, $i = 1,\ldots,l, \quad k_1,...,k_m = 0,1,2,...,n-1$.

□

Remark 3.2. The condition (c) in Theorem 3.2 is simply equivalent to that the composite matrix

$$\bar{O} = [C^T, A_1^TC^T, ...(A_1A_2)^TC^T, ..., (A_1A_2...A_m)^TC^T, ..., [(A_1A_2...A_m)^{n-1}]^TC^T]$$

is of rank n.

3.4 Canonical Structure

The controllability and observability are invariant under similarity transformation. From Theorem 3.1, the system (3.2.1) is uncontrollable when the controllability matrix is of rank $k < n$ i.e. $rank(\bar{C}) = k < n$. Hence for the nonlinear system, it's possible to obtain Kalman canonical decomposition which illuminate the basic structure for this system [104].

Lemma 3.2. *There exists a time invariant transformation matrix $U \in \mathbb{R}^{n\times n}$ that decomposes the system into the completely controllable and uncontrollable parts for all $t > 0$.*

Define $U = [q_1, q_2...q_k, q_{k+1}...q_n]$, in which $q_1...q_k$ are linear independent columns of $\bar{C}$ and $q_{k_1}...q_n$ are selected from any linear independent vectors such that U is invertible. Then

$$UB = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

Notice that $A_i q_i, i = 1, ..., k$ is linear combination of $q_1 ... q_k$ due to the Cayley-Halmilton theorem.

$$UA_iU^{-1} = \begin{bmatrix} A_{i1} & A_{i2} \\ 0 & A_{i3} \end{bmatrix},$$

for some matrix A_{i1} with dimension $k \times k$.

As each UA_iU^{-1} is still commutable, $UA_i^{k_i}U^{-1}, k_i = 0, \cdots, n-1$ is supposed have the form :

$$UA_i^{k_i}U^{-1} = \begin{bmatrix} A_{i1}^{k_i} & A_{p2} \\ 0 & A_{i3}^{k_i} \end{bmatrix},$$

$$U(A_1^{k_1}A_2^{k_2}\cdots A_m^{k_m})U^{-1} = UA_1^{k_1}U^{-1}...UA_m^{k_m}U^{-1} = \begin{bmatrix} A_{11}^{k_1}A_{21}^{k_2}\cdots A_{m1}^{k_m} & \bar{A}_{p2} \\ 0 & A_{13}^{k_1}A_{23}^{k_2}\cdots A_{m3}^{k_m} \end{bmatrix},$$

for some matrix A_{i1} with dimension $k \times k$. Therefore, the controllability matrix $\bar{C}$ under transformation

$$\hat{C} = U^{-1}\bar{C} = U^{-1}\begin{bmatrix} B_c & A_{11}B_c & ... & A_{m1}B_c & ... & (A_{11}A_{21}...A_{m1})^{n-1}B_c \\ 0 & 0 & ... & 0 & ... & 0 \end{bmatrix}.$$

As $A_{i1}^n, n \geq k$, is a linear combination of $A_{i1}^i, i = 0, 1, ..., k-1$ and rank $(\bar{C}) = k$, we have

$$\text{rank}[B_c, A_{11}B_c, ..., A_{m1}B_c, ..., (A_{11}A_{21}...A_{m1})^{k-1}B_c] = k.$$

Therefore, by introducing the state transformation $x(t) = Uz(t)$, where $z(t) = [z_1^T(t), \quad z_2^T(t)]^T$ the state equation of system (3.2.1) is transformed into:

$$\begin{cases} \dot{z_1}(t) = \sum_i^m a_i(t,x)A_{i1}z_1(t) + \sum_i^m a_i(t,x)A_{i2}z_2(t) + B_cu(t) \\ \dot{z_3}(t) = \sum_i^m a_i(t,x)A_{i3}z_3(t) \end{cases} \tag{3.4.1}$$

Remark 3.3. The state $z_1(t)$ is completely controllable while the state $z_2(t)$ is clearly uncontrollable. If the state $z_2(t)$ is asymptotically stable, then the system (3.4.1) is stabilisable.

By duality, we have the following decomposition if the system is not completely observable.

Lemma 3.3. *There exists a time invariant transformation matrix $U \in \mathbb{R}^{n\times n}$ that decomposes the system into the completely observable and unobservable parts.*

Proof. Using the same arguments, there's an invertible matrix U such that

$$CU^{-1} = \begin{bmatrix} C_o & 0 \end{bmatrix}$$

and

$$UA_iU^{-1} = \begin{bmatrix} A_{i1} & 0 \\ A_{2i} & A_{i3}. \end{bmatrix}$$

Then, the observability matrix $\bar{O}$ under transformation

$$\hat{O} = \bar{O}U = \begin{bmatrix} C_o^T & A_{11}^T C_o^T & ... & A_{m1}^T C_o^T & ... & (A_{11}A_{21}...A_{m1})^{n-1T} C_o^T \\ 0 & 0 & ... & 0 & ... & 0 \end{bmatrix}$$

□

3.5 Criteria for stability and feedback stabilisation

Consider the free system of (3.2.1) :

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{m} a_i(t, x(t)) A_i x(t), \\ x(0) = x_0 \end{cases} \tag{3.5.1}$$

where $A_i \in \mathbb{R}^{n \times n}$ and the state vector $x(\cdot) \in C_n[0, +\infty)$, the Banach space of continuous $\mathbb{R}^n$-valued functions on $[0, +\infty)$. The function $a_i : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ is continuous with respect to the state x and t.

Denote σ as the spectrum of the matrix A i.e. the collection of eigenvalues of A and $Re(\sigma(A)) < 0$ as all of the eigenvalues of A have negative real part.

Theorem 3.3. *The system (3.5.1) is globally asymptotically stable if the following conditions hold:*

1) $|\int_{t_0}^{t} a_i(s, x(s)) ds| \leq M$ *all* $x \in C_n T$, $t \in T$, $i = 1, 2...m$, *here* $M \in R+$.

2a) If there's $\lim_{t\to\infty} \int_{t_0}^{t} a_k(s, x(s)) ds = +\infty$ *for some* $1 \leq k \leq m$ *such that*

$\lim_{t\to\infty} \frac{\int_{t_0}^{t} a_i(s,x(s))ds}{\int_{t_0}^{t} a_k(s,x(s))ds} = c_i$ *and* $Re(\sigma(\sum_i c_i A_i)) < 0$, c_i *is a constant,* $i = 1, 2...m$.

2b) or If there's $\lim_{t\to\infty} \int_{t_0}^{t} a_k(s, x(s)) ds = -\infty$ *for some* $1 \leq k \leq m$ *such that*

$$\lim_{t\to\infty}\frac{\int_{t_0}^{t} a_i(s,x(s))ds}{\int_{t_0}^{t} a_k(s,x(s))ds} = c_i \text{ and } Re(\sigma(\sum_i c_iA_i)) > 0,\ c_i \text{ is a constant, } i = 1,2...m.$$

Proof. (1) Due to the assumption of commutative matrix A_i and the boundedness of $|\int_{t_0}^{t} a_i(s,x)|$, the Schauder's fixed-point principle still validate the following equation:

$$x(t) = \phi(t,0;z(t)x_0) = \exp\left[\sum_{i=1}^{m}\int_{t_0}^{t} a_i(s,x(s))dsA_i\right]x_0, \quad t\in T \tag{3.5.2}$$

For the convenience of notation, denote $g_k(t;x) = \int_{t_0}^{t} a_k(s,x(s))ds$. Under the assumption that $\lim_{t\to\infty} g_k(t;x) = +\infty, \forall x(\cdot) \in C_n([0,\infty])$, $g_k(t;x)$ will become positive beyond a certain time t_1.

So we have for $t > t_1$

$$x(t) = \exp\left[g_k(t;x)(\sum_i \frac{g_i(t;x)}{g_k(t;x)}A_i)\right]x_0. \tag{3.5.3}$$

When the time t approaches infinity,

$$\lim_{t\to\infty} x(t) = \exp\left[g_k(t;x)(\sum_i c_iA_i)\right]x_0 = 0. \tag{3.5.4}$$

The last equation holds as there exists some some positive constant D such that

$$\| \exp\left[g_k(t;x)(\sum_i c_iA_i)\right] \| \le D\exp\left[g_k(t;x)Re(\sigma(\sum_i c_iA_i))\right]. \tag{3.5.5}$$

If $Re(\sigma(\sum_i c_iA_i)) < 0$ and $\lim_{t\to\infty} g_k(t;x(t)) = +\infty$ or $Re(\sigma(\sum_i c_iA_i)) > 0$ and $\lim_{t\to\infty} g_k(t;x(t))$ $= -\infty$, thus $\exp[g_k(t)Re(\sigma(\sum_i c_iA_i))] \to 0$ as $t\to\infty$.

□

Remark 3.4. The work done by Langson & Alleyne [54] presenting a computable estimation of the Region of Attraction (ROA) for global stability analysis, it has serious difficulty in the application of the assessment of stability. Theorem 3.3 guarantees the global asymptotical stability by imposing conditions as listed above. The decision rule is very clear when the essential information is accessible.

Corollary 1. *The system 3.5.1 is globally asymptotically stable if the following conditions hold:*

1) $|\int_{t_0}^{t} a_i(s,x(s))ds| \le M$ *all* $x \in C_nT$, $t\in T$, $i = 1,2...m$, *here* $M > 0$.

2) If there's $\lim_{t\to\infty}\int_{t_0}^{t} a_k(s,x(s))ds = +\infty$ *for some* $1 \le k \le m$ *such that*

$\lim_{t\to\infty}\frac{\int_{t_0}^{t} a_i(s,x(s))ds}{\int_{t_0}^{t} a_k(s,x(s))ds} = c_i$ *and* $c_i Re(\sigma(A_i)) < 0$, c_i *is a constant,* $i = 1, 2...m.$

or

If there's $\lim_{t\to\infty}\int_{t_0}^{t} a_k(s,x(s))ds = -\infty$ *for some* $1 \le k \le m$ *such that*

$\lim_{t\to\infty}\frac{\int_{t_0}^{t} a_i(s,x(s))ds}{\int_{t_0}^{t} a_k(s,x(s))ds} = c_i$ *and* $c_i Re(\sigma(A_i)) > 0$, c_i *is a constant,* $i = 1, 2...m.$

Proof. (1) Proceeding in the same way as that of Theorem 3.3 and because of the community of matrix A_i, there's certain positive constants D_i such that

$$\parallel \exp[g_k(t;x)(\sum_i c_i A_i)] \parallel \le \prod_i \parallel \exp\left[g_k(t;x)(c_i A_i)\right] \parallel \le \prod_i D_i \exp\left[g_k(t;x) c_i Re(\sigma(A_i))\right]. \tag{3.5.6}$$

If $c_i Re(\sigma(\sum_i A_i)) < 0$ and $\lim_{t\to\infty} g_k(t;x(t)) = +\infty$ or $c_i Re(\sigma(\sum_i A_i)) > 0$ and $\lim_{t\to\infty} g_k(t;x(t))$ $= -\infty$, then $\lim_{t\to\infty} x(t) = \exp\left[g_k(t;x)(\sum_i c_i A_i)\right] x_0 = 0$ as $t \to \infty$.

□

Combining Theorem 3.3 and Lemma 3.2, the results of stabilisation of system (3.2.1) by means of nonlinear time-varying state-feedback are concluded in the following theorem.

Theorem 3.4. *Suppose for the system (3.2.1) with* $rank(\bar{C}) < n$ *is stabilisable if the following conditions hold:*

1) $|\int_{t_0}^{t} a_i(s,x)ds| \le M$ *all* $x \in C_n T$, $t \in T$, $i = 1, 2...m$, *here* $M > 0$.

2a) If there's $\lim_{t\to\infty}\int_{t_0}^{t} a_k(s,x)ds = +\infty$ *for some* $1 \le k \le m$ *such that*

$\lim_{t\to\infty}\frac{\int_{t_0}^{t} a_i(s,x)ds}{\int_{t_0}^{t} a_k(s,x)ds} = c_i$, $Re[\sigma(\sum_i c_i(A_{i1} + B_c K_{i1})] < 0$, *and* $Re(\sigma(\sum_i c_i A_{i3})) < 0$, $i \neq k$

or

2b) If there's $\lim_{t\to\infty}\int_{t_0}^{t} a_k(s,x)ds = -\infty$ *for some* $1 \le k \le m$ *such that*

$\lim_{t\to\infty}\frac{\int_{t_0}^{t} a_i(s,x)ds}{\int_{t_0}^{t} a_k(s,x)ds} = c_i$, $Re[\sigma(\sum_i c_i(A_{i1} + B_c K_{i1})] > 0$ *and* $Re(\sigma(\sum_i c_i A_{i3})) > 0$, *,*$i \neq k$.

Proof. Design the stabilising feedback controller as

$$u(t) = K(t,x)x(t). \tag{3.5.7}$$

The state equation turns out to be:

$$\dot{x}(t) = [\sum_{i=1}^{m} a_i(t,x)A_i + BK(t,x)]x(t). \tag{3.5.8}$$

Using the transformed state equation (3.4.1):

$$\begin{cases} \dot{z}_1(t) = \sum_i^m a_i(t,x)A_{i1}z_1(t) + \sum_i^m a_i(t,x)A_{i2}z_2(t) + B_cK(t,x)x(t) \\ \dot{z}_2(t) = \sum_i^m a_i(t,x)A_{i3}z_2(t) \end{cases}, \tag{3.5.9}$$

Selecting such a nonlinear time-varying state feedback gain: $K(t,x) = \sum_i^m a_i(t,x)K_{i1}z_1(t)$, the equation (3.5.9) becomes:

$$\dot{z}(t) = \sum_i^m a_i(t,x)\bar{A}_i z(t) \tag{3.5.10}$$

where

$$\bar{A}_i = \begin{bmatrix} A_{i1} + B_cK_{i1} & A_{i2} \\ 0 & A_{i3} \end{bmatrix}. \tag{3.5.11}$$

Due to the condition 1 and 2a, $Re(\sigma(\sum_i c_i\bar{A}_i)) > 0$. So the system (3.5.8) is asymptotically stable. The condition 1 and 2b would generate the same result according to the similar proof.

□

With lemma 3.3 and theorem 3.4, we would have the following theorem immediately:

Theorem 3.5. *Suppose for the system (3.5.1) is detectable and the observability matrix O has rank $k < n$ if the following conditions hold:*

1) $|\int_{t_0}^t a_i(s,x)ds| \leq M$ *all* $x \in C_nT$, $t \in T$, $i = 1, 2...m$, *here* $M > 0$.

2a) If there's $\lim_{t\to\infty} \int_{t_0}^t a_k(s,x)ds = +\infty$ *for some* $1 \leq k \leq m$ *such that*

$\lim_{t\to\infty} \frac{\int_{t_0}^t a_i(s,x)ds}{\int_{t_0}^t a_k(s,x)ds} = c_i$, $Re[\sigma(\sum_i c_i(A_{i1} - L_{i1}C_o)] < 0$ *and* $Re(\sigma(\sum_i c_iA_{i3})) < 0$, $i \neq k$.

or

2b) If there's $\lim_{t\to\infty} \int_{t_0}^t a_k(s,x)ds = -\infty$ *for some* $1 \leq k \leq m$ *such that*

$\lim_{t\to\infty} \frac{\int_{t_0}^t a_i(s,x)ds}{\int_{t_0}^t a_k(s,x)ds} = c_i,\ Re[\sigma(\sum_i c_i(A_{i1} - L_{i1}C_o)] > 0$ *and* $Re(\sigma(\sum_i c_i A_{i3})) > 0,$ $i \neq k.$

Proof. Consider the following nonlinear time-varying Luenberger-like observer:

$$\begin{cases} \dot{\tilde{x}}(t) = \sum_{i=1}^m a_i(t,x)A_i\tilde{x}(t) + L(t,x)(y - \tilde{y}) \\ \tilde{y} = C\tilde{x} \end{cases}, \tag{3.5.12}$$

Let the state estimation error be $e(t) = x(t) - \tilde{x}$. So the equation of the error is

$$\dot{e}(t) = \sum_{i=1}^m a_i(t,x)A_i e(t) - L(t,x)Ce(t) \tag{3.5.13}$$

Combining lemma 3.2 with a similar change of coordinates

$$Ue(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \tag{3.5.14}$$

and partition $L(t,x)$ in the form of $[L_1^T(t,x), L_2^T(t,x)]^T$, the system (3.5.13) becomes

$$\begin{cases} \dot{e_1}(t) = \sum_i^m a_i(t,x)A_{i1}e_1(t) - L_1(t,x)C_o e_1(t) \\ \dot{e_2}(t) = \sum_i^m a_i(t,x)A_{2i}e_1(t) + \sum_i^m a_i(t,x)A_{i3}e_2(t) - L_2(t,x)C_o e_1(t) \end{cases}. \tag{3.5.15}$$

Then the observer parameter $L_1(t,x), L_2(t,x)$ are determined to be in the nonlinear time-varying form of $\sum_i^m a_i(t,x)L_{i1}e_1(t)$and $\sum_i^m a_i(t,x)L_{i2}e_1(t)$ respectively. Using the same arguments as that in the Theorem 3.4,

$$\dot{e}(t) = \sum_i^m a_i(t,x)\hat{A}_i e(t) \tag{3.5.16}$$

where

$$\hat{A}_i = \begin{bmatrix} A_{i1} - L_{i1}C_o & 0 \\ A_{2i} - L_{i2}C_o & A_{i3} \end{bmatrix}. \tag{3.5.17}$$

Under the condition 1 and 2a, the system (3.5.13) is asymptotically stable. The condition 1 and 2b would generate similar results.

Next we would present that for the system consisting of the controller that feeds back the state of the observer, is also asymptotically stable.

We now focus on the design of the following nonlinear time-varying Luenberger observer for the system (3.2.1):

$$\begin{cases} \dot{\tilde{x}}(t) = \sum_{i=1}^{m} a_i(t,x)A_i\tilde{x}(t) + Bu(t) + L(t,x)(y-\tilde{y}), \\ \tilde{y} = C\tilde{x} \end{cases}. \tag{3.5.18}$$

Let the state estimation error be $e(t) = x(t) - \tilde{x}$. So the equation of the error is

$$\dot{e}(t) = [\sum_{i=1}^{m} a_i(t,x)A_i - L(t,x)C]e(t) \tag{3.5.19}$$

Let the controller be chosen based on the estimated state as:

$$u(t) = K(t,x)\tilde{x} = K(t,x)[x(t) - e(t)]. \tag{3.5.20}$$

The state equation turns out to be:

$$\dot{x}(t) = [\sum_{i=1}^{m} a_i(t,x)A_i + BK(t,x)]x(t) - BK(t,x)e(t).$$

Denoting by $s(t) = [x^T(t), e^T(t)]^T$, we have that

$$\begin{cases} \dot{s}(t) = H(t)s(t), \\ \text{where} \quad H(t) = \begin{bmatrix} \sum_{i=1}^{m} a_i(t,x)A_i + BK(t,x) & -BK(t,x) \\ 0 & \sum_{i=1}^{m} a_i(t,x)A_i - L(t)C \end{bmatrix}. \end{cases} \tag{3.5.21}$$

Suppose $K(t,x), L(t,x)$ are given as in Theorem 3.4 and Theorem 3.5, of which assumptions are satisfied, thus the whole system (3.5.21) is asymptotically stable. □

Remark 3.5. As we know, *separation principle* is available for linear systems in the design of controller. For nonlinear systems, the observer-based control problem becomes quite difficult. But here, this useful principle the observer-based stabilisation has been justified for this specified class of nonlinear time-varying systems.

3.6 Some numerical examples

We illustrate the obtained results about controllability and stability through several examples. Note that these conditions are sufficient for both properties. So it's straightforward for us to verify the effectiveness of the conclusion by the classical instances proposed in recent literature.

Example 1 (For globally complete controllability from example 6.1 in [19])

Consider the system:

$$\begin{aligned} \dot{x_1} &= x_2 + sin[g(x_1, x_2, t)]u, \\ \dot{x_1} &= -x_1 + sin[g(x_1, x_2, t)]u \end{aligned} \tag{3.6.1}$$

$sin[g(x_1, x_2, t)]$ is a continuous function of x_1, x_2 and a piecewise continuous function of t and satisfies the following inequality:

$0 < \varepsilon \leq g(x_1, x_2, t) \leq \pi - \varepsilon$ for all $x_1, x_2 \in C_n[t_0, t_f]$, $t \in [t_0, t_f]$.

In this case, we can rewrite the system in matrix form firstly:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + sin[g(x_1, x_2, t)] \begin{bmatrix} 1 \\ 1 \end{bmatrix} u. \tag{3.6.2}$$

After fulfilling first two conditions of the Theorem 3.1, it's easy to establish the rank test on the controllability matrix :

$$\bar{C} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad rank(\bar{C}) = 2.$$

The controllability matrix $\bar{C}$ has full rank, and the system 3.6.1 is therefore globally completely controllable, which coincides with the conclusion in that paper.

Example 2 (For stability from Example 4.5 in [27] and [60])

In this example, a nonlinear system in the SDC form is constructed as:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} a & b - c(x_1, x_2) \\ -b - c(x_1, x_2) & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_0 = x(0), \tag{3.6.3}$$

where $a.b \in R$ and $c(x_1, x_2)$ is assumed to be a smooth function: $\mathbb{R}^2 \to \mathbb{R}$.

It's shown in [60] that although the parameters $a = -0.1, b = 3$ and scalar function

$c(x_1, x_2) = -\frac{8}{\pi^2}\tan^{-1} x_1 \tan^{-1} x_2$ are chosen to meet the assumption of [54], the state of the system is gradually far away from the origin described by the integral curve with certain initial condition. And the portrait of the flow of the system is given in Figure 3.1.

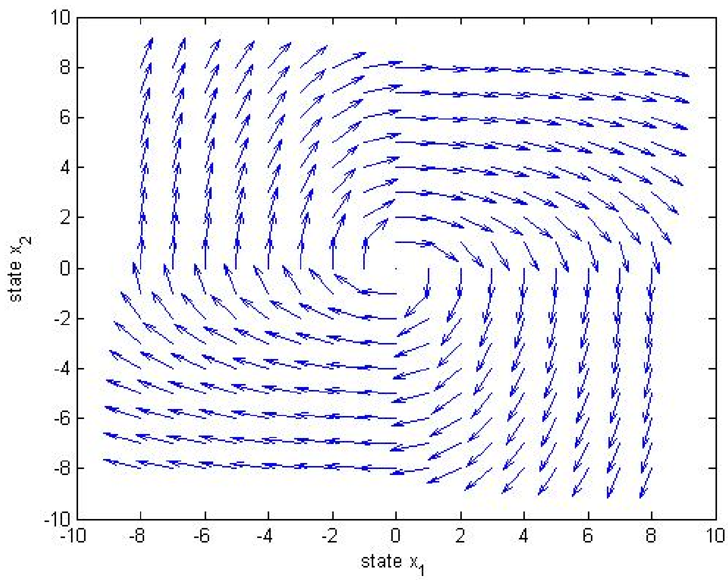

Figure 3.1: Phase portrait of the example in [60]

As pointed by Ghane & Menhaj [27] , the eigenstructure-based analysis fails to access the correct information about qualitative behaviour of this system. Hereby we take $a = -9$, $b = 0$ and $c(x_1, x_2) = -\frac{8}{\pi^2}\tan^{-1} x_1 \tan^{-1} x_2$ in accordance to the conditions of Theorem 3.2 leading to the conclusion of global asymptotical stable of the system, depicted by the Figure 3.2.

From Figure 3.2, it's seen that all the arrows are in the direction toward the origin. So the numerical depicted result of the system dynamics is in agreement with the theoretical conclusion.

3.7 Summary

We have derived basic results for the commutative class of nonlinear time-varying systems. These are: an algebraic criterion for complete controllability, canonical decomposition of the system and design approach to stabilising controllers and observers. The results are very explicit and have a strong link with the rich theory of linear

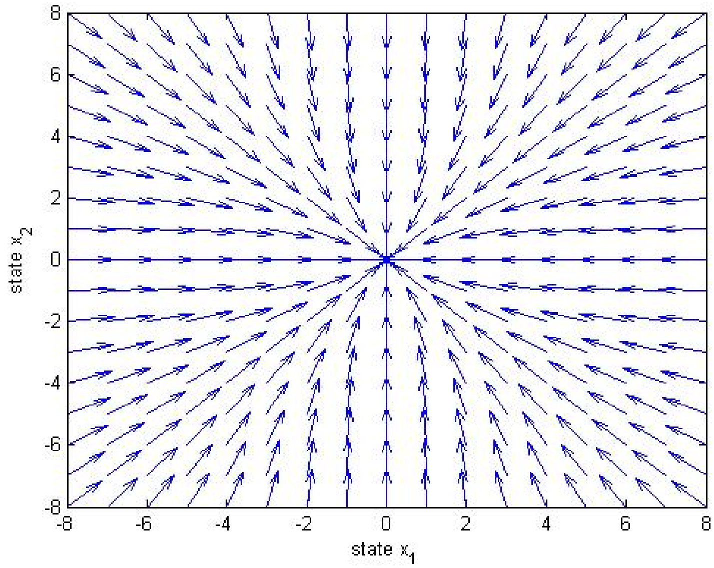

Figure 3.2: Phase portrait of the parameter-altered example

time-varying systems. It would be interesting to investigate if more general nonlinear time-varying systems can be approximated by or transformed into this class of systems.

Chapter 4

Observer design for one-sided Lipschitz nonlinear systems with time-varying output and state delays

4.1 Introduction

In this chapter, we design the observer for for one-sided Lipschitz nonlinear systems with time-varying output and state delays.Consider a class of one-sided Lipschitz nonlinear dynamical systems with time-varying output delays, given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau) + f(x,u) + \psi(t,y), \\ y(t) &= Cx(t) + C_d x(t-\tau), \end{aligned} \tag{4.1.1}$$

where $x \in \mathbb{R}^n$ $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $\psi(t,y) \in \mathbb{R}^n$ are the state vector, the control input, the output, and the nonlinear dynamics of the system, respectively. The linear constant matrices of the dynamical system are represented by A, A_d, $C$and C_d, and the nonlinear function is denoted by $f(x,u) \in \mathbb{R}^n$. The system given by (4.1.1) is assumed to be an observable system. The function $f(x,u)$ belongs to the one-sided Lipschitz nonlinearities owing to the equation (2.5.3) in Definition 2.5 and the equation (2.5.3) in Definition 2.6. Another concept employed for the observer design is quadratic inner-boundedness.

The continuous time-varying differentiable function τ refers to the time delay at the state and output, satisfying

$$\begin{aligned} &0 \le h_1 \le \tau \le h_2, \\ &\dot{\tau} \le \mu. \end{aligned} \tag{4.1.2}$$

The time delays, belonging to an interval, appearing in both state and output variables reformulate control problem in complex forms of engineering systems. The Lyapunov function used in the developed delay-dependent techniques ignores the lower bound of the time delay, conservatism remains; therefore, the range should be incorporated to establish less restricted results. And the delay-range-dependent techniques based on various Lyapunov-Krasovskii (LK) approaches have been proposed for nonlinear time-delay systems in [2]. It is motivated by the cutting-edge delay-range-dependent observer-design strategy and one-sided Lipschitz nonlinear observer construction methodologies to explore for the one-sided Lipschitz nonlinear systems with both measurement and state time-varying delays.

The aim of the present study is to propose and compare observer-design methodologies for a dynamic one-sided Lipschitz nonlinear system (4.1.1) subject to time-varying state and output delays varying in an interval.

4.2 Delay-range-dependent nonlinear observer design

Consider a Luenberger-like observer for a delayed one-sided Lipschitz nonlinear system (4.1.1) formulated as

$$\begin{aligned} &\dot{\hat{x}}(t) = A\hat{x}(t) + A_d\hat{x}(t-\tau) + f(\hat{x}, u) + \psi(t, y) + L((y(t) - \hat{y}(t))), \\ &\hat{y}(t) = C\hat{x}(t) + C_d\hat{x}(t-\tau), \end{aligned} \tag{4.2.1}$$

where $L \in \mathbb{R}^{n\times m}$ is the observer gain matrix. The state estimation error is given by

$$e = x - \hat{x}. \tag{4.2.2}$$

From (4.1.1) and (4.2.1) – (4.2.2), we have the error dynamics:

$$\dot{e}(t) = (A - LC)e(t) + f(x, u) - f(\hat{x}, u) - L((y(t) - \hat{y}(t))) + (A_d - LC_d)e(t - \tau), \tag{4.2.3}$$

which reduce further to

$$\dot{e}(t) = (A - LC)e(t) + \Phi(x, \hat{x}, u) + (A_d - LC_d)e(t - \tau), \tag{4.2.4}$$

by substitution of

$$\Phi(x, \hat{x}, u) = f(x, u) - f(\hat{x}, u). \tag{4.2.5}$$

Now, we provide an LMI-based sufficient condition to test the state-estimation ability of an observer (4.2.1) for a given observer gain matrix L. Note that the observer gain matrix obtained by using the traditional observer-design methodologies in [6, 97, 100, 101], etc. do not include time delays. Now delay-range-dependent technique is proposed to establish condition for asymptotic stability of system (4.1.1).

