Factorial hypersurfaces

A.V.Pukhlikov

In this paper the codimension of the complement to the set of factorial hypersurfaces of degree d in $\mathbb{P}^N$ is estimated for $d\geqslant 4$, $N\geqslant 7$.

Bibliography: 12 titles.

Key words: factoriality, hypersurface, singularity.

Introduction

0.1. Statement of the main result. A singular point o of an algebraic variety V is *factorial*, if every prime divisor $D\ni o$ in a neighborhood of this point is given by a single equation $f\in\mathcal{O}_{o,V}$. (For a non-singular point $o\in V$ this is a well known theorem of the classical algebraic geometry.) Majority of the modern techniques work for factorial or $\mathbb{Q}$-factorial varieties (in the latter case it is required that some multiple of every prime divisor $D\ni o$ were given by one equation, see, for instance, [1] or any paper on the minimal model program). A standard example of a non-$\mathbb{Q}$-factorial (and the more so, non-factorial) variety is the three-dimensional cone in $\mathbb{P}^4$ over a non-singular quadric in $\mathbb{P}^3$: no multiple of a plane passing through the vertex of the cone (and contained in that cone) can be given by one equation in the local ring of the vertex of the cone. The aim of this paper is to estimate from below the codimension of the complement to the set of factorial hypersurfaces of degree d in $\mathbb{P}^N$, $N\geqslant 7$, $d\geqslant 4$. More precisely, let $\mathcal{P}_{d,N+1}$ be the space of homogeneous polynomials of degree d in the coordinates $x_0,\dots,x_N$ on $\mathbb{P}^N$. Let $\mathcal{P}^{\rm fact}_{d,N+1}\subset\mathcal{P}_{d,N+1}$ be the subset, consisting of such $f\in\mathcal{P}_{d,N+1}$ that the hypersurface $\{f=0\}$ is irreducible, reduced and factorial.

Theorem 0.1. (i) *Assume that $4\leqslant d\leqslant N$ (that is, $\{f=0\}$ is a Fano hypersurface). Then the estimate*

$$\mathrm{codim}\left(\mathcal{P}_{d,N+1}\setminus\mathcal{P}^{\rm fact}_{d,N+1}\right)\geqslant\min\left[3\binom{d+N-5}{N-2}-N,\ 5\binom{d+N-6}{N-3}\right]$$

holds, and in the case $d=N$ (that is, $\{f=0\}$ is a Fano hypersurface of index one) in the right hand side one can leave only $5\binom{d+N-6}{N-3}$.

(ii) *Assume that $d\geqslant 2N$ (in particular, the hypersurface $\{f=0\}$ is a variety of* *general type). Then the following estimate holds:*

$$\mathrm{codim}\left(\mathcal{P}_{d,N+1}\setminus\mathcal{P}^{\mathrm{fact}}_{d,N+1}\right)\geqslant\binom{d+N-4}{N-4}+4\binom{d+N-5}{N-4}-4(N-3).$$

In fact, we will obtain an estimate for the codimension of the complement to the set $\mathcal{P}^{\mathrm{fact}}_{d,N+1}$ for any values $d\geqslant 4$ (Theorem 3.1), just in the Fano case ($d\leqslant N$) and in the case $d\geqslant 2N$ that estimate can be essentially simplified to the inequalities of Theorem 0.1.

0.2. The plan of the proof and the structure of the paper. Our proof is based on the famous Grothendieck's theorem [2] and the technique of estimating the codimension of the set of hypersurfaces in $\mathbb{P}^N$ with a singular set of prescribed dimension, developed in [3, 4]. Grothendieck's theorem claims that a variety with hypersurface singularities (in fact, with complete intersection singularities) is factorial, if the singular locus has codimension at least 4. For that reason, in order to estimate the codimension of the complement to $\mathcal{P}^{\mathrm{fact}}_{d,N+1}$, it is sufficient to estimate the codimension of the subset consisting of polynomials $f\in\mathcal{P}_{d,N+1}$ such that the singular locus of the hypersurface $\{f=0\}$ has codimension at most 3. This is what we will do in the present paper.

The paper is organized in the following way. In §1 we compute the codimensions of two sets of polynomials $f\in\mathcal{P}_{d,N+1}$: such that the hypersurface $\{f=0\}$ has a linear space of singular points and such that the hypersurface $\{f=0\}$ has a subvariety of singular points which is a hypersurface in a linear space. After that we set the general problem of estimating the codimension of the set of polynomials $f\in\mathcal{P}_{d,N+1}$, such that the singular locus of the hypersurface $\{f=0\}$ is of dimension at least $i\geqslant 1$, and state the main technical result — Theorem 1.1, solving this problem.

§2 contains the proof of Theorem 1.1 by means of the technique developed in [3, 4]. In §3 we obtain an estimate of the codimension of the set of non-factorial hypersurfaces, which implies Theorem 0.1.

0.3. Historical remarks and acknowledgements. Factoriality of algebraic varieties is a very old topic in Algebraic Geometry, with lots of papers written on the subject. We will only point out a few recent papers that demonstrate that the topic is still actively investigated today: [5, 6, 7, 8, 9]. Various technical points related to the constructions of this paper were discussed by the author in his talks given in 2009-2014 at Steklov Mathematical Institute. The author thanks the members of divisions of Algebraic Geometry and Algebra and Number Theory for the interest to his work. The author is also grateful to his colleagues in Algebraic Geometry group at the University of Liverpool for the creative atmosphere and general support.

The author thanks the referee for the useful comments.

1 Hypersurfaces with a large singular locus

In this section we consider the problem of estimating the codimension of the set of polynomials f, defining hypersurfaces with a large singular locus. As a first example we compute the codimension of the set of polynomials f, such that the hypersurface $\{f = 0\}$ has a linear subspace of singular points (Subsection 1.1). The next by complexity case, when the singular locus is a hypersurface in a linear subspace, is made in Subsection 1.2. In Subsection 1.3 we give a precise setting of the problem in the general case and state the main result.

1.1. Hypersurfaces with a linear subspace of singular points. Let $\mathbb{P}^N$ be the complex projective space with homogeneous coordinates $(x_0 : x_1 : \dots : x_N)$, $N \geqslant 3$ and

$$\mathcal{P}_{d,N+1} = H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$$

the linear space of homogeneous polynomials of degree d in x_*. For a polynomial $f \in \mathcal{P}_{d,N+1} \setminus \{0\}$ the set of singular points of the hypersurface $\{f = 0\}$ is denoted by the symbol $\mathrm{Sing}(f)$. Set

$$\mathcal{P}^{(i)}_{d,N+1} = \{f \in \mathcal{P}_{N,d} \mid \dim \mathrm{Sing}(f) \geqslant i\},$$

where for the identically zero polynomial $f \equiv 0$ we set $\mathrm{Sing}(0) = \mathbb{P}^N$. Obviously, the sets $\mathcal{P}^{(i)}_{d,N+1}$ are closed and for $i \geqslant j$ we have $\mathcal{P}^{(j)}_{d,N+1} \subset \mathcal{P}^{(i)}_{d,N+1}$.

