Pricing and simulating CAT bonds in a Markov-dependent environment

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Abstract

Insurance companies are seeking more adequate liquidity funds to cover the insured property losses related to nature and man-made disasters. Past experience shows that the losses caused by catastrophic events, such as earthquakes, tsunamis, floods or hurricanes, are extremely large. One of the alternative methods of covering these extreme losses is to transfer part of the risk to the financial markets, by issuing catastrophe-linked bonds. We derive bond price formulas in a stochastic interest rate environment, when the aggregate losses have compound forms in two ways. Firstly, the aggregate claims process is driven by a compound inhomogeneous Poisson perturbed by diffusion. Secondly, we consider the claim inter-arrival times to be dependent on the claim sizes by employing a two-dimensional semi-Markov process. For these two types of aggregate claims, we obtain zero-coupon CAT bond prices for multi threshold payoffs functions and classic payoffs functions. Finally, we estimate and calibrate the parameters in each model and use Monte Carlo simulations to obtain numerical results for the aforementioned CAT bonds pricing formulas.

Keywords. Catastrophe bonds, Markov-dependent environment, Pricing CAT bonds, Monte Carlo simulations.

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1 Introduction

Although catastrophe risk events, such as earthquakes, hurricanes, floods and other man-made disasters infrequently occur, massive claims in short period resulted in the insolvency of insurance company. The Insurance Service Office's (ISO's) Property Claim Service (PCS)[1] declared 254 catastrophes with approximately \$ 112bn between 1990 and 1996, while losses from Hurricane Andrew in 1992 reached \$ 26bn[2]. Potential enormous financial demands on the insurance (reinsurance) businesses and increasing difficulty to cover catastrophic losses by reinsurance make it realistic to introduce a securitization mechanic to protect vulnerable individuals.

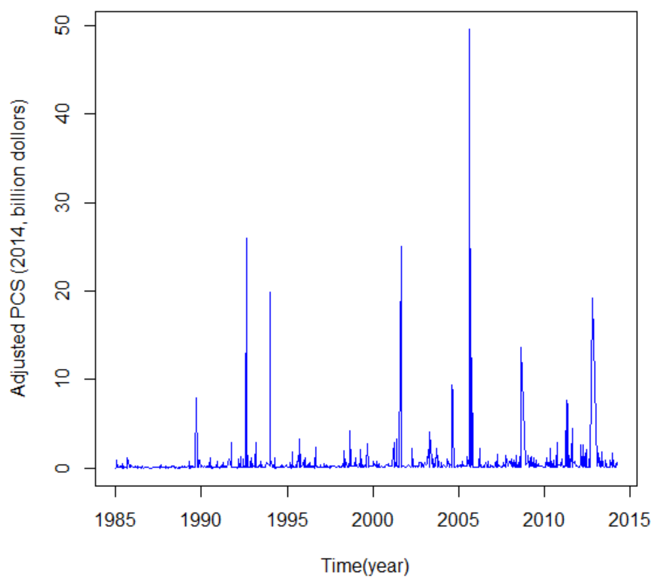


Figure 1: Graph of the PCS catastrophe loss data in US 1985-2013.

Catastrophe risk bond (CAT bond) or Act-of God bond is the most popular insurance-linked financial securities and has been accelerating throughout the last decade. It directly transfer the financial consequences of catastrophe events to capital markets, in contract to

[1]ISO's Property Claim Services (PCS) unit is the internationally recognized authority on insured property losses from catastrophes in the United States, Puerto Rico, and the U.S. Virgin Islands. It contains information on all historical catastrophes since 1949 including the states impacted, perils and associated loss estimates. http://www.verisk.com/property-claim-services/.

[2]An illustration of the PCS catastrophe loss data in US is given in Figure 1 and we can see Northridge earthquake (1994) with losses of \$ 20bn, 9/11 Terrorist Attacks (2001) with losses of \$ 25bn, Hurricane Katrina (2005) with losses of \$50 bn and Hurricane Sandy with losses of \$ 20bn. Data from PCS, converted to 2014 dollars using the CPI.

cover the possible huge liabilities through traditional reinsurance providers or governmental budgets. The first experimental transaction was completed in the mid-1990s after the Hurricane Andrew and Northridge earthquake with insurance losses of $\$15.5$ bn and $\$12.5$ bn, respectively, by a number of specialized catastrophe-oriented insurance and reinsurance companies in USA, such as AIG, Hannover Re, St. Paul Re, and USAA, GAO (2002). Catastrophe bond market reached historical best record at $7 bn in 2007 compared with $\$2$ bn placed during 2005, despiting the financial crisis in 2008 and 2009. Then, issuers raised approximately $7 bn worth of new catastrophe bonds in 2013, McGhee et al. (2008), Anger and Hum (2014). The CAT bonds are inherently risky, non-indemnity-based multi period deals that pay a regular coupon to investors at end of each period and a final principal payment at the maturity date if no predetermined catastrophic events have occurred. A major catastrophic event in the secured region before the CAT bond maturity date leads to full or partial loss of the capital. For bearing catastrophe risk, CAT bonds compensate a floating coupon of LIBOR plus a premium at rate between 2% and 20%, Cummins (2008), GAO (2002). Moreover, we call such a catastrophe a trigger event. In the literature, there are five types of triggering variable: indemnity, industry index, modeled loss indices, parametric indices and hybrid triggers, Hagedorn et al. (2009) and Burnecki et al. (2011). According to Lin and Wang (2009) and Ma and Ma (2013), PCS's estimates are widely accepted as reference index triggers in financial-market derivatives, including exchange-traded futures and options, catastrophe bonds, catastrophe swaps, industry loss warranties (ILWs), and other catastrophe linked instruments. Hence it is reasonable to use the PCS index losses from the entire property and casualty industry in the US to estimate the parameters related to aggregate losses for the pricing CAT bonds in this paper. We further assume that CAT loss industry indices are instantaneously measurable and updatable.

The amount of literature which is devoting to CAT bonds pricing is relatively limited. The presentence of the catastrophe risks requires an incomplete markets framework to evaluate the CAT bond price, because the catastrophe risks can not be replicated by a portfolio of primitive securities, see Harrison and Kreps (1979), Cox et al. (2000), Cox and Pedersen (2000), and Vaugirard (2003). For example, Froot and Posner (2000, 2002) derived an equilibrium pricing model for uncertain parameters of multi-events risks. Alternative common technique used in the literature of incomplete market setting is the principle of equivalent utility in order to obtain the indifferent pricing. Young (2004) calculated the price of a contingent claim under a stochastic interest rate for an exponential utility function. An extension was proposed by Egami and Young (2008), who introduced a more complex payment structure. Cox and Pedersen (2000) used a time-repeatable representative agent utility. Their approach is based on a model of the term structure of interest rates and a probability structure for catastrophe risks assuming that the agent uses the utility function to make choices about consumption streams. They applied their theoretical results to Morgan Stanley, Winterthur, USAA and Winterthur-style bonds. Extensions involving an n financial and m catastrophe risks framework were investigated by Shao et al. (2014), with applications to earthquakes data. Zimbidis et al. (2007) also adopted the Cox and Pedersen (2000) framework for pricing a Greek bond using equilibrium pricing theory with dynamic interest rates. Several other important alternative pricing mechanisms have been developed for catastrophe-linked securitization pricing models in different markets.

There are several other approaches using stochastic processes to price the CAT bonds. It is important to note that Vaugirard (2003) was the first to develop a simple arbitrage approach to evaluate catastrophe risk insurance-linked securities, notwithstanding the non-traded underlying framework. Lin et al. (2008) applied a Markov-modulated Poisson process for catastrophe occurrences using a similar approach to that of Vaugirard (2003). Baryshnikov et al. (2001) presented a continuous time no-arbitrage price incorporated a compound doubly stochastic Poisson process. Burnecki and Kukla (2003) corrected and then applied their results and calculated the arbitrage-free price of zero-coupon and coupon CAT bonds. Burnecki et al. (2011) illustrated the value of CAT bonds with loss data provided by PCS when the flow of events is inhomogeneous Poisson process. Those approaches are utilized by Härdle and Cabrera (2010) for calibrating CAT bonds prices for Mexican earthquakes. Lee and Yu (2002, 2007) additionally introduced default risk, moral hazard, and basis risk with stochastic interest rate. Ma and Ma (2013) proposed a mixed approximation method to find the numerical solution of CAT bonds with general pricing formulas. While Nowak and Romaniuk (2013) obtained CAT bond prices using Monte Carlo simulations with different payoff functions and spot interest rate.

In this paper, we derive the CAT bonds pricing formulas under the stochastic interest rate environment under the assumption that the occurrence of the localized catastrophe is independent of global financial market behaviour, Cox and Pedersen (2000). Our contribution to the literature of CAT bond pricing is three ways. Firstly, constructed two models of the aggregate claim process of the CAT bond as an extension of the approach of Ma and Ma (2013). We introduce extra uncertainly of the claims by perturbation model and include the dependency between the claims sizes and the claim inter-arrival times by Semi-Markov model. Secondly, we apply theoretical results to construct CAT bond and then use PCS data to estimate relevant parameters. Thirdly, we proved closed-form pricing formulas with respect to four different proposed payoffs functions. Section 2 presents the pricing model of CAT bonds. Section 3 provides the numerical analysis of the PCS data. Finally in Section 4 we provide a discussion on the results and suggest the future directions.

2 Modelling CAT bond

2.1 Modelling assumptions

In this section we introduce the preliminary presentation of the CAT bond structure, which generalizes and extends the CAT bond pricing approaches from Ma and Ma (2013). We price catastrophe risk bonds under the following assumptions:(i) there exists an arbitrage-free investment market with equivalent martingale measure $\mathbb{Q}$, (ii) financial market behaves independently with the occurrence of catastrophes, and (iii) the replicability of interest rate changes by existing financial instruments.

Let $0 < T < \infty$ be the maturity date of the continuous time trading interval $[0, T]$. The market uncertainty is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where $\mathcal{F}_t$ is an increasing family of σ-algebras given by $\mathcal{F}_t = \mathcal{F}_t^1 \times \mathcal{F}_t^2 \subset \mathcal{F}$, for $t \in [0, T]$. Here, $\mathcal{F}_t^1$ represents the investment information (e.g. past security prices, interest rates) available to the market at time t, and $\mathcal{F}_t^2$ represents the catastrophe risk information (e.g. insured property losses). Moreover, we define the following filtrations, namely $\mathcal{A}_t^1 = \mathcal{F}_t^1 \times \{\emptyset, \Omega_2\}$ for $t \in [0, T]$ and $\mathcal{A}_t^2 = \{\emptyset, \Omega_1\} \times \mathcal{F}_t^2$ for $t \in [0, T]$. Thus, an $\mathcal{A}_T^\kappa$ measurable random variable X on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ (or an $\mathcal{A}^\kappa$ adapted stochastic process Y) is said to be depend on financial risk variables ($\kappa = 1$) or catastrophic risk variables ($\kappa = 2$).