Theorem 4.1. *Consider the one-sided Lipschitz nonlinear system (4.1.1) satisfying the time-delay bounds given by condition (4.1.2), the one-sided Lipschitz condition eq(2.5.3), and the quadratic inner-boundedness criterion eq(2.5.4). Suppose there exist symmetric matrices* $P \in \mathbb{R}^{n\times n}$, $Q_i \in \mathbb{R}^{n\times n}$ *and* $Z_j \in \mathbb{R}^{n\times n}$ *for* $i = 1, 2, 3$ *and* $j = 1, 2$, *and scalars* ε_1 *and* ε_2, *such that the LMIs*

$$P > 0, \quad Q_i > 0, \quad Z_j > 0, \quad \varepsilon_1 > 0, \quad \varepsilon_2 > 0, \quad \forall i = 1, 2, 3 \quad j = 1, 2 \tag{4.2.6}$$

$$\begin{bmatrix} Y_1 + \rho\varepsilon_1 I + \beta\varepsilon_2 I & P\bar{A}_d & Z_1 & 0 & P - \dfrac{\varepsilon_1 I}{2} + \dfrac{\alpha\varepsilon_2 I}{2} & h_1\bar{A}^T Z_1 & h_{12}\bar{A}^T Z_2 \\ * & -\Lambda_1 & Z_2 & Z_2 & 0 & h_1\bar{A}_d^T Z_1 & h_{12}\bar{A}_d^T Z_2 \\ * & * & -\Lambda_2 & 0 & 0 & 0 & 0 \\ * & * & * & -\Lambda_3 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & h_1 Z_1 & h_{12} Z_2 \\ * & * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & * & -Z_2 \end{bmatrix} < 0 \tag{4.2.7}$$

are satisfied for a given matrix L, where

$$
\begin{aligned}
&Y_1 = P\bar{A} + \bar{A}^T P + \sum_{i=1}^{3} Q_i - Z_1,\\
&\Lambda_1 = (1-\mu)Q_3 + 2Z_2,\\
&\Lambda_2 = Q_1 + Z_1 + Z_2,\\
&\Lambda_3 = Q_2 + Z_2,\\
&\bar{A} = A - LC,\\
&\bar{A}_d = A_d - L_d C_d,\\
&h_{12} = h_2 - h_1.
\end{aligned} \tag{4.2.8}
$$

Then, there exists a Luenberger-type observer (4.2.1) such that the state-estimation error e asymptotically converges to the origin.

Proof. Define an LK functional candidate ([16, 80, 91]) as

$$
\begin{aligned}
V(e,t) \;=\;& e^T P e + \sum_{i=1}^{2} \int_{t-h_i}^{t} e^T(\alpha) Q_i e(\alpha) d\alpha + \int_{t-\tau}^{t} e^T(\alpha) Q_3 e(\alpha) d\alpha \\
&+ \int_{h_1}^{0} \int_{t+s}^{t} h_1 \dot{e}^T(\alpha) Z_1 \dot{e}(\alpha) d\alpha ds + \int_{h_2}^{h_1} \int_{t+s}^{t} h_{12} \dot{e}^T(\alpha) Z_2 \dot{e}(\alpha) d\alpha ds
\end{aligned} \tag{4.2.9}
$$

Acquiring the time derivative of (4.2.9) yields

$$
\begin{aligned}
\dot{V}(e,t) \;\leq\;& 2e^T P\dot{e} + \sum_{i=1}^{2} \{e^T Q_i e - e^T(t-h_i) Q_i e(t-h_i)\} + e^T Q_3 e \\
&-(1-\mu) e^T(t-\tau) Q_3 e(t-\tau) + \dot{e}^T(h_1^2 Z_1 + h_{12}^2 Z_2)\dot{e} \\
&- \int_{t-h_i}^{t} h_1 \dot{e}^T(\alpha) Z_1 \dot{e}(\alpha) d\alpha - \int_{t-h_2}^{t-h_1} h_{12} \dot{e}^T(\alpha) Z_2 \dot{e}(\alpha) d\alpha.
\end{aligned} \tag{4.2.10}
$$

Employing (4.2.4) and (4.2.10) and rearranging the terms, the upper bound on $\dot{V}(e,t)$

is obtained as

$$\begin{aligned}\dot{V}(e,t) \leq\; & 2e^T P(\bar{A}e + \Phi(x,\hat{x},u) + \bar{A}_d e(t-\tau)) + \sum_{i=1}^{3} e^T Q_i e \\ & - \sum_{i=1}^{2} e^T(t-h_i)Q_i e(t-h_i) - (1-\mu)e^T(t-\tau)Q_3 e(t-\tau) \\ & - \int_{t-h_i}^{t} h_1 \dot{e}^T(\alpha) Z_1 \dot{e}(\alpha)d\alpha + (\bar{A}e + \Phi(x,\hat{x},u) + \bar{A}_d e(t-\tau))^T \\ & \times(h_1^2 Z_1 + h_{12}^2 Z_2) \times (\bar{A}e + \Phi(x,\hat{x},u) + \bar{A}_d e(t-\tau)) - \int_{t-h_2}^{t-h_1} h_{12}\dot{e}^T(\alpha) Z_2 \dot{e}(\alpha)d\alpha.\end{aligned} \tag{4.2.11}$$

Applying Jensen's inequality reveals

$$\begin{aligned}-\int_{t-h_i}^{t} h_1 \dot{e}^T(\alpha) Z_1 \dot{e}(\alpha)d\alpha \;\leq\; & -\left(\int_{t-h_i}^{t} \dot{e}(\alpha)d\alpha\right)^T Z_1 \left(\int_{t-h_i}^{t} \dot{e}(\alpha)d\alpha\right) \\ \leq\; & -(e(t) - e(t-h_1))^T Z_1 (e(t) - e(t-h_1)).\end{aligned} \tag{4.2.12}$$

Similarly, we have

$$\begin{aligned}& -\int_{t-h_2}^{t-h_1} h_{12}\dot{e}^T(\alpha) Z_2 \dot{e}(\alpha)d\alpha \\ = & -\int_{t-h_2}^{t-\tau} h_{12}\dot{e}^T(\alpha) Z_2 \dot{e}(\alpha)d\alpha - \int_{t-\tau}^{t-h_1} h_{12}\dot{e}^T(\alpha) Z_2 \dot{e}(\alpha)d\alpha \\ \leq & -\left(\int_{t-h_2}^{t-\tau} \dot{e}(\alpha)d\alpha\right)^T Z_2 \left(\int_{t-h_2}^{t-\tau} \dot{e}(\alpha)d\alpha\right) - \left(\int_{t-\tau}^{t-h_1} \dot{e}(\alpha)d\alpha\right)^T Z_2 \left(\int_{t-\tau}^{t-h_1} \dot{e}(\alpha)d\alpha\right) \\ \leq & -(e(t-\tau) - e(t-h_2))^T Z_2 (e(t-\tau) - e(t-h_2)) \\ & -(e(t-h_1) - e(t-\tau))^T Z_2 (e(t-h_1) - e(t-\tau))\end{aligned} \tag{4.2.13}$$

Combining the results of (4.2.11)-(4.2.13), we have

$$\begin{aligned}
&\dot{V}(e,t) \\
\leq\;& e^T\left[P\bar{A} + \bar{A}^T P + \sum_{i=1}^{3} Q_i + \bar{A}^T(h_1^2 Z_1 + h_{12}^2 Z_2)\bar{A} - Z_1\right] e \\
&+2e^T\left[P\bar{A}_d + \bar{A}^T(h_1^2 Z_1 + h_{12}^2 Z_2)\bar{A}_d\right] e(t-\tau) + 2e^T Z_1 e(t-h_1) \\
&+e^T(t-\tau) \times \left[-(1-\mu)Q_3 - 2Z_2 + \bar{A}_d^T(h_1^2 Z_1 + h_{12}^2 Z_2)\bar{A}_d\right] \times e(t-\tau) \\
&2e^T(t-\tau)Z_2 e(t-h_1) + 2e^T(t-\tau)Z_2 e(t-h_2) + e^T(t-h_1)(-Q_1 - Z_1 - Z_2)e(t-h_1) \\
&+e^T(t-h_2)(-Q_2 - Z_2)e(t-h_2) + 2e^T\left[P + \bar{A}^T(h_1^2 Z_1 + h_{12}^2 Z_2)\right]\Phi(x,\hat{x},u) \\
&+\Phi^T(x,\hat{x},u)(h_1^2 Z_1 + h_{12}^2 Z_2)\Phi(x,\hat{x},u) - 2e^T(t-\tau)[\bar{A}_d^T(h_1^2 Z_1 + h_{12}^2)]\Phi(x,\hat{x},u).
\end{aligned} \tag{4.2.14}$$

From (4.2.14),

$$\dot{V}(e,t) \leq \Psi_1^T \Upsilon_1 \Psi_1, \tag{4.2.15}$$

where $\Psi_1^T = \begin{bmatrix} e^T & e^T(t-\tau) & e^T(t-h_1) & e^T(t-h_2) & \Phi^T(x,\hat{x},u) \end{bmatrix}$,

$$\Upsilon_1 = \begin{bmatrix} Y_1 + \bar{A}^T Y_4 \bar{A} & P\bar{A}_d + \bar{A}^T Y_4 \bar{A}_d & Z_1 & 0 & Y_2 \\ * & -(1-\mu)Q_3 - 2Z_2 + Y_3\bar{A}_d & Z_2 & Z_2 & Y_3 \\ * & * & -Q_1 - Z_1 - Z_2 & 0 & 0 \\ * & * & * & -Q_2 - Z_2 & 0 \\ * & * & * & * & Y_4 \end{bmatrix} < 0 \tag{4.2.16}$$

$$\begin{aligned}
Y_2 &= P + \bar{A}^T(h_1^2 Z_1 + h_{12}^2), \\
Y_3 &= \bar{A}_d^T(h_1^2 Z_1 + h_{12}^2), \\
Y_4 &= (h_1^2 Z_1 + h_{12}^2).
\end{aligned} \tag{4.2.17}$$

The one-sided Lipschitz condition given by eq(2.5.3) is equivalent to $\rho e^T e - e^T \Phi \geq 0$.

For a positive scalar ε_1, the expression can be written as

$$\Psi_1^T \begin{bmatrix} \rho\varepsilon_1 I & 0 & 0 & 0 & \frac{-\varepsilon_1 I}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{-\varepsilon_1 I}{2} & 0 & 0 & 0 & 0 \end{bmatrix} \Psi_1 \geq 0. \tag{4.2.18}$$

The quadratic inner-boundedness condition eq(2.5.4) implies $\Phi^T\Phi \leq \beta e^T e - \alpha e^T \Phi$ which for a positive scalar ε_2 results in

$$\Psi_1^T \begin{bmatrix} \beta\varepsilon_2 I & 0 & 0 & 0 & \alpha\frac{\varepsilon_2 I}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha\frac{\varepsilon_2 I}{2} & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} \Psi_1 \geq 0. \tag{4.2.19}$$

Merging (4.2.17), (4.2.18), and (4.2.19) and using the S-procedure entails

$$\begin{bmatrix} Y_1 + \bar{A}^T Y_4 \bar{A} + \rho\varepsilon_1 I + \beta\varepsilon_2 I & P\bar{A}_d + \bar{A}Y_4\bar{A}_d & Z_1 & 0 & Y_2 - \frac{\varepsilon_1 I}{2} + \alpha\frac{\varepsilon_2 I}{2} \\ * & -(1-\mu)Q_3 - 2Z_2 + Y_3\bar{A}_d & Z_2 & Z_2 & Y_3 \\ * & * & -Q_1 - Z_1 - Z_2 & 0 & 0 \\ * & * & * & -Q_2 - Z_2 & 0 \\ * & * & * & * & Y_4 - \varepsilon_2 I \end{bmatrix} < 0 \tag{4.2.20}$$

Applying the Schur complement lemma to inequality (4.2.20) produces LMI (4.2.7), which implies that $\dot{V}(e,t) \leq \Psi_1^T \Upsilon_1 \Psi_1 < 0$; That is, the error e asymptotically converges to the origin. This finishes the proof of Theorem 4.1. □

Theorem 4.1 ensures state estimation by means of an observer for a given gain matrix L. If a guess for the observer gain matrix L is unobtainable, the following Theorem 4.2 provides a solution in form of matrix inequalities.

Theorem 4.2. *Consider the one-sided Lipschitz nonlinear system (4.1.1) satisfying the time-delay bounds given by (4.1.2), the one-sided Lipschitz condition (2.5.3), and the* *quadratic inner-boundedness criterion (2.5.4). Suppose there exist symmetric matrices $P \in \mathbb{R}^{n\times n}$, $Q_i \in \mathbb{R}^{n\times n}$ and $Z_j \in \mathbb{R}^{n\times n}$ for $i = 1, 2, 3$ and $j = 1, 2$, matrix $X \in \mathbb{R}^{n\times m}$, and scalar ε_1 and ε_2 such that*

$$P > 0, \quad Q_i > 0, \quad Z_j > 0, \quad \varepsilon_1 > 0 \quad \varepsilon_2 > 0, \quad \forall i = 1, 2, 3 \quad j = 1, 2$$

$$\begin{bmatrix} Y_1 + \rho\varepsilon_1 I + \beta\varepsilon_2 I & -XC & Z_1 & 0 & P - \frac{\varepsilon_1 I}{2} + \frac{\alpha\varepsilon_2 I}{2} & h_1 A^T P & h_{12} A^T P \\ * & -\Lambda_1 & Z_2 & Z_2 & 0 & -h_1 C^T X^T & -h_{12} C^T X^T \\ * & * & -\Lambda_2 & 0 & 0 & 0 & 0 \\ * & * & * & -\Lambda_3 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & h_1 P & h_{12} P \\ * & * & * & * & * & -T_1 & 0 \\ * & * & * & * & * & * & -T_2 \end{bmatrix} < 0 \tag{4.2.21}$$

are satisfied, where $T_1 = PZ_1^{-1}P$ and $T_2 = PZ_2^{-1}P$. Then, there exists a Luenberger-type observer (4.2.1) such that the state estimation error e asymptotically converges to the origin.

Proof. Employing the congruence transform using $\text{diag}(I, I, I, I, I, PZ_1^{-1}, PZ_2^{-1})$ to the inequality (4.2.7) and defining $X = PL$ and $T_i = PZ_i^{-1}P$ for $i = 1, 2$, we obtain LMI (4.2.21). This completes the proof of Theorem 4. □

Remark 4.1. Note that the observer conditions in Theorem 4.2 for delay-range-dependent systems are difficult to convert into LMIs as the constraints include nonlinear terms $diag(-T_1, -T_2)$, where $T_1 = PZ_1^{-1}P$ and $T_2 = PZ_2^{-1}P$. To solve this problem, the cone complementary linearization technique has been adopted in [34] and [23]. But this certainly cause massive computations and extra time due to their iterative nature. Hence, we would provide an alternative algorithm which's easily tractable and computable to address the complexity of nonlinearity through the LMIs in the following section.

4.3 One-sided Lipschitz nonlinear system subject to state time-varying delays

Consider a class of one-sided Lipschitz nonlinear dynamical systems with time-varying output delays, given by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau) + f(x,u), \\ y(t) = Cx(t-\tau), \end{cases} \tag{4.3.1}$$

where $x \in \mathbb{R}^n$ $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are the state vector, the control input, the output of the system, respectively. The linear constant matrices of the dynamical system are represented by $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$, and the nonlinear function is denoted by $f(x,u) \in \mathbb{R}^n$. The system given by (4.3.1) is assumed to be an observable system. A continuous time-varying differentiable function τ refers to the time delay at the output, satisfying

$$\begin{aligned} &0 \le h_1 \le \tau \le h_2, \\ &\dot{\tau} \le \mu \end{aligned} \tag{4.3.2}$$

The function $f(x,u)$ is assumed to the one-sided Lipschitz nonlinearities owing to the equation (2.5.3). Another concept employed for the observer design is quadratic inner-boundedness condition like (2.5.4).

We design the following observer for the time delay one-sided Lipschitz nonlinear system (4.3.1)

$$\begin{cases} \dot{z}(t) = Fz(t) + Ez(t-\tau) + Gy(t) + Tf(K^{+}w,u), \\ w(t) = z(t) + Ny(t), \end{cases} \tag{4.3.3}$$

where $z(t) \in \mathbb{R}^r, 0 < r \le n$ representing the state vector of the observer and $w(t)$ denotes the estimate of $Kx(t)$. K^+ denotes the generalised inverse of K. $K \in \mathbb{R}^{r\times n}$ are known constant matrix. $F \in \mathbb{R}^{r\times r}$, $E \in \mathbb{R}^{r\times r}$, $G \in \mathbb{R}^{r\times p}$, $T \in \mathbb{R}^{r\times n}$ and $N \in \mathbb{R}^{r\times p}$ are unknown to be determined. Let the error be

$$e(t) = w(t) - Kx(t), \tag{4.3.4}$$

Then the dynamics of the error would be

$$\dot{e}(t) = \dot{z}(t) - (K - NC)\dot{x}(t), \tag{4.3.5}$$

and plug system (4.3.1) and (4.3.3) into (4.3.5) to obtain

$$\begin{aligned} \dot{e}(t) \quad = \quad & Fe(t) + Ee(t-\tau) - [F(K-NC) + GC - (K-NC)A]x(t) \\ & +[E(K-NC) - (K-NC)B]x(t-\tau) + Tf(K^{+}w, u) - (K-NC)f(x,u). \end{aligned} \tag{4.3.6}$$

If the matrix F, E, G, T, N can be selected to satisfy the conditions as:

$$K - NC = T, \tag{4.3.7}$$

$$FT + GC = TA, \tag{4.3.8}$$

$$ET - TB = 0, \tag{4.3.9}$$

then the system (4.3.5) becomes

$$\dot{e}(t) = Fe(t) + Ee(t-\tau)T\Delta f. \tag{4.3.10}$$

in which $\Delta f = f(K^{+}w, u) - (K - NC)f(x, u)$.

Rewrite (4.3.7) and (4.3.8) as

$$\begin{bmatrix} N & T \end{bmatrix} \begin{bmatrix} C \\ I_n \end{bmatrix} = K, \tag{4.3.11}$$

$$\begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} T \\ C \end{bmatrix} = TA. \tag{4.3.12}$$

In equation (4.3.11), the solution for N and T exist, because

$$rank \begin{bmatrix} C \\ I_n \\ K \end{bmatrix} = rank \begin{bmatrix} C \\ I_n \end{bmatrix} = n. \tag{4.3.13}$$

The equation (4.3.12) has a solution of F, G, if and only if

$$rank \begin{bmatrix} T \\ C \\ TA \end{bmatrix} = rank \begin{bmatrix} T \\ C \end{bmatrix} = n. \tag{4.3.14}$$

Theorem 4.3. *Consider the one-sided Lipschitz nonlinear system (4.3.1) satisfying the time-delay bounds given by (4.3.2), the one-sided Lipschitz condition (2.5.3), and the quadratic inner-boundedness criterion (2.5.4). The observer estimation error (4.3.4) is asymptotically stable, if there exist symmetric matrices X, $P \in \mathbb{R}^{n\times n}$, $Q_i \in \mathbb{R}^{n\times n}$ and $Z_j \in \mathbb{R}^{n\times n}$ for $i = 1, 2, 3$ and $j = 1, 2$, and scalars ε_1 and ε_2 as well as $N, E, T \in$ such that (4.3.7) - (4.3.9) and (4.3.14) hold, given a full column rank matrix K and are satisfied and the following matrix inequality is feasible:*

$$P > 0, \quad Q_i > 0, \quad Z_j > 0, \quad \varepsilon_1 > 0 \quad \varepsilon_2 > 0, \quad \forall i = 1, 2, 3 \quad j = 1, 2 \tag{4.3.15}$$

$$\begin{bmatrix} Y_1 + \rho\varepsilon_1\Gamma^T\Gamma + \beta\varepsilon_2\Gamma^T\Gamma & PE & Z_1 & 0 & P - \frac{\varepsilon_1\Gamma^T}{2} + \frac{\alpha\varepsilon_2\Gamma^T}{2} & h_1(M_1 + XN_1)^T Z_1 & h_{12}(M_1 + XN_1)^T Z_2 \\ * & -\Lambda_1 & Z_2 & Z_2 & 0 & h_1 E^T Z_1 & h_{12} E^T Z_2 \\ * & * & -\Lambda_2 & 0 & 0 & 0 & 0 \\ * & * & * & -\Lambda_3 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & h_1 Z_1 & h_{12} Z_2 \\ * & * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & * & -Z_2 \end{bmatrix} < 0, \tag{4.3.16}$$

$$\begin{aligned} Y_1 &= P(M_1 + XN_1) + (M_1 + XN_1)^T P + \sum_{i=1}^{3} Q_i - Z_1, \\ \Lambda_1 &= (1 - \mu)Q_3 + 2Z_2, \\ \Lambda_2 &= Q_1 + Z_1 + Z_2, \\ \Lambda_3 &= Q_2 + Z_2, \\ h_{12} &= h_2 - h_1. \\ \Gamma &= W \begin{bmatrix} I_r & 0 \end{bmatrix}^T Y, \quad W \in \mathbb{R}^{n\times n}, \quad Y \in \mathbb{R}^{r\times r} \end{aligned} \tag{4.3.17}$$

Proof. As shown in Theorem 4.2 Define an LK functional candidate as

$$\begin{aligned} V(e,t) &= e^T Pe + \sum \int_{t-h_i}^{t} e^T(\alpha) Q_i e(\alpha) d\alpha + \int_{t-\tau}^{t} e^T(\alpha) Q_3 e(\alpha) d\alpha \\ &\quad + \int_{h_1}^{0}\int_{t+s}^{t} h_1 \dot{e}^T(\alpha) Z_1 \dot{e}(\alpha) d\alpha ds + \int_{h_2}^{h_1}\int_{t+s}^{t} h_{12} \dot{e}^T(\alpha) Z_2 \dot{e}(\alpha) d\alpha ds \end{aligned} \tag{4.3.18}$$

Acquiring the time derivative of eq(4.3.18) yields

$$\begin{aligned} \dot{V}(e,t) &\leq 2e^T P\dot{e} + \sum_{i=1}^{2} \{e^T Q_i e - e^T(t-h_i) Q_i e(t-h_i)\} + e^T Q_3 e \\ &\quad -(1-\mu) e^T(t-\tau) Q_3 e(t-\tau) + \dot{e}^T (h_1^2 Z_1 + h_{12}^2 Z_2)\dot{e} \\ &\quad - \int_{t-h_i}^{t} h_1 \dot{e}^T(\alpha) Z_1 \dot{e}(\alpha) d\alpha - \int_{t-h_2}^{t-h_1} h_{12} \dot{e}^T(\alpha) Z_2 \dot{e}(\alpha) d\alpha. \end{aligned} \tag{4.3.19}$$

Employing (4.3.10) and (4.3.19) and rearranging the terms, the upper bound on $\dot{V}(e,t)$ is obtained as

$$\begin{aligned} \dot{V}(e,t) &\leq 2e^T P[Fe + T\Delta f + Ee(t-\tau)] + \sum_{i=1}^{3} e^T Q_i e \\ &\quad - \sum_{i=1}^{2} e^T(t-h_i) Q_i e(t-h_i) - (1-\mu) e^T(t-\tau) Q_3 e(t-\tau) \\ &\quad - \int_{t-h_i}^{t} h_1 \dot{e}^T(\alpha) Z_1 \dot{e}(\alpha) d\alpha + [Fe + T\Delta f + Ee(t-\tau)]^T \\ &\quad \times (h_1^2 Z_1 + h_{12}^2 Z_2) \times [Fe + T\Delta f + Ee(t-\tau)] - \int_{t-h_2}^{t-h_1} h_{12} \dot{e}^T(\alpha) Z_2 \dot{e}(\alpha) d\alpha. \end{aligned} \tag{4.3.20}$$

Applying Jensen's inequality reveals

$$\begin{aligned} - \int_{t-h_i}^{t} h_1 \dot{e}^T(\alpha) Z_1 \dot{e}(\alpha) d\alpha &\leq - \left(\int_{t-h_i}^{t} \dot{e}(\alpha) d\alpha \right)^T Z_1 \left(\int_{t-h_i}^{t} \dot{e}(\alpha) d\alpha \right) \\ &\leq -(e(t) - e(t-h_1))^T Z_1 (e(t) - e(t-h_1)). \end{aligned} \tag{4.3.21}$$

Similarly, we have

$$
\begin{aligned}
& -\int_{t-h_2}^{t-h_1} h_{12}\dot{e}^T(\alpha)Z_2\dot{e}(\alpha)d\alpha \\
= & -\int_{t-h_2}^{t-\tau} h_{12}\dot{e}^T(\alpha)Z_2\dot{e}(\alpha)d\alpha - \int_{t-\tau}^{t-h_1} h_{12}\dot{e}^T(\alpha)Z_2\dot{e}(\alpha)d\alpha \\
\leq & -\left(\int_{t-h_2}^{t-\tau} \dot{e}(\alpha)d\alpha\right)^T Z_2 \left(\int_{t-h_2}^{t-\tau} \dot{e}(\alpha)d\alpha\right) - \left(\int_{t-\tau}^{t-h_1} \dot{e}(\alpha)d\alpha\right)^T Z_2 \left(\int_{t-\tau}^{t-h_1} \dot{e}(\alpha)d\alpha\right) \\
\leq & -(e(t-\tau)-e(t-h_2))^T Z_2(e(t-\tau)-e(t-h_2)) \\
& -(e(t-h_1)-e(t-\tau))^T Z_2(e(t-h_1)-e(t-\tau)). \qquad (4.3.22)
\end{aligned}
$$

Combining the results of (4.3.20)-(4.3.22), we have

$$
\begin{aligned}
& \dot{V}(e,t) \\
\leq & \; e^T\left[PF + F^TP + \sum_{i=1}^{3} Q_i + F^T(h_1^2 Z_1 + h_{12}^2 Z_2)F - Z_1\right] e \\
& +2e^T\left[PE + F^T(h_1^2 Z_1 + h_{12}^2 Z_2)E\right] e(t-\tau) + 2e^T Z_1 e(t-h_1) \\
& +e^T(t-\tau) \times \left[-(1-\mu)Q_3 - 2Z_2 + E^T(h_1^2 Z_1 + h_{12}^2 Z_2)E\right] \times e(t-\tau) \\
& 2e^T(t-\tau)Z_2 e(t-h_1) + 2e^T(t-\tau)Z_2 e(t-h_2) + e^T(t-h_1)(-Q_1 - Z_1 - Z_2)e(t-h_1) \\
& +e^T(t-h_2)(-Q_2 - Z_2)e(t-h_2) + 2e^T\left[P + F^T(h_1^2 Z_1 + h_{12}^2 Z_2)\right] T\Delta f \\
& +T\Delta f(h_1^2 Z_1 + h_{12}^2 Z_2)T\Delta f - 2e^T(t-\tau)[E^T(h_1^2 Z_1 + h_{12}^2)]T\Delta f, u).
\end{aligned}
\qquad (4.3.23)
$$

Simplifying (4.3.23),

$$
\dot{V}(e,t) \leq \Psi_1^T \Upsilon_1 \Psi_1, \qquad (4.3.24)
$$

where $\Psi_1^T = \begin{bmatrix} e^T & e^T(t-\tau) & e^T(t-h_1) & e^T(t-h_2) & T\Delta f \end{bmatrix}$,

$$
\Upsilon = \begin{bmatrix}
PF + F^TP + \sum_{i=1}^{3} Q_i - Z_1 & PE & Z_1 & 0 & PT \\
* & -(1-\mu)Q_3 - 2Z_2 & Z_2 & Z_2 & 0 \\
* & * & -Q_1 - Z_1 - Z_2 & 0 & 0 \\
* & * & * & -Q_2 - Z_2 & 0 \\
* & * & * & * & 0
\end{bmatrix}
$$

$$+\begin{bmatrix}F & E & 0 & 0 & I\end{bmatrix}^T (h_1^2 Z_1 + h_{12}^2 Z_2) \begin{bmatrix}F & E & 0 & 0 & I\end{bmatrix} \tag{4.3.25}$$

Given C and A, when T is chosen such that equation (4.3.14) is satisfied, there exits matrices F and G for (4.3.8). Denote

$$C_d = \begin{bmatrix}T \\ C\end{bmatrix}. \tag{4.3.26}$$

Let

$$C_d^{\dagger} = \begin{bmatrix}T_G & C_G\end{bmatrix} \tag{4.3.27}$$

be any generalized inverse of C_d satisfying $C_d C_d^{\dagger} C_d = C_d$, where $T_G \in \mathbb{R}^{n\times q}$ and $C_G \in \mathbb{R}^{n\times p}$. Then the general solution to (4.3.8) is given by

$$\begin{bmatrix}F & G\end{bmatrix} = (TA)C_d^{\dagger} + X(I_{r+p} - C_d C_d^{\dagger}), \tag{4.3.28}$$

for some $X \in \mathbb{R}^{r\times r+p}$.

$$\begin{cases} M_1 = (TA)C_d^{\dagger} \begin{bmatrix}I_r \\ 0\end{bmatrix}, \\ N_1 = (I_{r+p} - C_d C_d^{\dagger}) \begin{bmatrix}I_r \\ 0\end{bmatrix}, \\ M_2 = (TA)C_d^{\dagger} \begin{bmatrix}0 \\ I_p\end{bmatrix}, \\ N_2 = (I_{r+p} - C_d C_d^{\dagger}) \begin{bmatrix}0 \\ I_p\end{bmatrix}. \end{cases} \tag{4.3.29}$$

$$F = M_1 + XN_1, \quad G = M_2 + XN_2 \tag{4.3.30}$$

Since $\text{rank}(K) = r$, there's invertible matrices $Y \in \mathbb{R}^{r\times r}$ and $W \in \mathbb{R}^{n\times n}$ such that

$$K^{\dagger} = Y^{-1} \begin{bmatrix}I_r & 0\end{bmatrix} W^{-1}. \tag{4.3.31}$$

Notice that K has full column rank, its unique Penrose inverse would be

$$K^{\dagger} = K^T (KK^T)^{-1} \tag{4.3.32}$$

Thus we find

$$K(K^{\dagger}w - x) = w - Kx, \tag{4.3.33}$$

and

$$K^{\dagger}w - x = W \begin{bmatrix} I_r \\ 0 \end{bmatrix} Y(w - Kx). \tag{4.3.34}$$

The one-sided Lipschitz condition given by eq(2.5.3) suggests

$$\rho e^T Y^T \begin{bmatrix} I_r & 0 \end{bmatrix} W^T W \begin{bmatrix} I_r \\ 0 \end{bmatrix} Ye - \Delta f W \begin{bmatrix} I_r \\ 0 \end{bmatrix} Ye \geq 0.$$

With a positive scalar ε_1, the expression can be written in matrix form

$$\Psi_1^T \begin{bmatrix} \rho\varepsilon_1\Gamma^T\Gamma & 0 & 0 & 0 & \frac{-\varepsilon_1\Gamma^T}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{-\varepsilon_1\Gamma}{2} & 0 & 0 & 0 & 0 \end{bmatrix} \Psi_1 \geq 0, \tag{4.3.35}$$

where $\Gamma = W \begin{bmatrix} I_r \\ 0 \end{bmatrix} Y$.

