EPQ for an unreliable production system with endogenous reliability and product deterioration

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Abstract

In this paper, we study an EPQ problem for a production line subjects to random shift from the in-control state (with high production rate) to out-of-control state (with low production rate). Different from previous research, we model the expected shift time as a controllable variable, based on the fact that, by investment on the resources, the reliability of the production line can be improved. In the mathematical model, we consider three possible scenarios: no shift, shift without demand shortage and shift with demand shortage, which are determined by the values of actual shift time and shifted production rate. Combining the three possible scenarios, the goal is to minimize the expected total cost per unit time by finding the optimal production time, as well as the optimal expected shift time. In addition, we extend the model to deterioration products and study the influence of product deterioration to the optimal decisions. Numerical examples are presented to illustrate the optimal solutions, followed by the sensitive analysis on important parameters. Comparing the optimal solutions under no reliability investment, reliability investment can help companies save more cost. Some other managerial insights are also proposed based on the numerical tests.

Keywords: EPQ; Unreliable production line; Endogenous reliability; Product deterioration

1. Introduction

The determination of the economic production quantity (EPQ) has been widely studied in the past few decades. By determining the optimal production batch size or production cycle, manufacturers can achieve their maximum profit or minimum cost. In the classical research on EPQ models, a common assumption is that the production process is reliable and the product quality is perfect. But in reality, it is almost impossible for companies to have a perfect reliable production line in a long run production process. The actual production line starts from an in-control state; then after running for some time, the production line may shift to an out-of-control state, and produce defective items or produce in a dropped production speed. For example, a large earthquake hitting Taiwan in

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the early morning of February 6, 2016, may impact iPhone 7 release. Taiwan Semiconductor Manufacturing Company (TSMC) is one of Apple's Ax processor suppliers. This earthquake may have caused far greater damage to TSMC manufacturing facilities than initially believed, which would result in more than 1% decrease in production in 2016 (Heisler, 2016).

In order to help companies deal with the inventory problems subject to production process unreliability, a lot of inventory models are studied. In most of studies on EPQ models, people treat the production line reliability as an exogenous factor and mainly concern about the scheduling, inventory or marketing problems (Ben-Daya et al., 2008; Chung et al., 2011; Jeang, 2012; Sarkar and Saren, 2016). However, in real industry, to mitigate the negative effects (e.g., raw material shortage, quality defect or productivity losses) on production line unreliability, companies have strong incentives to improve their production line reliability by investing in high quality machines, high skill workers, or advanced maintenance technologies.[1]

In this paper, we study an unreliable production system with endogenous reliability. Following Ben-Daya et al. (2008), the unreliability results in the drop of production rate when the system shifts from the in-control state to the out-of-control state. Firstly, we assume that the expected shift time can be delayed by investment in assets and technologies. And a longer in-control state means higher reliability, also results in higher investment cost. Secondly, comparing to the model proposed by Ben-Daya et al. (2008), we allow shortages in the model when the shifted production rate is smaller than the demand rate. In addition, the production rate is uncertain when the production line shifts to the out-of-control state. Finally, we take product deterioration into consideration.

Our research is closely related to two streams: (1) The inventory models for unreliable production line; (2) The inventory models for deterioration products.

(1) The inventory models for unreliable production line

In the EPQ models considering unreliable production line, some studies focused on the quality drop due to the state shift from in-control to out-of-control. Rosenblatt and Lee (1986) studied an inventory model considering that the machine will produce defective items after process breakdown. Their goal is to minimize the total average cost by finding the optimal cycle time and production quantity. In numerical examples, they also showed that the optimal cycle length is shorter than that of the case without process deterioration. Kim et al. (2001) extended their research by simultaneously considering the decisions of optimal inspection schedules and production cycle time. Chung and Hou (2003) considered the inventory model with random shifting time, defective products and shortages. Rahim and Al-Hajailan (2006) assumed that the defective rate in the out of control state is varying over time. Sana et al. (2007) developed an EPQ model with unreliable production process and assumed that some of the imperfect quality items can be sold at a lower price. Sana (2010a) assumed that the percentage of defective items produced in the out of control state follows a non-linearly trend, which is increasing in production rate and time. Defective products are restored to the original quality by rework. By setting the optimal production rate and production time length, the total cost is minimized. Sana (2010b) assumed that the defective rate of the products can be reduced by investing in some factors, e.g., labor, technology or resource. He developed a dynamic EMQ model to determine the optimal production path and the optimal investment over a finite decision horizon. Following Sana (2010b), Sarkar (2012) studied the simultaneous determination of price, production rate and investment when the production line is unreliable and demand is price

[1] http://www.lce.com/Introduction_to_Reliability_Excellence_Rx_78-item.html

linked. Atan and Snyder (2014) studied the EOQ models considering disruptions. Sarkar and Saren (2016) studied a more general model in which the defective rate in the in-control state is non-zero, then, in the out-of-control state, the defective rate rises. They also take the inspection errors and warranty cost into consideration. Their goal is to find an optimal inspection policy and production time to minimize the unit time total cost.

In addition to the research considers quality drop, another direction is the study on production rate drop. A common assumption of their research is that when the production line shifts to an out of control state, the production rate drops to zero and the production run stops, which is defined as 'full breakdown' (Glock, 2013). Abboud (1997) established an EMQ model by considering Poisson machine failure during production. He also proposed a simple approximation for the solution of the model. Then, Abboud et al. (2000) developed an economic lot sizing model with the consideration of random machine unavailability time and shortage. Later, Chakraborty et al. (2008) assumed that when the machine breaks down, a corrective and preventive repairing is conducted. But the repairing time length is a random variable. Chung et al. (2011) extended the model of Chakraborty et al. (2008) by considering deteriorating items with stochastic machine unavailability time and shortage. Gharbi et al. (2007) considered about the preventive maintenance policies to mitigate the effects of process breakdown. Widyadana and Wee (2012), Wee and Widyadana (2012) extended the model to deteriorating products. Giri et al. (2005) developed EPQ model with production rate related machine failure and random repair time. When the repairing time is long, demand shortage occurs. The goal is to minimize the annual expected total cost by determine the optimal production rate and production lot size. Jeang (2012) assumed that the process quality is controllable by setting a proper process mean. He consider about the optimal process quality design and the optimal production lot size under random machine breakdown and process deterioration.

The above literature studied the 'full breakdown' cases. However, in some situations, when negative impacts on the production line is not serious, the production line can be at any state between the 'fully-working' and 'full-breakdown' state. Gavish and Graves (1981) studied a production system with a single machine which has random production rates. A continuous review policy is conducted to minimize the expected total average cost per unit time. Iravani and Duenyas (2002) established a make-to-stock production/inventory system in which the production rate varies at several levels because of machine deterioration. They also considered about the maintenance policy to recover the production rate. They showed that under process deterioration, the maintenance policy is an efficient method to reduce the cost. Ben-Daya et al. (2008) considered about the EPQ problem with random shifting time and lower production speed after state shifting.

(2)The inventory models for deterioration products

According to Shah et al. (2013), deterioration is defined as decay, change or spoilage through which the items are not the same as its initial conditions. After a period of existence in market, the items lose the original economical value due to consumer preference, product quality or other reasons. Early research on deteriorating items can be dated back to 1963. An EOQ model with exponentially decaying inventory was initially proposed by Ghare and Schrader (1963). Based on the assumption of Ghare and Schrader (1963), more studies about deterioration inventory are conducted based on various realistic situations. Studies on models with different types of deterioration rate have been reported. The deterioration rate can be a constant parameter (He and He, 2010; He and Wang, 2010), exogenous time linked parameter (Balkhi and Tadj, 2008; Roy and Chaudhuri, 2009; Shah et al., 2013) or even a controllable parameter determined by preservation investment (Hsu et al., 2010; Dye and Hsieh, 2012). Some research assumed different types of demand rate. The demand types can be constant (Chung and Lin, 2001), ramp type (Mandal, 1998) price linked (Wee and Law, 1999) inventory level dependent (Wu et al., 2006) or the credit period linked (He and Huang, 2013, Jain and Aggarwal, 2012). Feng et al. (2015) studied a dynamic problem for perishable products. Also, in some research, the deterioration product models were developed under uncertain production process. Chung et al. (2011) studied an EPQ model for deteriorating items with random machine breakdown and shortage. Widyadana and Wee (2012) for deteriorating items with preventive maintenance policy and random machine breakdown .Wee and Widyadana (2012) studied an EPQ model for deteriorating items with rework of defective items and stochastic maintenance time.

The rest of the paper is organized as follows. Section 2 is the introduction of assumptions and notation. In Sections 3, we set up the mathematical model. In section 4, we extend the model to deterioration items. In section 5, we present numerical examples to illustrate managerial insights. Sensitive analysis is conducted and some results are obtained. The last section provides conclusions and directions for future research.

2. Assumptions and notation

We consider a single product production system in which the production system at any time can be either in the in-control state and out-of-control state. The production rate will drop to a lower level when the system shifts to the out-of-control state. The reliability can be characterized by the expectation of shifting time. When the expectation of the time before shifting to the out-of-control state is longer, so called shifting time in this research, the system is more reliable. By investment, the expected shifting time can be prolonged, so as to increase the reliability of the system. The goal is to find the optimal expected shifting time and production time length or production cycle to minimize the expected total cost per unit time in an infinite decision horizon.

