On the birational geometry of singular Fano varieties

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Ewan Johnstone

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Professor Aleksandr Pukhlikov

Department of Mathematical Sciences
University of Liverpool
United Kingdom
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Ewan Johnstone

Abstract

This thesis investigates the birational geometry of a class of higher dimensional Fano varieties of index 1 with quadratic hypersurface singularities. The main investigating question is, what structures of a rationally connected fibre space can these varieties have? Two cases are considered: double covers over a hypersurface of degree two, known as double quadrics and double covers over a hypersurface of degree three, known as double cubics. This thesis extends the study of double quadrics and cubics, first studied in the non-singular case by Iskovskikh and Pukhlikov, by showing that these varieties have the property of birational superrigidity, under certain conditions on the singularities of the branch divisor. This implies, amongst other things, that these varieties admit no non-trivial structures of a rationally connected fibre space and are thus non-rational. Additionally, the group of birational automorphisms coincides with the group of regular automorphisms. This is shown using the "Method of maximal singularities" of Iskovskikh and Manin, expanded upon by Pukhlikov and others, in conjunction with the connectedness principal of Shokurov and Kollár. These results are then used to give a lower bound on the codimension of the set of all double quadrics (and double cubics) which are either not factorial or not birationally superrigid, in the style of the joint work of Pukhlikov and Eckl on Fano hypersurfaces. Such a result has applications to the study of varieties which admit a fibration into double quadrics or cubics.

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Introduction

The field of birational geometry is one of the major current areas of research in algebraic geometry. Its primary goal is to classify and distinguish all algebraic varieties up to birational equivalence. This is done by studying properties of algebraic varieties that are invariant under birational maps; such properties are called birational invariants. In this thesis we study Fano varieties, which in the context of birational classification are members of the wider class of Fano-Mori fibre spaces, and show that a large family of them possess the property of birational rigidity. Birationally rigid Fano varieties have the property of not being birationally equivalent to any non-trivial Fano-Mori fibre space and so form a very special class in the birational classification of Fano varieties. In particular, they are birationally distinct from the class of varieties known as rational varieties, which are birationally equivalent to projective space.

The class of varieties that we will study in this thesis are Fano double hypersurfaces, which are varieties V with a morphism $\sigma : V \overset{2:1}{\to} Q_m \subset \mathbb{P}^{M+1}$ where Q_m is a smooth hypersurface of dimension $M \geq 3$ and degree m. The morphism σ is ramified with branch divisor $W \subset Q$, which is cut out on Q_m by a hypersurface W^* of degree $M - m - 1$. We consider the cases when m is equal to 2 and 3, which are called double quadrics and double cubics respectively. We extend the paper of Pukhlikov ([Puk00a]) to consider the case of double quadrics and double cubics (smooth double quadrics were considered earlier in [Puk89]) where the variety V has quadratic singularities coming from singularities on the branch divisor $W \subset Q_m$.

The original results of this thesis are the following:

Theorem 3.1.2. *Let $\sigma : V \to Q_2 \subset \mathbb{P}$ be a double quadric ramified over $W = W^* \cap Q_2$. Assume that $M \geq 6$ and W has at most quadratic singularities of rank at least 6. Then V is factorial and birationally superrigid.*

Theorem 3.1.3. *Let $\sigma : V \to Q_3 \subset \mathbb{P}$ be a double cubic ramified over $W = W^* \cap Q_3$. Assume that $M \geq 8$ and W has at most quadratic singularities of rank at least 8 and V satisfies the condition (*) (which we will introduce in Chapter 3). Then V is factorial and birationally superrigid.*

Let $\mathcal{I}_m = \mathcal{F}_m \times \mathcal{G}$ denote the parameter space for double hypersurfaces of L-degree $2m$. Let $S_m \subset \mathcal{I}_m$ be the set of pairs (Q_m, W^*) such that the corresponding double hypersurface V is a factorial variety with at most terminal singularities which is also birationally superrigid. The following holds:

Theorem 3.1.4. *The complement $\overline{\mathcal{I}_k \setminus S_k}$ has codimension at least $\binom{M-4}{2} + 1$ for $M \geq 6$ when $k = 2$ and at least $\binom{M-6}{2} + 1$ for $M \geq 8$ when $k = 3$.*

This estimate of the codimension follows in the style of the paper [EP14] which proved a similar result for Fano hypersurfaces.

To put this thesis into context we will now give a brief overview of the history of the rationality problem as well as the Lüroth problem followed by a survey of the literature on birational rigidity. After this, we will outline the structure of the thesis.

The rationality problem. One of the foundational questions of birational geometry is the following: given a projective algebraic variety V of dimension n over a field K, with function field $K(V)$, when is $K(V)$ isomorphic to $K(u_1, \ldots, u_n)$, the purely transcendental extension of K by n elements? Equivalently, is there a birational map from V to projective n-space $\mathbb{P}^n$? We call varieties with this property *rational*, and the question of determining if a given variety is rational or not is called the *rationality problem*.

To solve the rationality problem, mathematicians began looking for necessary and sufficient criteria for a given algebraic variety to be rational in the form of birational invariants.

We start with the simplest case when X is a smooth, projective curve and to this we associate the discrete birational invariant $g = g(C) \geq 0$, called the *genus* of C. It is known that C is rational (birationally equivalent to $\mathbb{P}^1$) if and only if $g(C) = 0$. An example of rational curves are conics in $\mathbb{P}^2$, and an example of non-rational curves are elliptic curves (smooth cubic plane curves, which have $g(C) = 1$). Thus the problem is fully solved for curves.

Moving on to the case of dimension 2 (algebraic surfaces), the problem was again fully solved (over $\mathbb{C}$) by the Italian mathematician Guido Castelnuovo. A smooth algebraic surface is rational if and only if the birational invariants $q(S) = h^0(S, \Omega_S)$ and $P_2(S) = h^0(S, K_S^{\otimes 2})$ are zero. An example of a rational surface is a smooth cubic surface in $\mathbb{P}^3$, whereas a surface of degree at least 5 in $\mathbb{P}^3$ is non-rational.

With these cases solved, attention turned to the case of dimension 3. Here, the approach used for curves and surfaces proved to be inadequate. This shortcoming is related to the following problem.

The Lüroth problem. We say an algebraic variety V of dimension n is *unirational* if there exists a rational dominant map $\mathbb{P}^n \dashrightarrow V$. Unirationality is equivalent to having a rational parameterisation, which were originally sought out to answer certain questions in number theory. For example, rational parameterisations can be used to obtain a formula for all integer solutions to the equation $x^2 + y^2 = z^2$, known as Pythagorian triples. Rational parameterisations are also useful for solving certain kinds of differential equations. Rational varieties are unirational by definition, and the Lüroth problem asks the following: does unirationality imply rationality? This problem has a positive answer for curves over any field, and surfaces over fields of characteristic zero. Thus in these cases being rational and having a rational parameterisation are equivalent. The simple reason for this is that unirational varieties have no non-zero differential forms (as any non-zero differential form could then be pulled back to $\mathbb{P}^n$, which has no regular differential forms) and so the above birational invariants are all zero, which implies rationality in the case of curves and surfaces. It turns out however that the absence of differential forms is only a necessary condition for rationality and in general is not sufficient.

In the 1970's, three separate approaches were used to produce examples of non-rational, unirational threefolds over $\mathbb{C}$: in Artin-Mumford [AM72] a double cover of $\mathbb{P}^3$ branched over a quartic was considered. Here non-rationality was proven by looking at the torsion group $T_2 \subset H^3(V, \mathbb{Z})$, which is a birational invariant and trivial if V is rational. This method has applications in the recent study of the stable Lüroth problem (see [Pir16] for a survey of the problem so far).

Clemens and Griffiths [CG72] looked at smooth cubic hypersurfaces in $\mathbb{P}^4$. Here, the argument was based around the indecomposibility of the intermediate Jacobian. So far this argument only works in dimension 3, however higher dimensional equivalents of the intermediate Jacobian have been proposed, for example the Griffiths component of the derived category [Kuz15].

In Iskovskikh and Manin's paper ([IM71]) the quartic hypersurface V_4 in $\mathbb{P}^4$ was considered. The argument used in this paper was based on an argument going back to the work of Max Noether on birational maps of the projective plane, which was later elaborated on by Gino Fano who use it to study $\mathrm{Bir}(V_4)$. In this paper, Iskovskikh and Manin showed that every birational automorphism of V_4 was in fact a regular automorphism and so $\mathrm{Bir}\, V_4 = \mathrm{Aut}\, V_4$. In modern terminology, they in fact proved that V_4 was birationally rigid (in fact, that it satisfied the stronger property of birational superrigidity). Since the latter group was known to be finite by [MM64] this was enough to prove non-rationality. The origins of birational rigidity come from this paper.

Literature review. After the seminal paper [IM71], the method of maximal singularities was then used to prove birational rigidity for numerous other non-singular Fano varieties. A general compete intersection of a quadric and a cubic in $\mathbb{P}^5$ (one of the examples first studied by Fano) was shown to be birationally rigid in [IP96]. General Fano hypersurfaces were studied in [Puk98a]. Generic complete intersections of index one were studied in [Puk01] and [Puk14]. Other examples include: double spaces and double quadrics [Puk89], general double hypersurfaces [Puk00a], iterated double covers [Puk03] and triple covers of $\mathbb{P}^n$ [Che04]. All of these cases can be realised as Fano complete intersections in weighted projective space, the $\mathbb{Q}$-Fano 3-fold hypersurface case was considered in generality in [CPR00].

Moving on to the singular case, quartic 3-folds with double point singularities were first studied in [Puk88] and in more generality in [Shr08] and [Mel04]. Quartics with more general terminal singularities were considered in [CM04]. Singular double spaces were studied in [Mul10] and [Che08].

Extending the work of [CPR00], quasi-smooth $\mathbb{Q}$-Fano weighted 3-fold hypersurfaces were studied in [CP06]. Fano weighted complete intersections of codimension 2 were considered in [IP96], [CM04] and [Gri11]. The remaining families (as listed in [IF00]) were comprehensively studied in [Oka14a], [Oka15] and [Oka14b]. General Fano hypersurfaces with quadratic singularities were studied in [EP14]. An example of birationally rigid varieties which are not weighted complete intersections are Pfaffian Fano 3-folds which were studied in [AO15].

The method of maximal singularities has been extended to also study Fano fibre spaces. One classical example that became tractable via the method were Del Pezzo fibrations. Fibrations into Del Pezzo surfaces of degree 1 and 2 were studied in [Puk98b] excluding some exceptional cases which were completed later in [Gri00], [Gri03] and [Gri04]. Varieties fibered over $\mathbb{P}^1$ into Fano hypersurfaces were studied in [Puk00b], fibrations into double hypersurfaces in [Puk04] and fibrations into Fano complete intersections in [Puk06].

Structure of the thesis. Chapter 1 introduces the definitions and theorems of algebraic geometry that will be used in the thesis. The definitions and objects used will mostly be classical in nature, following the texts [Hul03] and [Sha13]. This chapter serves to introduce the reader to all the necessary concepts for understanding the thesis and only a basic familiarity in algebra, commutative algebra and field theory is assumed.

In Chapter 2 we introduce the concept of birational (super)rigidity, maximal singularities and the Noether-Fano inequalities. We talk in detail about the method of maximal singularities and the various techniques that are used in its applications.

Chapter 3 states and proves the original result of the thesis. Section 1 serves as the introduction, introducing the variety to be studied and stating the theorems to be proven. The structure of the rest of the chapter is as follows: In section 2, Theorem 3.1.2 is proven via the method of maximal singularities outlined in Chapter 2. After this, Theorem 3.1.4 is proven using dimension counting arguments and Theorem 3.1.2 and Theorem 3.1.3, which are proven in Section 3. Section 3 is dedicated to proving Theorem 3.1.3, which uses the full extent of the methods introduced in Chapter 2. This proof is split into two parts; first, maximal singularities with center not contained in the singular locus are considered, with the second (and longer) part dedicated to the exclusion of maximal singularities with center contained in the singular locus.

Chapter 1

Background

In this chapter we will define the concepts and notation which will be used in this thesis. Since a full treatment of the subject would take several books, we will state only what is needed to understand the main chapter of the thesis. This chapter covers only a fraction of the material that would compose an introductory course on algebraic geometry. A good elementary introduction is [Hul03] , a more detailed but still classical in nature the reader is referred to [Sha13]. For a introduction to the modern theory involving schemes and cohomology the standard textbook is [Har77]. In Section 1, we briefly review the objects of projective algebraic geometry. In Section 2, we define rational and regular maps between algebraic varieties. In Section 3, we talk about the theory of singular and non-singular points. In Section 4, we recall the definitions of divisors and linear systems on a projective variety. In Section 5, we will explain what a Fano variety is and give examples. In Section 6, we conclude with a brief overview of intersection theory.

1 Varieties in projective space

We begin with the basic definitions.

Definition 1.1. $\mathbb{A}_k^n$ is the set of n-tuples with coordinates in the field k, known as affine n-space. Throughout, we will set $k = \mathbb{C}$, and write $\mathbb{A}^n$ for affine n-space over $\mathbb{C}$.

Definition 1.2. A subset $X \subset \mathbb{A}^n$ is *Zariski closed* if there exists a finite number of polynomials f_i in n variables such that

$$X = V(f_1, \dots, f_r) \coloneqq \{x \in \mathbb{A}^n \mid f_i(x) = 0, i = 1, \dots, r\}.$$

A subset $Y \subset X$ is a *Zariski closed subset* of X if there exists a Zariski closed set Z such that $Y = X \cap Z$.

Example 1.3. (i) Consider $\mathbb{A}^3$ with coordinates x, y, z. The set $V(x^2+y^2-z^2)$ defines the set of points $(x, y, z) \in \mathbb{A}^3$ such that $x^2 + y^2 = z^2$.
(ii) $V(x, y)$ in $\mathbb{A}^2$ defines the intersection of the x and y axis i.e. $o = (0, 0)$.

There is a mirror definition to this called the *ideal of an algebraic set* which we will refer to a handful of times. Let $X \subset \mathbb{A}^n$ be an algebraic subset. We define an ideal of the ring $\mathbb{C}[x_1, \dots, x_n]$ by

$$I(X) \coloneqq \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f(x) = 0, \forall x \in X\}.$$

Just as polynomials can be broken down into irreducible components, varieties can be decomposed into irreducible components.

Definition 1.4. We say a closed algebraic set X is *reducible* if $X = X_1 \cup X_2$ where X_1, X_2 are two distinct closed algebraic sets not equal to X. If X is not reducible then we say it is *irreducible*.

Zariski closed sets have the property that every X can be written as the union of a finite number of irreducible varieties X_i (Chapter 1, 3.1 Theorem 1, [Sha13]). If this list is irredundant, that is $X_i \neq X_j$ for $i \neq j$, then it is also unique (See Chapter 1, 3.1 Theorem 2, [Sha13]).

Example 1.5. The set $X = V(xy)$ in $\mathbb{A}^2$ is the union of the x-axis and y-axis. So $V(xy) = V(x) \cup V(y)$ is the unique factorisation of X into irreducible sets.

We now define an *affine variety* to be an irreducible Zariski closed set of $\mathbb{A}^n$. The majority of algebraic geometry is done over an extension of affine space. To define it, we first introduce an equivalence relation on $\mathbb{A}^n$. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two elements of affine space. We write $x \sim y$ if there exists a $\lambda \in \mathbb{C}^*$ such that $x_i = \lambda y_i$ for $i = 1, \ldots, n$. We now proceed to the definition.

Definition 1.6. We define projective n-space $\mathbb{P}^n$ as the set of equivalence classes of $\mathbb{A}^{n+1} \setminus \{0\}$, where $\{0\}$ denotes the origin $(0, \ldots, 0)$, under the relation $\sim$ defined above. For a point $(x_0, \ldots, x_n) \in \mathbb{A}^{n+1} \setminus \{0\}$ we denote its equivalence class in $\mathbb{P}^n$ by $(x_0 : \ldots : x_n)$. Any point $(x_0, \ldots, x_n)$ in $\mathbb{A}^{n+1}$ that is mapped to $x \in \mathbb{P}^n$ by the quotient map is called a set of homogeneous coordinates for x.

We can generalise the equivalence relation above to produce a space called weighted projective space.

Definition 1.7. We define weighted projective space $\mathbb{P}[a_0, \ldots, a_n]$ as the set of equivalence classes of $\mathbb{A}^{n+1} \setminus \{0\}$ under the following relation: Let $x = (x_0, \ldots, x_n)$ and $y = (y_0, \ldots, y_n)$, then $x \sim y$ if there exists a $\lambda \in \mathbb{C}^*$ such that $x_i = \lambda^{a_i} y_i$ for $i = 0, \ldots, n$. We say that $\mathbb{P}[a_0, \ldots, a_n]$ is *well-formed* if $gcd(a_0, \ldots, \hat{a}_i, \ldots, a_n) = 1$ for all $0 \leq i \leq n$. Throughout we will assume that each weighted projective space is well formed.

Example 1.8. Weighted projective space is a generalisation of projective space in the following sense, $\mathbb{P}[1^{n+1}] = \mathbb{P}[1, \ldots, 1] = \mathbb{P}^n$.

We can now define the fundamental object of study: an algebraic variety in projective space.

Definition 1.9. A *Zariski closed* subset X of $\mathbb{P}^n$ is a set of the form $X = V(f_1, \ldots, f_m) := \{x \in \mathbb{P}^n \mid f_i(x) = 0\}$, where $f_1, \ldots, f_m$ are homogeneous polynomials in the variables $x_0, \dots, x_n$. A *projective variety* is an irreducible, Zariski closed subset of $\mathbb{P}^n$. A subset $Y \subset X$ is a projective subvariety of X if Y is also a projective variety.

Example 1.10. Consider $\mathbb{P}^2$ with coordinates X, Y, Z. We define the projective curve $X = V(ZY^2 - X^3)$. This variety is irreducible, so it is a projective variety.

We note that affine and projective spaces are related in the following sense: for any coordinate x_i of $\mathbb{P}^n$, we consider the Zariski open subset

$$U_i := \{(x_1 : \dots : x_n) \in \mathbb{P}^n \mid x_i \neq 0\}.$$

This can be put in a 1-1 correspondence with $\mathbb{A}^n$ via the map $(x_1, \dots, x_n) \to (\frac{x_1}{x_i}, \dots, \hat{1}, \dots, \frac{x_n}{x_i})$. These affine subsets U_i are called the affine coordinate charts, and $\frac{x_j}{x_i} = z_j$ are the affine coordinates. These define an open cover of $\mathbb{P}^n$. For a projective variety $X \subset \mathbb{P}^n$, $X \cap U_i$ are called the affine pieces of X. We note that $X = \bigcup (X \cap U_i)$ so the affine pieces form an open cover of X.

For the reverse direction, given an affine variety $U \subset \mathbb{A}^n$ we have the projective closure $\overline{U} \subset \mathbb{P}^n$, defined as the intersection of all projective varieties containing U. Hence, $U = \overline{U} \cap \mathbb{A}^n$.

Example 1.11. (i) Consider the projective variety in Example 1.10. We consider its affine piece along the affine chart $U_0 = \{X \neq 0\}$. We define inhomogeneous coordinates $x_1 = \frac{Y}{X}$ and $x_2 = \frac{Z}{X}$, then $X \cap U_0$ is the affine variety in $\mathbb{A}^2$ defined by the equation $x_2 x_1^2 - 1 = 0$.
(ii) Consider the affine variety $V(x^2 + y^2 - 1)$ in $\mathbb{A}^2$. Its projective completion is the projective variety in $\mathbb{P}^2$ with homogeneous coordinates X, Y, Z given by the equation $X^2 + Y^2 = Z^2$. (iii) $\mathbb{P}[1,1,2]$ is the projective completion of the affine cone $xy = z^2$ in $\mathbb{A}^3$.

Example 1.12. Consider the weighted projective space $\mathbb{P}[1,1,a]$ with weighted coordinates x, y, z of weight $1, 1$ and a respectively (where $a \in \mathbb{Z}^+$). Then the equation $z^2 = f_{2a}(x, y)$ is homogeneous with respect to the weights and hence defines a weighted projective variety.

Remark 1.13. Weighted projective spaces are projective (Proposition 1.3.3. (i) of [Dol82]), hence weighted projective varieties are not a "larger" class of varieties but instead a different way of writing them. With this in mind, we now define a generalised notion which includes both projective and affine varieties.