The quadratic inner-boundedness condition eq(2.5.4) implies

$$\beta e^T \Gamma^T \Gamma e + \alpha e^T \Gamma^T \Delta f - \Delta f^T \Delta f \geq 0$$

which for a positive scalar ε_2 results in

$$\Psi_1^T \begin{bmatrix} \beta\varepsilon_2\Gamma^T\Gamma & 0 & 0 & 0 & \alpha\frac{\varepsilon_2\Gamma^T}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha\frac{\varepsilon_2\Gamma}{2} & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} \Psi_1 \geq 0. \tag{4.3.36}$$

Merging eq(4.3.29), eq(4.3.30) eq(4.3.25), eq(4.3.35), and eq(4.3.36), then using the S-procedure entails

$$
\begin{bmatrix}
Y_1 + (M_1 + XN_1)^T Y_4 (M_1 + XN_1) + \rho\varepsilon_1 \Gamma^T\Gamma + \beta\varepsilon_2\Gamma^T\Gamma & PE + FY_4E & Z_1 & 0 & Y_2 - \frac{\varepsilon_1\Gamma^T}{2} + \alpha\frac{\varepsilon_2\Gamma^T}{2} \\
* & -(1-\mu)Q_3 - 2Z_2 + Y_3E & Z_2 & Z_2 & Y_3 \\
* & * & -Q_1 - Z_1 - Z_2 & 0 & 0 \\
* & * & * & -Q_2 - Z_2 & 0 \\
* & * & * & * & Y_4 - \varepsilon_2 I
\end{bmatrix} < 0, \tag{4.3.37}
$$

$$
\begin{aligned}
Y_1 &= P(M_1 + XN_1) + (M_1 + XN_1)^T P + \sum_{i=1}^{3} Q_i - Z_1 \\
Y_2 &= P + (M_1 + XN_1)^T (h_1^2 Z_1 + h_{12}^2), \\
Y_3 &= E^T (h_1^2 Z_1 + h_{12}^2), \\
Y_4 &= (h_1^2 Z_1 + h_{12}^2).
\end{aligned} \tag{4.3.38}
$$

Applying the Schur complement produces (4.3.16), which implies that $\dot{V}(e,t) \leq \Psi_1^T \Upsilon_1 \Psi_1 < 0$. That is,when the time t goes to the infinity, the error $e(t)$ asymptotically converges to the origin and $w(t)$ asymptotically converges to $Kx(t)$. □

Theorem 4.4. *Given $K \in \mathbb{R}^{r\times n}$ and $C \in \mathbb{R}^{p\times n}$, if there exits matrix $Z \in \mathbb{R}^{r\times(r+p)}$ so that $T = J_2 + ZF_2$ is of full column rank, then there always exist matrix parameters N and T satisfying (4.3.7) and (4.3.9).*

Proof. Denote

$$
S_d = \begin{bmatrix} C \\ I_n \end{bmatrix}. \tag{4.3.39}
$$

So eq(4.3.13) becomes

$$
rank\begin{bmatrix} S_d \end{bmatrix} = rank\begin{bmatrix} S_d \\ K \end{bmatrix} = n. \tag{4.3.40}
$$

Then the general solution to (4.3.7) is given by

$$\begin{bmatrix} N & T \end{bmatrix} = KS_d^{\dagger} + Z(I_{n+p} - S_d S_d^{\dagger}). \tag{4.3.41}$$

$$\begin{cases} J_1 = (TC_d^{\dagger}) \begin{bmatrix} I_p \\ 0 \end{bmatrix}, \\ F_1 = (I_{p+n} - C_d C_d^{\dagger}) \begin{bmatrix} I_p \\ 0 \end{bmatrix}, \\ J_2 = (TC_d^{\dagger}) \begin{bmatrix} 0 \\ I_n \end{bmatrix}, \\ F_2 = (I_{p+n} - C_d C_d^{\dagger}) \begin{bmatrix} 0 \\ I_p \end{bmatrix}. \end{cases} \tag{4.3.42}$$

Thus,

$$N = J_1 + ZF_1, \tag{4.3.43}$$

$$T = J_2 + ZF_2. \tag{4.3.44}$$

With $Z \in \mathbb{R}^{r \times r+p}$ such that

$$rank(T) = rank(J_2 + ZF_2) = r. \tag{4.3.45}$$

Hence we can find the unique Penrose inverse of T as $T^{\dagger} = T^T(TT^T)^{-1}$.

From (4.3.9), then

$$E = TBT^{\dagger}. \tag{4.3.46}$$

□

Remark 4.2. From Theorem 4.3 and 4.4, a computational algorithm to design delay-range-dependent observer (4.3.3) is summarised as follows.

Step1: Given the matrix K, compute J_1, J_2, F_1, F_2 according to (4.3.42) and obtain (4.3.43), (4.3.44) in which Z is arbitrarily chosen with the dimension of $\mathbb{R}^{r \times (r+p)}$.

Step 2: If (4.3.14) holds with the chosen Z in Step 1, continue to the next. Otherwise, return the last step to adjust Z.

Step 3: If (4.3.45) holds with the chosen Z in Step 1, continue to the next. Otherwise, return the step 1 to adjust Z.

Step 4: Calculate E by 4.3.44 and 4.3.46. Compute M_1, M_2, N_1, N_2 by using 4.3.29.

Step 5: Solve the LMI in Theorem 4.3 by using the LMI toolbox.

Step 6: From Step 5, obtain F, G through 4.3.30.

4.4 Numerical Example

For a one-sided Lipschitz nonlinear system like (4.3.1) satisfying the time-delay bounds given by (4.3.2), the one-sided Lipschitz condition (2.5.3), and the quadratic inner-boundedness criterion (2.5.4), we assume

$$A = \begin{bmatrix} -10 & 1 & 1 & -10 \\ 10 & -10 & 10 & 1 \\ 1 & -20 & 12 & 10 \\ 10 & 20 & -10 & -50 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 1 & 21.6sin(t) & 0 & -x_4 \end{bmatrix}^T,$$

For the parameters in the one-sided Lipschitz condition (2.5.3), and the quadratic inner-boundedness criterion (2.5.4), we assume $\rho = -0.5, \quad \beta = 10, \quad \alpha = 9$.

We set $K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, so the observer (4.3.3) is full-order observer. Thus, assuming $Z = \begin{bmatrix} 1 & -0.5 & 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & -0.8 & 0 & 1 & 1 \end{bmatrix}$, we can design the observer according to Theorem 4.3 and Theorem 4.4. Following the algorithm in Theorem 4.3 and Theorem 4.4, we can get feasible solution for LMI (4.3.16) and parameters for observer (4.3.3) as below

$$\varepsilon_1 = 49.4272, \quad \varepsilon_2 = 4.6958$$

$$Z_1 = \begin{bmatrix} 0.6423 & 0.1029 & -0.5388 & -0.1938 \\ 0.1029 & 1.1286 & 0.1573 & -0.0889 \\ -0.5388 & 0.1573 & 0.5437 & 0.1682 \\ -0.1938 & -0.0889 & 0.1682 & 0.0781 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0.0856 & 0.0146 & -0.0723 & -0.0262 \\ 0.0146 & 0.1526 & 0.0207 & -0.0122 \\ -0.0723 & 0.0207 & 0.0723 & 0.0224 \\ -0.0262 & -0.0122 & 0.0224 & 0.0101 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 6.7180 & -0.0523 & 0.2160 & 0.0812 \\ -0.0523 & 6.4905 & -0.0608 & 0.0408 \\ 0.2160 & -0.0608 & 4.0179 & 0.4450 \\ 0.0812 & 0.0408 & 0.4450 & 7.1261 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 6.9437 & -0.0072 & 0.0341 & 0.0124 \\ -0.0072 & 6.9110 & -0.0097 & 0.0059 \\ 0.0341 & -0.0097 & 4.2026 & 0.5047 \\ 0.0124 & 0.0059 & 0.5047 & 7.1568 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 12.5740 & -0.5559 & -2.8055 & -0.4546 \\ -0.5559 & 11.1365 & 0.8006 & -0.3130 \\ -2.8055 & 0.8006 & 6.1847 & 0.9793 \\ -0.4546 & -0.3130 & 0.9793 & 7.3306 \end{bmatrix}, \quad P = \begin{bmatrix} 10.2696 & -1.8395 & -3.8555 & -0.1397 \\ -1.8395 & 4.3872 & 1.0602 & -0.1497 \\ -3.8555 & 1.0602 & 1.7254 & 0.2742 \\ -0.1397 & -0.1497 & 0.2742 & 1.0328 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} -196.7007 & -128.9196 & 12.5000 & 1.2500 \\ 114.3143 & 13.0423 & -4.5000 & -5.5000 \\ -574.7510 & -294.5811 & 16.5000 & 15.5000 \\ 173.0819 & 66.5234 & -10.9000 & -41.0000 \end{bmatrix}$$

$$G = \begin{bmatrix} -93.6377 & 105.1378 \\ -21.6205 & 182.1033 \\ -213.0542 & 495.4939 \\ 18.9773 & -196.7834 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & -1.25 \\ -0.5 & 0.5 \\ 0.5 & -0.5 \\ 0.9 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 1.25 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 \\ -0.5 & 0.5 & 1 & 0 \\ -0.9 & 0 & 0 & 1 \end{bmatrix}$$

.

4.5 Summary

A delay-range-dependent approach to the nonlinear observer-design dilemma for nonlinear systems subject to delayed output measurements and states was extended in this section. By application of Jensens inequality, LK functional, LMI tools, and appropriate matrix transformations, the delay-range-dependent conditions for observer synthesis of one-sided Lipschitz nonlinear systems with time-varying output delays, were derived. But the conditions converting into available LMIs require the cone complementary linearization algorithm at the cost of extra time and computation complexity. For this reason, we come up with more flexible and easily solvable algorithm to the nonlinear observer-design construction. The resultant observer-synthesis approach can be also applied and generalised to the estimation of the states of industrial nonlinear systems with fast time-varying delays in the system dynamics and outer disturbances.

Chapter 5

Controller and observer design for one-sided Lipschitz nonlinear time-varying system

5.1 Introduction

As seen in the last chapter, several observer-design problems for one-sided Lipschitz nonlinear systems have been investigated. The basic state observer-design scheme relies on the Lyapunov function for obtainment of simple linear matrix inequality (LMI) conditions for asymptotic stability of state estimation error was carried out.

Through this approach, the analysis and deduction problem in a unified LMI framework, which provides the condition for existence of a nonlinear state observer, is addressed by incorporating the one-sided Lipschitz the concept of quadratic inner-boundedness.

It should be noted that most of the above-mentioned references are focused on the same type of nonlinear system separated into two parts: time-invariant linearity and the nonlinearity. The influence of the nonlinearity is interpreted by the one-side Lipschitz condition. Then the system as a whole still complies with the one-side Lipschitz condition. One more challenging problem arises when the linear part becomes time variant. This would result in a more general system as it may not be one-side Lipschitz any more. Thus in this chapter, for the generalised system, based on Lyapunov stability theory, we study sufficient conditions for the existence of observers and resolve observer design problem with the Luenberger-type observer through the LMIs conditions and Riccati-type equations.

5.2 Preliminaries

Consider a class of nonlinear time-varying dynamical systems described by

$$\begin{cases} \dot{x} = A(t)x + f(x,u) \\ y = Cx \end{cases} \tag{5.2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output and $C \in \mathbb{R}^{p\times n}$. The elements $a_{jk}(t)$ of $A(t)$ $j,k = 1,2,...n$ are piecewise continuous functions with respect to time t and $f(x,u)$ is nolinear function satisfying the one-sided Lipschitz condition eq(2.5.3)and quadratic inner-bounded condition eq(2.5.4).

Hereby we set up a full-order Luenberger-like state observer with the time-varying gain matrix design for systems (5.2.1):

$$\dot{\tilde{x}} = A(t)\tilde{x} + f(\tilde{x},u) + L(t)(y - C\tilde{x}) \tag{5.2.2}$$

The time-varying matrix $A(t)$ can be decomposed as:

$$A(t) = A_0 + \sum_{i=1}^{m} a_i(t)A_i = A_0 + \sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)]A_i \tag{5.2.3}$$

in which matrices A_i are constant, the scalar coefficients $a_i^+ > 0$ and $a_i^- < 0$ for any $t > 0$.

The Luenberger-like gain matrix would be:

$$L(t) = L_0 + \sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)]L_i, \tag{5.2.4}$$

where L_i is constant matrix with compatible dimensions.

The error dynamics is given by $e := x - \tilde{x}$,

$$\dot{e} = [\bar{A}_0 + \sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)]\bar{A}_i]e + f - \tilde{f}, \tag{5.2.5}$$

where $\bar{A}_0 = A_0 - L_0C$, $\bar{A}_i := A_i - L_iC, i = 1,\cdots,m$ $f := f(x,u)$ and $\tilde{f} := f(\tilde{x},u)$.

For error dynamics (5.2.5), consider the Lyapunov function $V(t) = e^T(t)Pe(t)$. Then

$$\dot{V}(t) = e^T(t)\left\{[\bar{A}_0 + \sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)]\bar{A}_i]P + P[\bar{A}_0 + \sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)]\bar{A}_i]\right\}e(t) + 2e^T(t)P(f - \tilde{f}) \tag{5.2.6}$$

5.3 Full-order observer design

In this section, we first formulate conditions for the existence of observer design in 5.2.2 of the nonlinear time-varying system 5.2.1 and the following theorem will present them.

We now construct a new scalar function, called the Hamiltonian, is defined as

$$H = \dot{V} + \varepsilon_1 C_1 + \varepsilon_2 C_2 + \varepsilon_3 C_3 + C_4 \tag{5.3.1}$$

where $\varepsilon_1, \varepsilon_1$ and ε_3 are small positive integer and C_1, C_2 and C_3 are the cost functions given by (obtained by squaring and rearranging equation (2.5.3 & 2.5.4))

$$C_1 = \sum_{i=1}^{m}(a_i^+(t) - a_i^-(t))[\rho e(t)^T e(t) - (f - \tilde{f})^T e(t)] \tag{5.3.2}$$

$$C_2 = \sum_{i=1}^{m}(a_i^+(t) - a_i^-(t))[\beta e(t)^T e(t) - (f - \tilde{f})^T (f - \tilde{f}) + \gamma e(t)^T (f - \tilde{f})] \tag{5.3.3}$$

$$C_3 = [\rho e(t)^T e(t) - (f - \tilde{f})^T e(t)] + [\beta e(t)^T e(t) - (f - \tilde{f})^T (f - \tilde{f}) + \gamma e(t)^T (f - \tilde{f})] \tag{5.3.4}$$

$$C_4 = \sum_{i=1}^{m}(a_i^+(t) - a_i^-(t))[(f - \tilde{f})^T P (f - \tilde{f})] \tag{5.3.5}$$

such that

$$\dot{V} = H - \varepsilon_1 C_1 - \varepsilon_2 C_2 - \varepsilon_3 C_3 - C_4 \tag{5.3.6}$$

where the terms $\varepsilon_1 C_1, \varepsilon_2 C_2, \varepsilon_3 C_3$ and C_4 are always non-negative. The S-Procedure implies that if it can be proved H is negative definite, then $\dot{V} < 0$ is ensured.

Theorem 5.1. *Consider the nonlinear system 5.2.1 and the state observer holds the form of 5.2.2. The error dynamics 5.2.5 is asymptotically stable if there exists constants* $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$*and matrices* $P > 0$, R *such that the following LMIs are feasible* *for* $1 \le i \le m$*:*

$$\Xi_i < 0, \quad \Omega_i < 0, \quad \Theta < 0, \tag{5.3.7}$$

where

$$\Xi_i = \begin{bmatrix} {A_i}^T P + PA_i - C^T R_i - R_i^T C + (\varepsilon_1 \rho + \varepsilon_2 \beta) I & \dfrac{\gamma\varepsilon_2 - \varepsilon_1}{2} I \\ * & P - \varepsilon_2 I \end{bmatrix},$$

$$\Omega_i = \begin{bmatrix} -({A_i}^T P + PA_i - C^T R_i - R_i^T C) + (\varepsilon_1 \rho + \varepsilon_2 \beta) I & \dfrac{\gamma\varepsilon_2 - \varepsilon_1}{2} I \\ * & P - \varepsilon_2 I \end{bmatrix},$$

$$\Theta = \begin{bmatrix} ({A_0}^T P + PA_0 - C^T R_0 - R_0^T C) + \varepsilon_3 (\rho + \beta) I & P + \dfrac{\gamma\varepsilon_3 - \varepsilon_3}{2} I \\ * & -\varepsilon_3 I \end{bmatrix}.$$

The gain L_i, L_0 *can be selected as* $L_i = P^{-1} R_i^T$ *and* $L_0 = P^{-1} R_0^T$

Proof. By writing in matrix form, the equality (5.2.6) is equivalent to

$$\dot{V}(t) = \begin{bmatrix} e(t) \\ f - \tilde{f} \end{bmatrix}^T \begin{bmatrix} \sum_{i=1}^m [({\bar{A}_i}^T P + P\bar{A}_i) a_i^+(t) + ({\bar{A}_i}^T P + P\bar{A}_i) a_i^-(t)] & P \\ * & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ f - \tilde{f} \end{bmatrix} \tag{5.3.8}$$

From (5.3.2 to 5.3.5), $\varepsilon_1 C_1$ is equivalent to

$$\sum_{i=1}^m a_i^+(t) \varepsilon_1 \begin{bmatrix} e(t) \\ f - \tilde{f} \end{bmatrix}^T \begin{bmatrix} \rho I & -\dfrac{1}{2} \\ * & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ f - \tilde{f} \end{bmatrix} \ge 0, \tag{5.3.9}$$

and

$$-\sum_{i=1}^m a_i^-(t) \varepsilon_1 \begin{bmatrix} e(t) \\ f - \tilde{f} \end{bmatrix}^T \begin{bmatrix} \rho I & -\dfrac{1}{2} \\ * & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ f - \tilde{f} \end{bmatrix} \ge 0. \tag{5.3.10}$$

$\varepsilon_2 C_2$ is equivalent to

$$\sum_{i=1}^m a_i^+(t) \varepsilon_2 \begin{bmatrix} e(t) \\ f - \tilde{f} \end{bmatrix}^T \begin{bmatrix} \beta I & \dfrac{\gamma}{2} I \\ * & -I \end{bmatrix} \begin{bmatrix} e(t) \\ f - \tilde{f} \end{bmatrix} \ge 0 \tag{5.3.11}$$

and

$$-\sum_{i=1}^{m} a_i^-(t)\varepsilon_2 \begin{bmatrix} e(t) \\ f-\tilde{f} \end{bmatrix}^T \begin{bmatrix} \beta I & \frac{\gamma}{2} \\ * & I \end{bmatrix} \begin{bmatrix} e(t) \\ f-\tilde{f} \end{bmatrix} \geq 0. \tag{5.3.12}$$

Then $\varepsilon_3 C_3$ equals to

$$\begin{bmatrix} e(t) \\ f-\tilde{f} \end{bmatrix}^T \begin{bmatrix} \varepsilon_3(\rho+\beta)I & \frac{\gamma\varepsilon_3-\varepsilon_3}{2}I \\ * & -\varepsilon_3 I \end{bmatrix} \begin{bmatrix} e(t) \\ f-\tilde{f} \end{bmatrix} \geq 0.$$

And C_4 equals to

$$\sum_{i=1}^{m}(a_i^+(t)-a_i^-(t)) \begin{bmatrix} e(t) \\ f-\tilde{f} \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ * & P \end{bmatrix} \begin{bmatrix} e(t) \\ f-\tilde{f} \end{bmatrix} \geq 0.$$

Hence, the Hamiltonian function H yields

$$H = \begin{bmatrix} e(t) \\ f-\tilde{f} \end{bmatrix}^T \left\{ \sum_{i=1}^{m}[a_i^+(t)\Xi_i - a_i^-(t)\Omega_i] + \Theta \right\} \begin{bmatrix} e(t) \\ f-\tilde{f} \end{bmatrix} \tag{5.3.13}$$

in which

$$\Theta = \begin{bmatrix} {A_0}^T P + PA_0 - C^T L_0^T P - PL_0 C + \varepsilon_3(\rho+\beta)I & P + \frac{\gamma\varepsilon_3-\varepsilon_3}{2}I \\ * & -\varepsilon_3 I \end{bmatrix}, \tag{5.3.14}$$

$$\Xi_i = \begin{bmatrix} {A_i}^T P + PA_i - C^T L_i^T P - PL_i C + (\varepsilon_1\rho+\varepsilon_2\beta)I & \frac{\gamma\varepsilon_2-\varepsilon_1}{2}I \\ * & -\varepsilon_2 I \end{bmatrix}, \tag{5.3.15}$$

$$\Omega_i = \begin{bmatrix} -({A_i}^T P + PA_i - C^T L_i^T P - PL_i C) + (\varepsilon_1\rho+\varepsilon_2\beta)I & \frac{\gamma\varepsilon_2-\varepsilon_1}{2}I \\ * & -\varepsilon_2 I \end{bmatrix}. \tag{5.3.16}$$

In order to gain $H < 0$, it's sufficient to have that $\Xi_i < 0, \Omega_i < 0$ and $\Theta < 0$. But the inequalities seem to be nonconvex since each contains the product of two variables P and L_i. Thus a simple change of variables separating L_i from P needs to be done. Let $R_i = L_i^T P$. So the inequalities (5.3.15) (5.3.16) becomes

$$\Xi_i = \begin{bmatrix} {A_i}^T P + PA_i - C^T R_i - R_i^T C + (\varepsilon_1\rho+\varepsilon_2\beta)I & \frac{\gamma\varepsilon_2-\varepsilon_1}{2}I \\ * & P-\varepsilon_2 I \end{bmatrix} < 0, \tag{5.3.17}$$

and

$$\Omega_i = \begin{bmatrix} -(A_i{}^T P + PA_i - C^T R_i - R_i^T C) + (\varepsilon_1\rho + \varepsilon_2\beta)I & \frac{\gamma\varepsilon_2 - \varepsilon_1}{2} I \\ * & P - \varepsilon_2 I \end{bmatrix} < 0. \quad (5.3.18)$$

Searching for P and L_i satisfying the above inequalities 5.3.17 and 5.3.18 is equivalent to that of P and R_i for 5.3.15 and 5.3.16. Once a feasible set of P and R_i is found, L_i can be computed as $L_i = P^{-1}R_i$. Due to that P is invertible ($P > 0$) and that there exists a one to one mapping from R_i to L_i for a given P. Similarly, let $L_0 = P^{-1}R_0$. Then

$$\Theta = \begin{bmatrix} A_0{}^T P + PA_0 - C^T R_0 - R_0^T C + \varepsilon_3(\rho + \beta)I & P + \frac{\gamma\varepsilon_3 - \varepsilon_3}{2} I \\ * & -\varepsilon_3 I \end{bmatrix} < 0. \quad (5.3.19)$$

The proof is completed.

□

Remark 5.1. This theorem investigate the full-order observer design for one-sided Lipschitz nonlinear time-varying system by LMIs. A Riccati-type sufficient condition is proposed in the corollary below.

Corollary 2. *Consider the nonlinear system 5.2.1 and the state observer holds the form of 5.2.2. The error dynamics 5.2.5 is asymptotically stable if there exists constants* $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$ *and* $\sigma_i > 0$, $\sigma_0 > 0$ *such that the following Riccati-type inequality has a symmetric positive definite solution* P*:*

$$A_i{}^T P + PA_i - \sigma_i C^T C + (\varepsilon_1\rho + \varepsilon_2\beta)I - \left(\frac{\gamma\varepsilon_2 - \varepsilon_1}{2}\right)(P - \varepsilon_2 I)^{-1}\left(\frac{\gamma\varepsilon_2 - \varepsilon_1}{2}\right) < 0, \quad (5.3.20)$$

$$-[A_i{}^T P + PA_i - \sigma_i C^T C] + (\varepsilon_1\rho + \varepsilon_2\beta)I - \left(\frac{\gamma\varepsilon_2 - \varepsilon_1}{2}\right)(P - \varepsilon_2 I)^{-1}\left(\frac{\gamma\varepsilon_2 - \varepsilon_1}{2}\right) < 0, \quad (5.3.21)$$

$$A_0{}^T P + PA_0 - \sigma_0 C^T C + \varepsilon_3(\rho + \beta)I + \frac{1}{\varepsilon_3}\left[P + \frac{\gamma\varepsilon_3 - \varepsilon_3}{2} I\right]^2 < 0. \quad (5.3.22)$$

The observer gain can then be chosen as $L_i = \frac{\sigma_i}{2} P^{-1} C^T$ *and* $L_0 = P^{-1} R_0^T$.

Proof. As shown in Theorem 2, $\Xi_i < 0$ and $\Omega_i > 0$ with $L_i = P^{-1}R_i^T$ and $L_0 = P^{-1}R_0^T$.

Now replacing with $L_i = \dfrac{\sigma}{2} P^{-1} C^T$ to Ξ_i, Ω_i and $L_0 = P^{-1} R_0^T$ thus

$$\Xi_i = \begin{bmatrix} {A_i}^T P + P A_i - \sigma_i C^T C + (\varepsilon_1 \rho + \varepsilon_2 \beta) I & P + \dfrac{\gamma \varepsilon_2 - \varepsilon_1}{2} I \\ * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{5.3.23}$$

$$\Omega_i = \begin{bmatrix} -({A_i}^T P + P A_i - \sigma_i C^T C) + (\varepsilon_1 \rho + \varepsilon_2 \beta) I & P + \dfrac{\gamma \varepsilon_2 - \varepsilon_1}{2} I \\ * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{5.3.24}$$

$$\Theta = \begin{bmatrix} {A_0}^T P + P A_0 - \sigma_0 C^T C + \varepsilon_3 (\rho + \beta) I & P + \dfrac{\gamma \varepsilon_3 - \varepsilon_3}{2} I \\ * & -\varepsilon_3 I \end{bmatrix}. \tag{5.3.25}$$

Via Schur Complement, the conditions (5.3.23, 5.3.24, 5.3.25) equal to inequalities (5.3.20, 5.3.21, 5.3.22). Then the proof is finished. □

5.4 Reduced-order observer design

This part will show that the very conditions under which the full-order observer exists would also guarantee the existence of a reduced-order observer. The state vector is partitioned into two sub-states: $x = \begin{bmatrix} x_1 \\ ... \\ x_2 \end{bmatrix}$ such that $x_1 = \tilde{x}_1 = y = Cx$ where $C = [I_p, 0]$. We then decompose A_i and P into block matrices like

$$A_i = \begin{bmatrix} A_{i_{11}} & A_{i_{12}} \\ A_{i_{21}} & A_{i_{22}} \end{bmatrix}, \qquad P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3, \end{bmatrix}, \tag{5.4.1}$$

where $A_{i_{11}}, P_1 \in \mathbb{R}^{p\times p}$ and $A_{i_{22}}, P_3 \in \mathbb{R}^{(n-p)\times(n-p)}$. The reduced-order can be design in the form of:

$$\begin{cases} \dot{z}_2 = \sum_{i=0}^{m} a_i(t)(A_{i_{22}} + LA_{i_{12}})\hat{z}_2 + \sum_{i=1}^{m} a_i(t)[L(A_{i_{11}} - A_{i_{12}}L) + A_{i_{21}} - A_{i_{22}}L]y \\ \qquad + \begin{pmatrix} L & I_{n-p} \end{pmatrix} f\left(\begin{pmatrix} y \\ \hat{z}_2 - Ly \end{pmatrix}, u\right) \\ \hat{z}_1 = \hat{x}_1 = y \\ \hat{x}_2 = \hat{z}_2 - Ly \end{cases} \tag{5.4.2}$$

where $L = P_3^{-1}P_2^T \in \mathbb{R}^{(n-p)\times p}, a_0(t) = 1$.