2.1. The endogenous reliability

Following the assumption in Ben-Daya et al. (2008), in the in-control state, the production line is reliable with a constant production rate P . However, in the out-of-control state, the production rate drops to αP . The shifting time point s is a random variable, with probability density function $f(s)$ and cumulative distribution function $F(s)$. According to previous research, the shifting time point s follows an exponential distribution (Ben-Daya et al., 2008; Chung et al., 2011; Jeang, 2012; Sarkar and Saren, 2016). The probability density function is $f(s) = e^{-s/\lambda}/\lambda$. The corresponding cumulative distribution function is $F(s) = 1 - e^{-s/\lambda}$. And, the expectation of the shifting time s is $E[s] = \int_0^\infty sf(s)ds = \lambda$. Based on the previous research, people studied the influence of distribution parameter $1/\lambda$. When the expected in control time length $E[s] = \lambda$ is larger, the system is more reliable, which leads to more profit or less cost. **Figure 1** shows the patterns for the probability density function and the cumulative distribution function for different values of λ. When λ is smaller, the production system may shift from the in-control state to the out-of-control state at an earlier time. So here we use λ to denote the reliability of the production system, a larger λ indicates a more reliable system. We call the parameter λ as the reliability index.

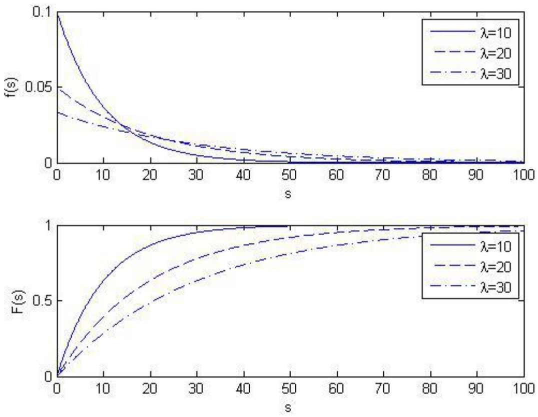


Figure 1. $f(s)$ and $F(s)$ w.r.t. λ.

A company can enhance the reliability of their production system by investing in technology, recruiting highly educated workers or cooperating with more reliable raw material suppliers. Therefore we assume that the reliability index λ is controllable by investment. If the initial production system reliability is λ_0, the reliability index can be increased to $\lambda_1(\lambda_1 > \lambda_0)$ with a unit time investment $C_I(\lambda_1)$. The pattern of the investment cost function is increasing and convex in λ_1, i.e., $C_I^{'}(\lambda_1) > 0$ and $C_I^{''}(\lambda_1) > 0$, which follows the law of diminishing marginal utility (DMU) and with condition $C_I(\lambda_1 \mid \lambda_1 = \lambda_0) = 0$. We denote the probability density function and the cumulative probability function under investment as $f_I(s)$ and $F_I(s)$, respectively.

2.2. The randomness of α

α denotes the proportion of the remaining production ability during the out-of-control state. According to previous research of inventory models on production reliability, when the production line shifts from in-control to out-of-control state, the production rate either drops to zero, i.e., $\alpha = 0$ (Jeang, 2012), or drops to a priori known lower production rate, i.e., $0 < \alpha < 1$ (Ben-Daya et al., 2008). However, in reality, the damage to the production line is affected by numerous factors, and the drop of the production rate cannot be known before the out-of-control state comes. Firms can only make predictions about the distribution of α based on previous data. To make the study more realistic, we set α as a random variable in interval $[\underline{\alpha}, \overline{\alpha}]$ with probability density function $g(\alpha)$ and cumulative probability function $G(\alpha)$.

2.3. Other assumptions

(1) Demand rate D is constant.

(2) Initial production rate P is constant and greater than the demand rate.

(3) Shortage is allowed and unfilled demand is totally lost with penalty cost.

(4) During the production run, production line cannot be restored.

(5) When production run ends, a fixed cost is paid to restore the production system.

(6) The restoration time is zero so that the damage of the production line has no effect to the production of the next period.

(7)The random variables α and s are not correlated.

There are two reasons that we study the unrecoverable system. Firstly, for some manufacturers, when the production period is relatively short, it is uneconomical to recover the production system with large amount of cost. Secondly, to recover the production system, workers have to stop the production line. For products produced by deteriorating raw materials, stopping the production line causes deterioration cost. So it is better to go on producing instead of stopping the production line.

2.4. Notation

We define the following notation:

Decision variables

T: Production time.

λ_1: The reliability index of shift time after investment, which equals to the expected shift time.

Parameters

P: Initial production rate.

D: Demand rate.

T_m: The time when inventory level drops to zero.

α: Proportion of the remaining production rate when shift to the out-of-control state, which is a random parameter.

$g(\alpha)$: Probability density function of α.

$G(\alpha)$: Cumulative probability function of α.

s: Shifting time point, a random variable follows exponential distribution.

$f(s)$: Probability density function of s.

$F(s)$: Cumulative probability function of s.

λ_0: The initial reliability parameter.

A: Fixed starting cost.

M: Restoration cost to transfer the out of control state to the in control state, which is linearly related to α, i.e., $M = M_0(1-\alpha)$. When the damage of the production line is low (α is large), the restoration cost is low; however, when the damage is high (α is small), the restoration cost is high.

M_0: The maximum restoration cost.

h: Inventory holding cost per unit item per unit time.

θ: Deterioration rate.

c_d: Per unit deterioration cost.

c_p: Per unit penalty cost for lost sales.

TC_i^j: The total cost for scenario i in case j.

T_i^j: The cycle length for scenario i in case j.

ETC^j: Expected total cost for a production cycle under case j.

ET^j: Expected time length for a production cycle under case j.

ATC^j: Expected unit time cost cycle under case j.

T_0: The optimal production time without reliability investment.

ATC_0^j: The unit time total profit without reliability investment.

j: Superscript $j=\{d,nd\}$ to denote the case with and without product deterioration.

i: Superscript $i=\{1,2,3\}$ to denote the without state shift ($i=1$), with state shift and no shortage ($i=2$), with state shift and shortage ($i=3$).

3. Model without product deterioration

In this section, based on the assumptions described above, we build the mathematical model without product deterioration.

3.1. Inventory patterns under different scenarios

In Ben-Daya et al. (2008), they assumed that the condition $\alpha P > D$ always holds, which avoids demand shortage during each production cycle. However, when the production rate in the out-of-control state is lower than the demand, shortage may occur. In addition to the two scenarios analyzed in Ben-Daya et al. (2008), when α is smaller than D/P, there is a third scenario in which shift occurs and demand shortage happens. In this scenario, companies will have penalty cost for the demand shortage. The inventory patterns for the three scenarios are shown in **Figure 2**.

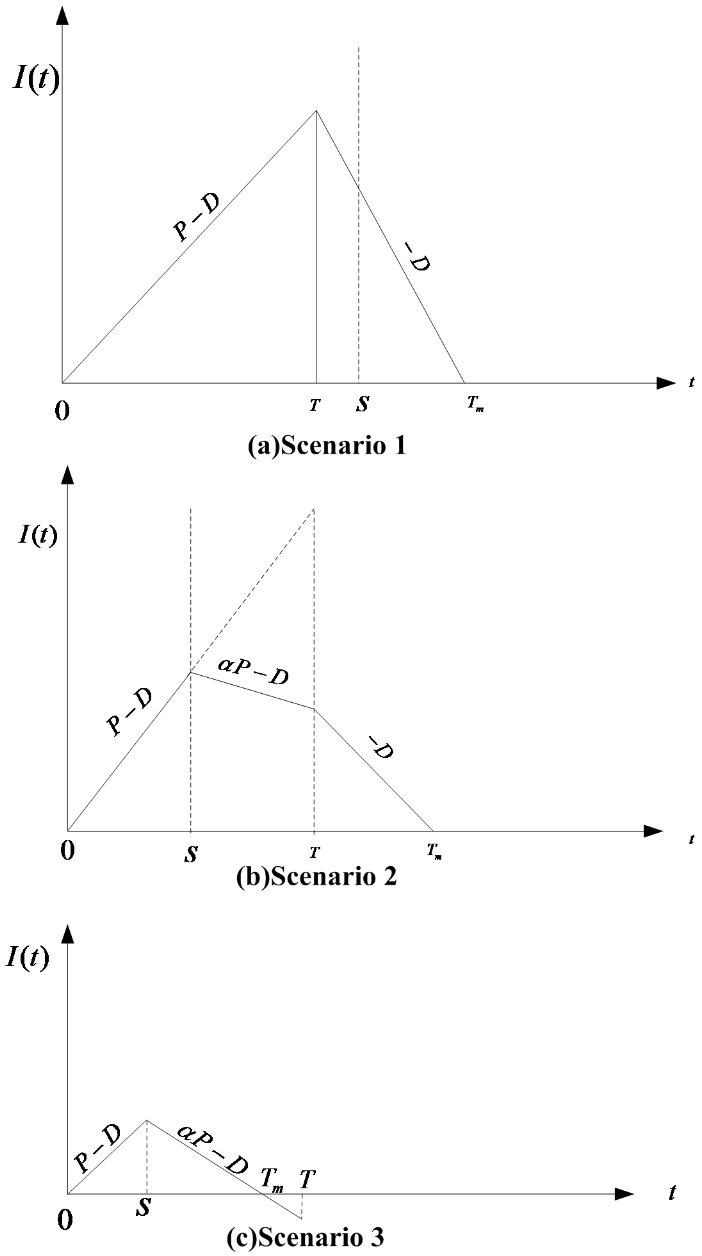


Figure 2. Inventory patterns for Scenarios 1-3 for non-deterioration products.

The three scenarios with different values of s and α can be summarized as follows:

Scenario 1**:** ($s > T, \underline{\alpha} < \alpha < \overline{\alpha}$). No shift during the production run.