Definition 1.14. A *quasi-projective* variety is an open subset of a projective variety. That is, $X = Y \cap U$, where Y is a projective variety and U is a Zariski open subset. A *subvariety* of a quasi-projective variety is any subset $Y \subset X$ which is also quasi-projective. The subvariety Y is said to be a *closed subvariety* if $Y = X \cap Z$ with Z a projective variety.

Projective varieties are trivially quasi-projective. Affine varieties are quasi-projective, as they are open subsets in their projective closure. While there do exist varieties that are quasi-projective but are neither projective nor affine, we will work exclusively with projective varieties. This notion is just to unify affine and projective varieties for ease of notation.

2 Rational maps on quasi-projective varieties

Now that we have our objects, we now introduce the kind of maps between these objects.

Definition 2.1. Let $X \subset \mathbb{P}^n$ be a quasi-projective variety. We define $\mathbb{C}(X)$ to be the *function field* of X, the field of fractions $\frac{F(x)}{G(x)}$, where F and G are homogeneous polynomials and G is not identically zero on X. Two functions $\frac{F(x)}{G(x)}$ and $\frac{F_1(x)}{G_1(x)}$ are equivalent if $FG_1 - F_1G$ is identically zero on X.

Example 2.2. The function field $\mathbb{C}(\mathbb{A}^n)$ is isomorphic to $\mathbb{C}(x_1, \dots, x_n)$, the field extension of $\mathbb{C}$ with n transcendental elements.

The elements of the function field of X are - strictly speaking - only partial functions on X. To get true functions we need rational forms which are defined at every point of X.

Definition 2.3. Let $\phi \in \mathbb{C}(X)$. We say ϕ is regular at x if there exists homogeneous polynomials F, G of the same degree such that $G(x) \neq 0$ and

$$\phi(x) = \frac{F(x)}{G(x)}$$

The set of points for which ϕ is regular is denoted by $\mathrm{dom}(\phi) \subset X$. It is a non-empty Zariski open subset of X. We say ϕ is a *regular function* on X if $\mathrm{dom}(\phi) = X$.

Proposition 2.4. *(i) If X is affine, then the ring of all regular functions on X is equal to $\mathbb{C}[X] := \mathbb{C}[x_1, \ldots, x_n]/I(X)$.*
(ii) If X is a projective variety then the only regular functions on X are the constant functions.

For the proof (i) see Chapter 1, 3.2. Theorem 4. [Sha13], for the proof of (ii) see Theorem 2.35 of [Hul03].

We now define maps between quasi-projective varieties.

Definition 2.5. Let $X \subset \mathbb{P}^n$ be a quasi-projective variety. A *rational map* $\phi : X \dashrightarrow \mathbb{P}^m$ is a map of the form

$$\phi(x) = (F_0(x) : \ldots : F_m(x))$$

where F_i are homogeneous polynomials of the same degree such that at least one of F_i does not vanish everywhere on X. Two maps $(F_0(x) : \ldots : F_m(x))$ and $(G_0(x) : \ldots : G_m(x))$ are equivalent if $F_i G_j = F_j G_i$ for all i, j.

We say a rational map $\phi : X \dashrightarrow Y$ is *regular* at a point $x \in X$ if the rational functions can be chosen such that F_i are all regular at x and $F_i(x) \neq 0$ for at least one i. The set of points for which a map is regular is called the domain of definition as is denoted by $\mathrm{dom}(\phi)$. It is an open subset of X. The image of a rational map is defined as $\phi(\mathrm{dom}(\phi)) \subset Y$.

Let $Y \subset \mathbb{P}^m$ be another quasi-projective variety. We say $\phi : X \dashrightarrow Y$ if in addition $\phi(x) \in Y$ for all $x \in \mathrm{dom}(\phi)$.

Example 2.6. The map from $\mathbb{P}^n$ to $\mathbb{P}^{n-1}$ given by

$$(x_0 : \ldots : x_n) \to (x_1 : \ldots : x_n)$$

is a rational map. It is defined at every point except for $p = (1 : 0 : \ldots : 0)$ (we note that $(0 : \ldots : 0)$ is not a "valid point" of $\mathbb{P}^{n-1}$). This is an example of a map called a linear projection. Geometrically, this map is obtained by sending a point $x \in \mathbb{P}^n$ to the unique intersection of the line L_{px} with the linear subspace $V(x_0 = 0) \cong \mathbb{P}^{n-1}$.

Definition 2.7. A rational map $\phi : X \dashrightarrow Y$ that is regular on the whole of X is called a *morphism* or a *regular map*. A regular map from X to itself with a regular inverse is called a regular automorphism, the set of which we denote by $\mathrm{Aut}(X)$. The set of birational maps from X to itself is denote by $\mathrm{Bir}(X)$, and is sometimes called the *Cemona group*.

Example 2.8. (i) The rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined as

$$\phi(x_0 : x_1 : x_2) = (x_0x_1 : x_0x_2 : x_1x_2)$$

is regular everywhere outside of the set $\{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$. One can also show that $\phi = \phi^{-1}$ and thus $\phi \in \mathrm{Bir}(X)$. (ii) Consider the map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$, given in homogeneous coordinates by

$$(x_0 : x_1 : x_2 : x_3) \dashrightarrow (x_1 : x_2 : x_3).$$

This is regular everywhere except for the point $p = (1 : 0 : 0 : 0)$. This map is regular on the subvariety $X \subset \mathbb{P}^3$ defined by the equation $x_0^2 = x_1x_2 + x_2x_3$, since p is not contained in X.

From this, we can now define our first notion of equivalence between varieties.

Definition 2.9. Two (quasi)-projective varieties X, Y are said to be *isomorphic* if there exists morphisms $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.

Example 2.10. Consider the curve in $\mathbb{A}^2$ defined by $V(xy - 1)$. This is isomorphic to the set $\mathbb{A}^1 \setminus \{0\}$, via the map $(x,y) \to (x)$. The inverse is given by $(x) \to (x, \frac{1}{x})$.

Remark 2.11. From the above example, we see that being a closed affine variety depends on the choice of embedding into affine space. The set $\mathbb{A}^1 \setminus \{0\}$ is not a closed set of $\mathbb{A}^1$ since any polynomial which vanishes on it must vanish on the whole of $\mathbb{A}^1$. However, it is isomorphic to a closed and irreducible affine variety. We say then that a quasi-projective variety is *affine* if it is isomorphic to an affine variety as defined in Section 1.

We now define an equivalence relation which forms the basis of birational geometry.

Definition 2.12. Two quasi-projective varieties X and Y are birationally equivalent if there exist rational maps $\phi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow X$ such that $\phi \circ \psi = id_Y$ and $\psi \circ \phi = id_X$. Equivalently, X and Y are birationally equivalent if there exist open subsets $U \subset X$ and $V \subset Y$ such that U and V are isomorphic.

Example 2.13. (i) The map from $\mathbb{P}^n$ to $\mathbb{A}^n$ given by

$$(x_0 : \ldots : x_n) \to (\frac{x_1}{x_0} : \ldots : \frac{x_n}{x_0})$$

is regular on U_0. The map $U_0 \to \mathbb{A}^n$ is an isomorphism with inverse given by $(x_1, \ldots, x_n) \to (1 : x_1 : \ldots x_n)$. Hence, $\mathbb{P}^n$ and $\mathbb{A}^n$ are birationally equivalent.
(ii) The variety X defined by $x^2 + y^2 = z^2$ in $\mathbb{P}^2$ is birationally equivalent to the projective line $\mathbb{P}^1$ via the following map. Take a point $p \in X$, and a projective line L not passing through p. For any point on X aside from p, there is a unique line L_{px}. We define $\pi(x)$ to be the unique point of intersection with L and L_{px}. This can be shown to be a birational map. This map gives a rational parameterisation of the circle which can be used to generate all Pythagorian triples (integer points of X).

Let $\phi : X \dashrightarrow Y$ be a rational map such that the image of ϕ is dense in Y, we call such rational maps *dominant*. Then there is a corresponding homomorphism of fields $\phi^* : \mathbb{C}(Y) \to \mathbb{C}(X)$. Conversely every homomorphism between function fields corresponds to a rational dominant map, leading to the following proposition

Proposition 2.14. *X and Y are birationally equivalent if and only if $\mathbb{C}(X) \cong \mathbb{C}(Y)$.*

We say that the correspondance between rational maps and $\mathbb{C}$-linear homomorphisms is functorial. In particular this gives us that $\mathrm{Bir}(X) = \mathrm{Aut}(\mathbb{C}(X))$. In this thesis, we investigate varieties which are birationally equivalent to projective space. These varieties are known as *rational varieties*.

We now finish the section by introducing a map of great importance in birational geometry.

Consider $\mathbb{P}^n$ with homogeneous coordinates $x_0, \dots, x_n$, and $\mathbb{P}^{n-1}$ with homogeneous coordinates $y_1, \dots, y_n$. Consider the variety

$$\mathbb{P}^n \times \mathbb{P}^{n-1} := ((x_0 : \dots x_n), (y_1 : \dots : y_n))$$

Consider the subvariety $\Pi \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ defined by the equations

$$x_i y_j = x_j y_i \text{ for } i, j = 1, \dots, n.$$

The morphism $\sigma : \Pi \to \mathbb{P}^n$ obtained by restricting the projection map $\mathbb{P}^n \times \mathbb{P}^{n-1} \to \mathbb{P}^n$ onto Π is called the *blow up* of $\mathbb{P}^n$ at the point $p = (1 : \dots : 0)$. $E = \sigma^{-1}(p)$ is called the exceptional divisor of the blow up. This map is an isomorphism between $\Pi \setminus E$ and $\mathbb{P}^n \setminus \{p\}$. Hence, Π is birationally equivalent to $\mathbb{P}^n$.

Now, let $X \subset \mathbb{P}^n$ be a projective variety and x a point on X. By a change of coordinates such that x is taken to the origin we define the blow up of $\mathbb{P}^n$ at the point x in the same way as before. The blow up $\tilde{X}$ of X at x is the Zariski closure of the set $\sigma^{-1}(X \setminus \{x\})$. The map $\sigma_X : \tilde{X} \to X$ is obtained by restricting the map σ to $\tilde{X}$. We note that it can be shown that $\tilde{X}$ does not depend on the embedding of X into $\mathbb{P}^n$.

We generalise this in the following way. Let $Z \subset \mathbb{A}^n$ be a subvariety. The ideal $I(Z)$ is generated by some functions $f_1, \ldots, f_m$. We define the blowup of $\mathbb{A}^n$ along Z to be the subvariety Bl_Z of $\mathbb{A}^n \times \mathbb{P}^{m-1}$ with coordinates $((x_1, \ldots, x_n), (t_1 : \ldots : t_m))$ defined by the equations

$$f_i(x)t_j = f_j(x)t_i$$

for $i = 1, \ldots, m$.

The map $\sigma : \mathrm{Bl}_Z \to \mathbb{A}^n$ is obtained by restricting the first projection map. Similarly to the case of a point, the blowup of a variety X along a non-singular subvariety Z is defined as the Zariski closure of $\sigma^{-1}(X \setminus Z)$. In the projective case, we first pass to an affine chart and then apply the above definition.

Example 2.15. Consider $\mathbb{A}^2$ with coordinates x, y. The blow up of $\mathbb{A}^2$ at the origin o is the variety

$$Y = \{(x,y)(t_0 : t_1) \mid t_0 y = t_1 x\} \subset \mathbb{A}^2 \times \mathbb{P}^1$$

Now consider the curve $C = V(y^2 - x^3)$ in $\mathbb{A}^2$. We consider the preimage of C on Y via the projection map $\pi : Y \to \mathbb{A}^2$.

$$\pi^{-1}(C) = \{(x,y)(t_0 : t_1) \mid y^2 = x^3, t_0 y = t_1 x\}$$

Considering this preimage on the affine chart $U_0 = \{t_0 = 1\}$, we see that it is defined locally by the equation $x^2(t_1^2 - x) = 0$. This splits into two irreducible components; the exceptional line $E = \pi^{-1}(\{0,0\})$ (given locally by $x = 0$) with multiplicity two, and the curve $t_1^2 = x$, which intersects E at the point $(0,0)(1:0)$ with multiplicity two. The closure of this second curve is denoted by C' and is called the strict transform of C. Hence $\pi^{-1}(C) = E \cup C'$.

3 Singularities of algebraic varieties

We now proceed to an important concept in algebraic geometry: that of singularities. This thesis extends a result to a certain class of singular varieties, hence we first define what it means for a variety to be singular. We first recall the definition of tangent space.

Definition 3.1. Let X be an affine variety. The tangent space at a point $p = (a_1, \dots, a_n) \in X$ is defined as

$$T_pX := \bigcap_{f \in I(X)} V(f_p^{(1)}),$$

where

$$f_p^{(1)} := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - a_i)$$

Example 3.2. Consider the affine variety $y^2 = x^2(x+1)$ in $\mathbb{A}^2$. For a point $p = (a, b)$, the tangent space is defined by the equation $(-3a^2 - 2a)(x - a) + 2b(y - b) = 0$. When $p \neq (0,0)$ this defines a line in $\mathbb{A}^2$. However, when $p = (0,0)$ the equation vanishes everywhere and instead $T_p(X) = \mathbb{A}^2$.

The tangent space is a $\mathbb{C}$-vector space with origin at p. We can use this notion of dimension to define the dimension of an algebraic variety.

Definition 3.3. We define the *dimension* of a variety X as

$$\dim X := \min_{p \in X} \dim T_pX.$$

Remark 3.4. It is known that the dimension of a variety is equal to the transcendence degree of $\mathbb{C}(X)$ (see Theorem 3.18 of [Hul03]). Since the function field is a birational invariant, this shows that dimension is also a birational invariant.

It is often more useful to express the dimension of a variety with respect to a variety containing it. Hence, we introduce the following notion.

Definition 3.5. Let $Y \subset X$ be varieties where Y has dimension m and X has dimension n. The *codimension* of Y in X, denoted by $\operatorname{codim}(Y \subset X)$ or $\operatorname{codim}_X Y$ is equal to $n - m$.

Example 3.6. (i) The variety of Example 3.2 has dimension 1 and thus the codimension is also 1.
(ii) As seen in Example 2.2. the function field of $\mathbb{A}^n$ (and also $\mathbb{P}^n$) is a purely trancendental field extension of $\mathbb{C}$ by n elements. Hence it has dimension n.
(iii) The dimension of a point x is always 0 as the function field at a point is $\mathbb{C}$.

From this definition of dimension it follows that $\dim T_pX \geq \dim X$ for every point. The points at which the inequality is strict are special and so have a distinct name.

Definition 3.7. Let X be an affine variety. We say $x \in X$ is a singular point if $\dim T_xX > \dim X$, else we say that x is non-singular (or smooth). An affine variety is non-singular (or smooth) if every point is non-singular.

Example 3.8. (i) The curve of Example 3.2 is singular at the origin since its tangent space is of dimension 2. (ii) $\mathbb{P}^n$ is non-singular for all $n \geq 1$. Weighted projective space $\mathbb{P}[a_0, \dots, a_n]$ can have singularities. A weighted projective variety which only has singularities arising from the singularities of the ambient space is called *quasi-smooth*.

The set of singular points of X forms a closed subset of X, which we denote by $\operatorname{Sing} X$. Since projective varieties are covered by affine open sets, and our definitions are local, we can extend these definitions to projective varieties. Let X be a projective variety, we say $x \in X$ is a non-singular point if there exists an affine open set (an affine neighbourhood) $U \ni x$ such that x is a non-singular point of U. Non-singular points have a lot of useful properties. Though in this thesis we must work with singular varieties, we can retain these useful properties provided that we can restrict ourselves to a certain kind of singularities which we define below. To define it,we first introduce the algebraic setting for singularities, namely the local ring at a point.

Definition 3.9. The *local ring* of X at a point $x \in X$, denoted by $\mathcal{O}_{x,X}$ is the subring of $\mathbb{C}(X)$ of all elements $\phi \in \mathbb{C}(X)$ which are regular at x. This ring has a maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{x,X}$, consisting of all elements which vanish at x.

Definition 3.10. A quasi-projective variety X is *normal* if every local ring $\mathcal{O}_{x,X}$ is an integrally closed domain.

Non-singular varieties are normal (Chapter 2, 5.1 Theorem 1 of [Sha13]). The converse is not true, however normal varieties have properties which limit the kind of singularities it can have.

Proposition 3.11.(Chapter 2, 5.1. Theorem 3 [Sha13]). *Let X be a normal variety. Then the set of singular points of X has codimension at least* 2

A variety with this property is said to be *non-singular in codimension* 1. The above proposition shows that all normal varieties are non-singular in codimension 1.

We conclude this section by first discussing an important result due to Hironaka [Hir64].

Theorem 3.12. *Let X be a projective variety over $\mathbb{C}$. There exists a non-singular variety Y and a birational morphism $\phi : Y \to X$ which is a composition of blow ups of smooth subvarieties Such a map is called a resolution of singularities.*

We know that for varieties of dimension 1 this non-singular model is unique but many non-isomorphic resolutions exist for higher dimensions.

We conclude by defining the kind of varieties we will be working with.

Definition 3.13. A *hypersurface* X in $\mathbb{P}^n$ is a projective variety defined as the vanishing locus of a single irreducible homogeneous form $f(x_0, \ldots, x_n)$. A weighted hypersurface X_d in $\mathbb{P}[a_0, \ldots, a_n]$ is defined by a single form which is homogeneous with respect to the weights a_i. We say X_d is *well formed* if

$$gcd(a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_n) \mid d$$

for all distinct i, j.

Hypersurfaces always have codimension 1. In general, we expect every equation to increase the codimension of the variety by 1. A variety X in $\mathbb{P}^n$ defined by $n - m$ equations will have dimension at least m. We have a special name for the case when the dimension of this intersection is as small as possible.

Definition 3.14. Let $X \subset \mathbb{P}^n$ be a variety of dimension m. We say X is a *complete intersection* if there exists homogeneous polynomials $f_1, \dots, f_{n-m}$ such that $I(X) = (f_1, \dots, f_{n-m})$. Similarly we can define a weighted complete intersection of degrees $d_1, \dots, d_k$ in $\mathbb{P}[a_0, \dots, a_n]$, and we say that it is *well formed* if for all $\mu = 1, \dots, k$, the highest common factor of any $(n - 1 - c + \mu)$ of the a_is must divide at least μ of the d_is.

Example 3.15. Consider the projective variety in $\mathbb{P}^3$ with homogeneous coordinates X, Y, Z, W, defined by the equations

$$XZ - Y^2, YW - Z^2, XW - YZ.$$

This is an example of a non-complete intersection. It is isomorphic to the curve in $\mathbb{P}^3$ defined parametrically as $\{(s^3 : s^2t : st^2 : t^3) \mid (s, t) \in \mathbb{P}^1)\}$, so it is of dimension 1 and thus of codimension 2. However, it can be shown that the above equations generate the ideal $I(X)$, and that all equations are needed.

Sometimes we can define varieties in terms of a morphism to another variety, as in the following case.

Definition 3.16. A *double cover* is a variety X which admits a morphism $\sigma : X \to Y$ which is two to one outside of a closed subset $W \subset Y$ called the branch locus. If $y \notin W$ then $\sigma^{-1}(y) = \{x, x'\}$ and when $y \in W$ we say y is a *branch point*, and $x = \sigma^{-1}(y) \in X$ is *ramified*.

Example 3.17. Consider a variety X with a morphism $\sigma : X \to \mathbb{P}^n$ which is branched over a smooth hypersurface $W \subset \mathbb{P}^n$ of degree $2m$. X can be realised as a hypersurface in the weighted projective space $\mathbb{P}[1^{n+1}, m]$ by the equation

$$x_{n+1}^2 = F(x_0 : \ldots : x_n)$$

where x_i have weight 1 for $i = 1, \ldots, n$ and x_{n+1} has weight m, F is a form of homogeneous degree $2m$. This is referred to as a double space.

In the next few sections, we will recall the more specialised theory we will make use of in this thesis.