Let us define stochastic processes and random variables with respect to probability measure $\mathbb{P}$. We denote the CAT bond price process by $\{V_\ell^{(\varrho)}(t) : t \in [0, T]\}$, which is characterized by aggregate loss process $\{L_\ell(t) : t \in [0, T]\}$, $\ell = 0, 1, 2, 3$, and payoffs functions $P_{CAT}^{(\varrho)}$, $\varrho = 1, 2, 3, 4$. For each $t \in [0, T]$, the process $\{N(t) : t \in [0, T]\}$ describes the number of claims occurred until the moment t. We also define $\{X_k : k \in \mathbb{N}^{+}$[3]$\}$ to be a sequence of i.i.d. random variables representing the size of individual claims and $\{T_k : k \in \mathbb{N}^+\}$ represents a sequence of epoch times of the claims. Additionally, we define the spot interest rate process by $\{r(t) : t \in [0, T]\}$ and let $\{W(t) : t \in [0, T]\}$ be a standard Brownian motion.

2.2 Valuation theory

The presence of catastrophic risks which are uncorrelated with the underlying financial risks leads us to consider an incomplete market. In this case, there is no universal theory that addresses all aspects of pricing, Vaugirard (2003). For valuation purposes, similar to Merton (1976), we assume that under the risk-neutral pricing measure $\mathbb{Q}$, events that depend on financial risks are independent of catastrophic events. It is a quite natural approximation since the global economic circumstances in terms of exchange and production are only marginally influenced by the localized catastrophes. For more information, see e.g. Merton (1976), Doherty (1997), Cox and Pedersen (2000), Lee and Yu (2007), Ma and Ma (2013) and Shao et al. (2014). According to Lemma 5.2 in Cox and Pedersen (2000), under the assumption that aggregate consumption is $\mathcal{A}^1$ adapted (assumption ii), for any random variable X that is $\mathcal{A}_T^2$ measurable we have ms1

$$E^{\mathbb{Q}}[X] = E^{\mathbb{P}}[X]. \tag{2.1}$$

That is to say, the aggregate loss process $\{L_\ell(t) : t \in [0, T]\}$, $\ell = 0, 1, 2, 3$, retain their original distributional characteristics after changing from the historical estimated actual probability measure $\mathbb{P}$ to the risk-neutral probability measure $\mathbb{Q}$. Moreover, under the risk-neutral

[3] $\mathbb{N}^+ = 1, 2, 3, ...$

probability measure $\mathbb{Q}$, the events $\mathcal{A}_T^1$ measurable and events $\mathcal{A}_T^2$ measurable are independent, see, e.g. Cox and Pedersen (2000), Ma and Ma (2013) and Shao et al. (2014). In an arbitrage free market (assumption i), at any time t, we can price a contingent claim with payoff $\{P(T) : T > t\}$ at time T by the fundamental theorem of asset pricing (Delbaen and Schachermayer (1994)). We can express the value of this claim by ms5

$$V(t) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r(s)ds} P(T)|\mathcal{F}_t), \tag{2.2}$$

where $\mathbb{E}^{\mathbb{Q}}$ is an equivalent expectation under the probability measure $\mathbb{Q}$.

2.3 Interest rate process

In this paper, the spot interest rate dynamic is assumed to be a Cox, Ingersol and Ross (CIR)[4] model Cox et al. (1985). The short-rate dynamics under the risk-neutral measure $\mathbb{Q}$ can be expressed as follows, va1

$$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t),$$

with $r(0), k, \theta$ and σ are positive constants. While the condition $2k\theta > \sigma^2$ guarantees the process $r(t)$ staying in the positive domain. Let us assume a stochastic process va2

$$\lambda_r^*(t) = \frac{\lambda_r}{\sigma}\sqrt{r(t)},\ t \in [0, T],$$

where constant λ_r represents the market risk. In order to price a zero-coupon bond, one can transfer the interest rate process from measure $\mathbb{P}$ to $\mathbb{Q}$ by λ_r^*. For detailed information of this transformation, check Ma and Ma (2013), Shirakawa (2002), Lee and Yu (2002), etc.

According to Brigo and Mercurio (2007), we can price a pure-discount T-bond at time t by the following equalities: va3

$$B_{CIR}(t,T) = A(t,T)e^{-B(t,T)r(t)}, \tag{2.3}$$

where va4-6

$$A(t,T) = \left[\frac{2\gamma e^{(k+\lambda_r+\gamma)(T-t)/2}}{2\gamma + (k+\lambda_r+\gamma)(e^{(T-t)h} - 1)}\right]^{\frac{2k\theta}{\sigma^2}}, \tag{2.4}$$

$$B(t,T) = \left[\frac{2(e^{(T-t)\gamma} - 1)}{2\gamma + (k+\lambda_r+\gamma)(e^{(T-t)\gamma} - 1)}\right], \tag{2.5}$$

$$\gamma = \sqrt{(k+\lambda_r)^2 + 2\sigma^2}. \tag{2.6}$$

[4]Nowak and Romaniuk (2013) compared the CAT bond prices under the assumption of spot interest rate described by Vasicek, Hull-White and CIR model. However, we are not interesting in the pricing process which affected by the interest rate dynamics and use the most popular model – CIR model – as an example. Readers can refer to Brigo and Mercurio (2007) for more options for interest rate dynamics.

2.4 Aggregate claims process

Let us describe the CAT bonds payment structure. CAT bonds investors receive premiums condition on a trigger event not occurring. In this paper, we utilize an insurance industry index trigger to price CAT bonds. This means that investors might loss their capital if the estimated aggregate losses from the whole industry exceed a predetermined level. We model the aggregate loss process by a compound distribution process which is characterized by the frequency (inter-arrival times) and the severity (claim sizes) of catastrophic events, (see e.g. Klugman et al. (2012), Tse (2009) and Ma and Ma (2013)). In this section, we introduce two models of aggregate claims process. Firstly, in Model I, frequency and the severity of catastrophe events are independent and the aggregate loss process perturbed by an additional diffusion process. Secondly, in Model II, dependency among the claim sizes and among the claim arrival times is introduced by a general Semi-Markov process. And finally we introduce a special case of Model II, where claim arrival process is a continuous time Markov process with exponential inter-arrival time.

2.4.1 Perturbed aggregate loss process (Model I)

In the classical actuarial literature (Bowers Jr. et al. (1986)), the claim number process $\{N(t) : t \in [0, T]\}$ follows a Poisson process with parameter $\lambda > 0$, describing the number of the future coming catastrophe occurs in the insured region. The claim sizes $\{X_k : k \in \mathbb{N}^+\}$, independent with the process $\{N(t) : t \in [0, T]\}$, are a sequence of positive i.i.d. random variables with common distribution function $F(x) = \mathbb{P}\{X_k < x\}$, which denote the amount of losses incurred by the kth event. Then, the aggregate loss process $\{L_0(t) : t \in [0, T]\}$ is modeled by compound Poisson process as follows: v1

$$L_0(t) = \sum_{k=1}^{N(t)} X_k, \tag{2.7}$$

with the convention that $L_0(t) = 0$ when $N(t) = 0$.

During the past twenty years, numerous papers are aiming to model market fluctuations incorporating Brownian motion and jump diffusion processes in finance, while this typical setting of aggregate loss dynamic given in Eq (2.7) can be extended by adding an independent diffusion process (e.g. Dufresne and Gerber (1991), Tsai (2001, 2003), Tsai and Willmot (2002)). This extra diffusion process reflects the uncertainty of the aggregate claims and is first introduced for CAT bond pricing. More precisely, the left-continuous aggregate claims process at time t ($t \geq 0$) is denoted by v2

$$L_1(t) = \sum_{k=1}^{N(t)} X_k + \sigma W(t), \tag{2.8}$$

where σ is a positive constant, $\{W(t) : t > 0\}$ is a standard Wiener process with drift coefficient 0 and variance t, and $\{N(t) : t \in [0, T]\}$, $\{X_k : k \in \mathbb{N}^+\}$ and $\{W(t) : t \in [0, T]\}$ are mutually independent. In order to operate a more general model, we employ the number-of-claims process $\{N(t) : t \in [0, T]\}$ by a nonhomogeneous Poisson process with parameters $\lambda(t) > 0$.

Proposition 2.1 produces the density function of the aggregate loss which is very useful in the pricing procedure of CAT bonds in the subsection 2.5. proposition2

Proposition 2.1. *Let $F_1(t, D)$ denotes the probability function that aggregate claims $L_1(t)$ less or equal to the threshold D at time t. Then* v9

$$F_1(t, D) = \frac{1}{\sqrt{2\pi t}} \sum_{n=0}^{\infty} e^{-\lambda(t)t} \frac{(\lambda(t)t)^n}{n!} \int_{-\infty}^{+\infty} e^{-\frac{w^2}{2t}} F^{*n}(D - \sigma w) dw, \tag{2.9}$$

where $F^{*n}(x) = \mathbb{P}(X_1 + X_2 + \cdots + X_n \leq x)$ denote the n-fold convolution of F.

Proof. Under the condition of $W(t), L_1(t), N(t), X_k$ are mutually independent, and according to the law of total probability,

$$\begin{aligned}
F_1(t, D) =& \mathbb{P}(\sum_{k=1}^{N(t)} X_k + \sigma W(t) \leq D) \\
=& \sum_{n=0}^{\infty} \mathbb{P}(\sum_{k=1}^{N(t)} X_k + \sigma W(t) \leq D | N(t) = n) \mathbb{P}(N(t) = n) \\
=& \sum_{n=0}^{\infty} \mathbb{P}(\sum_{k=1}^{n} X_k + \sigma W(t) \leq D) \mathbb{P}(N(t) = n) \\
=& \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} \mathbb{P}(\sum_{k=1}^{n} X_k + \sigma w \leq D) \frac{1}{\sqrt{2\pi t}} e^{-\frac{w^2}{2t}} dw \mathbb{P}(N(t) = n),
\end{aligned}$$

and the result follows. □ v10

remark1

Remark 2.1. *When $\sigma = 0$, Model I reduces to the case of nonhomogeneous Poisson process which is utilized by Ma and Ma (2013). We could easily get that the probability of aggregate claims $L_0(t)$ less or equal to the threshold D at time t equals to:* v43

$$F_0(t, D) = \sum_{n=0}^{\infty} e^{-\lambda(t)t} \frac{(\lambda(t)t)^n}{n!} F^{*n}(D), \tag{2.10}$$

where $F^{*n}(x) = \mathbb{P}(X_1 + X_2 + \cdots + X_n \leq x)$ denote the n-fold convolution of F.