In order work with the sets $\mathcal{P}^{(i)}_{d,N+1}$, it is convenient to represent them as a union of more special subsets that take into account more information about the set $\mathrm{Sing}(f)$, not only its dimension. For $k \geqslant i$ let

$$\mathcal{P}^{(i,k)}_{d,N+1} \subset \mathcal{P}^{(i)}_{d,N+1}$$

be the closure of the set $\mathcal{P}^{(i)}_{d,N+1}$, consisting of polynomials f, such that $\mathrm{Sing}(f)$ contains an irreducible component C of dimension $i \geqslant 1$, the linear span $\langle C \rangle$ of which is a k-plane in $\mathbb{P}^N$. For instance, $\mathcal{P}^{(i,i)}_{d,N+1}$ consists of polynomials f, such that $\mathrm{Sing}(f)$ contains a i-plane in $\mathbb{P}^N$. The closure in this case is not needed: the set $\mathcal{P}^{(i,i)}_{d,N+1}$ allows the following obvious explicit description.

Proposition 1.1. *The following equality holds:*

$$\mathrm{codim}\left(\mathcal{P}^{(i,i)}_{d,N+1} \subset \mathcal{P}_{d,N+1}\right) = \binom{d+i}{d} + (N-i)\binom{d+i-1}{d-1} - (i+1)(N-i).$$

Proof. For an i-plane $P \subset \mathbb{P}^N$ by the symbol $\mathcal{P}^{(i,i)}_{d,N+1}(P)$ we denote the closed set of polynomials f, such that $P \subset \mathrm{Sing}(f)$. Fixing P, we may assume that

$$P = \{x_{i+1} = \ldots = x_N = 0\},$$

so that the property $f \in \mathcal{P}^{(i,i)}_{d,N+1}(P)$ is defined by the set of identical equalities

$$f|_P \equiv \left.\frac{\partial f}{\partial x_{i+1}}\right|_P \equiv \ldots \equiv \left.\frac{\partial f}{\partial x_N}\right|_P \equiv 0.$$

It is easy to see that these equalities are equivalent to vanishing the coefficients at the monomials in $x_0,\dots,x_i$ and the monomials of the form

$$x_j x_0^{a_0} x_1^{a_1}\dots x_i^{a_i},$$

$j=i+1,\dots,N$, which are all distinct. Therefore, the codimension of the closed set $\mathcal{P}^{(i,i)}_{d,N+1}(P)$ is

$$\binom{d+i}{d}+(N-i)\binom{d+i-1}{d-1}.$$

Taking into account that for a general polynomial $f\in\mathcal{P}^{(i,i)}_{d,N+1}(P)$ the equality $\mathrm{Sing}(f)=P$ holds for some i-plane P, we obtain the claim of the proposition.

1.2. The singular locus is a hypersurface in a linear subspace. Let us consider now the next by complexity example: let us estimate the codimension of the set $\mathcal{P}^{(i,i+1)}_{d,N+1}$. This set is the closure of the set of polynomials f, such that for some $(i+1)$-plane $P\subset\mathbb{P}^N$ and an irreducible hypersurface $C\subset P$ of degree $q\geqslant 2$ we have the inclusion $C\subset\mathrm{Sing}(f)$. Fixing the linear subspace P, we obtain the closed subset $\mathcal{P}^{(i,i+1)}_{d,N+1}(P)\subset\mathcal{P}^{(i,i+1)}_{d,N+1}$, so that

$$\mathcal{P}^{(i,i+1)}_{d,N+1}=\bigcup_{P\subset\mathbb{P}^N}\mathcal{P}^{(i,i+1)}_{d,N+1}(P),$$

where the union is taken over all $(i+1)$-planes in $\mathbb{P}^N$. By Bertini's theorem (and the explicit description of the polynomials $f\in\mathcal{P}^{(i,i+1)}_{d,N+1}(P)$, given below in the proof of Proposition 1.2), for a general polynomial $f\in\mathcal{P}^{(i,i+1)}_{d,N+1}$ there is a unique $(i+1)$-plane $P\subset\mathbb{P}^N$, such that $f\in\mathcal{P}^{(i,i+1)}_{d,N+1}(P)$, and for that reason the equality

$$\mathrm{codim}\left(\mathcal{P}^{(i,i+1)}_{d,N+1}\subset\mathcal{P}_{d,N+1}\right)=\mathrm{codim}\left(\mathcal{P}^{(i,i+1)}_{d,N+1}(P)\subset\mathcal{P}_{d,N+1}\right)-(i+2)(N-i-1)$$

holds.

Now let us fix P: we may assume that

$$P=\{x_{i+2}=\dots=x_N=0\}.$$

It is clear that $C\subset\mathrm{Sing}(f|_P)$. If $f|_P\not\equiv 0$, then C is a multiple component of the hypersurface $\{f|_P=0\}$. There are at most $\left[\frac{d}{4}\right]$ such components and they are determined by the polynomial $f|_P$. However, there is also another option: $P=\mathrm{Sing}(f|_P)$, that is, $f|_P\equiv 0$. In that case the subvariety of singularities $C\subset P$ is determined by the polynomial f, but not by its restriction $f|_P$. In order to take both options into account, let us write

$$\mathcal{P}^{(i,i+1)}_{d,N+1}(P)=\mathcal{P}^{(i,i+1;i)}_{d,N+1}(P)\cup\mathcal{P}^{(i,i+1;i+1)}_{d,N+1}(P),$$

where $\mathcal{P}^{(i,i+1;l)}_{d,N+1}(P)$ is the closure of the set of polynomials $f\in\mathcal{P}^{(i,i+1)}_{d,N+1}(P)$, such that

$$\dim\mathrm{Sing}(f|_P)=l.$$

The codimension of the set $\mathcal{P}^{(i,i+1)}_{d,N+1}(P)$ is the minimum of the codimensions of those two sets. It is obvious from the explicit formulas for those codimensions that the minimum is attained at the first set.

Proposition 1.2. (i) *For $d \geqslant 4$, $d \neq 6$ or for $d = 6$, $i \leqslant N-2$ the following equality holds:* codim $\left(\mathcal{P}^{(i,i+1;i)}_{d,N+1}(P) \subset \mathcal{P}_{d,N+1}\right) =$

$$= \binom{d+i+1}{i+1} - \binom{d+i-3}{i+1} - \binom{i+3}{i+1} + (N-i-1)\left(\binom{d+i}{i+1} - \binom{d+i-2}{i+1}\right).$$

(ii) *For $d = 6$, $i = N-1$ the following equality holds*

$$\operatorname{codim}\left(\mathcal{P}^{(N-1,N;N-1)}_{6,N+1}(P) \subset \mathcal{P}_{6,N+1}\right) = \binom{N+6}{6} - \binom{N+3}{3} - 1.$$