4 Divisors and linear systems

A core technique of the method of maximal singularities (which we will cover in Chapter 2) is the "quadratic" method, which looks at the self-intersection of mobile linear systems. Hence it is necessary to recall the theory of divisors and linear systems, as well as give a very brief definition of the intersection product of divisors (which we will cover in Section 6). Throughout the section we assume that X is a normal and quasi-projective variety.

Definition 4.1. A *Weil divisor* D on X is an element of the free group $\operatorname{Div} X$ generated by all irreducible codimension 1 subvarieties of X i.e.
$D = \sum_i a_i C_i$, where $a_i \in \mathbb{Z}$ and C_i are distinct irreducible codimension 1 subvarieties, where all but a finite number of a_i are zero. An irreducible codimension 1 subvariety C_i with $a_i = 1$ is called a *prime divisor*. We define the support of a divisor to be $\operatorname{Supp} D \coloneqq \bigcup_i C_i$.

Definition 4.2. Let $D = \sum_i a_i C_i$ be a Weil divisor. If $a_i \geq 0$ for all i then D is said to be an *effective* divisor.

If we have two divisors $D = \sum_i a_i C_i$ and $D' = \sum_i b_i C_i$ (here the coefficients a_i, b_i can be zero), then the sum $D + D'$ is equal to the divisor $\sum_i (a_i + b_i) C_i$. This gives $\operatorname{Div} X$ the structure of an additive group where the zero element is $\sum_i a_i C_i$ where $a_i = 0$ for all i.

Example 4.3. We define $H = V(f = 0) \subset \mathbb{P}^n$, where f is a linear form. The divisor associated to this is called the *hyperplane* divisor.

Since we assumed at the beginning of the chapter that our variety is normal, and hence non-singular in codimension one, for every prime divisor C we can find an affine open set U of non-singular points that intersects C such that C is defined in U by a "local equation" π (Chapter 2, 3.1, Theorem 1 of [Sha13]). Then for any regular function f on U, there exists $k \geq 0$ such that $f \in (\pi^k)$ but $f \notin (\pi^{k+1})$. This is the order of zero of f along C, denoted as $\nu_C(f)$.

Since X is irreducible, any function $\phi \in \mathbb{C}(X)$ can be written as $\phi = \frac{f}{g}$ with f, g regular on U. We then define $\nu_C(\phi) = \nu_C(f) - \nu_C(g)$. See Chapter 3, 1.1 of [Sha13] for further details.

Definition 4.4. A *principal Weil divisor* is a divisor of the form

$$(f) \coloneqq \sum \nu_C(f) C$$

where $f \in \mathbb{C}(X)$ and the sum is taken over all prime divisors C.

Two Weil divisors D and D' are said to be *linearly equivalent* if $D - D'$ is principal. We denote this by $D \sim D'$. The group of Weil divisors modulo linear equivalence is denoted by $\operatorname{Cl} X$.

Example 4.5. Any two hyperplane divisors H, H' in $\mathbb{P}^n$ are linearly equivalent. In fact every divisor on $\mathbb{P}^n$ is linearly equivalent to a multiple of a hyperplane by the following argument. If D is an effective divisor, then it is defined by a homogeneous form f of degree d. The equation x_0^d defines the divisor dH, and so the rational function $\frac{f}{x_0^d}$ defines the divisor $D - dH$, so it is principal and therefore $D \sim dH$. For a general divisor D we need only observe that we can write it in the form $D' - D''$, where D' and D'' are effective divisors.

There is also another, more local, notion of divisors which we will make use of.

Definition 4.6. A *Cartier divisor* on X is defined as a collection (U_i, f_i), where U_i is an open cover of X, $f_i \in \mathbb{C}(X)$ and $\frac{f_i}{f_j}$ is regular on $U_i \cap U_j$ for all i, j. A collection (V_j, g_j) defines the same Cartier divisor if the functions f_i, g_j are equal on $U_i \cap V_j$. A *principal Cartier divisor* is a Cartier divisor of the form (f, U_i) for some $f \in \mathbb{C}(X)$.

The addition of Cartier divisors (U_i, f_i) and (V_j, g_j) is the Cartier divisor $(U_i \cap V_j, f_i g_j)$. Two Cartier divisors are said to be linearly equivalent if their difference is principal. We denote the group of Cartier divisors modulo linear equivalence by $\operatorname{Pic} X$.

Remark 4.7. Typically $\operatorname{Pic} X$ denotes the group of isomorphism classes of invertible line bundles on X. However in the case of algebraic varieties over $\mathbb{C}$ this group is isomorphic to the group of Cartier divisors, so we use the two notions interchangably.

Definition 4.8 Let $\psi : Y \to X$ be a morphism, and $D = (U_i, f_i)$ a Cartier divisor on X. We define $\psi^*(D)$ as the Cartier divisor $(\psi^{-1}(U_i), \psi^* f_i)$ on Y, where $\psi^* f_i = f_i \circ \psi$.

Let $\phi : Y \dashrightarrow X$ be a rational map, and D is as above. Let $[D]$ be the linear equivalence class of D. We define $\phi^*([D])$ to be the class of the Cartier divisor on Y associated to the divisor $(\phi^{-1}(U_i), \phi^* f_i)$ where $\phi^* f_i = f_i \circ \phi$.

In the above definition, for rational maps we consider the linear equivalence class since even though the pullback is not defined if the image of Y under ϕ is contained in the support of D, we can always pick a divisor linearly equivalent to D such that this does not occur (Chapter III, 1.3, Theorem 1 [Sha13]). In the case where the map is regular the pullback is always well defined. The main example of pulling back divisors is the pullback of a divisor onto the blow up. The study of the behaviour of this will be used in Chapter 2 when we look at the graph of a maximal singularity. To this end, it is necessary to introduce the following concept.

Definition 4.9. Let $x \in X$ be a point and let D be a prime divisor defined locally near x by f. We define the *multiplicity* of x on D to be

$$\operatorname{mult}_x D = \max\{k \in \mathbb{N} \mid f \in \mathfrak{m}_x^k\}.$$

Example 4.10. Consider the divisor $C = (y^2 - x^3 = 0)$ on $\mathbb{P}^2$. The maximal ideal $\mathfrak{m}_x$ at the point $x = (0,0)$ is the ideal (x, y). Setting $f = y^2 - x^3$ we see that $f \in \mathfrak{m}_x^2$ but $f \notin \mathfrak{m}_x^3$. Hence, $\operatorname{mult}_x C = 2$. Note that x is a cuspal singularity of C.

Example 4.11. Let X be a projective variety. Consider the blow up $\pi : \tilde{X} \to X$ along a non-singular point o. Let E be the exceptional divisor. We have the equality $\operatorname{Pic}(\tilde{X}) = \operatorname{Pic}(X) \oplus \mathbb{Z}E$. For a divisor D, $\pi^*(D) = D' + mE$, where $m = \operatorname{mult}_o D$. D' is called the strict transform of D on $\tilde{X}$.

In this thesis we will work with varieties which, while singular, have nice properties that ensure that most of our definitions and theorems still hold.

Definition 4.12. A normal variety X is *locally factorial* if every local ring is a unique factorisation domain. X is *factorial* if $\operatorname{Cl} X \cong \operatorname{Pic} X$.

One example of locally factorial varieties are non-singular ones, by the following theorem.

Proposition 4.13.(Chapter 2, 3.1. Theorem 2 [Sha13]). *Let X be a quasi-projective variety, $x \in X$ a non-singular point. Then $\mathcal{O}_{x,X}$ is a unique factorisation domain.*

Proposition 4.14.(Chapter 2 , Proposition 6.11 [Har77]) *If X is locally factorial, then it is factorial.*

We make use of a criterion for a variety to be factorial using the following theorem, originally due to Grothendieck [CL94].

Proposition 4.15. *Let X be a normal variety such that at every singular point x, X can be defined locally (in an open neighbourhood of x) as a complete intersection, and the codimension of the singular locus is at least* 4*. Then X is factorial.*

Example 4.16. (i) Consider the hypersurface $\sum_{i=1}^{5} x_i^2 = 0$ in $\mathbb{A}^5$. This is of

dimension 4 and it is singular at the origin. By Proposition 4.15. this variety is factorial.
(ii) The surface $x^2 + y^2 + z^2 = 0$ in $\mathbb{A}^3$. This is singular at the origin and everywhere else non-singular. This is a normal variety (Proposition 2, Chapter II, 7 of [Mum99]), but it fails to be factorial. Indeed, $x^2 + y^2$ is an element of the local ring at the origin that can be factorised in two distinct ways: $(x + iy)(x - iy)$ and as $(z) \cdot (-z)$.

We now introduce a divisor which will be of prime importance later.

Definition 4.17. Let X be a smooth variety of dimension n. Then the *canonical divisor* K_X is the unique (up to linear equivalence) Weil divisor associated to a non-zero rational differential n-form on X.

If X is a normal variety then by Proposition 3.11 its singular locus Z has codimension at least 2. Let $U = X \setminus Z$ denote the non-singular locus of X then by Chapter 2, Proposition 6.5 of [Har77] we have an isomorphism between $\mathrm{Cl}\, X$ and $\mathrm{Cl}\, U$, so we can define K_X by associating it to the canonical divisor defined on U.

Example 4.18. The canonical divisor of $\mathbb{P}^n$ is $-(n + 1)H$.

In Section 5 we will introduce some theorems which will allow us to calculate the canonical divisor for many examples.

Definition 4.19. A Weil divisor $D = \sum_i D_i$ on a smooth variety X of dimension n is a *simple normal crossing divisor* if every component D_i is smooth and for every point $p \in X$ D is defined locally around p by the equation $x_1 \cdot \ldots \cdot x_r$ for $r \leq n$, where $x_i \in \mathcal{O}_{p,X}$. Let $f : Y \to X$ be a resolution of singularities as defined in Theorem 3.12, we say f is a log resolution if the exceptional divisors of f on Y form a simple normal crossing divisor.

Definition 4.20. Let X be a normal, factorial projective variety. Let $\pi : Y \to X$ be a log resolution. Then

$$K_Y = \pi^* K_X + \sum_E a_E E$$

where the sum runs over all divisors E on Y such that $\dim \pi(E) < \dim E$ and $\pi(E) \subset \operatorname{Sing} X$. The number a_E is defined to be the *discrepancy* of E on Y, sometimes written as $a(E, Y)$. This number is independent of the choice of resolution. We say that X has *terminal* singularities if $a_E > 0$ for all π-exceptional divisors E. We say X has *canonical* singularities if $a_E \geq 0$ and *log-canonical* singularities if $a_E \geq -1$.

Example 4.21. (i) Let X be a projective variety. Let $\sigma_E : Y \to X$ be the blow up of X along a non-singular subvariety $B \subset X$ such that X is non-singular along B. Then Y has one exceptional divisor $E_B = \sigma_B^{-1}(B)$ which is a $\mathbb{P}^{r-1}$-bundle, where $r = \operatorname{codim} B$. Moreover by a standard calculation (see Exercise 8.5 in Chapter 2 of [Har77]) we obtain

$$K_Y = \sigma_B^* K_X + (\operatorname{codim} B - 1) E_B.$$

So $a(E_B, Y) = \operatorname{codim} B - 1$.
(ii) Consider X and B as above except now we assume that X can have quadratic singularities along B. The calculation in this case is mostly the same except now the exceptional divisor E is a fibration into quadrics, so we obtain

$$K_Y = \sigma_B^* K_X + (\operatorname{codim} B - 2) E_B.$$

and so $a(E_B, Y) = \operatorname{codim} B - 2$.

Throughout the thesis, we will consider not just divisors but whole families of divisors. Hence we introduce the following notation.

Definition 4.22. Let D be a divisor on X. $|D|$ is the set of all effective divisors linearly equivalent to D, and is called the complete linear system of D. This can be given the structure of a projective space. A projective subspace Σ of $|D|$ is called a *linear system* of X.

Definition 4.23. Consider a linear system Σ. We define the base locus as $\operatorname{Bs}(\Sigma) := \bigcap \operatorname{Supp} D$, where the intersection runs over all effective divisors in Σ. We say Σ is *free* (or base point free) if the base locus is empty. We say Σ

is *mobile* if $\mathrm{Bs}(\Sigma)$ does not contain any components of codimension 1.

Example 4.24. (i) Consider the complete linear system $|H|$ on $\mathbb{P}^3$ where H is any hyperplane section. This is a free linear system, since for any hyperplane H and any point $x \in H$, we can find a hyperplane H' linearly equivalent to H such that $x \notin H'$. The linear system Σ of hyperplanes containing a line L is contained in $|H|$. The base locus of Σ is exactly the line L, and so Σ is mobile but not free.
(ii) Consider a linear system Σ on a smooth projective variety X. Let x be a base point of the linear system and let $\pi : \tilde{X} \to X$ denote the blow up of X centered at this point. For any divisor $D \in \Sigma$ we know that the pullback $\pi^*(D) = D' + mE$, where $m > 0$ is the multiplicity of the general divisor in Σ at x. The linear system of divisors on $\tilde{X}$ which are pull backs of divisors of Σ has $E = \pi^{-1}\{x\}$ as a common component, hence it is not mobile. The divisors of the form $D' = \pi^*(D) - mE$ determines a mobile linear system of the same dimension, which is defined to be the strict transform of Σ.

5 Fano varieties

The varieties studied in this thesis are examples of Fano varieties, hence we recall the definition in this section. A Fano variety is defined as a variety such that the anticanonical divisor is ample,hence we first recall what it means for a divisor to be ample and then give various examples of Fano varieties and recall some of their properties.

Definition 5.1. We say a divisor D on X is *very ample* if there exists a morphism $f : X \to \mathbb{P}^N$ such that $X \cong f(X)$ and D is the hyperplane section of $f(X)$. That is to say, $D = f^*(H)$, where $H \subset \mathbb{P}^N$ is the divisor associated to a hyperplane. We say D is *ample* if mD is very ample for some $m \in \mathbb{N}$.

Definition 5.2. A projective, normal variety with at most terminal singularities is called $\mathbb{Q}$-*Fano* (or just *Fano*) if the anticanonical divisor $-K_X$ is ample and if some multiple of $-K_V$ is Cartier (we say in this case that $-K_V$ is $\mathbb{Q}$-Cartier). X is said to be *primitive Fano* if in addition the equality

$\operatorname{Pic} X = \mathbb{Z}K_X$ holds.

Before we give examples of Fano varieties we first introduce some theorems which will allow us to calculate the canonical divisor. The following proposition allows us to calculate the canonical divisor in many cases. It is commonly referred to as the *adjunction formula*.

Proposition 5.3. *Let D be a smooth divisor on a smooth variety X. Then the equality*

$$K_D = (K_X + D)|_D$$

holds.

Remark 5.4. The adjunction formula above can be applied to the case when D is an irreducible divisor which as a variety is factorial. Here K_D is defined as the Cartier divisor obtained from $K_X + D$ restricted onto the non-singular locus of D.

Example 5.5. (i) Let $V_d \subset \mathbb{P}^{n+1}$ be a smooth hypersurface defined by a homogeneous equation of degree d. Then by the adjunction formula,

$$K_{V_d} = (d - (n+2))H_d$$

where H_d is the class of $H \cap V_d$.
(ii) Let $V \subset \mathbb{P}^{n+1}$ be a complete intersection defined by $m \leq n$ equations $f_1, \dots, f_m$ of degree $d_1, \dots, d_m$ respectively. Then by repeated use of the adjunction formula we have

$$K_V = (\sum_{i=1}^{m} d_i - (n+2))H_V$$

(iii) Consider a quasi-smooth and well formed complete intersection V in

$\mathbb{P}[a_0, \ldots, a_n]$ with degrees $d_1, \ldots, d_m$. Then

$$K_V = (\sum_{i=1}^{m} d_i - \sum_{i=0}^{n} a_i)L$$

(Theorem 3.3.4 [Dol82]), where L is a very ample divisor on V. The double space of Example 3.17 has canonical class $K_X = -L$, where in this case $L = \sigma^*(H)$.

Example 5.6. (i) $\mathbb{P}^n$ is Fano. The anticanonical divisor of $\mathbb{P}^n$ is $(n+1)H$ which is very ample.
(ii) A complete intersection $V_d \subset \mathbb{P}^{n+1}$, where $d = \sum_i d_i$ is the sum of the degrees of the defining equations, is Fano if and only if $d \leq n+1$.
(iii) Weighted projective space is Fano. A (well formed) weighted complete intersection X in $\mathbb{P}[a_0, \ldots, a_n]$ with degrees $d_1, \ldots d_k$ is Fano if and only if $\sum d_i < \sum_i a_i$ (Chapter 5, 1.3.9, [Kol96]).

We finish the section by stating a result which shows that Fano varieties have the property known as rational connectedness.

Definition 5.7. We say a variety X is rationally connected if for any two points $x, y \in X$ there exists a rational curve $C \subset X$ containing x and y.

Rationally connected varieties form a birational class which includes rational and unirational varieties. The existence or non-existence of rationally connected, non-unirational varieties is one of the major open problems in the field as of writing. We finish by stating the following theorem, which is a special case of a more general theorem from ([Zha06]).

Theorem 5.8. *Let X be a primitive Fano variety which is factorial and has at most terminal singularities, then X is rationally connected.*

6 Algebraic cycles and intersection theory

Many of the proofs of Chapter 3 make use of the area known as intersection theory. For an in-depth treatment, the reader is direct towards [Ful98]. Here, we recall the definitions and theorems needed in this thesis.

Definition 6.1. Let X be a projective variety. A k-cycle of X is an element of the free group $Z_k(X)$ generated by all irreducible subvarieties of X of dimension k i.e. $C = \sum a_i C_i$, where C_i are distinct irreducible subvarieties of dimension k and $a_i \in \mathbb{Z}$. We write $\mathrm{mult}_{C_i} C = a_i$, this is referred to as the *geometric multiplicity* of C_i on C.

We now discuss the intersection product of cycles. We first consider the case when X is a non-singular variety. Let C and D be irreducible subvarieties of X with codimension i and j respectively. We say C and D have proper intersection if $C \cap D$ is a union of subvarieties of codimension $i + j$. We define the product $(C \circ D)$ to be the cycle $\sum_i m_i [Z_i]$, where Z_i are the irreducible components of $C \cap D$, m_i are the intersection multiplicities (Chapter 7 [Ful98]) of the components with respect to the cycle. This product can be extended to cycles with non-proper intersection, however an equivalence relation is needed. The equivalence relation must have the property that for any two cycles C and D, there are equivalent cycles C' and D' which intersect properly. One example of this is rational equivalence (see Chapter 1 of [Ful98] for a definition). We denote $CH_k(X)$ to be the group of k cycles modulo rational equivalence. We then define an intersection product $(C \cdot D)$ on the whole of $CH_k(X)$. This product is linear, that is for cycles C, D, E, we have $(C + D \cdot E) = (C \cdot E) + (D \cdot E)$. It is also commutative, that is, $(C \cdot D) = (D \cdot C)$.

Rational equivalence however is very difficult to work with, rational Chow groups are hard to compute in general. In our applications, we will be using a weaker equivalence called *numerical equivalence.* To define this we first must define the degree of a 0-cycle. Let C be a zero cycle on a projective variety X, that is, $C = \sum_i a_i x_i$, where $x_i \in X$. We can define a map $A_0(X) \to \mathbb{Z}$, by sending C to $\sum_i a_i$. This is referred to as the *degree* map, with the image $(\sum_i a_i)$ of C is the degree of C. We define numerical equivalence as follows.

Definition 6.3. Two k-cycles C and C' are said to be *numerically equivalent* if for any cycle Z of codimension k, we have $\deg(C \cdot Z) = \deg(C' \cdot Z)$.

Definition 6.4. The ring of numerical Chow cycles $A_k(X)$ is the ring of k-cycles on X modulo numerical equivalence with the intersection product. We will also use the notation $A^k(X)$ to denote the ring of cycles of codimension k.

Remark 6.5. For a cycle Z of dimension k on a non-singular variety X, there is a homomorphism sending Z to an element of the cohomology group $H^{2k}(X, \mathbb{Z})$, where X is considered as a n-dimensional complex manifold. From this we say that two k-cycles Z and Z' are *homologically equivalent* if their images in $H^{2k}(X, \mathbb{Z})$ are equal. These (co)homology groups are known when X is a complete intersection in (weighted) projective space (Example 19.3.10 [Ful98]) via use of the Lefschetz hyperplane theorem (Theorem 3.1.17. of [Laz04]). Homological equivalence implies numerical equivalence, so from this we obtain the following result.