Theorem 5.2. *Let $C = [I_p, 0]$. If there exist $P > 0$ and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$ such that the inequalities* 5.3.20*,* 5.3.21*,* 5.3.22 *are satisfied, then (*5.4.2*) is a reduced-order observer for the system* (5.2.1)*.*

Proof. Let $\omega_i = (A_{i_{22}} + LA_{i_{12}})^T P_3 + P_3(A_{i_{22}} + LA_{i_{12}})$, $\omega_0 = (A_{0_{22}} + LA_{0_{12}})^T P_3 + P_3(A_{0_{22}} + LA_{0_{12}})$ where $L = P_3^{-1}P_2^T \in \mathbb{R}^{(n-p)\times p}, i = 1, \cdots, m.$

From Theorem 6.1, $(P - \varepsilon_2 I) < 0$. We can decompose the inverse of $(P - \varepsilon_2 I)$.

$$(P - \varepsilon_2 I)^{-1} = \begin{bmatrix} p_1 & p_2 \\ p_2^T & p_3 \end{bmatrix}.$$

The block in the intersection of the second row and the second column in 5.3.20, 5.3.21 and 5.3.22 are

$$\omega_i + (\varepsilon_1\rho + \varepsilon_2\beta)I_{n-p} - p_3^{-1}(\frac{\gamma\varepsilon_2 - \varepsilon_1}{2})^2 < 0, \tag{5.4.3}$$

$$-\omega_i + (\varepsilon_1\rho + \varepsilon_2\beta)I_{n-p} - p_3^{-1}(\frac{\gamma\varepsilon_2 - \varepsilon_1}{2})^2 < 0, \tag{5.4.4}$$

$$\omega_0 + \varepsilon_3(\rho + \beta)I_{n-p} + \frac{P_2^T P_2}{\varepsilon_3} + \frac{1}{\varepsilon_3}(P_3 + \frac{\gamma\varepsilon_3 - \varepsilon_3}{2}I_{n-p})^2 < 0. \tag{5.4.5}$$

Take a coordinate transformation of $z = Tx$, where $T = \begin{bmatrix} I_p & 0 \\ L & I_{n-p} \end{bmatrix}$. Let $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, where $z_1 = y \in \mathbb{R}^p$ and $z_2 \in \mathbb{R}^{n-p}$. Then, from (5.2.1), z_2 satisfies the following equation:

$$\dot{z}_2 = \sum_{i=0}^{m} a_i(t)(A_{i_{22}} + LA_{i_{12}})z_2 + \sum_{i=0}^{m} a_i(t)[L(A_{i_{11}} - A_{i_{12}}L) + A_{i_{21}} - A_{i_{22}}L]y + \begin{pmatrix} L & I_{n-p} \end{pmatrix} f\left(\begin{pmatrix} y \\ z_2 - Ly \end{pmatrix}, u\right). \quad (5.4.6)$$

Subtracting the first equation of (5.4.2) from (5.4.6), the error $\tilde{z}_2 = z_2 - \hat{z}_2$ is then governed by

$$\dot{\tilde{z}}_2 = \sum_{i=0}^{m} a_i(t)(A_{i_{22}} + \sum_{i=0}^{m} a_i(t)LA_{i_{12}})\tilde{z}_2 + \begin{pmatrix} L & I_{n-p} \end{pmatrix} \Delta f, \quad (5.4.7)$$

where

$$\Delta f = f\left(\begin{pmatrix} y \\ z_2 - Ly \end{pmatrix}, u\right) - f\left(\begin{pmatrix} y \\ \tilde{z}_2 - Ly \end{pmatrix}, u\right). \quad (5.4.8)$$

Consider the Lyapunov function candidate

$$V_2(t) = \tilde{z}_2^T P_3 \tilde{z}_2, \quad (5.4.9)$$

then its time derivative along the trajectories of (5.4.9) is

$$\begin{aligned} \dot{V}_2(t) &= \sum_{i=0}^{m} a_i(t)\tilde{z}_2^T \left[(A_{i_{22}} + LA_{i_{12}})^T P_3 + P_3(A_{i_{22}} + LA_{i_{12}})\right]\tilde{z}_2 + 2\tilde{z}_2^T P_3 \begin{pmatrix} L & I_{n-p} \end{pmatrix} \Delta f \\ &= \sum_{i=0}^{m} a_i(t)\tilde{z}_2^T \left[(A_{i_{22}} + LA_{i_{12}})^T P_3 + P_3(A_{i_{22}} + LA_{i_{12}})\right]\tilde{z}_2 + 2\tilde{z}_2^T \left(P_2^T, P_3\right) \begin{bmatrix} \Delta f_1 \\ \Delta f_2 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix}^T \begin{bmatrix} \omega_0 + \sum_{i=1}^{m} \omega_i[(a_i^+(t) + a_i^-(t))] & P_2^T & P_3 \\ P_2 & 0 & 0 \\ P_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix}, \end{aligned} \quad (5.4.10)$$

where $\Delta f_1 \in \mathbb{R}^p$, $\Delta f_2 \in \mathbb{R}^{n-p}$. Using the one-sided Lipschitz condition (2.5.3), we have

$$\left\langle \Delta f, \begin{bmatrix} 0 \\ \tilde{z}_2 \end{bmatrix} \right\rangle \le \rho \| \begin{bmatrix} 0 \\ \tilde{z}_2 \end{bmatrix} \|^2, \quad (5.4.11)$$

The above inequality implies that $\Delta f_2^T \tilde{z}_2 \leq \rho \tilde{z}_2^T \tilde{z}_2$. Therefore, for any positive scalar ε_1, we have

$$\sum_{i=1}^{m} a_i^{+}(t)\varepsilon_1 \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix}^T \begin{bmatrix} \rho I_{n-p} & 0 & -\frac{I_{n-p}}{2} \\ 0 & 0 & 0 \\ (-\frac{I_{n-p}}{2})^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix} \geq 0, \tag{5.4.12}$$

$$-\sum_{i=1}^{m} a_i^{-}(t)\varepsilon_1 \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix}^T \begin{bmatrix} \rho I_{n-p} & 0 & -\frac{I_{n-p}}{2} \\ 0 & 0 & 0 \\ (-\frac{I_{n-p}}{2})^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix} \geq 0. \tag{5.4.13}$$

On the other hand, from the condition (2.5.4) of quadratic inner-boundedness, we get

$$\Delta f^T \Delta f \leq \beta \| \begin{bmatrix} 0 \\ \tilde{z}_2 \end{bmatrix} \|^2 + \gamma \left\langle \begin{bmatrix} 0 \\ \tilde{z}_2 \end{bmatrix}, \Delta f \right\rangle, \tag{5.4.14}$$

which implies that

$$\Delta f_1^T \Delta f_1 + \Delta f_2^T \Delta f_2 \leq \beta \tilde{z}_2^T \tilde{z}_2 + \gamma \tilde{z}_2^T \Delta f_2, \tag{5.4.15}$$

Thus, for any positive scalar ε_2, we have

$$\sum_{i=1}^{m} a_i^{+}(t)\varepsilon_2 \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix}^T \begin{bmatrix} \beta I_{n-p} & 0 & \frac{\gamma I_{n-p}}{2} \\ 0 & -I_p & 0 \\ (\frac{\gamma I_{n-p}}{2})^T & 0 & -I_{n-p} \end{bmatrix} \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix} \geq 0. \tag{5.4.16}$$

$$-\sum_{i=1}^{m} a_i^{-}(t)\varepsilon_2 \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix}^T \begin{bmatrix} \beta I_{n-p} & 0 & \frac{\gamma I_{n-p}}{2} \\ 0 & -I_p & 0 \\ (\frac{\gamma I_{n-p}}{2})^T & 0 & -I_{n-p} \end{bmatrix} \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix} \geq 0. \tag{5.4.17}$$

Besides, to combine 5.4.11 and 5.4.14

$$\varepsilon_3 \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix}^T \begin{bmatrix} (\rho+\beta) I_{n-p} & 0 & \frac{(\gamma-1) I_{n-p}}{2} \\ 0 & -I_p & 0 \\ (\frac{(\gamma-1) I_{n-p}}{2})^T & 0 & -I_{n-p} \end{bmatrix} \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix} \geq 0. \tag{5.4.18}$$

Also we have

$$\Delta f_2^T(-p_3^{-1} + \varepsilon_2 I_{n-p})\Delta f_2 \geq 0, \tag{5.4.19}$$

$$\sum_{i=1}^{m}[a_i^+(t) - a_i^-(t)] \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -p_3^{-1} + \varepsilon_2 I_{n-p} \end{bmatrix} \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix} \geq 0. \tag{5.4.20}$$

Adding the left terms of (5.4.12), (5.4.13), (5.4.16) and (5.4.17) to the right-hand side of (5.4.10) yields

$$\dot{V}_2(t) \leq \varepsilon_2 \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix}^T \left[\sum_{i=1}^{m} a_i^+(t)\Xi_i - \sum_{i=1}^{m} a_i^-(t)\Omega_i + \Theta\right] \begin{bmatrix} \tilde{z}_2 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix}. \tag{5.4.21}$$

For $\dot{V}_2(t) \leq 0$, it suffices to have

$$\Xi_i = \begin{bmatrix} \omega_i + (\varepsilon_1\rho + \varepsilon_2\beta)I_{n-p} & 0 & \frac{\gamma\varepsilon_2 - \varepsilon_1}{2} I_{n-p} \\ 0 & -\varepsilon_2 I_p & 0 \\ \frac{\gamma\varepsilon_2 - \varepsilon_1}{2} I_{n-p} & 0 & -p_3^{-1} \end{bmatrix} < 0, \tag{5.4.22}$$

$$\Omega_i = \begin{bmatrix} -\omega_i + (\varepsilon_1\rho + \varepsilon_2\beta)I_{n-p} & 0 & \frac{\gamma\varepsilon_2 - \varepsilon_1}{2} I_{n-p} \\ 0 & -\varepsilon_2 I_p & 0 \\ \frac{\gamma\varepsilon_2 - \varepsilon_1}{2} I_{n-p} & 0 & -p_3^{-1} \end{bmatrix} < 0, \tag{5.4.23}$$

$$\Theta = \begin{bmatrix} \omega_0 + \varepsilon_3(\rho + \beta)I_{n-p} & P_2^T & P_3 + \frac{\gamma\varepsilon_3 - \varepsilon_3}{2} I_{n-p} \\ P_2 & -\varepsilon_3 I_p & 0 \\ P_3 + \frac{\gamma\varepsilon_3 - \varepsilon_3}{2} I_{n-p} & 0 & -\varepsilon_3 I_{n-p} \end{bmatrix} < 0. \tag{5.4.24}$$

By Schur complement lemma, the condition (5.4.22) - (5.4.24) are equivalent to $\Xi < 0$ and $\Omega < 0$. Therefore, according to the standard Lyapunov stability theory, the error dynamics (5.4.7) is asymptotically stable. This indicates that (5.4.2) is an reduced-order observer of system (5.2.1) with the dimension of $n - p$.

□

5.5 Controller Design

Consider a class of nonlinear time-varying systems described by

$$\begin{cases} \dot{x}(t) = A(t)x(t) + Bu(t) + f(t,x) \\ y(t) = Cx(t) \end{cases} \tag{5.5.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state vector, the control input, and the measured output of the system, respectively. The matrices $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ are the known constant matrices. The vector-valued function $f(t,x) : (\mathbb{R}, \mathbb{R}^n) \to \mathbb{R}^n$ represents the nonlinearity of the system. The system (5.5.1) is assumed to be controllable. Throughout this section, without loss of generality, we assume that $f(0) = 0$, which implies that the unforced system (i.e. $u(t) \equiv 0$) has the origin as an equilibrium point. $f(t,x)$ is also assumed to satisfy (2.5.3) and (2.5.4) in Definition 2.4, 2.5. The time-varying matrix $A(t)$ can be rewritten as:

$$A(t) = A_0 + \sum_{i=1}^{m} a_i(t)A_i = A_0 + \sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)]A_i \tag{5.5.2}$$

in which matrices $A_i \in \mathbb{R}^{n\times n}$ are constant, the scalar coefficients $a_i^+ > 0$ and $a_i^- < 0$ for $i = 1, \cdots, m$. The system is to be stabilized by a state feedback law of the form:

$$u(t) = K(t)x(t). \tag{5.5.3}$$

The time-varying controller gain $K(t)$ is chosen as:

$$K(t) = K_0 + \sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)]K_i. \tag{5.5.4}$$

Then the closed-loop system becomes

$$\dot{x}(t) = A_0 + K_0 + \sum_{1}^{m}[a_i^+(t) + a_i^-(t)](A_i + K_i)X(t) + f(t,x). \tag{5.5.5}$$

Theorem 5.3. *If there exist scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$ and matrices Y and*

$Q = Q^T > 0$ *such that the LMIs given by*

$$\left.\begin{array}{l} \begin{bmatrix} \eta_i - Q & Q \\ Q & -[(\varepsilon_1\rho + \varepsilon_2\beta)I + Q - (\frac{\gamma\varepsilon_2 - \varepsilon_1}{2})^2(P - \varepsilon_2 I)^{-1}]^{-1} \end{bmatrix} < 0 \\ \begin{bmatrix} -\eta_i - Q & Q \\ Q & -[(\varepsilon_1\rho + \varepsilon_2\beta)I + Q - (\frac{\gamma\varepsilon_2 - \varepsilon_1}{2})^2(P - \varepsilon_2 I)^{-1}]^{-1} \end{bmatrix} < 0 \\ \begin{bmatrix} \eta_0 + (\gamma - 1)Q + \frac{1}{\varepsilon_3}Q & Q \\ Q & -[\varepsilon_3(\rho + \beta)I + \frac{1}{\varepsilon_3}(\dfrac{\gamma\varepsilon_3 - \varepsilon_3}{2})^2 I]^{-1} \end{bmatrix} < 0 \end{array}\right\} \tag{5.5.6}$$

the term η_i *is given by* $QA_i^T + A_iQ + Y_i^TB^T + BY_i$ *and* η_0 *is given by* $QA_0^T + A_0Q + Y_0^TB^T + BY_0$. *Then the controller* $K_0 = Y_0Q^{-1}$ $K_i = Y_iQ^{-1}, i = 1, \cdots, m$ *stabilizes the nonlinear system given by (5.5.1).*

Proof. We again proceed by differentiating Lyapunov function $V = x^TPx$ and substituting it in the Hamiltonian expression

$$H = \dot{V} + \varepsilon_1 C_1 + \varepsilon_2 C_2 + \varepsilon_3 C_3, \tag{5.5.7}$$

$$C_1 = \sum_{i=1}^{m}(a_i^+(t) - a_i^-(t))[\rho x(t)^T x(t) - f^T x(t)], \tag{5.5.8}$$

$$C_2 = \sum_{i=1}^{m}(a_i^+(t) - a_i^-(t))[\beta^2 x(t)^T x(t) - f^T f + \gamma x(t)^T f], \tag{5.5.9}$$

$$C_3 = [\rho x(t)^T x(t) - f^T x(t)] + [\beta^2 x(t)^T x(t) - f^T f + \gamma x(t)^T f]. \tag{5.5.10}$$

And we obtain

$$H = \sum_{i=1}^{m} \begin{bmatrix} x \\ f \end{bmatrix}^T [a_i^+(t)\Pi_i - a_i^-(t)\Lambda_i + \Theta] \begin{bmatrix} x \\ f \end{bmatrix} \tag{5.5.11}$$

where

$$\left.\begin{aligned}
\Theta &= \begin{bmatrix} {A_0}^T P + P A_0 + K_0^T B^T P + PBK_0 + \varepsilon_3(\rho+\beta)I & P + \dfrac{\gamma\varepsilon_3 - \varepsilon_3}{2} I \\ * & -\varepsilon_3 I \end{bmatrix} \\
\Pi_i &= \begin{bmatrix} {A_i}^T P + P A_i + K_i^T B^T P + PBK_i + (\varepsilon_1\rho+\varepsilon_2\beta)I & \dfrac{\gamma\varepsilon_2 - \varepsilon_1}{2} I \\ * & P - \varepsilon_2 I \end{bmatrix} \\
\Lambda_i &= \begin{bmatrix} -({A_i}^T P + P A_i + K_i^T B^T P + PBK_i) + (\varepsilon_1\rho+\varepsilon_2\beta)I & \dfrac{\gamma\varepsilon_2 - \varepsilon_1}{2} I \\ * & P - \varepsilon_2 I \end{bmatrix}
\end{aligned}\right\} \tag{5.5.12}$$

Obviously for stability of the closed loop system (5.5.5) it is necessary for the inequalities (5.5.12) to be negative definite, which implies Θ, Π_i, Λ_i to be negative definite.

However, inequalities (5.5.12) are nonconvex, since they include the unknowns K_0, K_i and P. It is essential to convexify the inequalities in order to convert it into LMIs via a change of variables.

Substituting $P = Q^{-1}$ in (5.5.12), and pre- and post-multiplying by

$$\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \tag{5.5.13}$$

we get

$$\begin{bmatrix} \eta_i + Q(\varepsilon_1\rho+\varepsilon_2\beta)Q & \dfrac{\gamma\varepsilon_2 - \varepsilon_1}{2} Q \\ * & P - \varepsilon_2 I \end{bmatrix} < 0, \tag{5.5.14}$$

$$\begin{bmatrix} -\eta_i + Q(\varepsilon_1\rho+\varepsilon_2\beta)Q & \dfrac{\gamma\varepsilon_2 - \varepsilon_1}{2} Q \\ * & P - \varepsilon_2 I \end{bmatrix} < 0, \tag{5.5.15}$$

$$\begin{bmatrix} \eta_0 + Q\varepsilon_3(\rho+\beta)Q & I + \dfrac{\gamma\varepsilon_3 - \varepsilon_3}{2} Q \\ * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{5.5.16}$$

where η_i is $QA_i^T + A_iQ + QK_i^TB^T + BK_iQ$ and η_0 is given by $QA_0^T + A_0Q + QK_0^TB^T + BK_0Q$.

Taking Schurs complement of (5.5.14) – (5.5.16), we

$$\eta_i + Q(\varepsilon_1\rho+\varepsilon_2\beta)Q - Q + Q - Q(\frac{\gamma\varepsilon_2 - \varepsilon_1}{2})^2(P - \varepsilon_2 I)^{-1}Q < 0, \tag{5.5.17}$$

$$-\eta_i + Q(\varepsilon_1\rho + \varepsilon_2\beta)Q - Q + Q - Q(\frac{\gamma\varepsilon_2 - \varepsilon_1}{2})^2(P - \varepsilon_2 I)^{-1}Q < 0, \tag{5.5.18}$$

$$\eta_0 + \varepsilon_3(\rho + \beta)I - [\frac{1}{\varepsilon_3}Q(\frac{\gamma\varepsilon_3 - \varepsilon_3}{2})^2 Q + (\frac{\gamma\varepsilon_3 - \varepsilon_3}{\varepsilon_3})Q + \frac{1}{\varepsilon_3}I] < 0. \tag{5.5.19}$$

Substituting $K_iQ = Y_i$ and $K_0Q = Y_0$, η_i is now given by $QA_i^T + A_iQ + Y_i^TB^T + BY_i$ and η_0 is $-(QA_0^T + A_0Q + Y_0^TB^T + BY_0)$

$$\eta_i + Q(\varepsilon_1\rho + \varepsilon_2\beta)Q - Q + Q - Q(\frac{\gamma\varepsilon_2 - \varepsilon_1}{2})^2(P - \varepsilon_2 I)^{-1}Q < 0, \tag{5.5.20}$$

$$-\eta_i + Q(\varepsilon_1\rho + \varepsilon_2\beta)Q - Q + Q - Q(\frac{\gamma\varepsilon_2 - \varepsilon_1}{2})^2(P - \varepsilon_2 I)^{-1}Q < 0, \tag{5.5.21}$$

$$\eta_0 + Q\varepsilon_3(\rho + \beta)Q - [\frac{1}{\varepsilon_3}Q(\frac{\gamma\varepsilon_3 - \varepsilon_3}{2})^2 Q + (\frac{\gamma\varepsilon_3 - \varepsilon_3}{\varepsilon_3})Q + \frac{1}{\varepsilon_3}I] < 0. \tag{5.5.22}$$

(5.5.20) – (5.5.22) are quadratic matrix inequalities (QMIs) because of the term $Q(\varepsilon_1\rho + \varepsilon_2\beta)Q + Q(\frac{\gamma\varepsilon_2 - \varepsilon_1}{2\varepsilon_2})Q$. Applying Schur complement again, (5.5.20) – (5.5.22) turn out to be (5.5.6)

Therefore, the system (5.5.1) can be stabilized by a state feedback controller (5.5.4) if the set of LMIs given by (5.5.6) are feasible. □

5.6 Observer-based stabilization design

In the design of feedback control systems, the knowledge of system state plays a key role. However, in engineering practice it may be quite difficult, sometimes even impossible, to directly measure all the system state variables through sensors.1 In those situations, a state observer is usually needed, and then the so-called observer-based control can be carried out using the estimated state. For linear systems, the observer-based control is readily achieved due to the separation principle. However, for nonlinear systems, the observer-based control problem becomes quite difficult. In fact, for a general nonlinear system, the state estimation by itself is still an open problem. In this section, we address the observer-based stabilization problem for one-sided Lipschitz time-varying systems.

Consider a class of continuous-time nonlinear systems described by

$$\begin{cases} \dot{x}(t) = A(t)x(t) + Bu(t) + f(t, x) \\ y(t) = Cx(t) \end{cases} \tag{5.6.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state vector, the control input, and the measured output of the system, respectively. The matrices $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ are the known constant matrices. The vector-valued function $f(t,x) : (\mathbb{R}, \mathbb{R}^n) \to \mathbb{R}^n$ represents the nonlinearity of the system. System (5.6.1) is controllable and observable. As we assume before in section 5.5 that $f(0) = 0$ and $f(t,x)$ satisfy conditions (2.5.3) and (2.5.4) in Definition 2.4, 2.5.

The time-varying matrix $A(t)$ can be decomposed to be:

$$A(t) = A_0 + \sum_{i=1}^{m} a_i(t)A_i = A_0 + \sum_{i=1}^{m} [a_i^+(t) + a_i^-(t)]A_i \tag{5.6.2}$$

in which matrices $A_0 \in \mathbb{R}^{n\times n}$ and $A_i \in \mathbb{R}^{n\times n}, i = 1, \cdots m$ are constant, and the scalar coefficients $a_i^+ > 0$ and $a_i^- < 0$ for $i = 1, \cdots, m$

As usual, we first employ the known Luenberger-like observer to estimate the state and then use the estimated state to design a linear time-varying output feedback. More precisely, we propose an observer-based controller as follows

$$\begin{cases} \dot{\hat{x}}(t) = A(t)\hat{x}(t) + Bu(t) + f(t,\hat{x}) + K(t)(y - C\hat{x}(t)) \\ u(t) = F(t)\hat{x}(t), \quad x(0) = \hat{x}_0 \end{cases} \tag{5.6.3}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimate of x, $\hat{x}(0) = \hat{x}_0$ is the initial value of the estimate. Here, $K(t)$ and $F(t)$ are chosen as:

$$K(t) = K_0 + \sum_{i=1}^{m} [a_i^+(t) + a_i^-(t)]K_i, \tag{5.6.4}$$

$$F(t) = F_0 + \sum_{i=1}^{m} [a_i^+(t) + a_i^-(t)]F_i. \tag{5.6.5}$$

$\{K_0 \quad K_i, \quad F_0 \quad F_i\} \in \mathbb{R}^{n\times n}$, for $i = 1, \cdots m$ are the constant matrices to be determined later.

Denote the estimation error by $e(t) := x(t) - \hat{x}(t)$. From equations (5.6.1) and (5.6.3), we have

$$\dot{e}(t) = (A_0 - K_0C)e(t) + \sum_{i=1}^{m} [a_i^+(t) + a_i^-(t)](A_i - K_iC)e(t) + f - \hat{f} \tag{5.6.6}$$

where $f := f(t,x)$ and $\hat{f} := f(t,\hat{x})$. Moreover, system (5.6.1) becomes

$$\dot{x} = (A_0 + BF_0)x + \sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)](A_i + BF_i)x - BF_0e - \sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)]BF_ie + f \tag{5.6.7}$$

The closed-loop system can be rewritten as

$$\overbrace{\begin{bmatrix} x \\ e \end{bmatrix}}^{\cdot} = \begin{bmatrix} (A_0 + BF_0) + \sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)](A_i + BF_i) & -BF_0 - \sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)]BF_i & I & 0 \\ 0 & (A_0 - K_0C) + \sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)](A_i - K_iC) & 0 & I \end{bmatrix} \times \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \tag{5.6.8}$$

where $\tilde{f} := f - \hat{f}$. For system (5.6.8), let us consider the following Lyapunov function candidate

$$V(x, e) = \begin{bmatrix} x \\ e \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} = x^TPx + e^TRe \tag{5.6.9}$$

Consequently, calculating the derivative of V along the state trajectories of equation (5.6.8) gives

$$\dot{V}(x, e) = \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \begin{bmatrix} \Sigma_{11} & -PBF_0 - \sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)]PBF_i & P & 0 \\ -\sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)](PBF_i)^T - (PBF_0)^T & \Sigma_{22} & 0 & R \\ P & 0 & 0 & 0 \\ 0 & R & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \tag{5.6.10}$$

where

$$\Sigma_{11} = (A_0+BF_0)^T P + P(A_0+BF_0) + \sum_{i=1}^{m}[a_i^+(t)+a_i^-(t)][(A_i+BF_i)^T P + P(A_i+BF_i)]$$

$$\Sigma_{22} = (A_0-K_0C)^T P + P(A_0-K_0C) + \sum_{i=1}^{m}[a_i^+(t)+a_i^-(t)][(A_i-K_iC)^T R + R(A_i-K_iC)]$$

Notice that $\dot{V}(x,e) \leq 0$ if the matrix inequality Σ holds. However, the matrix inequality $\Sigma \leq 0$ is a BMI since it involves the PBF term. As previously mentioned, up to now there is no efficient numerical algorithm to solve the BMI problem. We try to overcome this problem in the following theorem.