(iii) *The following equality holds* codim $\left(\mathcal{P}^{(i,i+1;i+1)}_{d,N+1}(P) \subset \mathcal{P}_{d,N+1}\right) =$

$$= \min\left\{\binom{d+i+1}{i+1} - \binom{i+3}{i+1} + (N-i-1)\left(\binom{d+i}{i+1} - \binom{d+i-2}{i+1}\right),\right.$$

$$\left.\binom{d+i+1}{i+1} + (N-i-2)\left(\binom{d+i}{i+1} - 1\right)\right\}.$$

Proof. Let us show the claim (i). For a general polynomial $f \in \mathcal{P}^{(i,i+1;i)}_{d,N+1}(P)$ we have: $f|_P \not\equiv 0$, the hypersurface $\{f|_P = 0\} \subset P$ has a multiple component C of degree $q \geqslant 2$, and moreover,

$$\left.\frac{\partial f}{\partial x_j}\right|_C \equiv 0 \tag{1}$$

for $j = i+2, \dots, N$. Note that the coefficients of the polynomials $f|_P$ and $\left.\frac{\partial f}{\partial x_j}\right|_P$, $j = i+2, \dots, N$, corrrespond to *distinct* coefficients of the original polynomial f. The requirement that the hypersurface $\{f|_P = 0\} \subset P$ has a double component C of degree $q \geqslant 2$, gives

$$E_q = \binom{d+i+1}{i+1} - \binom{d-2q+i+1}{i+1} - \binom{q+i+1}{i+1}$$

independent conditions for the coefficients of the polynomial $f|_P$.

Lemma 1.1. *For $d \neq 6$ the minimum of the numbers E_q, $q = 2, \dots, [d/2]$, is attained at $q = 2$.*

Proof. We have: $(i+1)!(E_{q+1} - E_q) =$

$$= [(d-2q+i+1)\dots(d-2q+1) - (d-2q+i-1)\dots(d-2q-1)]-$$

$$-[(q+i+2)\dots(q+2) - (q+i+1)\dots(q+1)],$$

whence after simplifications we get $i!(E_{q+1}-E_q)=$

$$=(2d-4q+i)(d-2q+i-1)\dots(d-2q+1)-(q+i+1)\dots(q+2).$$

The first product decreases when q is increasing, the second one is increasing. It is easy to check that for $d\geqslant 7$ the inequality $E_3>E_2$ holds. Therefore, for the sequence of integers $E_q, q=2,\dots,[d/2]$, there are two options:

- either it is increasing,
- or it is first increasing ($q=2,\dots,q^*$), and then decreasing ($q=q^*,\dots,[d/2]$).

In the first case the claim of the lemma is obvious. In the second case the minimum of the numbers E_q is attained either at $q=2$, or at $q=[d/2]$, and an easy check shows that the minimum corresponds precisely to the value $q=2$. Q.E.D. for the lemma.

Now let us fix the polynomial $f|_P\not\equiv 0$. Since the hypersurface $\{f|_P=0\}$ has finitely many components, we may assume that the irreducible hypersurface C of degree $q\geqslant 2$ is fixed. Now the requirement (1) imposes on the coefficients of the polynomial $(\partial f/\partial x_j)|_P$ precisely

$$\binom{d+i}{i+1}-\binom{d-q+i}{i+1} \tag{2}$$

independent conditions, and it is obvious, that the minimum of the last expression is attained at $q=2$. This completes the proof of the claim (i) (an explicit check shows that it is true for $d=6$, too, although for $d=6$ the claim of Lemma 1.1 is not true: $E_3<E_2$.) The claim (ii) is shown by explicit simple computations.

Let us show the claim (iii). In that case the hypersurface $\{f=0\}$ contains the entire subspace P. The closed subset of polynomials f, such that $f|_P\equiv 0$, has codimension $\binom{d+i+1}{i+1}$. Furthermore, either all partial derivatives $\frac{\partial f}{\partial x_j}, j=i+2.\dots,N$, vanish on P, so that $P\subset \mathrm{Sing}(f)$ and this gives an essentially higher codimension than what is claimed by (iii), and for that reason this option can be ignored, or $\frac{\partial f}{\partial x_j}|_P\not\equiv 0$ for some $j\in\{i+2,\dots,N\}$. Without loss of generality we may assume that $j=i+2$. Then all polynomials $\partial f/\partial x_j$, $j=i+3,\dots,N$, vanish on one of the components C of the hypersurface $\left\{\dfrac{\partial f}{\partial x_{i+2}}=0\right\}\subset P$, where $\deg C=q\geqslant 2$. This component can be assumed to be fixed and this gives for each of the $(N-i-2)$ polynomials $\partial f/\partial x_j$, $j=i+3,\dots,N$, the new independent conditions, the number of which is given by the formula (2). Taking into account that the hypersurface $\{\partial f/\partial x_{i+2}=0\}$ is reducible, we finally obtain

$$\binom{d+i+1}{i+1}+(N-i-1)\left(\binom{d+i}{i+1}-\binom{d-q+i}{i+1}\right)-\binom{i+q+1}{i+1}$$

independent conditions for the coefficients of the polynomial f. Using the same method as in the proof of the claim (i), it is easy to show that the minimum of the last expression for $q = 2, \ldots, d-1$ is attained at one of the end values of q: either at $q = 2$, or at $q = d-1$. Proof of Proposition 1.2 is complete. Q.E.D.

1.3. Statement of the main result for the general case. Now let us consider the general case. By the symbol $\mathcal{P}^{(i,k;l)}_{d,N+1}(P)$ for a k-plane $P \subset \mathbb{P}^N$ denote the closure of the set of polynomials $f \in \mathcal{P}_{d,N+1}$ such that:

- the set $\mathrm{Sing}(f)$ has an irreducible component $C \subset P$ of dimension i, and moreover $\langle C \rangle = P$,
- the set $\mathrm{Sing}(f|_P)$ has an irreducible component B of dimension $l \geqslant i$, and moreover $C \subset B$.

By the symbol $\mathcal{P}^{(i,k)}_{d,N+1}(P)$ for a k-plane $P \subset \mathbb{P}^N$ denote the closure of the set of polynomials $f \in \mathcal{P}_{d,N+1}$ such that the first of the two conditions stated above is satisfied. Obviously,

$$\mathcal{P}^{(i,k)}_{d,N+1}(P) = \bigcup_{l=i}^{k} \mathcal{P}^{(i,k;l)}_{d,N+1}(P).$$

Everywhere in the sequel the codimension of various closed sets in the space of polynomials $\mathcal{P}_{d,N+1}$ is meant to be with respect to that space, so that, for instance, $\mathrm{codim}\,\mathcal{P}^{(i,k)}_{d,N+1}(P)$ is the minimum of the codimensions $\mathrm{codim}\,\mathcal{P}^{(i,k;l)}_{d,N+1}(P)$, where $l = i, \ldots, k$, and the following estimate holds:

$$\mathrm{codim}\,\mathcal{P}^{(i,k)}_{d,N+1} \geqslant \mathrm{codim}\,\mathcal{P}^{(i,k)}_{d,N+1}(P) - (k+1)(N-k).$$

Remark 1.1. It is easy to see that for $N - k < l - i$ the singular set of a polynomial $f \in \mathcal{P}^{(i,k;l)}_{d,N+1}(P)$ is of dimension at least $i+1$. Indeed, $\mathrm{Sing}(f|_P)$ contains an l-dimensional irreducible component $C \subset P$, where $l > i$. If $k = N$, then $C \subset \mathrm{Sing}(f)$. If $k < N$, then

$$\left[C \cap \left\{ \left. \frac{\partial f}{\partial x_{k+1}} \right|_P = \ldots = \left. \frac{\partial f}{\partial x_N} \right|_P = 0 \right\} \right] \subset \mathrm{Sing}(f),$$

so that $\dim \mathrm{Sing}(f) \geqslant l - (N-k) > i$, as we claimed. For that reason everywhere below, whenever we consider the sets $\mathcal{P}^{(i,k;l)}_{d,N+1}(P)$, we assume that the inequality $N + i \geqslant k + l$ holds.

