# Special Geometry, Black Holes and Euclidean Supersymmetry[*]

**Thomas Mohaupt**

Theoretical Physics Division
Department of Mathematical Sciences
University of Liverpool
Liverpool L69 3BX, UK
Thomas.Mohaupt@liv.ac.uk

## ABSTRACT

We review recent developments in special geometry and explain its role in the theory of supersymmetric black holes. To make this article self-contained, a short introduction to black holes is given, with emphasis on the laws of black hole mechanics and black hole entropy. We also summarize the existing results on the para-complex version of special geometry, which occurs in Euclidean supersymmetry. The role of real coordinates in special geometry is illustrated, and we briefly indicate how Euclidean supersymmetry can be used to study stationary black hole solutions via dimensional reduction over time.

This article is an updated and substantially extended version of the previous review article 'New developments in special geometry', hep-th/0602171.

---

# 1 Introduction

Special geometry was discovered more than 20 years ago [1]. While the term special geometry originally referred to the geometry of vector multiplet scalars in four-dimensional $N = 2$ supergravity, today it is used more generally for the geometries encoding the scalar couplings of vector and hypermultiplets in theories with 8 real supercharges. It applies to rigidly and locally supersymmetric theories in $\leq 6$ space-time dimensions, both in Lorentzian and in Euclidean signature. The scalar geometries occuring in these cases are indeed closely related. In particular, they are all much more restricted than the Kähler geometry of scalars in theories with 4 supercharges, while still depending on arbitrary functions. In contrast, the scalar geometries of theories with 16 or more supercharges are completely fixed by their matter content. Theories with 8 supercharges have a rich dynamics, which is still constrained enough to allow one to answer many questions exactly. Special geometry lies at the heart of the Seiberg-Witten solution of $N = 2$ gauge theories [2] and of the non-perturbative dualities between $N = 2$ string compactifications [3, 4].

While the subject has now been studied for more than twenty years, there are still new aspects to be discovered. One, which will be the topic of this article, is the role of real coordinates. Many special geometries, in particular the special Kähler manifolds of four-dimensional vector multiplets and the hyper-Kähler geometries of rigid hypermultiplets are complex geometries. Nevertheless, they also possess distinguished real parametrizations, which are natural to use for certain physical problems. Our first example illustrates this in the context of special geometries in theories with Euclidean supersymmetry. This part reviews the results of [5, 6], and gives us the opportunity to explore another less studied aspect of special geometry, namely the scalar geometries of $N = 2$ supersymmetric theories in Euclidean space-time. It turns out that the relation between the scalar geometries of theories with Lorentzian and Euclidean space-time geometry is (roughly) given by replacing complex structures by para-complex structures. One technique for deriving the scalar geometry of a Euclidean theory in $D$ dimensions is to start with a Lorentzian theory in $D + 1$ dimensions and to perform a dimensional reduction along the time-like direction. The specific example we will review is to start with vector multiplets in four Lorentzian dimensions, which gives, by reduction over time, hypermultiplets in three Euclidean dimensions. This provides us with a Euclidean version of the so-called c-map. The original c-map [7, 8] maps any scalar manifold of four-dimensional vector multiplet scalars to a scalar manifold of hypermultiplets. For rigid supersymmetry,

this relates affine special Kähler manifolds to hyper-Kähler manifolds, while for local supersymmetry this relates projective special Kähler manifolds to quaternion-Kähler manifolds. By using dimensional reduction with respect to time rather than space, we will derive the scalar geometry of Euclidean hypermultiplets. As we will see, the underlying geometry is particularly transparent when using real scalar fields rather than complex ones. The geometries of Euclidean supermultiplets are relevant for the study of instantons, and, by 'dimensional oxidation over time' also for solitons, as outlined in [5]. In this article we will restrict ourselves to the geometrical aspects.

