Automatic balancing mechanisms for Notional Defined Contribution Accounts in the presence of uncertainty

December 2, 2016

Abstract

The notional defined contribution model combines pay-as-you-go financing and a defined contribution pension formula. This paper aims to demonstrate the extent to which liquidity and solvency indicators are affected by fluctuations in economic and demographic conditions and to explore the introduction of an automatic balancing mechanism into the pension scheme. We demonstrate that the introduction of an automatic balancing mechanism reduces the volatility of the buffer fund and that, in most cases, the automatic mechanism that re-establishes solvency produces the highest value of the risk-adjusted notional factor.

JEL: E62, H55, J26

Keywords: actuarial analysis, public pensions, retirement, solvency, stochastic processes, risk.

1 Introduction

A notional model is a pay-as-you-go (PAYG) system in which the amount of the pension depends on both contributions and returns, that is, the accumulated capital over the course of the participant's employment. The returns on contributions are calculated using a notional rate which is linked to an external index set by law, such as the growth rate of GDP, average wages, or contribution payments. The account balance is called notional because it is used only for record keeping, that is, the system does not invest funds in financial markets as the system is based on PAYG financing. When an individual reaches retirement age, the fictitious balance is converted into an annuity based on the indexation of benefits and technical interest rate as well as the life expectancy of the cohort.

The introduction of Notional (non-financial) Defined Contribution (NDC) pension accounts as components of modern, multi-pillar pension schemes in some countries has been a major innovation of the last two decades of pension reform. NDCs have been established in Italy (1995), Latvia (1996), Poland (1999) and Sweden (1999)[1], while Kyrgyzstan (1997), Mongolia (2000) and Russia (2002) have adopted some NDC features, such as life expectancy factors. According to Holzmann et al. (2012), countries such as China and Greece are also seriously considering introducing notional defined contribution pension schemes. Holzmann (2006) argues that notional accounts should provide the foundation for a unified pension scheme in the European Union.

The financial sustainability of a pension scheme is commonly analyzed in terms of their liquidity and/or solvency indicators. Liquidity, a cross-sectional equilibrium, implies that all that is received is paid. For example, in a pure PAYG pension scheme, pensions for retirees

[1] See Chłón-Domińczak et al. (2012) and Holzmann et al. (2012) for updated particularities of NDC countries.

in one particular period are paid for solely by the contributions of the working-age population. On the other hand, solvency corresponds to analyzing whether the liabilities towards all participants are backed by sufficient assets, which in a pay-as-you-go environment corresponds to an estimation of the 'pay-as-you-go asset' (Settergren and Mikula 2005). Also, there is a demand to re-establish the equity (also called actuarial fairness) in pension schemes. In this respect, NDCs pension schemes can be considered actuarially fair at some extent due to the direct link between contributions and their returns and the pensions paid at retirement (Palmer 2006; Queisser and Whitehouse 2006). Bommier and Lee (2003) argue that these indicators, liquidity, solvency and actuarial fairness, are not necessary compatible, i.e. they show that liquidity does not necessarily coexist with actuarial fairness.

Valdés-Prieto (2000) shows that NDCs cannot generally provide liquidity in the short run unless in the unrealistic steady state with a notional rate equal to the contribution base. Auerbach and Lee (2006) provide similar conclusions in regards of the solvency of NDCs. Therefore, in NDCs it is also necessary to introduce adjustment mechanisms, known as Automatic Balance Mechanisms (ABMs), just as in the traditional defined benefit PAYG schemes. Vidal-Meliá et al. (2009) and Vidal-Meliá et al. (2010) define ABMs as a set of pre-determined measures established by law to be applied immediately as required according to a sustainability indicator[2].

This paper contributes to fill a gap in the literature investigating how different ABMs, that re-establish liquidity or solvency, react to economic and demographic changes in NDC pension schemes in terms of expected value, variance and Sharpe ratio of the notional rate and the buffer fund. An excessive value of the liquidity or solvency indicator would not be welfare enhancing for the current cohorts as they do not get any extra benefit and this makes the system prone to political risk (Diamond 1994). A low value of the sustainability indicator leads the government to reduce pension benefits so as to ensure the pension scheme survival. Finally, an ideal automatic balancing mechanism provides a value of notional rate to provide adequate retirement income while ensuring limited variability. The ABMs considered affect to both the notional rate (in our case linked to the growth of the total contribution base) and indexation of pension benefits in actuarially fair schemes. This paper also aims to shed some light on how the system adapts to one-time exogenous demographic shocks and longevity improvements.

Following this introduction, this paper is structured as follows. The subsequent section 2 presents the population in a framework with four overlapping generations (OLG).The next section 3, provides two different methods of assessing the financial sustainability of a pension scheme. The section 4 focuses on the design of an automatic balancing mechanism to restore the liquidity or solvency of the system while the numerical illustration is presented in section 5. The final section of this paper provides the main conclusions and the two appendices provide additional details for the adjustment factor and the theoretical framework used.

2 A dynamic four OLG model

NDCs have been studied in 2 and 3-period OLG[3] in Valdés-Prieto (2000) and for multiple coexisting cohorts in Vidal-Meliá and Boado-Penas (2013), Boado-Penas and Vidal-Meliá (2014) and Vidal-Meliá et al. (2015). However, these papers consider that changes in the demographic and economic variables are constant which make their results only valid when the system is in steady state. Other authors study NDC systems in a multiple OLG setting when the variables are in stochastic steady state. In particular, Auerbach and Lee (2006) study the liquidity of

[2]For more details about ABMs please see Börsch-Supan et al. (2004) and Knell (2010).

[3]See Diamond (1977), Samuelson (1958) and Auerbach et al. (2013) for OLG models related to defined benefit schemes.

NDCs with calibrated US data while Auerbach and Lee (2011) analyze the performance of different NDC designs with regards to risk-spreading among generations.

This section examines NDCs using a four OLG model, that includes dynamics of population, where two generations of contributors, aged 1 and 2, and two generations of pensioners aged 3 and 4 coexist. At age 5 there are no survivors remaining.

The choice of four generations is not arbitrary; it introduces heterogeneity among contributors because these two generations are characterized by different demographic and wage histories as well as contributions to the system. It also introduces mortality and indexation during the retirement period. This is not possible in a 2 or 3-period model. A minimum of two retirement periods is needed to study explicitly the effect of mortality assumptions on the pension expenditures and to derive results on the 'indexation' that renders the system actuarially fair. Furthermore, it allows us to study different demographic scenarios without relying heavily on computational methods (Bhattacharya and Russell 2003).

The demographic structure at any time t is represented as follows:

Age:

$$x = \overbrace{1,2}^{\text{Contributors}},\ \underbrace{3,4}_{\text{Pensioners}}$$

Population at time t:

$$N_t^x = N_{t-x+1}^1 p_t^x = N_0^1 \prod_{j=1}^{t-x+1} (1+n_j)\, p_t^x \tag{2.1}$$

where

N_t^x denotes individuals aged x for x =1, 2, 3 and 4 who are alive at time $t > 0$ and entered the labor market at time $t-x+1$. New entrants at time t are denoted as N_t^1. At time $t=0$, the population structure is given at all ages x.

p_t^x is the time-dependent survival probability, that is, the probability that an individual will reach age x by time t. It is assumed that no mortality occurs prior to retirement; therefore, $p_t^x = 1$ for $x = 1$, 2, and 3. For simplicity[4], the probability that one individual is alive at the age of 4, p_t^4, is denoted by p_t. Furthermore, the survival probability p_t^5 is zero, that is, individuals cannot attain age 5.

n_j is the rate of population variation from period $j-1$ to period j and is a stochastic process defined in the probability space $(\Omega, \mathcal{F}, P)$.

Our population setting, as shown in (2.1), indicates that the population at time t only depends on the entrants who have survived by time t. This implies that the population is closed to migration[5]. Table 1 illustrates the population evolution over three periods by year of birth.

3 Liquidity and solvency indicators

There are various methods of assessing the sustainability of an unfunded pension scheme. This paper focuses on two methods. The first method assesses the liquidity ratio, which indicates the relationship between the income from contributions and the pension expenditures at a

[4]We ignore the mortality before retirement because it simplifies the notional rate paid to the contributions, based on the rate of increase of the income from contributions. It also highlights the effect of the survival probability on the annuity paid during retirement.

[5]This is a common assumption in the analysis of pay-as-you-go pension schemes (Settergren and Mikula 2005; OECD 2015).

Table 1: Population evolution in terms of the entry age x =1

Age[a]\ Time	t	$t+1$	$t+2$	$t+3$
1	N_t^1	$N_t^1(1+n_{t+1})$	$N_{t+1}^1(1+n_{t+2})$	$N_{t+2}^1(1+n_{t+3})$
2	N_{t-1}^1	N_t^1	$N_t^1(1+n_{t+1})$	$N_{t+1}^1(1+n_{t+2})$
3	N_{t-2}^1	N_{t-1}^1	N_t^1	$N_t^1(1+n_{t+1})$
4	$p_tN_{t-3}^1$	$p_{t+1}N_{t-2}^1$	$p_{t+2}N_{t-1}^1$	$p_{t+3}N_t^1$

[a] *Source:* the authors.

specific date. This method will be explained in Subsection 3.1. The second method assesses the solvency of the pension scheme by compiling an actuarial balance sheet, following the Swedish method, which compares assets and liabilities towards all participants in the system. This second method is developed in Subsection 3.2.

3.1 Liquidity ratio

As previously stated, NDCs are financed on a PAYG basis, that is, pensions for retirees are paid by the contributions of the working age population. The liquidity ratio indicates whether the current contributions and financial assets are sufficient to pay current retiree pensions. Formally, the ratio at time t, LR_t is represented as follows:

$$LR_t = \frac{C_t + F_t^-}{P_t} \tag{3.1}$$

where
C_t represents the income from contributions at time t,
P_t represents the total pension expenditures at time t,
and F_t^- represents the value of the (buffer) fund at time t, also called reserve fund, before new contributions and benefits payments are considered. F_t^- can be expressed as follows:

$$F_t^- = F_{t-1}^+(1+i_t) = F_0^- \prod_{j=1}^{t}(1+i_j) + \sum_{j=0}^{t-1}(C_j - P_j) \prod_{k=j+1}^{t}(1+i_k) \tag{3.2}$$

where
i_t represents the financial rate of return of the fund from period $t-1$ to t and is a stochastic process defined in the probability space $(\Omega, \mathcal{F}, P)$.