In order to give a compact statement of the main result about these codimensions, we introduce one notation more. For positive integers a, b, c, where $b \leqslant N$, set

$$\tau(a,b,c) = \max \left\{ \binom{a+c}{c},\ ab+1 \right\}.$$

If we fix c, then the first of the two numbers exceeds the second one for the values of a that are higher than a number of order $\dfrac{c}{e} N^{\frac{1}{c}}$, where e is the base of the natural logarithm. If we fix a, the first of the two numbers exceeds the second one for the values of c that are higher than a number of order $\dfrac{a}{e}N^{\frac{1}{a}}$. The meaning of the function τ will be clear in §2.

Theorem 1.1. *For $l \leqslant k-2$ the following estimate holds:*

$$\mathrm{codim}\,\mathcal{P}^{(i,k;l)}_{d,N+1}(P) \geqslant (k-l+1)\binom{d+l-2}{l+1} + (N+i-k-l)\,\tau(d-1,k,i).$$

Theorem 1.2. (i) *For $l=k-1$, $d\neq 6$ the following estimate holds:*

$$\mathrm{codim}\,\mathcal{P}^{(i,k;l)}_{d,N+1}(P) \geqslant \left[\binom{d+k}{k} - \binom{d-4+k}{k} - \binom{k+2}{k}\right] + (N+i-2k+1)\,\tau(d-1,k,i).$$

(ii) *For $l=k-1, d=6$ the following estimate holds:*

$$\mathrm{codim}\,\mathcal{P}^{(i,k;l)}_{d,N+1}(P) \geqslant \left[\binom{k+6}{k} - \binom{k+3}{k} - 1\right] + (N+i-2k+1)\,\tau(5,k,i).$$

Theorem 1.3. *For $l=k$ the following estimate holds:*

$$\mathrm{codim}\,\mathcal{P}^{(i,k;l)}_{d,N+1}(P) \geqslant \binom{d+k}{k} + (N+i-k-l)\,\tau(d-1,k,i).$$

Proof of these three theorems will be given in §2.

2 Good sequences and linear spans

In this section we prove Theorems 1.1-1.3. Theorem 1.1 is the hardest one. First (Subsection 2.1) we describe the strategy of the proof of this theorem and give the definition of good sequences and associated subvarieties: this technique amkes it possible to reconstruct the subvariety $C \subset \mathrm{Sing}(f)$ inside the, generally speaking, larger subvariety $B \subset \mathrm{Sing}(f|_P)$. After that we estimate the codimension of the subset of polynomials on P with a non-degenerate subvariety of singular points of dimension l (Subsection 2.2). Finally, in Subsection 2.3 we complete the proof of Theorem 1.1 by means of well known methods of estimating the codimension of the set of polynomials, vanishing on a given non-degenerate subvariety; after that we show Theorems 1.2 and 1.3, which is easy.

2.1. Plan and start of the proof of Theorem 1.1. Let us describe the strategy of the proof of Theorem 1.1. Fix a k-plane $P \subset \mathbb{P}^N$. We will assume that it is the coordinate plane

$$P = \{x_{k+1} = \ldots = x_N = 0\}.$$

As we noted in Subsection 1.2, the coefficients of the polynomials $f|_P$ and $\left.\frac{\partial f}{\partial x_j}\right|_P$, $j=k+1,\ldots,N$, correspond to distinct coefficients of the polynomial f. For that reason, considering the general polynomial $f \in \mathcal{P}^{(i,k;l)}_{d,N+1}(P)$, one has to solve three problems:

1) estimate the codimension of the closed subset $\mathcal{P}^{(l,k)}_{d,k+1}$ in the space $\mathcal{P}_{d,k+1}$ (since, obviously, $f|_P \in \mathcal{P}^{(l,k)}_{d,k+1}$),

2) using the $l-i$ polynomials $\left.\frac{\partial f}{\partial x_j}\right|_P$, where $j \in I \subset \{k+1,\dots,N\}$, so that $|I| = l-i$, reconstruct the variety of singular points $C \subset \mathrm{Sing}(f)$ as a subvariety of codimension $l-i$ of the variety of singular points $B \subset \mathrm{Sing}(f|_P)$, which depends on the restriction $f|_P$ only and for that reason can be assumed to be fixed, if we fix the polynomial $f|_P \in \mathcal{P}^{(l,k)}_{d,k+1}$,

3) estimate the codimension of the closed set of polynomials $h \in \mathcal{P}_{d-1,k+1}$, vanishing on a fixed non-degenerate subvariety $B \subset P$, and apply this estimate to the $(N+i-k-l)$ polynomials $\left.\frac{\partial f}{\partial x_j}\right|_P$, $j \in \{k+1,\dots,N\}$, $j \notin I$.

The sum of the estimate, obtained at the stage 1), with the $(N+i-k-l)$-multiple of the estimate, obtained at the stage 3), is precisely the inequality, claimed by Theorem 1.1.

Let us start to realize this programme.

First of all, recall the following definition (see [12, Section 3] or [11, Chapter 3]).

Definition 2.1. A sequence of homogeneous polynomials $g_1,\dots,g_m$ of arbitrary degrees on the projective space $\mathbb{P}^e$, $e \geqslant m+1$, is said to be a *good sequence*, and an irreducible subvariety $W \subset \mathbb{P}^e$ of codimension m is its *associated subvariety*, if there exists a sequence of irreducible subvarieties $W_j \subset \mathbb{P}^e$, $\mathrm{codim}\, W_j = j$ (in particular, $W_0 = \mathbb{P}^e$) such that:

- $g_{j+1}|_{W_j} \not\equiv 0$ for $j = 0,\dots,m+1$,
- W_{j+1} is an irreducible component of the closed algebraic set $g_{j+1}|_{W_j} = 0$,
- $W_m = W$.

A good sequence can have more than one associated subvarieties, but their number is bounded from above by a constant, depending on the degrees of the polynomials g_j only (see [12, Section 3]).