Our second example is taken from a different context, namely BPS black hole solutions of matter-coupled $N = 2$ supergravity. The laws of black hole mechanics suggest to assign an entropy to black holes, which is, at least to leading order, proportional to the area of the event horizon. Since (super-)gravity presumably is the low-energy effective theory of an underlying quantum theory of gravity, the black hole entropy is analogous to the macroscopic or thermodynamic entropy in thermodynamics. A quantum theory of gravity should provide the fundamental or microscopic level of description of a black hole and, in particular, should allow one to identify the microstates of a black hole and to compute the corresponding microscopic or statistical entropy. The microscopic entropy is the information missing if one only knows the macrostate but not the microstate of the black hole. In other words, if a black hole with given mass, charge(s) and angular momentum (which characterise the macrostate) can be in $d$ different microstates, then the microscopic entropy is $S_{\mathrm{micro}} = \log d$. If the area of the event horizon really is the corresponding macroscopic entropy, then these two quantities must be equal, at least to leading order in the semi-classical limit. In string theory it has been shown that the two entropies are indeed equal in this limit [9], at least for BPS states (also called supersymmetric states). These are states which sit in special representations of the supersymmetry algebra, where part of the generators act trivially. These BPS (also called short) representations saturate the lower bound set for the mass by the supersymmetry algebra, and, as a consequence, the mass is exactly equal to a central charge of the algebra.[1] In this article we will be interested in the macroscopic part of the story, which is the construction of BPS black hole solutions and the computation of their entropy. The near horizon limit of such solutions, which is all one needs to know in order to compute the entropy, is determined by the so-called black hole attractor equations [11], whose derivation is based on the special geometry of vector multiplets.

---

[1]See [10] Chapter 2.

The attractor equations are another example where real coordinates on the scalar manifold appear in a natural way. In the second part of the article we review how the attractor equations and the entropy can be obtained from a variational principle. When expressed in terms of real coordinates, the variational principle states that the black hole entropy is the Legendre transform of the Hesse potential of the scalar manifold. We also discuss how the black hole free energy introduced by Ooguri, Strominger and Vafa [12] fits into the picture, and indicate how higher curvature and non-holomorphic corrections to the effective action can be incorporated naturally. This part of the article is based on [13] and on older work including [14, 15, 16].

Let us now explain how our two subjects are connected to the second topic of this volume, pseudo-Riemannian geometry. In both parts of the article we have two relevant geometries, the geometry of space-time and the geometry of the target manifold of the scalar fields. In the first case, space-time is Euclidean, but, as we will see, the scalar manifold is pseudo-Riemannian with split signature. In the second case the scalar geometry is positive definite, but space-time is pseudo-Riemannian with Lorentz signature. In fact, our two subjects, the c-map and black holes, can be related in a rather direct fashion, as follows: for a static black hole one can perform a dimensional reduction along the time-like direction in complete analogy to the dimensional reduction of flat Minkowski space. Then one can dualize the vector multiplets into hypermultiplets, which gives rise to a 'local' version of the c-map.[2] This construction can be used to study time-independent four-dimensional geometries from a three dimensional perspective, which has the advantage that all bosonic degrees (metric, gauge fields and scalars) become scalars in the reduced theory and can then be combined into a non-linear sigma model. This method has been used in Einstein-Maxwell theory already a long time [18], and has been elaborated on both for black holes [19] and brane-type solutions [20]. We refer to [21] for a review. More recently, dimensional reduction to three Euclidean dimensions and the corresponding version of the c-map have been used by [22] to elaborate on the ideas of Ooguri, Strominger and Vafa [12] by quantizing static, spherically BPS black hole solutions. The three-dimensional solutions obtained by dimensional reduction for four-dimensional static black hole solutions can also be lifted to four Euclidean dimensions, where they desribe a wormhole solutions, which generalize the D-instanton of type-IIB string theory [17].

Let us finally mention two contributions to this volume which are closely

---

[2]Here local means that supersymmetry is realized as a local, i.e., space-time dependent symmetry.

related to our topics. The article [23] (which is based on [24]) disucsses new insights into the geometry of the c-map, which have been obtained by relating vector and hypermultiplets to tensor multiplets. The contribution of [25] discusses new developments in para-quaternionic geometry. While we only discuss the 'rigid' version of the Euclidean c-map in this article, its 'local' (supergravity) version maps projective special Kähler manifolds to para-Quaternionic manifolds.

