The $4n^2$-inequality for complete intersection singularities

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The famous $4n^2$-inequality is extended to generic complete intersection singularities: it is shown that the multiplicity of the self-intersection of a mobile linear system with a maximal singularity is greater than $4n^2\mu$, where μ is the multiplicity of the singular point.

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1. Statement of the result. Let (X,o) be a germ of a complete intersection singularity of codimension l and type $\underline{\mu}=(\mu_1,\dots,\mu_l)$, where

$$\dim X = M \geqslant l+\mu_1+\ldots+\mu_l+3.$$

We will assume the singularity to be generic in the sense of Sec. 2 below. The aim of this note is to prove the following claim.

Theorem. *Let Σ be a mobile linear system on X. Assume that for some positive $n\in\mathbb{Q}$ the pair $(X,\frac{1}{n}\Sigma)$ is not canonical at the point o but canonical outside this point. Then the self-intersection $Z=(D_1\circ D_2)$ of the system Σ satisfies the inequality*

$$\mathrm{mult}_o Z > 4n^2\,\mathrm{mult}_o X. \qquad (1)$$

Remark 1. (i) The assumption of the theorem means that the pair $(X,\frac{1}{n}\Sigma)$ has a non-canonical singularity with the centre at the point o. Explicitly, for some exceptional divisor R over X, the centre of which is the point o, the Noether-Fano inequality

$$\mathrm{ord}_R\,\Sigma > n\cdot a(R,X)$$

holds, where $a(R,X)$ is the discrepancy of R with respect to X.

(ii) The self-intersection $Z=(D_1\circ D_2)$ is the scheme-theoretic intersection of any two general divisors in Σ which is well defined as Σ is free from fixed components.

(iii) When $\mathrm{mult}_o X=1$, we get the standard $4n^2$-inequality, see [14, Chapter 2]. For that reason, we call the inequality (1) the $4n^2$-inequality as well. The standard $4n^2$-inequality (for the non-singular case) was first shown in [9] on the basis of the technique developed in [7]. Later a different proof was found by Corti [4] and various generalizations of the $4n^2$-inequality were investigated [3, 13], see [14, Chapter 2] for more details.

Note that in the smooth case (when $\mathrm{mult}_o X = 1$) the $4n^2$-inequality holds for $\dim X \geqslant 3$ without any additional assumptions. This is because the exceptional divisor of the blow up of the point o on X is just the projective space, and in the projective space it is very easy to bound multiplicities in terms of degrees. unfortunately, it is not so easy to do so (in the way we need) for hypersurfaces and complete intersections, which generate the need for additional assumptions.

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2. Generic complete intersection singularities. The germ (X, o) is given by a system of l analytic equations

$$\begin{array}{rcccccc} 0 & = & q_{1,\mu_1} & + & q_{1,\mu_1+1} & + & \dots \\ \dots & & & & & & \\ 0 & = & q_{l,\mu_l} & + & q_{l,\mu_l+1} & + & \dots \end{array}$$

in $\mathbb{C}^{M+l}$, where $2 \leqslant \mu_1 \leqslant \ldots \leqslant \mu_l$, $l \geqslant 1$ and the polynomials $q_{j,i}$ are homogeneous of degree i in the coordinates $z_1, \ldots, z_{M+l}$; the point $o = (0, \ldots, 0)$ is the origin. We denote by

$$\underline{\mu} = (\mu_1, \ldots, \mu_l)$$

the type of the singularity $o \in X$ and set

$$\mu = \mu_1 \cdots \mu_l = \mathrm{mult}_o X$$

to be the multiplicity of the point o (assuming the conditions of general position for the first polynomials $q_{1,\mu_1}, q_{2,\mu_2}, \ldots, q_{l,\mu_l}$, stated below). Set also

$$|\underline{\mu}| = \mu_1 + \ldots + \mu_l.$$

Recall that by assumption $M \geqslant l + |\underline{\mu}| + 3$. Let $P \ni o$ be a linear subspace in $\mathbb{C}^{M+l}$ of dimension $2l + |\underline{\mu}| + 3$. Denote by X_P the intersection $X \cap P$.