Theorem 6.6. *Let X be a non-singular (weighted) complete intersection of dimension at least 3 in (weighted) projective space. Then $A^i(X) \cong \mathbb{Z}$ for $i < \frac{1}{2}\dim X$.*

We obtain from this that $A^1(X) \cong \operatorname{Pic} X \cong \mathbb{Z}H$ when the dimension of X is at least 3, $A^2(X) \cong \mathbb{Z}H^2$ when X is of dimension at least 5, and so on.

Having first defined the product in the simpler non-singular case, we now consider a special case when X is factorial with terminal singularities. Here we define an intersection product between a Cartier divisor and a k-cycle in the following way.Let $C = \sum_i m_i C_i$ be a k-cycle and D a Cartier divisor on X. Suppose that $C_i \notin \operatorname{Supp} D$ for all i. Then restricting D onto C_i defines a Cartier divisor on C_i which we denote by $(C_i \cdot D)$. We can then define $(C \cdot D)$ to be the Cartier divisor $\sum m_i (C_i \cdot D)$. For the general intersection product (such as when $C_i \subset \operatorname{Supp} D$) the reader is referred to [Ful98]

We now proceed with some examples.

Example 6.7. Let D and D' be two curves in $\mathbb{P}^2$ of degree d and d' respectively. Then D is numerically equivalent to dH and D' to $d'H$ and their intersection product is $(D \cdot D') = \sum_i a_i x_i$, where x_i are the points of intersection and a_i are their multiplicities. This is numerically equivalent to the 0-cycle $dd'H^2$ and so $\deg(D \cdot D') = \sum_i a_i = dd'$.

Definition 6.8. Let $X \subset \mathbb{P}^n$ be a projective variety. Let H denote the class of the hyperplane. The *degree* of X is defined to be $\deg(X \circ [H]^k)$, where k is the codimension of X.

Geometrically this can be described as intersecting X with a sufficiently general hyperplane of codimension k and counting the number of points. We generalise this to all cycles as follows: given a cycle C, we write it as $\sum_i m_i C_i$ where C_i are cycles corresponding to irreducible subvarieties. Then the degree is equal to $\deg C = \sum_i m_i \deg C_i$.

We can also define the degree of a variety without referring to a given embedding in projective space, in the following way.

Definition 6.9. Let X be a projective variety of dimension n. Let L be an ample divisor on X. The L-degree of X is defined to be $\deg(L^n)$. For a subvariety $Y \subset X$ of dimension k, we define its L-degree to be the integer $\deg_L Y = \deg(Y \cdot L^k)$.

Example 6.10. Consider a double cover $\sigma : V \to Q_m \subset \mathbb{P}^N$ where Q_m is a smooth hypersurface of degree m. Let $L = \sigma^*(H)$, then the L degree of V is equal to $2m$.

When X is a Fano variety, we can take L to be $-K_X$. In this case, the L-degree is sometimes called the *anticanonical* degree. The following result is a generalisation of Example 6.7.

Proposition 6.11. Theorem 12.3 [Ful98]. *For $V_1, \dots, V_r$ subvarieties of $\mathbb{P}^n$, where $(V_1 \circ \dots \circ V_r) = \sum_i m_i Z_i$ is the scheme theoretic intersection, the inequality*

$$\prod_{i=1}^{r} \deg(V_i) \geq \sum_{i=1}^{r} m_i \deg Z_i$$

holds.

Conclusion

In this chapter we have introduced the objects of study for the remainder of the text. so the reader should now be familiar with what a projective algebraic variety is and what it means for such varieties to be rational. Section 3 defined the theory and properties of singularities on these varieties, which are one of the most important features of the varieties to be studied in Chapter 3. This chapter also outlined the theory of divisors and cycles and their intersections on these varieties, which will be of primary importance in the following chapter. With this done, we now proceed to the more specialised theory of birational rigidity which will form the theoretical basis of the thesis.

Chapter 2

Methods

In this chapter, we give an overview of the concept of birational rigidity and its method of proof known as the "method of maximal singularities". In Section 1, we introduce the definitions of birational rigidity in terms of the threshold of canonical adjunction. After this, we state the main geometric implications of rigidity, where the main application to birational geometry comes from. In Section 2 we go into more details about the "Noether-Fano" inequality, which arises from maximal singularities of a mobile linear system. Section 3 expands upon the theory outlined in Section 2 to prove several important results which we will make use of in Chapter 3. Section 4 introduces the more modern techniques which make use of the language of log pairs, and go over several of its applications to the method of maximal singularities. Section 5 outlines the cone technique, which serves as an invaluable tool in the exclusion of maximal singularities with center of codimension 2. The majority of the material covered in this chapter is quoted from [Puk13], which is the most comprehensive textbook on the subject of birational rigidity to date. For an alternative introduction to the concept of birational rigidity the reader is referred to the survey article [Che05].

1 Birational Rigidity

Definition 1.1. Let X be a projective algebraic variety. Set $A^1_{\mathbb{R}}X = A^1X \otimes \mathbb{R}$. We define A^1_+X to be the closed cone in $A^1_{\mathbb{R}}X$ generated by classes of effective divisors. A divisor D is said to be *pseudo-effective* if $D \in A^1_+X$.

The *threshold of canonical adjunction* of a divisor D on X is defined as

$$c(D, X) = \sup\{\epsilon \in \mathbb{Q}_+ \mid D + \epsilon K_X \in A^1_+X\}$$

And for a mobile linear system Σ we define $c(\Sigma, X) := c(D, \Sigma)$, where $D \in \Sigma$ is a general divisor of Σ.

We note that the value $\frac{1}{c(D,X)}$ is known as the Fujita invariant of the pair (D, X), first introduced in [HTT15].

Example 1.2. (i) Let X be a factorial Fano variety such that PicX $\cong \mathbb{Z}K_X$. For any effective divisor D we have $D \in |-nK_X|$ for some integer $n \geq 1$. Therefore, $c(D, X) = n$.
(ii) If X is a factorial variety satisfying the weaker condition of PicX $\cong \mathbb{Z}H$ and $K_X = -rH$, for $r \geq 2$. Then $D \in |nH|$ for some $n \geq 1$ and then $c(D, X) = \frac{n}{r}$.

The above value is not a birational invariant, as the following example shows.

Example 1.3. Consider the map $\pi : \mathbb{P}^M \dashrightarrow \mathbb{P}^m$ obtained via linear projection from a plane $P \subset \mathbb{P}^M$ of dimension $M - m - 1$. Take a mobile linear system $\Lambda \subset |nH|$ on $\mathbb{P}^m$ and let Σ be the strict transform of Λ on $\mathbb{P}^M$. As per example 1.2.(ii), we calculate the threshold to be $c(\Sigma, \mathbb{P}^M) = \frac{n}{M+1}$. Let $\sigma : \mathbb{P}^+ \to \mathbb{P}^M$ be the blow up along the plane P and Σ^+ the strict transform of Σ on $\mathbb{P}^+$. $\pi \circ \sigma : \mathbb{P}^+ \to \mathbb{P}^m$ is a $\mathbb{P}^{M-m}$ bundle. Now $c(\Sigma^+, \mathbb{P}^+) = 0$, since if $D^+ + \epsilon K_{\mathbb{P}^+}$ was pseudo-effecitve then it would be psuedo-effective when restricted onto any fibre $F_t \cong \mathbb{P}^{M-m}$ of $\pi \circ \sigma$. However, D^+ is pulled back from the base and so is trivial on F_t. In addition, $K_{\mathbb{P}^+}$ is not effective on F_t as it is not numerically effective (that is, $(C \cdot K_{F_t}) < 0$ for all effective curves C on F_t) because F_t is rationally connected.

To overcome birational non-invariance of the threshold, we simply introduce the following definition.

Definition 1.4. Let Σ be a mobile linear system on a variety X. The *virtual threshold of canonical adjunction* is defined by the formula

$$c_{vir}(\Sigma) = \inf\{c(\Sigma^+, X^+)\}$$

where the infimum runs over all birational morphisms $X^+ \to X$ where X^+ is a non-singular variety, Σ^+ is the strict transform of Σ on X^+.

This is by definition a birational invariant, however it is in general difficult to compute. We now introduce the main objects of study in this section.

Definition 1.5. A variety X is *birationally rigid* if for every mobile linear system Σ on X, there exists a birational automorphism $\chi \in \operatorname{Bir} X$ such that $c(\chi_*^{-1}(\Sigma), X) = c_{vir}(\Sigma)$, where $\chi_*^{-1}(\Sigma)$ denotes the strict transform of Σ with respect to χ. If X satisfies the stronger condition of $c(\Sigma, X) = c_{vir}(\Sigma)$ for all mobile linear systems, then X is said to be *birationally superrigid.*

Example 1.6. (i) A general smooth hypersurface $V_n \subset \mathbb{P}^n$ of degree $n \geq 4$ is birationally superrigid. [Puk98a]
(ii) A general complete intersection of a quadric and a cubic $V_{2\cdot 3} \subset \mathbb{P}^5$ is birationally rigid. [IP96]

Suppose a variety X is not birationally superigid, that is there exists a mobile linear system Σ such that $c_{vir}(\Sigma) < c(\Sigma, X)$. Then by the definition of the threshold there exists a birational morphism $\phi : X^* \to X$ such that $c(\phi_*^{-1}(\Sigma), X^*) < c(\Sigma, X)$. This implies that ϕ cannot be an isomorphism in codimension one because Σ and its strict transform would be isomorphic and the thresholds would be equal. Therefore there exists a codimension 1 subvariety $E \subset X^*$ which is contracted by ϕ i.e. a divisor E over X. We recall (see Chapter 1, Section 4) that to a divisor E over X we can associate a discrete valuation on $\mathbb{C}(X)$, which we denote as $\operatorname{ord}_E(f)$. Such a valuation is called a geometric valuation. We can apply this valuation on divisors D

by applying it to its local equations. We define $\mathrm{ord}_E(\Sigma) = \mathrm{ord}_E(D)$ where $D \in \Sigma$ is a generic divisor.

Definition 1.7. Let Σ be a mobile linear system on X and set $n = c(\Sigma, X)$. A divisor E over X with geometric valuation $\mathrm{ord}_E()$ is called a *maximal singularity* of Σ if the inequality

$$\mathrm{ord}_E(\Sigma) > n \cdot a(E, X) \tag{2.1}$$

holds, where $a(E, X)$ is the discrepancy of E with respect to X as defined in Definition 4.20 in Chapter 1.

Remark 1.8. We note that to define the quantities $\mathrm{ord}_E(D)$ and $a(E, V)$ we only need V^+ to be non-singular at a general point of E. If the singularities of V^+ are of codimension at least 2 then this is the case, such as when V^+ is a normal variety (Chapter 1, Proposition 3.11).

The core of the methods lies with the following proposition.

Proposition 1.9. *Suppose X is not birationally superrigid, then there exists a mobile linear system Σ which has a maximal singularity.*

Proof. If X is not birationally superrigid then by definition there exists a birational morphism $\phi : X^+ \to X$ and mobile linear system Σ on X such that $c(\Sigma^+, X^+) < c(\Sigma, X) = n$. We define $\mathcal{E}$ to be the set of prime divisors contracted by ϕ. Let D be a general divisor in Σ and D^+ denotes its strict transform on X^+. By the inequality $c(\Sigma^+, X^+) < c(\Sigma, X)$ it follows that $D^+ + nK_{X^+}$ is not pseudo effective. However we have the identity

$$D^+ + nK_{X^+} = \phi^*(D + nK_X) - \sum_{E \in \mathcal{E}} e(E)E$$

where $e(E) = \mathrm{ord}_E(D) - n \cdot a(E)$. Since $D + nK_X$ is pseudo-effective and the pullback of a pseudo-effective divisor is again psuedo-effective, there must exist at least one divisor E such that $e(E) = \mathrm{ord}_E(D) - n \cdot a(E) > 0$. E is then by definition a maximal singularity of Σ. $\square$

The structure of the method of maximal singularities is as follows: we assume that X is not birationally superrigid, then by Proposition 1.9 there exists a mobile linear system Σ with $n = c(\Sigma, X) > 0$ and a geometric valuation $\mathrm{ord}_E(-)$ such that (2.1) holds. We wish to either prove an inequality which contradicts this, thereby *excluding* the maximal singularity, or construct a birational automorphism $\chi_E \in \operatorname{Bir} X$ such that the inequality

$$c((\chi_E^{-1})_*\Sigma, X) < c(\Sigma, X)$$

holds. Here $(\chi_E^{-1})_*\Sigma$ is the strict transform of Σ with respect to χ_E, and E is no longer a maximal singularity of the system $(\chi_E^{-1})_*\Sigma$. Thus the map χ_E *untwists* the maximal singularity E while also decreasing the threshold of canonical adjunction. The following result is from Chapter 2, Theorem 1.1 of [Puk13]:

Theorem 1.10. Suppose X is a primitive Fano variety. Let $\mathcal{M}$ denote the set of geometric valuations which realise a maximal singularity on X. If for every $E \in \mathcal{M}$ we can associate a map $\chi_E \in \operatorname{Bir}(X)$ which untwists E, Then the following holds:

- The variety X is birationally rigid.
- The group of birational self-maps $\operatorname{Bir} X$ is generated by the subgroup of biregular automorphisms $\operatorname{Aut} X$ and the subgroup $B(X)$ is generated by untwisting automorphisms $\chi_E, E \in \mathcal{M}$.
- If $\mathcal{M} = \varnothing$ then X is birationally superrigid and $\operatorname{Bir} X = \operatorname{Aut} X$.

We now state the main geometric implications of birational (super)rigidity.

Proposition 1.11. *Let V be a rationally connected variety. If on V there are no mobile linear systems Σ such that $c(\Sigma, V) = 0$ then V admits no structures of a non-trivial rationally connected fibre space: there is no rational map $\rho : V \dashrightarrow S$, $\dim S \geq 1$ where S is a rationally connected variety and the generic fibre of ρ is rationally connected.*

Proof. Simply consider the mobile linear system $\Sigma = \rho^*(\Delta)$, where Δ is a mobile linear system on S. Since the general fibre of ρ is rationally connected then $c(\Sigma, V) = 0$ by the same argument used in Example 1.3, but this contradicts our assumption. □

Corollary 1.12. *Let V be a primitive Fano variety. Suppose V is birationally rigid, then it admits no structure of a rationally connected fibre space. In particular, V is non-rational.*

Theorem 1.13.(Chapter 2, Proposition 1.6 [Puk13]) *Let V be a primitive Fano variety and V' a Fano variety which is factorial, has at most terminal singularities and such that* $\operatorname{rk}\operatorname{Pic}(V') = 1$*. Let $\chi : V \dashrightarrow V'$ be a birational map. If V is birationally rigid then V and V' are isomorphic. In addition, if V is birationally superrigid then the map χ is a biregular isomorphism.*

Theorem 1.13 together with Corollary 1.12 implies that when you run the minimal model program on a birationally rigid primitive Fano variety V the output is equal to V, that is to say, V is its own unique structure of a Fano-Mori fibre space. In the literature this is sometimes taken to be the definition of a birationally rigid primitive Fano variety.

Although in this thesis no untwisting maps are needed, for completeness we provide some examples below.

Example 1.14.([Puk88]) We consider a quartic hypersurface $V = V_4 \subset \mathbb{P}^4$ with only one non-degenerate quadratic singularity p, then V is factorial (Theorem 2 of [Che06b]). To this singular point $p \in \operatorname{Sing} V$ we can associate a birational involution $\tau_p \in \operatorname{Bir} V$. Let $\pi_p : V \dashrightarrow \mathbb{P}^3$ be the projection from the point p, then this map is generically 2 to 1: for $y \in \mathbb{P}^3$ and $\pi_p^{-1}(y) = \{x, x'\}, x \neq x'$ for general choice of y. τ_p is the involution sending x to x'.
Now let $L \subset V_4$ be a line containing this unique singular point p, to this also we can associate a birational involution. Let $\pi : V \dashrightarrow \mathbb{P}^2$ be the projection from the line L. Let $\tilde{V}$ be the blow up of V along p and L. This has two exceptional divisors E_p and E_L. π extends to a regular map $\pi : \tilde{V} \to \mathbb{P}^2$. We can show that the projection map fibres V into cubic curves, the general

fibre $C_t, t \in \mathbb{P}^2$ being an elliptic curve. From this we obtain an involution τ_L which acts on the general fibre C_t by reflection from $0 = C_t \cap E_p$ in terms of the usual group law defined on elliptic curves.

We can now state the following: if V_4 is such that there are exactly $4! = 24$ lines on V_4 passing through p, then V_4 is birationally rigid and

$$\operatorname{Bir} V_4 = *_{i=0}^{24} \langle \tau_i \rangle \oplus \operatorname{Aut}(X)$$

which is the free product of 25 cyclic groups of order 2, where τ_0 is the reflection from the double point and $\tau_i, i = 1, \ldots 24$ are the involutions associated to the lines through p, as outlined above.

2 The Noether-Fano inequality

Recall that in the previous section we showed that if a variety X is not birationally superrigid then there exists a mobile linear system Σ on X with $n = c(\Sigma, X)$ and a geometric valuation $\operatorname{ord}_E(-)$ such that

$$\operatorname{ord}_E(\Sigma) > n \cdot a(E, X)$$

We call this the *Noether-Fano* inequality. We prove superrigidity by contradiction: we assume that our variety is not superrigid and thus has a mobile linear system with a maximal singularity. We then argue via various means that such a maximal singularity cannot occur, and thus our variety must be birationally superrigid.

To make use of this inequality we first consider the case when the center of E is not contained in the singular locus of X. We then wish to express this inequality in terms of a sequence of blow ups over non-singular centers.

Consider the general situation of a birational map $\psi : X^+ \dashrightarrow X$ to a projective (possibly singular) variety X which contracts a divisor $E \subset X^+$ to a subvariety

$B = \psi(E) \subset X$ of codimension ≥ 2 and $B \not\subset \operatorname{Sing} X$. Consider the blow up $\sigma_B : X(B) \to X$ with center B, $E(B) = \sigma_B^{-1}(B)$ the exceptional divisor. From the set up we obtain the following simple proposition.

Proposition 2.1. *Two cases hold: either $(\sigma_B^{-1} \circ \psi)(E) = E(B)$, and so the composition map $\sigma_B^{-1} \circ \psi : X^+ \dashrightarrow X(B)$ is an isomorphism in a neighbourhood of the generic point of E, or $B^+ := \sigma_B^{-1} \circ \psi(E)$ is an irreducible subvariety of codimension 2. Morever, $B^+ \not\subset \operatorname{Sing} X(B), B^+ \subset E(B)$ and $\sigma_B(B^+) = B$*

Iterating this proposition to the situation of a maximal singularity E lying over a birational morphism $\phi : X^+ \to X$, we obtain a sequence of blow ups

$$\begin{array}{rccc} \sigma_{i,i-1} : & X_i & \to & X_{i-1} \\ & \cup & & \cup \\ & E_i & \to & B_{i-1} \end{array}$$

where $X_0 = X, B_0 = \phi(E)$ and B_j is the center of E on X_j. $E_i = \sigma_{i,i-1}^{-1}(B_{i-1})$ is the exceptional divisor, B_{i-1} is the center of the blow up $\sigma_{i,i-1}$. Note that X_i can be singular but they are non-singular at a general point of the subvariety B_i. For $i > j$ we set

$$\sigma_{i,j} = \sigma_{j+1,j} \circ \ldots \circ \sigma_{i,i-1} : X_i \to X_j$$

By Proposition 2.1 $\sigma_{i,j}(B_i) = B_j$. For a subvariety Y on some X_j we denote its strict transform on X_i, when it is well defined (i.e. $Y \not\subset B_j$), by Y^i. This sequence of blow ups is called the resolution of the discrete valuation ord_E with respect to X. We will see later that this resolution always terminates in a finite number of steps.

Consider now the set of exceptional divisors $\{E_1, \ldots, E_K\}$ of such a resolution. We introduce the structure of a directed graph on this set in the following way: The verticies E_i and E_j are joined by an oriented edge, which we denote by $i \to j$, if $i > j$ and $B_{i-1} \subset E_j^{i-1}$. We also introduce the following notation, if $i > j$ then $p_{i,j}$ is equal to the number of paths from the vertex i to vertex j. We also set $p_{i,i} = 1$.

