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Robustness of Subset Simulation to Functional Forms of Limit State Functions in System Reliability Analysis: Revisiting and Improvement

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Abstract

With the development of computing technologies, computer-based simulation methods have gained increasing attention in reliability analysis of engineering systems, among which Subset Simulation (SS) provides a robust yet efficient tool for exploring system rare failure events and evaluating system reliability. However, the component limit state functions (LSFs) of a system can be formulated in different forms (e.g., linear, exponential, and scaled), depending on mathematical modelling of the engineering systems concerned. This affects the system LSF and the performance of SS, and may lead to inconsistent system reliability analysis results. This study systematically explores effects of the functional form of component LSFs on the performance of SS in system reliability analysis and accounts for such effects from the perspective of sampling procedures. It is found that the efficient generation of conditional samples during SS, which is pivotal to the success of SS, is affected by the functional form of component LSFs in the system concerned. The performance of SS can be sensitive to the functional form of component LSFs. Normalizing component LSFs eliminates the effects of scaled LSFs on the performance of SS, but it does not improve the robustness (insensitivity) of SS in system reliability analysis involving exponential LSFs that are nonlinear. Understanding the effects of component LSFs on the performance of SS, a generalized Subset Simulation (GSS) algorithm is proposed for system reliability analysis, which is robust to different functional forms of component LSFs provided that the functional transformation of component LSFs does not change their monotonicity and failure domains. Numerical examples and a real engineering example showed that the proposed algorithm is more robust to the functional form of component LSFs in system reliability analysis than standard SS.

Index Terms: System reliability analysis, Limit state functions, Subset Simulation, Robustness, Generalized Subset Simulation

I. INTRODUCTION

Uncertainties are inevitable in engineering systems, of which multiple component failure modes are often involved [1]. Their effects shall be rationally incorporated into evaluation of safety and reliability of engineering systems. With the rapid development of modern computing technology, simulation-based reliability analysis methods have gained increasing attention for system reliability analysis, such as direct Monte Carlo simulation (MCS) [2], importance sampling [3], line sampling [4], Subset Simulation (SS) [5], [6] and other variants of MCS [7], [8]. Among these methods, SS is considered as a recent advance of reliability analysis methods in the past two decades exhibiting a trade-off between computational efficiency and application robustness regardless of the number of uncertain parameters [9].

SS has been applied in a number of disciplines, including, e.g., civil engineering [5], [6], [10]–[14], nuclear engineering [15]–[17], aerospace engineering [18]–[20] and electronic engineering [21]. It provides a robust yet efficient tool for exploring rare failure events and calculating their probabilities.

SS stems from the idea that a rare failure event E with a small probability can be expressed as the product of the conditional probabilities of a sequence of intermediate failure events $\{E^{(j)}, j = 1, 2, \ldots, \mathrm{m}\}$ with larger probabilities [5], [10], [22], [23]. The intermediate failure events are usually determined adaptively during SS based on the system performance function (or limit state function (LSF)). For a given system, definition of LSF is pivotal to the implementation of SS and it can affect the accuracy and efficiency of SS [24], [25]. The formulation of component LSFs depends on mathematical modeling of component performance [26], which can be expressed in various functional forms (e.g., linear, exponential, and scaled) with the same failure domain. For example, the LSF (e.g., $f(\mathbf{X})$) of a component can be defined as a linear function (e.g., $f(\mathbf{X}) = \mathrm{X}_1 + \mathrm{C}$, where X_1 is an unknown parameter and C is a constant value) or, equivalently, a nonlinear function (e.g., $f(\mathbf{X})$ =

exp(X_1+C)-1) without changing its failure domain (e.g., $f(\mathbf{X}) < 0$ or $X_1 < -C$). In addition, it is not uncommon that geotechnical structure systems may contain various forms of LSFs of component performance, such as tunnel engineering system [26] and slope engineering system [27], [28]. This subsequently affects the definition of the system LSF and the performance of SS [29]. How the functional form of component LSFs affects the performance of SS in system reliability analysis is non-trivial and has not been adequately explored. Understanding this will help improve the robustness (i.e., insensitivity) of SS to the functional form of component LSFs in system reliability analysis.

This paper investigates the effects of the functional form of component LSFs on the performance of SS in system reliability analysis from the perspective of sampling procedures during SS. A generalized algorithm of SS, so-called Generalized Subset Simulation (GSS) [30], is modified to improve the robustness of SS to the functional form of component LSFs in system reliability analysis. GSS was originally developed to, simultaneously, estimate the failure probabilities of multiple components by a single simulation run [30]. The modified GSS in this study makes it feasible to system reliability analysis, which was not possible in the original version. The modification makes the algorithm more robust to the functional form of component LSFs in system reliability analysis than standard SS.

This paper starts with the definition of series and parallel systems, followed by explaining the effects of the functional form of component LSFs on the performance of SS in system reliability analysis in the context of SS. The modified GSS algorithm is then described for system reliability analysis. Finally, numerical examples and a real engineering example are used to illustrate the effects of the functional form of component LSFs on the performance of SS in system reliability analysis and the robustness of the modified GSS algorithm.

II. DEFINITION OF ENGINEERING SYSTEMS

Engineering systems can be broadly classified into three categories: series systems, parallel systems and combined systems, depending on how components are correlated with each other and affect system response [31], [32]. Since a combined system can be represented by a combination of sub-series systems and sub-parallel systems, this study focuses on exploring the performance of SS in reliability analyses of series and parallel systems. Fig. 1 illustrates a series system and a parallel system, each comprising n components. Let G_i, $i = 1, 2, \ldots, n$ denote the LSFs of the n components with corresponding failure events $E_i = \{G_i < 0\}$**.** For a series system, the failure of any one of the components leads to system failure $E_s = \{G_s < 0\}$, where G_s denotes the LSF of the series system. Without much loss of generality, G_s can be defined as

$$G_s = \min \{G_i, i = 1, 2, \ldots, n\} \tag{1}$$

where "min" indicates that G_s is taken as the minimum value among G_i , $i = 1, 2, \ldots, n$ in the series system. The failure domain of a series system is the union of failure domains of all the components, i.e., $E_s = \{E_1 \cup E_2 \cup \ldots \cup E_n\}$. That is, the occurrence of any event among E_i, $i = 1, 2, \ldots, n$ results in the occurrence of E_s.

On the other hand, for a parallel system as shown in Fig. 1(b), the system failure $E_p = \{G_p < 0\}$ occurs when all the components simultaneously reach their respective limit states. Here, G_p denotes the LSF of the parallel system and it can be expressed as

$$G_p = \max \{G_i, i = 1, 2, \ldots, n \} \tag{2}$$

where “max” indicates that G_p is taken as the maximum value among G_i , $i = 1, 2, \ldots, n$ in the parallel system. The failure domain of a parallel system is the intersection of failure domains of all the components, i.e., $E_s = \{E_1 \cap E_2 \cap \ldots \cap E_n\}$. As indicated in Eqs. (1) and (2), the system LSF relies on component LSFs. For a given random sample, the values of the component LSFs vary as their functional forms change. This affects the value of system LSF during Monte Carlo sampling and hence the performance of SS in system reliability analysis. The latter is discussed in the next section.

III. SUBSET SIMULATION

A. Algorithm of Subset Simulation

SS expresses a rare failure event E with a small probability as a sequence of intermediate failure events $\{E^{(j)}, j = 1, 2, \ldots, m\}$ with larger conditional failure probabilities [5], [6], [22]. Let G be the critical response of interest. Without loss of generality, define the rare failure event E as $E = \{G < b\}$, where b is a given threshold value (e.g., 0). The rare failure event E can be defined as component failure (i.e., $E = E_i$), series system failure (i.e., $E = E_s$) or parallel system failure (i.e., $E = E_p$). The corresponding responses are their respective values of LSFs, i.e., $G = G_i$ for $E = E_i$, $G = G_s$ for $E = E_s$ or $G= G_p$ for $E = E_p$. Let $b^{(1)} > b^{(2)} > \ldots > b^{(\mathrm{m})} = b$ be a decreasing sequence of intermediate threshold values. The intermediate failure events $\{E^{(j)}, j = 1, 2, \ldots, m\}$ are then defined as $E^{(j)}= \{G < b^{(j)}, j =1, 2, \ldots , m\}$. By sequentially conditioning on these intermediate events, the failure probability P_f of E is written as [22] :

$$P_f = P(E^{(m)}) = P(E^{(j)})\prod_{k=2}^{m} P(E^{(j)} \mid E^{(j-1)}) \quad (3)$$

where $P(E^{(1)})$ is equal to $P(G < b^{(1)})$, and $P(E^{(j)}| E^{(j-1)})$ is equal to $\{P(G < b^{(j)}| G < b^{(j-1)}), j = 2, 3, \ldots, m\}$. In implementations, $b^{(1)}$, $b^{2)}$, …, $b^{(m)}$ are generated adaptively using information from simulated samples so that the sample estimates of $P(E^{(1)})$ and $\{P(E^{(j)}| E^{(j-1)}), j = 2, 3, \ldots, m\}$ always correspond to a specified value of conditional probability p_0. The implementation procedures of SS are described below.

SS starts with direct MCS with N samples generated from their probability density functions specified in the problem. Their G values are then calculated and ranked in a descending order. The $(1-p_0)N$-th value in the descending list of G values is chosen as $b^{(1)}$, and hence, the sample estimate for $P(E^{(1)}) = P(G < b^{(1)})$ is always p_0. In other words, there are p_0N samples with $E^{(1)} = \{G < b^{(1)}\}$ among the samples generated by direct MCS. Starting from these p_0N "seed" samples, Markov Chain Monte Carlo Simulation (MCMCS) [22] is used to simulate additional $(1-p_0)N$ conditional samples given $E^{(1)} = \{G < b^{1)}\}$ so that there are a total of N samples with $E^{(1)} = \{G < b^{(1)}\}$. The G values of the N samples with $E^{(1)} = \{G < b^{(1)}\}$ are ranked again in a descending order, and the $(1-p_0)N$-th value in the descending list of G values is chosen as $b^{(2)}$, which defines the $E^{(2)} = \{G < b^{(2)}\}$. The sample estimate for $P(E^{(2)}|E^{(1)}) = P(G < b^{(2)}| G < b^{(1)})$ is also equal to p_0. Similarly, there are p_0N samples with $E^{(2)} = \{G < b^{(2)}\}$ and these samples provide "seeds" for the application of MCMCS to simulate additional $(1-p_0)N$ conditional samples with $E^{(2)} = \{G < b^{(2)}\}$ so that there are N conditional samples with $E^{(2)} = \{G < b^{(2)}\}$. The procedure is repeated m times until the probability space of interest (i.e., the failure domain with $G < b^{(m)}$, where $b^{(m)} = b$) is achieved. Finally, a total of m+1 levels of simulations (including one direct MCS level and m levels of MCMCS) are performed in this study, resulting in $N+m(1-p_0)N$ SS samples. Based on these SS samples, the P_f is estimated using Eq. (3).