Definition 1. We say that the complete intersection singularity (X, o) is *generic*, if for a general subspace P of dimension $2l + |\underline{\mu}| + 3$ the singularity $o \in X_P$ is an isolated singularity, $\dim X_P = l + |\underline{\mu}| + 3$ and for the blow up

$$\varphi_P \colon X_P^+ \to X_P$$

of the point o, the variety X_P^+ is non-singular in neighborhood of the exceptional divisor $Q_P = \varphi_P^{-1}(o)$, which is a non-singular complete intersection

$$Q_P = \{q_{1,\mu_1} = q_{2,\mu_2} = \ldots = q_{l,\mu_l} = 0\} \subset \mathbb{P}^{2l+|\underline{\mu}|+2}$$

of codimension l and type $\underline{\mu} = (\mu_1, \ldots, \mu_l)$.

From now on, we assume that the singularity $o \in X$ is generic. In particular, by Grothendieck's theorem on factoriality [1], X is a factorial variety near the point o.

3. Start of the proof. The idea of the proof is as follows. We use as a model the proof of the standard $4n^2$-inequality by means of the technique of counting multiplicities as it is given in [14, Chapter 2, Section 2.2]. First, we observe that by inversion of adjunction, the existence of a non-canonical singularity R implies the existence of another singularity E of the same pair $(X, \frac{1}{n}\Sigma)$ which satisfies a *Noether-Fano type* inequality. The latter is somewhat weaker (but sufficient for our purposes). However, the new singularity E has the crucial advantage that its centre on the blow up X^+ of the point o has a high dimension. This is done in the present section.

After that, in Section 4 we resolve the singularity E and use the assumptions on the singular point $o \in X$ to relate the multiplicities of the system Σ and its self-intersection at the point o with the multiplicities of the strict transforms of Σ and the self-intersection at the "higher storeys" of the resolution, at the centres of the singularity E on those "higher storeys".

This done, we apply the technique of counting multiplicities in word for word the same way as in [14, Chapter 2, Section 2.2] and complete the proof.

Let us realize this programme.

For a general $(2l + |\underline{\mu}| + 3)$-subspace P set $\Sigma_P = \Sigma|_P$ to be the restriction of Σ onto P. By inversion of adjunction [15, 8], the pair $(X_P, \frac{1}{n}\Sigma_P)$ is not canonical (for $M > l + |\underline{\mu}| + 3$, even non-log canonical, but we do not need that.) Obviously,

$$Z_P = Z|_P = (Z \circ X_P)$$

is the self-intersection of the system Σ_P and $\operatorname{mult}_o Z = \operatorname{mult}_o Z_P$. Therefore, we may (and will) assume from the beginning that $M = l + |\underline{\mu}| + 3$ and so $P = \mathbb{C}^{M+l}$, so that already the original singularity $o \in X$ is isolated. Now we omit the index P and write

$$\varphi : X^+ \to X$$

for the blow up of the point o and $Q = \varphi^{-1}(o)$ for the exceptional divisor, which is a non-singular complete intersection of type $\underline{\mu}$ in $\mathbb{P}^{2l+|\underline{\mu}|+2}$.

Now let $\Pi \ni o$ be a general linear subspace of dimension $|\underline{\mu}| + 3$. By the symbol X_Π we denote the intersection $X \cap \Pi$. Clearly, $o \in X_\Pi \subset \Pi = \mathbb{C}^{|\underline{\mu}|+3}$ is an isolated complete intersection singularity of codimension l. Let $\varphi_\Pi : X_\Pi^+ \to X_\Pi$ be the blow up of the point o and $Q_\Pi = \varphi_\Pi^{-1}(o)$ the exceptional divisor. Clearly $Q_\Pi \subset \mathbb{P}^{|\underline{\mu}|+2}$ is a non-singular complete intersection of type $\underline{\mu}$ (and codimension l).

Note that by the adjunction formula for the discrepancy we have the equality $a(Q_\Pi, X_\Pi) = 2$.