2.4.2 General Semi-Markov process (Model II)

Although the diffusion term in the perturbation model relaxes an extra uncertainty for the aggregate claims in the risk theory, however, the independent assumption is too restrictive in many applications. Therefore, a more appropriate option is to add the dependency between the claims sizes and inter-arrival times of the claim process when modeling the aggregate losses. In this subsection, we introduce semi-Markov risk model which is first introduced by Miller (1962) and fully developed by Janssen (1969); Janssen and Manca (2007).

Consider a semi-Markovian dependence structure in continuous time; the process $\{J_n, n \geq 0\}$ represents the successive type of claims or environment states taking their values in $J = \{1, ..., m\}$ $(m \in \mathbb{N}^+)$. Define $\{X_n, n \geq 1\}$ as a sequence of successive claim sizes, $X_0 = 0$ a.s. and $X_n > 0, \forall n$, and $\{T_n, n \in \mathbb{N}^+\}$ be the epoch times of the nth claim. We suppose that $0 < T_1 < T_2 < \ldots < T_n < T_{n+1} < \ldots$, $T_0 = U_0 = 0$ a.s., and $U_n = T_n - T_{n-1}$ $(n \in \mathbb{N}^+)$ denotes the sojourn time in state J_{n-1}. Suppose that tri-variate process $\{(J_n, U_n, X_n); n \geq 0\}$ is a semi-Markovian dependence process defined by the following matrix **Q**, v3

$$Q_{ij}(t, x) = \mathbb{P}(J_n = j, U_n \leq t, X_n \leq x | (J_k, U_k, X_k), k = 1, 2, ..., n-1, J_{n-1} = i), \tag{2.11}$$

where the process of successive claims $\{J_n\}$ is an irreducible homogeneous continuous time Markov chain with state space J and transition matrix $\mathbf{P} = ((p_{ij}), i, j \in J)$. The process changes its state at every instant of claim based on the transition matrix **Q** and an interpretation of this model in terms of CAT bond is that the arrival time before the next catastrophic event U_{k+1} is partially depending on the severity of the catastrophic event X_k, for all $k = 0, 1, 2, \ldots$.

Assuming that random variable $J_n, n \geq 0$ and the two-dimensional random variable $(U_n, X_n), n \geq 1$ are conditionally independent, then v24

$$\begin{aligned} G_{ij}(t, x) =& \mathbb{P}(U_n \leq t, X_n \leq x | J_0, ..., J_{n-1} = i, J_n = j) \\ =& \begin{cases} Q_{ij}(t, x)/p_{ij}, \text{ for } p_{ij} > 0, \\ \mathbb{1}\{t \geq 0\}\mathbb{1}\{x \geq 0\}, \text{ for } p_{ij} = 0. \end{cases} \end{aligned}$$

Here $\mathbb{1}\{\cdot\}$ represents an indicator function. The random variable $J_n, n \geq 0$ is conditionally dependent with the random variable $U_n, n \geq 1$ and similarly dependent with the random variable $X_n, n \geq 1$. We can get

$$\begin{aligned} G_{ij}(t, \infty) &= \mathbb{P}(U_n \leq t | J_0, ..., J_{n-1} = i, J_n = j), \\ G_{ij}(\infty, x) &= \mathbb{P}(X_n \leq x | J_0, ..., J_{n-1} = i, J_n = j). \end{aligned}$$

We can have the following equations by suppressing the condition J_n, v25

$$
\begin{aligned}
&H_i(t,x) = \mathbb{P}(U_n \le t, X_n \le x | J_0, ..., J_{n-1} = i) = \sum_{j=1}^{m} p_{ij} G_{ij}(t,x), \\
&H_i(t,\infty) = \mathbb{P}(U_n \le t | J_0, ..., J_{n-1} = i), \qquad (2.12)\\
&H_i(\infty,x) = \mathbb{P}(X_n \le x | J_0, ..., J_{n-1} = i).
\end{aligned}
$$

Assume that the sequences $\{U_n, n \ge 1\}$, $\{X_n, n \ge 1\}$ are conditionally independent given the sequence $\{J_n, n \ge 0\}$, therefore v26

$$G_{ij}(t,x) = G_{ij}(t,\infty) G_{ij}(\infty,x), \forall t, x \in \mathbb{R}, \forall i,j \in J.$$

Thus, the semi-Markov kernel **Q** can be expressed as v27

$$Q_{ij}(t,x) = p_{ij} G_{ij}(t,\infty) G_{ij}(\infty,x), \forall t, x \in \mathbb{R}, \forall i,j \in J.$$

Define ${}^{A}\mathbf{Q}$ be the kernel of the process $\{(J_n, U_n); n \ge 0\}$ and similarly ${}^{B}\mathbf{Q}$ be the kernel of the process $\{(J_n, X_n); n \ge 0\}$ v35

$$
\begin{aligned}
{}^{A}Q_{ij}(t) &= Q_{ij}(t,\infty) = p_{ij} G_{ij}(t,\infty), \forall t \in \mathbb{R}, \forall i,j \in J,\\
{}^{B}Q_{ij}(x) &= Q_{ij}(\infty,x) = p_{ij} G_{ij}(\infty,x), \forall x \in \mathbb{R}, \forall i,j \in J.
\end{aligned}
$$

In order to calculate the distribution function of the accumulated claims amount, we need to consider the following random walk process, as presented in Janssen and Manca (2007). Let L_n be the successive total claims amount after the arrival of the nth claim, defined as: v28

$$L_n = \sum_{k=1}^{n} X_k, \forall n \ge 1, \forall i,j \in J.$$

Then, we can express the joint probability of the process $\{(J_n, T_n, L_n); n \ge 0\}$, and define v37

$$\mathbb{P}[J_n = j, T_n \le t, L_n \le x | J_0 = i] = Q_{ij}^{*n}(t,x),$$

where $Q_{ij}^{*n}(\cdot,\cdot)$ is a form of n-fold convolution. This n-fold convolution matrix $\mathbf{Q}^{(n)}$ can be valued recursively by: v21

$$
\begin{aligned}
&Q_{ij}^{*0}(t,x) = \delta_{ij}(t,x) L_0(t,x) = \begin{cases} (1 - G_{ij}(0,\infty))(1 - G_{ij}(\infty,0)), \text{ if } i = j \\ 0, \text{ elsewhere,} \end{cases}\\
&Q_{ij}^{*1}(t,x) = Q_{ij}(t,x), \quad \ldots\\
&Q_{ij}^{*n}(t,x) = \sum_{l=1}^{m} \int_0^t \int_0^x Q_{lj}^{*(n-1)}(t-t', x-x') dQ_{il}(t',x').
\end{aligned}
$$

Similarly for the processes $\{(J_n, T_n); n \geq 0\}$ and the process $\{(J_n, X_n); n \geq 0\}$ we have v38v39

$$\mathbb{P}[J_n = j, T_n \leq t | J_0 = i] = {}^A Q_{ij}^{*n}(t) = \sum_{l=1}^{m} \int_0^t {}^A Q_{lj}^{*(n-1)}(t - t') d^A Q_{il}(t'), \tag{2.13}$$

$$\mathbb{P}[J_n = j, L_n \leq x | J_0 = i] = {}^B Q_{ij}^{*n}(x) = \sum_{l=1}^{m} \int_0^x {}^B Q_{lj}^{*(n-1)}(x - x') d^B Q_{il}(x').$$

We then have the following equations,

$$\mathbb{P}[J_n = j | J_0 = i] = p_{ij}^{*n} = \sum_{l=1}^{m} p_{lj}^{*(n-1)} p_{il},$$

$$\mathbb{P}[L_n \leq x | J_0 = i, J_n = j] = G_{ij}^{*n}(\infty, x) = \frac{{}^B Q_{ij}^{*n}(x)}{p_{ij}^{*n}}, \tag{2.14}$$

$$Q_{ij}^{*n}(t, x) = {}^A Q_{ij}^{*n}(t) G_{ij}^{*n}(\infty, x).$$

Let the counting process $\{N_i(t), t \geq 0\}$ denotes the total number of type i claims occurring in $(0, t]$, for all $i \in J$. Thus the total number of claims $\{N(t), t \geq 0\}$ occurring on $(0, t]$ is v20

$$N(t) = \sum_{i=1}^{m} N_i(t), \tag{2.15}$$

and $N(0) = 0, N_i(0) = 0$. Moreover, define $J_{N(t)}$ as the type of last claim occurred before or at t, thus the aggregate claims process can be expressed as: v4

$$L_2(t) = L_{N(t)} = \sum_{k=1}^{N(t)} X_k, \tag{2.16}$$

which is the same form of the classical aggregate claims process Eq (2.7). Moreover, we suppose that the embedded Markov Chain $\{J_n; n \geq 0\}$ is ergodic and there exist a sequence of unique probabilities $(\Pi_1, ..., \Pi_M)$ which represents the stationary probability distribution, $\Pi_1 + ... + \Pi_M = 1$ and $\Pi_1, ..., \Pi_M \in [0, 1]$.

The following proposition gives an explicit expression for the density function of aggregate loss. proposition3

v11

Proposition 2.2. *Let $F_2(t, D)$ denotes the probability function that aggregate claims $L_2(t)$ less than or equal to the threshold D, at time t. Then*

$$F_2(t, D) = \sum_{i=1}^{m} \sum_{j=1}^{m} \Pi_i \sum_{n=0}^{\infty} \int_0^t (1 - H_j(t - t', \infty)) d[{}^A Q_{ij}^{*n}(t') G_{ij}^{*n}(\infty, D)].$$

Proof. Starting with the stationary probability for J_0, it is given in Eq(4.2) of Janssen (1980) that v12

$$F_2(t,D) = \mathbb{P}(\sum_{k=1}^{N(t)} X_k \le D) = \sum_{i=1}^{m}\sum_{j=1}^{m} \Pi_i \mathbb{P}(\sum_{k=1}^{N(t)} X_k \le D, J_{N(t)} = j | J_0 = i).$$

Further, according to Chapter 7, Eq (3.32) in Janssen and Manca (2007), the following equality holds: v29

$$\mathbb{P}(\sum_{k=1}^{N(t)} X_k \le D, J_{N(t)} = j | J_0 = i) = \sum_{n=0}^{\infty} \int_0^t (1 - H_j(t-t', \infty)) dQ_{ij}^{*n}(t', D),$$

and the result follows by simple substitution. □

Using the results from Proposition 2.2, we introduce the SM'/SM model as a particular case of the Model II, and ${}^{A}\mathbf{G}$ is defined as: v40

$$G_{ij}(t,\infty) = \begin{cases} 0, \ t < 0 \\ 1 - e^{-\lambda_i t}, \ t \ge 0. \end{cases}$$