Note that the efficient generation of conditional failure samples is pivotal to the success of SS, and it is made possible through the machinery of MCMCS. The MCMCS generates a sequence of samples of random variables or a random vector (e.g., uncertain parameters $\mathbf{X}$=[X_1, X_2, ⋯, X_{Nd}] involved in the system reliability analysis) as states of Markov Chain with the probability density function (PDF) of random variables as the limiting stationary distribution of Markov Chain [33], [34]. During SS, a candidate sample for next state in the Markov Chain is first generated from a proposal PDF defined using the current Markov Chain state, and it is accepted or rejected to be the next state based on the acceptance ratio and the occurrence of intermediate failure events. However, the acceptance ratio often decreases exponentially in some original MCMCS algorithms (e.g., Metropolis algorithm) as the dimension (e.g., N_d) of uncertain parameters space increases, leading to many repeated samples and reduction of computational efficiency and accuracy in high dimensional problems [35]. To address this issue, a modified Metropolis algorithm (MMA) is developed to simulate conditional samples in SS [5], [22], [36], [37], which generates the candidate sample of a high dimensional random vector component by component. For example, using MMA to generate the candidate sample of $\mathbf{X}$ contains N_d steps. In each step, the candidate sample of X_j, j = 1, 2,…, N_d, is generated. After the candidate samples of all the components are obtained, they are collectively taken as the candidate sample of $\mathbf{X}$. If the $\mathbf{X}$'s candidate sample belongs to the intermediate failure event concerned, it is taken as the next state of $\mathbf{X}$ in the Markov Chain. Using MMA reduces the correlation among conditional samples generated by SS in high-dimensional space and, therefore, makes SS feasible in high-dimensional problems.

B. Revisiting performance of Subset Simulation in system reliability analysis

As described in the preceding subsection, selection of "seed" samples in each simulation level depends on the G values of simulated samples. Consider, for example, a system comprised of two components with LSFs $G_1(\mathbf{X})$ and $G_2(\mathbf{X})$ shown in Fig. 2, where $\mathbf{X}$ indicates the uncertain parameters in the system. For illustration, suppose that ten conditional samples (i.e., $\mathbf{x_1}$-$\mathbf{x_{10}}$) of $\mathbf{X}$ at the j-th simulation level are generated and p_0 is taken as 0.4. For a series system, its LSF is given by Eq. (1), which is taken as the minimum value of component LSFs. In the example shown in Fig.2(a), the LSF of the series system is taken as equal to G_2 when $\mathbf{X} > 0$, based on which the selected "seed" samples at the j-th simulation level are $\mathbf{X_1}$, $\mathbf{X_2}$, $\mathbf{X_3}$, and $\mathbf{X_5}$. On the other hand, for the parallel system, its LSF is given by Eq. (2), which is taken as the maximum value of component LSFs. In the example shown in Fig.2(a), the LSF of the parallel system is taken as equal to G_1 when $\mathbf{X} > 0$, based on which the selected "seed" samples at the j-th simulation level are $\mathbf{X_1}$, $\mathbf{X_2}$, $\mathbf{X_3}$, and $\mathbf{X_4}$.

Note that determination of the minimum or maximum value of component LSFs relies on the functional form of each LSF. For a given component LSF, its magnitude can be changed significantly as it is formulated as different functional forms even though its corresponding failure domain remain unchanged. This, subsequently, affects the value of system LSF and the sampling procedure of SS. For example, G_1 shown in Fig. 2(a) can be scaled by dividing a positive constant C due to some reason in formulation, leading a variant of G_1, namely $G_1' = G_1/C$, with the same failure domain (e.g., G_1 or $G_1' < 0$). As shown in Fig. 2(b), using G_1' and G_2 as component LSFs in the series system selects $\mathbf{X_1}$, $\mathbf{X_2}$, $\mathbf{X_3}$, and $\mathbf{X_4}$ as "seed" samples at the j-th simulation level while $\mathbf{X_1}$, $\mathbf{X_2}$, $\mathbf{X_3}$, and $\mathbf{X_5}$ are chosen as "seed" samples for the parallel system, which are different from those obtained according to G_1 and G_2. It is obvious that the functional form of component LSFs affect the system LSF and hence the sampling procedure of SS. This explains why the performance of SS depends on the functional form of component LSFs, which is reflected by the variation of P_f values estimated SS. Intuitively, effects of scaled LSFs (e.g., G_1') on the performance of SS can be eliminated through normalizing the component LSFs by their corresponding values evaluated at the mean value ($\overline{\mathbf{X}}$) of **X.** This may not be the case for other functional forms (e.g., exponential LSFs). Effects of the functional form of component LSFs and normalizing the component LSFs on the performance of SS in system reliability analysis will be further illustrated using numerical examples later. The next section proposes a modified GSS algorithm for system reliability analysis that is robust (insensitive) to the functional form of component LSFs.

IV. GENERALIZED SUBSET SIMULATION FOR SYSTEM RELIABILITY ANALYSIS

A. Original algorithm of Generalized Subset Simulation

GSS is originally developed by Li et al. [30] to efficiently estimate the respective failure probabilities of multiple components, which successfully avoids repeated simulation runs for each component required in original SS. Although the performance of GSS on simultaneous estimation of failure probabilities of multiple components has been investigated in [30], its performance on system reliability analysis remains unexplored.

The major difference between GSS and SS lies in determining intermediate failure events and selecting conditional "seed" samples during simulation. Using different component LSFs in SS, samples progressively populate different failure domains, yielding their corresponding failure probabilities. On the other hand, GSS simultaneously drives samples to failure domains of multiple components through unified intermediate failure events for them. Consider, for example, n component failure events E_i, i = 1, 2, …, n. Let $U^{(j)}$, j =1, 2, … , M denote the unified intermediate failure event at the j-th simulation level of

GSS, where M is the number of simulation levels in GSS. In the context of GSS, $U^{(j)}$ is defined as the union of intermediate failure events of E_i, $i = 1, 2, \ldots, n$, which is written as:

$$U^{(j)} = E_1^{(j)} \cup E_2^{(j)} \cup \cdots \cup E_n^{(j)} = \left\{G_1 < b_1^{(j)}\right\} \cup \left\{G_2 < b_2^{(j)}\right\} \cup \cdots \cup \left\{G_n < b_n^{(j)}\right\} \tag{4}$$

where $E_i^{(j)} = \left\{E_i < b_i^{(j)}\right\}$, $i = 1, 2, \ldots, n$, are intermediate failure events of E_i, respectively, at the j-th simulation level of GSS, and are defined by their respective intermediate threshold values $b_i^{(j)}$. Similar to SS, $b_i^{(j)}$, $i = 1, 2, \ldots, n$, are determined adaptively using information from simulated samples during GSS.

The failure probabilities of different component failure events (e.g., E_i, $i = 1, 2, \ldots, n$) concerned might be different, and their failure domains are, therefore, reached at different simulation levels during GSS. As the number of simulation levels increases, the component failure domains with relatively large failure probabilities are first arrived. Once some components reach their failure domains in the preceding simulation level, the unified intermediate failure event is redefined by dropping these components. The “seed” samples are then selected according to the newly-defined intermediate failure events in the current simulation level. As simulation level j increases, the component failure events (e.g., E_i) reach their respective target failure domains progressively, and the number of component failure events considered in the unified intermediate failure event decreases. Although the original GSS algorithm is able to provide occurrence probabilities of multiple component failure events by a single run of simulation, it fails to explore failure events (E_s and E_p) of series and parallel systems comprised of the n components due to the dropping machinery during GSS. The next section modifies the GSS algorithm to make it feasible to system reliability analysis.

B. A modified algorithm of Generalized Subset Simulation

As shown in Figure 3, the modified GSS algorithm also starts with direct MCS, in which N direct MCS samples are generated. The G_i, i = 1, 2, …, n, values of the N samples are calculated. For a given component, the N values of G_i are then ranked in a descending order. The (1-p_0)N-th value in the descending list of G_i values is chosen as $b_i^{(1)}$ so that the sample estimate for $P(E_i^{(1)}) = P(G_i < b_i^{(1)})$ is p_0. There are p_0N samples with $E_i^{(1)} = \{G_i < b_i^{(1)}\}$ among the samples generated by direct MCS. Such procedure is repeatedly performed for each component to determine their respective threshold values $b_i^{(1)}$ and to select p_0N "seeds" samples with $E_i^{(1)} = \{G_i < b_i^{(1)}\}$.

After the determination of $b_i^{(1)}$, i = 1, 2, …, n, the union of $E_i^{(1)}$ is defined as the intermediate failure event $U_s^{(1)}$ of a series system, i.e., $U_s^{(1)} = E_1^{(1)} \cup E_2^{(1)} \cup \cdots \cup E_n^{(1)}$. In other words, the samples in $U_s^{(1)}$ are those satisfying $E_i^{(1)} = \{G_i < b_i^{(1)}\}$ for any i = 1, 2, …, n. The definition of intermediate failure event of the series system is similar to that in original GSS algorithm (see Eq. (4)). However, the component intermediate failure event is not dropped during the simulation through the modified algorithm even though failure domains of some components have been arrived. Hence, the intermediate failure events of all the components are kept for construction of the system intermediate failure events before the system failure domain is reached. This benefits the exploration of system failure domain, particularly for parallel systems, as discussed below.

For parallel systems, the intermediate failure event $U_p^{(1)}$ is determined as the intersection of $E_i^{(1)}$, i.e., $U_p^{(1)} = E_1^{(1)} \cap E_2^{(1)} \cap \cdots \cap E_n^{(1)}$. The samples in $U_p^{(1)}$ belong to every $E_i^{(1)} = \{G_i < y_i^{(1)}\}$. The proposed intermediate failure event of parallel systems is different from that defined in original GSS algorithm (see Eq. (4)). Note that the failure probability of a parallel system is, in theory, smaller than any component failure probabilities since E_p is defined the intersection of E_i, i = 1, 2, …, n. Hence, the failure domain of E_p is not sufficiently explored until all the component failure domains have been reached. This requires not to drop any component intermediate failure events during the simulation so that the failure domain of parallel systems can be progressively approached.