## 2   Euclidean special geometry

### 2.1   Vector multiplets

We start by reviewing the geometry of vector multiplets in rigid four-dimensional $N = 2$ supersymmetry.[3] A vector multiplet consists of a gauge field $A_m$, ($m = 0, \ldots, 3$ is the Lorentz index), two Majorana spinors $\lambda^i$ ($i = 1, 2$) and one complex scalar $X$. We consider $n$ such multiplets, labeled by an index $I = 1, \ldots, n$. The field equations for the gauge fields are invariant under $Sp(2n, \mathbb{R})$ rotations which act linearly on the field strength $F_{mn}^I$ and the dual field strength $G_{I|mn} = \frac{\delta L}{\delta F_{mn}^I}$, where $L$ denotes the Lagrangian. These symplectic rotations generalize the electric-magnetic duality rotations of Maxwell theory and are in fact invariances of the full field equations. A thorough analysis shows that this has the important consequence that all vector multiplet couplings are encoded in a single holomorphic function of the scalars, $F(X^I)$, which is called the prepotential [1]. In superspace language the general action for vector multiplets can be written as a chiral superspace integral of the prepotential $F$, considered as a superspace function of $n$ so-called restricted chiral multiplets $(X^I, \lambda^{I+}, F_{mn}^{I-})$, which encode the gauge invariant quantities of the $n$ vector multiplets. Here $\lambda^{I+}$ are the positive chirality projections of the spinors and $F_{mn}^{I-}$ are the antiselfdual projections of the field strength. To be precise, the Lagrangian is the sum of a chiral and an antichiral superspace integral, the latter depending on the complex conjugated multiplets $(\overline{X}^I, \lambda^{I-}, F_{mn}^{I+})$. When working out the Lagrangian in components, all couplings can be expressed in terms of $F(X^I)$, its derivatives, which we denote $F_I, F_{IJ}, \ldots$ and their complex conjugates $\overline{F}_I, \overline{F}_{IJ}, \ldots$. For later use we specify the bosonic part of the Lagrangian:

$$L_{\text{bos}}^{\text{4d VM}} = -\tfrac{1}{2} N_{IJ} \partial_m X^I \partial^m \overline{X}^J - \tfrac{i}{2} \left( F_{IJ} F_{mn}^{I-} F^{J-mn} - \text{c.c.} \right) , \qquad (2.1)$$

---

[3]Some more background material and references on vector multiplets can be found in [26].

4

where

$$N_{IJ} = \partial_I \partial_{\overline{J}} \left( -i(X^I \overline{F}_I - F_I \overline{X}^I) \right) \qquad (2.2)$$

can be interpreted as a Riemannian metric on the target space $M_{VM}$ of the scalars $X^I$.[4] $N = 1$ supersymmetry requires this metric to be a Kähler metric, which is obviously the case, the Kähler potential being $K = -i(X^I \overline{F}_I - F_I \overline{X}^I)$. As a consequence of $N = 2$ supersymmetry the metric is not a generic Kähler metric, since the Kähler potential can be expressed in terms of the holomorphic prepotential $F(X^I)$. The resulting geometry is known as affine (also: rigid) special Kähler geometry. The intrinsic characterization of this geometry is the existence of a flat, torsionfree, symplectic connection $\nabla$, called the special connection, such that

$$(\nabla_U I)V = (\nabla_V I)U , \qquad (2.3)$$

where $I$ is the complex structure and $U, V$ are arbitrary vector fields [27]. It has been shown that all such manifolds can be constructed locally as holomorphic Langrangian immersions into the complex symplectic vector space $T^*\mathbb{C}^n \simeq \mathbb{C}^{2n}$ [29]. In this context $X^I, F_I$ are flat complex symplectic coordinates on $T^*\mathbb{C}^n$ and the prepotential is the generating function of the immersion $\Phi : M_{VM} \rightarrow T^*\mathbb{C}^n$, i.e., $\Phi = dF$. For generic choice of $\Phi$, the $X^I$ provide coordinates on the immersed $M_{VM}$, while $F_I = \partial_I F = F_I(X)$ along $M_{VM}$. The $X^I$ are non-generic coordinates, physically, because they are the lowest components of vector multiplets, mathematically, because they are adapted to the immersion. They are called special coordinates.