The value of the fund at time t after contributions and payments is given by $F_t^+ = F_t^- + C_t - P_t$.

The following subsections derive the necessary mathematical expressions needed to calculate the liquidity ratio presented in (3.1).

3.1.1 Income from contributions

The income from contributions received by the pension scheme at time t, C_t, corresponds to the sum of all contributions made by the cohorts aged 1 and 2 and is represented as follows:

$$C_t = \pi \cdot N_t^1 \cdot S_t^1 + \pi \cdot N_t^2 \cdot S_t^2$$
$$= \pi \cdot N_0^1 \prod_{i=1}^{t-1} (1+n_j) \prod_{i=1}^{t} (1+g_j) \underbrace{\left(S_0^1(1+n_t) + S_0^2\right)}_{K_t^C} \tag{3.3}$$

where

π is the fixed contribution rate of the NDC pension scheme,

K_t^C is a function representing the wage and population heterogeneity,

and S_t^x denotes the individual wages at time $t > 0$ which are earned by the active population and are assumed to be paid at the beginning of the calendar year. Wages are dependent on time t and age x in line with empirical evidence[6]. They are represented as follows:

$$S_t^x = S_0^x \prod_{j=1}^{t} (1+g_j) \tag{3.4}$$

where g_j is the rate of wage variation from the period $j-1$ to period j and is a stochastic process defined in the probability space $(\Omega, \mathcal{F}, P)$. Furthermore, S_0^x is the observed wage at time 0 for each active age x =1, 2. In our model, the wages coincide with the contribution bases.

3.1.2 Pension expenditures

In order to obtain the total pension expenditures we need to specify in the model how the pensions for the retirees aged 3 and 4 are calculated. First, we develop the accumulated notional capital which is needed to compute the initial pension for the retiring cohort - pensioners who have just retired. After this, we present the pension for the second cohort retirees and provide some insights on how the indexation on pension needs to be adjusted in order to ensure an actuarially fair scheme. Finally we show in Proposition 1 that the income from contributions is in general not equal to the pension expenditures, implying that the liquidity ratio (3.1) is in general not equal to 1 whenever the system is not in steady state.

Pensions for retirees aged 3

At retirement, the accumulated notional capital for each individual is converted into an annuity for the next two periods. In this model, all individuals in the same birth cohort have the same contribution history and mortality patterns. Therefore, the accumulated notional capital or notional pension wealth, NPW_t^x for x = 1, 2, 3, 4, will be calculated for each cohort[7].

To calculate the accumulated notional capital for each cohort we consider their past contributions[8] and assign them a return that corresponds to the notional rate until retirement. In this case, the total notional accumulated capital for the cohort aged 3 retiring at time t, NPW_t^3, is represented as follows:

[6] For instance the wage curve in some countries has an inverted U-shaped wage path that peaks in middle age and declines smoothly thereafter (Blanchflower and Oswald 1990; Groot et al. 1992; Sessions 1993).

[7] Note that this calculation method allows surviving contributors to receive eventual inheritance gains from the deceased (Boado-Penas and Vidal-Meliá 2014). In our setting, however, there are no inheritance gains because there is no mortality prior to retirement.

[8] Therefore the current cohort's notional capital is calculated using the accrual or retrospective method. The forecasted method produces the same results only if the adjustment A_t which ensures actuarial fairness is used. See Appendix A.2. in Alonso-García (2015) for a detailed proof.

$$NPW_t^3 = \pi S_{t-2}^1 N_{t-2}^1 (1+r_{t-1})(1+r_t) + \pi S_{t-1}^2 N_{t-1}^2 (1+r_t) \tag{3.5}$$

where

r_t is the notional rate for the period $t-1$ to t.

In our case, $1+r_t$, is chosen such that it reflects the sustainability of the system through changes in the total contribution base of the system[9], therefore is affected by both wage and population processes. It is represented as follows:

$$1 + r_t = \frac{C_t}{C_{t-1}} = (1+n_{t-1})(1+g_t)\frac{K_t^C}{K_{t-1}^C} \tag{3.6}$$

Note that if the population is in a steady state, the notional factor $1+r_t$ reduces to $(1+n)(1+g)$ which means that the notional rate, r_t equals $(1+n)(1+g)-1$. In our framework, steady state holds if n_t=n and g_t=g $\forall\ t$ and if the survival probabilities are constant over time.

In order to highlight the relation between the current income from contributions C_t and the notional capital for the retiring cohort, (3.5) can be also rewritten as follows:

$$NPW_t^3 = C_t K_t^N \tag{3.7}$$

where

$K_t^N = \frac{S_0^1(1+n_{t-2})}{S_0^1(1+n_{t-2})+S_0^2} + \frac{S_0^2}{S_0^1(1+n_{t-1})+S_0^2}$ is a function that depends on the notional rate during contributors' career.

The time-dependent factor K_t^N equals 1 when $n_{t-2} = n_{t-1}$, that is, the notional capital for the retiring cohort is equal to the system's income from contributions at t. It is remarkable that, in steady state, the income from contributions, which is a cross-sectional measure, is equivalent to the commitments the system takes on with pensioners who are just retired, which is a longitudinal measure[10].

The individual pension P_t^3 to be paid at time t to the new retirees aged 3 is then represented as follows:

$$P_t^3 = \frac{NPW_t^3}{a_t\ N_t^3} \tag{3.8}$$

with

$$a_t = \mathbb{E}_t\left[1 + p_{t+1}^* \frac{1+\lambda_{t+1}^*}{1+r_{t+1}^*}\right] \tag{3.9}$$

where

a_t represents the life annuity that transforms the accumulated notional capital into a pension considering ex-ante or theoretical pension indexation, λ_{t+1}^*, ex-ante or theoretical probability of survival, p_{t+1}^*, and ex-ante or theoretical discount (or notional rate), r_{t+1}^*, processes.

[9] In this case, the notional rate, r_t is also known as the 'natural rate' (Valdés-Prieto (2000) and Börsch-Supan (2006)) or 'canonical rate' of the NDC scheme (Gronchi and Nisticò 2006). It is also known as the 'biological rate' of the economy (Samuelson 1958). In practice, only Latvia and Poland apply this notional rate during the accumulation phase while Sweden applies the growth of the contribution base per capita, equivalent to the rate of increase of the salaries $1+g_t$ when wages are age-independent and Italy the three-year GDP growth average rate.

[10] Valdés-Prieto (2000) shows that the notional capital for the retiring cohort is proportional to the income from contributions in a continuous setting. Vidal-Meliá et al. (2015) show that the income from contributions coincides with both the accrued and forecasted debt in a discrete setting when the system is in steady state.

The pension scheme is exposed to risk in so far as we cannot know accurately the evolution of the indexation of pensions, survival rates and discount rates. In practice, as the calculation of the annuity at time t requires assumptions about the future processes, the theoretical values of the processes λ^*_{t+1}, p^*_{t+1} and r^*_{t+1} do not necessarily coincide with the ex-post or observed processes λ_{t+1}, p^{t+1} and r_{t+1}[11].

Pensions for the retirees aged 4

The pension for the second cohort of retirees aged 4, P_t^4, corresponds to the indexed first period pension, that is:

$$P_t^4 = P_{t-1}^3 \underbrace{(1+\lambda_t)\, A_t}_{\text{adjusted indexation}} \tag{3.10}$$

The first component of the adjusted indexation, λ_t, is the observed value at time t of the government's indexation process, λ_t^*, used in the annuity calculation.

The second component, A_t, corresponds to the actuarial adjustment applied to the observed indexation which ensures actuarial fairness or longitudinal equilibrium. We add this adjustment to the observed indexation here to later show that an actuarially fair system is not necessarily liquid or solvent[12]. The adjustment, A_t, is derived in Appendix A and is represented by the following expression:

$$A_t = \frac{\mathbb{E}_{t-1}\left[p_t^* \frac{1+\lambda_t^*}{1+r_t^*}\right]}{p_t \frac{1+\lambda_t}{1+r_t}} \tag{3.11}$$

As can be seen, this adjustment factor A_t, which captures the differences between the theoretical and the observed values, is a combination of an economic and a longevity factor. The economic adjustment arises due to the difference between the observed notional and indexation rates, that is, r_t and λ_t respectively, with the theoretical discount and indexation rates, r_t^* and λ_t^*[13].

The longevity adjustment compares the observed survival probability p_t with the theoretical survival probability p_t^*. So far, this adjustment is not found in any of the countries which have implemented NDCs. However Alho et al. (2013) argue that this kind of adjustment would be one of the solutions to the systematic miscalculation of the life expectancy in practice.

Aggregate pension expenditures

Finally, the pension expenditures, denoted by P_t, are described by the following expression:

[11] Please refer to Belloni and Maccheroni (2013), Chłón-Domińczak et al. (2012), Vanovska (2006) and Swedish Pension Agency (2015) for particularities on the annuity calculation in Italy, Poland, Latvia and Sweden respectively.

[12] Note that this adjustment would be unnecessary if the notional capital at retirement was paid as a lump sum as done in Valdés-Prieto (2000). Furthermore, the adjustment A_t, allows us to develop expressions which are independent of the government's assumptions regarding the annuity.

[13] In practice, this economic adjustment is used in Sweden. The annuity, as calculated by the Swedish Pension Agency, assumes a discounting rate of 1.6% (Swedish Pension Agency 2015). This front-loading decreases the value of the annuity and provides a higher initial pension at the expense of a potential lower indexation during retirement. It benefits the younger-than-average pensioner and creates a risk of relative poverty for the older elderly (Chłón-Domińczak et al. 2012). The indexation of pensions is therefore adjusted to the deviation of the actual notional rate from the 1.6% per year factored in the annuity divisor.

$$
\begin{aligned}
P_t &= P_t^3 N_t^3 + P_{t-1}^3 (1+\lambda_t) A_t N_t^4 = \frac{NPW_t^3}{a_t} + p_t \frac{NPW_{t-1}^3}{a_{t-1}} (1+\lambda_t) A_t \\
&= \underbrace{C_t \left(\frac{K_t^N}{a_t} + \frac{K_{t-1}^N}{a_{t-1}} \underbrace{\frac{p_t (1+\lambda_t) A_t}{1+r_t}}_{a_{t-1}-1} \right)}_{K_t^P}
\end{aligned} \tag{3.12}
$$

where

K_t^P is a function which accounts for the pension expenditures heterogeneity. As shown in (3.12), P_t, are also proportional to the total contribution C_t paid during the same period.