In this scenario, as depicted in **Figure 2 (a)** during the production time, the shift does not happen. Production rate is P during the production run. The total cost consists of inventory holding cost, starting cost and reliability investment cost. By calculation, we obtain the total cost for a cycle of scenario 1 as

$$TC_1^{nd}(T) = \{\text{Inventory holding cost+Starting cost+Reliability investment}\}$$
$$= \frac{h}{2}(\frac{P}{D} - 1)PT^2 + A + C_I(\lambda_1)PT/D \tag{1}$$

Then, the cycle length of scenario 1 as

$$T_1^{nd}(T) = PT/D \tag{2}$$

Scenario 2: ($\max\left\{\frac{D-\alpha P}{(1-\alpha)P}T, 0\right\} < s < T, \underline{\alpha} < \alpha < \bar{\alpha}$). Shift occurs without demand shortage.

In this scenario, as shown in **Figure 2 (b),** during the production run, shift happens at time s . Production rate in time interval $[0,s]$ is P , and in time interval $[s,T]$ is αP . All the demands in a cycle can be satisfied.

The threshold $s = \frac{D-\alpha P}{(1-\alpha)P}T$ can be obtained by equaling T_m in (A.8) or (A.11) to T , which means when $s = \frac{D-\alpha P}{(1-\alpha)P}T$, no demand shortage happens and all the products are consumed at the end of the production run. To make sure there is no demand shortage, the shift time should satisfy $s > \frac{D-\alpha P}{(1-\alpha)P}T$. Recall that $s > 0$, so when $\max\left\{\frac{D-\alpha P}{(1-\alpha)P}T, 0\right\} < s < T$, there is no shortage. On the contrary, when $s < \max\left\{\frac{D-\alpha P}{(1-\alpha)P}T, 0\right\}$ in scenario 3, demand shortage happens.

The total cost for scenario 2 consists of inventory holding cost, starting cost, restoration cost, and the reliability investment cost.

By calculation, we obtain the total cost in a cycle for scenario 2:

$$TC_2^{nd}(T,s) = \{\text{Inventory holding cost+Starting cost+Restoration cost+Reliability cost}\}$$
$$= \left[\frac{h(P-D)P}{2D}s^2 + \frac{h(\alpha P - D)\alpha P}{2D}(T-s)^2 + \frac{h(P-D)\alpha P}{D}(T-s)s\right] + A + M_0(1-\alpha)$$
$$+ C_I(\lambda_1)\left[\frac{(1-\alpha)Ps}{D} + \frac{\alpha PT}{D}\right] \tag{3}$$

As well as the cycle length

$$T_2^{nd}(T,s) = \frac{(1-\alpha)Ps}{D} + \frac{\alpha PT}{D}. \tag{4}$$

***Scenario 3*:** ($s < \max\left\{\frac{D-\alpha P}{(1-\alpha)P}T, 0\right\}, \underline{\alpha} < \alpha < \bar{\alpha}$). Shift occurs with demand shortage.

As shown in **Figure 2 (c)**, production rate in time interval $[0,s]$ is P, and for time $[s,T]$ is αP. Comparing to scenario 2, when $s < \max\left\{\frac{D-\alpha P}{(1-\alpha)P}T, 0\right\}$, quantity of the produced products can not satisfy all the demands even in the production run. In this scenario, the total cost consists of inventory holding cost, starting cost, restoration cost, reliability investment cost, as well as the shortage penalty cost.

The total cost for a cycle of scenario 3 can be expressed as

$$TC_3^{nd}(T,s) = \begin{Bmatrix} \text{Inventory holding cost+Starting cost+Restoration cost} \\ \text{+Shortage cost+Reliability investment} \end{Bmatrix}$$
$$= \frac{h(P-D)(1-\alpha)P}{2(D-\alpha P)}s^2 + A + M_0(1-\alpha) + c_p(D-\alpha P)\left[T - \frac{(1-\alpha)Ps}{D-\alpha P}\right] + C_I(\lambda_1)T \tag{5}$$

The cycle length for scenario 3 is

$$T_3^{nd}(T) = T\ . \tag{6}$$

Calculations of the inventory level, total cost and cycle length for scenarios 1-3 are presented in **Appendix A**.

3.2. Expected unit time total cost for the production system

After obtaining the cost functions and cycle length functions under the three scenarios, we can get the expected unit time total cost based on the probability theories.

The corresponding expected profit function for a cycle and the expected production cycle length are as follows

$$ETC^{nd}(T,\lambda_1) = \int_{\underline{\alpha}}^{\bar{\alpha}}\int_T^{\infty} TC_1^{nd}(T)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha + \int_{\underline{\alpha}}^{\bar{\alpha}}\int_{\max\left\{\frac{D-\alpha P}{(1-\alpha)P}T,0\right\}}^{T} TC_2^{nd}(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha$$
$$+\int_{\underline{\alpha}}^{\bar{\alpha}}\int_0^{\max\left\{\frac{D-\alpha P}{(1-\alpha)P}T,0\right\}} TC_3^{nd}(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha \tag{7}$$

$$ET^{nd}(T,\lambda_1) = \int_{\underline{\alpha}}^{\bar{\alpha}}\int_T^{\infty} T_1^{nd}(T)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha + \int_{\underline{\alpha}}^{\bar{\alpha}}\int_{\max\left\{\frac{D-\alpha P}{(1-\alpha)P}T,0\right\}}^{T} T_2^{nd}(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha$$
$$+\int_{\underline{\alpha}}^{\bar{\alpha}}\int_0^{\max\left\{\frac{D-\alpha P}{(1-\alpha)P}T,0\right\}} T_3^{nd}(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha \tag{8}$$

So, following the approach in Jeang (2012), we derive the expected unit time total cost function as

$$ATC^{nd}(T,\lambda_1) = \frac{ETC^{nd}(T,\lambda_1)}{ET^{nd}(T,\lambda_1)}\ . \tag{9}$$

The final optimization problem is to minimize the function $ATC^{nd}(T,\lambda_1)$.

P1: $\underset{T,\lambda_1}{Min}\left\{ATC^{nd}(T,\lambda_1)\right\}$,

$$S.T. \quad T>0, \lambda_1 \ge \lambda_0. \tag{10}$$

Lemma 1. *In the model without deterioration, for the distribution interval* $[\underline{\alpha},\bar{\alpha}]$ *of* α *, when the lower bound* $\underline{\alpha} > D/P$ *, there is no possibility of scenario 3; otherwise, scenario 3 exists.*

Proof: When $\underline{\alpha} > D/P$, $\max\left\{\frac{D-\alpha P}{(1-\alpha)P}T, 0\right\} = 0$, then

$$\int_{\underline{\alpha}}^{\bar{\alpha}}\int_0^{\max\left\{\frac{D-\alpha P}{(1-\alpha)P}T,0\right\}} TC_3^{nd}(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha = \int_{\underline{\alpha}}^{\bar{\alpha}}\int_0^0 TC_3^{nd}(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha = 0$$

and

$$\int_{\underline{\alpha}}^{\bar{\alpha}}\int_0^{\max\left\{\frac{D-\alpha P}{(1-\alpha)P}T,0\right\}} T_3^{nd}(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha = \int_{\underline{\alpha}}^{\bar{\alpha}}\int_0^0 T_3^{nd}(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha = 0 .$$

So scenario 3 doesn't exist.

When $\underline{\alpha} < D/P$, for $\alpha \in [\underline{\alpha}, \min\{D/P, \bar{\alpha}\}]$ $\max\left\{\frac{D-\alpha P}{(1-\alpha)P}T, 0\right\} > 0$, then

$$\int_{\underline{\alpha}}^{\bar{\alpha}}\int_0^{\max\left\{\frac{D-\alpha P}{(1-\alpha)P}T,0\right\}} TC_3^{nd}(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha > 0 \quad \text{and}$$

$$\int_{\underline{\alpha}}^{\bar{\alpha}}\int_0^{\max\left\{\frac{D-\alpha P}{(1-\alpha)P}T,0\right\}} T_3^{nd}(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha > 0 .$$

So, the possibility of scenario 3 is positive and scenario 3 exists. □

4. Extension to deterioration products

In this section, we extend the model to products with deterioration.

4.1. Inventory patterns under different scenarios

The inventory patterns for Scenarios 1-3 are shown in **Figure 3**.