Assuming the polynomial $f|_P$ and the subvariety B to be fixed, let us construct a good sequence of polynomials on $P = \mathbb{P}^k$ with the subvariety C as one of its associated subvarieties. This sequence starts with $g_1 = f|_P \not\equiv 0$. Since B is an irreducible l-dimensional component of the closed set $\mathrm{Sing}(f|_P)$, for some $(k-l-1)$ polynomials $\left.\frac{\partial f}{\partial x_j}\right|_P$, $j \in \{0,\dots,k\}$, we obtain a good sequence of polynomials with B as one of its associated subvarieties. If $l = i$, then there is nothingmore to construct. Assume that $l \geqslant i+1$. Then among the polynomials $\left.\frac{\partial f}{\partial x_j}\right|_P$, $j \in \{k+1,\dots,N\}$, there is one which does not vanish on B (otherwise, $B \subset \mathrm{Sing}(f)$, so that $f \in \mathcal{P}^{(l,k;l)}_{d,N+1}(P)$, and this contradicts to the assumption that $C \subset B$, $C \neq B$ is an irreducible component of the set $\mathrm{Sing}(f)$). We add this polynomial to already constructed sequence. Continuing in this way (using at every step the fact that C is an irreducible component of the set $\mathrm{Sing}(f)$), we complete the construction of a good sequence. Assuming the polynomials of the good sequence to be fixed, we may assume that the variety $C \subset P$ to be fixed as well. This solves the problem 2), stated above. Now let us consider the most difficult problem 1).

2.2. Linearly independent singular points. The problem 1), stated above, is solved in the following claim.

Proposition 2.1. *The following estimate holds:*

$$\mathrm{codim}\left(\mathcal{P}^{(l,k)}_{d,k+1} \subset \mathcal{P}_{d,k+1}\right) \geqslant (k-l+1)\binom{d+l-2}{l+1}. \tag{3}$$

Proof. In order to simplify the notations, we assume that $k = N$. Let us describe the technique of estimating the codimension of the closed subset of the space $\mathcal{P}_{d,N+1}$, consisting of polynomials with many singular points. The following claim is true.

Lemma 2.1. *Assume that $d \geqslant 3$. For any set of m linearly independent points $p_1, \dots, p_m \in \mathbb{P}^N$, $m \leqslant N+1$, the condition*

$$\{p_1, \dots, p_m\} \subset \mathrm{Sing}(g),$$

$g \in \mathcal{P}_{d,N+1}$, defines a linear subspace of codimension $m(N+1)$ in $\mathcal{P}_{d,N+1}$.

Proof. We may assume that

$$p_1 = (1:0:0\dots:0), \quad p_2 = (0:1:0:\dots:0)$$

and so on correspond to the first m vectors of the standard basis of the linear space $\mathbb{C}^{N+1}$. The condition $p_i \in \mathrm{Sing}(g)$ means vanishing of the coefficients at the monomials $x^d_{i-1}, x^{d-1}_{i-1}x_j$, for all $j \neq i-1$. For $d \geqslant 3$ all these $m(N+1)$ monomials are distinct. Q.E.D. for the lemma.

Now let us consider an arbitrary linear subspace $\Pi \subset \mathbb{P}^N$ of codimension $r+1$, where $r \geqslant 1$, given by a system of $r+1$ equations

$$l_0(x) = 0,\ l_1(x) = 0, \dots, l_r(x) = 0,$$

where $l_0, \dots, l_r$ are linearly independent forms. For each $i = 1, \dots, r$ fix an arbitrary set of distinct constants $\lambda_{i,0}, \dots, \lambda_{i,d-1} \in \mathbb{C}$; we assume that $\lambda_{i,0} = 0$ for all $i = 1, \dots, r$. Now for any integer valued point

$$\underline{e} = (e_1, \dots, e_r) \in \mathbb{Z}^r_+, \quad e_i \leqslant d-1,$$

by the symbol $\Theta(\underline{e})$ we denote the linear subspace

$$\{l_i(x) - \lambda_{i,e_i} l_0(x) = 0 \,|\, i = 1, \dots, r\} \subset \mathbb{P}^N$$

of codimension r. Obviously, $\Theta(\underline{e}) \supset \Pi$. Set

$$|\underline{e}| = e_1 + \ldots + e_r \in \mathbb{Z}_+.$$

For every tuple $\underline{e} \in \mathbb{Z}_+^r$ with $|\underline{e}| \leqslant d-3$ consider an arbitrary set

$$S(\underline{e}) = \{p_1(\underline{e}), \ldots, p_m(\underline{e})\} \subset \Theta(\underline{e}) \backslash \Pi$$

of m linearly independent points (so that $m \leqslant N-r+1$).

Proposition 2.2. *The set of conditions*

$$S(\underline{e}) \subset \mathrm{Sing}(\mathrm{g}|_{\Theta(\underline{\mathrm{e}})}),$$

$\underline{e} \in \mathbb{Z}_+^r$, $|\underline{e}| \leqslant d-3$, *defines a linear subspace of codimension*

$$m(N-r+1)|\Delta|$$

in $\mathcal{P}_{d,N+1}$, *where*

$$\Delta = \{e_1 \geqslant 0, \ldots, e_r \geqslant 0, e_1 + \ldots + e_r \leqslant d-3\} \subset \mathbb{R}^r$$

is an integer valued simplex and $|\Delta|$ *is the number of integral points in that simplex,* $|\Delta| = \sharp(\Delta \cap \mathbb{Z}^r)$.

Proof. We may assume that $l_0 = x_0,\ l_1 = x_1, \ldots, l_r = x_r$. In order to simplify the formulas we will prove the affine version of the proposition: set $v_1 = x_1/x_0, \ldots, v_r = x_r/x_0$ and $u_i = x_{r+i}/x_0,\ i = 1, \ldots, N-r$. In the affine space $\mathbb{A}^N \subset \mathbb{P}^N$, $\mathbb{A}^N = \mathbb{P}^N \backslash \{x_0 = 0\}$ with coordinates $(u,v) = (u_1, \ldots, u_{N-r},\ v_1, \ldots, v_r)$ the affine spaces $A(\underline{e}) = \Theta(\underline{e}) \backslash \Pi$ are contained entirely:

$$A(\underline{e}) = \Theta(\underline{e}) \cap \mathbb{A}^N,$$

so that $S(\underline{e}) \subset A(\underline{e})$ for all $\underline{e}$. Obviously,

$$A(\underline{e}) = \{v_1 = \lambda_{1,e_1}, \ldots, v_r = \lambda_{r,e_r}\} \subset \mathbb{A}^N$$

is a $(N-r)$-plane, which is parallel to the coordinate $(N-r)$-plane $(u_1, \ldots, u_{N-r}, 0, \ldots, 0)$. Now let us write the polynomial g in terms of the affine coordinates (u,v) in the following way:

$$g(u,v) = \sum_{\underline{e} \in \mathbb{Z}_+^r, |\underline{e}| \leqslant d} g_{e_1,\ldots,e_r}(u) \prod_{i=1}^{r} \prod_{j=0}^{e_i - 1} (v_i - \lambda_{i,j})$$