So far we have considered vector multiplets in a four-dimensional Minkowski space-time. In four-dimensional Euclidean space the theory has the same form, except that the complex structure $I$, $I^2 = -\mathbb{1}$ is replaced by a para-complex structure $J$. This is an endomorphism of $TM_{VM}$ such that $J^2 = \mathbb{1}$, with the eigendistributions corresponding to the eigenvalues $\pm 1$ having equal rank. Many notions of complex geometry, including Kähler and special Kähler geometry can be adapted to the para-complex realm. We refer to [5, 6] for the details. In particular, it can be shown that the target space geometry of rigid Euclidean vector multiplets is affine special para-Kähler. Such manifolds are the para-complex analogues of affine special Kähler manifolds. When using an appropriate notation, the expressions for the Lagrangian, the equations of motion and the supersymmetry transformation rules take the same form as for Lorentzian supersymmetry, except

---

[4] The scalar fields $X^I$ might only provide local coordinates. We will work in a single coordinate patch throughout.

that complex quantities have to be re-interpreted as para-complex ones. For example, the analogue of complex coordinates $X^I = x^I + iu^I$, where $x^I, u^I$ are real and $i$ is the imaginary unit, are para-complex coordinates $X^I = x^I + eu^I$, where $e$ is the para-complex unit characterized by $e^2 = 1$ and $\overline{e} = -e$, where the 'bar' denotes para-complex conjugation.[5] While in Lorentzian signature the selfdual and antiselfdual projections of the field strength are related by complex conjugation, in the Euclidean theory one can re-define the selfdual and antiselfdual projections by appropriate factors of $e$ such that they are related by para-complex conjugation. One can also define para-complex spinor fields such that the fermionic terms of the Euclidean theory take the same form as in the Lorentzian one. The Euclidean bosonic Lagrangian takes the same form (2.1) as the Lorentzian one, with (2.2) replaced by

$$N_{IJ} = \partial_I \partial_{\overline{J}} \left( -e(X^I \overline{F}_I - F_I \overline{X}^I) \right) . \tag{2.4}$$

Note that the Euclidean Lagrangian is real-valued, although the fields $X^I$ and $F^{I-}_{mn}$ are para-complex. We also remark that a para-Kähler metric always has split signature. The full Lagrangian, including fermionic terms, and the supersymmetry transformation rules can be found in [5]. There we also verified that it is related to the rigid limit of the general Lorentzian signature vector multiplet Lagrangian [31, 32] by replacing $i \to e$ (together with additional field redefinitions, which account for different normalizations and conventions).

## 2.2 Hypermultiplets

Our next step is to derive the geometry of Euclidean hypermultiplets. This can be done by either reducing the Lorentzian vector multiplet Lagrangian with respect to time or the Euclidean vector multiplet Lagrangian with respect to space [6]. Here we start from the Lorentzian Lagrangian and perform the reduction over space and over time in parallel. This is instructive, because the reduction over space corresponds to the standard $c$-map and gives us hypermultiplets in three-dimensional Minkowski space-time, while the reduction over time is the new para-$c$-map and gives us hypermultiplets in three-dimensional Euclidean space.

Before performing the reduction, we rewrite the Lorentzian vector multiplet Lagrangian in terms of real fields. Above we noted that the intrinsic

---

[5]It has been known for quite a while that the Euclidean version of a supersymmetric theory can sometimes be obtained by replacing $i \to e$ [30].