Theorem 5.4. *For the system (5.6.1), let the observer-based output feedback controller be constructed in the form of equation (5.6.3). Then the closed-loop system (5.6.8) is asymptotically stable if there exist matrices* $Q > 0$, $R > 0$, $\hat{K}_0$, $\hat{F}_0$, $\hat{K}_i$, *and* $\hat{F}_i$ *for* $i = 1, \cdots m$ *with appropriate dimensions and scalars* $\varepsilon_1 > 0$ $\varepsilon_2 > 0$ $\varepsilon_3 > 0$ $\varepsilon_4 > 0$ *and* $\phi_1 > 0$ *such that*

$$\begin{bmatrix} \bar{\Delta}_i & S_2 \\ S_2^T & W \end{bmatrix} < 0 \tag{5.6.11}$$

$$\begin{bmatrix} \bar{\Upsilon}_i & S_2 \\ S_2^T & W \end{bmatrix} < 0 \tag{5.6.12}$$

$$\begin{bmatrix} \bar{\Psi} & S_2 \\ S_2^T & W \end{bmatrix} < 0 \tag{5.6.13}$$

where

$$\bar{\Delta}_i = \begin{bmatrix} \hat{\Sigma}_{i11} - Q + \frac{1}{\alpha_1} I & 0 & 0 & 0 \\ 0 & \Sigma_{i22} + 2(\varepsilon_3\rho + \varepsilon_4\beta)I & 0 & (\varepsilon_4\gamma - \varepsilon_3)I \\ 0 & 0 & -2\varepsilon_2 I & 0 \\ 0 & * & 0 & -2\varepsilon_4 I \end{bmatrix}$$

$$\bar{\Upsilon}_i = \begin{bmatrix} -\hat{\Sigma}_{i11} - Q + \frac{1}{\alpha_1} I & 0 & 0 & 0 \\ 0 & -\Sigma_{i22} + 2(\varepsilon_3\rho + \varepsilon_4\beta)I & 0 & (\varepsilon_4\gamma - \varepsilon_3)I \\ 0 & 0 & -2\varepsilon_2 I & 0 \\ 0 & * & 0 & -2\varepsilon_4 I \end{bmatrix}$$

$$\bar{\Psi} = \begin{bmatrix} \hat{\Sigma}_{011} - 2\varepsilon_5(\rho + \beta)I & 0 & I & 0 \\ 0 & \Sigma_{022} + 2\varepsilon_6(\rho + \beta)I & 0 & R + \varepsilon_6(\gamma - 1)I \\ 0 & 0 & -2\varepsilon_5 I & 0 \\ 0 & * & 0 & -2\varepsilon_6 I \end{bmatrix}$$

$$S_2 = \begin{bmatrix} -B\hat{F_i}^T & 0 & Q & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.6.14}$$

$$W = diag\{-\frac{Q}{\phi_1}, -\phi_1 Q, -\psi(\varepsilon_1, \varepsilon_2)I, -\zeta(\varepsilon_1, \varepsilon_2)I\} \tag{5.6.15}$$

$$\psi(\varepsilon_1, \varepsilon_2) = \frac{1}{[(2\rho - 1)\varepsilon_1 + (2\beta + \gamma)\varepsilon_2]}$$

$$\zeta(\varepsilon_1, \varepsilon_2) = \frac{1}{(\varepsilon_2\gamma - \varepsilon_1)}$$

$$\hat{\Sigma}_{011} = (A_0 + B\hat{F}_0)^T P + P(A_0 + B\hat{F}_0)$$

$$\hat{\Sigma}_{i11} = QA^T + AQ + \hat{F}_i B^T + B\hat{F_i}^T$$

$$\Sigma_{022} = A_0^T R + RA_0 - C^T \hat{K}_0 - \hat{K_0}^T C$$

$$\Sigma_{i22} = A_i^T R + RA_i - C^T \hat{K}_i - \hat{K_i}^T C$$

Furthermore, the resulting observer gain matrix K and the output feedback gain matrix F are, respectively, given by $K_0 = R^{-1}\hat{K}_0^T$ and $F_0 = \hat{F}_0^T Q^{-1}$, $K_i = R^{-1}\hat{K}_i^T$ and $F_i = \hat{F}_i^T Q^{-1}$ for $i = 1, \cdots m$.

Proof. To begin with, let the Lyapunov function candidate $V(x, e)$ be defined in the form of equation (5.6.9). Notice that $f(0) = 0$. Then from conditions (2.5.3) and (2.5.4), for any positive scalars ε_1 and ε_2, we can obtain

$$\varepsilon_1 C_1 = \sum_{i=1}^{m}(a_i^+(t) - a_i^-(t))\varepsilon_1 \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \begin{bmatrix} 2\rho I & 0 & -I & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \geq 0 \tag{5.6.16}$$

and

$$\varepsilon_2 C_2 = \sum_{i=1}^{m}(a_i^+(t) - a_i^-(t))\varepsilon_2 \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \begin{bmatrix} 2\beta I & 0 & \gamma I & 0 \\ 0 & 0 & 0 & 0 \\ \gamma I & 0 & -2I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \geq 0. \tag{5.6.17}$$

Similarly in Theorem 6.3, we get

$$\varepsilon_3 C_3 = \sum_{i=1}^{m}(a_i^+(t) - a_i^-(t))\varepsilon_3 \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\rho I & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \geq 0 \tag{5.6.18}$$

and

$$\varepsilon_4 C_4 = \sum_{i=1}^{m}(a_i^+(t) - a_i^-(t))\varepsilon_4 \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\beta I & 0 & \gamma I \\ 0 & 0 & 0 & 0 \\ 0 & \gamma I & 0 & -2I \end{bmatrix} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \geq 0 \tag{5.6.19}$$

where ε_3 and ε_4 are the two positive scalars.

$$\varepsilon_5 C_5 = \varepsilon_5 \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \begin{bmatrix} (2\rho+2\beta)I & 0 & (\gamma-1)I & 0 \\ 0 & 0 & 0 & 0 \\ (\gamma-1)I & 0 & -2I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \geq 0 \tag{5.6.20}$$

$$\varepsilon_6 C_6 = \varepsilon_6 \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (2\rho+2\beta)I & 0 & (\gamma-1)I \\ 0 & 0 & 0 & 0 \\ 0 & (\gamma-1)I & 0 & -2I \end{bmatrix} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \geq 0 \tag{5.6.21}$$

$$\varepsilon_7 C_7 = \sum_{i=1}^{m}(a_i^+(t) - a_i^-(t))\varepsilon_7 \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \geq 0. \tag{5.6.22}$$

Consequently, adding the left sides of equations (5.6.16) (5.6.17) (5.6.18) (5.6.19) (5.6.20) (5.6.21) (5.6.22) to the right side of equation (5.6.9) gives

$$\dot{H} = \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \underbrace{\begin{bmatrix} \Sigma_{11} + \eta_1 I & \bar{\Sigma}_{21} & P + \eta_2 I + \varepsilon_5(\gamma-1)I & 0 \\ * & \bar{\Sigma}_{22} + \varepsilon_6(2\rho+2\beta)I & 0 & R + \eta_3 I + \varepsilon_6(\gamma-1)I \\ * & 0 & -2\varepsilon_2 I - 2\varepsilon_5 I & 0 \\ 0 & * & 0 & -2\varepsilon_4 I - 2\varepsilon_6 I \end{bmatrix}}_{\bar{\Sigma}} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \tag{5.6.23}$$

where

$$\bar{\Sigma}_{21} = -PBF_0 - \sum_{i=1}^{m}[a_i^+(t) + a_i^-(t)]PBF_i$$

$$\bar{\Sigma}_{22} = \sum_{i=1}^{m}[a_i^+(t)+a_i^-(t)](A_i^T R + RA_i - C^T\hat{K}_i - \hat{K}_i^T C) + \sum_{i=1}^{m}[a_i^+(t)-a_i^-(t)]2(\varepsilon_3\rho+\varepsilon_4\beta)I)$$

$$\hat{K}_i = K_i^T R, \quad \eta_1 = \sum_{i=1}^{m}[a_i^+(t) - a_i^-(t)]2(\varepsilon_1\rho + \varepsilon_2\beta) + \varepsilon_5(2\rho + 2\beta)$$

$$\eta_2 = \sum_{i=1}^{m}[a_i^+(t) - a_i^-(t)](\varepsilon_2\gamma - \varepsilon_1), \quad \eta_3 = \sum_{i=1}^{m}[a_i^+(t) - a_i^-(t)](\varepsilon_4\gamma - \varepsilon_3)$$

Reorganise equation(5.6.23) we get

$$H = \sum_{i=1}^{m}\left[a_i^+(t)\Delta_i - a_i^-(t)\Upsilon_i + \Psi\right] \tag{5.6.24}$$

$$\Delta_i = \begin{bmatrix} \Sigma_{i11} + 2(\varepsilon_1\rho + \varepsilon_2\beta)I & -PBF_i & (\varepsilon_2\gamma - \varepsilon_1)I & 0 \\ * & \Sigma_{i22} + 2(\varepsilon_3\rho + \varepsilon_4\beta)I & 0 & (\varepsilon_2\gamma - \varepsilon_1)I \\ * & 0 & -2\varepsilon_2 I & 0 \\ 0 & * & 0 & -2\varepsilon_4 I \end{bmatrix} \tag{5.6.25}$$

$$\Upsilon_i = \begin{bmatrix} -\Sigma_{i11} + 2(\varepsilon_1\rho + \varepsilon_2\beta)I & -PBF_i & (\varepsilon_2\gamma - \varepsilon_1)I & 0 \\ * & -\Sigma_{i22} + 2(\varepsilon_3\rho + \varepsilon_4\beta)I & 0 & (\varepsilon_2\gamma - \varepsilon_1)I \\ * & 0 & -2\varepsilon_2 I & 0 \\ 0 & * & 0 & -2\varepsilon_4 I \end{bmatrix} \tag{5.6.26}$$

$$\Psi = \begin{bmatrix} \Sigma_{011} + \varepsilon_5(2\rho + 2\beta)I & -PBF_0 & P + \varepsilon_5(\gamma - 1)I & 0 \\ * & \Sigma_{022} + \varepsilon_6(2\rho + 2\beta)I & 0 & R + \varepsilon_6(\gamma - 1)I \\ * & 0 & -2\varepsilon_5 I & 0 \\ 0 & * & 0 & -2\varepsilon_6 I \end{bmatrix} \tag{5.6.27}$$

where

$$\Sigma_{i11} = (A_i + BF_i)^T P + P(A_i + BF_i)$$

$$\Sigma_{i22} = A_i^T R + RA_i - C^T\hat{K}_i - \hat{K}_i^T C$$

and

$$\Sigma_{011} = (A_0 + BF_0)^T P + P(A_0 + BF_0)$$

$$\Sigma_{022} = A_0^T R + RA_0 - C^T\hat{K}_0 - \hat{K}_0^T C$$

From equation (5.6.24), we know that $\dot{V} < 0$ if the condition $\Delta_i < 0$, $\Upsilon_i < 0$, $\Psi < 0$ hold. Let us define $Q : P^{-1}$. Pre- and post- multiplying Δ_i, Υ_i, Ψ by matrix $diag(Q, I, I, I)$ yields

$$\tilde{\Delta}_i = \begin{bmatrix} \tilde{\Sigma}_{i11} + 2Q(\varepsilon_1\rho + \varepsilon_2\beta)Q & -BF_i & (\varepsilon_2\gamma - \varepsilon_1)Q & 0 \\ * & \Sigma_{i22} + 2(\varepsilon_3\rho + \varepsilon_4\beta)I & 0 & (\varepsilon_2\gamma - \varepsilon_1)I \\ * & 0 & -2\varepsilon_2 I & 0 \\ 0 & * & 0 & -2\varepsilon_4 I \end{bmatrix} < 0 \tag{5.6.28}$$

where

$$\hat{F}_i = QF_i^T, \quad \tilde{\Sigma}_{i11} = (A_i + B\hat{F}_i)^T P + P(A_i + B\hat{F}_i).$$

Consequently, by developing $\tilde{\Delta}_i < 0$, we get

$$\begin{aligned} \tilde{\Delta}_i \;=\; & \hat{\Delta}_i + \frac{1}{2}\begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \frac{1}{2}\begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + 2(\varepsilon_1\rho + \varepsilon_2\beta)\begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} -B\hat{F}_i \\ 0 \\ 0 \\ 0 \end{bmatrix} Q^{-1}\begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}^T \\ & + \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} Q^{-1}\begin{bmatrix} -B\hat{F}_i \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + (\varepsilon_2\gamma - \varepsilon_1)\begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T + (\varepsilon_2\gamma - \varepsilon_1)\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}\begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}^T < 0 \end{aligned} \tag{5.6.29}$$

where

$$\hat{\Delta}_i = \begin{bmatrix} \hat{\Sigma}_{i11} - Q & 0 & 0 & 0 \\ 0 & \Sigma_{i22} + 2(\varepsilon_3\rho + \varepsilon_4\beta)I & 0 & (\varepsilon_4\gamma - \varepsilon_3)I \\ 0 & 0 & -2\varepsilon_2 I & 0 \\ 0 & * & 0 & -2\varepsilon_4 I \end{bmatrix}$$

$$\hat{\Sigma}_{i11} = QA^T + AQ + \hat{F}_i B^T + B\hat{F}_i^T.$$

Using the Young's relation, we get the following inequality

$$\tilde{\Delta}_i \leq \hat{\Delta}_i + \begin{bmatrix} -B\hat{F}_i^T \\ 0 \\ 0 \\ 0 \end{bmatrix} (\phi_1 Q) \begin{bmatrix} -B\hat{F}_i^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} \frac{1}{\phi_1} Q^{-1} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}^T$$
$$+ [2(\varepsilon_1\rho + \varepsilon_2\beta) + \alpha_1] \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \frac{1}{\alpha_1} \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + (\varepsilon_2\gamma - \varepsilon_1) \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + (\varepsilon_2\gamma - \varepsilon_1) \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T < 0 \tag{5.6.30}$$

where ϕ_1, α_1 are positive scalar.

Reorganising from equation (5.6.30) we get the following inequality

$$\tilde{\Delta}_i \leq \bar{\Delta}_i - \begin{bmatrix} -B\hat{F}_i^T \\ 0 \\ 0 \\ 0 \end{bmatrix} (-\phi_1 Q) \begin{bmatrix} -B\hat{F}_i^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} (\frac{1}{\phi_1}) Q^{-1} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}^T$$
$$-[-(2\rho - 1)\varepsilon_1 - (2\beta + \gamma)\varepsilon_2 - \alpha_1] \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}^T - (-\varepsilon_2\gamma + \varepsilon_1) \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T < 0 \tag{5.6.31}$$

where

$$\bar{\Delta}_i = \begin{bmatrix} \hat{\Sigma}_{i11} - Q + \frac{1}{\alpha_1} I & 0 & 0 & 0 \\ 0 & \Sigma_{i22} + 2(\varepsilon_3\rho + \varepsilon_4\beta) I & 0 & (\varepsilon_4\gamma - \varepsilon_3) I \\ 0 & 0 & -2\varepsilon_2 I & 0 \\ 0 & * & 0 & -2\varepsilon_4 I \end{bmatrix}.$$

Similarly,

$$\tilde{\Upsilon}_i \leq \bar{\Upsilon}_i - \begin{bmatrix} -B\hat{F}_i^T \\ 0 \\ 0 \\ 0 \end{bmatrix} (-\phi_1 Q) \begin{bmatrix} -B\hat{F}_i^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} (\frac{1}{\phi_1}) Q^{-1} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}^T$$
$$-[-(2\rho-1)\varepsilon_1 - (2\beta+\gamma)\varepsilon_2 - \alpha_1] \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}^T - (-\varepsilon_2\gamma + \varepsilon_1) \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T < 0 \tag{5.6.32}$$

$$\bar{\Upsilon}_i = \begin{bmatrix} -\hat{\Sigma}_{i11} - Q + \frac{1}{\alpha_1} I & 0 & 0 & 0 \\ 0 & -\Sigma_{i22} + 2(\varepsilon_3\rho + \varepsilon_4\beta) I & 0 & (\varepsilon_4\gamma - \varepsilon_3) I \\ 0 & 0 & -2\varepsilon_2 I & 0 \\ 0 & * & 0 & -2\varepsilon_4 I \end{bmatrix}$$

and

$$\tilde{\Psi} \leq \bar{\Psi} - \begin{bmatrix} -B\hat{F}_0^T \\ 0 \\ 0 \\ 0 \end{bmatrix} (-\phi_1 Q) \begin{bmatrix} -B\hat{F}_0^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} (\frac{1}{\phi_1}) Q^{-1} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}^T$$
$$-[-(2\rho-1)\varepsilon_1 - (2\beta+\gamma)\varepsilon_2] \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}^T - (-\varepsilon_2\gamma + \varepsilon_1) \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T < 0 \tag{5.6.33}$$

$$\bar{\Psi} = \begin{bmatrix} \hat{\Sigma}_{011} - 2\varepsilon_5(\rho+\beta)I & 0 & I & 0 \\ 0 & \Sigma_{022} + 2\varepsilon_6(\rho+\beta)I & 0 & R + \varepsilon_6(\gamma-1)I \\ 0 & 0 & -2\varepsilon_5 I & 0 \\ 0 & * & 0 & -2\varepsilon_6 I \end{bmatrix}$$

$$\hat{F}_0 = QF_0^T, \quad \hat{\Sigma}_{011} = (A_0 + B\hat{F}_0)^T P + P(A_0 + B\hat{F}_0).$$

Apply Schur complement to (5.6.31), (5.6.32) and (5.6.34), we obtain the conditions (5.6.11), (5.6.12) and (5.6.13) in Theorem 5.4. □

Remark 5.2. It should be mentioned that equation (5.6.15) is not a strict LMI form because in its blocks there exist some terms like $\frac{Q}{\phi_1}$ and $\phi_1 Q$. We will employ additional constrains of ϕ and Q in order to transform to LMIs, which is $Q > cI$ and $\frac{I}{\phi_1} \leq -(2-\phi_1)I$. Therefore, equation (5.6.11 –5.6.13) can be formulated into an LMI with respect to c and $d := c\phi_1$, where c is a positive scalar. In fact, we have

$$-\frac{Q}{\phi_1} \leq -(2-\phi_1)Q \leq -(2-\phi_1)cI = -(2c-d)I \qquad (5.6.34)$$

and

$$-\phi_1 Q \leq -\phi_1 cI = -dI. \qquad (5.6.35)$$

Hence, by equations (5.6.34) and (5.6.35), $(2c-d)I$, dI can replace the blocks $\frac{Q}{\phi_1}$, $\phi_1 Q$ in (5.6.15), and $2c - d > 0$, $d > 0$

5.7 Summary

In this chapter. the system under consideration is an extension of the general family of nonlinear functions, known as one-side Lipschitz functions. For such system, sufficient conditions for the existence of observers are discussed and we also construct the full-order and reduced order Luenberger-type observer through the standard LMI approach and Riccati equation based approach respectively. Further, the observer-based output feedback stabilization problem is investigated. Consequently, one solution to the controller problem is established in the numerically efficient form of linear matrix inequalities.

Chapter 6

Robust H_∞ control of class of time-varying nonlinear discrete time stochastic systems

6.1 Introduction

The stability analysis of both continuous and discrete-time stochastic systems has attracted many researchers in system science area for decades. Stochastic systems arise in a wide area of applications in control engineering such as filtering, adaptive systems and identification, and learning etc. Meanwhile, control theory for stochastic system is very significant and could be widely applied to the economic and financial problems such as development and planning of production and inventory, growth models and portfolio selections [4, 40]. Most of these application is in short run stabilisation and uses discrete time models [40].

The problem of robust quadratic stabilization of systems under nonlinear perturbation was studied for continuous time systems in [82] and for discrete time systems in [83]. The solutions provided in [82, 83] are for quadratically bounded nonlinear perturbations but only available for deterministic systems. Attempts have been made in Yaz et al. [95], Sathananthan et al. [79] and Zhang et al. [99] for nonlinear stochastic discrete-time systems. Although a larger foundation has been laid out for stability and stabilization of discrete time stochastic systems, the problem of robust stabilization of discrete time stochastic systems under nonlinear perturbation should be given more attention.

The research of Chapter 3, 4, 5 in the thesis focus on continuous time deterministic system. So in Chapter 6, we intend to do some investigation on the system control problem of discrete time stochastic system with bounded time-varying nonlinear perturbation origin from one-sided Lipschitz property. Our results are different from the results in the literature (Sathananthan et al. [79] and Zhang et al. [99]), because of the fact that a more general nonlinear uncertainties structure is proposed in this chapter. The objective of this chapter is to show how a control law that stabilizes such complex stochastic system can be solved. Unlike the results in (Sathananthan et al. [79] and Zhang et al. [99]), problem is solved by solving the nonconvex feasibility problem.

6.2 System Descriptions and Definitions

Consider the time-varying nonlinear discrete stochastic system described by the following equation:

$$\begin{cases} \underline{x}_{t+1} = A\underline{x}_t + h_1(t,\underline{x}_t) + B\underline{u}_t + (C\underline{x}_t + h_2(t,\underline{x}_t) + D\underline{u}_t)\, v_t, \\ \underline{x}(0) = x_0 \in \mathbb{R}^n, t \in N, \end{cases} \tag{6.2.1}$$

where $\underline{x}_t \in \mathbb{R}^n$ is the n-dimensional state vector and $\underline{u}_t \in \mathbb{R}^m$ is the control input. Now, we assume that v_t is a sequence of one-dimensional independent white noise processes defined on the complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathscr{P})$, where $\mathscr{F}_t = \sigma\{v_0, v_1, v_2, \ldots, v_t\}$. We assume that

$$\mathbb{E}\{v_t\} = 0, \quad \mathbb{E}\{v_t v_j\} = \delta_{tj}. \tag{6.2.2}$$

where δ_{tj} is a Kronecker function defined by $\delta_{tj} = 0$ for $t \neq j$ and $\delta_{tj} = 1$ for $t = j$.

Assumption 6.1. The time-varying nonlinear functions $h_1(t,\underline{x}_t)$, $h_2(t,\underline{x}_t)$ describe uncertainties of the system and satisfy the following quadratic inequalities:

$$h_1^T(t,\underline{x}_t)\underline{x}_t \leq \rho_1 \underline{x}_t^T \underline{x}_t, \tag{6.2.3}$$

$$h_1^T(t,\underline{x}_t)h_1(t,\underline{x}_t) \leq \beta_1 \underline{x}_t^T \underline{x}_t + \gamma_1 x^T(t) h_1(t,\underline{x}_t), \tag{6.2.4}$$

$$h_2^T(t,\underline{x}_t)\underline{x}_t \leq \rho_2 \underline{x}_t^T \underline{x}_t, \tag{6.2.5}$$

$$h_2^T(t, \underline{x}_t)h_2(t, \underline{x}_t) \leq \beta_2 \underline{x}_t^T \underline{x}_t + \gamma_2 x^T(t) h_2(t, \underline{x}_t). \tag{6.2.6}$$

for all $t \in N$, where $\beta_i, \gamma_i, \rho_i$ are constants related to the function h_i for $i = 1, 2$. $\beta_i, \gamma_i, \rho_i$ are constant defining structure of h_i. Assumption 6.1 can be regarded as a type of one-sided Lipschitz condition (2.5.3) and quadratic boundedness condition (2.5.4).

As the generalized version of the system in (Sathananthan [79], Zhang [99]), the system state, control input, and uncertain terms in the system (6.2.1) depend on noise simultaneously, which makes this typy of nonlinear system more useful in describing many practical phenomena.

In the following sections, we give our main results on stochastic stability, stochastic stabilization, and robust control via LMI based approach. Firstly, we introduce the following lemma which will be used in the proof of our main results.

Lemma 6.1. *[99] For any real matrices U, $N = N^T > 0$ and W with appropriate dimensions, we have*

$$U^T N W + W^T N U \leq U^T N U + W^T N W. \tag{6.2.7}$$

6.3 Robust Stability Criteria

Consider the following stochastic discrete time system:

$$\begin{cases} \underline{x}_{t+1} = A\underline{x}_t + h_1(t, \underline{x}_t) + (C\underline{x}_t + h_2(t, \underline{x}_t))\, v_t, \\ \underline{x}_0 = x_0 \in \mathbb{R}^n, t \in N, \end{cases} \tag{6.3.1}$$

where $h_1(t, \underline{x}_t)$ and $h_2(t, \underline{x}_t)$ are satisfied (6.2.6) and (6.2.5). We have the defintion of robust stochastic stability as below:

Definition 6.1. The unforced system (6.3.1) is said to be robustly stochastically stable with margins ρ_1, ρ_2, β_1, β_2, γ_1, γ_2 if there exists a constant $\delta(x_0, \rho_1, \rho_2, \beta_1, \beta_2)$ such that

$$\mathbb{E}\left[\sum_{t=0}^{\infty} x^T(t)\underline{x}_t\right] \leq \delta(x_0, \rho_1, \rho_2, \beta_1, \beta_2). \tag{6.3.2}$$

The following theorem gives a sufficient condition of robust stochastic stability for system (6.3.1)

Theorem 6.1. *System (6.3.1) with margins $\rho_1, \rho_2, \beta_1, \beta_2$ is said to be robustly stochastically stable, if there exists a symmetric positive definite matrix $Q > 0$ and real scalar $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > 0$ such that the following conditions (6.3.3) are satisfied.*

$$\begin{bmatrix} -Q + 2I + 2(\varepsilon_1\rho_1 + \varepsilon_2\beta_1)I + 2(\varepsilon_3\rho_2 + \varepsilon_4\beta_2)I & A^TQ & C^TQ \\ * & -\frac{1}{2}Q & 0 \\ * & * & -\frac{1}{2}Q \end{bmatrix} < 0. \tag{6.3.3}$$

$$\begin{bmatrix} 2Q - 2\varepsilon_2 I & \varepsilon_2\gamma_1 I - \varepsilon_1 I \\ \varepsilon_2\gamma_1 I - \varepsilon_1 I & -I. \end{bmatrix} < 0, \tag{6.3.4}$$

$$\begin{bmatrix} 2Q - 2\varepsilon_4 I & \varepsilon_4\gamma_2 I - \varepsilon_3 I \\ \varepsilon_4\gamma_2 I - \varepsilon_3 I & -I \end{bmatrix} < 0. \tag{6.3.5}$$

Proof. If (6.3.3), (6.3.4), (6.3.5) hold, we set $V(\underline{x}_t) = \underline{x}_t^T Q \underline{x}_t$ as a Lyapunov function candidate of system (6.3.1). Note that $\underline{x}_t$ and v_t are independent, so the difference generator is

$$\begin{aligned} \mathbb{E}\Delta V(\underline{x}_t) &= \mathbb{E}\left[V(\underline{x}_{t+1}) - V(\underline{x}_t)\right] \\ &\leq \mathbb{E}[V(\underline{x}_{t+1}) - V(\underline{x}_t) + \underline{x}_t^T(2\varepsilon_1\rho_1 I + 2\varepsilon_2\beta_1 I + 2\varepsilon_3\rho_2 I + 2\varepsilon_4\beta_2 I)\underline{x}_t \\ &\quad + x^T(t)2(\varepsilon_2\gamma_1 - \varepsilon_1)Ih_1(t, \underline{x}_t) + x^T(t)2(\varepsilon_4\gamma_2 - \varepsilon_3)Ih_2(t, \underline{x}_t) \\ &\quad - 2\varepsilon_2 h_1(t, \underline{x}_t)^T h_1(t, \underline{x}_t) - 2\varepsilon_4 h_2(t, \underline{x}_t)^T h_2(t, \underline{x}_t)] \\ &= \mathbb{E}\{\underline{x}_t^T[A^TQA + C^TQC - Q + 2(\varepsilon_1\rho_1 + \varepsilon_2\beta_1)I + 2(\varepsilon_3\rho_2 + \varepsilon_4\beta_2)I]\underline{x}_t \\ &\quad + \underline{x}_t^T[A^TQ + (\varepsilon_2\gamma_1 - \varepsilon_1)I]h_1(t, \underline{x}_t) + h_1^T(t, \underline{x}_t)[QA + (\varepsilon_2\gamma_1 - \varepsilon_1)I]]\underline{x}_t \\ &\quad + h_2^T(t, \underline{x}_t)[QC + (\varepsilon_4\gamma_2 - \varepsilon_3)I]\underline{x}_t + \underline{x}_t^T[C^TQ + (\varepsilon_4\gamma_2 - \varepsilon_3)I]h_2(t, \underline{x}_t) \\ &\quad + h_1^T(t, \underline{x}_t)(Q - 2\varepsilon_2 I)h_1(t, \underline{x}_t) + h_2^T(t, \underline{x}_t)(Q - 2\varepsilon_4 I)h_2(t, \underline{x}_t)\}. \end{aligned} \tag{6.3.6}$$

Applying Young's inequality, Using lemma 6.1, we have

$$\begin{aligned}
& \underline{x}_t^T A^T Q h_1(t,\underline{x}_t) + h_1^T(t,\underline{x}_t) Q A \underline{x}_t \\
\leq\ & \underline{x}_t^T A^T Q A \underline{x}_t + h_1^T(t,\underline{x}_t) Q h_1(t,\underline{x}_t). \\
& \underline{x}_t^T (\varepsilon_2\gamma_1 - \varepsilon_1) I h_1(t,\underline{x}_t) + h_1^T(t,\underline{x}_t)(\varepsilon_2\gamma_1 - \varepsilon_1) I \underline{x}_t \\
\leq\ & \underline{x}_t^T \underline{x}_t + h_1^T(t,\underline{x}_t)(\varepsilon_2\gamma_1 - \varepsilon_1)^2 I h_1(t,\underline{x}_t). \\
& \underline{x}_t^T C^T Q h_2(t,\underline{x}_t) + h_2^T(t,\underline{x}_t) Q C \underline{x}_t \\
\leq\ & \underline{x}_t^T C^T Q C \underline{x}_t + h_2^T(t,\underline{x}_t) Q h_2(t,\underline{x}_t). \qquad (6.3.7) \\
& \underline{x}_t^T (\varepsilon_4\gamma_2 - \varepsilon_3) I h_2(t,\underline{x}_t) + h_2^T(t,\underline{x}_t)(\varepsilon_4\gamma_2 - \varepsilon_3) I \underline{x}_t \\
\leq\ & \underline{x}_t^T \underline{x}_t + h_2^T(t,\underline{x}_t)(\varepsilon_4\gamma_2 - \varepsilon_3)^2 I h_2(t,\underline{x}_t).
\end{aligned}$$