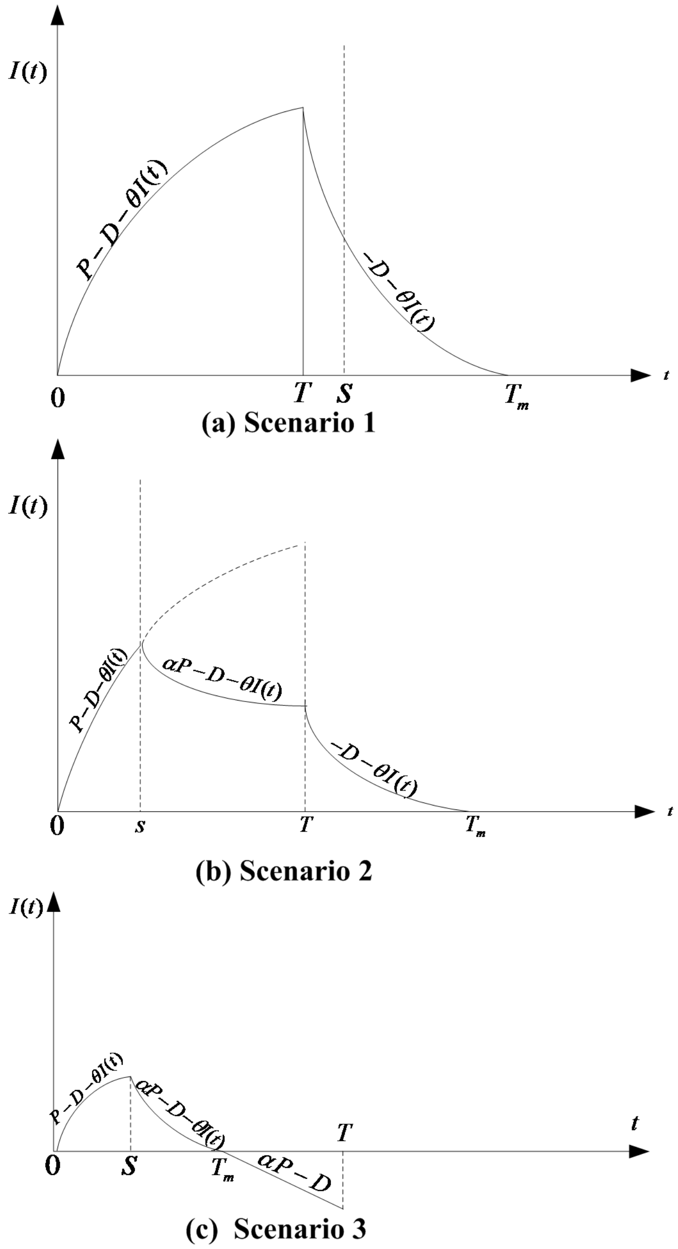


Figure 3. Inventory patterns of Scenarios 1-3 for deterioration products.

Also, following the approach in section 3, we summarize the three scenarios as follows w.r.t. different values of s and α .

Scenario 1**:** ($s > T, \underline{\alpha} < \alpha < \overline{\alpha}$). No shift during the production run.

Production rate during the whole production run is P (See **Figure 3 (a)**).

The total cost during a production cycle for scenario 1 is

$$
\begin{aligned}
TC_1^d(T) &= \left\{ \text{Inventory holding cost+Starting cost+Deterioration cost+Reliability investment} \right\} \\
&= h\left\{ \frac{P-D}{\theta^2}\left[\theta T + e^{-\theta T} - 1 \right] + \frac{D}{\theta^2}\left[e^{\theta(T_m - T)} - \theta(T_m - T) - 1 \right] \right\} + A + c_d(pT - dT_m) + C_I(\lambda_1)T_m
\end{aligned}
\tag{11}
$$

The total cycle length for scenario1 is

$$T_1^d(T) = T_m = \frac{1}{\theta}\ln[\frac{P-D}{D}(1-e^{-\theta T})+1]+T . \quad (12)$$

Scenario 2: ($\max\left\{\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P},0\right\} < s < T, \underline{\alpha} < \alpha < \bar{\alpha}$). Shift occurs without shortage.

Shift occurs during the production run and all the demand during the production run can be satisfied. Production rate in time interval $[0,s]$ is P, and for time $[s,T]$ is αP (See **Figure 3 (b)**). The threshold value $\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P}$ can be obtained by equaling T_m in (B.8) to T. When $s > \max\left\{0,\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P}\right\}$, $T_m > T$, which implies that there is no shortage during the production run. On the contrary, when $s < \max\left\{0,\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P}\right\}$, shortage happens.

The total cost during a production cycle for scenario 2 is

$$\begin{aligned} TC_2^d(T,s) &= \begin{Bmatrix} \text{Inventory holding cost+Starting cost+Restoration cost} \\ \text{+Deterioration cost+Reliability investment} \end{Bmatrix} \\ &= h\begin{Bmatrix} \frac{P-D}{\theta^2}(\theta s+e^{-\theta s}-1)+\frac{\alpha P-D}{\theta}(T-s) \\ -\frac{(1-\alpha)Pe^{\theta s}-P+D}{\theta^2}(e^{-\theta T}-e^{-\theta s})+\frac{D}{\theta^2}\left[e^{\theta(T_m-T)}-\theta(T_m-T)-1\right] \end{Bmatrix} \\ &\quad +A+M_0(1-\alpha)+c_d(Ps+\alpha P(T-s)-DT_m)+C_I(\beta)T_m \end{aligned} \quad (13)$$

The total cycle length for scenario 2 is

$$T_2^d(T,s) = T_m = \frac{1}{\theta}\ln[\frac{\alpha P-D}{D}+\frac{(1-\alpha)P}{D}e^{\theta(s-T)}-\frac{P-D}{D}e^{-\theta T}+1]+T . \quad (14)$$

Scenario 3: ($0 < s < \max\left\{\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P},0\right\}, \underline{\alpha} < \alpha < \bar{\alpha}$). Shift occurs with shortage.

In this scenario, shift occurs during the production run and the produced products can not satisfy the order in the production run (See **Figure 3 (c)**). Shortage will occur with shortage penalty cost. Production rate in time interval $[0,s]$ is P, and for time $[s,T]$ is αP.

The total cost during a production cycle for scenario 3 is

$$TC_3^d(T,s)=\begin{Bmatrix}\text{Inventory holding cost+Starting cost+Restoration cost}\\ \text{+Deterioration cost+Shortage cost+Reliability investment}\end{Bmatrix}$$

$$=h\left\{\frac{P-D}{\theta^2}(\theta s+e^{-\theta s}-1)+\frac{\alpha P-D}{\theta}(T_m-s)-\frac{(1-\alpha)Pe^{\theta s}-P+D}{\theta^2}(e^{-\theta T_m}-e^{-\theta s})\right\} \quad (15)$$

$$+A+M_0(1-\alpha)+c_d\left[Ps+\alpha P(T_m-s)-DT_m\right]+c_p(D-\alpha P)(T-T_m)+C_I(\lambda_1)T_m$$

The cycle length for scenario 3 is

$$T_3^d(T)=T\ . \quad (16)$$

The calculations of the total cost and cycle length for scenarios 1-3 for deterioration products are presented in **Appendix B**.

4.2. Expected unit time total cost for the production system

After obtaining the cost functions under different scenarios, we model the expected unit time total cost based on the probability theories. The corresponding expected profit function and the expected production cycle length are:

$$ETC^d(T,\lambda_1)=\int_{\underline{\alpha}}^{\overline{\alpha}}\int_T^{\infty}TC_1^d(T)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha+\int_{\underline{\alpha}}^{\overline{\alpha}}\int_{\max\left\{\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P},0\right\}}^{T}TC_2^d(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha$$

$$+\int_{\underline{\alpha}}^{\overline{\alpha}}\int_0^{\max\left\{\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P},0\right\}}TC_3^d(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha \quad (17)$$

$$ET^d(T,\lambda_1)=\int_{\underline{\alpha}}^{\overline{\alpha}}\int_T^{\infty}T_1^d(T)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha+\int_{\underline{\alpha}}^{\overline{\alpha}}\int_{\max\left\{\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P},0\right\}}^{T}T_2^d(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha$$

$$+\int_{\underline{\alpha}}^{\overline{\alpha}}\int_0^{\max\left\{\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P},0\right\}}T_3^d(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha \quad (18)$$

The expected total average cost is

$$ATC^d(T,\lambda_1)=\frac{ETC^d(T,\lambda_1)}{ET^d(T,\lambda_1)}\ . \quad (19)$$

The optimization problem is to minimize the function $ATC^d(T,\lambda_1)$.

P2: $$\underset{T,\lambda_1}{Min}\left\{ATC^d(T,\lambda_1)\right\},$$

$$S.T.\qquad T>0,\lambda_1\geq\lambda_0\ . \quad (20)$$

For the distribution interval $[\underline{\alpha},\overline{\alpha}]$ of α, when the lower bound $\underline{\alpha}>D/P$, there is no possibility of scenario 3; otherwise, scenario 3 exists.

Lemma 2. *In the model with deterioration, for the distribution interval* $[\underline{\alpha},\overline{\alpha}]$ *of* α*, when the lower bound* $\underline{\alpha}>D/P$*, there is no possibility of scenario 3; otherwise, scenario 3 exists.*

Proof. When $\underline{\alpha} > D/P$, for all $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $\max\left\{\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P}, 0\right\} = 0$, then

$$\int_{\underline{\alpha}}^{\overline{\alpha}}\int_0^{\max\left\{\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P},0\right\}} TC_3^d(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha = \int_{\underline{\alpha}}^{\overline{\alpha}}\int_0^0 TC_3^d(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha = 0$$

and $\int_{\underline{\alpha}}^{\overline{\alpha}}\int_0^{\max\left\{\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P},0\right\}} T_3^d(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha = \int_{\underline{\alpha}}^{\overline{\alpha}}\int_0^0 T_3^d(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha = 0$.

So scenario 3 doesn't exist.

When $\underline{\alpha} < D/P$, for all $\alpha \in [\underline{\alpha}, \min\{D/P, \overline{\alpha}\}]$ $\max\left\{\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P}, 0\right\} > 0$, then

$$\int_{\underline{\alpha}}^{\overline{\alpha}}\int_0^{\max\left\{\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P},0\right\}} TC_3^d(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha > 0$$

and $\int_{\underline{\alpha}}^{\overline{\alpha}}\int_0^{\max\left\{\frac{1}{\theta}\ln\frac{(D-\alpha P)e^{\theta T}+P-D}{(1-\alpha)P},0\right\}} T_3^d(T,s)f_I(s)g(\alpha)\mathrm{d}s\mathrm{d}\alpha > 0$.