Let N_1 denote the number of samples of $U^{(1)}$ (i.e., $U_s^{(1)}$ for series systems and $U_p^{(1)}$ for parallel systems). The probability $P(U^{(1)})$ of $U^{(1)}$ is estimated as $P(U^{(1)}) \approx N_1/N$. The N_1 samples in $U^{(1)}$ are used as "seed" samples for MCMCS to simulate additional N-N_1 conditional samples in $U^{(1)}$. This results in N conditional samples in $U^{(1)}$, based on which $b_i^{(2)}$, i = 1, 2, …, n, are determined so that the sample estimates of $P(E_i^{(2)}|U^{(1)})$ are equal to p_0. Next, $U^{(2)}$ (i.e., $E_1^{(2)} \cup E_2^{(2)} \cup \cdots \cup E_n^{(2)}$ for series systems and $E_1^{(2)} \cap E_2^{(2)} \cap \cdots \cap E_n^{(2)}$ for parallel systems) is constructed, and N_2 samples in $U^{(2)}$ are identified as "seed" samples for MCMCS to generate conditional samples in the next simulation level. This procedure is repeatedly performed until system failure domains concerned are reached. The samples provide estimates of system failure probability:

$$P_f = P(U^{(1)})P(U^{(2)}|U^{(1)})\cdots P(U^{(M_F-1)}|U^{(M_F-2)})P(E|U^{(M_F-1)}) = \prod_{k=1}^{M_F-1}\frac{N_k}{N}\times\frac{N_F}{N} \tag{5}$$

where M_F denote the number of simulation levels needed to reach the failure domain of E (i.e., E_s for series systems or E_p for parallel systems); $P(U^{(j)}|\ U^{(j\text{-}1)})$, j = 2, 3…, M_F-1, is conditional probability of $U^{(j)}$ given sampling in $U^{(j\text{-}1)}$, and is calculated as the ratio of the number N_j of "seed" samples selected at the j-th MCMCS level among N samples with $U^{(j\text{-}1)}$

over N; $P(E|U^{(M_F\text{-}1)})$ is the conditional probability of system failure event E given sampling in $U^{(M_F\text{-}1)}$, and is estimated as the ratio of the number N_F of system failure samples among N samples generated in $U^{(M_F\text{-}1)}$ over N.

Note that the main difference among SS, GSS and the modified GSS is the definition of intermediate failure event $U^{(j)}$. SS uses the system event E_s or E_p to directly define intermediate failure event (i.e., $U^{(j)} = \{G_s < b^{(j)}\}$ for series systems and $U^{(j)} = \{G_p < b^{(j)}\}$ for parallel systems) in the system reliability analysis while the modified GSS defines the intermediate failure events by the unified and intersected event of component failure events for series systems and parallel systems, respectively. The original GSS defined the intermediate failure event as the unified event of multiple component events and these component events contained in the unified event change as the simulation level increases. As a result, the original GSS is not feasible to parallel systems. However, the number of component intermediate failure events contained in the intermediate failure event defined for the series system and parallel system in modified GSS do not change with the simulation level. This allows the modified GSS to efficiently explore failure domains of both series and parallel systems.

C. Robustness of the modified GSS algorithm to the functional form of component LSFs

As described above, selection of system "seed" samples in the modified GSS algorithm is achieved through determination of "seed" samples for each component, which only depends on the G_i values for each component and is irrespective of relative magnitudes of G_i for different components. This allows the selected system "seed" samples in the modified GSS algorithm to be insensitive to the functional form of component LSFs assuming that the functional transformation of component LSFs does not change their monotonicity and failure domains. This assumption is reasonable in engineering system analysis since each component shall have the same actual behaviors (e.g., monotonicity and plausibility of failure) without regard to the mathematical modeling (i.e., LSFs) of its performance. Reasonable LSFs of a component shall reflect the same behavior of the component no matter in which functional forms they are formulated as.

Consider, again, the illustrative example shown in Fig. 2. Using the modified GSS algorithm, the selected "seed" samples for a series system comprised of G_1 and G_2 shown in Fig. 2 (a) at the j-th simulation level are $\mathbf{X_1}$, $\mathbf{X_2}$, $\mathbf{X_3}$, $\mathbf{X_4}$ and $\mathbf{X_5}$. The "seed" samples remain the same for a series system comprised of G_1' and G_2 shown in Fig. 2 (b). Similarly, using the modified GSS algorithm, the selected "seed" samples for a parallel system comprised of G_1 and G_2 shown in Fig. 2 (a) at the j-th simulation level are $\mathbf{X_1}$, $\mathbf{X_2}$, and $\mathbf{X_3}$, which also remain the same after G_1 is scaled to G_1'. The sampling procedure of the modified GSS algorithm is generally not affected by the functional form of component LSFs if the functional transformation of component LSFs does not change their monotonicity and corresponding failure domains. Compared with SS, the modified GSS algorithm improves the robustness of system reliability analysis to the functional form of component LSFs. This is illustrated using numerical examples in the following sections.

V. COMPONENT LIMIT STATE FUNCTIONS USED IN INVESTIGATION

This section presents two component LSFs $f(\mathbf{X})$ and $g(\mathbf{X})$ used to explore effects of the functional form of component LSFs on the performance of SS in system reliability analysis and to illustrate the robustness of the modified GSS algorithm to the functional form of component LSFs. For the purposes, $f(\mathbf{X})$ and $g(\mathbf{X})$ are assumed to have three different functional forms [29]:

$$\begin{cases} f_{\mathrm{L}}(\mathbf{X}) = \mathrm{X}_1 + 80 \\ f_{\mathrm{E}}(\mathbf{X}) = \exp\left(\mathrm{X}_1 + 80\right) - 1 \\ f_{\mathrm{S}}(\mathbf{X}) = 1000\left(\mathrm{X}_1 + 80\right) \end{cases} \quad (6)$$

$$\begin{cases} g_{\mathrm{L}}(\mathbf{X}) = \mathrm{X}_1 + \mathrm{X}_2 + 80 \\ g_{\mathrm{E}}(\mathbf{X}) = \exp\left(\mathrm{X}_1 + \mathrm{X}_2 + 80\right) - 1 \\ g_{\mathrm{S}}(\mathbf{X}) = \left(\mathrm{X}_1 + \mathrm{X}_2 + 80\right)/1000 \end{cases} \quad (7)$$

where both X_1 and X_2 are normally distributed random variables with a mean of 120 and a standard deviation of 60; and the subscripts "L", "E", and "S" indicate "Linear", "Exponential (or Nonlinear)", and "Scaled", respectively. Figs. 4 and 5 show different functional forms of *f*(**X**) and *g*(**X**), respectively. It shall be noted that although *f*(**X**) and *g*(**X**) can be linear, exponential, and scaled functions, their respective monotonicity and failure domains remain unchanged. For example, the failure domain for *f*(**X**) is $X_1 < -80$ no matter which functional form is adopted in reliability analysis. Fig. 6 shows failure domains (i.e., *f*(**X**) < 0 and *g*(**X**) <0) corresponding to *f*(**X**) and *g*(**X**) by areas with horizontal and vertical lines, respectively. Their union and intersection are the respective failure domains of series and parallel systems comprised of the two components with *f*(**X**) and *g*(**X**) as LSFs. Based on *f*(**X**) and *g*(**X**), the performance of SS in system reliability analysis is revisited in Section VI, and the robustness of the modified GSS algorithm to the functional form of component LSFs is illustrated in Section VII.

VI. PERFORMANCE OF SUBESET SIMULATION IN SYSTEM RELIABILITY ANALYSIS

To explore effects of the functional form of component LSFs on system reliability analysis, SS runs with $p_0 = 0.1$ and $N = 1000$ are performed to evaluate P_f values of 9 series systems and 9 parallel systems comprised of the two components with different functional forms of

LSFs given by Eqs. (6) and (7). The LSFs of the 9 series systems and 9 parallel systems are taken as min{f(**X**), g(**X**)} and max{f(**X**), g(**X**)}, respectively, where f(**X**) and g(**X**) have three different functional forms. For each system, 100 SS runs are performed to evaluate the mean value and coefficient of variation COV_p of P_f. The P_f estimates from SS in the following discussions are referring to its mean value obtained from 100 SS runs. In addition, a direct MCS run with 10^7 samples is also performed to validate SS results, and its corresponding COV_p is estimated as $\sqrt{(1-P_f)/N_{MCS}P_f}$, where N_{MCS} is the number of random samples generated by direct MCS. For comparison, the P_f and COV_p corresponding to each component LSF are also calculated using SS and direct MCS. The system and component reliability analysis results are provided below.

A. Effects of the functional form of LSFs on component reliability analysis

Table I summarizes reliability analysis results for different component LSFs obtained from direct MCS and SS. For all the component LSF, the COV_p values of P_f obtained from direct MCS are less than 0.1, which indicates using 10^7 random samples in direct MCS gives sufficiently accurate P_f estimates (i.e., 4.37×10^{-4} for f(**X**) and 7.93×10^{-5} for g(**X**)) for component LSFs. The P_f estimates from direct MCS are favorably comparable with those obtained from SS. This validates the P_f values estimated from SS. For different functional forms of each component LSF, the P_f and COV_p obtained SS remain almost unchanged. Hence, the functional form of LSFs has minimal effects on the performance of SS in component reliability analysis. This is attributed to the fact the functional transformation of component LSFs in Eqs. (6) and (7) does not affect the monotonicity and failure domains of component LSFs and the sampling procedure of SS in component reliability analysis. For example, as shown in Fig. 7, failure samples generated by SS for two variants f_L(**X**) and f_E(**X**) of $f(\mathbf{X})$ are distributed similarly. It is hence not surprising to see that the P_f estimates from SS and their accuracy (quantified by COV_p) for $f_L(\mathbf{X})$ and $f_E(\mathbf{X})$ are similar.

Compared with SS, the P_f and its COV value obtained from direct MCS for components with various functional forms are almost unchanged, which indicates the performance of MCS is not affected by the functional form of LSFs. This can be attributed to the fact that the components associated with different forms of a LSF have the same failure domain. Hence, random samples that fall into the failure domain of a component also fall into the failure domain of other components with different functional forms of the same LSF in this study. In addition, the number (i.e., 10^7) of samples used in MCS in this study is large enough to obtain a quite small COV value (i.e., 0.015) of P_f estimates, as shown in Tables I. It is, hence, not surprise to see that the P_f estimates from MCS of components with different functional forms remain almost unchanged. Similar observation can be also obtained from reliability analyses of series and parallel systems in the next two subsections.

B. Effects of the functional form of component LSFs on series system reliability analysis

Table II summarizes reliability analysis results obtained from direct MCS and SS for the 9 series systems with different combinations of component LSFs given by Eqs.(6) and(7). Again, the P_f estimates from SS are validated against that (i.e., 4.97×10^{-4}) obtained from direct MCS. It is shown that P_f estimates from SS vary slightly around that from direct MCS, but their corresponding COV_p values vary considerably from 0.523 to 0.704 as the functional form of component LSFs in series systems changes. This indicates that the accuracy of P_f estimates of series systems from SS is sensitive to the functional form of component LSFs. As discussed in Subsection IV.C, the functional transformation of component LSFs leads to variation of LSF of series systems given by Eq. (1), which changes the selection of “seed” samples and the subsequent sampling procedure of SS. This affects the accuracy of P_f

estimates of series systems from SS. For example, Fig. 8 shows random samples generated in different simulation levels of SS for reliability analysis of series systems with component LSFs $\{f_L(\mathbf{X}), g_L(\mathbf{X})\}$ and $\{f_L(\mathbf{X}), g_S(\mathbf{X})\}$ by circles and squares, respectively. For the series system with component LSFs $f_L(\mathbf{X})$ and $g_L(\mathbf{X})$, $f_L(\mathbf{X})$ dominates the sampling in early simulation levels of SS; however, for the series system with component LSFs $f_L(\mathbf{X})$ and $g_S(\mathbf{X})$, $g_S(\mathbf{X})$ dominates the sampling in early simulation levels of SS. Although the two series systems have, in theory, the same value of P_f, their sampling procedures in SS are different. This leads to different numbers of failure samples in the last simulation level of SS (see Fig. 8(e)), which affects the accuracy of P_f estimates of series systems from SS.