Despite the fact that the notional rate reflects the growth of the income from contributions, the pension expenditures depend on the past wages and demographic evolutions as well as longevity developments. For instance, a sudden one-off increase in the population entries (such as a 'baby boom') in the past would have risen the notional rate paid to contributions, increasing the pension expenditures one or two periods later when the baby boom' cohort retires.

The following proposition shows that the NDC scheme may not be liquid even when the indexation ensures actuarial fairness and/or the notional rate incorporates demographic and wages changes. Our result extends Valdés-Prieto (2000) when the notional capital is converted to a payment stream paid during retirement.

Proposition 1. *In general, contributions are not equal to pension expenditures in this 4-period OLG unfunded dynamic model, that is $C_t \neq P_t$ $\forall t$.*

Proof. This result is proven by counterexample. Let us assume that $n_t = n$=cte and $g_t = g$=cte $\forall t$ and deterministic. Let us assume further that $p_t^* \neq p_s^*$ for $s \neq t$ and that the theoretical indexation and discount interest rate coincide, that is, $\lambda_s^* = r_s^*$ $\forall s$. Then the notional capital of the retiring cohort is represented as follows:

$$
NPW_t^3 = C_t K_t^N = C_t \frac{S_0^1 \cdot (1+n)}{S_0^1 \cdot (1+n) + S_0^2} + \frac{S_0^2}{S_0^1 \cdot (1+n) + S_0^2} = C_t
$$

Under this setting the expressions of the annuities are:

$$
\begin{aligned}
a_t &= 1 + p_{t+1}^* \\
a_{t-1} &= 1 + p_t^*
\end{aligned}
$$

The pension expenditures are then represented as follows:

$$
P_t = C_t \left(\frac{1}{a_t} + \frac{a_{t-1} - 1}{a_{t-1}} \right) = C_t \left(\frac{1 + 2p_t^* + p_t^* p_{t+1}^*}{1 + p_t^* + p_{t+1}^* + p_t^* p_{t+1}^*} \right) \neq C_t
$$

The income from contributions at t, C_t does not equal the pension expenditures, P_t when the parameters are chosen as above. □

3.2 Solvency ratio

The solvency of a pension scheme can be assessed by compiling an actuarial balance. There are typically two ways for Social Security departments to compile an actuarial balance, namely the 'Aggregate accounting projection' and the 'Swedish' method[14]. The aggregate accounting projection method compares the net present value of the expenditure on pensions and the income from contribution in a long time horizon. This balance uses a forecasted demographic scenario while the macroeconomic scenario is exogenous. In practice, this kind of balance is compiled, on a regular bases, in countries such as U.S (Boards of Trustees of the Federal Old-Age and Survivors Insurance and Disability Insurance Trust Funds 2015), Japan (Sakamoto 2005) or Canada (Office of the Superintendent of Financial Institutions Canada (OSFIC) (2007, 2008)), amongst others.

In Sweden, compiling an official actuarial balance sheet has been normal practice since 2001. This actuarial balance sheet is a traditional accounting balance sheet that lists assets and liabilities over a definite horizon (see Boado-Penas et al. (2008) and Swedish Pension Agency (2015)). The actuarial balance sheet is a financial statement listing the pension scheme obligations to contributors and pensioners at a particular time and the various assets (financial and contributions) that back these obligations. The solvency ratio, defined as the ratio of assets to liabilities, calculated from the actuarial balance sheet serves the following two purposes in Sweden: to measure whether the system can fulfil its obligations to participants and to decide whether an ABM should be applied. This paper follows the Swedish method[15], based on cross-sectional verifiable facts, that does not need any projections. This should not be interpreted as a belief that all the basic parameters determining the items on the balance sheet will remain constant but as a conscious policy to prefer cross-sectional data. Changes are not included until they occur and can be verified. Then, these changes are incorporated into the balance sheet on an annual basis. Swedish authorities note that system solvency does not depend on either assets or liabilities but on the relationship between these two via the solvency ratio. Therefore, valuing assets and liabilities with cross-sectional data is adequate if applied consistently[16].

The solvency ratio (also called balance ratio in Sweden), that emerges from the actuarial balance sheet ("Swedish method") is defined as the relationship between assets and liabilities as follows:

$$SR_t = \frac{CA_t + F_t^-}{V_t} \tag{3.13}$$

where

CA_t corresponds to the contribution asset at time t and is described in Section 3.2.1.

F_t^- is given by (3.2).

V_t represents the liabilities to all participants in the pension scheme at time t and is described in Section 3.2.2.

3.2.1 Contribution Asset

The contribution asset was firstly defined by Settergren and Mikula (2005) as the product of the current income from contributions C_t and a function called the 'turnover duration'. The contribution asset, CA_t, is represented as follows:

[14] Other methodologies to analyze the sustainability of pension schemes include micro-simulation models, general equilibrium models and indirect models. On this aspect see the papers by Lefebvre (2007) and Economic Policy Committee European Comission (2008).

[15] As Sweden is the only NDC country that compiles a balance on an annual basis, we will follow this approach to analyze how different ABMs applied to an NDC country react to uncertainty. To compile different types of actuarial balances are outside the scope of this paper.

[16] See Settergren and Mikula (2005) for the notional model, Boado-Penas et al. (2008) and Boado-Penas and Vidal-Meliá (2013) for the defined benefit PAYG system and Bommier and Lee (2003) for a general transfer model under the golden rule.

$$CA_t = C_t \times TD_t \tag{3.14}$$

where

$$\begin{aligned} TD_t = A_t^R - A_t^C &= \frac{3 \cdot P_t^3 N_t^3 + 4 \cdot P_{t-1}^3 (1+\lambda_t) A_t N_t^4}{P_t^3 N_t^3 + P_{t-1}^3 (1+\lambda_t) A_t N_t^4} - \frac{1 \cdot \pi S_t^1 N_t^1 + 2 \cdot \pi S_t^2 N_t^2}{\pi S_t^1 N_t^1 + \pi S_t^2 N_t^2} \\ &= 2 + \frac{K_{t-1}^N}{K_t^P} \frac{a_{t-1} - 1}{a_{t-1}} - \frac{S_0^2}{K_t^C} \end{aligned} \tag{3.15}$$

The first component of the turnover duration, A_t^R, is the weighted average age of pensioners and A_t^C is the weighted average age of contributors. The turnover duration represents the 'the average time a unit of money is in the system' as stated in Palmer (2006) assuming that economic, demographic and legal conditions remain constant, or, in other words, the number of years that is expected to elapse before all system liabilities are renewed or rotated[17].

The structure of the turnover duration as shown in (3.15) can be described as follows: the number 2 is the difference between the initial retirement age, 3, and the age of the entrants into the system, 1. The second component is the proportion of pensions paid to the second cohort retirees to the total pension expenditures and the third component corresponds to the proportion of the contributions received by the second cohort contributors to the total income from contributions

3.2.2 Liabilities

The sum of the liabilities of all participants at time t, V_t, can be split into the liabilities to contributors, V_t^1 and V_t^2, and the liabilities towards the retirees, V_t^3 and V_t^4. Note that the liabilities towards the cohort who just retired, those aged 3, have been already developed in Subsection 3.1. The total liabilities for each cohort are represented as follows:

$$\begin{aligned} V_t^1 &= NPW_t^1 = 0 \\ V_t^2 &= NPW_t^2 = \pi S_{t-1}^1 N_{t-1}^1 (1+r_t) = C_t \frac{S_0^1 (1+n_{t-1})}{K_{t-1}^C} \\ V_t^3 &= NPW_t^3 = C_t K_t^N \\ V_t^4 &= NPW_t^4 = \left(NPW_{t-1}^3 - P_{t-1}^3\right)(1+r_t) = C_t K_{t-1}^N \frac{a_{t-1} - 1}{a_{t-1}} \end{aligned}$$

Then the total liabilities, V_t, are:

$$\begin{aligned} V_t &= C_t \left(\frac{S_0^1 (1+n_{t-1})}{K_{t-1}^C} + K_t^N + K_{t-1}^N \frac{a_{t-1} - 1}{a_{t-1}} \right) \\ &= C_t \underbrace{\left(2 + \frac{S_0^1 (1+n_{t-3})}{K_{t-3}^C} - \frac{K_{t-1}^N}{a_{t-1}} \right)}_{K_t^V} \end{aligned} \tag{3.16}$$

where

K_t^V is a factor which represent the heterogeneity of the liabilities. The liabilities are proportional to the current contributions made by the active population.

[17] For more details, see Boado-Penas et al. (2008) and Settergren and Mikula (2005).

The number 2 in (3.16) appears from the sum of K_t^N and K_{t-1}^N which after rearranging provide the same number 2 that appears in the expression of the turnover duration (3.15). We observe that the expressions, (3.15) and (3.16), would coincide if the following equality holds:

$$\frac{K_{t-1}^N}{K_t^P}\frac{a_{t-1}-1}{a_{t-1}} - \frac{S_0^2}{K_t^C} = \frac{S_0^1(1+n_{t-3})}{K_{t-3}^C} - \frac{K_{t-1}^N}{a_{t-1}}$$

However, as shown in Proposition 2 presented below, this does not generally hold in a dynamic 4-OLG model. The following result demonstrates that the pension scheme is not generally solvent in absence of a buffer fund.

Proposition 2. *In general, the contribution asset does not equal liabilities in this four-period OLG unfunded dynamic model, that is $CA_t \neq V_t$ $\forall t$.*

Proof. This result is proven by counterexample. Let us assume that $n_t = n$=cte and $g_t = g$=cte $\forall t$ and deterministic. The working population and wages are then not time dependent and are denoted by $(1+n)$ and S_0^j for $j = 1, 2$. Let us assume further that $p_t^* \neq p_s^*$ for $t \neq s$ and that the theoretical indexation and discount rates coincide, that is, $\lambda_s^* = r_s^*$ $\forall s$. The notional capital of the retiring cohort at time t is then equal to C_t as shown in Proposition 1.