That is to say, the distribution function of sojourn time depends uniquely on the current state i, which is exponentially distributed with parameter λ_i. Further assume that Markov chain jumps to state j at each instant of a claim with claim size distribution $F_j(D) = \mathbb{P}_j(X_k \le D)$. This has a practical meaning as a bigger catastrophic event can trigger many other events as side effects. We can formally have the following assumptions:

$$G_{ij}(t,\infty) = G_i(t,\infty), G_{ij}(\infty, D) = G_j(\infty, D) = F_j(D), i, j \in J, t, x > 0.$$

More precisely, the process $\{J_n, U_n, X_n; n \ge 0\}$ has the following probabilistic structure, v30

$$\begin{aligned} Q_{ij}(t,D) &= \mathbb{P}[J_n = j, U_n \le t, X_n \le D | (J_k, U_k, X_k), k = 1, 2, ..., n-1, J_{n-1} = i] \\ &= \mathbb{P}[J_1 = j, U_1 \le t, X_1 \le D | J_0 = i] \\ &= p_{ij} F_j(D)(1 - e^{-\lambda_i t}), \end{aligned}$$

$\forall t, x \in \mathbb{R}, \forall i, j \in J$. Thus J_n, W_n, and X_n are independent of the past given J_{n-1}, and the sequences $\{U_n, n \ge 1\}$, $\{X_n, n \ge 1\}$ are conditionally independent given the sequence

$\{J_n, n \geq 0\}$. We could rewrite the equations Eq (2.13) Eq (2.14) and Eq (2.12) as:

$$
\begin{aligned}
{}^{A}Q_{ij}^{*n}(t) &= (p_{ij}(1 - e^{-\lambda_i t}))^{*n}, \\
G_{ij}^{*n}(\infty, D) &= \frac{(p_{ij}F_j(D))^{*n}}{p_{ij}^{*n}}, \\
H_j(t, \infty) &= \sum_{i=1}^{m} p_{ji}(1 - e^{-\lambda_j t}) = 1 - e^{-\lambda_j t}.
\end{aligned}
$$

Substitute in Proposition 2.2 and obtain the following corollary. corollary1

Corollary 2.1. *At time t, the probability of total loss amount $L_3(t) = \sum_{k=1}^{N(t)} X_k$ no more than the pre-defined level D can be computed as:*

$$
F_3(t, D) = \sum_{i=1}^{m} \sum_{j=1}^{m} \Pi_i \sum_{n=0}^{\infty} \int_0^t e^{-\lambda_j (t-t')} d \left[\frac{(p_{ij}(1 - e^{-\lambda_i t'}))^{*n} (p_{ij}F_j(D))^{*n}}{p_{ij}^{*n}} \right]. \quad (2.17)
$$

v31

Remark 2.2. *For $m = 1$, this model is the classical Poisson process model with parameter λ.*

Remark 2.3. *One can easily combine Model I and Model II together, which is to say modelling the aggregate loss process with both uncertainty and also dependency between the severity and intensity of the claims.*

2.5 Pricing model for the CAT bonds

In this subsection, we show how one can price the CAT bonds using the standard tool of a risk-neutral valuation measure with the following payoffs functions for a T time maturity zero-coupon CAT bonds[5]. Their valuation is a consequence of Eq (2.2).

We start by defining a hypothetical zero coupon CAT bond at the maturity date as follows:

v5

$$
P_{CAT}^{(1)} = \begin{cases} Z, \text{ for } L_\ell(T) \leq D, \\ pZ, \text{ for } L_\ell(T) > D, \end{cases} \quad (2.18)
$$

where $\ell = 0, 1, 2, 3$; $L_\ell(T)$ is the total insured loss value at the expiry date T; D denotes the threshold value agreed in the bond contract; p ($p \in [0, 1)$) is the fraction of the principle Z which is obligated to be paid to the bondholders when the trigger event occurs.

The next payoffs function with multi-threshold value is given by the equation v6

[5] We will only discuss zero-coupon bonds in this paper, since coupon bonds can be treated as a portfolio of zero-coupon bonds with different maturity.

$$P_{CAT}^{(2)} = p_k Z \quad \forall D_{k-1} < L_\ell(T) \le D_k, \tag{2.19}$$

where $\ell = 0, 1, 2, 3$; $k = 1, 2, \ldots, r$ with $p_1 = 1 > p_2 > \cdots > p_r \ge 0$ and $D_0 = 0 < D_1 < \cdots < D_r$. Generally speaking, investors rate of return is inversely proportional to the total catastrophe claims. An application of multi-threshold payoffs function CAT bonds can be found in Shao et al. (2014).

Another payoffs function with a coupon payment at the maturity date if the trigger not been pulled, is in the form v41

$$P_{CAT}^{(3)} = \begin{cases} Z + C, \text{ for } L_\ell(T) \le D, \\ Z, \text{ for } L_\ell(T) > D, \end{cases} \tag{2.20}$$

where $\ell = 0, 1, 2, 3$; $C > 0$ is the coupon payment level.

In order to introduce the final payoffs function, we consider a CAT bond issuer have the asset value A_{issue} and debt value B_{issue} at the bond maturity time. We know that default risk exists when the sponsor unable to pay the obligations (e.g. premium of the CAT bond). Thus, a CAT bondholder would not receive full amount capital even if the aggregate loss is less than the predetermined level. Let $\{N_{issue} : N_{issue} \ge 0\}$ be the number of this CAT bond issued. Further assume that the issuer's financial situation independent of the aggregate loss process. If the issuing company obtains enough funds to pay bondholders at the maturity date T, face value Z will be paid condition on trigger event has not been pulled, else portion of the principle. If the issuing company fail to meet its obligation, bondholders will loss all of their capital. More precisely, the structure of the defaultable payoffs function equals to v23

$$P_{CAT}^{(4)} = \begin{cases} Z, \text{ if } L_\ell(T) \le D \text{ and } A_{issue} > B_{issue} + ZN_{issue} \\ pZ, \text{ if } L_\ell(T) > D \text{ and } A_{issue} > B_{issue} + pZN_{issue} \\ 0, \text{ otherwise,} \end{cases} \tag{2.21}$$

where $\ell = 0, 1, 2, 3$. In this case, the payoffs of the CAT bond depend not only on the listed catastrophe events, but also the issuer's financial position. However in this stage, we are not interesting in the performance of the issuing company over the trading period.

According to the payoff structures Eq (2.18)-(2.21) of the CAT bonds, interest rate dynamic Eq (2.3), and aggregate loss processes Eq (2.7),(2.8) and (2.16), we present the prices of the CAT bonds in Theorem 2.1-2.4. And these are the main results of this paper.

We give the zero-coupon CAT bond prices at time t paying principal Z at time to maturity T in following Theorem 2.1.

Theorem 2.1. *Let $V_\ell^{(1)}(t)$ ($\ell = 0, 1, 2, 3$) be the prices of T-maturity zero-coupon CAT bond under the risk-neutral measure $\mathbb{Q}$ at time t with payoffs function $P_{CAT}^{(1)}$ defined in Eq* (2.18)*. Then,* v32

$$V_\ell^{(1)}(t) = B_{CIR}(t,T)Z(p + (1-p)F_\ell(T-t,D)),\ \ell = 0,1,2,3,$$

where $F_\ell(T-t,D)$ represents the cumulated function of aggregate loss in alternative models given in Proposition 2.1-2.2, Remark 2.1 and Corollary 2.1, respectively, and pure discounted bond price $B_{CIR}(t,T)$ with CIR interest rate model is given by Eq (2.3)-(2.6).

Proof. Cox and Pedersen (2000) suggests that the payoffs function is independent with financial risks variable (interest rate) under the risk-neutral measure $\mathbb{Q}$. then, according to Eq (2.2) we have

$$V_\ell^{(1)}(t) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r_s ds} P_{CAT}^{(1)}(T)|\mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r_s ds}|\mathcal{F}_t)\mathbb{E}^{\mathbb{Q}}(P_{CAT}^{(1)}(T)|\mathcal{F}_t)$$

We use the result of zero-coupon bond price with the CIR interest rate model (see Brigo and Mercurio (2007)) which discussed in Section 2.3, and have $\mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r_s ds}) = B_{CIR}(t,T)$. Addition with Eq (2.1), the above equation can be written as

$$B_{CIR}(t,T)\mathbb{E}^{\mathbb{P}}(P_{CAT}^{(1)}(T)|\mathcal{F}_t).$$

Simply apply the payoffs function Eq (2.19) and rearrange the formula, the CAT bond price can be formulated by

$$\begin{aligned}
V_\ell^{(1)}(t) =& B_{CIR}(t,T)\mathbb{E}^{\mathbb{P}}(Z\mathbb{1}\{L_\ell(T) \le D\} + pZ\mathbb{1}\{L_\ell(T) > D\}|\mathcal{F}_t)\\
=& B_{CIR}(t,T)(Z\mathbb{P}(L_\ell(T) \le D) + pZ\mathbb{P}(L_\ell(T) \ge D))\\
=& B_{CIR}(t,T)Z(F_\ell(T,D) + p(1 - F_\ell(T,D)))\\
=& B_{CIR}(t,T)Z(p + (1-p)F_\ell(T,D)),
\end{aligned}$$

where $\ell = 0,1,2,3,$ and $F_\ell(T,x) = \mathbb{P}(L_\ell(T) \le x)$ denotes the probability function of aggregate loss process less or equal to x at time T. $\square$

Similarly, in the next theorem, we have the prices of zero-coupon CAT bond at time t paying principal Z at time to maturity T depend on the amount of the aggregate claims.

Theorem 2.2. *Let $V_\ell^{(2)}(t)$ ($\ell = 0,1,2,3$) be the prices of T-maturity zero-coupon CAT bond under the risk-neutral measure $\mathbb{Q}$ at time t with payoffs function $P_{CAT}^{(2)}$ defined in Eq* (2.19). *Then,* v33

$$V_\ell^{(2)}(t) = B_{CIR}(t,T)Z\sum_{k=1}^{r} p_k(F_\ell(T-t,D_k) - F_\ell(T-t,D_{k-1})),\ \ell = 0,1,2,3,$$

where $F_\ell(T-t,x)$ represents the cumulated function of aggregate loss in alternative models given in Proposition 2.1-2.2, Remark 2.1 and Corollary 2.1, respectively, and pure discounted bond price $B_{CIR}(t,T)$ with CIR interest rate model is given by Eq (2.3)-(2.6).