As shown by the simplified example provided in the Subsection III. B, the performance of SS in system reliability analysis is affected by the scale effects among different LSFs of components. Normalization is an effective and straightforward way to reduce these scale effects [29], [38]. Therefore, normalizing the component LSFs by their corresponding values evaluated at the mean value ($\overline{\mathbf{X}}$) of $\mathbf{X}$ is used to improve the robustness of SS to different functional forms of component LSFs in system reliability analysis, by which the component LSFs of the 9 series systems are re-written as $f_L(\mathbf{X})/f_L(\overline{\mathbf{X}})$, $f_E(\mathbf{X})/f_E(\overline{\mathbf{X}})$, $f_S(\mathbf{X})/f_S(\overline{\mathbf{X}})$, $g_L(\mathbf{X})/g_L(\overline{\mathbf{X}})$, $g_E(\mathbf{X})/g_E(\overline{\mathbf{X}})$ and $g_S(\mathbf{X})/g_S(\overline{\mathbf{X}})$. Using the normalized LSFs, the P_f and COV_p values for each series system are re-evaluated using 100 SS runs. The results are also included in Table II. It is shown that the series systems with linear and scaled LSFs (i.e., $\{f_L(\mathbf{X}), g_L(\mathbf{X})\}$, $\{f_L(\mathbf{X}), g_S(\mathbf{X})\}$, $\{f_S(\mathbf{X}), g_L(\mathbf{X})\}$, and $\{f_S(\mathbf{X}), g_S(\mathbf{X})\}$) have the same values of P_f and COV_p. Normalizing LSFs eliminates effects of scaled LSFs on the performance of SS in series system reliability analysis because the scaling constant C is canceled by normalization. However, such an observation is not the case for the series systems with exponential LSFs, whose COV_p values vary from 0.598 to 0.698 and are greater than that (i.e., 0.433) of series systems with linear and scaled LSFs. Normalizing LSFs does not improve the robustness of SS in series system reliability analysis involving exponential LSFs.

C. Effects of the functional form of component LSFs on parallel system reliability analysis

Table III summarizes reliability analysis results obtained from direct MCS and SS for the 9 parallel systems with different combinations of component LSFs given by Eqs.(6) and(7). Similar to series systems, the P_f estimates of parallel systems from SS are generally favorably comparable with that (i.e., 1.97×10^{-5}) from direct MCS. The variation of the P_f estimates of parallel systems is, again, quantified by their corresponding COV_p values, which range from 0.832 to 0.978. The accuracy of P_f estimates of parallel systems from SS also depends on the functional form of component LSFs. This is attributed to effects of the functional form of component LSFs on parallel system LSFs given by Eq. (2), which affects the selection of "seed" samples and the subsequent sampling procedure of SS. Fig. 9 shows random samples generated in different simulation levels of SS for reliability analysis of parallel systems with component LSFs $\{f_L(\mathbf{X}), g_L(\mathbf{X})\}$ and $\{f_L(\mathbf{X}), g_S(\mathbf{X})\}$ by circles and squares, respectively. In contrast to series systems, $g_L(\mathbf{X})$ dominates the sampling in early simulation levels of SS for the parallel system with component LSFs $f_L(\mathbf{X})$ and $g_L(\mathbf{X})$, but $f_L(\mathbf{X})$ dominates the sampling in early simulation levels of SS for the parallel system with component LSFs $f_L(\mathbf{X})$ and $g_S(\mathbf{X})$. Different sampling procedures of SS for the two parallel systems lead to different numbers of failure samples in the last simulation level of SS (see Fig. 9(f)) and, then, affects the accuracy of P_f estimates of parallel systems from SS.

To explore effects of normalizing component LSFs on the performance of SS in parallel system reliability analysis, the P_f and COV_p values for each parallel system are re-evaluated based on normalized component LSFs using 100 SS runs. The results are also included in Table III. Compared with the reliability analysis results of series systems using normalized component LSFs, similar observations are obtained for parallel systems. Normalizing LSFs eliminates effects of scaled LSFs on the performance of SS in parallel system reliability analysis, but this does not work well for parallel systems involving exponential LSFs that are nonlinear.

VII. SYSTEM RELIABILITY ANALYSIS RESULTS FROM THE MODIFIED GSS ALGORITHM

This section re-evaluates the P_f and COV_p values of the 9 series systems and 9 parallel systems with different combinations of component LSFs given by Eqs. (6) and (7) using the modified GSS algorithm. For each system, 100 GSS runs with p_0 = 0.1 and N = 1000 are performed to evaluate the mean value and COV_p of P_f values using the proposed algorithm. The P_f estimates from the modified GSS in the following discussions are, again, referring to its mean value obtained from 100 GSS runs.

Table IV summarizes reliability analysis results obtained from the modified GSS algorithm for the 9 series systems and 9 parallel systems. The P_f estimates of the series and parallel systems obtained from GSS are 4.92×10^{-4} and 1.19×10^{-5}, respectively, and remain the same no matter which functional forms of component LSFs are adopted in the system. In general, these values are favorably comparable with those (i.e., 4.97×10^{-4} and 1.97×10^{-5}) estimated from direct MCS with 10^7 samples (see Tables II and III). This validates the modified GSS algorithm. In addition, it is also observed that COV_p values of P_f estimates of series and parallel systems from the modified GSS algorithm remains almost unchanged at around 0.544 and 1.000, respectively, without regard to the functional form of component LSFs. The COV_p reflects the performance of the probabilistic analysis algorithm [29]. Unchanged COV_p shown in Table IV indicates that the performance of modified GSS algorithm is not insensitive to different functional forms of component LSFs in reliability analyses of series and parallel systems compared with SS. Such an improvement is attributed to the fact that selection of "seed" samples in the modified GSS algorithm only relies on the magnitude of each component LSF and is irrespective of relative magnitudes of LSFs of different components, as discussed in Subsection IV.C. The functional transformation of component LSFs does not affect the sampling procedure of conditional samples during GSS provided that it does not change the monotonicity and failure domains of component LSFs. For example, Fig. 10 shows random samples generated in different simulation levels of the modified GSS algorithm for reliability analysis of series systems with component LSFs $\{f_L(\mathbf{X}), g_L(\mathbf{X})\}$ and $\{f_L(\mathbf{X}), g_S(\mathbf{X})\}$ by circles and squares, respectively. The two sets of random samples are distributed similarly in each simulation level of GSS using the proposed algorithm. Similar observations are also obtained for parallel systems with component LSFs $\{f_L(\mathbf{X}), g_L(\mathbf{X})\}$ and $\{f_L(\mathbf{X}), g_S(\mathbf{X})\}$, as shown in Fig. 11.

VIII. APPLICATION OF THE MODIFIED GSS TO SYSTEM RELIABILITY ANALYSIS OF CONGRESS STREET CUT

This section uses a real engineering example, namely Congress Street cut in Chicago, to further demonstrate the performance of the modified GSS. The cut slope has been adopted to investigate the slope system reliability analysis problem by numerous researchers [39]–[42] and its geometry is shown in Fig. 12. As shown in Fig. 12, the cut slope contains one fill layer and three clay layers. The internal friction angle of the fill ϕ is characterized as a deterministic parameter with a value of 30° and the undrained shear strength for the three clay layers c_{u1}, c_{u2}, and c_{u3} are taken as random parameters with mean values of 136, 80, and 102 kPa and standard deviations of 50, 15 and 24 kPa, respectively. Previous studies [42] indicated that the slope system can be effectively represented by three representative failure modes (RFMs) shown in Fig. 12. This study uses these RFMs to estimate the slope system reliability. Generally, the limit state function of the slope sliding along a slip surface can be defined as f

= R – S or f = (R/S) – 1, where the resistance force R and sliding force S are estimated by Bishop's simplified method. In other words, each of these three RFMs has two different forms of LSF, i.e. f_1, and f_2 for the first RFM, g_1 and g_2 for the second RFM and h_1 and h_2 for the third RFM. Although other forms might exist in the slope system reliability problem, these forms shown in Table V are considered for illustration in this section.

Table V summarizes reliability analysis results obtained from the MCS, SS and the modified GSS for the cut slope system where eight forms of the system LSF are considered. The settings of MCS, SS and modified GSS are the same as that used in the numerical example. The P_f estimated by MCS is 1.18×10^{-2}, which agrees well with that of 1.19×10^{-2} calculated by the modified GSS. SS also gives favorably comparable estimates of P_f, however, the COV_p value of P_f estimates for SS varies with the forms of LSFs contained in the slope system. In contrast, the COV_p value of P_f estimates for modified GSS remains almost unchanged at the value of 0.202. This further indicates that the proposed algorithm is insensitive to the functional form of component LFSs in the system reliability analysis.

IX. SUMMARY AND CONCLUDING REMARKS

This paper revisited the performance of Subset Simulation (SS) in system reliability analysis and revealed effects of the functional form of component limit state functions (LSFs) on the performance of SS from the perspective of sampling procedures. It was shown that the functional transformation of component LSFs results in the variation of system LSF given by Eqs. (1) and (2), which affects the selection of conditional "seed" samples during SS and the subsequent sampling procedure. This makes the performance (or accuracy) of SS sensitive to the functional form of component LSFs in system reliability analysis. Normalizing component LSFs eliminates effects of scaled LSFs on the performance of SS in system reliability analysis, but it does not improve the robustness (or insensitivity) of SS in system reliability analysis involving exponential LSFs that are nonlinear.

With the understanding of effects of the functional form of component LSFs on the performance of SS in system reliability analysis, a modified GSS algorithm was proposed for system reliability analysis, which is robust (or insensitive) to different functional forms of component LSFs provided that the functional transformation of component LSFs does not change their monotonicity and failure domains. The modification of GSS in this study lies in construction of intermediate failure events during the simulation. In contrast to the original GSS algorithm, the modified algorithm uses all the component intermediate failure events to construct the system intermediate failure events without dropping any one during the whole simulation, and it adopts the union and intersection of component intermediate failure events as respective intermediate failure events of series and parallel systems. This allows sufficient exploration of system failure domains using the modified algorithm and makes GSS feasible in reliability analyses of both series and parallel systems. The modified GSS algorithm was illustrated using numerical examples used to explore the performance of SS in system reliability analysis. In addition, a practical engineering slope system, namely Congress Street cut in Chicago, is also employed to demonstrate the performance of the modified GSS algorithm. Results showed that the performance of proposed algorithm is insensitive to the functional form of component LSFs in system reliability analysis. It is more robust to the functional form of component LSFs in system reliability analysis than SS.