Example 2.2.

$$\begin{tikzcd} 4 \arrow[r] & 3 \arrow[r] \arrow[rr, bend left] & 2 \arrow[r] & 1 \end{tikzcd}$$

This graph corresponds to a resolution of size $K = 4$, where the center ($B_2 \subset E_2$) of the blow up $\sigma_{3,2}$ is contained in the strict transform E_1^2. We see that $p_{3,1} = 2$ and $p_{4,1} = 2$

We have the identity

$$p_{i,j} = \sum_{i \to k} p_{k,j}$$

to obtain this, for each path from i to j mark the first vertex of the graph after i i.e. $i \to k \to \ldots \to j$. We can use this notation to express taking pullbacks of exceptional divisors on this sequence of blow ups.

$$E_j^i = \sigma_{i,j}^* E_j - \sum_{j \leftarrow k \leq i} \sigma_{i,k}^* E_k.$$

A simple argument by induction shows the following identity, after noting that $\sigma_{j+1,j}^* E_j = E_j^{j+1} + E_{j+1}$ since $B_j \subset E_j$ and the divisor E_j is non-singular at a general point of B_j.

Proposition 2.3. *The following identity holds*

$$\sigma_{i,j}^* E_j = \sum_{k=j}^{i} p_{k,j} E_k^i.$$

The combinatorial invariants $p_{i,j}$ allow us to give explicit formulas for multiplicities and discrepancies. Let Σ^j denote the strict transform of the linear system Σ on X_j. Set $\nu_j = \operatorname{mult}_{B_{j-1}} \Sigma^{j-1}$ and $\beta_j = \operatorname{codim} B_{j-1} - 1$. Again arguing by induction we obtain the equalities

$$\operatorname{mult}_{E_i}(\Sigma) = \sum_{j=1}^{i} p_{i,j} \nu_j$$

$$a(E_i, V) = \sum_{j=1}^{i} p_{i,j}\beta_j$$

from this we can now show the following proposition.

Proposition 2.4. *The sequence of blow ups always terminates: there exists some* $K \geq 1$ *such that* $\sigma_{K,0}^{-1} \circ \psi(E) = E_K$.

Proof. From the second formula above and the fact that $p_{i,j} \geq 1$ for all $i \geq j$ and $\beta_j \geq 1$ for all j, we see that $a(E_i, X) \geq i$. However we note that $a(E_i, X) \leq a(E, X)$ because the center of E on X_i is contained in E_i. So i is bounded above, and hence the proposition holds. □

So now to any maximal singularity E we can associated a sequence of blow ups and thus a graph with verticies $\{E_1, \ldots E_K\}$, such that $\text{mult}_{E_K}(\Sigma) = \text{mult}_E(\Sigma)$. Hence, we can write the Noether-Fano inequality as follows:

$$\sum_{i=1}^{K} p_i\nu_i > n\sum_{i=1}^{K} p_i\beta_i \tag{2.2}$$

Our ultimate aim is to use this inequality to arrive at a contradiction and thus show that no maximal singularities can occur, proving birational super-rigidity. To this end we can use this resolution to split maximal singularities into two types. We consider a maximal singularity $E \subset V^+$ and look at its resolution. We observe that the centers B_i of the blow ups have non decreasing dimension. Therefore, two cases can occur: either $\dim B_0 = \ldots = \dim B_{K-1}$ or $\dim B_0 < \dim B_{K-1}$. In the latter case we say that E is an *infinitely near maximal singularity* of Σ.

Proposition 2.5. *Suppose* E *is a maximal singularity of a mobile linear system* Σ, $c(\Sigma, X) = n$ *and* $\dim B_0 = \dim B_{K-1}$. *Then for the center of the singularity* $B = B_0$, *the inequality*

$$\text{mult}_B \Sigma > n(\text{codim}\, B - 1)$$

holds.

Note that if B is a subvariety of X such that X is smooth along B, then blowing up along B we see that the exceptional divisor E_1 of this blow up satisfies

$$\operatorname*{mult}_{E_1} \Sigma = \operatorname{mult}_B \Sigma > n(\operatorname{codim} B - 1) = n \cdot a(E_1, X)$$

and therefore E_1 is also a maximal singularity of Σ.

Here, we define $\operatorname{mult}_B \Sigma = \operatorname{mult}_B D$ for a general $D \in \Sigma$. Since X is smooth at the general point of B we can define

$$\operatorname*{mult}_B(D) := \operatorname*{mult}_x(D)$$

for a generic point $x \in B$. We note that $\operatorname{mult}_B \Sigma > 0$ if and only if $B \subset \operatorname{Bs}(\Sigma)$. In the case above we say B is a *maximal subvariety* of Σ.

Proof. Since E is a maximal singularity of Σ, we have the Noether-Fano inequality

$$\sum_{i=1}^{K} p_i \nu_i > n \sum_{i=1}^{K} p_i \beta_i$$

Since by assumption $\dim B_0 = \dim B_{K-1}$ then $\beta_i = \operatorname{codim} B - 1$ for all i. We also observe that $\nu_1 \geq \ldots\ldots \nu_K$. Substituting these inequalities into the above formula we obtain

$$\nu_1 \geq n(\operatorname{codim} B - 1)$$

which is the precise inequality we need. □

Corollary 2.6. *If the center B of the maximal singularity E is not a maximal subvariety, that is the inequality*

$$\operatorname{mult}_B \Sigma > n(\operatorname{codim} B - 1)$$

does not hold, then E is an infinitely near maximal singularity.

We also note that if the codimension of the center B is 2 then we get equality of the codimensions of the centers and thus B is automatically a maximal subvariety.

3 The technique of counting multiplicities

In this section we examine the graph of the blow up introduced in the previous section to prove a series of important inequalities, starting with the following.

Theorem 3.1. **(The $4n^2$-inequality)** *Let X be a non-singular, projective variety. Let $B \subset X$ be the center of a maximal singularity of Σ. Let $n = c(\Sigma, X)$ and define $Z := (D_1 \circ D_2)$, where $D_1, D_2 \in \Sigma$ are general divisors. Suppose* $\operatorname{codim} B \geq 3$, *then the inequality*

$$\operatorname{mult}_B Z > 4n^2$$

holds, where $\operatorname{mult}_B Z$ *is the intersection multiplicity of B on Z.*

Before we state the proof (which is from Theorem 2.1, Chapter 2 of [Puk13]), we consider first a more general situation. Let $B \subset X, B \notin \operatorname{Sing} X$ be an irreducible cycle of codimension ≥ 2 and $\sigma_B : X(B) \to X$ be the blow up with center B, and $E(B) = \sigma_B^{-1}(B)$ the exceptional divisor. Let $W = \sum m_i W_i, W_i \subset E(B)$ be a k-cycle, $k \geq \dim B$. We define the *degree* of W to be

$$\deg W = \sum_i m_i \deg(W_i \cap \sigma_B^{-1}(b))$$

where $b \in B$ is a generic point, $\sigma_B^{-1}(b) \cong \mathbb{P}^{\operatorname{codim} B - 1}$ and the right-hand side degree is as defined for varieties in projective space in Chapter 1. We note that $\deg W_i = 0$ if and only if $\sigma_B(W_i)$ is a proper closed subset of B.

We now recall an important piece of intersection theory which will be central to the proof. Let D and Q be two distinct Weil divisors on X, D^B and Q^B denote their strict transforms on $X(B)$.

Lemma 3.2. *Assume that* $\operatorname{codim} B \geq 2$. *Then*

$$D^B \circ Q^B = (D \circ Q)^B + Z$$

where Z is a divisor whose support is contained in $E(B)$. Moreover,

$$\operatorname{mult}_B(D \circ Q) = (\operatorname{mult}_B D)(\operatorname{mult}_B Q) + \deg Z$$

In particular if $\operatorname{codim} B = 2$. *Then*

$$D^B \circ Q^B = Z + Z_1$$

where $\operatorname{Supp} Z \subset E(B)$, $\operatorname{Supp} \sigma_B(Z_1)$ *does not contain B. In addition the equality*

$$D \circ Q = [(\operatorname{mult}_B D)(\operatorname{mult}_B Q) + \deg Z]B + (\sigma_B)_* Z_1$$

holds.

See [Ful98] Chapter 12 for more details. With the setting established, we proceed with the proof.

Proof of Theorem 3.1.

Consider a maximal singularity $\operatorname{ord}_E(-)$ of Σ and its corresponding resolution $\sigma_{i,i-1} : X_i \to X_{i-1}$. We divide this resolution into two parts: the lower part with indicies $i = 1, \dots L \leq K$ corresponds to the blow ups where $\operatorname{codim} B_{i-1} \geq 3$, and the upper part with indicies $i = L+1, \dots K$ corresponding to the blow ups where $\operatorname{codim} B_{i-1} = 2$. Note that it is possible that $L = K$ and thus the upper part is empty.

Let $D_1, D_2 \in \Sigma$ be two different general divisors. We define a sequence of codimension 2 cycles, one for each X_i , as follows

$$
\begin{aligned}
&D_1 \circ D_2 = Z_0 = Z \\
&D_1^1 \circ D_2^1 = Z_0^1 + Z_1 \\
&\vdots \\
&D_1^i \circ D_2^i = (D_1^{i-1} \circ D_2^{i-1})^i + Z_i \\
&\vdots
\end{aligned}
$$

where $Z_i \subset E_i$. Thus for any $i \leq L$ we get

$$D_1^i \circ D_2^i = Z_0^i + Z_1^i + \ldots + Z_{i-1}^i + Z_i$$

For any $j > i$ where $j \leq L$ define $m_{i,j} := \mathrm{mult}_{B_{j-1}}(Z_i^{j-1})$ and $d_i := \deg Z_i$.

From the above sequence we get the following system of equalities:

$$
\begin{aligned}
&\nu_1^2 + d_1 = m_{0,1} \\
&\nu_2^2 + d_2 = m_{0,2,} + m_{1,2} \\
&\vdots \\
&\upsilon_i^2 + d_i = m_{0,i} + \ldots + m_{i-1,i} \\
&\vdots \\
&\upsilon_L^2 + d_L = m_{0,L} + \ldots + m_{L-1,L}
\end{aligned}
$$

Now we obtain

$$d_L \geq \sum_{i=L+1}^{K} \nu_i^2 \deg(\sigma_{i-1,L})_* B_{i-1} \geq \sum_{i=L+1}^{K} \nu_i^2$$

and by Proposition 2.2.4. of [Puk13] we have the inequality

$$\sum_{i=1}^{L} p_i m_{0,i} \geq \sum_{i=1}^{L} p_i \nu_i^2 + p_L \sum_{i=L+1}^{K} \nu_i^2$$

Setting $m := m_{0,1} = \mathrm{mult}_B(D_1 \circ D_2)$ and using the fact that $m_{0,i} \leq m_{0,1}$ we get

$$m\left(\sum_{i=1}^{L} p_i\right) \geq \sum_{i=1}^{L} p_i \nu_i^2 + p_L \sum_{i=L+1}^{K} \nu_i^2$$

Using this inequality and the fact that for $i \geq L+1$ we have $p_i \leq p_L$ we obtain the inequality

$$m\left(\sum_{i=1}^{L} p_i\right) \geq \sum_{i=1}^{K} p_i \nu_i^2$$

We consider now the quadratic form $\sum_{i=1}^{K} p_i \nu_i^2$ and see where it is minimised. Recall the explicit Noether-Fano inequality

$$\sum_{i=1}^{K} p_i \nu_i > n \sum_{i=1}^{K} p_i \beta_i$$

using this we see that this quadratic form is minimised at the point

$$\nu_1 = \ldots = \nu_K = \frac{\sum_{i=1}^{K} p_i \beta_i n}{\sum_{i=1}^{K} p_i}$$

Now define $\Sigma_l = \sum_{\beta_j \geq 2} p_j, \Sigma_u = \sum_{\beta_j = 1} p_j$. Then we obtain the inequality

$$\mathrm{mult}_B\, Z > \frac{(2\Sigma_l + \Sigma_u)^2}{\Sigma_l(\Sigma_l + \Sigma_u)} n^2$$

A simple computation shows that the right hand side is strictly greater than $4n^2$, thus concluding the proof of Theorem 3.1.

The above result also holds when X has a certain kind of singularities, which we define below:

Definition 3.3. Let $X \subset Y$ be a subvariety of codimension 1 inside a smooth projective complex variety Y of dimension n. A point $P \in X$ is called a *quadratic point of rank r* if there exist local analytic coordinates $z = (z_1, \dots z_n)$ of Y around P and a quadratic form $q_2(z)$ of rank r such that the germ of X in P can be defined as

$$(P \in X) \cong \{q_2(z) + g(z) = 0\} \subset Y$$

where $g(z)$ is a polynomial with cubic and higher homogeneous components.

One useful property of quadratic singularities which will be of great importance in Chapter 3 is the following:

Lemma 3.4. *Suppose V is a projective variety with at most quadratic singularities of rank $\geq r$, then* $\operatorname{codim}(\operatorname{Sing} V \subset V) \geq r - 1$.

Proof. Let $x \in \operatorname{Sing} V$ and consider the blow up $\sigma : V^+ \to V$ at the point x. V^+ has one exceptional divisor E which is a quadric of rank $r_1 \geq r$ and $\operatorname{codim}(\operatorname{Sing} E \subset E) = r_1 - 1$. Since $\operatorname{Sing} V^+ \cap E \subset \operatorname{Sing} E$ we obtain the bound $\operatorname{codim}(\operatorname{Sing}(V^+) \cap E \subset V^+) \geq r$. Since cutting by a codimension 1 subvariety increases the codimension by at most 1 then $\operatorname{codim}(\operatorname{Sing} V^+ \subset V^+) \geq r-1$. We then end the proof by noting that $\operatorname{codim}(\operatorname{Sing} V^+ \subset V^+) = \operatorname{codim}(\operatorname{Sing} V \subset V)$ (unless, of course if $\operatorname{Sing} V = \{x\}$ but in this case the result is trivial). $\square$

From this Lemma and Chapter 1, Proposition 4.15 it follows that if a variety has at most quadratic singularities (which are by definition locally hypersurface singularities) of rank at least 5 then V is factorial. This fact will be important for the results in Chapter 3.

The following extension of the $4n^2$-inequality was first shown in [EP14].

Theorem 3.5. *Let X be a projective variety such that every point is either smooth or a quadratic singularity of rank $r \geq 5$. Let Σ be a mobile linear system and E a maximal singularity of Σ, B is the center of the maximal* *singularity E. Let $Z = (D_1 \circ D_2)$ be the self intersection of Σ. Suppose* $\operatorname{codim} B \geq 3$*, then the inequality*

$$\operatorname{mult}_B Z > 4n^2$$

holds.

Remark 3.6. The condition of rank at least 5 is necessary for this result for the following reason: If we consider a 3-fold X with a non-degenerate quadratic point p, if this quadratic point is the center of a maximal singularity E of a mobile linear system Σ with threshold $n = c(\Sigma, X)$ then the Noether-Fano inequality states that

$$\operatorname{mult}_p \Sigma > n$$

therefore for the self intersection Z of Σ, using the technique of counting multiplicities we can at most obtain the bound

$$\operatorname{mult}_p Z > 2n^2.$$

Before we prove this theorem, we must first revisit the situation of Proposition 2.1 and consider the case where the center of E is contained in the singular locus $\operatorname{Sing} X$. Once again we can construct the resolution of the maximal singularity E and the disrepancies are still bounded so the resolution still terminates. However, in this case the centers B_i can be contained in the singular locus of V_i but, the singularities are controlled in the following sense.

Lemma 3.7. *Let $X \subset Y$ be a subvariety of codimension 1 in a smooth projective complex variety Y of dimension n with at most quadratic singularities of rank $\geq r$. Let $B \subset X$ be an irreducible subvariety. Then there exists an open subset $U \subset Y$ such that*

(i) $B \cap U$ is smooth

(ii) the blow up $\tilde{X}_U$ of $X \cap U$ along $B \cap U$ has at most quadratic singularities

of rank $\geq r$.

See Theorem 4 of [EP14] for a proof. We now construct the resolution of the maximal singularity E

$$\begin{array}{cccc} \sigma_{i,i-1}: & X_i & \to & X_{i-1} \\ & \cup & & \cup \\ & E_i & \to & B_{i-1} \end{array}$$

for $i = 1, \ldots, K$. By Lemma 3.7 for $i = 0, \ldots, K-1$ there is a Zariski open subset $U_i \subset X_i$ such that $U_i \cap B_i \neq \varnothing$ is smooth and either V_i is smooth along $B_i \cap V_i$ or every point $p \in B_i \cap V_i$ is a quadratic singularity of rank at least 5. In particular, the quasi-projective varieties $\sigma_{i,i-1}^{-1}(U_{i-1}), i = 1, \ldots, K$ are factorial and the exceptional divisor $E_i^* := E_i \cap \sigma_{i,i-1}^{-1}(U_{i-1})$ is irreducible and either a projective bundle over $U_{i-1} \cap B_{i-1}$ or a fibration over $U_{i-1} \cap B_{i-1}$ into quadrics of rank at least 5. We may assume that $U_i \subset E_i \cap \sigma_{i,i-1}^{-1}(U_{i-1})$ for $i = 1, \ldots K-1$.

We separate this resolution into a lower part $(i = 1, \ldots, L)$ and an upper half $(i = L+1, \ldots, K)$, where $\operatorname{codim} B_{i-1} \geq 3$ if and only if $i \leq L$. It may occur that $L = K$ and thus the upper half is empty. We define

$$L_* := \max\{i = 1, \ldots, K \mid \operatorname{mult}_{B_{i-1}} X_{i-1} = 2\}$$

Here $\operatorname{mult}_{B_{i-1}} X_{i-1} = 2$ means that B_{i-1} is contained in the singular locus of X_{i-1}. From the definition we see that $L_* \leq L$. We also define

$$\delta_i = \begin{cases} \operatorname{codim} B_{i-1} - 2 \, \text{for} \, 1 \leq i \leq L_* \\ \operatorname{codim} B_{i-1} - 1 \, \text{for} \, L_* + 1 \leq i \leq K \end{cases}$$

Note that δ_i are the discrepancies of exceptional divisors when blowing up smooth and quadratic points as in Example 4.21(i) and (ii) in Chapter 1.

Similarly we define

$$\nu_i = \begin{cases} \frac{1}{2}\,\mathrm{mult}_{B_{i-1}}\,\Sigma^{i-1} \text{ for } 1 \leq i \leq L_* \\ \mathrm{mult}_{B_{i-1}}\,\Sigma^{i-1} \text{ for } L_* + 1 \leq i \leq K \end{cases}$$

Let $D \in \Sigma$ be a generic divisor. Then

$$D^i\big|_{U_i} = \sigma^*_{i,i-1}(\,D^{i-1}\big|_{U_{i-1}}) - \nu_i E^*_i.$$

Now we can write the Noether-Fano inequality as follows:

$$\sum_{i=1}^{K} p_i \nu_i > n\left(\sum_{i=1}^{K} p_i \delta_i\right)$$

We now sketch out a proof of Theorem 3.5 since the proof is very similar to that of Theorem 3.1. We refer the reader to [EP14] for the full details.

We may assume that $\nu_1 < \sqrt{2}n$, else otherwise $\mathrm{mult}_B\,Z \geq 2\nu_1^2 > 4n^2$ and we are done. We also have the inequality $\nu_1 > n$ (a fact which we prove in Proposition 4.5). The multiplicities ν_i also satisfy the following inequalities:

$$\nu_1 \geq \ldots \geq \nu_{L_*} \tag{2.3}$$

and, providing $K \geq L_* + 1$

$$2\nu_{L_*} \geq \ldots \geq \nu_K \tag{2.4}$$

Recall that $Z = (D_1 \circ D_2)$ is the self intersection of Σ, we write $m_i = \mathrm{mult}_{B_{i-1}}\,Z^{i-1}$ for $i = 1, \ldots, L$. Now using the same techniques as used in the proof of Theorem 3.1 we obtain the estimate

$$\sum_{i=1}^{L} p_i m_i \geq 2\sum_{i=1}^{L_*} p_i \nu^2 + \sum_{i=L_*+1}^{K} p_i \nu^2$$

Denote the right hand side by $q(\nu_*)$. We see that

$$\sum_{i=1}^{L} p_i m_i \geq \mu$$

where μ is the minimum of the positive definite quadratic form $q(\nu_*)$ on the compact, convex polytope Δ defined on the hyperplane

$$\Pi := \left\{ \sum_{i=1}^{K} p_i \nu_i = n \left(\sum_{i=1}^{K} p_i \delta_i \right) \right\}$$

by the inequalities (2.3) and (2.4). We optimise in a similar way as in the proof of Theorem 3.1 and so we end the sketch proof here.