The liabilities and contribution asset at time t can be represented respectively as follows:

$$V_t = C_t\left(2 + \frac{S_0^1\cdot(1+n)}{S_0^1\cdot(1+n)+S_0^2} - \frac{1}{a_{t-1}}\right)$$
$$CA_t = C_t\left(2 + \frac{1}{K_t^P}\frac{a_{t-1}-1}{a_{t-1}} - \frac{S_0^2}{S_0^1\cdot(1+n)+S_0^2}\right)$$

The liabilities at time t, V_t, are equal to the contribution asset, CA_t, when the following expression holds:

$$1 - \frac{1}{a_{t-1}} = \frac{1}{K_t^P}\frac{a_{t-1}-1}{a_{t-1}}$$

Rearranging and replacing the annuities by their explicit expressions we obtain:

$$1 - \frac{1}{1+p_t^*} = \frac{(1+p_{t+1}^*)(1+p_t^*)}{(1+p_t^*)+p_t^*(1+p_{t+1}^*)}\frac{p_t^*}{1+p_t^*} \tag{3.17}$$

which only holds when $p_{t+1}^* = p_t^*$. □

4 Design of automatic balancing mechanisms (ABMs)

As demonstrated in the previous section, a notional defined contribution pension scheme does not guarantee liquidity or solvency by design in a dynamic environment. The desired liquidity and solvency objectives can be reached through automatic balancing mechanisms (ABMs). The purpose of successive application of ABMs is to provide automatic sustainability, which can be defined as 'the capacity of a pension scheme to adapt to financial turbulence without legislative intervention' (Settergren 2013).

This section will focus on two types of ABMs. The first mechanism re-establishes liquidity to the system, whereas the second mechanism restores solvency as defined in the previous sections. Both mechanisms will be considered for symmetric and asymmetric designs, although symmetric cases are rarely[18] applied in practice. Alho et al. (2013) and Palmer (2013) state that the mechanism can be symmetric in the sense that these adjust for both positive and negative deviations according to the sustainability indicator.

ABMs adjust the notional rate with a time-dependent variable to obtain liquidity or solvency ratios of at least unity at all times, depending on whether it is symmetric or asymmetric. The adjusted notional rate r_t^z applied to the system to attain the desired equilibrium will be $(1+r_t^z) = (1+r_t)\,B_t^z = I_t^z$, where $z = LR$ when the ABM for liquidity equilibrium is used and $z = SR$ when the ABM for solvency equilibrium is used. ABMs affect contributors by adapting the notional rate by which their notional capital is accumulated and retirees through changes in pension indexation.

The introduction of an ABM affects the notional rate and therefore has an impact in all derivations made in Section 3 which depend on the level of this rate. For instance, after introduction of an ABM z, the expression of the notional capital of the retiring cohort at time T, NPW_t^3, changes to $NPW_t^{3,z}$ as follows:

$$\begin{aligned} NPW_t^{3,z} &= \pi S_{t-2}^1 N_{t-2}^1 \left(1+r_{t-1}\right) B_{t-1}^z \left(1+r_t\right) B_t^z + \pi S_{t-1}^2 N_{t-1}^2 \left(1+r_t\right) B_t^z \\ &= C_t B_t^z K_t^{N,z} \end{aligned}$$

with

$$K_t^{N,z} = \frac{S_0^1(1+n_{t-2})}{K_{t-2}^C} B_{t-1}^z + \frac{S_0^2}{K_{t-1}^C}$$

Similarly, the other expressions also need to be adjusted after the introduction of the ABM[19]. The detailed formulae are presented in Appendix B. Section 4.1 and 4.2 provide details of the two ABMs considered.

4.1 ABM for the liquidity ratio

This ABM is designed to attain a liquidity ratio of 1. The notional rate is therefore affected by B_t^{LR}; therefore, $(1+r_t^{LR}) = (1+r_t)\,B_t^{LR} = I_t^{LR}$. The automatic mechanism B_t^{LR} (4.1), taking into account expression (3.1), can be easily derived as follows:

$$\begin{aligned} C_t + F_t^{-,LR} &= \overbrace{P_t^{LR}}^{C_t B_t^{LR} K_t^{P,LR}} \\ B_t^{LR} &= \frac{C_t + F_t^{-,LR}}{C_t K_t^{P,LR}} \end{aligned} \tag{4.1}$$

where $K_t^{P,LR}$ and $F_t^{-,LR}$ are given by (B.2) and (B.3) respectively for $z = LR$.

If the ABM is symmetric the surplus (deficit) is shared with participants in good (bad) times. Furthermore, in this case we observe that (4.1) is simplified to $B_t^{LR} = \frac{1}{K_t^{P,LR}}$ because a symmetric ABM associated to a liquidity ratio equal to unity does not accumulate buffer funds. On the other hand, if the ABM is asymmetric, participants do not benefit from positive economic shocks and funds are accumulated according to formula (B.3) for $z = LR$.

[18]Only Sweden, among the four NDC countries has put in place an ABM to restore the health of the system (Chłón-Domińczak et al. 2012). The ABM is asymmetric, as it is only triggered when the solvency ratio is lower than 1. However, the Swedish ABM allows for recovery. After a period of low returns as a consequence of the mechanism, a period of higher-than-normal returns follows. See Barr and Diamond (2011).

[19]Please note that the formula (3.3) corresponding to the income from contributions C_t remains unchanged as it is not affected by the notional rate.

4.2 ABM for the solvency ratio

This ABM is designed to restore the system's solvency (solvency indicator equal to 1) by reducing the growth of the pension liabilities, that is, pension payments and contributor's notional capital. The adjusted notional rate is then $(1 + r_t^{SR}) = (1 + r_t) B_t^{SR} = I_t^{SR}$. The contribution asset at time t, CA_t^{SR} does not depend on the ABM B_t^{SR} at time t, but depends on those applied before t, as noted in formula (B.4) and (B.9). Only the liabilities, V_t^{SR} (B.5) depend on the ABM factor at time t. The automatic mechanism, B_t^{SR}, can be represented as follows:

$$
\begin{aligned}
CA_t^{SR} + F_t^{-,SR} &= \overbrace{V_t^{SR}}^{C_t B_t^{SR} K_t^{V,z}} \\
B_t^{SR} &= \frac{CA_t^{SR} + F_t^{-,SR}}{C_t K_t^{V,z}}
\end{aligned}
\tag{4.2}
$$

This ABM affects the solvency of the pension scheme in two ways: it affects the rate of increase of the liabilities of both contributors and pensioners and the pension expenditures. The latter affects the contribution asset via the expenditure-based component of the turnover duration. Please note that, contrary to the ABM for the liquidity ratio, this ABM might accumulate some funds even when it is symmetric because solvency does not necessarily imply liquidity in a dynamic setting (Alonso-García 2015).

5 Numerical illustration

This section presents a numerical example and analyzes the behavior of the notional factor in a stochastic environment whenever an ABM is applied. The effects of these ABMs are evaluated in terms of the expected value and variance of the notional factor and ratio[20] of the buffer fund to contributions. Please note that the results of an introduction of the ABM are compared to those with the no-introduction of an ABM.The figures in this section plot the notional compounding factor $1 + r_t$ for presentation purposes and we term this as the notional factor.

We assume NDC was fully in force before the first application of the ABM at time t. Before this moment t, no funds have been accumulated.

5.1 Model and simulation setup

The wage and population processes are given by the following expressions:

$$1 + n_s = e^{n - \frac{\sigma_P^2}{2} + \sigma_P (W_s^P - W_{s-1}^P)} \tag{5.1}$$

$$1 + g_s = e^{g - \frac{\sigma_S^2}{2} + \sigma_S (W_s^S - W_{s-1}^S)} \tag{5.2}$$

The stochastic process $1 + n_s$ (resp. $1 + g_s$) is distributed log-normally with mean $n - \sigma_P^2/2$ and standard deviation σ_P (resp. with mean $g - \sigma_S^2/2$ and standard deviation σ_S) $\forall t$. The processes W^P and W^S are correlated Wiener processes, that is, the expected value of $W_s^P W_s^S$ is equal to $\rho \cdot s$.

[20]The ratio provides more insights than examining the nominal value of the fund because we can easily compare to values of contributions.

Table 2 shows the main assumptions for demographic, economic and financial variables and sensitivity analysis that we carry out in our numerical model. As shown in the column 'Survival probability', we study three different trends of the future survival probability. The 'base' scenario corresponds to constant mortality rates, whereas the 'up' (resp. 'down') scenario correspond to an increasing (resp. decreasing) survival probability at retirement.

Table 2: Population, wages and market assumptions

Population	Wages	Market	Survival probability	
$n = 0.25\%$	$g = 1.5\%$	$i = 0\%$	Base	$p_s = 0.5\ \forall t > s$
$\sigma_P = 5\%$	$\sigma_S = 10\%$	$F_t = 0$	Up	$p_s = 0.5 + 0.005(s - t)$ for $t > s$
$N_t^x = 100\ \forall x$	$S_t^1 = 30,000$		Down	$p_t = 0.5 - 0.005(s - t)$ for $t > s$
$\rho = -0.25$[a]	$S_t^2 = 45,000$			

[a]The negative correlation reflects the fact that cohort size negatively affects earnings (see Brunello (2009)).

The results are obtained through Monte Carlo simulations for 1 million randomly generated paths. The analysis is conducted for 8 periods, which represents two full population renewals and for the various scenarios presented in Table 3.

Table 3: Scenarios analyzed

ABM	Symmetric	Mortality	Life table	Baby Boom
LR	Yes	Base	Case 1: prospective	Yes
SR	No	Up	Case 2: current	No
		Down		

To calculate the expected value and variance of the notional factor and the expected value of the ratio of the buffer fund to the contributions, we use the closed-form expressions developed in Section 2, 3 and 4 and the tools described in Appendix B.