characteristic of an affine special Kähler manifold is the existence of the special connection $\nabla$, which is, in particular, flat, torsionfree and symplectic [27]. The corresponding flat symplectic coordinates are

$$x^I = \mathrm{Re}X^I \quad y_I = \mathrm{Re}F_I .\tag{2.5}$$

Note that since $F$ is an arbitrary holomorphic function, these real coordinates are related in a complicated way to the special coordinates $X^I$. The real coordinates $x^I, y_I$ are flat (or affine) coordinates with respect to $\nabla$, i.e., $\nabla dx^I = 0 = \nabla dy_I$, and they are symplectic (or Darboux coordinates), because the symplectic form on $M_{VM}$ is $\omega = 2dx^I \wedge dy_I$. While in special coordinates the metric of $M_{VM}$ can be expressed in terms of the prepotential by (2.2), the metric has a Hesse potential when using the real coordinates $q^a = (x^I, y_I)$, where $a = 1, \ldots, 2n$ [27, 28]:

$$g_{ab} = \frac{\partial^2 H}{\partial q^a \partial q^b} .\tag{2.6}$$

The Hesse potential is related to the imaginary part of the prepotential by a Legendre transform [44]:

$$H(x, y) = 2\mathrm{Im}F(x + iu) - 2u^I y_I .\tag{2.7}$$

The two parametrizations of the metric on $M_{VM}$ are related by

$$ds^2 = -\tfrac{1}{2}N_{IJ}dX^I d\overline{X}^J = -g_{ab}dq^a dq^b .\tag{2.8}$$

In order to rewrite the Lagrangian (2.1) completely in terms of real fields, we express the (anti)selfdual field strength $F_{mn}^{I\pm}$ in terms of the field strength $F_{mn}^I = F_{mn}^{I+} + F_{mn}^{I-}$ and their Hodge-duals $\tilde{F}_{mn}^I = i(F_{mn}^{I+} - F_{mn}^{I-})$. The result is

$$L_{\mathrm{bos}}^{\mathrm{4d\ VM}} = -g_{ab}\partial_m q^a \partial^m q^b - \tfrac{1}{4}N_{IJ}F_{mn}^I F^{Jmn} + \tfrac{1}{4}R_{IJ}F_{mn}^I \tilde{F}^{Jmn} ,\tag{2.9}$$

where

$$\begin{aligned}
R_{IJ} &= F_{IJ} + \overline{F}_{IJ} , \\
N_{IJ} &= i(F_{IJ} - \overline{F}_{IJ}) = \partial_I \partial_{\overline{J}}\left(-i(X^I\overline{F}_I - F_I\overline{X}^I)\right) .
\end{aligned}\tag{2.10}$$

We now perform the reduction of the Lagrangian (2.9) from four to three dimensions. We treat the reduction over space and over time in parallel. In the following formulae, $\epsilon = 1$ refers to reduction over time, which gives a

Euclidean three-dimensional theory, while $\epsilon = -1$ refers to reduction over space. By reduction, one component of each gauge field becomes a scalar. We define:

$$p^I = A^{I|0} \text{ for } \epsilon = 1 \,, \quad p^I = A^{I|3} \text{ for } \epsilon = -1 \,. \tag{2.11}$$

Moreover, the $n$ three-dimensional gauge fields $A^{I|\hat{m}}$ obtained from dimensional reduction[6] can be dualized into $n$ further real scalars $s_I$. Denoting the new scalars by

$$(\hat{q}_a) = (s_I, 2p^I) \,, \tag{2.12}$$

the reduced bosonic Lagrangian takes the following, remarkably simple form:

$$L_{HM} = -g_{ab}(q)\partial_i q^a \partial^i q_b + \epsilon g^{ab}(q)\partial_i \hat{q}_a \partial^i \hat{q}_b \,, \tag{2.13}$$

where $g^{ab}(q)$ is the inverse of $g_{ab}(q)$. In this parametrization it is manifest that the hypermultiplet target space with metric $(g_{ab}(q)) \oplus (-\epsilon g^{ab}(q))$ is $N = M_{HM} = T^* M_{VM}$. The geometry underlying this Lagrangian was presented in detail in [6] for $\epsilon = 1$, and works analogously for $\epsilon = -1$. Here we give a brief summary. The special connection $\nabla$ on $M = M_{VM}$, can be used to define a decomposition

$$T_\xi N = \mathcal{H}_\xi^\nabla \oplus T_\xi^v N \simeq T_q M \oplus T_q^* M \,, \tag{2.14}$$