We end this section by stating a version of the $4n^2$-inequality for surfaces which we will make use of in the next section and in Chapter 3.

Let $o \in X$ be a point on a smooth, projective surface, C a smooth curve containing o and Σ a mobile linear system on X. Let $Z = (D_1 \circ D_2)$ be the self-intersection of the linear system Σ, which in this case is an effective 0-cycle. We assume that $\operatorname{Supp} Z = \{o\}$ and so $\deg Z = \operatorname{mult}_o Z$.

Proposition 3.8.(Theorem 3.1, [Cor00]) *If there exists a real (possibly negative) number $a < 1$ such that the pair*

$$\left(X, \frac{1}{n}\Sigma + aC \right)$$

is not log-canonical, where $n > 0$. Then the inequality

$$\deg Z > 4(1-a)n^2$$

holds.

A further extension of the $4n^2$-inequality for general complete intersection singularities was recently show in [Puk16].

4 Inversion of adjunction

The next technique comes from the following interpretation of the Noether-Fano inequality in terms of singularities of log pairs, an approach first seen in [Cor00].

Definition 4.1. A *log pair* (D, X) is a variety X and D a formal sum of prime divisors with rational coefficients (i.e. a $\mathbb{Q}$-divisor) such that $K_X + D$ is $\mathbb{Q}$-Cartier. Consider a log pair (X, D) where D is effective. We say that (X, D) is *canonical* if for any geometeric valuation ν_E the inequality $\nu_E(D) \leq a(E, X)$ holds and *log-canonical* if $\nu_E(D) \leq a(E, X) + 1$.

Definition 4.2. Let (D, X) be a log pair. A *log resolution* of the pair (D, X) is a birational morphism $f : Y \to X$ such that $f^{-1}(D) \cup \bigcup_i E_i$ is a divisor with global normal crossings, where E_i are the exceptional divisors of f.

Now let Σ be a linear system on X. An equivalent way of saying that Σ has a maximal singularity is to say that for a general divisor $D \in \Sigma$ the pair $(X, \frac{1}{n}D)$ is non-canonical i.e. there is a geometric valuation ν_E such that $\nu_E(\Sigma) > na(E, X)$.

The application of this approach lies in the usage of the following theorem (Theorem 7.4 ,[Kol97]).

Theorem 4.3. (The Connectedness theorem) *Let X be a normal variety and $D = \sum d_i D$ an effective $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $g : Y \to X$ be a log resolution of the pair (X, D). $K_Y = g^*(K_X + D) + \sum e_i E_i$. Define $F = -\sum_{e_i \leq -1} e_i E_i$, then* $\operatorname{Supp} F$ *is connected in a neighbourhood of every fibre of g.*

The main utility of this theorem is in proving a result commonly referred to as inversion of adjunction. We state a version of this theorem which is stated and proven in Chapter 4, Section 3, Theorem 3.4 of [Puk13].

Theorem 4.4. (Inversion of adjunction) *Let $x \in X$ be a germ of a $\mathbb{Q}$-factorial variety with terminal singularities, D an effective $\mathbb{Q}$ divisor the support of which contains x. Let $R \subset X$ be an irreducible codimension* 1 *subvariety which is Cartier and not contained in the support of D. Assume that the point x is an isolated centre of a non-canonical singularity of the pair (X, D). Then, the pair $(R, D_R = D\,|_R)$ is not log canonical at the point x.*

This allows us to restrict our attention onto irreducible subvarieties of X, which in many cases can vastly simplify our analysis.

We will use Theorem 4.4 directly several times in Chapter 3, but another application is proving the following useful inequality. Suppose $x \in X$ is the germ of an isolated hypersurface singularity, and $\dim X \geq 4$. More precisely, if $\phi : X^+ \to X$ is the blow up of X along x, $E \subset X^+$ is the exceptional divisor, X^+ and E are smooth with E isomorphic to a smooth hypersurface of degree $\mu = \operatorname{mult}_x X \leq \dim(X) - 1$ in $\mathbb{P}^M$. Furthermore, for any prime divisor $D \ni x$, D^+ its strict transform is of the form $D^+ \sim -\nu E$ for $\nu \in \mathbb{Z}_+$, so that $\operatorname{mult}_x D = \mu\nu$.

Proposition 4.5. *Assume the pair $(X, \frac{1}{n}D)$ is not canonical at x, but canonical away from x. Then, the inequality*

$$\nu > n$$

holds.

Proof. Assume the converse: $\nu \leq n$. Then the pair $(X^+, \frac{1}{n}D^+)$ is non-canonical and the center of any non-canonical singularity of this pair (that is, of any maximal singularity of D^+) is contained in E. By Theorem 4.4 the pair $(E, \frac{1}{n}D^+_E)$, where $D^+_E = D^+|_E$ is not log-canonical. Let $H_E = -\,E|_E$ be the generator of the Picard group $\operatorname{Pic} E$, which is the hyperplane section of E with respect to its embedding in $\mathbb{P}^M$. Then

$$D^+_E \sim -\nu\; E|_E = \nu H_E.$$

Since $\nu \leq n$, the pair $(E, \frac{1}{n}D^+_E)$ cannot be log-canonical by Proposition 7.2.6

of ([Puk13]).

□

Another application of Theorem 4.4 is proving the following theorem which was first introduced in ([Che00]). We first set up the problem. Let $o \in X$ be a germ of a smooth variety where $\dim X \geq 4$. Let Σ be a mobile linear system on X and $Z = (D_1 \circ D_2)$,where $D_1, D_2 \in \Sigma$ are generic divisors, its self intersection. Let $\phi : X^+ \to X$ denote the blow up of X at the point x, $E = \phi^{-1}(o) \cong \mathbb{P}^{\dim X - 1}$ the exceptional divisor. We denote the strict transform of Z and Σ on X^+ by Z^+ and Σ^+ respectively.

Theorem 4.6. (the $8n^2$ inequality). *Assume the pair*

$$\left(X, \frac{1}{n}\Sigma\right)$$

is not canonical, but canonical outside the point o, where $n > 0$. Then, there exists a linear subspace $\Phi \subset E$ of codimension 2 (with respect to E), such that the inequality

$$\operatorname{mult}_o Z + \operatorname{mult}_\Phi Z^+ > 8n^2$$

holds.

Parts of the proof of the result in Chapter 3 follow closely the proof of this theorem, so we will give a partial proof here.

Proof. We first can restrict the system Σ onto a generic smooth subvariety of dimension 4 containing o, hence we can assume that $\dim X = 4$. Moreover, we can assume that $\nu = \operatorname{mult}_o \Sigma \leq 2\sqrt{2}n < 3n$, since otherwise $\operatorname{mult}_o Z \geq \nu^2 > 8n^2$ and the claim already holds.

Lemma 4.7. *The pair*

$$(X^+, \frac{1}{n}\Sigma^+ + \frac{(\nu - 2n)}{n}E) \tag{2.5}$$

is not log-canonical, and the center of any of its non-log canonical singularities is contained in E*.*

Proof of Lemma 4.7. Let $\lambda : \tilde{X} \to X$ be a resolution of singularities of the pair $(X, \frac{1}{n}\Sigma)$ and $E^* \subset \tilde{X}$ a prime exceptional divisor, realising a non-canonical singularity of the pair. Then $\lambda(E^*) = o$ and the Noether-Fano inequality

$$\mathrm{ord}_{E^*}(\Sigma) > na(E^*)$$

holds. For a generic divisor $D \in \Sigma$ we get $\phi^* D = D^+ + \nu E$, so that

$$\mathrm{ord}_{E^*}(\Sigma) = \mathrm{ord}_{E^*}(\Sigma^+) + \nu \cdot \mathrm{ord}_{E^*}(E)$$

and

$$a(E^*, X) = a(E^*, X^+) + 3\,\mathrm{ord}_{E^*}(E).$$

Using this we get

$$\begin{aligned} \mathrm{ord}_{E^*}\left(\frac{1}{n}\Sigma^+ + \frac{(\nu - 2n)}{n}E\right) &= \mathrm{ord}_{E^*}\left(\frac{1}{n}\Sigma\right) = 2\,\mathrm{ord}_{E^*}(E) \\ &> a(E^*, X^+) + \mathrm{ord}_{E^*}(E) \geq a(E^*, X^+) + 1 \end{aligned}$$

which is precisely the condition for being non-canonical. QED.

Let $R \ni o$ be a generic 3-dimensional subvariety and $R^+ \subset X^+$ its strict transform on the blow up of the point o. For a small $\epsilon > 0$ the pair

$$\left(X^+, \frac{1}{1+\epsilon}\frac{1}{n}\Sigma^+ + \frac{(\nu - 2n)}{n}E + R^+\right)$$

still satisfies the connectedness principal (Theorem 4.3) with respect to the morphism $\phi : X^+ \to X$, so that the set of centers of non-log canonical singularities of this pair is connected. Since R^+ is itself a non-log canonical singularity, we conclude that the pair (2.5) has a non-log canonical singularity, the center of which on X^+ has positive dimension, since it intersects R^+. Let $Y \subset E$ be a center of a non-log canonical singularity of (2.5) with maximal dimension. We have two cases: Either $\dim Y = 2$ or $\dim Y = 1$.

We start with the case $\dim Y = 2$. Let S be a generic linear subvariety of X^+ of dimension 2 which intersects Y transversally at a point of general position.

The pair (2.5) restricted onto S is not log-canonical at this point so applying Proposition 3.8 we see that

$$\mathrm{mult}_Y(D_1^+ \circ D_2^+) > 4(3 - \frac{\nu}{n})n^2$$

so that

$$\mathrm{mult}_o Z \geq \nu^2 + \mathrm{mult}_Y(D_1^+ \circ D_2^+)\deg Y$$

$$> (\nu - 2n)^2 + 8n^2,$$

which proves the claim.

We now consider the case $\dim Y = 1$. Then we consider the pair

$$\left(R^+, \frac{1}{1+\epsilon}\frac{1}{n}\Sigma_R^+ + \frac{\nu - 2n}{n}E_R\right) \tag{2.6}$$

where $\Sigma_R^+ = \Sigma^+|_{R^+}$ and $E_R = E|_{R^+}$. Since this pair satisfies the conditions of the connectedness principal and R^+ intersects Y at $\deg Y$ distinct points, we conclude that $Y \subset E$ is a line in $\mathbb{P}^3$.

We now distinguish between two cases: when $\nu \geq 2n$ and when $\nu < 2n$. The latter case is historically the most troublesome case, and we will only cite its proof. We will now consider the case when $\nu \geq 2n$.

Let us choose a generic three dimensional subvariety $R \ni o$ such that $R^+ \supset Y$. Since by our assumption (2.5) is an effective cycle, we can apply Theorem 4.3 and conclude that (2.6) is not log-canonical at Y.

We apply Proposition 3.8 to the pair (2.6) in a similar manner as above, giving us the inequality

$$\operatorname{mult}_Y(D_1^+|_{R^+} \circ D_1^+|_{R^+}) > 4\left(3 - \frac{\nu}{n}\right)n^2.$$

On the left in brackets we have the self-intersection of the mobile linear system Σ_R^+, which separates into two components:

$$(D_1^+|_{R^+} \circ D_1^+|_{R^+}) = Z_R^+ + Z_R^{(1)}$$

where Z_R^+ is the strict transform of the cycle $Z_R = Z|_R$ on R^+ and the support of the cycle $Z_R^{(1)}$ is contained in E_R. The line Y is a component of the effective 1-cycle $Z_R^{(1)}$.

On the other hand, the self-intersection of the mobile linear system Σ^+ we get

$$(D_1^+ \circ D_2^+) = Z^+ + Z_1$$

where the support of the cycle Z_1 is contained in E. From the genericity of R it follows that outside Y the cycles $Z_R^{(1)}$ and $Z_1|_{R^+}$ coincide. At Y we get the equality

$$\operatorname{mult}_Y Z_R^{(1)} = \operatorname{mult}_Y Z^+ + \operatorname{mult}_Y Z_1$$

However, $\operatorname{mult}_Y Z_1 \leq \deg Z_1$ so that

$$\begin{gathered}\operatorname{mult}_o Z + \operatorname{mult}_Y Z^+ \\ = \nu^2 + \deg Z_1 + \operatorname{mult}_Y Z^+ \\ \geq \nu^2 + \operatorname{mult}_Y Z_R^{(1)} > 8n^2\end{gathered}$$

which proves the claim in the case $\nu \geq 2n$. For the case $\nu < 2n$ we refer the reader to Chapter 2, Section 4.2 of [Puk13]. The original proof in [Che00] contained a gap in the case $\nu < 2n$, this case was then proved first in [Puk10].

5 The cone technique

In this chapter we include a technique which is very useful for excluding maximal subvarieties.

Proposition 5.1. *Let $X \subset \mathbb{P}^m$ be a smooth hypersurface and $\Sigma \subset |nH|$ a mobile linear system on X. Suppose $C \subset X$ is an irreducible curve, then the inequality*

$$\operatorname{mult}_C \Sigma \leq n$$

holds.

Before we prove this, we first prove the following proposition.

Proposition 5.2. Let $W \subset \mathbb{P}^m$ be a smooth hypersurface, and $C \subset W$ a curve. Let $x \in \mathbb{P}^m \setminus W$ be a general point and $C(x)$ the cone with base C and vertex x. Then

$$C(x) \cap W = C \cup R(x)$$

where $R(x)$ is a curve which intersects C in $\deg R(x)$ distinct points.

Proof. Consider the projection map $\pi : W \to \mathbb{P}^m$ from the point x with ramification divisor $W_x \subset W$. A point $y \in C$ is a ramified point of this map precisely when the line $L_{xy} \subset C(x)$ has multiple intersection with W which happens when $y \in C \cap R(x)$ and so $C \cap R(x) = C \cap W_x$. On W the equation for W_x is

$$F_x = \sum_{i=0}^{m} \frac{\partial F}{\partial z_i} x_i = 0,$$

where $(z_0 : \ldots : z_m)$ are homogeneous coordinates on $\mathbb{P}^m$, $F(x_0, \ldots, x_M)$ is the equation of W in terms of these coordinates and $x = (x_0 : \ldots : x_m)$. From this we see that W_x is numerically equivalent to $\deg W - 1$ times a hyperplane section on W. Since W is non-singular the linear system

$$\left| \sum_{i=0}^{m} \lambda_i \frac{\partial F}{\partial z_i} \right|$$

is base point free. Thus for a general point x the intersection $C \cap W_x$ meets in $(\deg W - 1) \deg C = \deg R(x)$ distinct points. □

We now proceed to the proof of Proposition 5.1. Take a general point $x \in \mathbb{P}^m$ away from X and let $Z(x)$ denote the cone with vertex x and base C. Then $Z(x) \cap X = C \cup R(x)$, where $R(x)$ is the residual curve. Now by Proposition 5.2, $R(x)$ and C intersect at $\deg R(x)$ distinct points. Now consider $\Sigma|_{R(x)}$. The general element of this has degree $n \deg R(x)$, but also contains $\deg R(x)$ points of multiplicity $\operatorname{mult}_C \Sigma$. So $n \deg R(x) \geq \deg R(x) \operatorname{mult}_C \Sigma$ and Proposition 5.1 follows.

As an example, we apply it to prove the following theorem.

Theorem 5.3. *Let $V = V_4 \subset \mathbb{P}^4$ be a smooth quartic hypersurface. Then V is birationally superrigid.*

Proof. Suppose for a contradiction that there is a mobile linear system $\Sigma \subset |nH|$ on V. Let $B \subset V$ be the center of this maximal singularity. We have two cases to consider: either $\dim B = 1$ or $\dim B = 0$. In the first case, then B is a maximal subvariety of V and we have the inequality $\operatorname{mult}_B \Sigma > n$. Applying Proposition 5.1 we immediately arrive at a contradiction. We consider the case when $B = p$ is a point. Consider the 1-cycle $Z = (D_1 \circ D_2)$ on V, where $D_1, D_2 \in \Sigma$ are general divisors. Then by Theorem 3.1 the inequality

$$\operatorname{mult}_p Z > 4n^2$$

holds. However, Z has degree $4n^2$, so the above inequality cannot hold. Having excluded all possible cases, we conclude that V is birationally superrigid. $\square$

This result was first shown in [IM71], though the statement and proof have been updated since the original paper.

Conclusion

In this chapter we have introduced the concept of birational rigidity and outlined its application to the rationality problem discussed in the introduction. We have explained how the property of rigidity is shown via the method of maximal singularities and the Noether-Fano inequality. We have also familiarised the reader with the connectedness theorem and how it is applied to the method of maximal singularities. After having introduced the method and its many components, we will now apply it to a new class of $\mathbb{Q}$-Factorial Fano varieties in Chapter 3.

Chapter 3

Birationally rigid singular double quadrics and cubics

In this chapter we prove the core results of this thesis. We will prove that Fano double quadrics of index 1 and dimension at least 6 are birationally superrigid if the branch divisor has at most quadratic singularities of rank at least 6. Fano double cubics of index 1 and dimension at least 8 are birationally superrigid if the branch divisor has at most quadratic singularities of rank at least 8 and another minor condition of general position is satisfied. Hence, in the parameter spaces of these varieties the complement to the set of factorial and birationally superrigid varieties is of codimension at least $\binom{M-4}{2}+1$ and $\binom{M-6}{2}+1$ respectively. In Section 1 we introduce the varieties to be studied and state the main results. In Section 2 we prove birational superrigidity for Fano double quadrics and perform the calculation of the codimension of the set of varieties not satisfying factoriality or superrigidity. In Section 3 we prove birational superrigidity for Fano double cubics. The results of this chapter have been put together into a paper [Joh17].

1 Introduction

We start this introduction with a definition.

Definition 1.1. A *Fano double hypersurface* is defined as a projective algebraic variety V equipped with a morphism $\sigma : V \to Q \subset \mathbb{P}^{M+1}$, where $M \geq 4$, Q is an irreducible hypersurface of degree m, where $2 \leq m \leq M-2$, and σ is a double cover ramified over a divisor $W \subset Q$ which is cut out on Q by a hypersurface $W^* \subset \mathbb{P}^{M+1}$ of degree $2M - 2m + 2$.

As is the case for double spaces considered in Chapter 1, Example 3.17, a Fano double hypersurface can be written as a complete intersection in weighted projective space. Indeed, the variety V can be realized as a complete intersection of codimension 2 in the weighted projective space $\mathbb{P}(1^{M+2}, M - m + 1)$, given by the equations

$$f(x_0, \ldots, x_{M+1}) = 0, \quad y^2 = g(x_0, \ldots, x_{M+1})$$

where $x_0, \ldots, x_{M+1}$ have weight 1 and y has weight $M - m + 1$, and f and g are of degree m and $2(M - m + 1)$ respectively. When $m = 2$ we call V a double quadric and when $m = 3$ we call it a double cubic. Throughout this chapter we assume that Q is non-singular. By the Riemann-Hurwitz formula we have $K_V = \sigma^*(K_Q) + R$, where $2R = \sigma^*(W)$ and by Chapter 1, Example 5.5(iii) and Chapter 1, Theorem 6.6 and we see that $\operatorname{Pic} V = \mathbb{Z}L$ and $K_V = -L$, where $L = \sigma^*(H)$ and $H \subset Q$ is the divisor associated to a hyperplane section $Q \cap H$. Hence, it is a primitive Fano variety of index 1.