For a general divisor $D \in \Sigma$ and its strict transform $D^+ \in \Sigma^+$ on X^+ we have

$$D^+ \sim -\nu Q$$

for some positive integer ν (recall that we consider a local situation: $o \in X$ is a germ). Obviously, if $\nu > 2n$, then

$$\operatorname{mult}_o Z \geqslant \nu^2 \mu > 4n^2 \mu$$

and the $4n^2$-inequality holds. For that reason, from now on we assume that

$$\nu \leqslant 2n.$$

Setting $D_\Pi = D|_{X_\Pi}$, we get $D_\Pi^+ \sim -\nu\, Q_\Pi$. By the inversion of adjunction the pair $\left(X_\Pi, \frac{1}{n} D_\Pi\right)$ is not log canonical at the point o, the more so not canonical, so for some exceptional divisor E_Π over X_Π the Noether-Fano inequality

$$\mathrm{ord}_{E_\Pi} \Sigma_\Pi > na(E_\Pi, X_\Pi)$$

is satisfied. As $\nu \leqslant 2n$ and $a(Q_\Pi, X_\Pi) = 2$, we see that $E_\Pi \neq Q_\Pi$ and E_Π is a non log canonical (and so not canonical) singularity of the pair

$$\left(X_\Pi^+, \frac{1}{n} D_\Pi^+ + \frac{(\nu - 2n)}{n} Q_\Pi \right)$$

(the more so, of the pair $\left(X_\Pi^+, \frac{1}{n} D_\Pi^+\right)$). Denote by $\Delta_\Pi \subset Q_\Pi$ the centre of E_Π on X_Π^+, an irreducible subvariety in Q_Π.

Proposition 1. *If* $\mathrm{codim}\,(\Delta_\Pi \subset Q_\Pi) = 1$*, then the estimate*

$$\mathrm{mult}_o\, Z \geqslant 8n^2\mu$$

holds.

Proof. We note that $\mathrm{mult}_o\, Z = \mathrm{mult}_o\, Z_\Pi$. Arguing as in the proof of Proposition 4.1 in [14, Chapter 2] (see also [3, Section 1.7]), we get the following estimate:

$$\mathrm{mult}_o\, Z_\Pi \geqslant \nu^2\mu + 4\left(3 - \frac{\nu}{n}\right) n^2\mu,$$

and easy calculations complete the proof. Q.E.D.

Therefore we may assume that $\mathrm{codim}(\Delta_\Pi \subset Q_\Pi) \geqslant 2$.

Coming back to the variety X, we conclude that for some exceptional divisor E over X with the centre at o the Noether-Fano type inequality

$$\mathrm{ord}_E\, \Sigma > n(2\,\mathrm{ord}_E\, Q + a(E, X^+))$$

is satisfied. Moreover, the centre $\Delta \subset Q$ of E on X has codimension at least 2 and dimension at least $2l$.

4. Resolution of the singularity E**.** Consider the sequence of blow ups

$$X_0 = X \leftarrow X_1 = X^+ \leftarrow X_2 \leftarrow \ldots \leftarrow X_K,$$

where $\varphi_{i,i-1}\colon X_i \to X_{i-1}$ is the blow up of the centre $B_{i-1} \subset X_{i-1}$ of the exceptional divisor E on X_{i-1}. In particular, $B_0 = o$ and $B_1 = \Delta$. Using the notations, identical to those in [14, Chapter 2, Section 2.2], we set

$$E_i = \varphi_{i,i-1}^{-1}(B_{i-1}) \subset X_i$$

to be the exceptional divisor, so that $E_1 = Q$. As $X_1 = X^+$ is non-singular in a neighborhood of E_1, all subsequent varieties X_i are non-singular at the generic point of B_i and all constructions of [14, Chapter 2, Section 2.2] work automatically for the blow ups $\varphi_{i,i-1}$ with $i \geqslant 2$.

The last exceptional divisor E_K defines the discrete valuation ord_E.