where $\xi \in N$ is a point on $N$ (with local coordinates $(q^a, \hat{q}_a)$), $q = \pi(\xi) \in M$ is its projection onto $M$, $\mathcal{H}_\xi^\nabla$ is the horizontal subspace with respect to the connection $\nabla$ and $T_\xi^v N$ is the vertical subspace. The identification with $T_q M \oplus T_q^* M$ is canonical, and the scalar fields $q^a, \hat{q}_a$ obtained by dimensional reduction are adapted to the decomposition. One can then define a complex structure $J_1$ on $N$, which acts on $T_\xi N \simeq T_q M \oplus T_q^* M$ by multiplication with

$$J_1 := J_1^\nabla = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix} \,, \tag{2.15}$$

where $J$, $J^*$ denote the action of the complex structure $J$ of $M$ on $TM$ and $T^* M$, respectively. Let us now consider the Euclidean case $\epsilon = 1$ for definiteness. Using the Kähler form $\omega$ on $M$, one can further define

$$J_2 = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix} \,, \tag{2.16}$$

where $\omega$ is interpreted as a map $T_q M \to T_q^* M$. This is a para-complex structure, $J_2^2 = \mathbb{1}$. Moreover, $J_3 = J_1 J_2$ is a second para-complex structure,

---

[6] The three-dimensional vector index takes values $\hat{m} = 0, 1, 2$ for $\epsilon = -1$ and $\hat{m} = 1, 2, 3$ for $\epsilon = 1$.

and $J_1, J_2, J_3$ satisfy a modified version of the quaternionic algebra known as the para-quaternionic algebra. Thus, $(J_1, J_2, J_3)$ is a para-hyper-complex structure on $N$. When defining, as in (2.13), the metric on $N$ by

$$g_N = \begin{pmatrix} g & 0 \\ 0 & -g^{-1} \end{pmatrix} , \qquad (2.17)$$

where $g$ is the metric on $M$, then $J_1$ is an isometry, while $J_2, J_3$ are anti-isometries. This means that $(J_1, J_2, J_3, g_N)$ is a para-hyper Hermitian structure.[7] Moreover, the structures $J_\alpha$, $\alpha = 1, 2, 3$ are parallel with respect to the Levi-Civita connection on $N$. Thus the metric $g_N$ is para-hyper Kähler, meaning that it is Kähler with respect to $J_1$ and para-Kähler with respect to $J_2, J_3$. The case $\epsilon = -1$ works analogously. Here one finds three complex structures satisfying the quaternionic algebra, and the metric defined by (2.13) is hyper-Kähler.

One can introduce (para-)complex fields such that one of the complex or (para-)complex structures becomes manifest in the three-dimensional Lagrangian [7, 6]. In these coordinates the Lagrangian is more complicated, and the geometrical structure reviewed above is less clear. Moreover one has singled out one of the three (para-)complex structures. Thus working in real coordinates has advantages, which should be exploited further in the future. Note in particular that for the c-map in local supersymmetry, the target space of hypermultiplets is quaternion-Kähler for Lorentzian space-time, while it is expected to be para-quaternion-Kähler for Euclidean space-time. In general, the almost (para-)complex structures of a (para-)quaternion-Kähler manifold need not be integrable. Then combining real scalar fields into (para-)complex fields is not natural, as these fields do not define local (para-)complex coordinates.

## 3 Black holes

In order to prepare for our second application of special geometry, we now give a brief self-contained introduction to certain aspects of black holes.[8] Somewhat surprisingly, one can associate thermodynamic properties to black holes. The so called laws of black hole mechanics, which have been derived in the framework of classical, matter-coupled Einstein gravity, formally have the same structure as the laws of thermodynamics [36]. While this was originally suspected to be a coincidence, the (theoretical) discovery of the

---

[7] Also note that $J_1, J_2, J_3$ are integrable, which follows from the integrability of $J$.

[8] See [33, 34, 35] for a detailed discussion.

Hawking effect [38] strongly suggested to take this observation seriously. More recently, developments in string theory have provided additional insights. Let us now briefly review this, starting with the laws of black hole mechanics in classical gravity.