(if $e_i = 0$, then the corresponding product is meant to be equal to 1). Here $g_{\underline{e}}(u) = g_{e_1,\ldots,e_r}(u)$ is an affine polynomial in $u_1, \ldots, u_{N-r}$ of degree $\deg g_{\underline{e}} \leqslant d - |e|$. When $\lambda_{i,j}$ are fixed, this expression is unique. By Lemma 2.1, the condition

$$S(\underline{0}) = S(0, \ldots, 0) \subset \mathrm{Sing}(g|_{A(\underline{0})})$$

defines a linear subspace of codimension $m(N-r+1)$ in the space of polynomials $\mathcal{P}_{d,N-r+1}$. However it is easy to see that

$$g|_{A(\underline{0})} = g_{0,\dots,0}(u),$$

since for $\underline{e} \neq \underline{0}$ in the product

$$\prod_{i=1}^{r} \prod_{j=0}^{e_i-1} (v_i - \lambda_{i,j})$$

there is at least one factor $(v_i - \lambda_{i,0}) = v_i$, which vanishes when we restrict it onto the $(N-r)$-plane $A(\underline{0})$. Therefore, the condition $S(\underline{0}) \subset \mathrm{Sing}(g|_{A(\underline{0})})$ imposes on the coefficients of the polynomial $g_{0,\dots,0}(u)$ precisely $m(N-r+1)$ independent conditions, whereas the polynomials $g_{\underline{e}}(u)$ for $\underline{e} \neq \underline{0}$ can be arbitrary.

Now let us complete the proof of Proposition 2.2 by induction on $|\underline{e}|$. More precisely, for every $a \in \mathbb{Z}_+$ set

$$\Delta_a = \{e_1 \geqslant 0, \dots, e_r \geqslant 0,\ e_1 + \dots + e_r \leqslant a\} \subset \mathbb{R}^r,$$

so that $\Delta = \Delta_{d-3}$. Let us prove the claim of Proposition 2.2 in the following form: for every $a = 0, \dots, d-3$

$(*)_a$ the set of conditions

$$S(\underline{e}) \subset \mathrm{Sing}(g|_{\Theta(\underline{e})}),$$

$\underline{e} \in \mathbb{Z}_+^r$, $|\underline{e}| \leqslant a$, defines a linear subspace of codimension $m(N-r+1)|\Delta_a|$ in $\mathcal{P}_{d,N+1}$, and, moreover, the linear conditions are imposed on the coefficients of the polynomials $g_{\underline{e}}(u)$ for $\underline{e} \in \Delta_a$, whereas for $\underline{e} \notin \Delta_a$ the polynomials $g_{\underline{e}}(u)$ can be arbitrary.

The case $a = 0$ has already been considered, so that assume that $a \leqslant d-4$ and the claims $(*)_j$ for $j = 0, \dots, a$ have been shown. Let us show the claim $(*)_{a+1}$. Let $\underline{e} \in \mathbb{Z}_+^r$ be an arbitrary multi-index, $|\underline{e}| = a+1$. The restriction onto the affine subspace $A(\underline{e})$ means the substitution $v_1 = \lambda_{1,e_1}, \dots, v_r = \lambda_{r,e_r}$. Therefore the polynomial $g_{\underline{e}(u)}$ comes into the restriction $g|_{A(\underline{e})}$ with the non-zero coefficient

$$\alpha_{\underline{e}} = \prod_{i=1}^{r} \prod_{j=0}^{e_i-1} (\lambda_{i,e_i} - \lambda_{i,j}).$$

On the other hand, for $\underline{e}' \neq \underline{e}$, $|\underline{e}'| \geqslant a+1$ the product

$$\prod_{i=1}^{r} \prod_{j=0}^{e'_i-1} (\lambda_{i,e_i} - \lambda_{i,j}).$$

is equal to zero, since for at least one index $i \in \{1, \ldots, r\}$ we have $e'_i > e_i$ and so this product contains a factor equal to zero. Therefore, $g|_{A(\underline{e})}$ is the sum of the polynomial $\alpha_{\underline{e}} g_{\underline{e}}$ and a linear combination of the polynomials $g_{\underline{e}'}$ with $|\underline{e}'| \leqslant a$ with constant coefficients. Now, fixing the polynomials $g_{\underline{e}'}$ with $|\underline{e}'| \leqslant a$, we obtain that the condition

$$S(\underline{e}) \subset \mathrm{Sing}(g|_{A(\underline{e})})$$

defines an *affine* (generally speaking, not linear) subspace of codimension $m(N - r + 1)$ of the space of polynomials $g_{\underline{e}}(u_1, \ldots, u_{N-r})$ of degree at most $d - |e|$, the corresponding linear subspace of which is given by the condition

$$S(\underline{e}) \subset \mathrm{Sing}\, g_{\underline{e}}(u).$$

Moreover, no restrictions are imposed on the coefficients of other polynomials $g_{\underline{e}'}$ with $|\underline{e}'| = a + 1$.

This proves the claim $(*)_a$ for all $a = 0, \ldots, d-3$. Proof of Proposition 2.2 is complete.

Now let

$$\Theta = \Theta[l_0, \ldots, l_r; \lambda_{i,j}, i = 1, \ldots, r, j = 0, \ldots, d-1] = \{\Theta(\underline{e}) \,|\, \underline{e} \in \Delta\}$$

be some set of linear subspaces of codimension r in $\mathbb{P}^N$, considered in Proposition 2.2. We define the subset

$$\mathcal{P}_{d,N+1}(\Theta) \subset \mathcal{P}_{d,N+1}$$

by the following condition: for every subspace $\Theta(\underline{e})$ with $|\underline{e}| \leqslant d - 3$ there is a set $S(\underline{e}) \subset \Theta(\underline{e}) \backslash \Pi$, consisting of m linearly independent points, such that $S(\underline{e}) \subset \mathrm{Sing}(g|_{\Theta(\underline{e})})$.

Proposition 2.3. *The following inequality holds:*

$$\mathrm{codim}(\mathcal{P}_{d,N+1}(\Theta) \subset \mathcal{P}_{d,N+1}) \geqslant m|\Delta|.$$

Proof is obtained by means of the obvious dimension count: the subspaces $\Theta(\underline{e})$ are fixed, so that every point $p_i(\underline{e})$ varies in a $(N-r)$-dimensional family. Q.E.D. for the proposition.

Finally, let us complete the proof of Proposition 2.1. Set $N = k$, so that the space of polynomials of degree d is $\mathcal{P}_{d,k+1}$. For an arbitrary set $\Theta = \{\Theta(\underline{e}) \,|\, \underline{e} \in \Delta\}$ of linear subspaces of codimension l in $P = \mathbb{P}^k$ let

$$\mathcal{P}^{(l,k)}_{d,k+1}(P, \Theta) \subset \mathcal{P}^{(l,k)}_{d,k+1}$$

be the set of polynomials $h \in \mathcal{P}^{(l,k)}_{d,k+1}$ such that the set $\mathrm{Sing}(h)$ has an irreducible component Q of dimension l, where $\langle Q \rangle = P$ and the variety Q is in general position with the subspaces from the set Θ: for all $\underline{e} \in \Delta$ the set $\Theta(\underline{e}) \cap Q$ contains $(k-l+1)$ linearly independent points. Since $\langle Q \rangle = P$, the subset $\mathcal{P}^{(l,k)}_{d,k+1}(P, \Theta)$ is a Zariski open subset of the set $\mathcal{P}^{(l,k)}_{d,k+1}$, so that the inequality (3) will be shown, if we prove it for $\mathcal{P}^{(l,k)}_{d,k+1}(\Theta)$ instead of $\mathcal{P}^{(l,k)}_{d,k+1}$. However, for $\mathcal{P}^{(l,k)}_{d,k+1}(\Theta)$ this inequality follows immediately from Proposition 2.3, since in that case $m = k-l+1$ and $|\Delta| = \binom{d+l-2}{l+1}$.