We achieve that, by substituting above inequalities in (6.3.7) into (6.3.6)

$$\begin{aligned}
& \mathbb{E}\Delta V(\underline{x}_t) \\
\leq\ & \mathbb{E}\{\underline{x}_t^T \left(2A^T Q A + 2C^T Q C - Q + 2I + 2(\varepsilon_1\rho_1 + \varepsilon_2\beta_1)I + 2(\varepsilon_3\rho_2 + \varepsilon_4\beta_2)I\right) \underline{x}_t \\
& + h_1^T(t,\underline{x}_t)[2Q + (\varepsilon_2\gamma_1 - \varepsilon_1)^2 I - 2\varepsilon_2 I] h_1(t,\underline{x}_t) \\
& + h_2^T(t,\underline{x}_t)[2Q + (\varepsilon_4\gamma_2 - \varepsilon_3)^2 I - 2\varepsilon_4 I] h_2(t,\underline{x}_t)\}. \qquad (6.3.8)
\end{aligned}$$

By Schur's complement, if we let

$$\Omega_1 = 2A^T Q A + 2C^T Q C - Q + 2I + 2(\varepsilon_1\rho_1 + \varepsilon_2\beta_1)I + 2(\varepsilon_3\rho_2 + \varepsilon_4\beta_2)I, \qquad (6.3.9)$$

and

$$\Omega_2 = 2Q + (\varepsilon_2\gamma_1 - \varepsilon_1)^2 I - 2\varepsilon_2 I < 0$$

$$\Omega_3 = 2Q + (\varepsilon_4\gamma_2 - \varepsilon_3)^2 I - 2\varepsilon_4 I < 0,$$

then, $\Omega_1 < 0$ is equivalent to

$$\begin{bmatrix} -Q + 2I + 2(\varepsilon_1\rho_1 + \varepsilon_2\beta_1)I + 2(\varepsilon_3\rho_2 + \varepsilon_4\beta_2)I & A^T Q & C^T Q \\ * & -\frac{1}{2}Q & 0 \\ * & * & -\frac{1}{2}Q \end{bmatrix} < 0.$$

$\Omega_2 < 0$ is equivalent to

$$\begin{bmatrix} 2Q - 2\varepsilon_2 I & \varepsilon_2\gamma_1 I - \varepsilon_1 I \\ \varepsilon_2\gamma_1 I - \varepsilon_1 I & -I. \end{bmatrix} < 0,$$

$\Omega_3 < 0$ is equivalent to

$$\begin{bmatrix} 2Q - 2\varepsilon_4 I & \varepsilon_4\gamma_2 I - \varepsilon_3 I \\ \varepsilon_4\gamma_2 I - \varepsilon_3 I & -I \end{bmatrix} < 0$$

which are hold by LMI (6.3.3), (6.3.4), (6.3.5). We denote $\lambda_{\max}(\Omega)$ and $\lambda_{\min}(\Omega)$ to be the largest and the minimum eigenvalues of the matrix Ω, respectively; then inequality (6.3.8) yields

$$\mathbb{E}\Delta V(\underline{x}_t) \leq \lambda_{\max}(\Omega_1)\mathbb{E}\|\underline{x}_t\|^2 + \lambda_{\max}(\Omega_2)\mathbb{E}\|h_1(t,\underline{x}_t)\|^2 + \lambda_{\max}(\Omega_3)\mathbb{E}\|h_2(t,\underline{x}_t)\|^2 \quad (6.3.10)$$

We sum up both side of (6.3.10) from $t = 0$ to $t = T_0 > 0$, we get

$$\begin{aligned} \mathbb{E}[V(\underline{x}_{T_0})] - V(x_0) &= \mathbb{E}\left[\sum_{t=0}^{T_0} \Delta V(\underline{x}_t)\right]. \\ &\leq \lambda_{\max}(\Omega_1)\mathbb{E}\left[\sum_{t=0}^{T_0} \underline{x}_t^T \underline{x}_t\right] + \lambda_{\max}(\Omega_2)\mathbb{E}\left[\sum_{t=0}^{T_0} h_1(t,\underline{x}_t)^T h_1(t,\underline{x}_t)\right] \\ &\quad + \lambda_{\max}(\Omega_3)\mathbb{E}\left[\sum_{t=0}^{T_0} h_2(t,\underline{x}_t)^T h_2(t,\underline{x}_t)\right]. \qquad (6.3.11) \end{aligned}$$

Therefore,

$$\begin{aligned} &\lambda_{\min}(-\Omega_1)\mathbb{E}\left[\sum_{t=0}^{T_0} \underline{x}_t^T \underline{x}_t\right] + \lambda_{\min}(-\Omega_2)\mathbb{E}\left[\sum_{t=0}^{T_0} h_1(t,\underline{x}_t)^T h_1(t,\underline{x}_t)\right] \\ &+ \lambda_{\min}(-\Omega_3)\mathbb{E}\left[\sum_{t=0}^{T_0} h_2(t,\underline{x}_t)^T h_2(t,\underline{x}_t)\right] \leq V(x_0), \qquad (6.3.12) \end{aligned}$$

which leads to

$$\mathbb{E}\left[\sum_{t=0}^{T_0} \underline{x}_t^T \underline{x}_t\right] \leq \delta(x_0, \rho_1, \rho_2, \beta_1, \beta_2) = \frac{V(x_0)}{\lambda_{\min}(-\Omega_1)}. \qquad (6.3.13)$$

Hence, letting $T \to \infty$, we have

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \underline{x}_t^T \underline{x}_t\right] \leq \delta(x_0, \rho_1, \rho_2, \beta_1, \beta_2). \tag{6.3.14}$$

Then, system (6.3.1) is robust stochastic stable. □

6.4 Robust Stabilization of system

$$\begin{cases} \underline{x}_{t+1} = A\underline{x}_t + h_1(t, \underline{x}_t) + B\underline{u}_t + (C\underline{x}_t + h_2(t, \underline{x}_t) + D\underline{u}_t)\, v_t, \\ \underline{x}_0 = x_0 \in \mathbb{R}^n, t \in N \end{cases} \tag{6.4.1}$$

Theorem 6.2. *System (6.4.1) with given constant ρ_1, ρ_2, β_1, β_2, γ_1, γ_2 is robustly stochastically stabilizable if there exist real matrices Y, $X > 0$, $Q > 0$ and real scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > 0$, $\kappa_1 > 0$, $\kappa_2 > 0$, $\kappa_3 > 0$, $\kappa_4 > 0$ such that the following conditions hold:*

$$\begin{bmatrix} -X & X & X & X & X & X & (AX+BY)^T & (CX+DY)^T \\ * & -\frac{1}{2}I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\frac{\kappa_1}{2\rho_1}I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\frac{\kappa_2}{2\beta_1}I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\frac{\kappa_3}{2\rho_2}I & 0 & 0 & 0 \\ * & * & * & * & * & -\frac{\kappa_4}{2\beta_2}I & 0 & 0 \\ * & * & * & * & * & * & -\frac{1}{2}X & 0 \\ * & * & * & * & * & * & * & -\frac{1}{2}X \end{bmatrix} < 0. \tag{6.4.2}$$

$$\Omega_2 = \begin{bmatrix} 2Q - 2\varepsilon_2 I & \varepsilon_2\gamma_1 I - \varepsilon_1 I \\ \varepsilon_2\gamma_1 I - \varepsilon_1 I & -I. \end{bmatrix} < 0, \tag{6.4.3}$$

$$\Omega_3 = \begin{bmatrix} 2Q - 2\varepsilon_4 I & \varepsilon_4\gamma_2 I - \varepsilon_3 I \\ \varepsilon_4\gamma_2 I - \varepsilon_3 I & -I \end{bmatrix} < 0, \tag{6.4.4}$$

and

$$QX = I, \quad \kappa_1 I \times \varepsilon_1 I = I, \quad \kappa_2 I \times \varepsilon_2 I = I, \quad \kappa_3 I \times \varepsilon_3 I = I, \quad \kappa_4 I \times \varepsilon_4 I = I. \tag{6.4.5}$$

In this case, $\underline{u}_t = K\underline{x}_t = YX^{-1}(t)$ is a robustly stochastically stabilizing control law.

Proof. We consider synthesizing a state feedback controller $\underline{u}_t = K\underline{x}_t$ to stabilize system (6.4.1). Substituting $\underline{u}_t = K\underline{x}_t$ into system (6.4.1) yields the closed loop system described by

$$\begin{cases} \underline{x}_{t+1} = \bar{A}\underline{x}_t + h_1(t, \underline{x}_t) + \left(\bar{C}\underline{x}_t + h_2(t, \underline{x}_t)\right) v_t, \\ \underline{x}_0 = x_0 \in \mathbb{R}^n, t \in N, \end{cases} \tag{6.4.6}$$

with $\bar{A} = A + BK$ and $\bar{C} = C + DK$. By theorem 6.1, system (6.4.6) is robustly stochastically stable if there exists a matrix Q such that the following LMIs

$$\bar{\Omega_1} = \begin{bmatrix} -Q + 2I + 2(\varepsilon_1\rho_1 + \varepsilon_2\beta_1)I + 2(\varepsilon_3\rho_2 + \varepsilon_4\beta_2)I & \bar{A}^T Q & \bar{C}^T Q \\ * & -\frac{1}{2}Q & 0 \\ * & * & -\frac{1}{2}Q \end{bmatrix} < 0, \tag{6.4.7}$$

and

$$\Omega_2 = 2Q + (\varepsilon_2\gamma_1 - \varepsilon_1)^2 I - 2\varepsilon_2 I < 0$$

$$\Omega_3 = 2Q + (\varepsilon_4\gamma_2 - \varepsilon_3)^2 I - 2\varepsilon_4 I < 0$$

holds.

$\Omega_2 < 0$ is equivalent to

$$\begin{bmatrix} 2Q - 2\varepsilon_2 I & \varepsilon_2\gamma_1 - \varepsilon_1 \\ \varepsilon_2\gamma_1 - \varepsilon_1 & -I. \end{bmatrix} < 0, \tag{6.4.8}$$

$\Omega_3 < 0$ is equivalent to

$$\begin{bmatrix} 2Q - 2\varepsilon_4 I & \varepsilon_4\gamma_2 - \varepsilon_3 \\ \varepsilon_4\gamma_2 - \varepsilon_3 & -I \end{bmatrix} < 0. \tag{6.4.9}$$

Let $Q^{-1} = X$ and set $K = YX^{-1}$, then pre and post multiply

$$diag[X, X, X]$$

on both sides of inequality (6.4.7), and apply schur complement. It yields

$$\begin{bmatrix} -X & X & X & X & X & X & (AX+BY)^T & (CX+DY)^T \\ * & -\frac{1}{2}I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\frac{1}{2\varepsilon_1\rho_1}I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\frac{1}{2\varepsilon_2\beta_1}I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\frac{1}{2\varepsilon_3\rho_2}I & 0 & 0 & 0 \\ * & * & * & * & * & -\frac{1}{2\varepsilon_4\beta_2}I & 0 & 0 \\ * & * & * & * & * & * & -\frac{1}{2}X & 0 \\ * & * & * & * & * & * & * & -\frac{1}{2}X \end{bmatrix} < 0. \tag{6.4.10}$$

We combine (6.4.10) and $\kappa_1 I = \frac{1}{\varepsilon_1}I$, $\kappa_2 I = \frac{1}{\varepsilon_2}I$, $\kappa_3 I = \frac{1}{\varepsilon_3}I$, $\kappa_4 I = \frac{1}{\varepsilon_4}I$, then conditions (6.4.2) are obtained. The proof is completed. □

Remark 6.1. It should be noted that although the resulting conditions (6.4.2),(6.4.3),(6.4.4) and (6.4.5) in Theorem 6.2 are not strict LMI conditions due to (6.4.5). We can cope with this nonconvex feasibility problem using the cone complementary linearization algorithm developed in El Ghaoui et al. (1997) [23] and Zhang et al. (2008) [98].

First, we transform the nonconvex feasibility problem in Theorem 6.2 into the following nonlinear minimisation problem subject to LMI constraints.

$$\begin{cases} \text{Minimise} \\ Trace\left(QX + \kappa_1 I\varepsilon_1 I + \kappa_2 I\varepsilon_2 I + \kappa_3 I\varepsilon_3 I + \kappa_4 I\varepsilon_4 I\right), \\ \text{subject to conditions(6.4.2), (6.4.3), (6.4.4)and(6.4.12).} \end{cases} \tag{6.4.11}$$

$$\begin{bmatrix} Q & I \\ I & X \end{bmatrix} \geq 0, \quad \begin{bmatrix} \kappa_1 I & I \\ I & \varepsilon_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} \kappa_2 I & I \\ I & \varepsilon_2 \end{bmatrix} \geq 0, \quad \begin{bmatrix} \kappa_3 I & I \\ I & \varepsilon_3 \end{bmatrix} \geq 0, \quad \begin{bmatrix} \kappa_4 I & I \\ I & \varepsilon_4 \end{bmatrix} \geq 0. \tag{6.4.12}$$

Then as illustrated in Zhang et al. (2008) [98], if the solution of the above minimisation problem is $5n$, that is

$$Tr\left(QX + \kappa_1 I\varepsilon_1 I + \kappa_2 I\varepsilon_2 I + \kappa_3 I\varepsilon_3 I + \kappa_4 I\varepsilon_4 I\right) = 5n,$$

then the conditions of Theorem 6.2 are solvable. Although it is yet not always possible to find the global optimal solution, the proposed nonlinear minimisation problem is easier than the original nonconvex feasibility problem. In fact, we can modify algorithm in Zhang et al. (2008) [98] to solve the above nonlinear problem as follows:

Step 1: Find a feasible set $(Q, X, Y, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4)_0$ satisfying (6.4.2),(6.4.3),(6.4.4) and (6.4.12). Set $k = 0$.

Step 2: Solve the following LMI problem

$$\begin{aligned} \text{Minimise} \quad & Tr[Q_k X + Q X_k + \kappa_{1_k} I * \varepsilon_1 I + \kappa_1 I * \varepsilon_{1_k} I + \kappa_{2_k} I * \varepsilon_2 I + \kappa_2 I * \varepsilon_{2_k} I \\ & + \kappa_{3_k} I * \varepsilon_3 I + \kappa_3 I * \varepsilon_{3_k} I + \kappa_{4_k} I * \varepsilon_4 I + \kappa_4 I * \varepsilon_{4_k} I] \qquad (6.4.13) \\ & \text{subject to conditions}(6.4.2), (6.4.3), (6.4.4) and (6.4.12) \end{aligned}$$

Step 3: Substitute the obtained variables $(Q, X, Y, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ into (6.4.8)(6.4.9)(6.4.10). If conditions (6.4.8)(6.4.9)(6.4.10) are satisfied with

$$|Tr[QX + \kappa_1 I \varepsilon_1 I + \kappa_2 I \varepsilon_2 I + \kappa_3 I \varepsilon_3 I + \kappa_4 I \varepsilon_4 I] - 5n| \leq \delta$$

for some sufficiently small scalar $\delta > 0$, then output the feasible

solutions $(Q, X, Y, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$, exit, else Step 4.

Step 4: If $k > N$, where N is the maximum number of iterations allowed, exit, else Step 5.

Step 5: Set $k = k{+}1$, $(Q, X, Y, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4)_k = (Q, X, Y, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$, and go to Step 2.

The algorithm above aims to find a feasible solution of desired controller for system (6.4.1) with given constant.

6.5 H_∞ control

In this section, we will describe the result about robust H_∞ control.

For system (6.2.1), if there exists the external disturbance $\underline{w}_t$ in it. Then it will becomes

$$\begin{cases} \underline{x}_{t+1} = A\underline{x}_t + A_0\underline{w}_t + h_1(t,\underline{x}_t) + B\underline{u}_t \\ \qquad\qquad + \left(C\underline{x}_t + C_0\underline{w}_t + h_2(t,\underline{x}_t) + D\underline{u}_t\right) v_t, \\ \underline{z}_t = \begin{bmatrix} L\underline{x}_t \\ M\underline{w}_t \end{bmatrix} \\ \underline{x}_0 = x_0 \in \mathbb{R}^n, \quad t \in N, \end{cases} \tag{6.5.1}$$

where $\underline{w}_t \in \mathbb{R}^q$ is the outside disturbance and is independent of v_t. $\underline{z}_t \in \mathbb{R}^p$ is the controlled output.

Definition 6.2. For a given disturbance attenuation level $\gamma > 0$, $\underline{u}_t = K\underline{x}_t$ is an H_∞ controller of system (6.5.1), if (i). system (6.5.1) is internally stochastically stabilizable for $\underline{u}_t = K\underline{x}_t$ in the absence of external disturbance $\underline{w}_t$;
(ii). The H_∞ norm of system (6.5.1) is less than $\gamma > 0$ with zero initial condition $x_0 = 0$, which is

$$\begin{aligned} \|H\| &= \sup_{w\in l_w^2(N,\mathbb{R}^q),\underline{w}_t\neq 0} \frac{\|\underline{z}_t\|_{l_w^2(N,\mathbb{R}^q)}}{\|\underline{w}_t\|_{l_w^2(N,\mathbb{R}^q)}} \\ &= \sup_{w\in l_w^2(N,\mathbb{R}^q),\underline{w}_t\neq 0} \frac{\left(\sum_{t=0}^{\infty}\mathbb{E}\|\underline{z}_t\|^2\right)^{\frac{1}{2}}}{\left(\sum_{t=0}^{\infty}\mathbb{E}\|\underline{w}_t\|^2\right)^{\frac{1}{2}}} < \gamma \end{aligned}$$

Theorem 6.3. *For system (6.5.1) with given constant ρ_1, ρ_2, β_1, β_2, γ_1, γ_2, if there exist real matrices Y, $X > 0$, $Q > 0$ and real scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > 0$, $\kappa_1 > 0$, $\kappa_2 > 0$, $\kappa_3 > 0$, $\kappa_4 > 0$ such that the following conditions hold:*

$$
\left[\begin{matrix}
-X & 0 & L^T & 0 & (AX+BY)^T & (CX+DY)^T \\
* & -\gamma^2 I & 0 & M^T & A_0 & C_0 \\
* & * & -I & 0 & 0 & 0 \\
* & * & * & -I & 0 & 0 \\
* & * & * & * & -X & 0 \\
* & * & * & * & * & -X \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & *
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
(AX+BY)^T & (CX+DY)^T & 0 & 0 & X & X & X & X \\
0 & 0 & A_0^T & C_0^T & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & -X & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -X & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -X & 0 & 0 & 0 & 0 \\
* & * & * & * & -\frac{\kappa_1}{2\rho_1} I & 0 & 0 & 0 \\
* & * & * & * & * & -\frac{\kappa_2}{2\beta_1} I & 0 & 0 \\
* & * & * & * & * & * & -\frac{\kappa_3}{2\rho_2} I & 0 \\
* & * & * & * & * & * & 0 & -\frac{\kappa_4}{2\beta_2} I
\end{matrix}\right] < 0, \tag{6.5.2}
$$

$$
\begin{bmatrix}
3Q - 2\varepsilon_2 I & \varepsilon_2\gamma_2 I - \varepsilon_1 I \\
\varepsilon_2\gamma_2 I - \varepsilon_1 I & -I.
\end{bmatrix} < 0, \tag{6.5.3}
$$

$$\begin{bmatrix} 3Q - 2\varepsilon_4 I & \varepsilon_4\gamma_2 I - \varepsilon_3 I \\ \varepsilon_4\gamma_2 I - \varepsilon_3 I & -I \end{bmatrix} < 0, \tag{6.5.4}$$

$$QX = I, \quad \kappa_1 I * \varepsilon_1 I = I, \quad \kappa_2 I * \varepsilon_2 I = I, \quad \kappa_3 I * \varepsilon_3 I = I, \quad \kappa_4 I * \varepsilon_4 I = I. \tag{6.5.5}$$

Then system(6.5.1) is H_∞ controllable for the given $\gamma > 0$, and the robust H_∞ controller $\underline{U}_t = K\underline{x}_t = YX^{-1}\underline{x}_t$ for $t \in N$.

Remark 6.2. K is parametrised by YX^{-1} so that we can implement Schur complement to generate LMI (6.5.2). Matrix X and Y are the feasible solution of LMI (6.5.2).

Proof. Since LMI (6.5.2) implies LMI (6.4.2), system (6.5.1) is stablizable through controller $\underline{u}_t = K\underline{x}_t$ by theorem 6.2 when $\underline{w}_t = 0$. Then, we only need to show $\|H\| < \gamma$. Take $\underline{u}_t = K\underline{x}_t$ and choose the Lyapunov function $V(\underline{x}_t) = \underline{x}_t^T Q\underline{x}_t$.

$$\begin{cases} \underline{x}_{t+1} = \bar{A}\underline{x}_t + A_0(t)\underline{w}_t + h_1(t,\underline{x}_t) + \left(\bar{C}\underline{x}_t + C_0(t)\underline{w}_t + h_2(t,\underline{x}_t)\right) v_t, \\ \underline{x}_0 = x_0 \in \mathbb{R}^n, t \in N, \end{cases} \tag{6.5.6}$$

with $\bar{A} = A + BK$ and $\bar{C} = C + DK$. Since we have $\underline{x}_t$ and $\underline{w}_t$ independent of v_t, then we can derive

$$\begin{aligned}
& \mathbb{E}\Delta V(\underline{x}_t) \\
= \; & \mathbb{E}\left[V(\underline{x}_{t+1}) - V(\underline{x}_t)\right] \\
\leq \; & \mathbb{E}\left[\underline{x}_{t+1}^T Q\underline{x}_{t+1} - \underline{x}_t^T Q\underline{x}_t\right] + \underline{x}_t^T[2(\varepsilon_1\rho_1 + \varepsilon_2\beta_1)I + 2(\varepsilon_3\rho_2 + \varepsilon_4\beta_2)I]\underline{x}_t \\
& + x^T(t)2(\varepsilon_2\gamma_1 - \varepsilon_1)Ih_1(t,\underline{x}_t) + x^T(t)2(\varepsilon_4\gamma_2 - \varepsilon_3)Ih_2(t,\underline{x}_t) \\
& - 2\varepsilon_2 h_1(t,\underline{x}_t)^T h_1(t,\underline{x}_t) - 2\varepsilon_4 h_2(t,\underline{x}_t)^T h_2(t,\underline{x}_t) \\
= \; & \mathbb{E}\{\underline{x}_t^T[\bar{A}^T Q\bar{A} + \bar{C}^T Q\bar{C} - Q + 2(\varepsilon_1\rho_1 + \varepsilon_2\beta_1)I + 2(\varepsilon_3\rho_2 + \varepsilon_4\beta_2)I]\underline{x}_t \\
& + \underline{x}_t^T[\bar{A}^T Q + (\varepsilon_2\gamma_1 - \varepsilon_1)I]h_1(t,\underline{x}_t) + \underline{x}_t^T[\bar{C}^T Q + (\varepsilon_4\gamma_2 - \varepsilon_3)I]h_2(t,\underline{x}_t) \\
& + \underline{x}_t^T\left[\bar{A}^T QA_0 + \bar{C}^T QC_0\right]\underline{w}_t + h_1^T(t,\underline{x}_t)[Q\bar{A} + (\varepsilon_2\gamma_1 - \varepsilon_1)I]\underline{x}_t \\
& + h_1^T(t,\underline{x}_t)[Q - 2\varepsilon_2 I]h_1(t,\underline{x}_t) + h_1^T(t,\underline{x}_t)QA_0\underline{w}_t + h_2^T(t,\underline{x}_t)[Q\bar{C} + (\varepsilon_4\gamma_2 - \varepsilon_3)I]\underline{x}_t \\
& + h_2^T(t,\underline{x}_t)[Q - 2\varepsilon_4 I]h_2(t,\underline{x}_t) + h_2^T(t,\underline{x}_t)QC_0\underline{w}_t + w^T(t)(A_0^T QA_0 + C_0^T QC_0)\underline{w}_t \\
& + \underline{w}_t^T(A_0^T Q\bar{A} + C_0^T Q\bar{C})\underline{x}_t + \underline{w}_t^T A_0^T Qh_1(t,\underline{x}_t) + \underline{w}_t^T C_0^T Qh_2(t,\underline{x}_t)\}.
\end{aligned} \tag{6.5.7}$$

With the zero initial condition, for any $\underline{w}_t \in l_w^2(N, \mathbb{R}^q)$,

$$
\begin{aligned}
& \|\underline{z}_t\|^2 - \gamma^2 \|\underline{w}_t\|^2 \\
= \; & \mathbb{E} \sum_{t=0}^{T_0} \left[\underline{x}_t^T L^T L \underline{x}_t + \underline{w}_t^T M^T M \underline{w}_t - \gamma^2 \underline{w}_t^T \underline{w}_t + \Delta V(\underline{x}_t) \right] - \underline{x}_{T_0}^T Q \underline{x}_{T_0} \\
\leq \; & \mathbb{E} \sum_{t=0}^{T_0} \{ \underline{x}_t^T L^T L \underline{x}_t + \underline{w}_t^T M^T M \underline{w}_t - \gamma^2 \underline{w}_t^T \underline{w}_t + \underline{w}_t^T (A_0^T Q A_0 + C_0^T Q C_0) \underline{w}_t \\
& + \underline{x}_t^T [\bar{A}^T Q \bar{A} + \bar{C}^T Q \bar{C} - Q + 2(\varepsilon_1 \rho_1 + \varepsilon_2 \beta_1) I + 2(\varepsilon_3 \rho_2 + \varepsilon_4 \beta_2) I] \underline{x}_t \\
& + \underline{x}_t^T [\bar{A}^T Q + (\varepsilon_2 \gamma_1 - \varepsilon_1) I] h_1(t, \underline{x}_t) + \underline{x}_t^T [\bar{C}^T Q + (\varepsilon_4 \gamma_2 - \varepsilon_3) I] h_2(t, \underline{x}_t) \\
& + \underline{x}_t^T \left[\bar{A}^T Q A_0 + \bar{C}^T Q C_0 \right] \underline{w}_t + h_1^T(t, \underline{x}_t) [Q \bar{A} + (\varepsilon_2 \gamma_1 - \varepsilon_1) I] \underline{x}_t \\
& + h_1^T(t, \underline{x}_t) [Q - 2\varepsilon_2 I] h_1(t, \underline{x}_t) + h_1^T(t, \underline{x}_t) Q A_0 \underline{w}_t + h_2^T(t, \underline{x}_t) [Q \bar{C} + (\varepsilon_4 \gamma_2 - \varepsilon_3) I] \underline{x}_t \\
& + h_2^T(t, \underline{x}_t) [Q - 2\varepsilon_4 I] h_2(t, \underline{x}_t) + h_2^T(t, \underline{x}_t) Q C_0 \underline{w}_t + \underline{w}_t^T (A_0^T Q A_0 + C_0^T Q C_0) \underline{w}_t \\
& + \underline{w}_t^T (A_0^T Q \bar{A} + C_0^T Q \bar{C}) \underline{x}_t + \underline{w}_t^T A_0^T Q h_1(t, \underline{x}_t) + \underline{w}_t^T C_0^T Q h_2(t, \underline{x}_t) \}. \qquad (6.5.8)
\end{aligned}
$$

Applying Young's inequality, Using lemma 6.1, we have

$$
\begin{aligned}
& \underline{x}_t^T \bar{A}^T Q h_1(t, \underline{x}_t) + h_1^T(t, \underline{x}_t) Q \bar{A} \underline{x}_t \\
\leq \; & \underline{x}_t^T \bar{A}^T Q \bar{A} \underline{x}_t + h_1^T(t, \underline{x}_t) Q h_1(t, \underline{x}_t). \qquad (6.5.9) \\
& \underline{x}_t^T (\varepsilon_2 \gamma_1 - \varepsilon_1) I h_1(t, \underline{x}_t) + h_1^T(t, \underline{x}_t) (\varepsilon_2 \gamma_1 - \varepsilon_1) I \underline{x}_t \\
\leq \; & \underline{x}_t^T \underline{x}_t + h_1^T(t, \underline{x}_t) (\varepsilon_2 \gamma_1 - \varepsilon_1)^2 I h_1(t, \underline{x}_t). \qquad (6.5.10) \\
& \underline{x}_t^T \bar{C}^T Q h_2(t, \underline{x}_t) + h_2^T(t, \underline{x}_t) Q \bar{C} \underline{x}_t \\
\leq \; & \underline{x}_t^T \bar{C}^T Q \bar{C} \underline{x}_t + h_2^T(t, \underline{x}_t) Q h_2(t, \underline{x}_t). \qquad (6.5.11) \\
& \underline{x}_t^T (\varepsilon_4 \gamma_2 - \varepsilon_3) I h_2(t, \underline{x}_t) + h_2^T(t, \underline{x}_t) (\varepsilon_4 \gamma_2 - \varepsilon_3) I \underline{x}_t \\
\leq \; & \underline{x}_t^T \underline{x}_t + h_2^T(t, \underline{x}_t) (\varepsilon_4 \gamma_2 - \varepsilon_3)^2 I h_2(t, \underline{x}_t). \qquad (6.5.12)
\end{aligned}
$$

$$
\begin{aligned}
& h_1^T(t, \underline{x}_t) Q A_0 \underline{w}_t + w^T(t) A_0^T Q h_1(t, \underline{x}_t) \\
\leq \; & h_1^T(t, \underline{x}_t) Q h_1(t, \underline{x}_t) + w^T(t) A_0^T Q A_0 \underline{w}_t, \qquad (6.5.13)
\end{aligned}
$$

$$
\begin{aligned}
& h_2^T(t,\underline{x}_t)QC_0\underline{w}_t + w^T(t)C_0^TQh_2(t,\underline{x}_t) \\
\leq\ & h_2^T(t,\underline{x}_t)Qh_2(t,\underline{x}_t) + w^T(t)C_0^TQC_0\underline{w}_t.
\end{aligned} \tag{6.5.14}
$$