It is worthwhile to point out that, although only the performance of the proposed approach on series system and parallel system is demonstrated in this study, it is generally applicable to more sophisticated systems, such as combined systems. Further research on performance of the proposed approach on combined systems is warranted.

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LIST OF TABLES:

TABLE I COMPONENT RELIABILITY ANALYSIS RESULTS FROM SUBSET SIMULATION

Component LSF	MCS		SS*	
	P_f	COV_p	P_f	COV_p
$f_L(\mathbf{X})$	4.37×10^{-4}	0.015	4.02×10^{-4}	0.623
$f_E(\mathbf{X})$	4.37×10^{-4}	0.015	4.02×10^{-4}	0.625
$f_S(\mathbf{X})$	4.37×10^{-4}	0.015	4.02×10^{-4}	0.623
$g_L(\mathbf{X})$	7.93×10^{-5}	0.035	8.45×10^{-5}	0.508
$g_E(\mathbf{X})$	7.93×10^{-5}	0.035	8.45×10^{-5}	0.500
$g_S(\mathbf{X})$	7.93×10^{-5}	0.035	8.45×10^{-5}	0.508

*：Based on 100 SS runs

TABLE II RELIABILITY ANALYSIS RESULTS FROM SUBSET SIMULATION FOR SERIES SYSTEMS

LSFs of Series System	Based on **original** form of LSFs				Based on the **normalized** form of LSFs			
	MCS		SS*		MCS		SS*	
	P_f	COV_p	P_f	COV_p	P_f	COV_p	P_f	COV_p
$\min\{f_L, g_L\}$	4.97×10^{-4}	0.014	4.75×10^{-4}	0.523	4.97×10^{-4}	0.014	4.98×10^{-4}	0.433
$\min\{f_L, g_E\}$	4.97×10^{-4}	0.014	4.57×10^{-4}	0.587	4.97×10^{-4}	0.014	5.09×10^{-4}	0.698
$\min\{f_L, g_S\}$	4.97×10^{-4}	0.014	5.09×10^{-4}	0.704	4.97×10^{-4}	0.014	4.98×10^{-4}	0.433
$\min\{f_E, g_L\}$	4.97×10^{-4}	0.014	4.99×10^{-4}	0.630	4.97×10^{-4}	0.014	4.58×10^{-4}	0.598
$\min\{f_E, g_E\}$	4.97×10^{-4}	0.014	4.75×10^{-4}	0.523	4.97×10^{-4}	0.014	5.02×10^{-4}	0.614
$\min\{f_E, g_S\}$	4.97×10^{-4}	0.014	5.09×10^{-4}	0.703	4.97×10^{-4}	0.014	4.58×10^{-4}	0.598
$\min\{f_S, g_L\}$	4.97×10^{-4}	0.014	5.09×10^{-4}	0.704	4.97×10^{-4}	0.014	4.98×10^{-4}	0.433
$\min\{f_S, g_E\}$	4.97×10^{-4}	0.014	4.60×10^{-4}	0.563	4.97×10^{-4}	0.014	5.09×10^{-4}	0.698
$\min\{f_S, g_S\}$	4.97×10^{-4}	0.014	5.09×10^{-4}	0.699	4.97×10^{-4}	0.014	4.98×10^{-4}	0.433

*：Based on 100 SS runs

TABLE III RELIABILITY ANALYSIS RESULTS FROM SUBSET SIMULATION FOR PARALLEL SYSTEMS

LSFs of Parallel System	Based on **original** form of LSFs				Based on the **normalized** form of LSFs			
	MCS		SS*		MCS		SS*	
	P_f	COV_p	P_f	COV_p	P_f	COV_p	P_f	COV_p
max $\{f_L, g_L\}$	1.97×10^{-5}	0.071	2.06×10^{-5}	0.978	1.97×10^{-5}	0.071	1.88×10^{-5}	0.827
max $\{f_L, g_E\}$	1.97×10^{-5}	0.071	2.17×10^{-5}	0.929	1.97×10^{-5}	0.071	1.68×10^{-5}	0.812
max $\{f_L, g_S\}$	1.97×10^{-5}	0.071	1.64×10^{-5}	0.843	1.97×10^{-5}	0.071	1.88×10^{-5}	0.827
max $\{f_E, g_L\}$	1.97×10^{-5}	0.071	1.67×10^{-5}	0.844	1.97×10^{-5}	0.071	2.06×10^{-5}	0.887
max $\{f_E, g_E\}$	1.97×10^{-5}	0.071	2.06×10^{-5}	0.972	1.97×10^{-5}	0.071	1.72×10^{-5}	0.990
max $\{f_E, g_S\}$	1.97×10^{-5}	0.071	1.64×10^{-5}	0.842	1.97×10^{-5}	0.071	2.06×10^{-5}	0.887
max $\{f_S, g_L\}$	1.97×10^{-5}	0.071	1.64×10^{-5}	0.843	1.97×10^{-5}	0.071	1.88×10^{-5}	0.827
max $\{f_S, g_E\}$	1.97×10^{-5}	0.071	2.18×10^{-5}	0.922	1.97×10^{-5}	0.071	1.68×10^{-5}	0.813
max $\{f_S, g_S\}$	1.97×10^{-5}	0.071	1.64×10^{-5}	0.832	1.97×10^{-5}	0.071	1.88×10^{-5}	0.827

*：Based on 100 SS runs

TABLE IV SYSTEM RELIABILITY ANALYSIS RESULTS FROM GENERALIZED SUBSET SIMULATION

Component LSFs	Series Systems*		Parallel Systems*	
	P_f	COV_p	P_f	COV_p
$\{f_L, g_L\}$	4.92×10^{-4}	0.544	1.19×10^{-5}	0.999
$\{f_L, g_E\}$	4.92×10^{-4}	0.544	1.19×10^{-5}	0.999
$\{f_L, g_S\}$	4.92×10^{-4}	0.544	1.19×10^{-5}	0.999
$\{f_E, g_L\}$	4.92×10^{-4}	0.544	1.19×10^{-5}	1.006
$\{f_E, g_E\}$	4.92×10^{-4}	0.544	1.19×10^{-5}	1.006
$\{f_E, g_S\}$	4.92×10^{-4}	0.544	1.19×10^{-5}	1.006
$\{f_S, g_L\}$	4.92×10^{-4}	0.544	1.19×10^{-5}	0.999
$\{f_S, g_E\}$	4.92×10^{-4}	0.544	1.19×10^{-5}	0.999
$\{f_S, g_S\}$	4.92×10^{-4}	0.544	1.19×10^{-5}	0.999

*：Based on 100 GSS runs

TABLE V COMPARISON OF SYSTEM RELIABILITY ANALYSIS RESULTS FOR THE CONGRESS STREET CUT EXAMPLE

Component LSFs	MCS		SS		GSS	
	P_f	COV_p	P_f	COV_p	P_f	COV_p
$\min\{f_1, g_1, h_1\}$	1.18×10^{-2}	0.003	1.17×10^{-2}	0.225	1.19×10^{-2}	0.202
$\min\{f_1, g_1, h_2\}$	1.18×10^{-2}	0.003	1.19×10^{-2}	0.248	1.19×10^{-2}	0.202
$\min\{f_1, g_2, h_1\}$	1.18×10^{-2}	0.003	1.21×10^{-2}	0.224	1.19×10^{-2}	0.202
$\min\{f_1, g_2, h_2\}$	1.18×10^{-2}	0.003	1.19×10^{-2}	0.235	1.19×10^{-2}	0.202
$\min\{f_2, g_1, h_1\}$	1.18×10^{-2}	0.003	1.20×10^{-2}	0.223	1.19×10^{-2}	0.202
$\min\{f_2, g_1, h_2\}$	1.18×10^{-2}	0.003	1.20×10^{-2}	0.264	1.19×10^{-2}	0.202
$\min\{f_2, g_2, h_1\}$	1.18×10^{-2}	0.003	1.19×10^{-2}	0.246	1.19×10^{-2}	0.202
$\min\{f_2, g_2, h_2\}$	1.18×10^{-2}	0.003	1.19×10^{-2}	0.207	1.19×10^{-2}	0.202

LIST OF FIGURES:

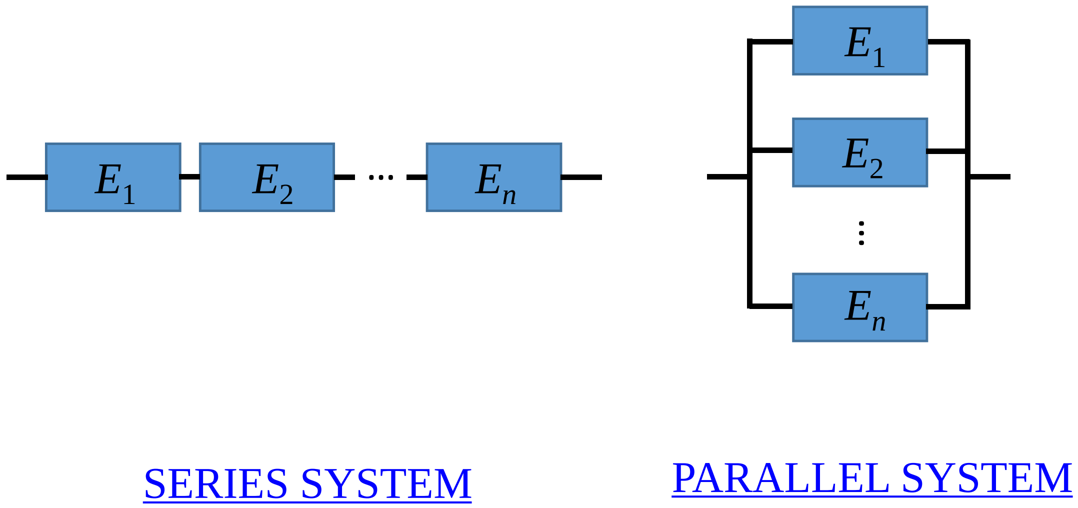
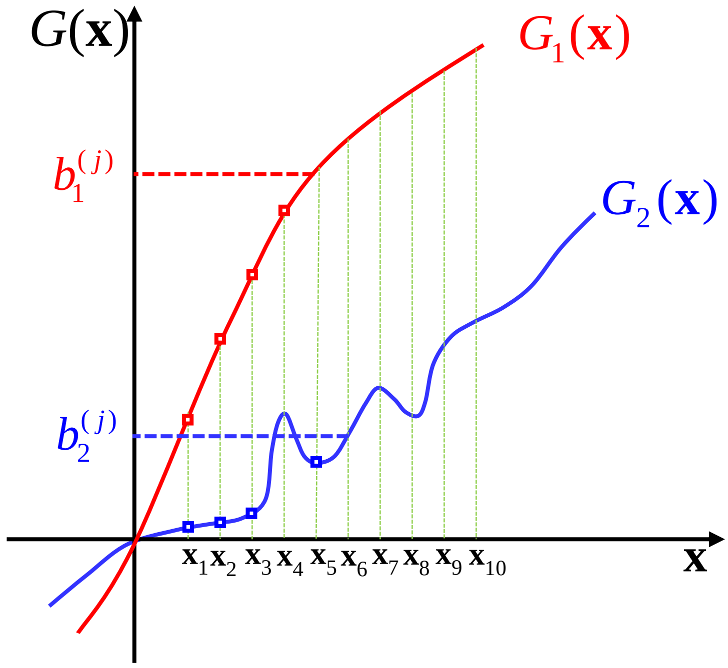
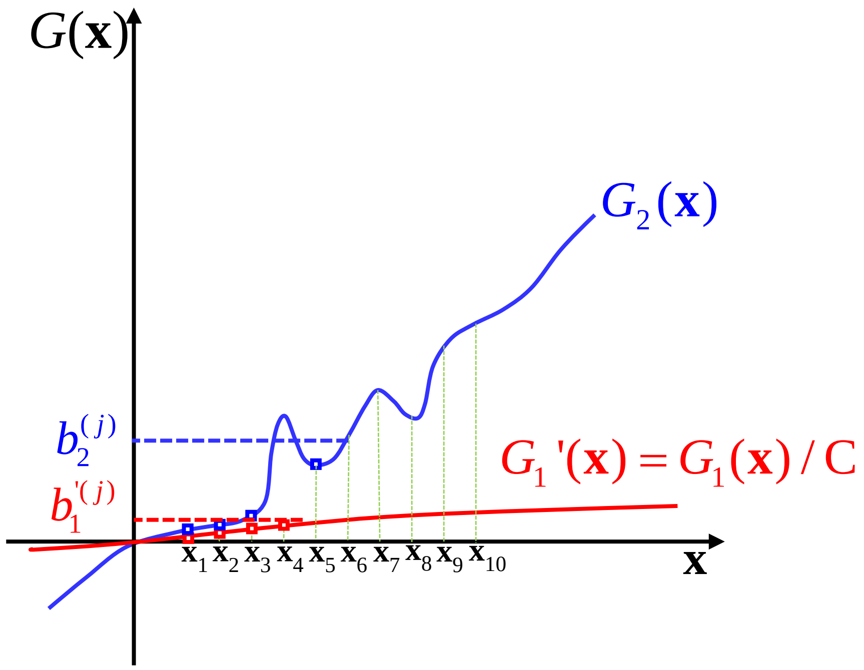


Fig. 1. Illustration of series and parallel systems.



(a)



(b)

Fig. 2. Effects of the functional form of component LSFs on system LSF and selection of "seed" samples during SS

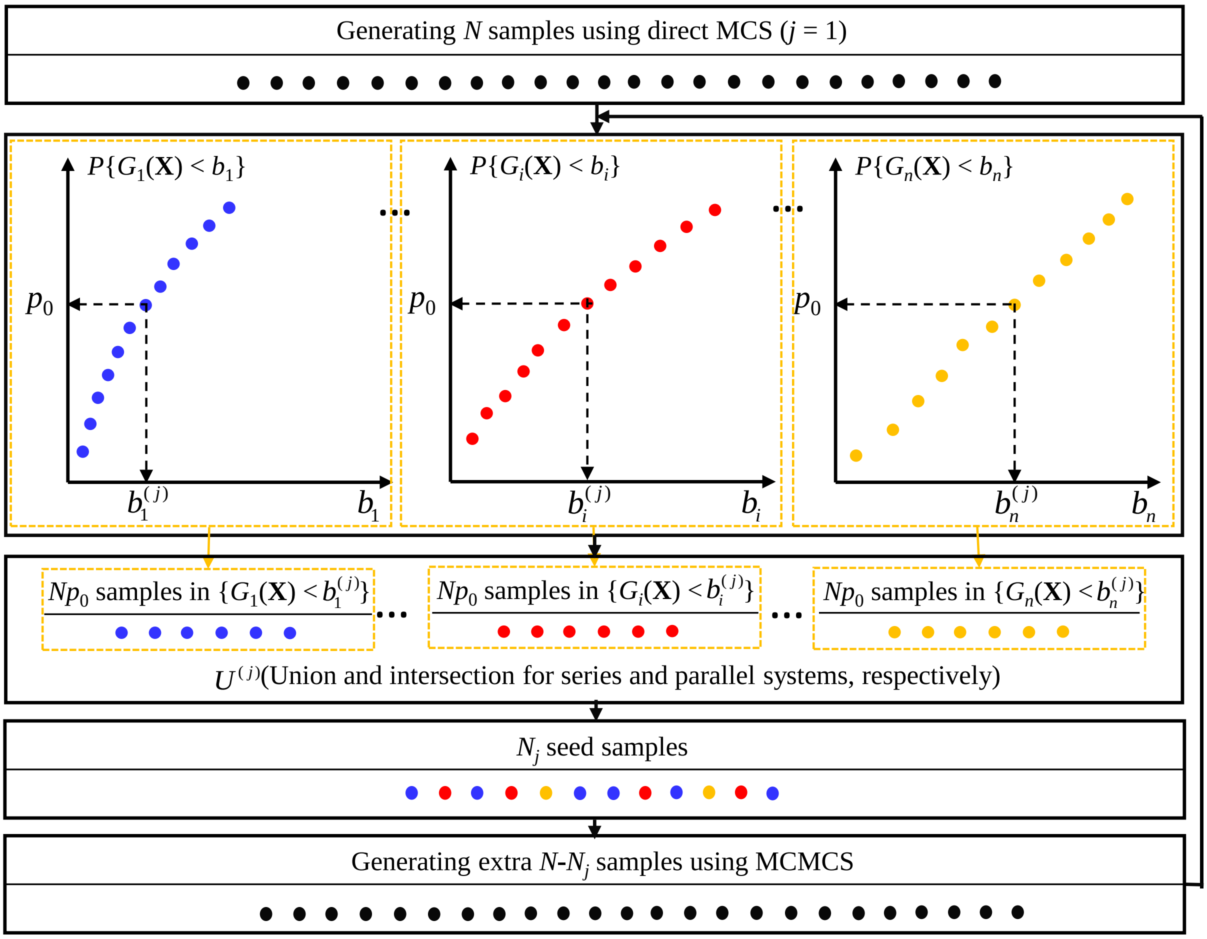


Fig. 3. Implementation procedures of the modified GSS algorithm for system reliability analysis

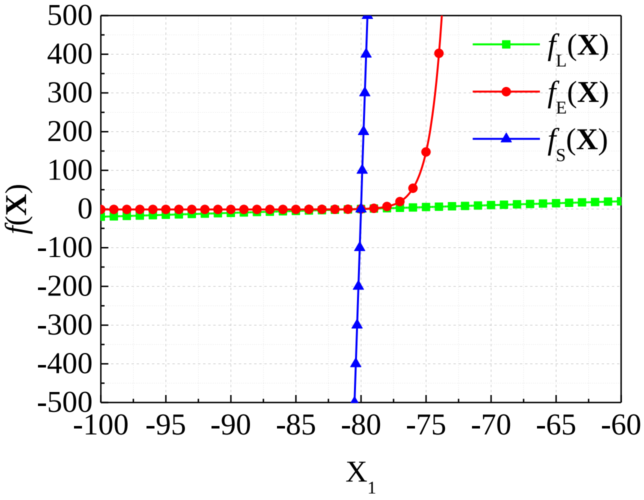
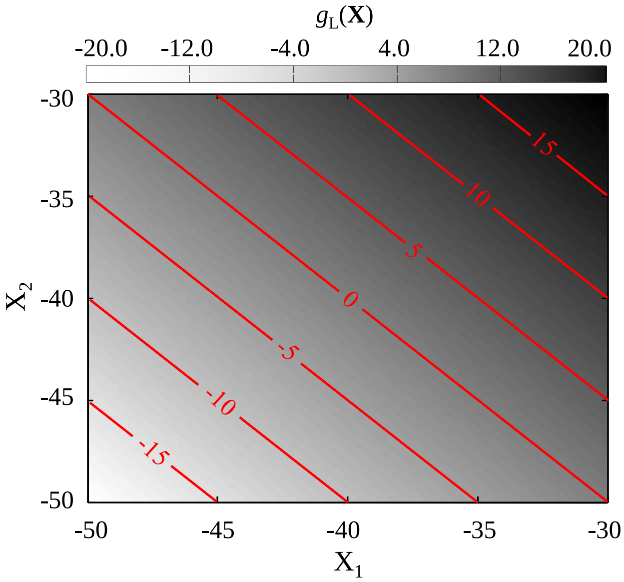
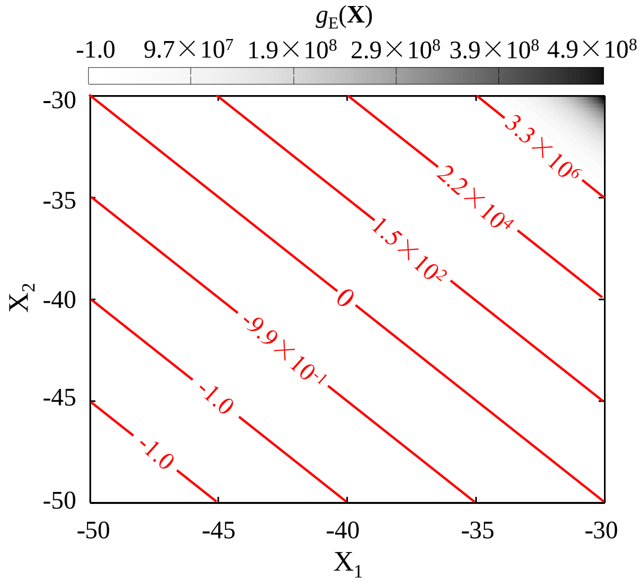