We divide the sequence $\varphi_{i,i-1}$, $i = 1, \ldots, K$, of blow ups into the *lower part*, $i = 1, \ldots, L \leqslant K$, corresponding to the centres B_{i-1} of codimensions at least 3, and the *upper part*, $i = L+1, \ldots, K$, corresponding to the centres B_{i-1} of codimension two. As usual, we denote the strict transform of any geometric object on X_i by adding the upper index i and set:

$$\nu_i = \mathrm{mult}_{B_{i-1}} \Sigma^i$$

for any $i = 2, \ldots, K$ to be the elementary multiplicities. Let Γ be the oriented graph of the resolution of the singularity E and p_{ij} the number of paths from the vertex i to the vertex j, $p_{ii} = 1$ by definition (see [14, Chapter 2, Section 2.2] for the standard details). We also set $p_i = p_{Ki}$, $i = 1, \ldots, K$. Now the Noether-Fano type inequality takes the form

$$\sum_{i=1}^{K} p_i \nu_i > \left(2p_1 + \sum_{i=2}^{K} p_i \delta_i \right), \tag{2}$$

where $\nu_1 = \nu$ and $\delta_i = \mathrm{codim}(B_{i-1} \subset X_{i-1})$ are the elementary discrepancies. By the linearity of the Noether-Fano type inequality (2) and the standard properties of the numbers p_{ij} we may assume that $\nu_K > n$ (replacing, if necessary, E_K by a lower singularity E_j for some $j < K$). In order to proceed, we need the following known fact.

Proposition 2. *Let $Y \subset \mathbb{P}^N$ be a non-singular complete intersection of codimension $l \geqslant 1$, $S \subset Y$ an irreducible subvariety of codimension $a \geqslant 1$ and $B \subset Y$ an irreducible subvariety of dimension al, where the estimate $N \geqslant (l+1)(a+1)$ is satisfied. Then the inequality*

$$\mathrm{mult}_B S \leqslant m$$

holds, where $m \geqslant 1$ is defined by the condition $S \sim mH_Y^a$ and $H_Y \in A^1 Y$ is the class of a hyperplane section of Y.

Proof for the case $l = 1$ was given in [11]. The argument extends directly to the general case of an arbitrary l, see [16] (also [12, 2]). Q.E.D.

Applying Proposition 2 to a divisor in the linear system $\Sigma^1|_Q$, we conclude that $\nu_1 \geqslant \nu_2$, since $\dim B_1 = \dim \Delta \geqslant 2l$. The inequalities

$$\nu_2 \geqslant \nu_3 \geqslant \ldots \geqslant \nu_K$$

are standard. We deduce that the upper part of the resolution of E is non-empty (that is to say, $L < K$) and the upper part of the graph Γ is a chain:

$$L \leftarrow (L+1) \leftarrow \ldots \leftarrow K;$$

moreover, there are no arrows connecting either of the vertices $L+1,\dots,K$ with any of vertices $1,2,\dots,L-1$. (These are the standard consequences of inequalities $\nu_K > n$ and $\nu_1 \leqslant 2n$, see [14, Chapter 2, Section 2.2].) We do not need this additional information for the proof of our theorem, but in particular geometric problems it might be useful.

5. The technique of counting multiplicities. Now everything is ready for the proof of the desired inequality (1). Take a general pair of divisors $D_1, D_2 \in \Sigma$ and set

$$Z = Z_0 = (D_1 \circ D_2)$$

to be their scheme-theoretic intersection, the self-intersection of the mobile linear system Σ. Recall that the strict transform of an irreducible subvariety or an effective cycle, or a linear system on some X_i is denoted by adding the upper index i. (This notation silently implies that the irreducible subvariety or the effective cycle etc. is sitting on a lower storey X_j, $j \leqslant i$, of the resolution and that the operation of taking the strict transform is well defined for that particular subvariety etc.) For $i \geqslant 1$ write

$$(D_1^i \circ D_2^i) = (D_1^{i-1} \circ D_2^{i-1})^i + Z_i,$$

where the effective cycle Z_i of codimension 2 is supported on E_i and so may be viewed as an effective divisor on E_i. Thus for any $i \leqslant L$ we obtain the presentation

$$(D_1^i \circ D_2^i) = Z_0^i + Z_1^i + \ldots + Z_{i-1}^i + Z_i.$$

For any $j > i$, where $j \leqslant L$, set

$$m_{i,j} = \mathrm{mult}_{B_{j-1}} Z_i^{j-1}$$

and for $i = 2,\dots,L$ set $d_i = \deg Z_i$ in the same sense as in [14, Chapter 2, Section 2.2]. For the effective divisor Z_1 on $E_1 = Q$ we have the relation