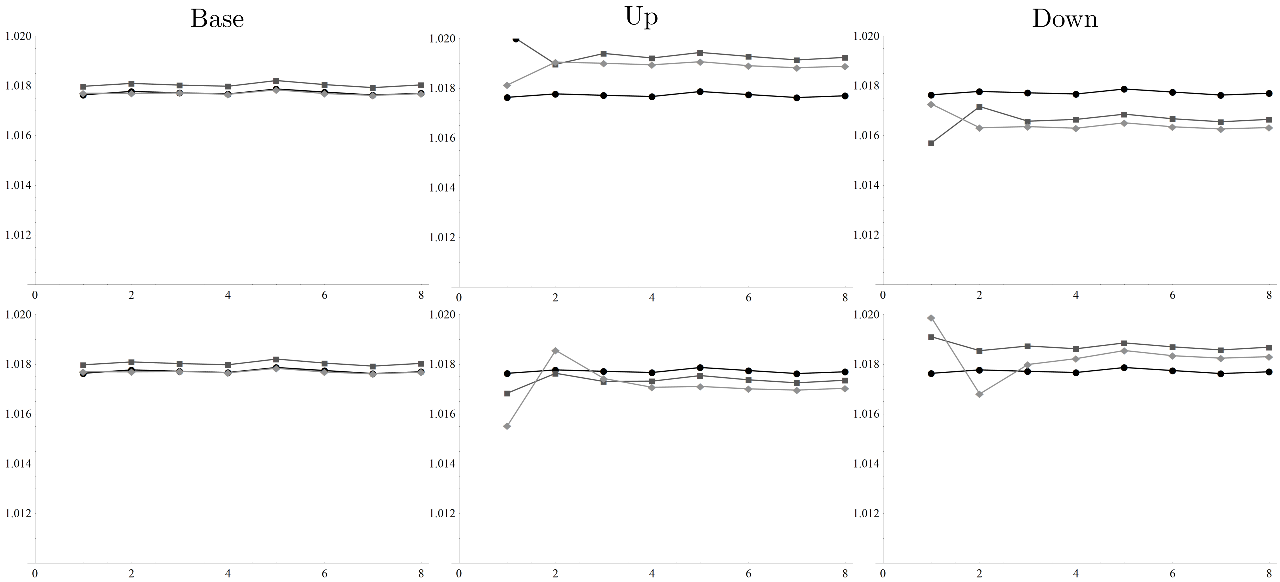
The following Subsection 5.2 analyzes the impact of the choice of the life table used in the calculation of the annuity, as well as the impact of the longevity trend. In Subsection 5.3 the effect of a one-time exogenous shock ('baby boom') is assessed.

5.2 Mortality risk

The system is exposed to mortality risk to the extent that we do not know the survival probability of individuals. It is well understood that this risk has an impact on the sustainability of pension schemes. Hence, we study the system when the future longevity patterns are known and incorporated in the annuity calculation (Case 1). This corresponds to $p_t^*{=}p_t$ in the annuity formula (3.9). Case 2 uses current historical values of mortality and evaluates to what extent mortality risk affects our results. In this latter case, current longevity is used, even though it is expected to change, and does not allow for any longevity adjustment in the pension revaluation which makes the system no longer actuarially fair. This corresponds to $p_t^*{=}p_{t-1}$ in the annuity formula (3.9). Mathematically, the annuity[21] for the Case 1 is $a_t^1 = \mathbb{E}_t\left[1 + p_{t+1}\right]$, whereas the annuity for the Case 2 is $a_t^2 = \mathbb{E}_t\left[1 + p_t\right]$.

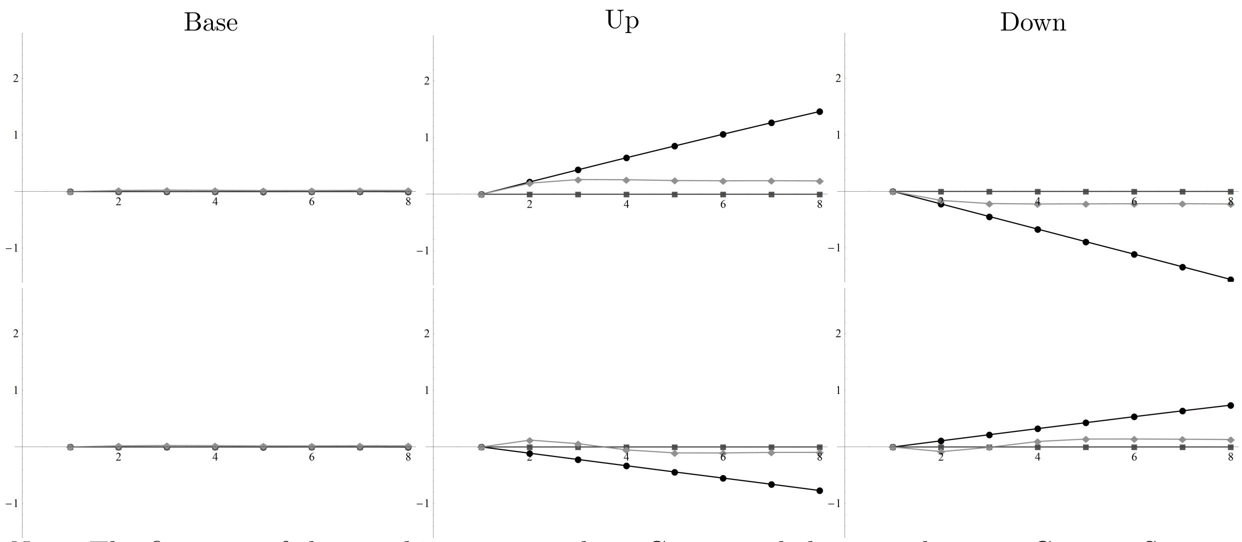
[21]The annuity in our framework is calculated under the assumption that the theoretical indexation rate λ_t^* is equal to the theoretical discounting rate r_t^*

Figure 1: The expected value of the notional factor after the introduction of an ABM for the symmetric case: without ABM (dotted black line), liquidity ratio ABM (dark gray squares) and solvency ratio ABM (light gray rhombuses).



Note: The first row of the graphic corresponds to Case 1 and the second row to Case 2. *Source:* the authors.

Figure 2: Expected value of the ratio between the fund and contributions after the introduction of an ABM for the symmetric case: without ABM (dotted black line), liquidity ratio ABM (dark gray squares) and solvency ratio ABM (light gray rhombuses).



Note: The first row of the graphic corresponds to Case 1 and the second row to Case 2. *Source:* the authors.

The figures depicted in this section compare the introduction of an ABM in various scenarios. In terms of design, the usage of prospective tables (Case 1, Figures 1 and 2, first row) for the symmetric case does not guarantee either liquidity or solvency and ABMs are still needed when the survival probability is dynamic. In fact, in the Base case (Figure 1, first column), we observe that the effect of the liquidity ratio ABM is negligible and the effect of the solvency ratio ABM is non-existent as it is not triggered under this scenario.

More importantly, based on our simulations, prospective life tables may be welfare enhancing for old-age retirees when the longevity is increasing. In fact, the inclusion of future longevity increases in the annuity reduces the initial pension but increases the corrected indexation. The

Table 4: The variance of the Notional Factor: Aggregated Values

	Case 1			Case 2		
	Base	Up	Down	Base	Up	Down
	Symmetric - No baby boom					
LR	6.11%	6.25%	5.96%	6.11%	5.83%	6.39%
SR	-1.98%	-1.94%	-2.03%	-1.98%	-2.02%	-1.95%
	Symmetric- Baby boom					
LR	6.20%	6.33%	6.04%	6.20%	5.92%	6.48%
SR	-1.92%	-1.87%	-1.96%	-1.92%	-1.96%	-1.88%
	No Symmetric - No baby boom					
LR	0.61%	0.37%	0.98%	0.61%	0.77%	0.48%
SR	-0.57%	-0.39%	-0.80%	-0.57%	-0.66%	-0.49%
	No Symmetric - Baby boom					
LR	0.54%	0.30%	0.86%	0.54%	0.68%	0.41%
SR	-0.76%	-0.57%	-0.97%	-0.76%	-0.85%	-0.67%

Note: The values presented correspond to the relative value of the sum of the variance of the 8 periods compared to the sum of the variance of the 8 periods in the scenario without ABM. The values indicate that the total variance associated to the LR (resp. SR) is higher (resp. lower) in all scenarios.

ABMs also lower the wealth accumulation or debt (Figure 2, first row) leading to governments investing their resources on current participants. Please note that this welfare increase for old-age retirees does not hold when longevity is decreasing (Case 1, Down scenario). In this case, the application of an ABM corrects fund imbalances (Figure 2) but favors the younger-than-average retirees who have higher initial pensions and are less affected by subsequent ABM applications.

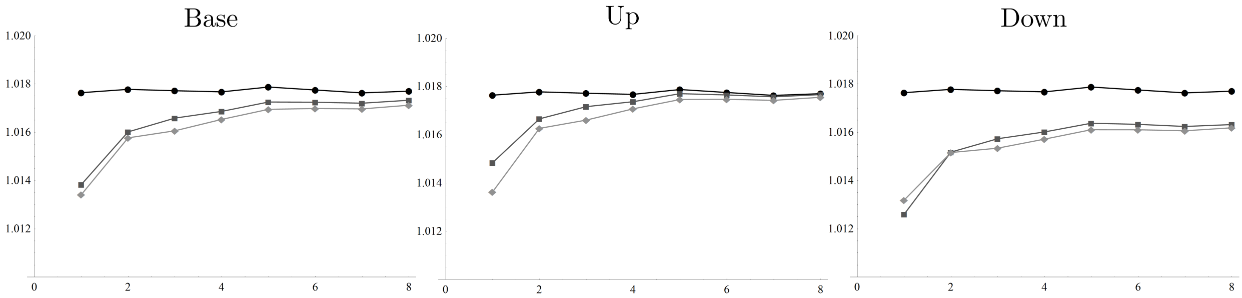
As indicated in the second row of Figures 1 and 2 the results vary if current mortality rates are used (Case 2). When the mortality is increasing the introduction of the ABMs lowers the notional rate paid to contributions and retirees because the ABMs correct errors introduced by using current mortality rates, which resulted in higher initial pensions. The contrary occurs, Figure 1, under decreasing longevity for a symmetric mechanism, that is, the ABMs increase the notional rate because of the over-estimation of the survival probability in the computed annuity.