Proof. Similar to the proof in Proposition 2.1, one can easily obtain

$$V_\ell^{(2)}(t) = B_{CIR}(t,T)\mathbb{E}^{\mathbb{P}}(P_{CAT}^{(2)}(T)|\mathcal{F}_t).$$

Let payoffs function follows Eq (2.19), then

$$\begin{aligned} V_\ell^{(2)}(t) =& B_{CIR}(t,T)\mathbb{E}^{\mathbb{P}}(\sum_{k=1}^{r} Zp_k\mathbb{1}\{D_{k-1} < L_\ell(T) \leq D_k\}|\mathcal{F}_t) \\ =& B_{CIR}(t,T)(Z\sum_{k=1}^{r} p_k\mathbb{P}(D_{k-1} < L_\ell(T) \leq D_k)) \\ =& B_{CIR}(t,T)Z\sum_{k=1}^{r} p_k(F_\ell(T,D_k) - F_\ell(T,D_{k-1})) \end{aligned}$$

where $\ell = 0,1,2,3$ and $F_\ell(T,x) = \mathbb{P}(L_\ell(T) \leq x)$ denotes the probability function of aggregate loss process less or equal to x at time T. □

In the next theorem, we have the prices of coupon CAT bond at time t paying principal Z and a coupon C at time to maturity T depend on the amount of the aggregate claims.

Theorem 2.3. *Let $V_\ell^{(3)}(t)$ ($\ell = 0,1,2,3$) be the prices of T-maturity coupon CAT bond under the risk-neutral measure $\mathbb{Q}$ at time t with payoffs function $P_{CAT}^{(3)}$ defined in Eq* (2.20). *Then,* v42

$$V_\ell^{(3)}(t) = B_{CIR}(t,T)(Z + CF_\ell(T,D)),\ \ell = 0,1,2,3,$$

where $F_\ell(T-t,x)$ represents the cumulated function of aggregate loss in alternative models given in Proposition 2.1-2.2, Remark 2.1 and Corollary 2.1, respectively, and pure discounted bond price $B_{CIR}(t,T)$ with CIR interest rate model is given by Eq (2.3)-(2.6).

Proof. Similar to the proof in Proposition 2.1, one can easily obtain

$$V_\ell^{(3)}(t) = B_{CIR}(t,T)\mathbb{E}^{\mathbb{P}}(P_{CAT}^{(3)}(T)|\mathcal{F}_t).$$

Let payoffs function follows Eq (2.20), then

$$\begin{aligned} V_\ell^{(3)}(t) =& B_{CIR}(t,T)\mathbb{E}^{\mathbb{P}}((Z+C)\mathbb{1}\{L_\ell(T) \leq D\} + Z\mathbb{1}\{L_\ell(T) > D\}|\mathcal{F}_t) \\ =& B_{CIR}(t,T)((Z+C)\mathbb{P}(L_\ell(T) \leq D) + Z\mathbb{P}(L_\ell(T) \geq D)) \\ =& B_{CIR}(t,T)((Z+C)F_\ell(T,D) + Z(1 - F_\ell(T,D))) \end{aligned}$$

where $\ell = 0,1,2,3$ and the result follows. □

In the next theorem, we have the prices of zero-coupon CAT bond at time t paying principal Z at time to maturity T depend on the amount of the aggregate claims which also associate with the probability of issuing company default at time T.

Theorem 2.4. *Let $V_\ell^{(4)}(t)$ ($\ell = 0, 1, 2, 3$) be the prices of T-maturity zero-coupon CAT bond under the risk-neutral measure $\mathbb{Q}$ at time t with payoffs function $P_{CAT}^{(3)}$ defined in Eq* (2.21). *Then,* v34

$$V_\ell^{(4)}(t) = B_{CIR}(t,T)Z[p + (1 - p - \tilde{F}(Z) - p\tilde{F}(pZ))F_\ell(T - t, D)) + p\tilde{F}(pZ)],$$

where $\ell = 0, 1, 2, 3$ and $F_\ell(T - t, D)$ represents the cumulated function of aggregate loss in alternative models given in Proposition 2.1-2.2, Remark 2.1 and Corollary 2.1, respectively, and pure discounted bond price $B_{CIR}(t, T)$ with CIR interest rate model is given by Eq (2.3)-(2.6). *$\tilde{F}(x)$ denotes the issuing company default probability at time T, and* v16

$$\tilde{F}(x) = \mathbb{P}(\frac{A_{issue} - B_{issue}}{N_{issue}} \leq x).$$

Proof. Similar to the proof in Proposition 2.1, we have

$$V_\ell^{(4)}(t) = B_{CIR}(t,T)\mathbb{E}^{\mathbb{P}}(P_{CAT}^{(3)}(T)|\mathcal{F}_t).$$

Let payoffs function follows Eq (2.21) and denote $M = \frac{A_{issue} - B_{issue}}{N_{issue}}$. According to the assumption that default risk and catastrophe risk are independent, that is $L_\ell(T)$ and M are independent under the measure $\mathbb{P}$, the following equalities hold

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}(P_{CAT}^{(4)}(T)|\mathcal{F}_t) =& \mathbb{E}^{\mathbb{P}}[Z\mathbb{1}\{L_\ell(T) \leq D, A_{issue} > B_{issue} + ZN_{issue}\} \\
& + pZ\mathbb{1}\{L_\ell(T) > D, A_{issue} > B_{issue} + pZN_{issue} + 0\} \\
=& Z\mathbb{P}(L_\ell(T) \leq D, M > Z) + pZ\mathbb{P}(L_\ell(T) > D, M > pZ) \\
=& Z\mathbb{P}(L_\ell(T) \leq D)\mathbb{P}(M > Z) + pZ\mathbb{P}(L_\ell(T) > D)\mathbb{P}(M > pZ),
\end{aligned}$$

where $\ell = 0, 1, 2, 3$. And the result follows. □

3 Numerical analysis

In this section, we compute the value of the zero-coupon CAT bonds modeled in Section 2, with face value $Z = \$1$ at time $t = 0$. In order to apply pricing formulas to the real world and obtain the CAT bonds prices, we need to compute the exact of the distribution of aggregate loss $F_\ell(T, D)$ ($\ell = 0, 1, 2, 3$). However, this is extremely difficult to be calculate because the closed-form solutions of those high-order convolutions are not available. Therefore, we employ Monte Carlo simulations to and conduct the approximated CAT bonds prices via numerical computation.

We analyze the the CAT bond price when the spot interest rate process follows CIR model, see Section 2.3. In this experiment, we employ the 3 month maturity US monthly Treasure bill data (1994-2013) to estimate the parameters of the CIR model. Due to the fact that we do not have closed-form formulas of the maximum likelihood estimate for the parameters in CIR model, therefore we find the MLE by numerical optimization of the log-likelihood function[6]. By implicating the MLE method, we conclude that both the initial short term interest r_0 and the long-term mean interest rate θ are 2.04% annually, the mean-reverting force $k = 0.0984$ and the volatility parameter $\sigma = 4.77\%$. Further assume the market price of risk λ_r be a constant -0.01.

In the actuarial literature, we called an event catastrophic if it occurs with a low probability and with severity damage. We conducted empirical studies for the data provided by the Insurance Service Office's (ISO's) Property Claim Services (PCS) unit, which describes insured property loss in United States caused by catastrophic events over a predetermined threshold occurring between 1985 and 2013. We adjust the inflation for a set of 870 original loss data by using the Consumer Price Index (CPI). Figure 2 illustrate annual adjusted PCS loss and annual total number of qualified catastrophes between 1980 and 2014. The twenty most costly insured CAT losses are listed in Tabel 1. An illustration of individual CAT loss can be refer back to Figure 1 and we can see peaks in the figure represent most costly events. Then, we can easily conclude that the PCS loss data is heavy-tailed.

In this paper, we omit the detailed process of parameters estimation and non-parametric tests [7]. We fit the distribution of PCS loss by General Extreme Value (GEV) distribution with parameters: shape parameter equals to 0.9273133, location parameter equals to 10.2718058 and scale parameter equals to 10.6295782; and the next best fit lognormal distribution with parameters $\mu_2 = 2.858557$ and $\sigma_2 = 1.26377$. In Model I, by applying the Nonlinear Least Squares procedure, we conclude that the quantity of loss process can be modeled by a inhomogeneous Possion process with intensity $\lambda(s) = 31.067647 - 1.122352\sin^2(s-0.473033)+1.167737\exp\{\cos(\frac{2\pi s}{7.704062})\}$. This allows us to model catastrophic data in changing economic or nature environment. According to detrended fluctuation analysis Peng et al. (1994), we can estimate the variance coefficient of the diffusion process $\sigma = 0.7038$. In Figure 3 we see a real catastrophe loss trajectory (in green) and sample trajectories of the aggregate claims process generated under the assumptions of GEV distribution (red) and lognormal distribution (blue) with non-homogeneous Possion intensity, respectively, between 1985 and 2013. This may suggest that GEV distribution would

[6] For detailed information, check Brigo and Mercurio (2007) and Dagıstan (2010).

[7] The choice of the distribution is very important because it varies the bond price. Readers can refer to Ma and Ma (2013) for using MLE to estimate parameters and choosing the best fit model by non-parametric tests.

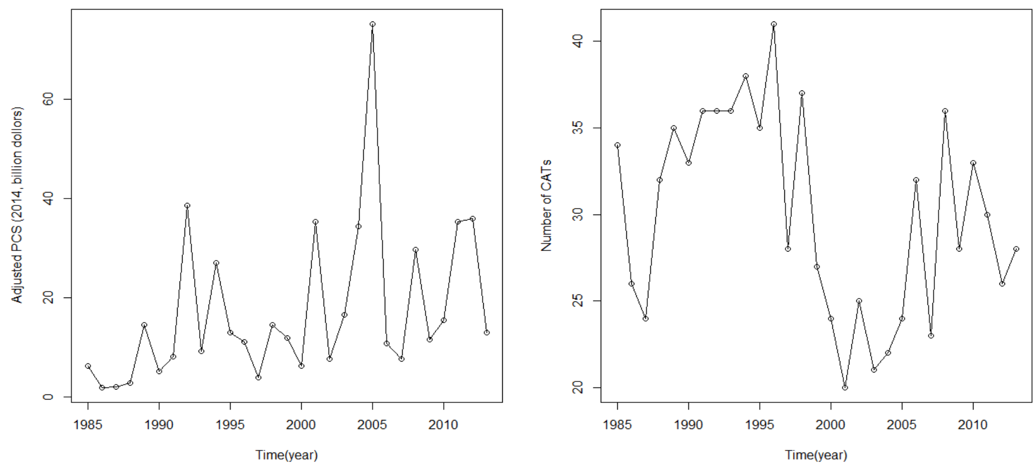


Figure 2: PCS annual catastrophe loss (left) and number of catastrophes (right) in US 1985-2013.

Table 1: The twenty most costly insured CAT losses in US, 1985-2014.