Proof of Proposition 2.1 is complete. Q.E.D.

2.3. Polynomials, vanishing on a given variety. By the symbol $\mathcal{P}_{d-1,k+1}(B)$ we denote the closed subset of polynomials $h \in \mathcal{P}_{d-1,k+1}$ such that $h|_B \equiv 0$ for a fixed irreducible subvariety B. In our case $\dim B = i$ and $\langle B \rangle = \mathbb{P}^k$. There are two methods of estimating the codimension of the set $\mathcal{P}_{d-1,k+1}(B)$.

The first method was developed in [10] (see also [11, Chapter 3]). Consider a general linear projection $\mathbb{P}^k \dashrightarrow \mathbb{P}^i$, so that $\pi|_B$ is a regular surjective map. For any non-zero polynomial $g \in \mathcal{P}_{d-1,i+1}$ we have $\pi^* g|_B \not\equiv 0$, so that

$$\operatorname{codim} \mathcal{P}_{d-1,k+1}(B) \geqslant \mathcal{P}_{d-1,i+1} = \binom{d-1+i}{i}.$$

The second method was developed in [12] (see also [11, Chapter 3]). Since $\langle B \rangle = \mathbb{P}^k$, a non-zero linear form can not vanish on B. Therefore, the closed subset of decomposable forms

$$\underbrace{\mathcal{P}_{1,k+1} \cdot \mathcal{P}_{1,k+1} \cdot \dots \cdot \mathcal{P}_{1,k+1}}_{d-1} \subset \mathcal{P}_{d-1,k+1}$$

intersects with $\mathcal{P}_{d-1,k+1}(B)$ by zero only, so that

$$\operatorname{codim} \mathcal{P}_{d-1,k+1}(B) \geqslant (d-1)k + 1.$$

Finally we get:

$$\operatorname{codim}(\mathcal{P}_{d-1,k+1}(B) \subset \mathcal{P}_{d-1,k+1}) \geqslant \tau(d-1,k,i).$$

By the arguments of Subsection 2.1 about good sequences and Proposition 2.1, this completes the proof of Theorem 1.1.

Let us show Theorems 1.2 and 1.3. By the arguments given above, we only need to prove the inequalities

$$\operatorname{codim}(\mathcal{P}^{(k-1,k)}_{d,k+1} \subset \mathcal{P}_{d,k+1}) \geqslant \binom{d+k}{k} - \binom{d-4+k}{k} - \binom{k+2}{k}$$

for $d \neq 6$ and

$$\operatorname{codim}(\mathcal{P}^{(k-1,k)}_{6,k+1} \subset \mathcal{P}_{6,k+1}) \geqslant \binom{k+6}{k} - \binom{k+3}{k} - 1$$

(both are in fact equalities), which are obtained by repeating the arguments that were used in the proof of Proposition 1.2 word for word. The claim of Theorem 1.3 is obvious.

3 The codimension of the set of non-factorial hypersurfaces

In this section we prove Theorem 0.1. First we give the list of possible values of the parameters $i = N-4, k, l$, so that the complement to the set of polynomials $\mathcal{P}^{\rm fact}_{d,N+1}$ is contained in the union of the sets $\mathcal{P}^{(i,k;l)}_{d,N+1}$. After that, the estimates obtained in §§1-2 are applied in order to estimate the codimension of the set of non-factorial hypersurfaces (Subsection 3.1); this estimate is used for proving part (ii) of Theorem 0.1 in Subsection 3.2 and part (i) of Theorem 0.1 in Subsection 3.3.

3.1. Hypersurfaces with the singular locus of codimension three. Note the following simple

Proposition 3.1. *For $N \geqslant 7$, $d \geqslant 4$, $k \leqslant N$ and $(d,k,N) \neq (4,7,7)$ the following equality holds:*

$$\tau(d-1,k,N-4) = \binom{d+N-5}{N-4}.$$

Proof: simple computations. The case $d=4$ is easy to do, considering the cubic (in N) polynomial

$$\binom{N-1}{N-4} = \binom{N-1}{3}.$$

Therefore assume that $d \geqslant 5$. The claim of the proposition is equivalent to the inequality

$$(N-3)((N-3)+1)\dots((N-3)+(d-2)) > N(d-1)(d-1)!.$$

Since for $N \geqslant 7$ we have $N < 2(N-3)$, the last inequality follows from the estimate

$$\xi(N-3) \geqslant 2(d-1)(d-1)!,$$

where $\xi(t) = (t+1)(t+2)\dots(t+(d-2))$. Since the function $\xi(t)$ is increasing, it is sufficient to set $N=7$ and show the inequality

$$\xi(4) = \frac{(d+2)!}{4!} \geqslant 2(d-1)(d-1)!,$$

which for $d \geqslant 5$ is equivalent to the inequality

$$d(d+1)(d+2) \geqslant 48(d-1).$$

This inequality holds for $d \geqslant 5$ in an obvious way. Q.E.D. for the proposition.

By Grothendieck's theorem,

$$\mathcal{P}_{d,N+1} \setminus \mathcal{P}^{\rm fact}_{d,N+1} \subset \mathcal{P}^{(N-4)}_{d,N+1},$$

so that in order to estimate the codimension of the set of non-factorial hypersurfaces, we will estimate the codimension of the set of hypersurfaces with the singular locus of codimension three. In the notations of §1, the set $\mathcal{P}^{(N-4)}_{d,N+1}$ is the union of the following eight sets:

$$\mathcal{P}^{(N-4,N-4)}_{d,N+1},\quad \mathcal{P}^{(N-4,N-3)}_{d,N+1},\quad \mathcal{P}^{(N-4,N-2;N-4)}_{d,N+1},\quad \mathcal{P}^{(N-4,N-2;N-3)}_{d,N+1},$$

$$\mathcal{P}^{(N-4,N-2;N-2)}_{d,N+1},\quad \mathcal{P}^{(N-4,N-1;N-4)}_{d,N+1},\quad \mathcal{P}^{(N-4,N-1;N-3)}_{d,N+1},\quad \mathcal{P}^{(N-4,N)}_{d,N+1}.$$