In Section 2 the following result is shown:

Theorem 1.2. *Let $\sigma : V \to Q \subset \mathbb{P}^{M+1}$ be a double quadric ramified over $W = W^* \cap Q$. Assume that $M \geq 6$ and W has at most quadratic singularities of rank at least 6. Then V is factorial and birationally superrigid.*

In Section 3 we consider rigidity of the double cubic case. In the proof a small regularity condition is required which we will now introduce. Let

$\sigma : V \to Q \subset \mathbb{P}^{M+1}$ be a double cubic, branched over $W = W^* \cap Q$. We say that V satisfies the condition (*) if for any non-singular point $p \in W$ of the branch divisor in a system $(z_1, \dots, z_{M+1})$ of affine coordinates with the origin at p the hypersurface Q is given by the equation

$$0 = q_1(z_*) + q_2(z_*) + \dots$$

where $q_i(z_*)$ are homogeneous of degree i and $q_2|_{T_pW} \not\equiv 0$. In Section 3 the following result is proven:

Theorem 1.3. *Assume that $M \geq 8$, $W = W^* \cap Q$ has at most quadratic singularities of rank at least 8 and V satisfies the condition (*). Then V is factorial and birationally superrigid.*

In Section 2 we prove a result on the parameter space of double quadrics and cubics which are neither factorial or birationally superrigid, in the same vein as [EP14]. We set up the result as follows. We set

$$\mathcal{F}_m \subset \mathbb{P}(H^0(\mathbb{P}^{M+1}, \mathcal{O}_{\mathbb{P}^{M+1}}(m)))$$

to be the open set of non-singular hypersurfaces Q of degree m and

$$\mathcal{G} = \mathbb{P}(H^0(\mathbb{P}^{M+1}, \mathcal{O}_{\mathbb{P}^{M+1}}(2M - 2m + 2)))$$

the parameter space of hypersurfaces W^*. Then $\mathcal{I}_m = \mathcal{F}_m \times \mathcal{G}$ is a natural parameter space for double hypersurfaces of dimension M and L-degree $2m$. Let $S_m \subset \mathcal{I}_m$ be the set of pairs (Q, W^*) such that the corresponding double hypersurface V is a factorial variety with at most terminal singularities (and therefore, a Fano variety with $\operatorname{Pic} V = \mathbb{Z}K_V, K_V = -L$) which is birationally superrigid. The result shown in Section 3 is the following:

Theorem 1.4. *The complement $\overline{\mathcal{I}_k \setminus S_k}$ has codimension at least $\binom{M-4}{2} + 1$ for $M \geq 6$ when $k = 2$ and at least $\binom{M-6}{2} + 1$ for $M \geq 8$ when $k = 3$.*

This thesis continues a number of previous works; superrigidity of generic (in particular, non-singular) double hypersurfaces was first shown in [Puk00a].

Certain singular cases were investigated in [Che06a], see also [Che05]. Cyclic covers of degree 3 and higher were studied in [Puk09], triple spaces with isolated quadratic points in [Che04]. Double spaces of index 1 with higher-dimensional singular locus were shown to be birationally superrigid in [Mul10].

Here we work in the style of [EP14], not only showing birational superrigidity of a certain class of Fano varieties but also estimating the codimension of the complement to the set of factorial and superrigid varieties. Such estimates are important due to applications to the theory of birational rigidity of Fano fibre spaces, which we will explain in more detail in Section 3.

The method of proof used for Theorem 1.2 and Theorem 1.3 is the method of maximal singularities as outlined in Chapter 2. Theorem 1.4 is shown using Theorem 1.2, Theorem 1.3 and a dimension counting argument.

2 Fano double quadrics

2.1. Double quadrics. Let us prove Theorem 1.2. We first note that since W has at most quadratic singularities of rank at least 6 then V has at most quadratic singularities of rank at least 7 (the term y^2 increases the rank by 1), and therefore by Chapter 2, Lemma 3.4, $\operatorname{codim} \operatorname{Sing} V \geq 6$. Thus, by Chapter 1, Proposition 4.15, V is factorial. Hence, it now remains to prove that V is birationally superrigid. Assume that $\Sigma \subset |nL|$ is a mobile linear system with a maximal singularity $E \subset V^*$, where $\phi : V^* \to V$ is a birational morphism from a non-singular projective variety V^*. Let $B = \phi(E)$ be the centre of E on V. Assume first that $\operatorname{codim} B \geq 3$.

Recall by Theorem 3.5 of Chapter 2 the following inequality

$$\operatorname{mult}_B Z > 4n^2$$

holds. Now the linear system $|L| = |-K_V|$ has basis $\operatorname{div}(x_0), \ldots, \operatorname{div}(x_{M+1})$. Let $x = (x_0 : \ldots : x_{M+1} : y) \in V$ then $|L|$ defines a map onto $\mathbb{P}^{M+1}$ by sending

$x \to (x_0 : \ldots : x_{M+1})$. This map is precisely σ, so then for any point $o \in B$ we get the inequality

$$\mathrm{mult}_o Z \leq \deg_L Z = 4n^2$$

We conclude that the case when $\operatorname{codim} B \geq 3$ is impossible.

We now consider the case when B is a subvariety of codimension 2. Then it is a maximal subvariety of the linear system Σ and by Chapter 2, Proposition 2.5 the inequality

$$\mathrm{mult}_B \Sigma > n$$

holds. We define $V_H = V \cap H$, where H is a generic linear subvariety of dimension 6. Since the codimension of the singular set of V is at least 6 we conclude that V_H is non-singular. We define $B_H = B \cap H$ and note that it satisfies the same inequality with respect to $\Sigma_H = \Sigma|_{V_H}$. Since $\dim V_H \geq 5$ by the Lefschetz hyperplane theorem (Chapter 1, Theorem 6.6) B_H is numerically equivalent to a multiple kL^2. Set $\nu = \mathrm{mult}_{B_H} \Sigma_H > n$. Then for the cycle Z we see that $Z \sim n^2 L^2$ and

$$Z = \nu^2 B_H + Z_1$$

with Z_1 an effective cycle. Comparing the L-degrees on the left and right hand sides, we get a contradiction. Thus concluding the proof of Theorem 1.2.

Remark 2.1. More subtle arguments could be used to give a proof of Theorem 1.2 for $M = 4$ and 5 under the slightly weaker assumption that the quadratic singularities are of rank at least 4. Only the arguments for the case $\operatorname{codim} B = 2$ need to be modified in a way similar to [Puk89]. Here we can not assume that $A^2 V = \mathbb{Z} K_V^2$. However, we can still get some information on codimension 2 cycles. Recall the identity $Z = \alpha B + Y$ where $\alpha > n^2$.

Computing $\deg Z = n^2 \deg V \geq \alpha \deg B$ we obtain the inequality

$$\deg B < \deg V$$

Since in this case, $\deg V = 4$, we see that $\deg B \leq 3$. Here we could exclude each case in turn by use of the "test class" method (for examples, see [Isk77]).

2.3. Codimension in the parameter space. Let us prove Theorem 1.4, assuming Theorems 1.2 and 1.3. First, we consider double quadrics. Fix a non-singular hypersurface $Q \in \mathcal{F}_m$. Given the claim of Theorem 1.2, it is sufficient to show that the set of hypersurfaces $W^* \in \mathcal{G}$ violating the assumptions of Theorem 1.2 is of codimension at least $\binom{M-4}{2} + 1$ in $\mathcal{G}$. Fix a point $p \in Q$. The hypersurface Q is given by an equation

$$0 = q_1 + q_2 + \dots$$

in some system of affine coordinates with origin at p. The hypersurface W^* is given by an equation

$$0 = w_1 + w_2 + \dots$$

where w_i are homogeneous of degree i (we assume that $p \in W$ - otherwise the case is trivial). Violation of the assumptions of Theorem 1.2 at p means that $w_1 = \lambda q_1$ for some $\lambda \in \mathbb{C}$ and $w_2|_{q_1=0}$ is a quadratic form of rank at most 5. The first imposes $M + 1$ independent conditions while the second imposes $\binom{M+1}{2} - \binom{r+1}{2} - r(M-r)$, where in this case $r = 5$. Thus, these conditions impose in total

$$\binom{M-r-1}{2} + M = \binom{M-4}{2} + M$$

independent conditions on the coefficients of the polynomial g. Since the point p varies in the intersection $W = W^* \cap Q)$ which has dimension $M - 1$, the codimension of the set of W^* who violate the conditions is at least

$$\binom{M-4}{2} + M - (M-1) = \binom{M-4}{2} + 1.$$

This completes the proof of Theorem 1.1 in the case of double quadrics. In the case of double cubics we obtain the lower bound $\binom{M-6}{2} + 1$ by the same argument (in this case $r = 7$). However, we also have the condition (*). Again, we consider first the violation of (*) at a fixed point $p \in W$. The condition

$$q_2|_{T_pW} \equiv 0$$

for fixed q_1, w_1 (in the notations above) imposes $\frac{M(M-1)}{2}$ independent conditions on q_2. Therefore, the set of pairs (f, g) such that in a system of affine coordinates with the origin $p \in \{f = g = 0\}$ the condition (*) is violated is of codimension at least $\frac{M(M-1)}{2} - M = \frac{M(M-3)}{2}$ in the parameter space. As this number is higher than $\binom{M-6}{2} + 1$, the proof of Theorem 1.4 is complete.

Remark 2.2. We will now explain one potential application of this estimate, in the style of [Puk15]. Consider a variety $\pi : V \to S$, where V is factorial and has at most terminal singularities, such that $-K_V$ is ample on the fibres of π and S is a non-singular and rationally connected variety. Moreover, the condition,

$$\operatorname{Pic} V = \mathbb{Z}K_V \bigoplus \pi^* \operatorname{Pic} S.$$

holds. This is called a *standard Fano-Mori fibre space.* Let $\chi : V' \to S'$ be a rationally connected fibre space. We say that $\pi : V \to S$ is a *birationally rigid Fano fibre space* if for any birational map $\phi : V \dashrightarrow V'$ there exists a rational dominant map $\beta : S \dashrightarrow S'$ such that the diagram

$$\begin{tikzcd} V \arrow[r, dashed, "\phi"] \arrow[d, "\pi"] & V' \arrow[d, "\chi"] \\ S \arrow[r, dashed, "\beta"] & S' \end{tikzcd}$$

commutes.

The technique of proving that a given Fano-Mori fibre spaces satisfies this condition is essentially a generalisation of the method of maximal singularities discussed in this thesis. However, the conditions that need to be satisfied by such a fibre space are more complex. We not only have to consider what kind of varieties the fibres are allowed to be but also how the fibres are allowed to vary over the base S. In [Puk15] a set of conditions which together form a sufficient criterion for birational rigidity are outlined. Three of which we will focus on here.

1. Every fibre F_s is an irreducible factorial Fano variety with terminal singularities and Picard group $\operatorname{Pic} F_s = \mathbb{Z}K_{F_s}$.
2. Given any fibre F_s, for every effective divisor $D \in |-nK_{F_s}|$, the pair $(F_s, \frac{1}{n}D)$ is log-canonical. (Divisorial log-canonicity)
3. For any mobile linear system $\Sigma \subset |-nK_{F_s}|$ on F_s the pair $(F_s, \frac{1}{n}D)$ is canonical, where $D \in \Sigma$ is a general divisor. (Mobile canonicity).

We note that the second condition is equivalent to $\alpha(F_s) \geq 1$, where $\alpha(F_s)$ is the α-invariant of Tian, first introduced in [Tia87]. The last condition implies birational superrigidity of F_s.

Consider again our fibre space $\pi : V \to S$. Consider the specific example when F_s are all Fano hypersurfaces of index 1 and dimension M, this is an example considered in ([Puk15]). We can construct an example of this as follows: Let $S = \mathbb{P}^m$, and $X = \mathbb{P}^M \times \mathbb{P}^m$, and $V \subset X$ a hypersurface of bidegree (M, l). Then $V \to \mathbb{P}^m$ is a fibration into Fano hypersurfaces of index 1. Let $\mathcal{W}$ denote the parameter space of all Fano hypersurfaces of fixed dimension and $\mathcal{W}_{reg}$ the subset of Fano hypersurfaces which satisfy the above criterion (that is every, $W \in \mathcal{W}$ satisfies the properties of F_s above).

Now, assume we have a bound $\operatorname{codim}((\mathcal{W} \setminus \mathcal{W}_{reg}) \subset \mathcal{W}) \geq \delta$. then every fibre $F_s \in \mathcal{W}_{reg}$. Now in the example above, if $m < \delta$ and the hypersurface V is general (also l satisfies a technical condition which we do not discuss here) then $V \to S$ satisfies the above conditions. This illustrates the utility of the estimate δ.

We now consider the case of fibrations into Fano double quadrics and cubics. By Theorem 1.4 we have some information about double quadrics and cubics that satisfy criterion (1) and (3) and so we deduce that if we consider fibrations as above, the dimension of the base S should be bounded above by $\binom{M-4}{2}+1$ and $\binom{M-6}{2}+1$ respectively. To require that divisorial log-canonicity is also satisfied imposes further restrictions, so we expect that we would need a lower upper bound on the dimension of S to guarantee the above criterion. Thus, a natural extension to the result in this thesis would be to try and prove divisorial log-canonicity for double quadrics and cubics and give an estimate for $\operatorname{codim}((\mathcal{W}\setminus\mathcal{W}_{reg})\subset\mathcal{W})$ for double quadrics and cubics.

3 Fano double cubics

In this section we prove Theorem 1.3. Since we are working with a variety with singularities, we must carefully consider the cases when the center of a maximal singularity is contained in the singular locus $\operatorname{Sing} V$ and when the center is not contained in it. Since our variety is a double cover, we must also distinguish the cases when the center contains ramified points or not. We begin by considering the comparatively simpler case of a maximal singularity with center outside the singular locus. Subsections 3.2 and 3.3 are then dedicated to considering the case of centers contained in the singular locus.

3.1. Maximal singularities outside the singular locus. Recall that we are working with a double cover $\sigma: V\to Q$, where $Q\subset\mathbb{P}^{M+1}$ is a non-singular cubic hypersurface, σ is branched over $W=W^*\cap Q$ and the assumptions of Theorem 1.3 holds. In particular, V is factorial by the same argument as for double quadrics and $\operatorname{Pic} V=\mathbb{Z}L$ where L is the σ-pullback of the hyper-plane section of Q. Assume that V is not birationally superrigid. Then there is a mobile linear system $\Sigma\subset|nL|$ with a maximal singularity $E\subset V^*$, where $\phi: V^*\to V$ is a birational morphism from a non-singular projective V^*. We start with the following observation.

Lemma 3.1. *Suppose $B = \phi(E)$ is the center of a maximal singularity on V, then* $\operatorname{codim} B \geq 3$.

Proof. If $\operatorname{codim} B = 2$, we argue as for double quadrics (Subsection 2.2) and come to a contradiction. □

Therefore for the rest of the section we will consider when $\operatorname{codim} B \geq 3$. Assume first that $B \not\subset \operatorname{Sing} V$. Note that this implies $\sigma(B) \not\subset \operatorname{Sing} W$. The self-intersection $Z = (D_1 \circ D_2)$ of the system Σ (where D_1, D_2 are general divisors) satisfies the $4n^2$-inequality $\operatorname{mult}_B Z > 4n^2$.

Lemma 3.2. *The case $\sigma(B) \subset W$ cannot occur.*

Proof. Take a point $p \in \sigma(B) \setminus \operatorname{Sing} W$. Consider the intersection $T_pW \cap Q$, which near p is given by the equation $0 = q_2|_{T_pW} + q_3|_{T_pW}$. By the condition (*), p has multiplicity 2 at $T_pW \cap Q$ and therefore the irreducible subvariety $\sigma^{-1}(T_pW \cap Q) \sim L^2$ has multiplicity precisely 4 at $o = \sigma^{-1}(p)$. Hence, there exists an irreducible component Y of the cycle Z, such that $Y \sim lL^2$, $\operatorname{mult}_o Y > 4l$ and $Y \neq \sigma^{-1}(T_pW \cap Q)$. So Y is not contained in both divisors $\sigma^{-1}(T_pQ \cap Q)$ and $\sigma^{-1}(T_pW^* \cap Q)$. Intersecting Y with one which does not contain Y, we obtain an effective cycle $Y^* \sim lL^3$ of codimension 3 such that $\operatorname{mult}_o Y^* > 8l$. As $\deg_L Y^* = 6l$, we obtain a contradiction. □

The same argument works in the case of B such that $\sigma(B) \not\subset W$ but also $\sigma(B) \cap W \not\subset \operatorname{Sing} W$, provided we take p such that $p \in \sigma(B) \cap W$, else $\sigma^{-1}(p)$ is not well defined. Therefore, the last case to consider in this section is a maximal singularity such that its centre B satisfies the property $\sigma(B) \not\subset W$ and $\sigma(B) \cap W \subset \operatorname{Sing} W$ (this also covers the case when $\sigma(B) \cap W = \varnothing$).

We take a general point $o \in B$ such that $\bar{o} = \sigma(o) \notin W$ (we note that $\bar{o}$ is non-singular). Let $\psi : V^+ \to V$ be the blow up of the point o and let $E = \psi^{-1}(o)$ be the exceptional divisor. By the $8n^2$-inequality (Chapter 2, Theorem 4.6) there is a linear subspace $\Psi \subset E \cong \mathbb{P}^{M-1}$ of codimension 2, such that the self-intersection Z of Σ satisfies the property

$$\mathrm{mult}_o Z + \mathrm{mult}_\Psi Z^+ > 8n^2$$

Z^+ being its strict transform on V^+. As σ gives an isomorphism of local rings $\mathcal{O}_{o,V}$ and $\mathcal{O}_{\bar{o},Q}$, we can find a hyperplane $H \subset \mathbb{P}$, such that $\sigma^{-1}(H_Q)^+$ contains Ψ, where $H_Q = H \cap Q$ is the corresponding hyperplane section. As such hypersurfaces form a 2-dimensional linear system, we may assume that $\sigma^{-1}(H_Q)$ contains none of the components of Z, so the cycle $(Z \circ H_Q)$ is well defined. Obviously, $(Z \circ H_Q) \sim n^2 L^3$, so $\deg(Z \circ H_Q) = 6n^2$. On the other hand

$$\mathrm{mult}_o(Z \circ H_Q) \geq \mathrm{mult}_o Z + \mathrm{mult}_\Psi Z^+ > 8n^2$$

which is a contradiction.

We have inspected all options for the case $B \not\subset \mathrm{Sing}\, V$. Therefore we may assume that $B \subset \mathrm{Sing}\, V$ and so $\sigma(B) \subset \mathrm{Sing}\, W$. We consider this case in the next section.

3.2. Inversion of adjunction. Fix a general point $o \in B$. As the singularities of W are quadratic of rank at least 8, the singularities of V are quadratic of rank at least 9. Near the point o we may consider the germ $o \in V$ analytically as a germ of a hypersurface in $\mathbb{C}^{M+1}$. We now work locally near o by defining X to be a generic section of V by $dim(V) - 5$ general, very ample divisors passing through o. Then X has dimension 4 and $o \in X$ is a germ of an isolated quadratic singularity of rank 5. Let $\pi : V^+ \to V$ and $\pi_X : X^+ \to X$ be the blow ups of the point o, and $E = \pi^{-1}(o), E_X = \pi_X^{-1}(o)$ the exceptional divisors. In an obvious sense, the non-singular 3-dimensional quadric E_X is the section of the quadric E by $dim(V) - 4$ general very ample divisors. For a general divisor $D \in \Sigma$ set

$$\pi^* D = D^+ + \nu E$$

Let $D_1, D_2 \in \Sigma$ be generic divisors, then $D_i = D_i^+ + \nu E$. We now quote a result from [Ful98] (Theorem 12.4.8) which we will make extensive use of (this is a generalisation of Chapter 2, Lemma 3.2)

Proposition 3.3. *Let D_i be defined as above, and set $(D_1^+ \circ D_2^+) = Z^+ + Z^*$, then the inequality*

$$\operatorname{mult}_o Z \geq 2\nu^2 + \deg Z^*$$

holds.

where D_X denotes the restriction of D onto X and D_X^+ its strict transform on X^+. By inversion of adjunction, the pair $(X, \frac{1}{n}D_X)$ is not log canonical at o. By our assumption about B and what was shown in Subsection 3.1, the point o is an isolated centre of a non-canonical singularity of the pair $(X, \frac{1}{n}D_X)$.