Event	Date	PCS loss (US$ bn)	2014 dollars (US$ bn)
Hurricane Katrina	25/08/2005	41.10	49.56
Hurricane Andrew	24/08/1992	15.50	26.02
Terrorist attacks	11/09/2001	18.78	24.97
Northridge earthquake	17/01/1994	12.50	19.86
Hurricane Sandy	28/10/2012	18.75	19.23
Hurricane Ike	12/09/2008	12.50	13.67
Hurricane Wilma	24/10/2005	10.30	12.42
Hurricane Charley	13/08/2004	7.47	9.32
Hurricane Ivan	15/09/2004	7.11	8.86
Hurricane Hugo	17/09/1989	4.20	7.97
Wind and Thunderstorm Event	22/04/2011	7.30	7.64
Wind and Thunderstorm Event	20/05/2011	6.90	7.22
Hurricane Rita	20/09/2005	5.63	6.79
Hurricane Frances	03/09/2004	4.59	5.73
Hurricane Jeanne	15/09/2004	3.65	4.56
Hurricane Irene	26/08/2011	4.30	4.50
Hurricane Georges	21/09/1998	2.96	4.27
Wind and Thunderstorm Event	02/05/2003	3.21	4.10
Tropical Storm Allison	05/06/2001	2.50	3.32
Hurricane Opal	04/10/1995	2.10	3.25

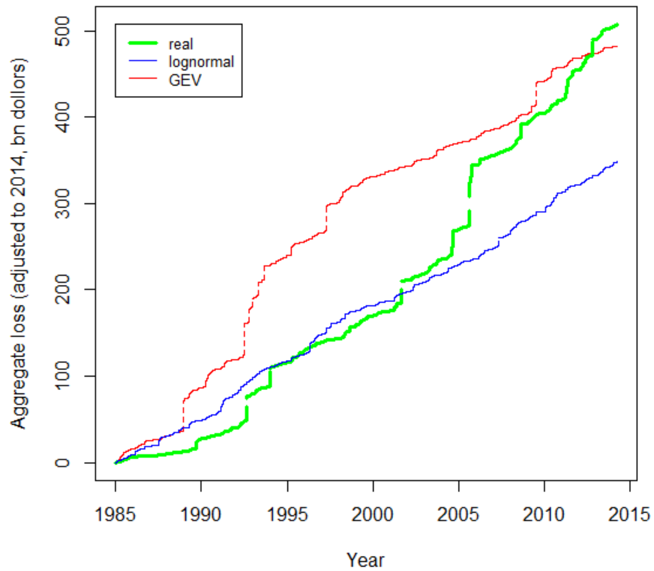


Figure 3: Sample trajectories of the aggregate loss process in US 1985-2013.

be better to fit of the real-life aggregate loss process in long-term.

In order to analysis Model II[8], we assume that we work in a two states (m=2) environment: many claims period (state 1, according to Siegl and Tichy (1999), they have the storm season with claim frequency λ_1) and few claims period (state 2 with claim frequency λ_2). We define a period to be a storm season (or many claims period) with the following conditions:

1. more than one claim per month during the entire period;
2. the next claim after storm season occurs at least 10 days after the last claim of the storm season;
3. the first claim of storm season occurs at least 10 days after the pervious claim;
4. the gap between two storm seasons (i.e. non-storm season or few claims period) lasts at least 3 months;
5. less than one claims per month during the non-storm seasons.

By analysing the occurring dates of the PCS loss data, we conclude that there are 19 storm seasons and the parameters of Model II can be found in table 2.

[8]By the fact that SM'/SM model is a special case of the general Markov model, in this part of the application we consider SM'/SM model as an example.

Table 2: Parameters of Model II.

Parameters		State 1	State 2
GEV distribution	k	0.9417813	0.7368964
	σ	10.5249838	12.8791786
	μ	10.0954148	11.6702931
Lognormal distribution	μ	2.84877246	2.9874937
	σ	1.27307413	1.1261473
Intensity of HPP	λ	34.967753	10.520688
Transition probabilities	p_{1j} for $j \in 1, 2$	0.9767442	0.02325581
	p_{2j} for $j \in 1, 2$	0.3064516	0.6935484

In the experiment of each model, we generate $N = 100,000$ simulations to obtain the $T \in [0.25, 2.25]$ years maturity zero-coupon CAT bond prices by Monte-Carlo simulations. For the case of payoffs functions Eq (2.18) and Eq (2.21), we assume that $p = 0.5$ when the aggregate loss $L_\ell(T)$ exceed the threshold level $D \in [434, 5210]$ \$10 million, that is the threshold level in the interval of quarterly to three times of annual average loss. Additionally, for payoffs function Eq (2.19), we arbitrarily set the following parameters: $r = 3$, $p_1 = 1$, $p_2 = 0.5$, $p_3 = 0.25$, $D_1 = 434$ \$ten million, $D_2 \in [434, 5210]$ \$10 million and $D_3 = \infty$. For a real world CAT bond, issuing company might adapt a multi-threshold payoffs structure to reduce the risk of investment and attract more investors. Furthermore, let the probabilities of issuing company default at time T are $\tilde{F}(Z) = 0.1$ and $\tilde{F}(pZ) = 0.05$. And finally, assume the coupon payment rate in Eq (2.20) equals to \$ 0.1.

Figure 4 illustrate the CAT bonds prices for payoffs functions $P^{(1)}_{CAT}$ with respect to threshold level D and time to maturity T under stochastic interest rates assumptions. We present the CAT bond prices of which c.d.f. of classical aggregate loss process $F_0(t, D)$ given in Eq (2.10), loss distribution follows GEV distribution and intensity of the claims is an non-homogenous Poisson process in the sub Figure 4a. While in sub Figure 4b we have the CAT bond prices with c.d.f. of perturbed aggregate loss process $F_1(t, D)$ given in Eq (2.9), and in sub Figure 4c the CAT bond prices with c.d.f. of SM'/SM model $F_3(t, D)$ given in Eq (2.17). And finally in sub Figure 4d shows CAT bond prices with $F_1(t, D)$ and loss distribution follows lognormal distribution. Following similar setting for sub figures in Figure 4, Figure 5–Figure 7 illustrate the CAT bonds prices for payoffs functions $P^{(2)}_{CAT}$, $P^{(3)}_{CAT}$ and $P^{(4)}_{CAT}$, respectively.

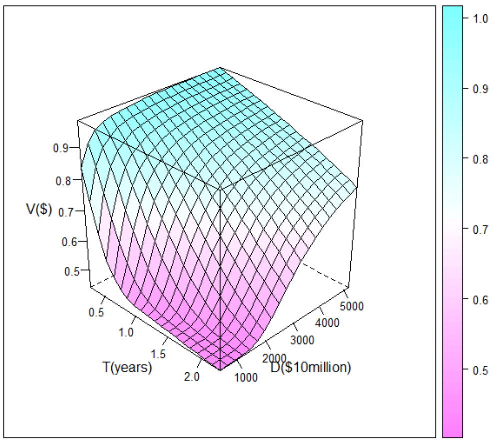
We can see from Figure 4–Figure 7 that there are not much difference between different aggregate loss model, since we are using the same data set. Additionally, as the maturity time increases, the CAT bond price decreases while the GEV case decreases with a faster rate than the lognormal case. This is a promising result because the trajectory of the aggregate loss process GEV distribution always larger than the lognormal distribution process and will be shown in Figure 10. Moreover, increasing threshold level leads to higher bond prices. We can observe that CAT bond prices decreases with additional threshold and default risk adding to the payoffs function, while coupon CAT bonds have higher prices compared to zero-coupon CAT bonds.

The price differences between the CAT bond prices of classical and perturbed aggregate loss process is given in Figure 8a, Figure 8c, Figure 9a and Figure 9c, respectively, under the GEV, the NHPP and stochastic interest rates assumptions. While The price differences between the CAT bond prices of classical aggregate loss process and SM'/SM model is given in Figure 8b, Figure 8d, Figure 9b and Figure 9d, respectively, under the GEV, the NHPP and stochastic interest rates assumptions. We observed that the uncertainty in the aggregate loss process make small impact to the CAT bond prices with price differences less than $4 * 10^{-4}$. The choice between the classical or SM'/SM model influences the CAT bond prices more than the perturbed model. We clearly observe that the differences of the bond price significantly changes as much as 5%. This further validate that aggregate loss process has impact on the bond prices. In Figure 10, we illustrate how the bond prices are affected by loss severity distribution (lognormal distribution and GEV distribution). The differences is especially marked in the tails (larger threshold level), therefore, heavy-tailed distribution is a more appropriate choice for modelling catastrophe loss; this is also demonstrated by Ma and Ma (2013).

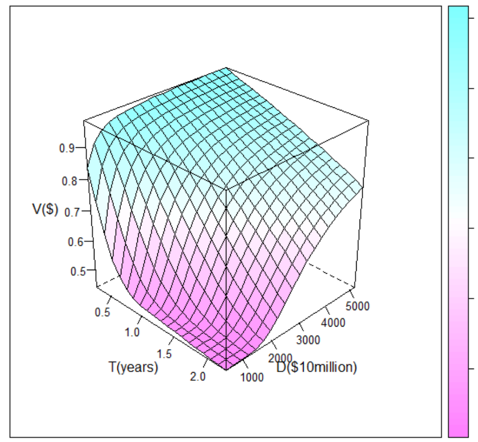
4 Conclusions

In this paper, we developed a contingent claim process to price catastrophe risk bonds applying models of the risk-free spot interest rate under the assumptions of no-arbitrage market, independent of the financial risks and catastrophe risks and the possibility of replicate interest rate changes by existing financial instruments. Under the risk-neutral pricing measure, we derive bond price formulas for the CIR interest rate model with four types of payoffs functions (the classic zero coupon, the multi-threshold zero coupon, the defaultable zero coupon, and the coupon payoffs functions) for two models of aggregate loss process (compound inhomogeneous Poisson perturbed by diffusion, and claim inter-arrival times dependent on the claim sizes with the particular case when inter-arrival time follows an exponential distribution).

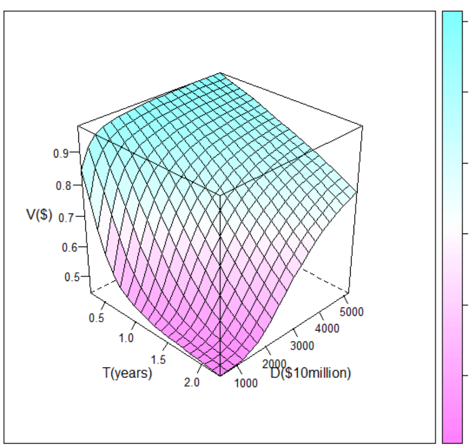
Then numerical experiments are taken by utilizing Monte Carlo simulation with the data from PCS loss index in US occurring from 1985 to 2014. From the numerical analyse we can see that the CAT bonds prices decreases as the threshold level decreases, the time to maturity increases and the existence of default probability. While CAT bonds prices increases with the introduce of coupons. It further shows that the choices of fitted loss severity distribution



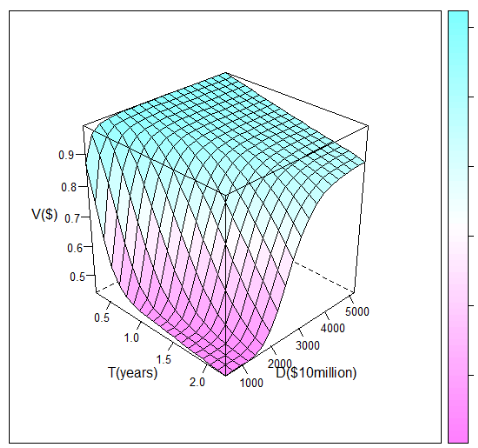
(a) $V_0^{(1)}(t)$ with GEV distribution.