We set respectively:

$$\alpha_1 = \binom{d+N-4}{N-4} + 4\binom{d+N-5}{N-4} - 4(N-3),$$

$\alpha_2 = \min\{\alpha_{2a}, \alpha_{2b}\}$, where

$$\alpha_{2a} = \binom{d+N-3}{N-3} - \binom{d+N-7}{N-3} + 3\left[\binom{d+N-4}{N-3} - \binom{d+N-6}{N-3}\right] - \frac{(N+5)(N-2)}{2},$$

$$\alpha_{2b} = \binom{d+N-3}{N-3} + 2\binom{d+N-4}{N-3} - 3N + 4.$$

Furthermore,

$$\alpha_3 = 3\binom{d+N-6}{N-3} + 2\binom{d+N-5}{N-4} - 2(N-1),$$

$$\alpha_4 = \binom{d+N-2}{N-2} - \binom{d+N-6}{N-2} + \binom{d+N-5}{N-4} - \frac{(N+4)(N-1)}{2}$$

and

$$\alpha'_4 = \binom{N+4}{6} - \binom{N+1}{3} + \binom{N-1}{3} - 2N + 1$$

depends on the dimension N only. Finally,

$$\alpha_5 = \binom{d+N-2}{N-2} - 2(N-1),$$

$$\alpha_6 = 4\binom{d+N-6}{N-3} + \binom{d+N-5}{N-4} - N,$$

$$\alpha_7 = 3\binom{d+N-5}{N-2} - N$$

and

$$\alpha_8 = 5\binom{d+N-6}{N-3}.$$

Now Propositions 1.1, 1.2 and Theorems 1.1-1.3, taking into account Proposition 3.1, immediately imply

Theorem 3.1. *For $N \geqslant 7$ the following inequality holds:*

$$\mathrm{codim}\left(\mathcal{P}_{d,N+1} \setminus \mathcal{P}^{\mathrm{fact}}_{d,N+1}\right) \geqslant \min\{\alpha_i \,|\, i = 1, \ldots, 8\}.$$

Remark 3.1. For $d = 6$ in this inequality one should replace α_4 by α_4', however, the minimum of the right hand side is attained at α_8 all the same, so that the claim of Theorem 3.1 remains correct in this case as well.

3.2. Hypersurfaces of general type. In oredr to prove the claim (ii) of Theorem 0.1, one needs to check that $\alpha_i \geqslant \alpha_1$ for $i \geqslant 2$, if $d \geqslant 2N$. This check is elementary and we do not perform it here, giving only one example: setting $d = 2N + a$, write

$$\alpha_8 - \alpha_1 = 4(N-3) + \frac{(3N+a-6)!}{(N-3)!(2N+a)!} \times$$

$$\times [10N(2N+a-1)(2N+a-2) - (N-3)(3N+a-5)(11N+5a-4)].$$

It is easy to see that for $a \geqslant 0$ the expression in the square brackets is positive, which implies that $\alpha_8 > \alpha_1$. (In fact, the difference $\alpha_8 - \alpha_1$ is quite large, but we do not need that.) The remaining inequalities $\alpha_i > \alpha_1$ for $i \neq 2$ are shown in a similar way. The reason why α_1 realizes the minimum of α_i, $i = 1, \ldots, 8$, is the polynomiality of the functions α_i in d when the dimension N is fixed: α_1 is a polynomial of the least degree $N - 4$. The polynomial α_{2a} also has the degree $N - 4$, but its senior coefficient is much higher. The claim (ii) of Theorem 0.1 is shown.

We see that for $d \geqslant 2N$ the irreducible component of the maximal dimension of the closed set $\mathcal{P}^{(N-4)}_{d,N+1}$ is $\mathcal{P}^{(N-4,N-4)}_{d,N+1}$, that is, the set of polynomials $f \in \mathcal{P}_{d,N+1}$ such that the hypersurface $\{f = 0\}$ has a $(N-4)$-lane of singular points.

3.3. Fano hypersurfaces. Let us prove the claim (i) of Theorem 0.1. Again an elementary (but tiresome) check shows that for $i = 1, \ldots, 6$ the inequality

$$\alpha_i \geqslant \min\{\alpha_7, \alpha_8\}$$

holds (for $d = 6$ with α_4 replaced by α_4'), which implies the claim (i). We do not give the tiresome computations here, except for one example:

$$\alpha_6 - \alpha_8 = -N + \frac{(d+N-6)!}{(N-3)!(d-1)!}[-d^2 + d(N-1) + (N^2 - 9N + 18)].$$

It is easy to see that for $d = 4, \ldots, N$ the difference $\alpha_6 - \alpha_8$ is positive. In a similar way the other inequalities $\alpha_i > \alpha_8$ for $i = 1, \ldots, 5$ are checked. Proof of the claim (i) of Theorem 0.1 is complete.

Remark 3.2. Elementary computations, which we do not give here, show that for $d = 4, \ldots, d_*(N)$ the inequality $\alpha_7 \leqslant \alpha_8$ holds, and for $d = d_*(N) + 1, \ldots, N$ the opposite inequality $\alpha_7 > \alpha_8$ holds. Here $d_*(N) \sim \frac{2}{3}N$. More precisely, if $N = 3m + e$, $e \in \{0, 1, 2\}$, then $d_*(N) = 2m + e + 1$.

References

[1] Mella M., Birational geometry of quartic 3-folds. II. The importance of being $\mathbb{Q}$-factorial. Math. Ann. **330** (2004), No. 1, 107-126.

[2] Call F. and Lyubeznik G., A simple proof of Grothendieck's theorem on the parafactoriality of local rings, Contemp. Math. **159** (1994), 15-18.

[3] A.V.Pukhlikov, Birational geometry of Fano hypersurfaces of index two. Math. Ann. Published online 26 December 2015. DOI 10.1007/s00208-015-1345-2.

[4] A.V.Pukhlikov, Birational geometry of algebraic varieties fibred into Fano double spaces. Izvestiya: Mathematics (in press). ArXiv:1512.05681.

[5] I.Cheltsov, The factoriality of nodal threefolds and he connectedness of the set of log canonical singularities. Sbornik: Mathematics. **197** (2006), No. 3-4, 387-414.

[6] I.Cheltsov, Factorial threefold hypersurfaces, J. Alg. Geom. **19** (2010), No. 4, 781-791.

[7] I.Cheltsov and J.Park, Factorial hypersurfaces in $\mathbb{P}^4$ with nodes. Geom. Dedicata **121** (2006), 205-219.

[8] C.Ciliberto and V. Di Gennaro, Factoriality of certain threefold complete intersections in $\mathbb{P}^5$ with ordinary double points. Comm. Algebra **32** (2004), No. 7, 2705-2710.

[9] F.Polizzi, A.Rapagnetta and P.Sabatino, On factoriality of threefolds with isolated singularities, Michigan Math. J. **63** (2014), No. 4, 781-801.

[10] Pukhlikov A. V., Birational automorphisms of Fano hypersurfaces, Invent. Math. **134** (1998), no. 2, 401-426.

[11] Pukhlikov Aleksandr, Birationally Rigid Varieties. Mathematical Surveys and Monographs **190**, AMS, 2013.

[12] Pukhlikov A. V., Birationally rigid Fano complete intersections, Crelle J. für die reine und angew. Math. **541** (2001), 55-79.

Department of Mathematical Sciences,
The University of Liverpool

pukh@liv.ac.uk