So, the possibility of scenario 3 is positive and scenario 3 exists. □

5. Numerical examples and sensitive analysis

Due to the complexity of the function, it is hard to obtain the explicit mathematical solutions. So we use the software Matlab R2015a as a tool to obtain the optimal solutions in the numerical examples. We set the unit time investment cost function as

$$C_I(\lambda_1) = k(\lambda_1 - \lambda_0)^2/2, \lambda_1 \in [\lambda_0, \infty), \tag{21}$$

which is an increasing and convex function of λ_1. k is the investment cost coefficient parameter.

The parameter α follows a uniform distribution with density function

$$g(\alpha) = \begin{cases} 1/(\overline{\alpha} - \underline{\alpha}), & 0 < \underline{\alpha} < \alpha < \overline{\alpha} < 1 \\ 0, & \text{Otherwise} \end{cases}.$$

The values of related parameters are listed in **Table 1.**

Table 1. Values of system parameters.

$P = 600$ units per time	$A = \$500$ per cycle
$D = 300$ units per time	$h = \$1.0$ per unit time per item
$c_p = \$200$ per unit	$\lambda_0 = 3$ months
$M_0 = \$5000$ per cycle	$k = 10$

5.1. Illustrative examples

Example 1: $\{\underline{\alpha}, \overline{\alpha}\} = \{0.6, 0.8\}$, no deterioration.

Substitute the data into the equation (7), (8) and (9), we have

$$ATC^{nd}(T, \lambda_1) = \frac{\int_{0.6}^{0.8}\int_{T}^{\infty}\{15T^2+500\}\frac{5e^{-s/\lambda_1}}{\lambda_1}\mathrm{d}s\mathrm{d}\alpha + \int_{0.6}^{0.8}\int_{0}^{T}\begin{Bmatrix}15s^2+(60\alpha^2-30\alpha)(T-s)^2+ \\ 60\alpha(T-s)s+500+5000(1-\alpha)\end{Bmatrix}\frac{5e^{-s/\lambda_1}}{\lambda_1}\mathrm{d}s\mathrm{d}\alpha}{\int_{0.6}^{0.8}\int_{T}^{\infty}\{2T\}\frac{5e^{-s/\lambda_1}}{\lambda_1}\mathrm{d}s\mathrm{d}\alpha + \int_{0.6}^{0.8}\int_{0}^{T}\{2(1-\alpha)s+2\alpha T\}\frac{5e^{-s/\lambda_1}}{\lambda_1}\mathrm{d}s\mathrm{d}\alpha} + \frac{(\lambda_1-3)^2}{200}$$

In this example, the lower bound of α is larger than $D/P = 0.5$. So there is no possibility of demand shortage. As is shown in **Figure 4 (a)**, the unit time profit function is jointly convex in T and λ_1. The optimal decision is $T^{nd*} = 1.42 months$, $\lambda_1^{nd*} = 5.21$, and $ATC^{nd*} = \$540.791$.

Example 2: $\{\underline{\alpha}, \overline{\alpha}\} = \{0.4, 0.6\}$, no deterioration.

Substitute the values of parameters into equation (7), (8) and (9).

$$ATC^{nd}(T, \beta) = \frac{\left\{\begin{array}{l}\int_{0.4}^{0.6}\int_{T}^{\infty}\{15T^2+500\}\frac{5e^{-s/\lambda_1}}{\lambda_1}\mathrm{d}s\mathrm{d}\alpha + \int_{0.4}^{0.6}\int_{\max\left\{\frac{1-2\alpha}{2(1-\alpha)}T,0\right\}}^{T}\begin{Bmatrix}15s^2+(60\alpha^2-30\alpha)(T-s)^2+ \\ 60\alpha(T-s)s+500+5000(1-\alpha)\end{Bmatrix}\frac{5e^{-s/\lambda_1}}{\lambda_1}\mathrm{d}s\mathrm{d}\alpha \\ +\int_{0.4}^{0.5}\int_{0}^{\frac{1-2\alpha}{2(1-\alpha)}T}\left\{\frac{30(1-\alpha)}{1-2\alpha}s^2+6000[(1-2\alpha)T-2(1-\alpha)s]+500+5000(1-\alpha)\right\}\frac{5e^{-s/\lambda_1}}{\lambda_1}dsd\alpha\end{array}\right\}}{\int_{0.4}^{0.6}\int_{T}^{\infty}\{2T\}\frac{5e^{-s/\lambda_1}}{\lambda_1}\mathrm{d}s\mathrm{d}\alpha + \int_{0.4}^{0.6}\int_{\max\left\{\frac{1-2\alpha}{2(1-\alpha)}T,0\right\}}^{T}\{2(1-\alpha)s+2\alpha T\}\frac{5e^{-s/\lambda_1}}{\lambda_1}\mathrm{d}s\mathrm{d}\alpha + \int_{0.4}^{0.5}\int_{0}^{\frac{1-2\alpha}{2(1-\alpha)}T}\{T\}\frac{5e^{-s/\lambda_1}}{\lambda_1}dsd\alpha} + \frac{(\lambda_1-3)^2}{200}$$

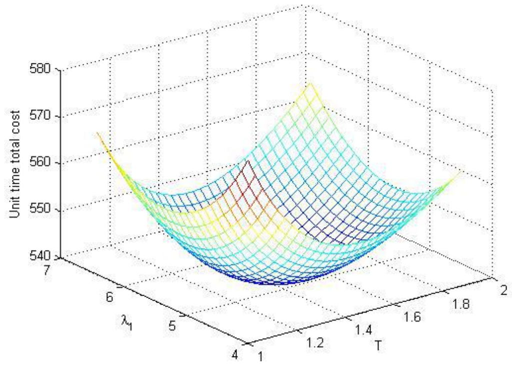
In this example, the lower bound of α is less than $D/P = 0.5$. So there is possibility of demand shortage. As is shown in **Figure 4 (b)**, the unit time profit function is jointly convex in T and λ_1. The optimal production time $T^{nd*} = 1.35 month$, $\lambda_1^{nd*} = 6.28$, and $ATC^{nd*} = \$656.361$.

Example 3: $\{\underline{\alpha}, \overline{\alpha}\} = \{0.6, 0.8\}$, $\theta = 0.02$, $c_d = \$20$.

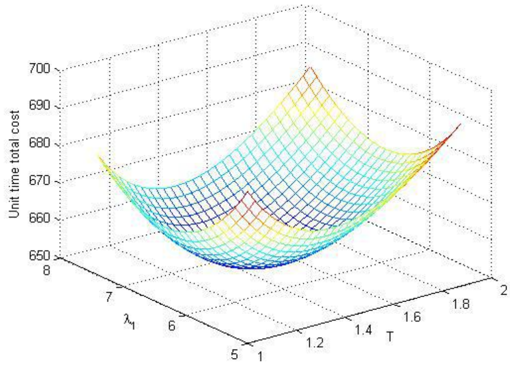
Similar to example 1, when the lower bound of α is higher than $D/P = 0.5$, no shortage occurs. Substitute the data into equation (17), (18) and (19). As shown in **Figure 4 (c)**, the cost function is jointly convex in production time and reliability increment. And the optimal decision is $T^{d*} = 1.13 month$, $\lambda_1^{d*} = 5.40$, and $ATC^{d*} = \$626.669$.

Example 4: $\{\underline{\alpha}, \overline{\alpha}\} = \{0.4, 0.6\}$, $\theta = 0.02$, $c_d = \$20$.

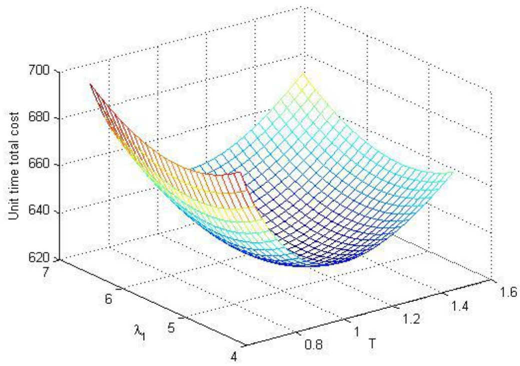
Similar to example 2, we substitute the data into equations (17), (18) and (19). When the lower bound of α is lower than $D/P = 0.5$, shortage occurs. As shown in **Figure 4 (d)**, the cost function is jointly convex in production time and reliability increment. And the optimal decision is $T^{d*} = 1.10 month$, $\lambda_1^{d*} = 6.30$, and $ATC^{d*} = \$728.987$.



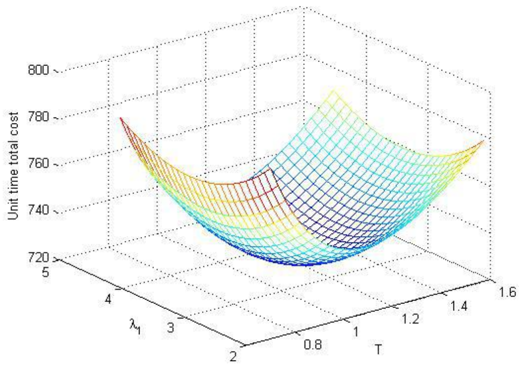
(a) Example 1: $\alpha \sim U[0.6, 0.8]$, $\theta = 0$



(b) Example 2: $\alpha \sim U[0.4, 0.6]$, $\theta = 0$



(c) Example 3: $\alpha \sim U[0.6, 0.8]$, $\theta = 0.02$



(d) Example 4: $\alpha \sim U[0.4, 0.6]$, $\theta = 0.02$

Figure 4. Unit profit function w.r.t. T and λ_1 for Examples 1-4.