Substituting inequalities in (6.5.9)-(6.5.14) into inequality (6.5.8) , it yields

$$
\begin{aligned}
& \|\underline{z}_t\|^2 - \gamma^2\|\underline{w}_t\|^2 \\
\leq\ & \mathbb{E}\sum_{t=0}^{T_0}\{\underline{x}_t^T[-Q + 2\bar{A}^TQ\bar{A} + 2\bar{C}^TQ\bar{C} + L^TL + 2(\varepsilon_1\rho_1 + \varepsilon_2\beta_1)I + 2(\varepsilon_3\rho_2 + \varepsilon_4\beta_2)I]\underline{x}_t \\
& + \underline{x}_t^T[\bar{A}^TQA_0 + \bar{C}^TQC_0]\underline{w}_t + w^T(t)[A_0^TQ\bar{A} + C_0^TQ\bar{C}]\underline{x}_t \\
& + \underline{w}_t^T(2A_0^TQA_0 + 2C_0^TQC_0 - \gamma^2I + M^TM)\underline{w}_t \\
& + h_1^T(t,\underline{x}_t)[3Q + (\varepsilon_2\gamma_1 - \varepsilon_1)^2I - 2\varepsilon_2I]h_1(t,\underline{x}_t) \\
& + h_2^T(t,\underline{x}_t)[3Q + (\varepsilon_4\gamma_2 - \varepsilon_3)^2I - 2\varepsilon_4I]h_2(t,\underline{x}_t)\} \\
=\ & \mathbb{E}\sum_{t=0}^{T_0}\begin{bmatrix}\underline{x}_t \\ \underline{w}_t\end{bmatrix}^T \Xi_1 \begin{bmatrix}\underline{x}_t \\ \underline{w}_t\end{bmatrix} + \mathbb{E}\sum_{t=0}^{T_0}\left[h_1^T(t,\underline{x}_t)\Xi_2h_1(t,\underline{x}_t)\right] + \mathbb{E}\sum_{t=0}^{T_0}\left[h_2^T(t,\underline{x}_t)\Xi_3h_2(t,\underline{x}_t)\right]
\end{aligned} \tag{6.5.15}
$$

where

$$
\Xi_1 = \begin{bmatrix}\Xi_{11} & \Xi_{12} \\ * & \Xi_{22}\end{bmatrix}, \tag{6.5.16}
$$

$$
\Xi_2 = 3Q + (\varepsilon_2\gamma_2 - \varepsilon_1)^2I - 2\varepsilon_2I, \tag{6.5.17}
$$

$$
\Xi_3 = 3Q + (\varepsilon_4\gamma_2 - \varepsilon_3)^2I - 2\varepsilon_4I. \tag{6.5.18}
$$

with

$$
\begin{aligned}
\Xi_{11} &= -Q + 2\bar{A}^TQ\bar{A} + 2\bar{C}^TQ\bar{C} + L^TL + 2(\varepsilon_1\rho_1 + \varepsilon_2\beta_1)I + 2(\varepsilon_3\rho_2 + \varepsilon_4\beta_2)I, \\
\Xi_{12} &= \bar{A}^TQA_0 + \bar{C}^TQC_0, \\
\Xi_{22} &= 2A_0^TQA_0 + 2C_0^TQC_0 - \gamma^2I + M^TM.
\end{aligned} \tag{6.5.19}
$$

Let $T \to \infty$ in (6.5.15); then we have

$$
\|\underline{z}_t\|^2 - \gamma^2\|\underline{w}_t\|^2 \leq \mathbb{E}\sum_{t=0}^{\infty}\begin{bmatrix}\underline{x}_t \\ \underline{w}_t\end{bmatrix}^T \Xi \begin{bmatrix}\underline{x}_t \\ \underline{w}_t\end{bmatrix}. \tag{6.5.20}
$$

$\Xi < 0$ is satisfied when $\Xi_1 < 0, \Xi_2 < 0$ and $\Xi_3 < 0$. By Schur complement, $\Xi_1 < 0$ is equivalent to

$$\begin{bmatrix} -Q + 2(\varepsilon_1\rho_1 + \varepsilon_2\beta_1)I + 2(\varepsilon_3\rho_2 + \varepsilon_4\beta_2)I & 0 & L^T & 0 & \bar{A}^T & \bar{C}^T & \bar{A}^T Q & \bar{C}^T Q & 0 & 0 \\ * & -\gamma^2 I & 0 & M^T & A_0 & C_0 & 0 & 0 & A_0^T Q & C_0^T Q \\ * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -Q^{-1} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q^{-1} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -Q & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -Q & 0 & 0 \\ * & * & * & * & * & * & * & * & -Q & 0 \\ * & * & * & * & * & * & * & * & * & -Q \end{bmatrix} < \quad 0. \tag{6.5.21}$$

$\Xi_2 < 0$ is equivalent to

$$\begin{bmatrix} 3Q - 2\varepsilon_2 I & \varepsilon_2\gamma_2 I - \varepsilon_1 I \\ \varepsilon_2\gamma_2 I - \varepsilon_1 I & -I. \end{bmatrix} < 0. \tag{6.5.22}$$

and $\Xi_3 < 0$ is equivalent to

$$\begin{bmatrix} 3Q - 2\varepsilon_4 I & \varepsilon_4\gamma_2 I - \varepsilon_3 I \\ \varepsilon_4\gamma_2 I - \varepsilon_3 I & -I \end{bmatrix} < 0. \tag{6.5.23}$$

In order to derive the matrix K, we set $Q^{-1} = X$ and pre- and post multiply $diag[X \quad I \quad I \quad I \quad I \quad I \quad X \quad X \quad X \quad X]$ on both side of (6.5.21). Then we have

$$\begin{bmatrix} -X + 2X(\varepsilon_1\rho_1 + \varepsilon_2\beta_1)IX + 2X(\varepsilon_3\rho_2 + \varepsilon_4\beta_2)IX & 0 & L^T & 0 & X\bar{A}^T & X\bar{C}^T & X\bar{A}^T & X\bar{C}^T & 0 & 0 \\ * & -\gamma^2 I & 0 & M^T & A_0 & C_0 & 0 & 0 & A_0^T & C_0^T \\ * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -X & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -X & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -X & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -X & 0 & 0 \\ * & * & * & * & * & * & * & * & -X & 0 \\ * & * & * & * & * & * & * & * & * & -X \end{bmatrix} < 0, \tag{6.5.24}$$

Then, using Schur complement we can derive LMI

$$
\left[\begin{array}{cccccc}
-X & 0 & L^T & 0 & X\bar{A}^T & X\bar{C}^T \\
* & -\gamma^2 I & 0 & M^T & A_0 & C_0 \\
* & * & -I & 0 & 0 & 0 \\
* & * & * & -I & 0 & 0 \\
* & * & * & * & -X & 0 \\
* & * & * & * & * & -X \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & *
\end{array}\right.
$$

$$
\left.\begin{array}{cccccccc}
X\bar{A}^T & X\bar{C}^T & 0 & 0 & X & X & X & X \\
0 & 0 & A_0^T & C_0^T & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & -X & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -X & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -X & 0 & 0 & 0 & 0 \\
* & * & * & * & -\frac{1}{2\varepsilon_1\rho_1}I & 0 & 0 & 0 \\
* & * & * & * & * & -\frac{1}{2\varepsilon_2\beta_1}I & 0 & 0 \\
* & * & * & * & * & * & -\frac{1}{2\varepsilon_3\rho_2}I & 0 \\
* & * & * & * & * & * & 0 & -\frac{1}{2\varepsilon_4\beta_2}I
\end{array}\right] < 0. \quad (6.5.25)
$$

We combine (6.5.25) and $\kappa_1 I = \frac{1}{\varepsilon_1}I$, $\kappa_2 I = \frac{1}{\varepsilon_2}I$, $\kappa_3 I = \frac{1}{\varepsilon_3}I$, $\kappa_4 I = \frac{1}{\varepsilon_4}I$, then conditions (6.5.2) are obtained. The proof is completed. □

Remark 6.3. It should be noted that although the resulting conditions (6.5.2),(6.5.3) and (6.5.4) in Theorem 6.3 are not strict LMI conditions due to (6.5.5), we can cope with this nonconvex feasibility problem using similar algorithm introduced in the previous section.

First, we transform the nonconvex feasibility problem in Theorem 6.2 into the following nonlinear minimisation problem subject to LMI constraints.

$$\begin{cases} \text{Minimise} \\ Trace\left(QX + \kappa_1 I \varepsilon_1 I + \kappa_2 I \varepsilon_2 I + \kappa_3 I \varepsilon_3 I + \kappa_4 I \varepsilon_4 I\right), \\ \text{subject to conditions(6.5.2), (6.5.3)(6.5.4)and(6.5.27).} \end{cases} \tag{6.5.26}$$

$$\begin{bmatrix} Q & I \\ I & X \end{bmatrix} \geq 0, \quad \begin{bmatrix} \kappa_1 I & I \\ I & \varepsilon_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} \kappa_2 I & I \\ I & \varepsilon_2 \end{bmatrix} \geq 0, \quad \begin{bmatrix} \kappa_3 I & I \\ I & \varepsilon_3 \end{bmatrix} \geq 0, \quad \begin{bmatrix} \kappa_4 I & I \\ I & \varepsilon_4 \end{bmatrix} \geq 0. \tag{6.5.27}$$

If the solution of the above minimisation problem is $5n$, that is

$$Tr\left(QX + \kappa_1 I \varepsilon_1 I + \kappa_2 I \varepsilon_2 I + \kappa_3 I \varepsilon_3 I + \kappa_4 I \varepsilon_4 I\right) = 5n$$

then the conditions of Theorem 6.2 are solvable. As discussed in Remark 6.1, although it is yet not always possible to find the global optimal solution, the proposed nonlinear minimisation problem is easier than the original nonconvex feasibility problem. Then, we follow similar method in Remark 6.1 to solve the above nonlinear problem as follows:

Step 1: Find a feasible set $(Q, X, Y, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4)_0$ satisfying (6.5.22),(6.5.23),(6.5.25) and (6.5.27). Set $k = 0$.

Step 2: Solve the following LMI problem

$$\begin{aligned} \text{Minimise} \quad & Tr[Q_k X + Q X_k + \kappa_{1_k} I * \varepsilon_1 I + \kappa_1 I * \varepsilon_{1_k} I + \kappa_{2_k} I * \varepsilon_2 I + \kappa_2 I * \varepsilon_{2_k} I \\ & + \kappa_{3_k} I * \varepsilon_3 I + \kappa_3 I * \varepsilon_{3_k} I + \kappa_{4_k} I * \varepsilon_4 I + \kappa_4 I * \varepsilon_{4_k} I] \qquad (6.5.28) \\ & \text{subject to conditions(6.5.2), (6.5.3)(6.5.4)and(6.5.27)} \end{aligned}$$

Step 3: Substitute the obtained variables $(Q, X, Y, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ into (6.5.22),(6.5.23),(6.5.25). If conditions (6.5.22),(6.5.23) and (6.5.25) are satisfied with

$$|Tr[QX + \kappa_1 I \varepsilon_1 I + \kappa_2 I \varepsilon_2 I + \kappa_3 I \varepsilon_3 I + \kappa_4 I \varepsilon_4 I] - 5n| \leq \delta$$

for some sufficiently small scalar $\delta > 0$, then output the feasible solutions $(Q, X, Y, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$, exit, else Step 4.

Step 4: If $k > N$, where N is the maximum number of iterations allowed, exit, else Step 5.

Step 5: Set $k = k+1$, $(Q, X, Y, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4)_k = (Q, X, Y, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$, and go to Step 2.

The algorithm above aims to find a feasible solution of desired robust H_∞ controller for system (6.5.1) with given constant.

6.6 Summary

In this chapter, we have investigated the stabilisation problem of stochastic discrete-time quadratic bounded time-varying nonlinear system. The algorithm in Remark 6.3 could provide a feasible robust H_∞ controller. In the next chapter, we will apply results in this chapter in a non-life insurance problem.

Chapter 7

Robust H_∞ control for classes of time-varying nonlinear discrete time stochastic P-R systems

7.1 Introduction

In this chapter, we intend to further investigate the problem of P-R system which has been discussed in paper Pantelous and Yang (2014) [66] and Yang et al. (2016) [94]. Previous developments of control theory in P-R system are largely focused in the linear discrete time framework. All linear control methods are based on the assumption that the system to be controlled can be accurately described or approximated by linear discrete time system with or without uncertainties. In Pantelous and Yang (2014) [66] and Yang et al. (2016) [94], the P-R systems are modelled by a linear discrete time system. The P-R system in Pantelous and Yang (2014) [66] considers a linear stochastic P-R system with admissible parameter uncertainties. In practice, serveral factors may make nonlinear effect on the process of accumulated reserve. For example, the investment return generated by accumulated reserve could obey some nonlinearity due to taxation and decreasing marginal investment rate, if accumulated reserve exceed a limit. Therefore, the linear model for P-R system could not be an accurate description of the real system. And under nonlinear modelling framework, the properties of P-R systems will be better described.

In this chapter, the P-R system will be modelled by a nonlinear uncertain system with Lipschitz-type and quadratic bounded conditions respectively. The parameter uncertainties and model uncertainties can follow specified nonlinear conditions. Then, we have to consider the consequential impact of these nonlinear uncertainties on the stability of P-R system.

In other words, in this chapter, we will present first a time-varying nonlinear discrete stochastic P-R systems subject to Lipschitz-type condition. Then the problem of stabilization and controlability is investigated for a general class of discrete time nonlinear stochastic P-R systems. A H_∞ controller for the P-R system is designed which guarantees the stability of system, and methodology for the designing of a stabilizing feedback controller for discrete-time nonlinear stochastic system with structured parameter uncertainties is proposed. In the second part, we will present a P-R system subject to Lipschitz-type condition. We will investigate the stablity and H_∞ controller design for one-sided Lipschitz-type nonlinear P-R system based on the theorem we derived in Chapter 6.

7.2 Model formulation

7.2.1 The Reserve Process

$\underline{R}_t = (R_{1,t}R_{2,t}\cdots R_{m,t})^T$ is the vector of the accumulated reserves, where $R_{i,t}$ is the accumulated reserve of i^{th} product at time t. The accumulated reserve $\underline{R}_t$ is defined by

$$\underline{R}_{t+1} = J\underline{R}_t + h_1(t, \underline{R}_t) + e\underline{P}_{t+1} - \underline{C}_{t+1} + [J\underline{R}_t + h_1(t, \underline{R}_t)]v_t, \tag{7.2.1}$$

where J is the base investment return matrix. Now, we assume that v_t is a sequence of one-dimensional independent white noise processes defined on the complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathscr{P})$, where $\mathscr{F}_t = \sigma\{v_0, v_1, v_2, \ldots, v_t\}$. v_t is used to model different types of financial uncertainties such as inflation, taxation policy etc. We assume that

$$\mathbb{E}\{v_t\} = 0, \quad \mathbb{E}\{v_t v_j\} = \delta_{tj}, \tag{7.2.2}$$

where δ_{tj} is a Kronecker function defined by $\delta_{tj} = 0$ for $t \neq j$ and $\delta_{tj} = 1$ for $t = j$.

Moreover, we assume that the investment strategy is to invest all of accumulated reserves to risk-free asset, and J could be a risk-free interest rate. In insurance industry, it's a common practice for insurer to invest majorities of accumulated reserves of short term insurance product to some short term fixed income investment such as bonds with duration at most 6 month (Pantelous and Yang (2014) [66])

7.2.2 The Premium Rating Rule

In this chapter, the premium rating rule is designed to be a feedback mechanism. the premium process is formulated as follows:

$$\underline{P}_{t+1} = \underline{\hat{C}}_{t+1} - Z\underline{U}_t(1 + v(t)). \tag{7.2.3}$$

$\underline{U}_t$ is the controller element to premium.

where $\underline{\hat{C}}$ is the 'claim estimator', which is proposed in Zimbidis and Harberman (2001)[108] and will be explained in more details in the next section 7.2.3. $\underline{U}_t \in \mathbb{R}^m$ is the control input that has been added in the original system. However, for simplicity, without loss of generality, the state feedback controller is considered to be $\underline{U}_t = K\underline{R}_t$, where the matrix K should be determined. In practice, we can assume Z is an identity matrix such that the controller is derived and have impact on the premium directly.

As we can see in equation(7.2.3), the stochastic parameter v_t can be implemented similar with that in Pantelous and Yang (2014)[66]. As it becomes clearer later in this chapter, the appropriate robust stabilizing controllers $\underline{U}_t$ for the P-R process are constructed by solving appropriate LMI or non-strict LMI problems.

In this model, the insurer can control its financial position. A suitable control of premiums can lead to a stable and realistic evolution of the accumulated reserve as well as solvency margin.

7.2.3 Claim's Estimator

The claims have been incurred by the end of the accounting year. Since usually a substantial part of the incurred claims is unknown when the balance sheet is compiled, their total value has to be estimated. This estimate is for the claims incurred which is subject to a considerable degree of errors. Meanwhile, the amount of claims in one year would be cleared not until many years in the future, in some insurance lines or cases even in one decade.

The premium P_{t+1} for the $(t + 1)$ year is calculated by *claim estimator* $\hat{C}_{t+1}$. As in Zimbidis and Haberman (2001) [108] $\hat{C}_{t+1}$ is determined by the inflation-weighted average of the most recent available claim experience of the f years $[C_{t-\tau_t-f}, C_{t-\tau_t-f+1}, \cdots, C_{t-\tau_t}]$ and a feedback mechanism using the past reserve value of $R_{t-\tau}$.

$$\hat{C}_{t+1} = \frac{1}{Me}[(1+j)^{f+\tau_t}C_{t-\tau_t-f} + (1+j)^{f+\tau_t-1}C_{t-\tau_t-f+1} + \cdots + (1+j)^{\tau_t}C_{t-\tau_t},$$

$$M = \sum_{k=0}^{f}(1+j)^{f+\tau_t-k}.$$

where j is the inflation rate. An inaccurate claims estimation is misleading in many ways and can have fatal consequences. For instance, an underestimation of the claims incurred can result in unprofitable premium level. Underestimation of the claims also lead to a higher probability of insolvency, which can delay corrective action by the management. In this paper, $\underline{w}_{t+1}$ is one of the disturbance to system which is caused by the error between estimated claim value and actual incurred value.

$$\underline{w}_{t+1} = e\underline{\hat{C}}_{t+1} - \underline{C}_{t+1} \in l_{e_2}([\mathbb{N}; \mathbb{R}^m),$$

$\underline{C}_t = (C_{1,t}, C_{2,t} \cdots C_{m,t})^T$ for $t \in \mathbb{N}$ is the vector of the incurred claims which is assumed to follow a stochastic process.

As described in Zimbidis and Haberman (2001) [108], Pantelous and Papageorgiou (2013) [65], Pantelous and Yang (2014) [66] and Yang et al. (2016) [94], the relationship among the administration expenses, the relative operation costs, the desired profit margin and corresponding premium can be expressed by the equation:

$$\textbf{Operation Costs} + \textbf{Profit Margin} = (1-e)P_t$$

7.2.4 P-R system

In this chapter, the P-R system is developed into a nonlinear stochastic, discrete-time framework. And the case that the system is affected by external disturbances $\underline{w}_{t+1}$ is also considered as well. In P-R systems, the existence of external disturbance $\underline{w}_{t+1} \neq 0$ means actual incurred claims are not exactly the same with the claim estimator. The P-R systems is described by a class of Lipschitz-type or one-sied Lipschitz-type time-varying nonlinear system, which makes significant difference from Pantelous and Yang

(2014)[66] research work. In their paper, the P-R systems are assumed to be a linear discrete time system with an admissible parameter uncertainties. Therefore, theorems in Pantelous and Yang (2014) [66] are extended.

In practice, it is often difficult and unnecessary to obtain a precise linear relationships for the dynamics of accumulated reserves in P-R system. And a premium which is sufficient enough to cover the expected claims and to keep the derived reserves (surplus) stable is always required. Therefore, the nonlinear discrete stochastic accumulated reserve process described by the following equation:

$$\begin{cases} \underline{R}_{t+1} = J\underline{R}_t + h_1(t, \underline{R}_t) + e\underline{P}_{t+1} - \underline{C}_{t+1} + [J\underline{R}_t + h_2(t, \underline{R}_t)]v_t, \\ \underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0]. \end{cases} \tag{7.2.4}$$

After substituing eq.(7.2.3) into eq.(7.2.4), we derive the time-varying nonlinear discrete stochastic P-R system which is:

$$\Theta_1 : \begin{cases} \underline{R}_{t+1} = J\underline{R}_t + h_1(t, \underline{R}_t) - eZ\underline{U}_t + [J\underline{R}_t + h_2(t, \underline{R}_t) + -eZ\underline{U}_t]v_t + \underline{w}_{t+1} \\ \underline{z}_t = \underline{R}_t \\ \underline{R}_t = \underline{\varphi}_t \; for \; t \in [-\tau_{\max}, 0], \end{cases} \tag{7.2.5}$$

where $\underline{w}_{t+1} = e\underline{\hat{C}}_{t+1} - \underline{C}_{t+1} \in l_{e_2}(N; \mathbb{R}^m)$ and $z(t) \in \mathbb{R}^p$ is the controlled output.. We denote the above system as Θ_1. The stochastic disturbance parameter $v(t)$ is defined by eq. (7.3.6).

Also, substituting the control input $\underline{U}_t = K\underline{R}_t$, our new closed loop system becomes

$$\begin{cases} \underline{R}_{t+1} = [J - eZK]\underline{R}_t + h_1(t, \underline{R}_t) + \{[J - eZK]\underline{R}_t + h_2(t, \underline{R}_t)\}v_t + \underline{w}_{t+1} \\ \underline{z}_t = \underline{R}_t \\ \underline{R}_t = \underline{\varphi}_t \; for \; t \in [-\tau_{\max}, 0], \end{cases} \tag{7.2.6}$$

with initial conditions $\underline{R}_t = \underline{\varphi_t} \; for \; t \in [-\tau_{\max}, 0]$. We denote the above system with feedback controller $\underline{U}_t$ as Θ_1.

7.3 Lipschitz-type time-varying nonlinear P-R system

In this section, we will focus on P-R systems with Lipschitz-type time-varying nonlinearity. First, we will give the specific expression of Lipschitz-type time-varying condition which is assumed to hold throughout section 7.3.

Lipschitz-type nonlinear condition: The nonlinear functions $h_1(t, \underline{R}_t), h_2(t, \underline{R}_t)$ describe uncertainties of the system and satisfy the following quadratic inequalities:

$$h_1^T(t, \underline{R}_t)h_1(t, \underline{R}_t) \leq \alpha_1^2 \underline{R}_t^T H_1^T H_1 \underline{R}_t, \tag{7.3.1}$$

$$h_2^T(t, \underline{R}_t)h_2(t, \underline{R}_t) \leq \alpha_2^2 \underline{R}_t^T H_2^T H_2 \underline{R}_t, \tag{7.3.2}$$

for all $t \in N$, where α_i is a constant related to the function h_i for $i = 1, 2$. H_i is a constant matrix reflecting structure of h_i.

We note that inequalities (7.3.1) and (7.3.2) can be written as a matrix form:

$$\begin{bmatrix} \underline{R}_t \\ h_1 \\ h_2 \end{bmatrix}^T \begin{bmatrix} -\alpha_1^2 H_1^T H_1 - \alpha_2^2 H_2^T H_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \underline{R}_t \\ h_1 \\ h_2 \end{bmatrix} \leq 0. \tag{7.3.3}$$

7.3.1 Robust stability of the system

Considering the following basic P-R system Θ_2 without disturbance and controller,

$$\Theta_2 : \begin{cases} \underline{R}_{t+1} = J\underline{R}_t + h_1(t, \underline{R}_t) + \{J\underline{R}_t + h_2(t, \underline{R}_t)\}v_t, \\ \underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0], \end{cases} \tag{7.3.4}$$

Definition 7.1. The Lipschitz-type time-varying nonlinear P-R system Θ_2 is said to be robustly stochastically stable with margins $\alpha_1 > 0$, $\alpha_2 > 0$ if there exists a constant $\delta(x_0, \alpha)$ such that

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \underline{R}_t^T \underline{R}_t\right] \leq \delta(x_0, \alpha_1 > 0, \alpha_2). \tag{7.3.5}$$

We can derive the following theorem about the system stability.

Theorem 7.1. *System Θ_2 with margins $\alpha_1 > 0$ and $\alpha_2 > 0$ is said to be robustly stochastically stable, if there exists a symmetric positive definite matrix $Q > 0$ and a* *scalar $\alpha > 0$ such that*

$$\begin{bmatrix} -Q + 2\alpha_1^2 \alpha H^T H + 2\alpha_2^2 \alpha H^T H & A^T Q & C^T Q & 0 \\ * & -\frac{1}{2}Q & 0 & 0 \\ * & * & -\frac{1}{2}Q & 0 \\ * & * & * & Q - \alpha I \end{bmatrix} < 0. \tag{7.3.6}$$

Proof. Lipschitz-type time-varying nonlinear P-R system Θ_2 is the special case of nonlinear system (6.3.1). The proof of Theorem 7.1 can refer to Theorem 6.1 and the result in Zhang et al. (2016)[99]. □

7.3.2 Robust Stabilization of the system

So far we gave the sufficient condition for the robust stability of the P-R system Θ_1 with $\underline{w}_{t+1} = 0$ and $\underline{U}_t = K\underline{R}_t = 0$. In practice, it is possible that the P-R process can be unstable; however it can be stabilized eventually with the appropriate choice of the premium strategy.

Consequently, the nonlinear P-R system Θ_1 with $\underline{w}_{t+1} = 0$ is considered. The new system has an additional input controller $\underline{U}_t = K\underline{R}_t$. In order to confirm that the new closed-loop system is robust stochastically stable, the previous feedback controller is developed and discussed.

$$\Theta_3 : \begin{cases} \underline{R}_{t+1} = [J - eZK]\underline{R}_t + h_1(t, \underline{R}_t) + \{[J - eZK]\underline{R}_t + h_2(t, \underline{R}_t)\}v_t, \\ \underline{R}_t = \underline{\varphi}_t \ for \ t \in [-\tau_{\max}, 0]. \end{cases} \tag{7.3.7}$$

Now, we can derive the following theorem:

Theorem 7.2. *System Θ_3 with margins $\alpha_1 > 0$ and $\alpha_2 > 0$ is said to be robustly stochastically stabilizable, if there exists matrics Y and $X > 0$ and a real scalar $\beta > 0$*

such that

$$\begin{bmatrix} -X & \alpha_1 XH^T & \alpha_2 XH^T & A^T & A^T & 0 \\ * & -\frac{1}{2}\beta I & 0 & 0 & 0 & 0 \\ * & * & -\frac{1}{2}\beta I & 0 & 0 & 0 \\ * & * & * & -\frac{1}{2}X & 0 & 0 \\ * & * & * & * & -\frac{1}{2}X & 0 \\ * & * & * & * & * & \beta I - X \end{bmatrix} < 0. \tag{7.3.8}$$

holds, where where $A = JX - eZY$

In this case, $\underline{U}_t = K\underline{R}_t = YX^{-1}\underline{R}_t$ *is a robustly stochastically stabilizing controller.*

Proof. Lipschitz-type time-varying nonlinear P-R system Θ_3 is the special case of nonlinear system (6.4.1). The proof of Theorem 7.2 can refer to Theorem 6.2 and the result in Zhang et al. (2016)[99]. □

7.3.3 Robust H_∞ control

In previous sections, the external disturbance of the nonlinear P-R system is assumed to be zero. In Pantelous and Yang (2014)[66], the disturbance is first time assumed to be non-zero, i.e. $\underline{w}_t \neq 0$. Here, since we focus a more general nonlinear P-R system in this chapter, the state feedback controller $\underline{U}_t = K\underline{R}_t$ is determined such that the resulting closed-loop system Θ_1 is robust stochastically stable with disturbance attenuation level γ which is a given constant performance level. For nonlinear P-R system, the disturbance attenuation γ is a parameter which measures the accumulated impact of the outside disturbance on the system output. In the insurance industry, as indicated by Pantelous and Yang (2014) [66], we can consider γ as a parameter which measures the influence of the disturbance in the market for the accumulated reserve.