Comparing the symmetric ABMs for the Case 1 and 2 we can state that whenever the longevity is increasing the younger-than-average benefit from using current tables (Case 2) because of the higher initial pensions paid while older-than-average benefit from using prospective tables because of the higher indexation rate paid after introduction of ABMs. In both cases the surplus or debt of the buffer fund will be brought close to zero after implementation of ABMs. In terms of the variance, Table 4, the variance of the liquidity ratio ABM is always higher than both the no ABM scenario and solvency ratio ABM, in this order.

Under an asymmetric design (Figure 3), it is not surprising that the activation of an ABM reduces the expected value of the notional factor in all longevity scenarios because the asymmetric ABMs are designed to be capped at 1. When we calculate the average value of the million scenarios, values less than 1 will have higher relative weights, which significantly decrease the expected value of the notional factor in presence of an ABM. This reduction is greater for the solvency ratio ABM. In all scenarios, the expected value and variance of the notional factor under the liquidity ratio ABM is higher than in absence of an ABM or under a solvency ratio ABM as depicted in Table 4.

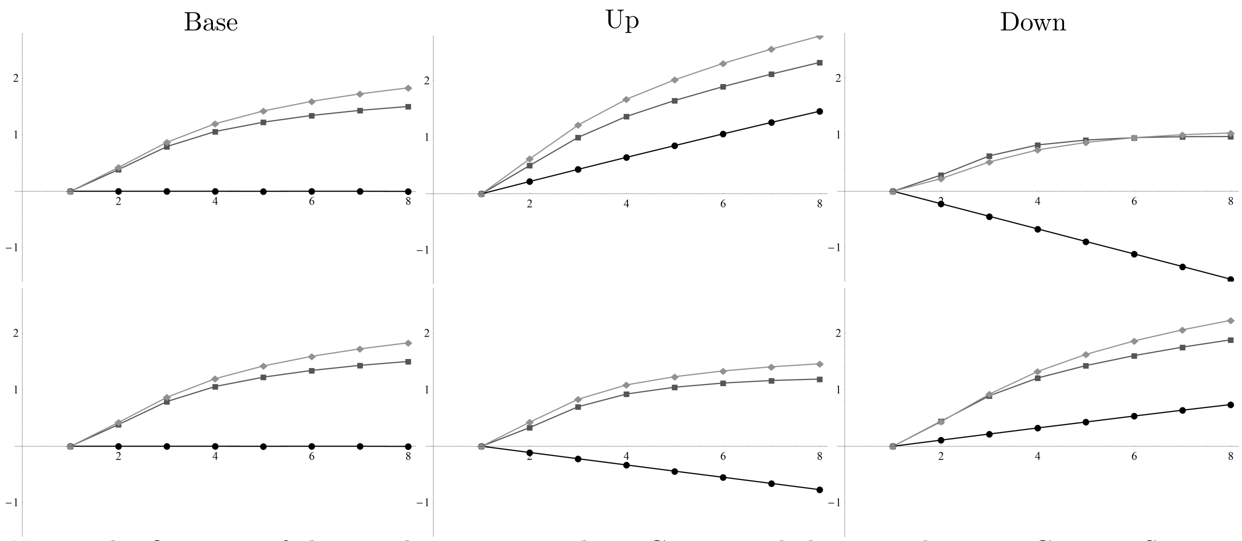
As shown in Figure 4, ABMs under the asymmetric desing increase the expected value of

Figure 3: The expected value of the notional factor after the introduction of an asymmetric ABM (Case 1): without ABM (dotted black line), liquidity ratio ABM (dark gray squares) and solvency ratio ABM (light gray rhombuses).



Note: The graphics corresponding to Case 2 provide very similar results. *Source:* the authors.

Figure 4: Expected value of the ratio between the fund and contributions after the introduction of an ABM for the asymmetric case: without ABM (dotted black line), liquidity ratio ABM (dark gray squares) and solvency ratio ABM (light gray rhombuses).

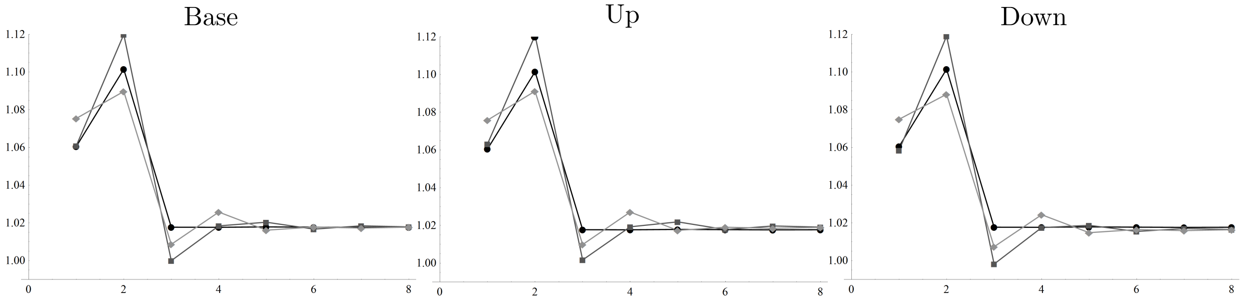


Note: The first row of the graphic corresponds to Case 1 and the second row to Case 2. *Source:* the authors.

the fund. For instance, if longevity increases but current mortality rates are used in the pension calculation (Case 2), the amount of the initial pension increases. This results in higher pension expenditures and a corresponding decrease in the value of the fund. The ABMs consequently decrease the notional factor to re-establish liquidity or solvency.

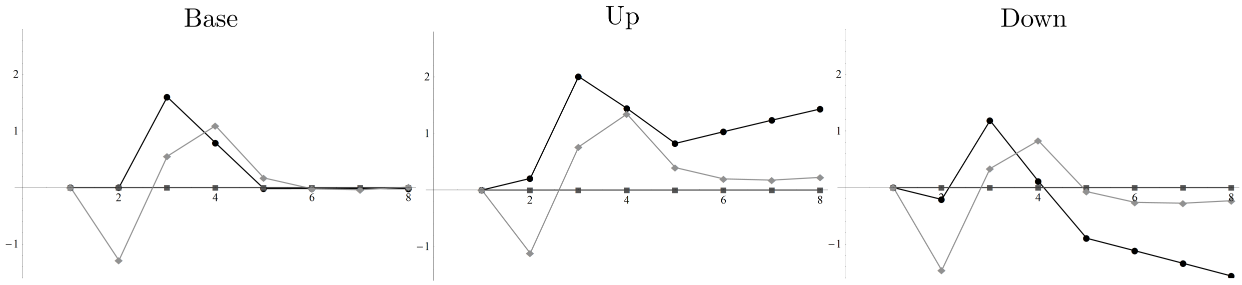
Finally, we observe that for asymmetric designs (Case 1 and 2), the expected value of the fund increases more than under the symmetric design due to the fact that a lower average notional factor systematically increases the buffer fund. Note that the final values of the expected value of the fund in the asymmetric design approach a value of 2, which means that the value of the fund is two times greater than the value of contributions at the same time. This value implies that asymmetric ABM may be too conservative and lead to excessive capital accumulation under our assumptions. As above mentioned, this may be undesirable for participants because the resources will not be used for them. However, an asymmetric ABM may be worth considering if the government plans to reform its pension scheme and needs some capital injections to finance the transition between reforms.

Figure 5: Expected Value of the notional factor after a baby boom with an ABM: no ABM (dotted black line), liquidity ratio ABM (dark gray squares) and solvency ratio ABM (light gray rhombuses).



Note: The graphics corresponding to Case 2 provide very similar results. *Source:* the authors.

Figure 6: Expected Value of the ratio between the fund and contributions after a baby boom with an ABM: no ABM (dotted black line), liquidity ratio ABM (dark gray squares) and solvency ratio ABM (light gray rhombuses).



Note: The graphics corresponding to Case 2 provide very similar results. *Source:* the authors.

5.3 Baby boom

The demographic and the wage processes, $1+n_s$ (5.1) and $1+g_s$ (5.2), respectively, are stationary stochastic processes, which means that their distributions do not depend on time. This holds in the absence of exogenous shocks, such as a drop in fertility or a baby boom. For instance, if the demographic process at time t is affected by a one-time exogenous shock δ, then it is represented by $1 + n_t^* = (1 + n_t)(1 + \delta)$. To evaluate a demographic shock, we examine the impact of one-time positive exogenous shock that affects the entrant population at time t. This shock is assumed to have a value of $\delta = 10$ percent. The effects of this shock are illustrated by a so-called baby boom case, which replicates the effect of a fertility boost.

We evaluate the sustainability of the pension scheme after a baby boom and observe the effects on the ABM. Figure 5 indicates the expected value of the notional factor and Figure 6 indicates the ratio of the fund to the contribution level for the symmetric design[22].

We observe that the notional rate increases substantially from the average value of approximately 1.8% due to the sudden entrant population increase, which increases the total contribution base substantially. Once the baby boom cohort retires, there is no longer an effect on the notional factor because the entry population returns to the relative stationary level after the exogenous shock. Note that the notional factor in absence of ABM does not depend on the retired population but on the contributors, which explains that the level of the notional factor goes back to the pre-shock level once that the shocked cohort retires.

The liquidity ratio ABM increases the notional rate during the first two periods due to the surplus created by the higher relative contribution base. However, once the baby boom cohort retires, this factor decreases significantly to avoid debt. However, the solvency ratio balance

[22]The asymmetric case results are omitted because the explanation is similar to the non-baby boom case.

mechanism responds differently to this shock due to its construction. When the baby boom cohort enters the pension scheme, the turnover duration increases because it assumes that the future population will have the same composition, that is, it behaves as if the current shock is permanent. This anticipation increases the level of the notional factor during the first period, affecting both contributors and retirees. This leads to debt as the expenditures increase above its sustainable levels. This increase affects the solvency balance mechanism during the second period, which decreases the notional factor. The decrease is sufficient to restore a surplus level, which will positively affect the notional factor. The notional rate affected by both balance mechanisms approaches the level in absence of adjustments once the shock cohort exits the pension scheme.