5.2. Sensitive analysis

In this subsection, we conduct sensitive analysis based on the examples. We also compare the optimal decisions and costs in the proposed model to that in the model without reliability investment. The following finding can be drawn from the analysis results in Tables 2 and 3 for both non-deterioration and deterioration products.

(1) Sensitive results of inventory holding cost h.

The optimal production cycle time, and minimum cost are decreasing in inventory holding cost. However, the optimal reliability index is increasing in inventory holding cost. This implies that, for a higher inventory holding cost, companies should set a shorter production cycle time, and increase the reliability of the production line.

(2) Sensitive results of restoration cost M_0.

The optimal reliability index, production cycle time and minimum cost are increasing in restoration cost. Because the restoration cost is fixed in a cycle, when the cycle time is longer, the unit time restoration cost is lower. Also, to lower down the possibility of production process breakdown, company should make the production system more reliable. So, when the restoration cost increases, to achieve a lower unit time total cost, company should set a longer production time and increase the reliability of the production line.

(3) Sensitive results of investment cost coefficient k .

When the cost coefficient of the investment increases, the optimal expected shifting time decreases while the production cycle time and minimum cost increases. Because a higher k means that company should invest more cost to achieve the same reliability. To balance the operational cost with the investment cost, the company may set a reasonable reliability and a longer production cycle time.

Table 2. Sensitive results for parameters $h, M_0, k, \lambda_0, c_p$ when $\alpha \sim U[0.4, 0.6]$.

		$\theta = 0$					$\theta = 0.02$				
		λ_1^{nd*}	T^{nd*}	ATC^{nd*}	T_0^{nd*}	ATC_0^{nd*}	λ_1^{d*}	T^{d*}	ATC^{d*}	T_0^{d*}	ATC_0^{d*}
h	0.6	6.22	1.72	572.052	1.92	730.104	6.29	1.30	658.279	1.34	820.235
	0.8	6.26	1.50	616.646	1.65	775.578	6.30	1.19	695.134	1.22	858.675
	1	6.28	1.35	656.361	1.47	816.085	6.30	1.10	728.987	1.12	893.715
	1.2	6.29	1.23	692.466	1.33	852.890	6.31	1.03	760.495	1.05	926.196
	1.4	6.30	1.14	725.815	1.23	886.850	6.32	0.97	790.078	0.99	956.576
M_0	1000	4.63	1.32	478.610	1.33	505.864	4.71	1.08	556.234	1.06	586.639
	3000	5.60	1.34	573.624	1.39	661.405	5.64	1.09	648.367	1.09	740.355
	5000	6.28	1.35	656.361	1.47	816.085	6.30	1.10	728.987	1.12	893.715
	7000	6.83	1.35	731.815	1.56	969.729	6.83	1.11	802.691	1.15	1046.691
	9000	7.27	1.36	802.226	1.67	1122.138	7.27	1.11	871.500	1.19	1199.270
k	6	7.20	1.34	629.010	1.47	816.085	7.23	1.10	701.323	1.12	893.715
	8	6.66	1.34	644.354	1.47	816.085	6.68	1.10	716.847	1.12	893.715
	10	6.28	1.35	656.361	1.47	816.085	6.30	1.11	728.987	1.12	893.715
	12	5.99	1.35	666.152	1.47	816.085	6.01	1.11	738.917	1.12	893.715
	14	5.76	1.35	674.398	1.47	816.085	5.78	1.10	747.285	1.12	893.715
λ_0	1	5.39	1.36	732.679	3.22	1351.809	5.41	1.11	805.781	1.62	1539.674
	2	5.81	1.35	691.739	1.73	995.705	5.83	1.10	764.600	1.18	1084.838
	3	6.28	1.35	656.361	1.47	816.085	6.30	1.10	728.987	1.12	893.715
	4	6.81	1.34	625.950	1.4	716.8708	6.83	1.10	698.390	1.11	791.524
	5	7.40	1.34	599.976	1.37	654.681	7.41	1.10	672.233	1.1	728.078
c_p	160	6.24	1.36	652.483	1.52	806.740	6.27	1.11	725.829	1.14	886.693
	180	6.26	1.35	654.421	1.49	811.449	6.29	1.11	727.434	1.13	890.220
	200	6.28	1.35	656.361	1.47	816.085	6.30	1.10	728.987	1.12	893.715
	220	6.30	1.34	658.262	1.45	820.643	6.31	1.10	730.570	1.11	897.178
	240	6.31	1.33	660.149	1.43	825.125	6.33	1.09	732.109	1.1	900.611

(4) Sensitive results of initial expected shift time λ_0 .

When the initial expected shift time increases, the production cycle time and minimum cost decreases while the optimal reliability index increases. This implies that, when the initial reliability is high, it may be much easier for the company to achieve a higher reliability. So the company should set a higher reliability index when initial reliability increases.

(5) Sensitive results of shortage penalty cost c_p.

When the remaining proportion of production rate is large, i.e., $\alpha \sim U[0.6, 0.8]$, company does not need to consider about the shortage penalty cost. However, when the proportion is relatively small, i.e., $\alpha \sim U[0.4, 0.6]$, shortage penalty is an important factor that affects the optimal decisions. From **Table 2**, it can be seen that the expected shifting time and total cost are increasing, while the optimal production time is decreasing with the penalty cost. This is because company can mitigate the possibility of shortage by increasing the reliability of the production line and set a shorter production cycle.

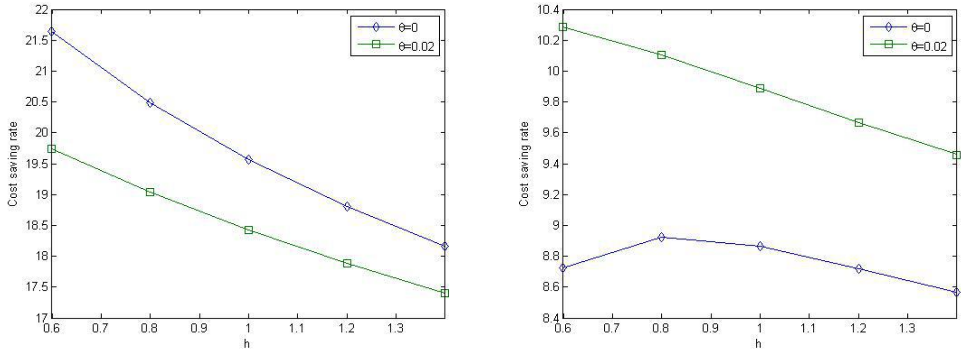
Table 3. Sensitive results for parameters $h, M_0, k, \lambda_0, c_p$ when $\alpha \sim U[0.6, 0.8]$.

		$\theta = 0$					$\theta = 0.02$				
		λ_1^{nd*}	T^{nd*}	ATC^{nd*}	T_0^{nd*}	ATC_0^{nd*}	λ_1^{d*}	T^{d*}	ATC^{d*}	T_0^{d*}	ATC_0^{d*}
h	0.6	5.12	1.91	449.432	2.34	492.384	5.35	1.36	552.343	1.44	615.683
	0.8	5.19	1.61	498.292	1.87	547.110	5.38	1.23	591.258	1.29	657.736
	1	5.21	1.42	540.791	1.60	593.397	5.40	1.13	626.669	1.17	695.452
	1.2	5.24	1.29	578.935	1.42	634.228	5.42	1.06	659.437	1.09	730.019
	1.4	5.26	1.18	613.804	1.32	671.316	5.43	0.99	689.993	1.02	762.072
M_0	1000	3.71	1.42	422.016	1.46	425.434	4.04	1.12	509.531	1.12	518.147
	3000	4.62	1.42	485.660	1.53	509.767	4.84	1.13	571.886	1.15	606.947
	5000	5.21	1.42	540.791	1.60	593.397	5.40	1.13	626.669	1.17	695.452
	7000	5.67	1.43	591.104	1.69	676.250	5.84	1.14	676.961	1.20	783.713
	9000	6.06	1.43	638.071	1.80	758.201	6.22	1.14	724.025	1.24	871.707
k	6	5.93	1.40	527.886	1.60	593.397	6.15	1.13	611.686	1.17	695.452
	8	5.51	1.41	535.236	1.60	593.397	5.71	1.13	620.180	1.17	695.452
	10	5.21	1.42	540.791	1.60	593.397	5.40	1.13	626.669	1.17	695.452
	12	4.99	1.43	545.202	1.60	593.397	5.17	1.13	631.870	1.17	695.452
	14	4.81	1.44	548.823	1.60	593.397	4.99	1.14	636.210	1.17	695.452
λ_0	1	4.20	1.47	594.537	4.01	688.411	4.44	1.14	684.635	1.76	974.996
	2	4.68	1.44	565.188	2.04	659.641	4.88	1.14	653.076	1.26	790.341
	3	5.21	1.42	540.791	1.60	593.397	5.40	1.13	626.669	1.17	695.452
	4	5.82	1.41	520.678	1.48	550.011	5.98	1.13	604.809	1.15	642.103
	5	6.49	1.39	504.136	1.43	521.203	6.63	1.13	586.774	1.14	608.212

5.3. Value of reliability investment

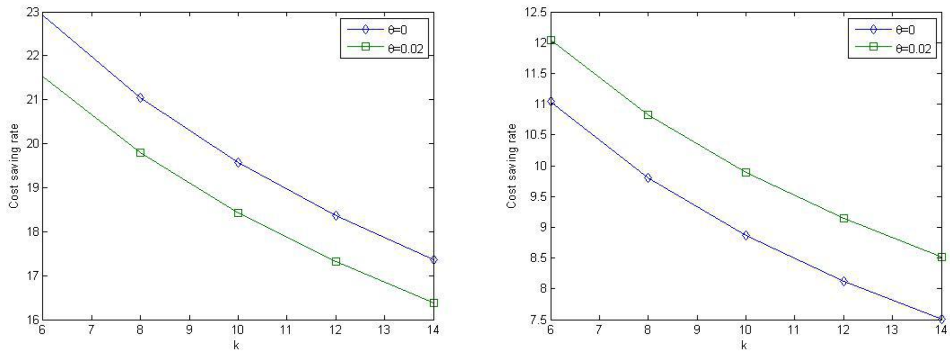
To study the value of the reliability investment, we define $\Delta\% = ATC_0^k - ATC^k / ATC_0^k \times 100\%$ as the cost saving rate, in which $k = \{d, nd\}$, ATC_0^k is the minimum cost without investment (i.e.,

$\lambda_1 = \lambda_0$). We presented our results in **Figures 5-9**.