Definition 7.2. For a given disturbance attenuation level $\gamma > 0$, $\underline{U}_t = K\underline{R}_t$ is an H_∞ controller of system Θ_1,
if (i). System Θ_1 is internally stochastically stabilizable for $\underline{U}_t = K\underline{R}_t$ in the absence of external disturbance $\underline{w}_t$;
(ii). The H_∞ norm of system Θ_1 is less than disturbance attenuation constant level

$\gamma > 0$ with zero initial condition $\underline{R}_0 = 0$, which is

$$\begin{aligned} \|H\| &= \sup_{w \in l_w^2(N,\mathbb{R}^q), \underline{w}_t \neq 0} \frac{\|\underline{z}_t\|_{l_w^2(N,\mathbb{R}^q)}}{\|\underline{w}_t\|_{l_w^2(N,\mathbb{R}^q)}} \\ &= \sup_{w \in l_w^2(N,\mathbb{R}^q), \underline{w}_t \neq 0} \frac{\left(\sum_{t=0}^{\infty} \mathbb{E}\|\underline{z}_t\|^2\right)^{\frac{1}{2}}}{\left(\sum_{t=0}^{\infty} \mathbb{E}\|\underline{w}_t\|^2\right)^{\frac{1}{2}}} < \gamma. \end{aligned}$$

Theorem 7.3. *For the given $\gamma > 0$, system Θ_1 with margins $\alpha_1 > 0$ and $\alpha_2 > 0$ is said to be H_∞ controllable, if there exists matrics Y and $X > 0$ and a real scalar $\beta > 0$ such that the following LMI is satisfied,*

$$\left[\begin{array}{cccccccc} -X & I & \alpha_1^2 X H_1^T & \alpha_2^2 X H_2^T & A^T & A^T & 0 & 0 \\ * & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\frac{1}{3}\beta I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\frac{1}{3}\beta I & 0 & 0 & 0 & 0 \\ * & * & * & * & -X & 0 & 0 & 0 \\ * & * & * & * & * & -X & 0 & 0 \\ * & * & * & * & * & * & -\gamma^2 I & I \\ * & * & * & * & * & * & * & -X \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{array} \right.$$

$$
\left.\begin{array}{ccccc}
0 & 0 & A^T & A^T & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-X & 0 & 0 & 0 & 0 \\
* & I & 0 & 0 & 0 \\
* & * & -X & 0 & 0 \\
* & * & * & -X & 0 \\
* & * & * & * & \beta I - X
\end{array}\right] < 0, \tag{7.3.9}
$$

where $A = JX - eZY$ *and the robust* H_∞ *controller* $\underline{U}_t = K\underline{R}_t = YX^{-1}\underline{R}_t$ *for* $t \in N$*. Then, a robust stabilizing state feedback controller is given by*

$$\underline{U}_t = K\underline{R}_t = YX^{-1}\underline{R}_t.$$

Proof. The proof of Theorem 7.3 can refer to Theorem 6.3 and the result in Zhang et al. (2016)[99]. □

7.3.4 Numerical Application

In this section, a numerical application for illustrating the applicability of the theoretical results for an insurance company is formulated. We assume that it runs three different insurance lines which are mutually correlated. Then, we use the LMI sufficient condition from the result in Theorem 7.3 to find out the H_∞ controller for the P-R system Θ_1 with Lipschitz-type nonlinear condition (7.3.1, 7.3.2). Then we apply Theorem 7.2 to stabilise system Θ_3 without outside disturbance, and present the result in two figures.

- To design the H_∞ controller, we first assume the value of the reserve accounts at

$t = 0$ is given by the following zero value matrix,

$$\underline{R}_0 = \begin{bmatrix} R_0(1) \\ R_0(2) \\ R_0(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

i.e. at time $t = 0$, we assume that the reserve account for each insurance lines is £ 0 pounds, respectively.

- The uncertainties of the system are formulated by the nonlinear function and is satisfied the inequalities (7.3.1),(7.3.2). We assume the constant a_1, a_2 and constant matrix H_1, H_2 defining the structure of nonlinear functions are:

$$a_1 = 0.3, \quad a_2 = 0.4$$

$$H_1 = H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7.3.10}$$

- In our model, it is assumed that the insurer can invest the reserve in risk-free investments (T-bills). We assume that the corresponding rate of income is given from the following matrix:

$$J = \begin{bmatrix} 1.021 * 0.86 & 1.021 * 0.10 & 1.021 * 0.08 \\ 1.021 * 0.07 & 1.021 * 0.87 & 1.021 * 0.09 \\ 1.021 * 0.07 & 1.021 * 0.03 & 1.021 * 0.83 \end{bmatrix}.$$

- The weight ratios w_{nm} which demonstrates the solvency relation between each line have the following values:

  $w_{1,1} = 0.86$, $w_{1,2} = 0.10$ and $w_{1,3} = 0.08$,

  $w_{2,1} = 0.07$, $w_{2,2} = 0.87$ and $w_{2,3} = 0.09$,

  $w_{3,1} = 0.07$, $w_{3,2} = 0.03$ and $w_{3,3} = 0.83$.

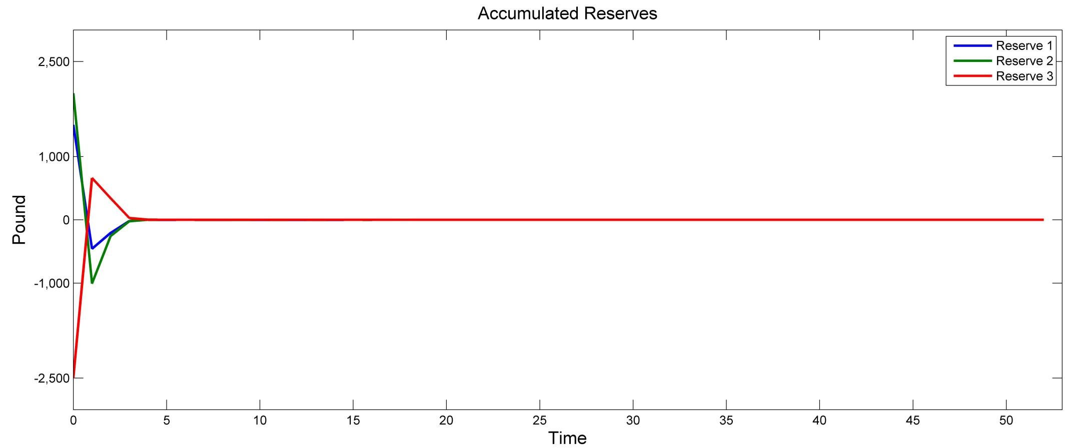

Figure 7.1: Accumulated Reserves for 3 products: with controller; without external disturbance

- For the parameter e, we let $e = 0.8$, which means that $1 - 0.8 = 0.2$ (or 20%) of the premium revenue is used to cover the administration and operating cost and give to the company a reasonable profit margin.
- $\gamma = 2$. This is the given value (not optimal) which measures the maximum impact level of the disturbance on the reserves.

Here, the performance of system under different markovian switching signals is presented. The simulation results are provided for the time-period of $t = 52$ weeks.

By applying the result of the Theorem 7.3, the H_∞ controller is derived, and we get the feedback controller for nonlinear P-R system are as below:

$$K = \begin{bmatrix} 0.5248 & 0.0425 & 0.0426 \\ 0.0609 & 0.5309 & 0.0182 \\ 0.0487 & 0.0548 & 0.5066 \end{bmatrix}.$$

We provide the simulation results for the time-period of $t = 52$, and the Figures 7.1 and Figures 7.2 are derived.

Figure 7.1 shows the trajectory of the accumulated reserves with initial state values $\underline{R}_0 = [R_{1,0} \quad R_{2,0} \quad R_{3,0}]^T = [1,500 \quad 2,000 \quad -2,500]^T$. By using Theorem 7.2 to derive the feedback controller $\underline{U}_t$ for the P-R system Θ_3 ((7.3.7) with $\underline{w}_t = 0$, the accumulated reserve process can be stabilized. We can see the value of 3 accumulated reserve accounts converge the a fixed (zero) level after several time periods.

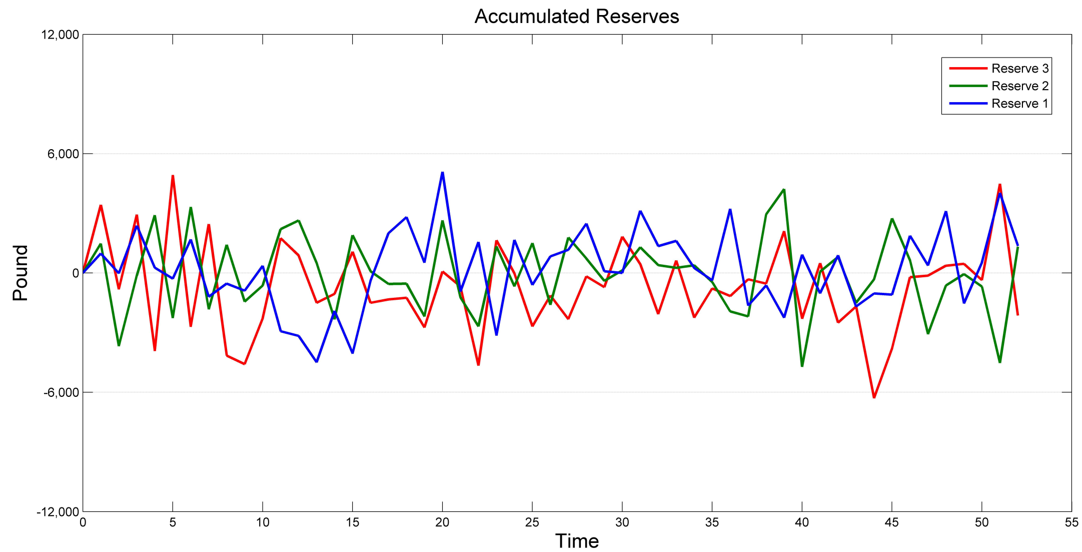

Figure 7.2: Accumulated Reserves for 3 products: with external disturbance

Figure 7.2 shows the movement of the accumulated reserve for each dependent insurance product with the effect of outside disturbance $\underline{w}_t$. In this case, the stability of the system can be achieved by using the robust H_∞ tool to generate the state feedback controller $\underline{U}_t$, even though the system disturbance $\underline{w}_t \neq 0$ exist. Compared Figure 7.2 to Figure 7.1, we can see the disturbance $\underline{w}_t$ affect significantly the trajectory of accumulated reserves. However, the state feedback controller $\underline{U}_t$ ensure the fluctuation of the accumulated reserves are bounded with a certain level γ and stable.

To calculated the most suitable feasible solution to complex LMI (7.3.8) & (7.3.9) , we use the feasp algorithm in LMI toolbox in Matlab, see Gahinet et al. (1995) [25]). With proper setting, this toolbox will directly give us the feasible solution when it does exists feasible solution.

7.4 One-sided Lipschitz-type time-varying nonlinear P-R system

The following is assumed to hold throughout section 7.4.

One-sided Lipschitz-type time-varying nonlinear condition: The nonlinear functions $h_1(t, \underline{R}_t), h_2(t, \underline{R}_t)$ describe parameter uncertainty of the system and satisfy the following quadratic inequalities:

$$h_1^T(\underline{R}_t)\underline{R}_t \leq \rho_1 \underline{R}_t^T \underline{R}_t, \tag{7.4.1}$$

$$h_1^T(\underline{R}_t)h_1(\underline{R}_t) \leq \beta_1 \underline{R}_t^T \underline{R}_t + \gamma_1 \underline{R}_t^T h_1(\underline{R}_t), \tag{7.4.2}$$

$$h_2^T(\underline{R}_t)\underline{R}_t \leq \rho_2 \underline{R}_t^T \underline{R}_t, \tag{7.4.3}$$

$$h_2^T(\underline{R}_t)h_2(\underline{R}_t) \leq \beta_2 \underline{R}_t^T \underline{R}_t + \gamma_2 \underline{R}_t^T h_2(\underline{R}_t). \tag{7.4.4}$$

for all $t \in N$, where $\beta_i, \gamma_i, \rho_i$ are constants related to the function h_i for $i = 1, 2$. $\beta_i, \gamma_i, \rho_i$ are constant defining structure of h_i.

7.4.1 Robust stability of the system

Considering the following basic P-R system Θ_2 without disturbance and controller,

$$\Theta_2 : \begin{cases} \underline{R}_{t+1} = J\underline{R}_t + h_1(t, \underline{R}_t) + \{J\underline{R}_t + h_2(t, \underline{R}_t)\}v_t, \\ \underline{R}_t = \underline{\varphi}_t \ for \ t \in [-\tau_{\max}, 0]. \end{cases}$$

we can first define the definition of stability of P-R system Θ_2 and then derive the following theorem about the system stability.

Definition 7.3. The one-sided Lipschitz-type time-varying nonlinear P-R system Θ_2 is said to be robustly stochastically stable with margins ρ_1, ρ_2, β_1, β_2, γ_1, γ_2 if there exists a constant $\delta(x_0, \rho_1, \rho_2, \beta_1, \beta_2)$ such that

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \underline{R}_t^T \underline{R}_t\right] \leq \delta(x_0, \rho_1, \rho_2, \beta_1, \beta_2). \tag{7.4.5}$$

The following theorem gives a sufficient condition of robust stochastic stability for system Θ_2.

Theorem 7.4. *One-sided Lipschitz-type time-varying nonlinear P-R system Θ_2 with margins $\rho_1, \rho_2, \beta_1, \beta_2$ is said to be robustly stochastically stable, if there exists a symmetric positive definite matrix $Q > 0$ and real scalar $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > 0$ such that the following conditions are satisfied.*

$$\begin{bmatrix} -Q + 2I + 2(\varepsilon_1\rho_1 + \varepsilon_2\beta_1)I + 2(\varepsilon_3\rho_2 + \varepsilon_4\beta_2)I & J^TQ & J^TQ \\ * & -\frac{1}{2}Q & 0 \\ * & * & -\frac{1}{2}Q \end{bmatrix} < 0. \tag{7.4.6}$$

$$\begin{bmatrix} 2Q - 2\varepsilon_2 I & \varepsilon_2\gamma_1 I - \varepsilon_1 I \\ \varepsilon_2\gamma_1 I - \varepsilon_1 I & -I. \end{bmatrix} < 0, \tag{7.4.7}$$

$$\begin{bmatrix} 2Q - 2\varepsilon_4 I & \varepsilon_4\gamma_2 I - \varepsilon_3 I \\ \varepsilon_4\gamma_2 I - \varepsilon_3 I & -I \end{bmatrix} < 0. \tag{7.4.8}$$

Proof. The One-sided Lipschitz-type time-varying nonlinear P-R system Θ_2 is the special case of system (6.3.1), and it has a same nonlinear property with system (6.3.1). So the proof of Theorem 7.4 can refer to Theorem 6.1. □

7.4.2 Robust H_∞ control

For the nonlinear discrete stochastic P-R system with disturbance $\underline{w}_t$ and controller $\underline{U}_t$ which is:

$$\Theta_1 : \begin{cases} \underline{R}_{t+1} = J\underline{R}_t + h_1(t, \underline{R}_t) - eZ\underline{U}_t + [J\underline{R}_t + h_2(t, \underline{R}_t) + -eZU_t]v_t + \underline{w}_{t+1} \\ \underline{z}_t = \underline{R}_t \\ \underline{R}_t = \underline{\varphi}_t \ for \ t \in [-\tau_{\max}, 0], \end{cases}$$

where $\underline{w}_{t+1} = e\hat{\underline{C}}_{t+1} - \underline{C}_{t+1} \in l_{e_2}(N; \mathbb{R}^m)$.

If it follows one-sided Lipschitz-type time-varying nonlinear condition, then we can possibly design a robust H_∞ controller based on theorem for this P-R system.

Theorem 7.5. *For system Θ_1 with given one-sided Lipschitz-type time-varying nonlinear condition constant ρ_1, ρ_2, β_1, β_2, γ_1, γ_2, if there exist real matrices Y, $X > 0$, $Q > 0$ and real scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > 0$, $\kappa_1 > 0$, $\kappa_2 > 0$, $\kappa_3 > 0$, $\kappa_4 > 0$ such that the following conditions hold:*

$$
\left[\begin{array}{ccccc}
-X & 0 & I & (JX-eZY)^T & (JX-eZY)^T \\
* & -\gamma^2 I & 0 & I & I \\
* & * & -I & 0 & 0 \\
* & * & * & 0 & 0 \\
* & * & * & -X & 0 \\
* & * & * & * & -X \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{array}\right.
$$

$$
\left.\begin{array}{cccccccc}
(JX-eZY)^T & (JX-eZY)^T & 0 & 0 & X & X & X & X \\
0 & 0 & I & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & -X & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -X & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -X & 0 & 0 & 0 & 0 \\
* & * & * & * & -\frac{\kappa_1}{2\rho_1}I & 0 & 0 & 0 \\
* & * & * & * & * & -\frac{\kappa_2}{2\beta_1}I & 0 & 0 \\
* & * & * & * & * & * & -\frac{\kappa_3}{2\rho_2}I & 0 \\
* & * & * & * & * & * & 0 & -\frac{\kappa_4}{2\beta_2}I
\end{array}\right] < 0, \tag{7.4.9}
$$

$$
\begin{bmatrix} 3Q - 2\varepsilon_2 I & \varepsilon_2\gamma_2 I - \varepsilon_1 I \\ \varepsilon_2\gamma_2 I - \varepsilon_1 I & -I. \end{bmatrix} < 0, \tag{7.4.10}
$$

$$\begin{bmatrix} 3Q - 2\varepsilon_4 I & \varepsilon_4\gamma_2 I - \varepsilon_3 I \\ \varepsilon_4\gamma_2 I - \varepsilon_3 I & -I \end{bmatrix} < 0, \tag{7.4.11}$$

$$QX = I, \quad \kappa_1 I * \varepsilon_1 I = I, \quad \kappa_2 I * \varepsilon_2 I = I, \quad \kappa_3 I * \varepsilon_3 I = I, \quad \kappa_4 I * \varepsilon_4 I = I. \tag{7.4.12}$$

Then system Θ_1 is H_∞ controllable for the given $\gamma > 0$, and the robust H_∞ controller $\underline{U}_t = Kx(t) = YX^{-1}x(t)$ for $t \in N$.

Proof. The one-sided Lipschitz-type time-varying nonlinear P-R system Θ_1 is the special case of system (6.5.1), and it has a same nonlinear property with system (6.5.1). So the proof of Theorem 7.5 can refer to Theorem 6.3. □

Remark 7.1. The conditions to be solved in Theorem 7.5 construct a nonconvex feasibility problem. This nonconvex feasibility problem using similar algorithm introduced in the Remark 6.3.

7.4.3 Numerical Application

In this sub-section, we extend the numerical example that has been presented previously in section 7.3 to show how the robust H_∞ technique can be used in the one-sided Lipschitz-type time-varying nonlinear stochastic discrete time P-R system process. Thus, the portfolio we simulate is the same with the portfolio assumed in section 7.3. However, we should give values to some new parameters involved.

- The uncertainties of the system are formulated by the nonlinear functions and are satisfied the inequalities (7.4.1), (7.4.2), (7.4.3) and (7.4.4). We assume the one-sided Lipschitz-type time-varying nonlinear condition constant ρ_1, ρ_2, β_1, β_2, γ_1, γ_2 defining the structure of nonlinear functions $h_1(t, \underline{R}_t)$, $h_2(t, \underline{R}_t)$ are:

$$\rho_1 = 1.5; \quad \rho_2 = 1.6; \quad \beta_1 = 2.3; \quad \beta_2 = 3; \quad \gamma_1 = -1.2; \quad \gamma_2 = -2;$$

- Since the algorithm described in Remark 6.3 is used to solve nonconvex feasibility problem in Theorem 7.5, we assume the following parameter value in the algorithm:

  The maximum number of iterations allowed is $N = 3$;
  The sufficient small positive scalar $\delta = 0.3$

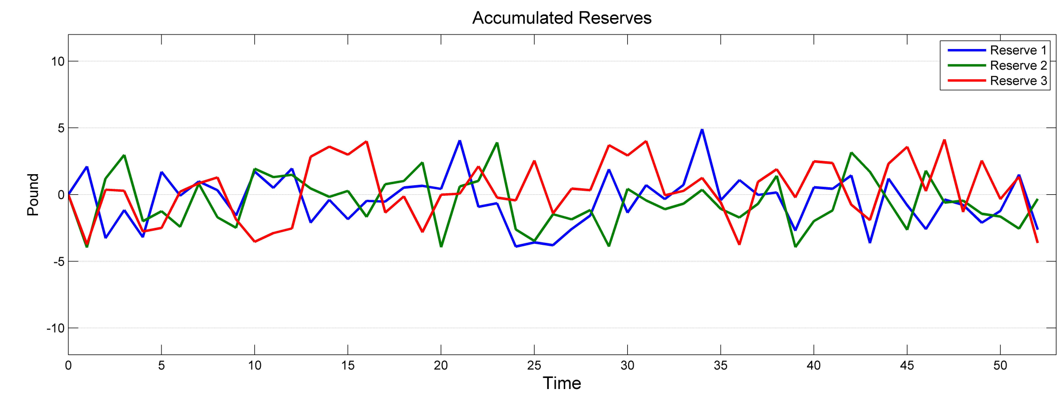

Figure 7.3: Accumulated Reserves for 3 products: with external disturbance

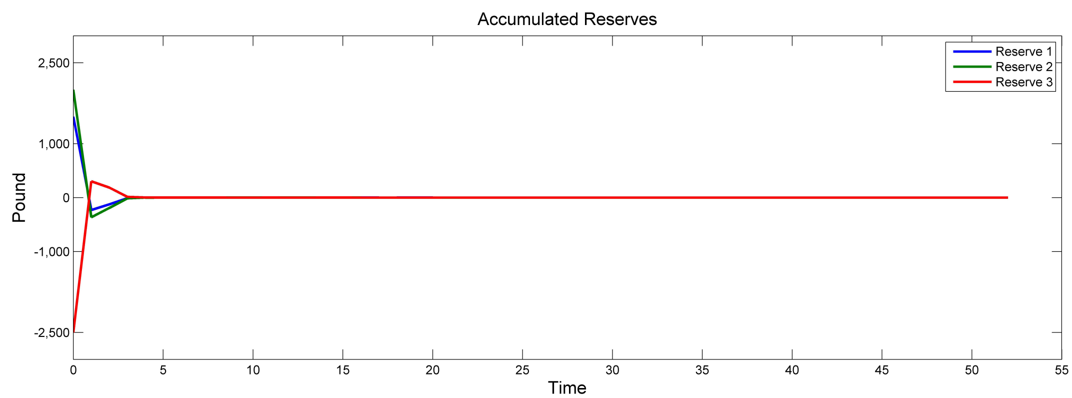

Figure 7.4: Accumulated Reserves for 3 products: with controller; without external disturbance

Here, the performance of system under different markovian switching signals is presented. The simulation results are provided for the time-period of $t = 52$ weeks.

By applying the result of the Theorem 7.5 and the algorithm in Remark 6.3, the H_∞ controller is derived, and we get the feedback controller for nonlinear P-R system are as below:

$$K = \begin{bmatrix} 0.4577 & 0.0722 & 0.0606 \\ 0.0568 & 0.4661 & 0.0612 \\ 0.0543 & 0.0296 & 0.4354 \end{bmatrix}.$$

We provide the simulation results for the time-period of $t = 52$, and the Figures 7.3, Figures 7.4 and Figures 7.5 are derived. Figure 7.4 shows the trajectory of the accumulated reserves process with the effect of controller when external disturbance doesn't

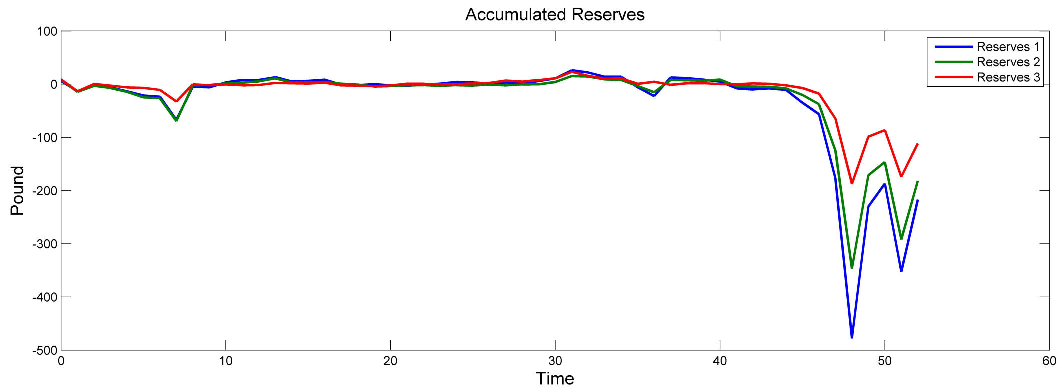

Figure 7.5: Accumulated Reserves for 3 products: without controller; with external disturbance

exist. The initial state values $\underline{R}_0 = [R_{1,0} \quad R_{2,0} \quad R_{3,0}]^T = [1,500 \quad 2,000 \quad -2,500]^T$. The process of accumulated reserves converge to zero in this case.

Figure 7.3 shows the trajectory of the accumulated reserves with zero initial state values. Figure 7.5 shows a situation when system doesn't have controller which could be derived by theorem 7.5. In Figure 7.5, we can see path of the accumulated resevers are more volatile and the level of accumulated reservers are beyond our desired bound at some points.

By using Theorem 7.5 to derive the feedback controller $\underline{U}_t$ for the P-R system Θ_3 ((7.3.7) with $\underline{w}_t \neq 0$, the accumulated reserve process can be stabilized. Obviously, the reason that the reserve can not exactly converge into zero level is the effect of external random disturbances on the system. However, as we can also obtain the state feedback controller $\underline{U}_t$ to restrict the impact of the disturbance and eventually stabilizes the system.

7.4.4 Summary

In Chapter 7, the nonlinear control theory is applied in the classic non-life P-R system in insurance. By using the theorem derived in Chapter 6, the P-R system is furthur developed to a time-varying nonlinear discrete stochastic system. In this chapter, the uncertainties of P-R systems are modelled by Lipschitz-type time-varying nonlinear condition in section 7.3 and one-sided Lipschitz-type time-varying nonlinear condition in section 7.4. Then, we provide the method to generate robust H_∞ controller for these nonlinear P-R systems and present numerical examples respectively.

Chapter 8

Conclusion and future research

The objective of this thesis is to study control theory for classes of nonlinear systems with application in the non-life reserve management and premium rating policy discussed in the end. As an extension of previous literature, we have shown the beautiful properties of the linear time-varying system holding for the commutative class of nonlinear time-varying system as well. However, quite different from linear systems, it is very difficult to obtain the precise physical model for nonlinear system and more distinct structural models can be chosen. To serve certain purpose, it is classified based on description of some of its properties. So we solve the problem of observer design and feedback stabilization for linear time-varying systems under One-sided Lipschitz nonlinear perturbation. The corresponding results for stochastic discrete-time systems have been worked out so as to present in more comprehensive structure of the thesis, together with application in premium-reserve model. The applicability of those theorems is demonstrated by numerical examples. In numerical examples, we assume an insurance company runs a non-life insurance portfolio containing multiple products, which may be exposed to outside financial and economic disturbances, parameter uncertainties, etc.

Much effort in this area has been relied mainly on improving the bounding techniques for example and Jensens Inequality for use in guaranteeing the negative definiteness of the derivative of the Lyapunov-Krasovskii functionals. While the introducing of slack variables can also significantly increase the computational complexity. Therefore, how to develop new methods in order to further reduce the conservatism in existing stability results while keeping a reasonably low computational complexity is an important issue to be investigated in the future. On the other hand, by comparing the stability results obtained through the use of various Lyapunov-type functions or functionals , there exists invariant trade-off between theoretical conservatism and computational complexity. Much of the research has been focussed on reducing the conservatism of the stability conditions.

In future, we plan to extend the results in Chapter 4 and 5 for one-sided Lipschitz nonlinear time-varying system incorporating time delays in controls.

The essence of those theorems is based on sufficient LMI criteria. It should be noted that the robust controller is designed without any constraint and the P-R system do not consider the effect of time delays. Therefore we can do further research on the robust guaranteed cost control approach and could possibly incorporate different time delays in this nonlinear system.

In Chapter 6 and 7, we give an attempt to consider classes nonlinear system in discrete-time stochastic framework and their application in non-life P-R model.

During the last two decades, applications of regime switching models in finance and macroeconomics have received a great attention among researchers and particularly, market practitioners. Thus, we could further think about how a nonlinear system with regime switching in discrete-time could be used to model the medium- and long- term reserves and the premiums of an insurer.

Last but importantly, as pointed in previous researcher (see e.g. Kendrick [40], Yang et al. [94]), when we try to apply the models to solve real world problems such as P-R system process, we should always keep in mind that we need to translate the real world problem in an appropriate way. That not only means we should give a reasonable approach to define the practical meaning and determine specific value of uncertain parameters, but also a thorough understanding of the mathematical ramifications of these concepts enables one to formulate a theoretical control problem meaningfully and to make the necessary assumptions at the outset of the analysis.

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