In absence of ABMs, the system only partially regulates itself after the baby boom population exits the system, as indicated by the triangle in Figure 6. For instance, in Case 1 of the longevity base scenario, we observe a triangle that comes back to liquidity and solvency after the shocked cohort leaves the scheme. The triangle form can be explained as follows: the pre-contribution fund equals 0 during the second period because we assume an initial fund of 0. The level of the notional rate, although higher than its normal levels, is insufficient to attain fund neutrality. Therefore, a surplus occurs. However, this surplus is used to pay the pensions to the higher number of baby boom pensioners until they leave the pension scheme, decreasing the fund to a level approaching but not reaching zero. This interpretation is independent of the market interest rate and is due to the simple nature of the shock. Please note that our results are robust to alternative assumptions[23]. For simplicity, we provide only some of the representative results.

5.4 Sharpe ratio

The introduction of an ABM reduces the amount of the buffer fund and corrects for dynamic deviation. However, it is not clear whether the liquidity or the solvency ratio ABM should be preferred. A common feature of the figures and tables presented is that the expected value and variance of the notional factor increases after the introduction of the liquidity ratio ABM, while it decreases if the solvency ratio ABM is triggered. A way to assess whether the increase in average return outweighs the increase in variance is to analyze the ratio between the expected value and its standard deviation, that is, the Sharpe ratio[24].

Table 5 shows that the Sharpe ratio decreases whenever the liquidity ratio ABM is applied. The contrary holds whenever the solvency ratio ABM is considered. The values presented in Table 5 are calculated based on the Sharpe ratio in presence of ABMs for the different scenarios compared to the Sharpe ratio of 9.829 without ABM for the no baby boom case and to the Sharpe ratio of 9.834 without ABM for the baby boom case. We also note that the decrease in the Sharpe ratio when the liquidity ratio ABM is symmetric is almost 3 times the increase for the SR case. We observe that the variation related to the no ABM case is much lower when the ABM considered is asymmetric. However, the ranking "ABM LR $<$ no ABM $<$ ABM SR" always holds in all scenarios.

The results obtained indicate that despite the higher expected value for the liquidity ratio ABM in most scenarios, its proportional increase in variance makes it a less attractive choice than the solvency ratio ABM. In terms of 'reward-to-variability' ratio we would therefore choose the solvency ratio ABM. However, while this ABM restores solvency, it may not re-establish liquidity (Alonso-García 2015).

[23] We also addressed the possibility that contributors aged 2 earn less than contributors aged 1 do. Furthermore, we considered the possibility of a negative notional rate with a negative sum of g and n. Finally, we examined a case with negative parameter n as well as a case with a negative exogenous shock δ.

[24] See Sharpe (1966, 1994) and references therein.

Table 5: The Sharpe ratio after the introduction of an ABM: relative percentage increase or decrease with respect to the case without ABM

	Case 1			Case 2		
	Base	Up	Down	Base	Up	Down
	Symmetric - No baby boom					
LR	-2.89%	-2.83%	-2.96%	-2.89%	-2.83%	-2.95%
SR	1.01%	1.10%	0.91%	1.01%	0.97%	1.05%
	Symmetric- Baby boom					
LR	-2.89%	-2.83%	-2.96%	-2.89%	-2.83%	-2.96%
SR	1.01%	1.10%	0.91%	1.01%	0.97%	1.05%
	No Symmetric - No baby boom					
LR	-0.42%	-0.24%	-0.69%	-0.42%	-0.54%	-0.32%
SR	0.14%	0.09%	0.18%	0.14%	0.15%	0.13%
	No Symmetric - Baby boom					
LR	-0.38%	-0.20%	-0.64%	-0.38%	-0.49%	-0.28%
SR	0.23%	0.18%	0.27%	0.23%	0.25%	0.21%

6 Conclusion

This paper describes how liquidity and solvency indicators in notional defined contribution accounts are affected by fluctuations in economic and demographic conditions in a stochastic environment. The analysis uses a four-generation model in which two generations are contributors and two generations are pensioners. The consideration of four cohorts introduces heterogeneity to the pension scheme while maintaining a tractable dependence structure. Furthermore, we can draw interesting conclusions without the drawback of heavier computation.

Under this scenario, the notional rate, defined as the rate of increase of the contribution base (Börsch-Supan 2006), that indexes the accumulated notional capital is affected by both wage and population processes. The pension is revalued by an adjusted indexation process that guarantees longitudinal equilibrium among the cohorts inspired by Alho et al. (2013). However, this equilibrium, also referred as actuarial fairness, does not guarantee liquidity or solvency of the pension scheme in a dynamic framework. Consequently, some ABMs should be designed and triggered.

This paper considers two ABMs. The first mechanism makes income from contributions equal pension expenditures (liquidity ratio ABM), and the second mechanism, based on the current Swedish adjustment, makes the assets of the pension scheme equal to liabilities (solvency ratio ABM). The results are evaluated in terms of the expected values, variances and Sharpe ratios of the notional factor and buffer fund and are calculated under both symmetric and asymmetric designs. In all cases, the introduction of an ABM reduces the volatility of the fund. The solvency ratio ABM yields the lowest value of the notional factor's variance and the lowest expected value. In terms of 'reward-to-variability', also known as Sharpe ratio, the solvency ratio ABM provides the highest value.

The paper also analyzes how longevity risk affects our analysis. We find that there are clear advantages of introducing a symmetric ABM in the case of unanticipated longevity increase. We show that whenever longevity is increasing the younger-than-average retirees will benefit from using current tables because of the higher initial pensions paid while the older-than-average will benefit from using prospectives tables because of the higher indexation rate paid after the introduction of ABMs. Furthermore, these symmetric adjustments avoid debt accumulation. However, note that asymmetric ABMs can lead to significant capital accumulation after two population renewals. This is consistent with the fact that an 'NDC scheme does little to prevent significant asset accumulation (...) on average' as argued by Auerbach and Lee (2006).

These conclusions are robust to a one-time exogenous demographic shock. Additionally, the notional system that uses the rate of increase of the total contribution base as the compounding factor can regulate itself almost completely in presence of such a shock, that is, once the shocked cohort exits the pension scheme, the level of debt or surplus approaches the level in absence of a shock. Lastly, this paper contributes to the NDC literature by providing closed-form solutions of the main components of the pension scheme and ABMs in a dynamic setting.

Finally, based on the two ABMs presented in this paper, at least two important directions for future research can be identified:

- To extend the four OLG model to a real society and assess the impact of economic, financial and demographic conditions on the ABM.
- To look at inter-generational fairness after the introduction of automatic balancing mechanisms.

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A Derivation of the longitudinal adjustment factor

This appendix shows that adjustments to the pension indexation can be made in order to attain longitudinal equilibrium, that is, to ensure that individuals receive no more or less than what they have contributed. However, it implies that indexation of pensions in practice differs among retired cohorts, which may not be desirable when various cohorts coexist. The processes r_t^*, p_t^* and λ_t^* represent respectively the theoretical notional rate, mortality rate and indexation of pensions for period t. The processes r_t, p_t and λ_t denote, respectively, the observed notional factor, which could be affected by some ABM B_t^z, mortality rate and indexation rate at time t. In this framework, the annuity is given by (3.9).

The aim is to produce equivalence between accrual and forecasted liabilities for all cohorts, in particular, for the cohort aged 3 at time t. The accrual liability $V_t^{3,A}$ is denoted by (A.1) and the forecasted liability $V_t^{3,F}$ is denoted by (A.2). The adjustment factor at time $t+1$ by A_{t+1} is represented as follows:

$$
\begin{aligned}
V_t^{3,A} &= NPW_t^3 && \text{(A.1)} \\
V_t^{3,F} &= P_t^3 N_t^3 + P_t^3 N_{t+1}^4 \frac{1+\lambda_{t+1}}{1+r_{t+1}} A_{t+1} \\
&= NPW_t^3 \frac{1}{a_t}\left(1 + p_{t+1}\frac{1+\lambda_{t+1}}{1+r_{t+1}} A_{t+1}\right) && \text{(A.2)}
\end{aligned}
$$

The following adjustment factor makes both liabilities equivalent:

$$
A_{t+1} = \left(\frac{1+r_{t+1}}{1+\lambda_{t+1}}\right)\frac{1}{p_{t+1}}\mathbb{E}_t\left[p_{t+1}^*\left(\frac{1+\lambda_{t+1}^*}{1+r_{t+1}^*}\right)\right] \tag{A.3}
$$

When the mortality, demographic and economic processes are independent, the adjustment factor A_{t+1} (A.3) can be rewritten as follows:

$$
A_{t+1} = \underbrace{\frac{\mathbb{E}_t\left[\frac{1+\lambda_{t+1}^*}{1+r_{t+1}^*}\right]}{\frac{1+\lambda_{t+1}}{1+r_{t+1}}}}_{\text{Economic adjustment}} \quad \underbrace{\frac{\mathbb{E}_t\left[p_{t+1}^*\right]}{p_{t+1}}}_{\text{Longevity adjustment}} \tag{A.4}
$$

The economic adjustment affects all retirees because the calculation only considers the difference between the theoretical and the observed notional and indexation rates. However, the second part, which corresponds to the longevity adjustment, would differ by retired cohort because it depends on the theoretical mortality at the time of annuity calculation and the experienced mortality of the same cohort during retirement.