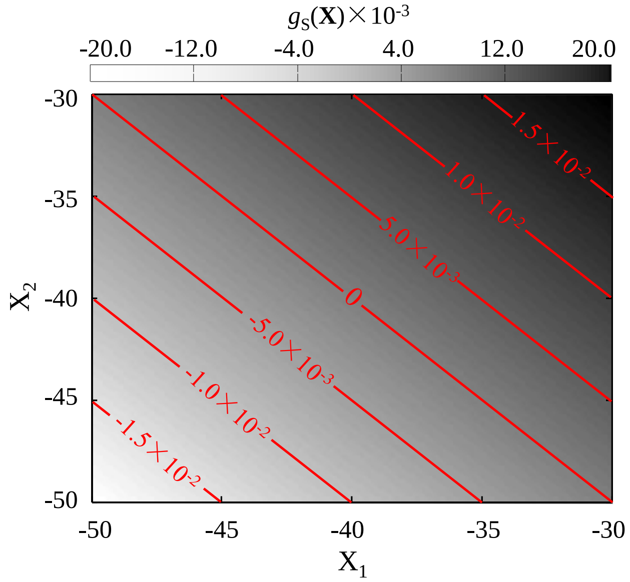
Fig. 4. Three different functional forms of the limit state function $f(\mathbf{X})$



(a) Contour of $g_L(\mathbf{X})$



(b) Contour of $g_E(\mathbf{X})$



(c) Contour of $g_S(\mathbf{X})$

Fig. 5. Contours of limit state function $g(\mathbf{X})$ for its three different functional forms

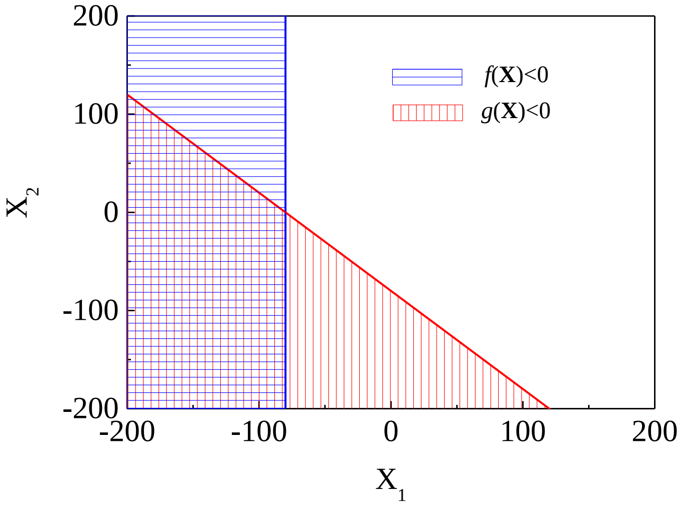


Fig. 6. Failure domains of two limit state functions $f(\mathbf{X})$ and $g(\mathbf{X})$

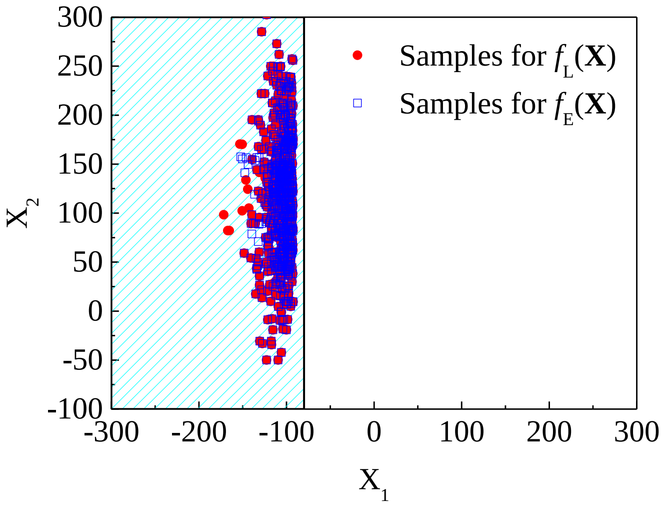


Fig. 7. Failure samples generated by SS for component limit state functions $f_{\mathrm{L}}(\mathbf{X})$ and $f_{\mathrm{E}}(\mathbf{X})$

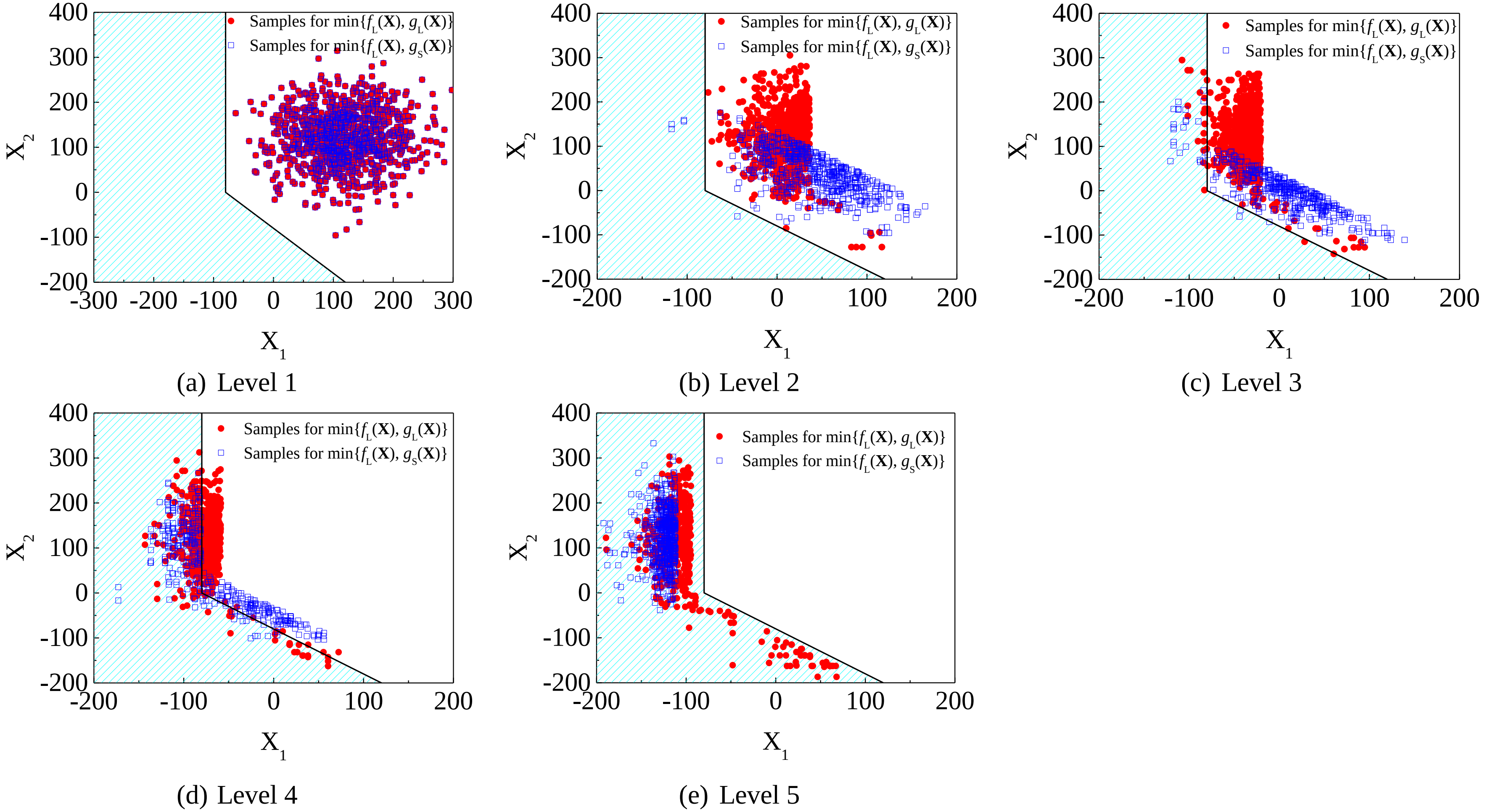


Fig. 8. Random samples generated by SS for series systems with component limit state functions $\{f_L(\mathbf{X}), g_L(\mathbf{X})\}$ and $\{f_L(\mathbf{X}), g_S(\mathbf{X})\}$

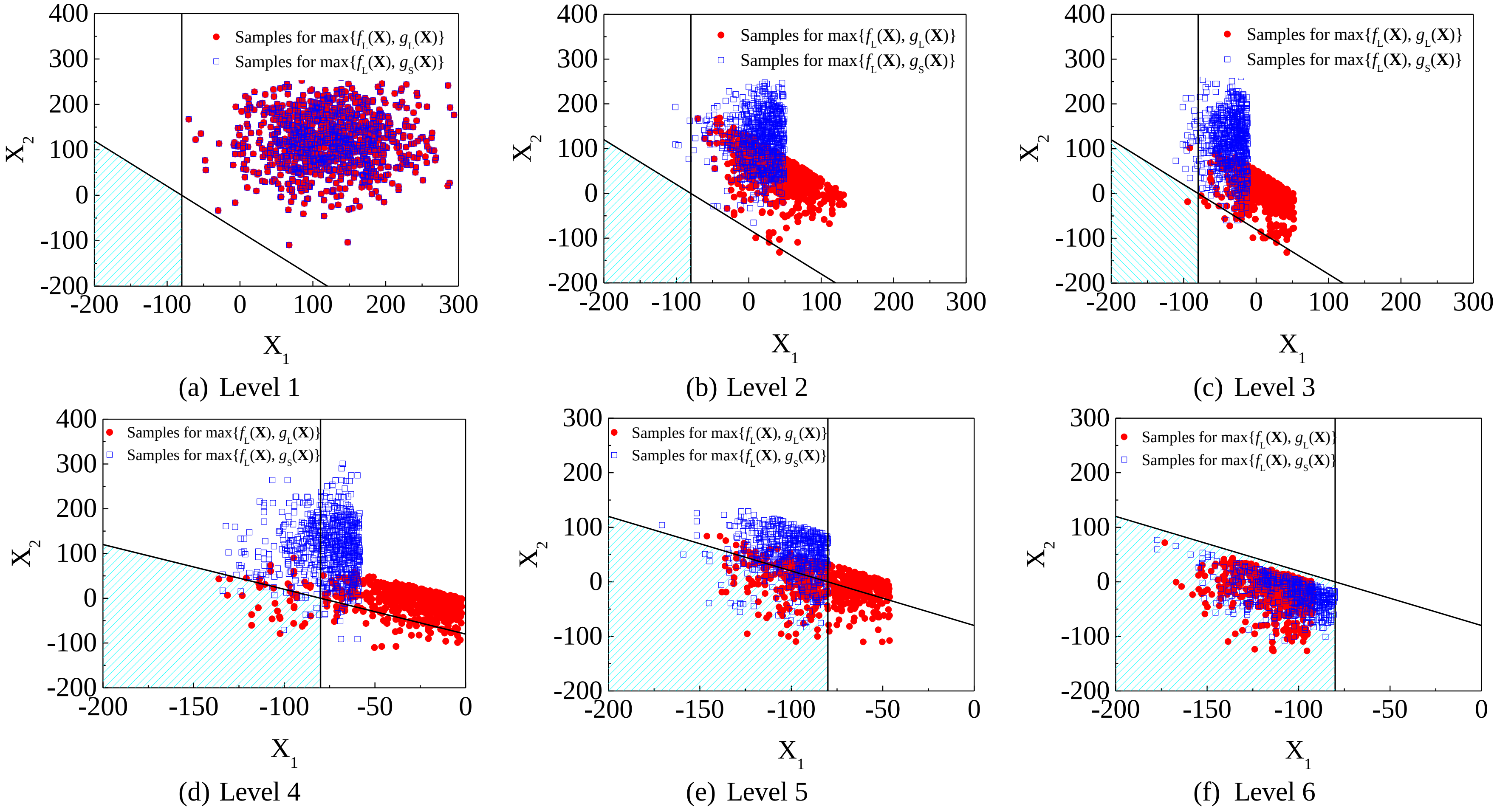


Fig. 9. Random samples generated by SS for parallel systems with component limit state functions $\{f_L(\mathbf{X}), g_L(\mathbf{X})\}$ and $\{f_L(\mathbf{X}), g_S(\mathbf{X})\}$