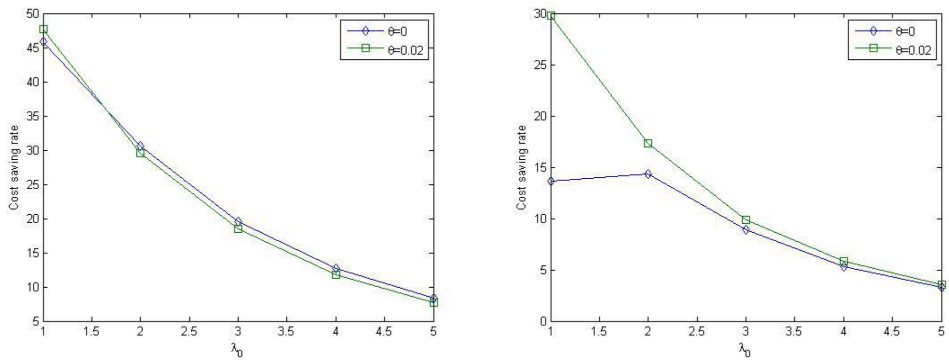
(a) $\alpha \sim U[0.4, 0.6]$ (b) $\alpha \sim U[0.6, 0.8]$

Figure 5. Cost saving rate (%) w.r.t. h .



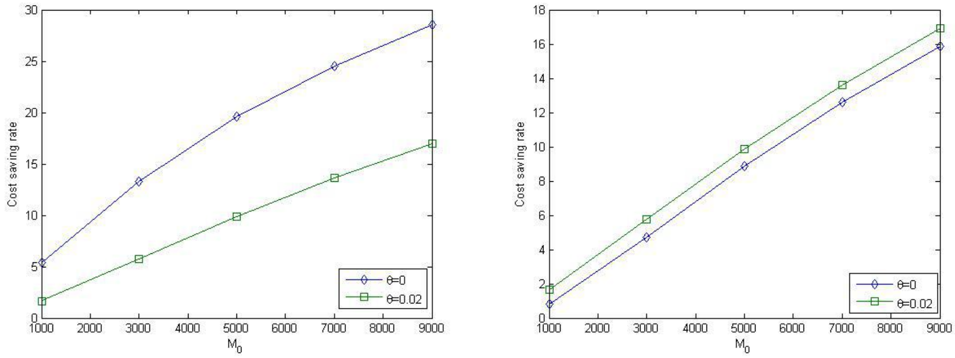
(a) $\alpha \sim U[0.4, 0.6]$ (b) $\alpha \sim U[0.6, 0.8]$

Figure 6. Cost saving rate (%) w.r.t. k .



(a) $\alpha \sim U[0.4, 0.6]$ (b) $\alpha \sim U[0.6, 0.8]$

Figure 7. Cost saving rate (%) w.r.t. λ_0 .



(a) $\alpha \sim U[0.4, 0.6]$ (b) $\alpha \sim U[0.6, 0.8]$

Figure 8. Cost saving rate (%) w.r.t. M_0.

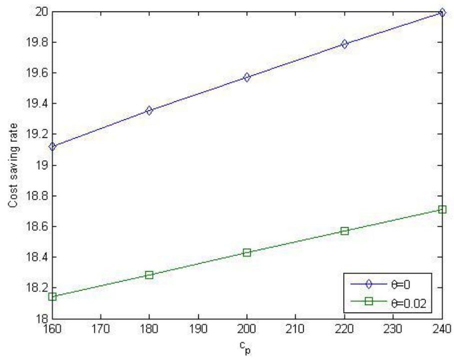
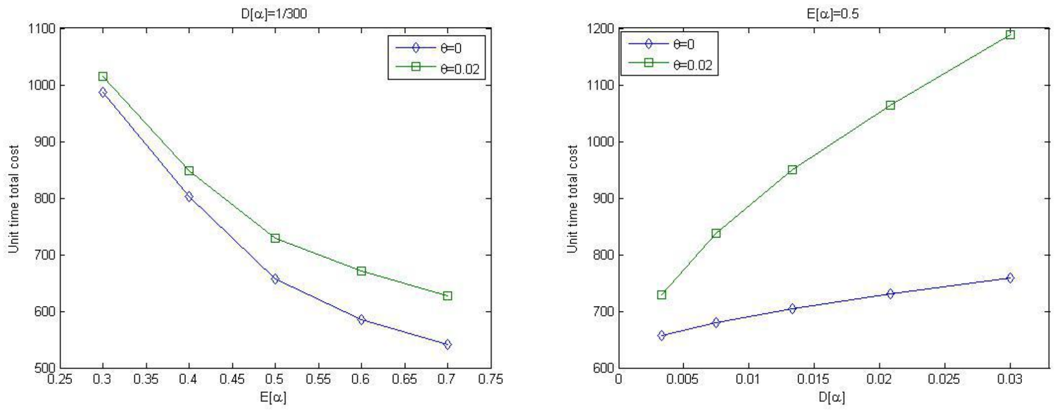


Figure 9. Cost saving rate (%) w.r.t. c_p.

From **Figure 5**, we can see that, when the inventory holding cost increases, the cost saving rate for deterioration products is decreasing. However, for the non-deterioration products, when $\alpha \sim U[0.6, 0.8]$, the cost saving rate is concave in inventory holding cost. As shown in **Figures 5-7**, the cost saving rate is decreasing in the initial shifting time λ_0 and the investment cost coefficient k. This implies that lower initial reliability index and investment cost can offer more incentives for companies to improve the production reliability. However, as shown in **Figures 8-9**, the cost saving rate is increasing in the restoration cost M_0 and penalty cost c_p. So, for companies with unreliable production line, when the restoration cost is high, or shortage penalty cost is high, it is more beneficial to improve the reliability of the production line.

5.4. Sensitive results of α 's randomness



(a) **(b)**

Figure 10. Unit time total cost change w.r.t. α 's (a) mean and (b) variance.

The parameter α indicates the extent of damage of the production line. In **Figure 10 (a),** We fix the variance of α ($D[\alpha]=1/300$) and test the impacts of different mean values ($E[\alpha]=\{0.3,0.4,0.5,0.6,0.7\}$) to the minimum cost. We find that for models with and without deterioration, the minimum cost is decreasing in the mean value of α .

Then we fix the mean value of α ($E[\alpha]=0.5$) and test the impacts of variance values ($D[\alpha]=\{4,9,16,25,36\}/1200$). It is shown in **Figure 10 (b)** that higher variance of α leads to higher cost for products with or without deterioration.

6. Conclusions and future research

In this paper, we studied the economic production quantity problem for an imperfect supply chain subject to random shift from the in-control state to the out-of-control state. The shift of state results in the production speed drop in the out-of-control state. Different from previous research of Ben-Daya et al. (2008), we treat the reliability index of the production line as a decision variable, which corresponds to the facts that, by investing in some assets and resources (e.g. high quality machine or highly educated workers, etc.), the reliability of the production line can be improved. Also we extend the model to deterioration products.

The numerical results show that, comparing to the case without reliability investment, investing in reliability can help companies save more cost. And, for the different values of system parameters, e.g., higher restoration cost, penalty cost and lower inventory holding cost, investment cost coefficient, initial shifting time, companies may have more willingness to invest in production line reliability.

For companies with higher deterioration rate, when the inventory holding cost increases, the cost saving rate for deterioration products is decreasing. Also, for companies with lower initial reliability index and investment cost, companies are more willing to improve the production line reliability index. However, the cost saving rate is increasing in the restoration cost and penalty cost. And, for companies with unreliable production line, when the restoration cost is high, or shortage penalty cost is high, it is more beneficial to improve the reliability of the production line.

Also, from the numerical tests, we can see that the mean and variance of production line damage rate has a significant impact on companies' profitability. When the mean value is large, the damage of the production line is small and the total average cost is low. However, when the variance value is large, companies have to deal with the uncertainty with a higher cost.

The model can be extended in three ways in future research. Firstly, we can consider marketing strategies and inventory decisions simultaneously to maximize the total profit. In this paper, we assume the demand of the product is a constant. However, in real markets, demand is linked to price, promotion efforts, product quality (Pal et al., 2015) or freshness. In addition to the inventory decisions, considering the variable demand is more realistic. Secondly, we can extend the model to a multiple level supply chains, and study gaming problems among supply chain members. Supply chain always consists of multi-stages. As part of the supply chain, the manufacturer's decision is not only determined by the internal factors, but also affected by the upstream supplier and the downstream retailer. Studying the interactions between different members in a supply chain is more realistic (Wang et al., 2016). Lastly, we can also study the assortment planning problem for a manufacturer producing multiple products. As customers' preference are changing rapidly, companies should offer varies kinds of products to attract more customers. How to set its assortment under unreliable production line is another interesting problem.