(b) $V_1^{(1)}(t)$ with GEV distribution.

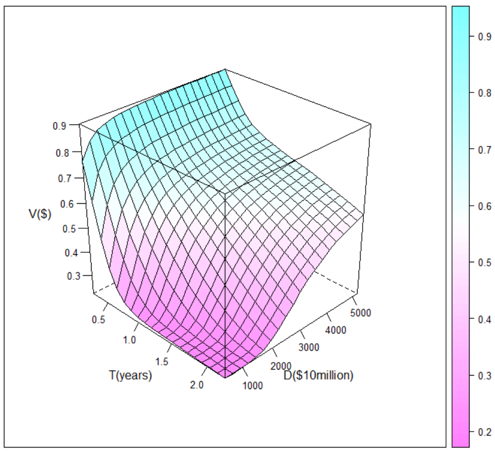


(c) $V_2^{(1)}(t)$ with GEV distribution.

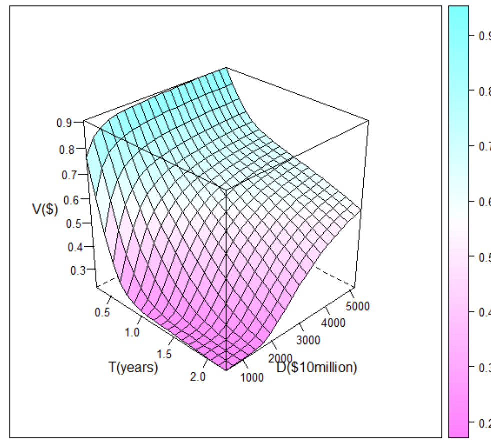


(d) $V_1^{(1)}(t)$ with lognormal distribution.

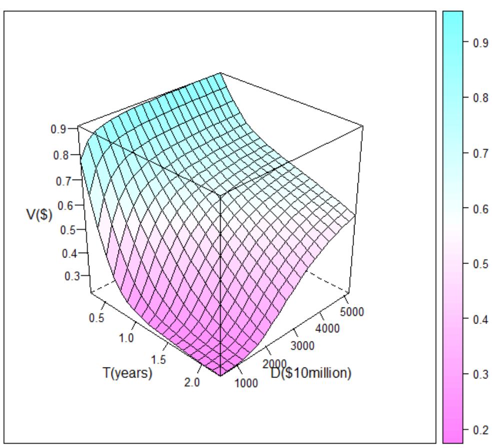
Figure 4: CAT bonds prices (z-coordinate axes) for payoffs functions $P_{CAT}^{(1)}$ under the GEV (or lognormal), the NHPP and stochastic interest rates assumptions. Here, time to the maturity (T) decreases by the left axes and threshold level (D) increases by the right axes.



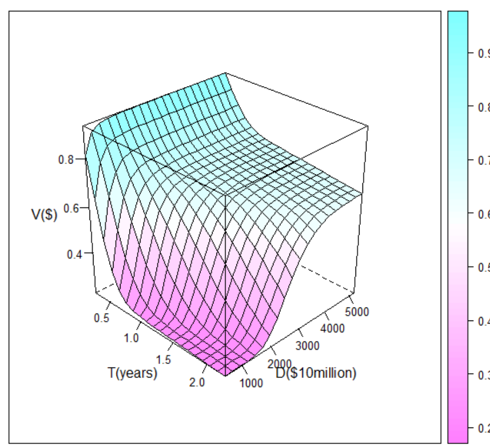
(a) $V_0^{(2)}(t)$ with GEV distribution.



(b) $V_1^{(2)}(t)$ with GEV distribution.

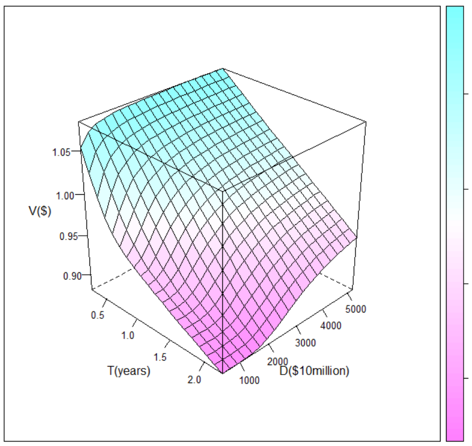


(c) $V_2^{(2)}(t)$ with GEV distribution.

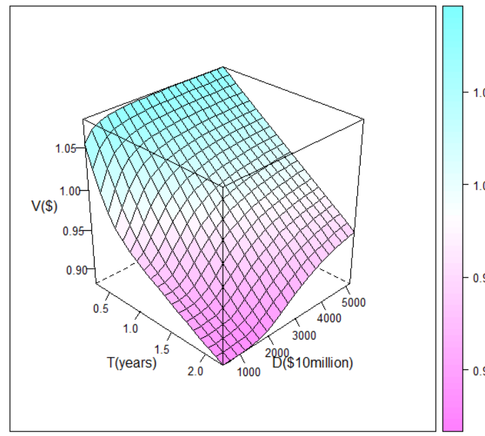


(d) $V_1^{(2)}(t)$ with lognormal distribution.

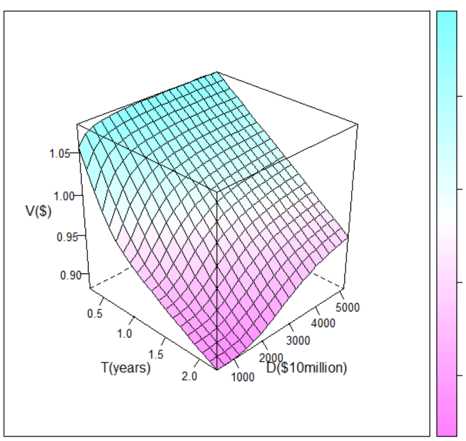
Figure 5: CAT bonds prices (z-coordinate axes) for payoffs functions $P_{CAT}^{(2)}$ under the GEV (or lognormal), the NHPP and stochastic interest rates assumptions. Here, time to the maturity (T) decreases by the left axes and threshold level (D) increases by the right axes.



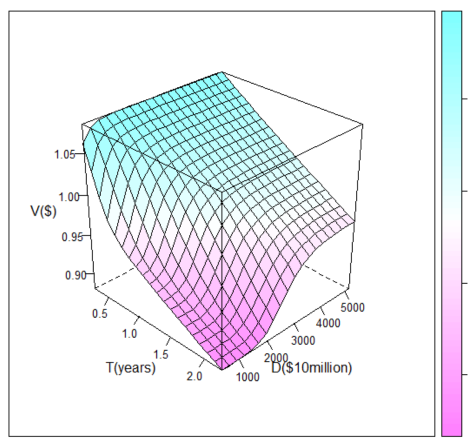
(a) $V_0^{(3)}(t)$ with GEV distribution.



(b) $V_1^{(3)}(t)$ with GEV distribution.

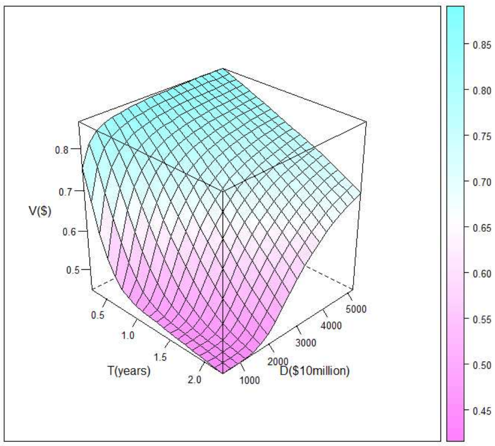


(c) $V_2^{(3)}(t)$ with GEV distribution.

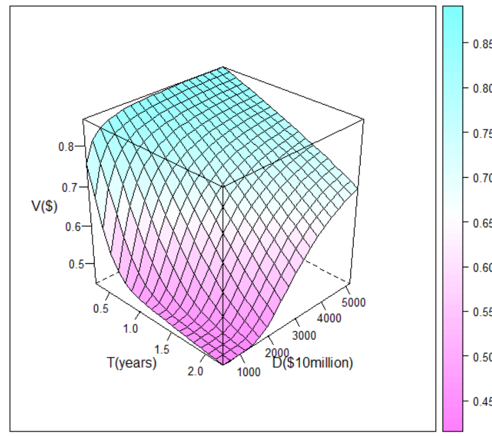


(d) $V_1^{(3)}(t)$ with lognormal distribution.

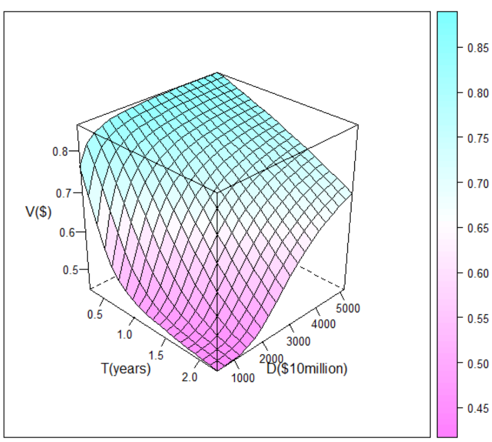
Figure 6: CAT bonds prices (z-coordinate axes) for payoffs functions $P_{CAT}^{(3)}$ under the GEV (or lognormal), the NHPP and stochastic interest rates assumptions. Here, time to the maturity (T) decreases by the left axes and threshold level (D) increases by the right axes.



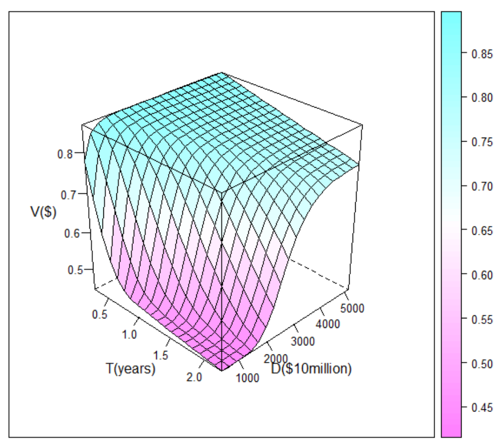
(a) $V_0^{(4)}(t)$ with GEV distribution.



(b) $V_1^{(4)}(t)$ with GEV distribution.