Proposition 3.4. *The multiplicity ν satisfies the inequalities*

$$n < \nu \leq \sqrt{3}n$$

Proof. The first inequality is Proposition 4.5 in Chapter 2. For the inequality $\nu \leq \sqrt{3}n$, we note that $\deg_L Z = 6n^2 \geq \operatorname{mult}_o Z \geq 2\nu^2$. □

In particular, $\nu < 2n$, which implies that the pair $(X^+, \frac{1}{n}D_X^+)$ is non log-canonical and the centers of log-canonical singularities are contained in E_X. Moreover, by the connectedness principal (Chapter 2, Theorem 4.3) the union of centres of all non log-canonical singularities is a connected subset of E_X. We therefore have 3 cases to consider:

Case 1. $(X^+, \frac{1}{n}D_X^+)$ is non log-canonical at a surface in E_X.

Case 2. $(X^+, \frac{1}{n}D_X^+)$ is non log-canonical at a curve in E_X.

Case 3. $(X^+, \frac{1}{n}D_X^+)$ is non log-canonical at a point $p_X \in E_X$.

Lemma 3.5. *Case 1 cannot occur.*

Proof. In this case the pair $(X^+, \frac{1}{n}\Sigma_X^+)$ is non log-canonical at an irreducible divisor $R_X \subset E_X$, which is a section of an irreducible divisor $R \subset E \subset \mathbb{P}^M$. Therefore, the degree of R is at least 2. Since $(X^+, \frac{1}{n}\Sigma_X^+)$ is non log-canonical, for generic divisors $D_1, D_2 \in \Sigma$ we can use Proposition 3.8 of Chapter 2 to obtain the inequality

$$\operatorname{mult}_R(D_1^+ \circ D_2^+) > 4n^2$$

so using Proposition 6.11 of Chapter 1, Proposition 3.3 and the fact that R is an irreducible component of the cycle $(D_1^+ \circ D_2^+)$ we obtain $\operatorname{mult}_o Z \geq 2\nu^2 + \deg R \operatorname{mult}_R(D_1^+ \circ D_2^+) > 2\nu^2 + 2(4n^2) > 10n^2$, a contradiction. □

Lemma 3.6. *Case 3 cannot occur.*

Proof. We note that $p_X = S \cap E_X$ where S must be a linear subspace of codimension 3 on E, as E_X is a generic section of E. Since a quadric of rank at least 9 cannot contain a linear subspace of codimension 3, we arrive at a contradiction. □

Therefore the only case left is Case 2: $(X^+, \frac{1}{n}D_X^+)$ is non log-canonical at an irreducible curve Y_X which is a section of an irreducible subvariety $Y \subset E$ of codimension 2. We consider this final case in the next section.

3.3. Centre at an irreducible curve. Write the self-intersection of the linear system Σ_X^+ as

$$(D_1^+|_X \circ D_2^+|_X) = Z_X^+ + Z_X^*$$

where Z_X^* is an effective divisor on E_X, which is the restriction onto E_X of an effective divisor Z^* on E, where Z^* is the component of the self intersection of Σ^+ which is contained in E. Since $(X^+, \frac{1}{n}D_X^+)$ is non log-canonical (and thus non-canonical), we have the $4n^2$-inequality (Chapter 2, Theorem 3.1)

$$\operatorname{mult}_{Y_X} Z_X^+ + \operatorname{mult}_{Y_X} Z_X^* > 4n^2$$

so that $\operatorname{mult}_Y Z^+ + \operatorname{mult}_Y Z^* > 4n^2$. We first give a bound on the multiplicity of $\operatorname{mult}_Y Z^*$.

Lemma 3.7. *The inequality* $\operatorname{mult}_Y Z^* = \operatorname{mult}_{Y_X} Z_X^* \le 2n^2$ *holds.*

Proof. Assume the converse: $\operatorname{mult}_Y Z^* > 2n^2$. As Z_X^* is an effective divisor on a non-singular quadric, see that

$$Z_X^* \sim \alpha H_{E_X}$$

where H_E is the hyperplane section of E. Since E_X is a smooth hypersurface and $Z_X^* \in |\alpha H_{E_X}|$ and $Y_X \subset E_X$ is a curve, by Proposition 5.1 of Chapter 2 we obtain

$$2n^2 < \operatorname{mult}_{Y_X} Z_X^* \le \alpha.$$

Therefore $\deg Z^* > \deg H_E \cdot (2n^2) = 4n^2$. However, this implies that $\operatorname{mult}_o Z > 2\nu^2 + 4n^2 > 6n^2$, a contradiction. □

From this we conclude that $\operatorname{mult}_Y Z^+ = \operatorname{mult}_{Y_X} Z_X^+ > 2n^2$. Our next observation is that Y must be a section of E by a linear subspace of codimension 2 (recall that E is a quadric with a natural embedding in projective space). Indeed, since E is a quadric of rank at least 9, Y is numerically equivalent to βH_E^2 for some $\beta \ge 1$.

Lemma 3.8. *The equality* $\beta = 1$ *holds.*

Proof. Assume the converse: $\beta \ge 2$. Then $\deg Y \ge 4$ and therefore $\operatorname{mult}_o Z = \deg(Z^+ \circ E) \ge \operatorname{mult}_Y Z^+ \deg Y > 8n^2$. A contradiction. So then $\beta = 1$ and Y is a section of E by a linear subspace as claimed. □

Set $\bar{o} = \sigma(o) \in Q$. Let $\pi_Q : Q^+ \to Q$ be the blow up of $\bar{o}$, $\overline{E} = \pi_Q^{-1}(\bar{o})$ the exceptional divisor, $\overline{E} \cong \mathbb{P}^{M-1}$. Note that $\sigma : V \to Q$ extends to $\sigma^+ : V^+ \to Q^+$, where $\sigma^+|_E : E \to \overline{E}$ is a double cover branched over the quadric $W^+ \cap \overline{E}$. We now have two cases:

- $\sigma^+(Y)$ is a linear subspace in $\overline{E}$ of codimension 2 and $\sigma^+|_Y$ is a double cover
- $\sigma^+(Y) = \overline{Y}$ is a quadric in a hyperplane in $\overline{E}$ and $\sigma^+|_Y$ is birational.

The first case is excluded by similar arguments used in subsection 3.1 where the $8n^2$-inequality was applied: there exists a linear subspace $\Delta \subset \mathbb{P}^{M+1}$ such that its strict transform Δ^+ cuts out B. Take a general hyperplane $T \supset \Delta$ and consider the cycle $H = \sigma^*(Q \circ T)$ on V. Intersecting with Z we obtain the inequality

$$\operatorname{mult}_o(Z \circ H) \geq 2\nu^2 + 2 \operatorname{mult}_Y(D_1^+ \circ D_2^+) > 8n^2$$

which contradicts the fact that $\deg(Z \circ H) = \deg(Z) = 6n^2$.

We now consider the second case. For a quadric hypersurface cone $\Lambda \subset \mathbb{P}$ with vertex $\bar{o}$, set $\Lambda_Q = \Lambda|_Q = \Lambda \cap Q$. Take a general cone Λ such that $\Lambda_Q^+ \cap \overline{E}$ contains $\overline{Y}$, so that $\sigma^{-1}(\Lambda_Q)^+$ contains Y. By generality, $\sigma^{-1}(\Lambda_Q)$ contains none of the components of Z so the cycle

$$Z_\Lambda = (Z \circ \sigma^{-1}(\Lambda_Q))$$

is well defined. Its L-degree is $12n^2$. Now by Proposition 3.3 we obtain

$$\operatorname{mult}_o Z_\Lambda \geq 2 \operatorname{mult}_o Z + 2 \operatorname{mult}_Y Z^+ > 8n^2 + 4n^2 = 12n^2$$

which gives the final contradiction. This concludes the proof of Theorem 1.3.

Conclusions and Further Work

In this thesis we have investigated the birational properties of a large class of high dimensional, $\mathbb{Q}$-factorial Fano varieties of index 1, which are realised as weighted complete intersections of codimension 2. We have shown that these varieties have the property of birational superrigidity and thus have all the properties that superrigidity entails, such as being non-rational and admitting no non-trivial structures of a rationally connected fibre space. This thesis thus contributes to the larger project of birational classification of $\mathbb{Q}$-factorial, terminal weighted projective complete intersections. In addition, the estimates for the codimension on the locus of non-superrigid or non-factorial double quadrics and cubics forms an important step towards the investigation of varieties fibered into double quadrics and cubics, as outlined in Chapter 3.

The results of Chapter 3 could most likely be improved upon; for example, we expect that the conditions on the singularities of the branch divisor could be relaxed to quadratic singularities of rank at least 4. This would further improve the bound given in Theorem 3.1.4. Any rank lower than this would be more difficult to achieve as the variety is no longer factorial in general. In addition, the assumptions on dimension could probably be relaxed to dimension 4 and higher. This would likely require a lot of work since there would be a lot more cases of maximal singularities to exclude, as the numerical Chow group becomes more complex. In general however, we expect the behaviour of factorial double quadrics and cubics with quadratic singularities to mirror that of the non-singular case, as is the case for Fano hypersurfaces of dimension at least 4.

Regarding further results which build on this thesis, one could extend the type of singularities to isolated singularities of multiplicity at least 3, as done for Fano hypersurfaces in [Puk02]. The recent extension of the $4n^2$-inequality for complete intersection singularities shown in [Puk16] could be of use in this case. Another potential area of study would be Fano double hypersurfaces of higher degree. Double quartics might prove surmountable with the methods used in this thesis, as the degree of this variety is 8 which would mean that the $8n^2$-inequality should still be of use. Any higher degree than this would require the use of the method of hypertangent divisors, which would impose further restrictions on the variety in the form of regularity conditions. This would potentially weaken any bounds on the codimension as in Theorem 3.1.4 . It is possible however to give estimates on the codimension of the set of non-regular hypersurfaces (see Proposition 3 of [EP14]) so calculating an estimate in the case of higher degree double hypersurfaces should be possible with current methods. Another potential generalisation would be to consider K-degree cyclic covers of Fano hypersurfaces, starting with triple covers.

Finally, another potential direction of further research would be to prove divisorial canonicity of double quadrics and cubics with quadratic singularities, which would extend the work done on smooth double covers in [Puk08].

Bibliography

[AM72] Michael. Artin and David Mumford. Some elementary examples of unirational varieties which are not rational. *Proc. London Math. Soc. (3)*, 25:75–95, 1972.

[AO15] Hamid. Ahmadinezhad and Takuzo Okada. Birationally rigid Pfaffian Fano 3-folds. *ArXiv e-prints*, August 2015. arXiv:1508.02974.

[CG72] Herbert Clemens and Phillip Griffiths. The intermediate Jacobian of the cubic threefold. *Ann. of Math. (2)*, 95:281–356, 1972.

[Che00] Ivan Cheltsov. On a smooth quintic 4-fold. *Sbornik: Mathematics*, 191(9):1399, 2000.

[Che04] Ivan Cheltsov. Birationally super-rigid cyclic triple spaces. *Izv. Math*, 68(6):1229–1275, 2004.

[Che05] Ivan Cheltsov. Birationally rigid Fano varieties. *Russian Math. Surveys*, 60(5):875–965, 2005.

[Che06a] Ivan Cheltsov. Double cubics and double quartics. *Math. Z.*, 253(1):75–86, 2006.

[Che06b] Ivan Cheltsov. Nonrational nodal quartic threefolds. *Pacific journal of mathematics*, 226(1):65–81, 2006.

[Che08] Ivan Cheltsov. Double spaces with singularities. *Mat. Sb.*, 199(2):131–148, 2008.

[CL94] Frederick Call and Gennady Lyubeznik. A simple proof of Grothendieck's theorem on the parafactoriality of local rings. In *Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992)*, volume 159 of *Contemp. Math.*, pages 15–18. Amer. Math. Soc., Providence, RI, 1994.

[CM04] Alessio Corti and Massimiliano Mella. Birational geometry of terminal quartic 3-folds. I. *Amer. J. Math.*, 126(4):739–761, 2004.

[Cor00] Alessio Corti. Singularities of linear systems and 3-fold birational geometry. In *Explicit birational geometry of 3-folds*, volume 281 of *London Math. Soc. Lecture Note Ser.*, pages 259–312. Cambridge Univ. Press, Cambridge, 2000.

[CP06] Ivan Cheltsov and Jihun Park. Weighted Fano threefold hypersurfaces. *J. Reine Angew. Math.*, 600:81–116, 2006.

[CPR00] Alessio Corti, Aleksandr Pukhlikov, and Miles Reid. Fano 3-fold hypersurfaces. In *Explicit birational geometry of 3-folds*, volume 281 of *London Math. Soc. Lecture Note Ser.*, pages 175–258. Cambridge Univ. Press, Cambridge, 2000.

[Dol82] Igor Dolgachev. Weighted projective varieties. In *Group actions and vector fields (Vancouver, B.C., 1981)*, volume 956 of *Lecture Notes in Math.*, pages 34–71. Springer, Berlin, 1982.

[EP14] Thomas Eckl and Aleksandr Pukhlikov. On the locus of non-rigid hypersurfaces. In *Automorphisms in birational and affine geometry*, volume 79 of *Springer Proc. Math. Stat.*, pages 121–139. Springer, Cham, 2014.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, second edition, 1998.

[Gri00] Mikhail Grinenko. Birational properties of pencils of del Pezzo surfaces of degrees 1 and 2. *Sb. Math*, 191(5-6):633–653, 2000.

[Gri03] Mikhail Grinenko. On the double cone over the Veronese surface. *Izv. math*, 67(3):421–438, 2003.

[Gri04] Mikhail Grinenko. Mori structures on a Fano threefold of index 2 and degree 1. *Proc. Steklov Inst. Math.*, 246(3):103–128, 2004.

[Gri11] Mikhail Grinenko. Three-dimensional quartics containing a plane. *Mat. Sb.*, 202(2):57–92, 2011.

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Hir64] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2)* **79** *(1964), 109–203; ibid. (2)*, 79:205–326, 1964.

[HTT15] Brendan Hassett, Sho Tanimoto, and Yuri Tschinkel. Balanced line bundles and equivariant compactifications of homogeneous spaces. *International Mathematics Research Notices*, 2015(15):6375–6410, 2015.

[Hul03] Klaus Hulek. *Elementary algebraic geometry*, volume 20 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2003. Translated from the 2000 German original by Helena Verrill.

[IF00] Anthony Iano-Fletcher. Working with weighted complete intersections. In *Explicit birational geometry of 3-folds*, volume 281 of *London Math. Soc. Lecture Note Ser.*, pages 101–173. Cambridge Univ. Press, Cambridge, 2000.

[IM71] Vasili Iskovskih and Yuri Manin. Three-dimensional quartics and counterexamples to the Lüroth problem. *Mat. USSR-Sb.*, 15:141–166, 1971.

[IP96] Vasili Iskovskikh and Aleksandr Pukhlikov. Birational automorphisms of multidimensional algebraic manifolds. *J. Math. Sci.*, 82(4):3528–3613, 1996. Algebraic geometry, 1.

[Isk77] Vasili Iskovskih. Birational automorphisms of three-dimensional algebraic varieties. *Soviet Math. Dokl.*, 18(3):748–750, 1977.

[Joh17] Ewan Johnstone. Birationally rigid singular double quadrics and double cubics. *Mathematical Notes*, (in press) 2017.

[Kol96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.

[Kol97] János Kollár. Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287. Amer. Math. Soc., Providence, RI, 1997.

[Kuz15] Alexander Kuznetsov. Derived categories view on rationality problems. *ArXiv e-prints*, September 2015. arXiv:1509.09115.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[Mel04] Massimiliano Mella. Birational geometry of quartic 3-folds. II. The importance of being $\mathbb{Q}$-factorial. *Math. Ann.*, 330(1):107–126, 2004.

[MM64] Hideyuki Matsumura and Paul Monsky. On the automorphisms of hypersurfaces. *J. Math. Kyoto Univ.*, 3:347–361, 1963/1964.

[Mul10] R. Dzh. Mullani. Fano double spaces with a large singular locus. *Math. Notes*, 87(3-4):444–448, 2010.

[Mum99] David Mumford. *The red book of varieties and schemes*, volume 1358 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 1999. Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello.

[Oka14a] Takuzo Okada. Birational Mori fiber structures of $\mathbb{Q}$-Fano 3-fold weighted complete intersections. *Proc. Lond. Math. Soc. (3)*, 109(6):1549–1600, 2014.

[Oka14b] Takuzo Okada. Birational Mori fiber structures of $\mathbb{Q}$-Fano 3-fold weighted complete intersections, III. *ArXiv e-prints*, September 2014. arXiv:1409.1506.

[Oka15] Takuzo Okada. Birational mori fiber structures of $\mathbb{Q}$-fano 3-fold weighted complete intersections, ii. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2015.

[Pir16] Alena Pirutka. Varieties that are not stably rational, zero-cycles and unramified cohomology. *ArXiv e-prints*, March 2016. arXiv:1603.09261.

[Puk88] Aleksandr Pukhlikov. Birational automorphisms of a three-dimensional quartic with a simple singularity. *Mat. Sb. (N.S.)*, 135(177)(4):472–496, 559, 1988.

[Puk89] Aleksandr Pukhlikov. Birational automorphisms of a double space and a double quadric. *Math. USSR-Izv*, 32(1):233–243, 1989.

[Puk98a] Aleksandr Pukhlikov. Birational automorphisms of Fano hypersurfaces. *Invent. Math.*, 134(2):401–426, 1998.

[Puk98b] Aleksandr Pukhlikov. Birational automorphisms of three-dimensional algebraic varieties with a pencil of del Pezzo surfaces. *Izv. Math*, 62(1):115–155, 1998.

[Puk00a] Aleksandr Pukhlikov. Birationally rigid double Fano hypersurfaces. *Sb. Math*, 191(5-6):883–908, 2000.

[Puk00b] Aleksandr Pukhlikov. Birationally rigid Fano fibrations. *Izv. Math*, 64(3):563–581, 2000.

[Puk01] Aleksandr Pukhlikov. Birationally rigid Fano complete intersections. *J. Reine Angew. Math.*, 541:55–79, 2001.

[Puk02] Aleksandr Pukhlikov. Birationally rigid Fano hypersurfaces with isolated singularities. *Sb. Math*, 193(3-4):445–471, 2002.

[Puk03] Aleksandr Pukhlikov. Birationally rigid Fano iterated double coverings. *Izv. Ross. Akad. Nauk Ser. Mat.*, 67(3):139–182, 2003.

[Puk04] Aleksandr Pukhlikov. Birationally rigid varieties with a pencil of Fano double covers. I. *Sb. Math*, 195(7-8):1039–1071, 2004.

[Puk06] Aleksandr Pukhlikov. Birational geometry of algebraic varieties with a pencil of Fano complete intersections. *Manuscripta Math.*, 121(4):491–526, 2006.

[Puk08] Aleksandr Pukhlikov. Birational geometry of Fano double coverings. *Sb. Math.*, 199(7-8):1225–1250, 2008.

[Puk09] Aleksandr Pukhlikov. Birational geometry of algebraic varieties with a pencil of Fano cyclic covers. *Pure Appl. Math. Q.*, 5(2, Special Issue: In honor of Friedrich Hirzebruch. Part 1):641–700, 2009.

[Puk10] Aleksandr Pukhlikov. On the self-intersection of a movable linear system. *J. Math. Sci. (N. Y.)*, 164(1):119–130, 2010.

[Puk13] Aleksandr Pukhlikov. *Birationally rigid varieties*, volume 190 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013.

[Puk14] Aleksandr Pukhlikov. Birationally rigid Fano complete intersections. II. *J. Reine Angew. Math.*, 688:209–218, 2014.

[Puk15] Aleksandr Pukhlikov. Birationally rigid Fano fibrations. II. *Izv. Math.*, 79(4):809–837, 2015.

[Puk16] Aleksandr Pukhlikov. The $4n^2$-inequality for complete intersection singularities. *ArXiv e-prints*, July 2016. arXiv:1607.02921.

[Sha13] Igor Shafarevich. *Basic algebraic geometry. 1.* Springer, Heidelberg, third edition, 2013. Varieties in projective space.

[Shr08] Constantin Shramov. Birational automorphisms of nodal quartic threefolds. *ArXiv e-prints*, March 2008. arXiv:0803.4348.

[Tia87] Gang Tian. On kähler-einstein metrics on certain kähler manifolds with $c_1(m) > 0$. *Inventiones mathematicae*, 89(2):225–246, 1987.

[Zha06] De-Qi Zhang. Rational connectedness of log $\mathbf{Q}$-Fano varieties. *J. Reine Angew. Math.*, 590:131–142, 2006.