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Appendix A

Scenario 1: According to **Figure2 (a)**, the inventory level $I(t)$ satisfies

$$\dot{I}(t)=\begin{cases}P-D, 0<t<T\\ -D, T<t<T_m\end{cases}, \tag{A.1}$$

and continuous at $t = T$ with boundary conditions $I(0)=I(T_m)=0$.

We can derive the inventory level as

$$I(t)=\begin{cases}(P-D)t, 0<t<T\\ D(T_m-t), T<t<T_m\end{cases} \tag{A.2}$$

The total cost and cycle time for scenario 1 can be calculated as

$$TC_1(T) = h\int_0^T I(t)dt + h\int_T^{T_m} I(t)dt + A = \frac{h}{2}(\frac{P}{D} - 1)PT^2 + A \tag{A.3}$$

$$T_1(T) = T_m = \frac{P}{D}T \tag{A.4}$$

Scenario 2: According to **Figure 2 (b)**, the inventory level $I(t)$ satisfies

$$\dot{I}(t) = \begin{cases} P - D, 0 < t < s \\ \alpha P - D, s < t < T \\ -D, T < t < T_m \end{cases}, \tag{A.5}$$

and continuous at $t = s$, $t = T$ with boundary conditions $I(0) = I(T_m) = 0$.

We can derive the inventory level as

$$I(t) = \begin{cases} (P - D)t, 0 < t < s \\ (P - D)s + (\alpha P - D)t, s < t < T \\ D(T_m - t), T < t < T_m \end{cases} \tag{A.6}$$

The total cost and cycle time for scenario 1 can be calculated as

$$\begin{aligned} TC_2(T, s) &= h\int_0^s I(t)dt + h\int_s^T I(t) + h\int_T^{T_m} I(t)dt + A + M \\ &= \frac{h}{2}(P - D)s^2 + \frac{h}{2}(\alpha P - D)(T - s)^2 + h(P - D)(T - s)s \\ &+ \frac{hD}{2}[(\frac{P}{D} - 1)s + (\frac{\alpha P}{D} - 1)(T - s)]^2 + A + M \end{aligned} \tag{A.7}$$

$$T_2(T, s) = T_m = \frac{(1 - \alpha)Ps}{D} + \frac{\alpha PT}{D} \tag{A.8}$$

Scenario 3: According to **Figure 2 (c)**, the inventory level $I(t)$ satisfies

$$\dot{I}(t) = \begin{cases} P - D, 0 < t < s \\ \alpha P - D, s < t < T \end{cases}, \tag{A.9}$$

and continuous at $t = s$ with boundary conditions $I(0) = I(T_m) = 0$.

We can derive the inventory level as

$$I(t) = \begin{cases} (P - D)t, 0 < t < s \\ (P - D)s + (\alpha P - D)(t - s), s < t < T \end{cases}. \tag{A.10}$$

in which

$$T_m = \frac{(1 - \alpha)Ps}{D - \alpha P}. \tag{A.11}$$

The total cost and cycle time for scenario 1 can be calculated as

$$\begin{aligned} TC_3(T,s) &= h\int_0^s I(t)dt + h\int_s^{T_m} I(t)dt + c_p(D-\alpha P)[I(T_m)-I(T)] + A + M \\ &= \frac{h(P-D)(1-\alpha)P}{2(D-\alpha P)}s^2 + c_p(D-\alpha P)[T - \frac{(1-\alpha)Ps}{D-\alpha P}] + A + M \end{aligned} \tag{A.12}$$

$$T_3(T) = T\ . \tag{A.13}$$

Appendix B

Scenario 1: According to **Figure 5 (a)**, the inventory level $I(t)$ satisfies

$$\dot{I}(t) = \begin{cases} P - D - \theta I(t), 0 < t < T \\ -D - \theta I(t), T < t < T_m \end{cases}, \tag{B.1}$$

and continuous at $t = T$ with boundary conditions $I(0) = I(T_m) = 0$.

We can derive the inventory level as

$$I(t) = \begin{cases} \frac{P-D}{\theta}(1-e^{-\theta t}), 0 < t < T \\ \frac{D}{\theta}(e^{\theta(T_m - t)} - 1), T < t < T_m \end{cases}, \tag{B.2}$$

in which

$$T_m = \frac{1}{\theta}\ln[\frac{P-D}{D}(1-e^{-\theta T}) + 1] + T\ . \tag{B.3}$$

The total cost and cycle time for scenario 1 can be calculated as

$$\begin{aligned} TC_1(T_1) &= h\int_0^T I(t)dt + h\int_T^{T_m} I_2(t)dt + A + c_d(pT - dT_m) \\ &= h\left\{\frac{P-D}{\theta^2}\left[\theta T + e^{-\theta T} - 1\right] + \frac{D}{\theta^2}\left[e^{\theta(T_m - T)} - \theta(T_m - T) - 1\right]\right\} + A + c_d(pT - dT_m) \end{aligned} \tag{B.4}$$

$$T_1(T) = T_m = \frac{1}{\theta}\ln[\frac{P-D}{D}(1-e^{-\theta T}) + 1] + T\ . \tag{B.5}$$

Scenario 2: According to **Figure 2 (b)**, the inventory level $I(t)$ satisfies

$$\dot{I}(t) = \begin{cases} P - D - \theta I(t), 0 < t < s \\ \alpha P - D - \theta I(t), s < t < T\ , \\ -D - \theta I(t), T < t < T_m \end{cases} \tag{B.6}$$

and continuous at $t = s$, $t = T$ with boundary conditions $I(0) = I(T_m) = 0$.

We can derive the inventory level as

$$I(t)=\begin{cases}\dfrac{P-D}{\theta}(1-e^{-\theta t}),0<t<s\\ \dfrac{\alpha P-D}{\theta}+[\dfrac{(1-\alpha)P}{\theta}e^{\theta s}-\dfrac{P-D}{\theta}]e^{-\theta t},s<t<T\\ \dfrac{D}{\theta}(e^{\theta(T_m-t)}-1),T<t<T_m\end{cases},\tag{B.7}$$

in which

$$T_m=\frac{1}{\theta}\ln[\frac{\alpha P-D}{D}+\frac{(1-\alpha)P}{D}e^{\theta(s-T)}-\frac{P-D}{D}e^{-\theta T}+1]+T.\tag{B.8}$$

The total cost and cycle time for scenario 1 can be calculated as

$$\begin{aligned}TC_2(T,s)&=h\int_0^s I(t)dt+h\int_s^T I(t)dt+h\int_T^{T_m} I(t)dt+A+M\\&\quad+c_d(Ps+\alpha P(T-s)-DT_m)\\&=h\begin{Bmatrix}\dfrac{P-D}{\theta^2}(\theta s+e^{-\theta s}-1)+\dfrac{\alpha P-D}{\theta}(T-s)\\-\dfrac{(1-\alpha)Pe^{\theta s}-P+D}{\theta^2}(e^{-\theta T}-e^{-\theta s})+\dfrac{D}{\theta^2}\left[e^{\theta(T_m-T)}-\theta(T_m-T)-1\right]\end{Bmatrix}\\&\quad+A+M+c_d(Ps+\alpha P(T-s)-DT_m)\end{aligned}\tag{B.9}$$

$$T_2(T,s)=T_m=\frac{1}{\theta}\ln[\frac{\alpha P-D}{D}+\frac{(1-\alpha)P}{D}e^{\theta(s-T)}-\frac{P-D}{D}e^{-\theta T}+1]+T.\tag{B.10}$$

Scenario 3: According to **Figure 5 (c)**, the inventory level $I(t)$ satisfies

$$\dot{I}(t)=\begin{cases}P-D-\theta I(t),0<t<s\\ \alpha P-D-\theta I(t),s<t<T_m\\ \alpha P-D,T_m<t<T\end{cases},\tag{B.11}$$

and continuous at $t=s$, $t=T_m$ with boundary conditions $I(0)=I(T_m)=0$.

We can derive the inventory level as

$$I(t)=\begin{cases}\dfrac{P-D}{\theta}(1-e^{-\theta t}),0<t<s\\ \dfrac{\alpha P-D}{\theta}+[\dfrac{(1-\alpha)P}{\theta}e^{\theta s}-\dfrac{P-D}{\theta}]e^{-\theta t},s<t<T_m\\ (\alpha P-D)(t-T_m),T_m<t<T\end{cases},\tag{B.12}$$

in which

$$T_m=\frac{1}{\theta}\ln[\frac{(1-\alpha)Pe^{\theta s}-P+D}{D-\alpha P}].\tag{B.13}$$

The total cost and cycle time for scenario 1 can be calculated as

$$
\begin{aligned}
TC_3 &= h\int_0^s I(t)dt + h\int_s^{T_m} I(t)dt + h\int_{T_m}^{T} I(t)dt + A + M \\
&+c_d(Ps+\alpha P(T_m - s) - DT_m) + c_p(D-\alpha P)(T-T_m) \\
&= h(\frac{P-D}{\theta^2}(\theta s + e^{-\theta s} - 1) + \frac{\alpha P - D}{\theta}(T_m - s) - \frac{(1-\alpha)Pe^{\theta s} - P + D}{\theta^2}(e^{-\theta T_m} - e^{-\theta s}) \\
&+A + M + c_d(Ps + \alpha P(T_m - s) - DT_m) + c_p(D-\alpha P)(T-T_m)
\end{aligned}
\quad (B.14)
$$

$$
T_3(T) = T \ . \quad (B.15)
$$

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