$$Z_1 \sim d_1 H_Q$$

for some $d_1 \in \mathbb{Z}_+$, where H_Q is the class of a hyperplane section of the complete intersection $Q \subset \mathbb{P}^{4l+2}$. Following the procedure of [14, Chapter 2], we obtain the system of equalities

$$\begin{array}{rcl} \mu(\nu_1^2 + d_1) & = & m_{0,1}, \\ \nu_2^2 + d_2 & = & m_{0,2} + m_{1,2}, \\ & \cdots & \\ \nu_i^2 + d_i & = & m_{0,i} + \ldots + m_{i-1,i}, \\ & \cdots & \end{array}$$

$i = 2,\dots,L$, where the estimate

$$d_L \geqslant \sum_{i=L+1}^{K} \nu_i^2$$

holds as usual, see [14, p. 53].

Proposition 3. (i) *The inequality*

$$d_1 \geqslant m_{1,2}$$

holds.

(ii) *The inequality*

$$m_{0,1} \geqslant \mu m_{0,2}$$

holds.

Proof. Part (i) follows from Proposition 2 as $Z_1 \sim d_1 H_Q$ and $\dim B_1 \geqslant 2l$. In order to show part (ii), we note that (numerically)

$$(Z^1 \circ E_1) \sim \frac{1}{\mu} m_{0,1} H_Q^2$$

as $m_{0,1} = \deg(Z^1 \circ E_1)$, the cycle $(Z^1 \circ E_1) = (Z^1 \circ Q)$ being of pure codimension 2 on Q. Applying Proposition 2 to the cycle $(Z^1 \circ Q)$, we get the inequality

$$m_{0,2} \leqslant \mathrm{mult}_\Delta (Z^1 \circ Q) \leqslant \frac{1}{\mu} m_{0,1},$$

which completes the proof of the proposition. Q.E.D.

The more so, $m_{0,1} \geqslant \mu m_{0,i}$ for $i \geqslant 3$ as $m_{0,2} \geqslant m_{0,3} \geqslant \ldots \geqslant m_{0,L}$.

Now set

$$m^*_{i,j} = \mu m_{i,j}$$

for $(i,j) \neq (0,1)$ and $m^*_{0,1} = m_{0,1}$. Also set

$$d^*_i = \mu d_i$$

for $i = 1, \ldots, L$. We obtain the following system of equalities:

$$\begin{array}{rcccccc}
\mu\nu_1^2 + d_1^* & = & m^*_{0,1}, & & & & \\
\mu\nu_2^2 + d_2^* & = & m^*_{0,2} & + & m^*_{1,2}, & & \\
 & \ldots & & & & & \\
\mu\nu_i^2 + d_i^* & = & m^*_{0,i} & + & \ldots & + & m^*_{i-1,i}, \\
 & \ldots & & & & &
\end{array}$$

where $i = 1, \ldots, L$, and

$$d^*_L \geqslant \mu \sum_{i=L+1}^{K} \nu_i^2,$$

where the integers $m^*_{i,j}$ and d^*_i satisfy precisely the same properties, as the integers $m_{i,j}$ and d_i in the non-singular case considered in [14, Chapter 2, p. 52-53]. Now repeating the arguments of [14, Chapter 2, p. 52-53] word for word, we obtain the inequality

$$\left(\sum_{i=1}^{L} p_i \right) \mathrm{mult}_o Z \geqslant \mu \sum_{i=1}^{K} p_i \nu_i^2,$$

which in the standard way implies the desired estimate

$$\mathrm{mult}_o Z > \mu \cdot 4n^2.$$

Proof of the theorem is completed.

Remark 2. The inequality (1) essentially simplifies the proof of birational superrigidity of Fano hypersurfaces with isolated singularities of general position given in [10]. The cases of singular points of multiplicity $\mu = 3$ and 4 in that paper are really hard. The inequality (1) gives for the multiplicity $\mathrm{mult}_o Z$ at such points the lower bound $12n^2$ and $16n^2$, respectively, which is more than enough to exclude the maximal singularities over such points by the standard (in fact, relaxed) technique of hypertangent divisors. More applications of the inequality (1) in the spirit of [5, 6] will be given separately.

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