B Theoretical framework

Afer the introduction of an ABM in a pension scheme, the formulae presented in Section 3 needs to be adjusted. The superscript z denotes the kind of ABM that is implemented, with $z = LR$ representing the liquidity ratio and $z = SR$ representing the solvency ratio. Note that the notional factor and the income from contributions are not affected by the ABMs.

$$
\begin{aligned}
NPW_t^{3,z} &= C_t B_t^z K_t^{N,z} && \text{(B.1)} \\
P_t^z &= C_t B_t^z K_t^{P,z} && \text{(B.2)} \\
F_t^{-,z} &= F_{t-1}^{+,z}(1+i_t) = F_0^- \prod_{j=1}^{t}(1+i_j) + \sum_{j=0}^{t-1}(C_j - P_j^z)\prod_{k=j+1}^{t}(1+i_k) && \text{(B.3)} \\
CA_t^z &= C_t TD_t^z && \text{(B.4)} \\
V_t^z &= C_t B_t^z K_t^{V,z} && \text{(B.5)} \\
A_t^z &= \frac{\mathbb{E}_{t-1}\left[p_t^* \frac{1+\lambda_t^*}{1+r_t^*}\right]}{p_t \frac{1+\lambda_t}{1+r_t}} B_t^z && \text{(B.6)}
\end{aligned}
$$

where

$$K_t^{N,z} = \frac{S_0^1(1+n_{t-2})}{K_{t-2}^C} B_{t-1}^z + \frac{S_0^2}{K_{t-1}^C} \tag{B.7}$$

$$K_t^{P,z} = \frac{K_t^{N,z}}{a_t} + B_{t-1}^z K_{t-1}^{N,z} \frac{a_{t-1}-1}{a_{t-1}} \tag{B.8}$$

$$TD_t^z = 2 + B_{t-1}^z \frac{K_{t-1}^{N,z}}{K_t^{P,z}} \frac{a_{t-1}-1}{a_{t-1}} - \frac{S_0^2}{K_t^C} \tag{B.9}$$

$$K_t^{V,z} = 1 + B_{t-1}^z \left\{ \frac{S_0^1(1+n_{t-2})}{K_{t-2}^C} + K_{t-1}^{N,z} \frac{a_{t-1}-1}{a_{t-1}} \right\} \tag{B.10}$$

Due to the structure of the ABMs B_t^{LR} (4.1) and B_t^{SR} (4.2), it is not possible to specify their probability distributions. Therefore, the expected value and variance cannot be calculated through analytical formulas. However, given two random variables X and Y and a function $h(X,Y)$, the expected value of this function can be expressed directly in terms of the transformation function $h(X,Y)$ and the joint density $f_{X,Y}(x,y)$ of X and Y: $\mathrm{E}[h(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) f_{X,Y}(x,y)dxdy$ (see Papoulis (1991)). Furthermore, if the two random variables X and Y are independent, the joint density is $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. This property can be applied to our problem. In the remainder of the appendix we will present the recursive formulas used in our framework.

Denote the transformation of the multivariate random vector X_s at time $s > t$ for the ABM $z = LR, SR$ as $h_{t,s}^z(x_s)$. Note that t indicates the time when the ABM is first implemented. The entries' and wages' rate of increase, $1+n_s$ and $1+g_s$ respectively, will be rewritten in the remainder of this section in terms of D_s and G_s respectively in order to shorten the lengthy notation. The following formulae presents the explicit dependence of (B.7-B.9) to the different demographic and wage processes D_s and G_s:

$$K_s^C = f_{K_s^C}(D_s) \tag{B.11}$$

$$K_s^{N,z} = f_{K_s^{N,z}}(D_{s-1}, D_{s-2}, h_{t,s-1}^z(x_{s-1})) \tag{B.12}$$

$$K_s^{P,z} = f_{K_s^{P,z}}(D_{s-1}, D_{s-2}, D_{s-3}, h_{t,s-1}^z(x_{s-1}), h_{t,s-2}^z(x_{s-2})) \tag{B.13}$$

$$TD_s^z = f_{TD_s^z}(D_s, D_{s-1}, D_{s-2}, D_{s-3}, h_{t,s-1}^z(x_{s-1}), h_{t,s-2}^z(x_{s-2})) \tag{B.14}$$

$$K_s^{V,z} = f_{K_s^{V,z}}(D_{s-2}, D_{s-3}, h_{t,s-1}^z(x_{s-1}), h_{t,s-2}^z(x_{s-2})) \tag{B.15}$$

$$I_s^z = f_{I_s^z}(D_s, D_{s-1}, G_s) \tag{B.16}$$

$$\frac{F_s^{-,z}}{C_s} = f_{t,s}^z = f_{f_{t,s}^z}(D_{t-1}, .., D_s, G_{t+1}, .., G_s) \tag{B.17}$$

Furthermore, the δ affecting the demographic process at time t represents a one-time exogenous shock affecting the population, as presented in Section 5. This shock will vary the notation of the notional factor during the first two periods t, $t+1$ as follows:

$$1 + r_t = G_t D_{t-1} \frac{S_0^1 D_t (1+\delta) + S_0^2}{S_0^1 D_{t-1} + S_0^2} \tag{B.18}$$

$$1 + r_{t+1} = G_{t+1} D_t (1+\delta) \frac{S_0^1 D_{t+1} + S_0^2}{S_0^1 D_t + S_0^2} \tag{B.19}$$

The notional factor for $j \geq t+2$ will be given by:

$$1 + r_j = G_j D_{j-1} \frac{S_0^1 D_j + S_0^2}{S_0^1 D_{j-1} + S_0^2} \tag{B.20}$$

Furthermore, we assume that the individual increments have zero expected value, i.e., $\mathrm{E}[B_x(j) - B_x(k)] = 0$ for $x = P, W$ and $j \neq k$, and, finally, that the increments of both processes are independent, that is:

$$\mathrm{E}[(W^P(j) - W^P(k))(W^S(j) - W^S(k))] = 0 \text{ for j} \neq \text{k}.$$

These assumptions lead to $Cov(D_j, D_k) = 0$ and $Cov(D_j, G_k) = 0$ for $j \neq k$ and to a non-zero covariance when the processes interact during the same period $Cov(G_s, D_s) = e^{R+\gamma+\frac{\sigma_P^2+\sigma_S^2}{2}} (e^{\rho\sigma_P\sigma_S} - 1)$.

The ABM at time $s = t$ depends on the random variable vector $X_t = (D_{t-3}, D_{t-2}, D_{t-1})$ for $z = LR$ and $X_t = (D_{t-3}, D_{t-2}, D_{t-1}, D_t)$ for $z = SR$, and on the random variable vector $X_s = (D_{t-3}, ..., D_t, G_t, ..., G_s)$ for $s \geq t + 1$ for both $z = LR, SR$ with $n = 2(s - t) + 4$ length of the vector.

The following represents the joint distribution of a random vector X_s:

$$f_{X_s}(x) = \prod_{j=t-3}^{t} f_{D_j}(d_j) \prod_{j=t+1}^{s} f_{G_j,D_j}(g_j, d_j) \tag{B.21}$$

for $s \geq t+1$ where the distribution of the product of the demographic process D_s and wage process G_s $\forall s$ is denoted by $G_s D_s \sim logN(R+\gamma-\frac{\sigma_P^2+\sigma_S^2}{2}, \sigma_{P,W}^2)$ with $\sigma_{P,W}^2 = \sigma_P^2+\sigma_S^2+2\rho\sigma_P\sigma_S$. Then the joint density function of (G_s, D_s) is represented as follows:

$$f_{G_s,D_s}(x, y) = \frac{1}{xy\sqrt{|\Sigma|}} e^{-\frac{1}{2|\Sigma|}\left((\log z - \mu)' \Sigma^{-1} (\log z - \mu)\right)} \text{ for } xy > 0 \tag{B.22}$$

with:

$$\log z = \begin{pmatrix} \log x \\ \log y \end{pmatrix}, \ \mu = \begin{pmatrix} R - \frac{\sigma_P^2}{2} \\ \gamma - \frac{\sigma_S^2}{2} \end{pmatrix} \tag{B.23}$$

$$\Sigma = \begin{pmatrix} \sigma_S & \rho\sigma_S\sigma_P \\ \rho\sigma_S\sigma_P & \sigma_P \end{pmatrix} \tag{B.24}$$

$$|\Sigma| = \text{determinant of variance-covariance matrix } \Sigma \tag{B.25}$$

Finally, the k^{th} raw moment of the the ABM for $z = LR, SR$ is given by the following:

$$\begin{aligned} \mathrm{E}[(B_s^z)^k] &= \mathrm{E}[\left(h_{t,s}^z(x_s)\right)^k] \\ &= \int_0^\infty ... \int_0^\infty \left(h_{t,s}^z(x_1, ..., x_n)\right)^k f_X(x_1, ..., x_n) dx_1 ... dx_n \end{aligned} \tag{B.26}$$

The transformation $h_{t,s}^z(x_s)$ when the ABM sought is symmetric is given by the following expressions:

$$h_{t,s}^{LR}(x_s) = \frac{1 + f_{t,s}^{LR}}{K_s^{P,LR}} \tag{B.27}$$

$$h_{t,s}^{SR}(x_s) = \frac{TD_s^{SR} + f_{t,s}^{SR}}{K_s^{V,LR}} \tag{B.28}$$

However, if the ABM is asymmetric, the transformation is given by the following:

$$h_{t,s}^{LR}(x_s) = \text{Min}\left[\frac{1 + f_{t,s}^{LR}}{K_s^{P,LR}}, 1\right] \tag{B.29}$$

$$h_{t,s}^{SR}(x_s) = \text{Min}\left[\frac{TD_s^{SR} + f_{t,s}^{SR}}{K_s^{V,LR}}, 1\right] \tag{B.30}$$

As denoted in the section 2, the formulas were developed for cases in which longitudinal equilibrium, also known as actuarial fairness, is a constraint. This constraint implies that indexation differs between retired cohorts according to the realized mortality and economic outcomes. Nevertheless, in practice, this solution is not politically viable. Therefore, we consider a second case without the actuarial fairness constraint by using current mortality data rather than projected mortality values. Throughout the paper, this is called Case 2. In this case, the expressions representing the outcome P_t^z (B.2), turnover duration TD_t^z (B.9) and liabilities V_t^z (B.5) slightly change as follows:

$$K_t^{P,z} = \frac{K_t^{N,z}}{a_t} + B_{t-1}^z K_{t-1}^{N,z} \frac{a_t - 1}{a_{t-1}} \tag{B.31}$$

$$TD_t^z = 2 + B_{t-1}^z \frac{K_{t-1}^{N,z}}{K_t^{P,z}} \frac{a_t - 1}{a_{t-1}} - \frac{S_0^2}{K_t^C} \tag{B.32}$$

$$K_t^{V,z} = 1 + B_{t-1}^z \left\{ \frac{S_0^1(1 + n_{t-2})}{K_{t-2}^C} + K_{t-1}^{N,z} \frac{a_t - 1}{a_{t-1}} \right\} \tag{B.33}$$

where $K_t^{N,z}$ is the same as in (B.7). Note that the main difference between the Case 1 and Case 2 is that the ratio $\frac{a_{t-1}-1}{a_{t-1}}$ becomes $\frac{a_t-1}{a_{t-1}}$. The calculation of the variances and expected values participants parallels Case 1.