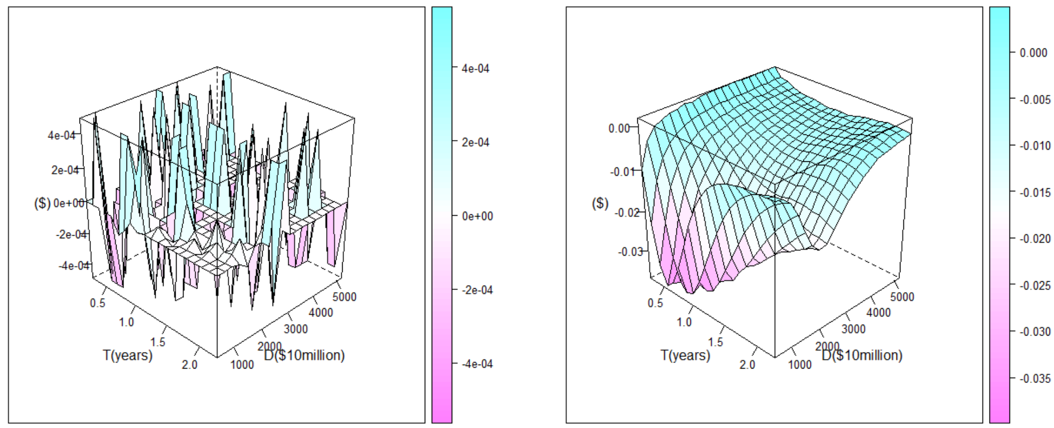


(c) $V_2^{(4)}(t)$ with GEV distribution.



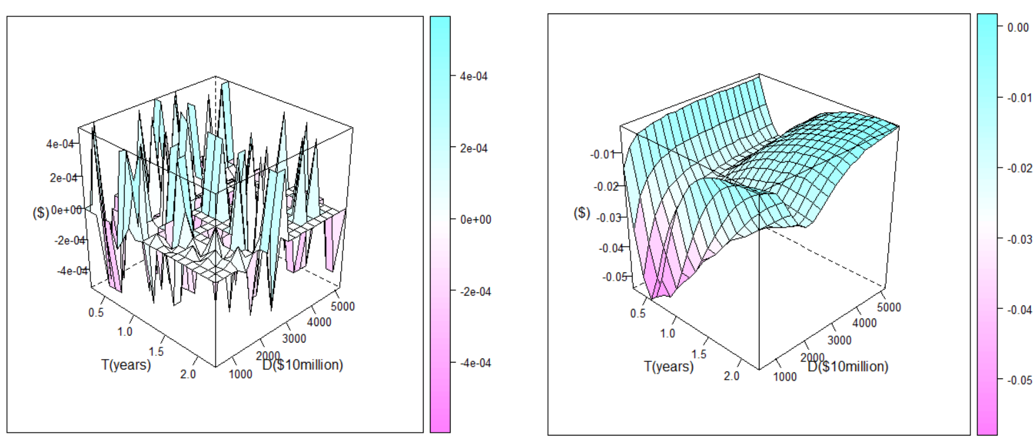
(d) $V_1^{(4)}(t)$ with lognormal distribution.

Figure 7: CAT bonds prices (z-coordinate axes) for payoffs functions $P_{CAT}^{(4)}$ under the GEV (or lognormal), the NHPP and stochastic interest rates assumptions. Here, time to the maturity (T) decreases by the left axes and threshold level (D) increases by the right axes.



(a) differences between $V_0^{(1)}$ and $V_1^{(1)}$

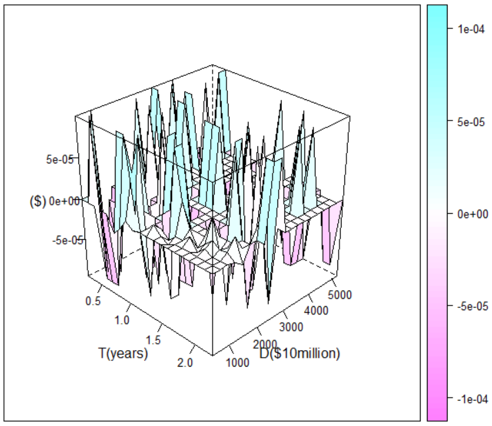
(b) differences between $V_0^{(1)}$ and $V_2^{(1)}$



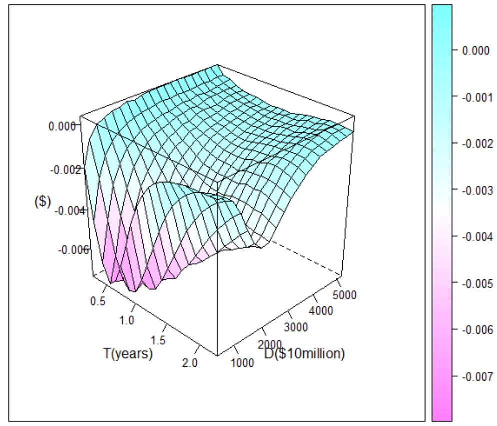
(c) differences between $V_0^{(2)}$ and $V_1^{(2)}$

(d) differences between $V_0^{(2)}$ and $V_2^{(2)}$

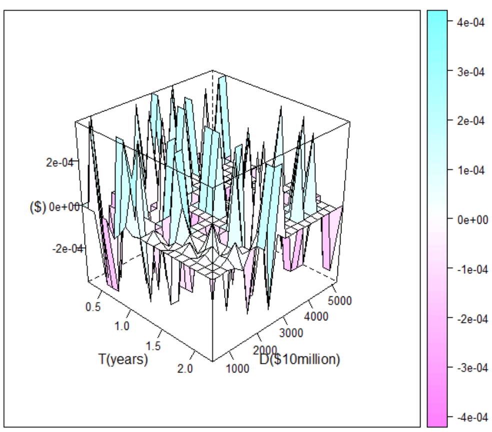
Figure 8: The price differences (z-coordinate axes) between the CAT bond prices under the GEV, the NHPP and stochastic interest rates assumptions. Here, time to the maturity (T) decreases by the left axes and threshold level (D) increases by the right axes.



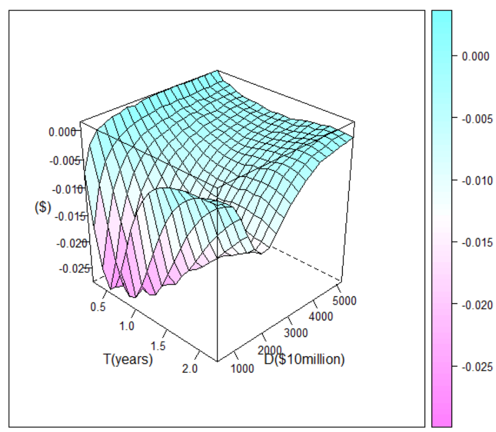
(a) differences between $V_0^{(3)}$ and $V_1^{(3)}$



(b) differences between $V_0^{(3)}$ and $V_2^{(3)}$

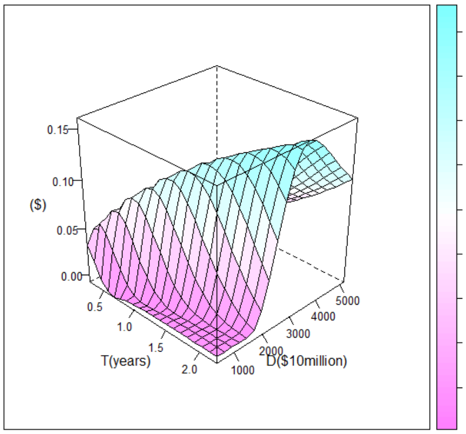


(c) differences between $V_0^{(4)}$ and $V_1^{(4)}$

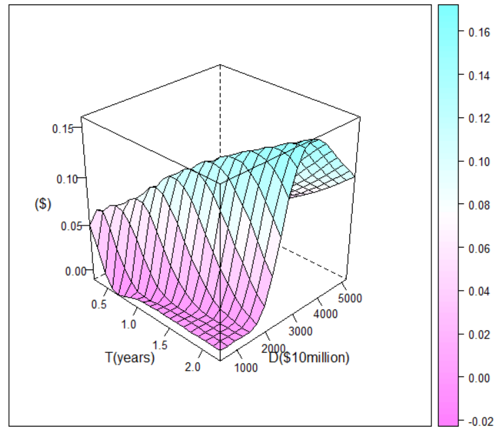


(d) differences between $V_0^{(4)}$ and $V_2^{(4)}$

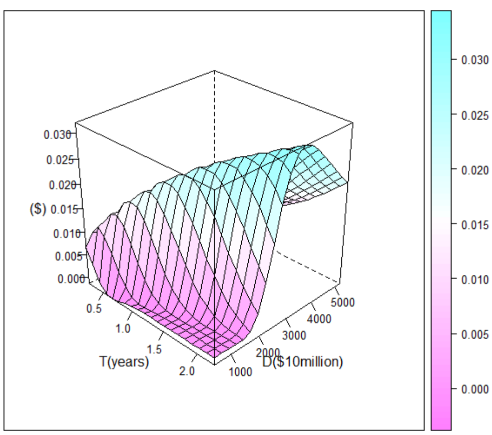
Figure 9: The price differences (z-coordinate axes) between the CAT bond prices under the GEV, the NHPP and stochastic interest rates assumptions. Here, time to the maturity (T) decreases by the left axes and threshold level (D) increases by the right axes.



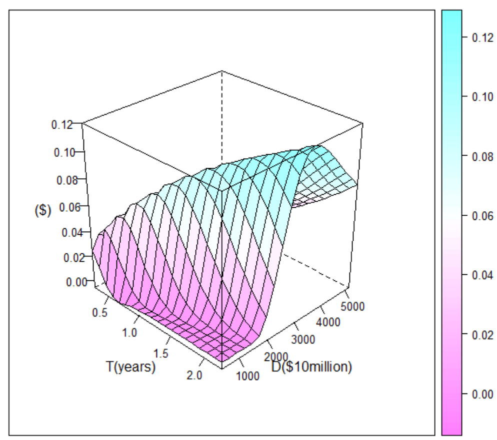
(a) differences between lognormal and GEV distribution of $V_1^{(1)}$



(b) differences between lognormal and GEV distribution of $V_1^{(2)}$



(c) differences between lognormal and GEV distribution of $V_1^{(3)}$



(d) differences between lognormal and GEV distribution of $V_1^{(4)}$

Figure 10: The price differences (z-coordinate axes) between lognormal distribution and GEV distribution for the CAT bond prices, where aggregate claims process is a perturbed (or SM'/SM) model. Here, time to the maturity (T) decreases by the left axes and threshold level (D) increases by the right axes.

have great impart on the bond prices. Furthermore, the choices of the aggregate losses process have great impact when pricing CAT bonds.

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