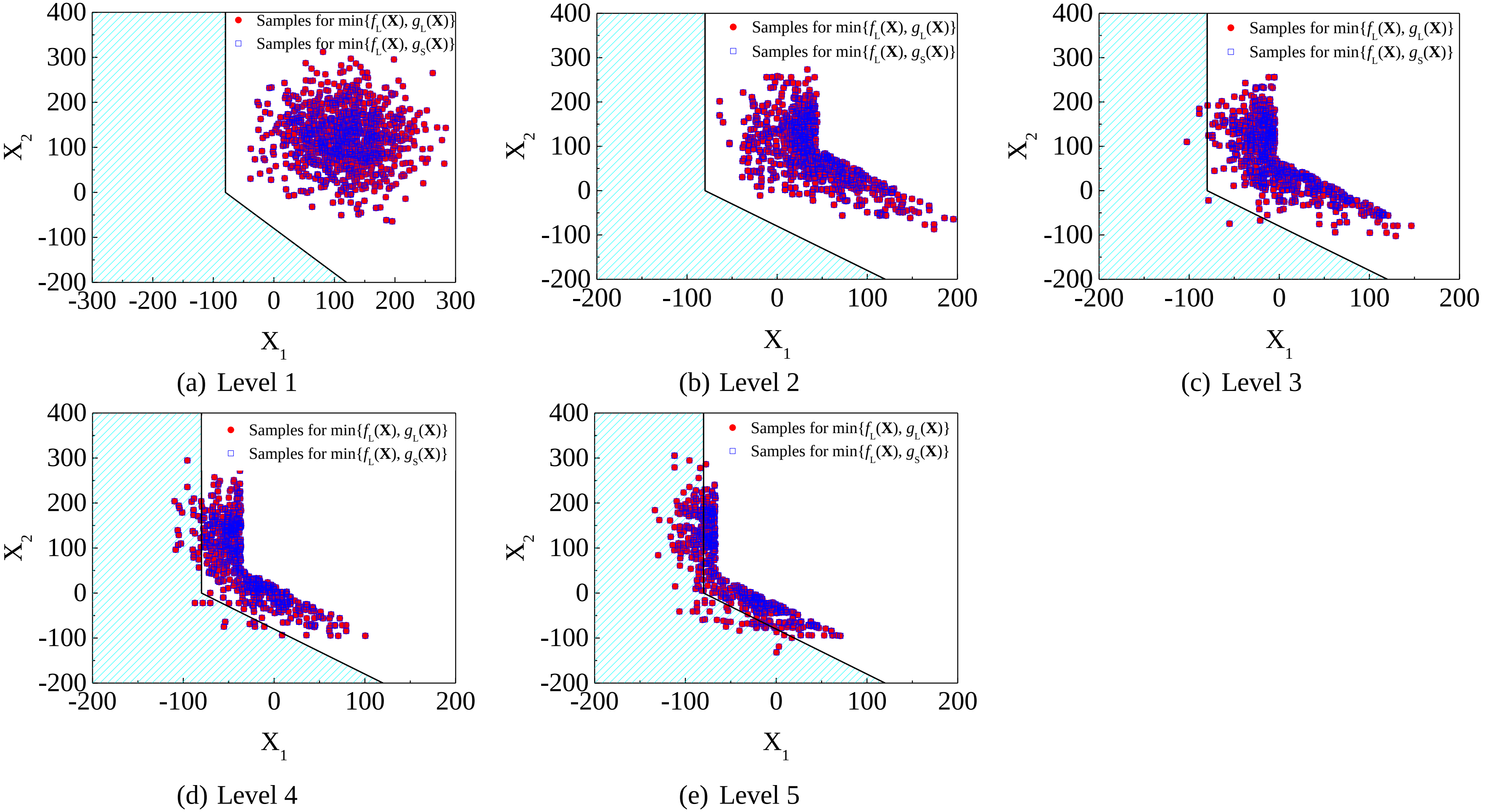


Fig. 10. Random samples generated by the modified GSS algorithm for series systems with component limit state functions {f_L(**X**), g_L(**X**)} and {f_L(**X**), g_S(**X**)}

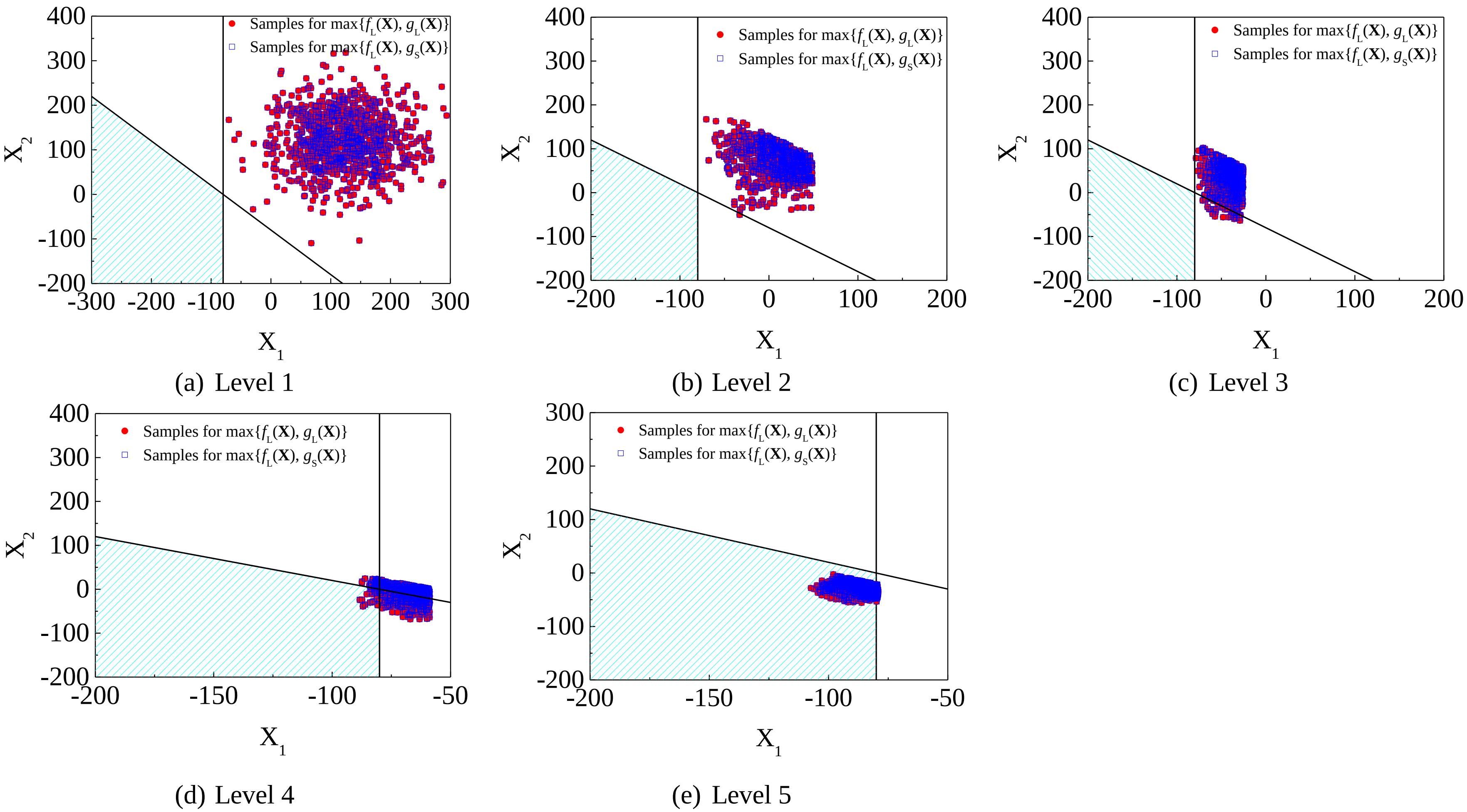


Fig. 11. Random samples generated by the modified GSS algorithm for parallel systems with component limit state functions $\{f_L(\mathbf{X}), g_L(\mathbf{X})\}$ and $\{f_L(\mathbf{X}), g_S(\mathbf{X})\}$

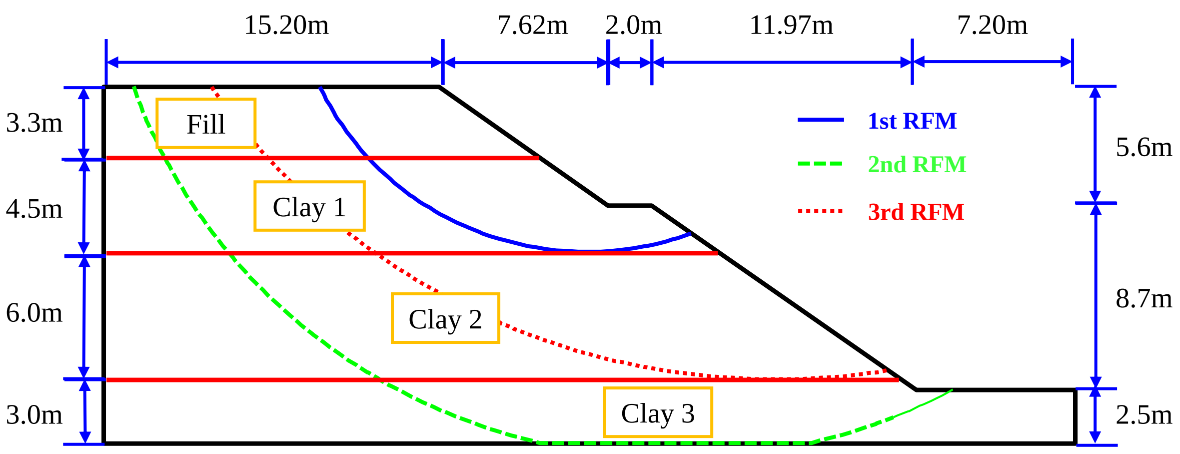


Fig. 12 Geometry of congress cut slope