## 3.1 The laws of black hole mechanics

The zeroth laws of black hole mechanics states that the surface gravity $\kappa_S$ of a black hole is constant over the event horizon $\Delta$.[9] The surface gravity can be defined if the event horizon is a Killing horizon, which is the case for all stationary black hole solutions of matter-coupled Einstein gravity. A Killing horizon is a hypersurface in space-time where a Killing vector field $\xi$ becomes null: $\xi^\nu \xi_\nu = 0$. One can show that a Killing horizon is generated by the integral lines of the Killing vector fields, which are null geodesics. There are two natural normal vector fields: the Killing vector field $\xi$ itself and the gradient of its normed-squared, $\nabla(\xi^\nu \xi_\nu)$. Both vector fields must be proportional, and the factor of proportionality is defined to be the surface gravity:

$$\nabla^\mu(\xi^\nu \xi_\nu) = -2\kappa_S \xi^\mu \ . \tag{3.18}$$

While this implies that $\kappa_S$ is a function on the horizon, the zeroth law states that this function is constant. The physical interpretation of the surface gravity is that it measures the force which an observer outside the black hole must apply in order to keep a unit test mass fixed at the horizon. Thus it measures the strength of gravity at the horizon. Since the zeroth law of thermodynamics is that temperature is constant in thermodynamical equilibrium, this suggests to interprete surface gravity as temperature and stationary black holes as equilibrium states. At the classical level this interpretation cannot be defended against the obvious problem that a black hole does not emit radiation, a fact which is explicitly alluded to in the term 'black' hole. As we will review below this changes once quantum effects are taken into account. For the time being we focus on the assumptions needed to prove the zeroth law. The classical proof uses the explicit form of the Einstein equations, while the effects of matter are controlled by imposing a suitable condition on the energy-momentum tensor. Moreover the solution must be stationary. It then follows from the field equations that the event

---

[9]The term 'horizon' is unfortunately used for two different but closely related concepts. We will use $\Delta$ to denote the null hypersurface which is the boundary between the exterior and interior of the black hole in space-time, and $H$ for the space-like surface which is the boundary in space, at given time. Thus $H$ is a spatial section of $\Delta$ while $\Delta$ is the 'worldline' of $H$.

horizon is a Killing horizon. However, it was realized later that the zeroth law does not depend on the details of the gravitational field equations. Instead, it can be proved for any covariant (diffeomorphism invariant) action, including actions which contain higher derivative and in particular higher curvature terms [37]. The relevance of such actions will be discussed below. The prize for not specifying the field equations is that one needs to make the following assumptions: (i) the field equations admit stationary black hole solutions with a Cauchy hypersurface, (ii) the event horizon is a Killing horizon, (iii) if the black hole is stationary but not static, then certain symmetry properties, which in Einstein geometry are consequences of the field equations, need to be imposed.[10]

Before proceeding, let us explain why it is desirable to admit actions containing higher derivative terms. The reason is that we would like to include so-called effective actions which incorporate quantum effects. In quantum field theory the effective action is defined to be the generating functional of the correlation functions. Since the classical action generates the classical contribution to the correlation functions (the leading part in an expansion in $\hbar$) the effective action might be considered to be its quantum version. Unfortunately the exact effective action is usually a rather formal and inaccessible object. However, certain approximations can be computed, and string theory provides a framework where quantum corrections to the gravitational action can and have been computed.[11] As expected on general grounds, quantum gravity manifests itself in the form of higher derivative terms in the effective action, in particular terms which contain higher powers of the Riemann tensor and its contractions. We will discuss a particular class of such terms in the next section.

Let us next turn to the first law of black hole mechanics, which states that for a stationary black hole an infinitesimal change of the mass $M$ is related to infinitesimal changes of the horizon area $A$, of the angular momentum $J$ and of the electric charge $Q$ by:

$$\delta M = \frac{\kappa_S}{8\pi}\delta A + \omega\delta J + \phi\delta Q \; , \qquad (3.19)$$

where $\omega$ is the angular velocity and $\phi$ the electrostatic potential. This should be compared to the first law of thermodynamics (for a grand canonical

---

[10]A space-time is stationary if it admits a time-like Killing vector field. In a static space-time this Killing vector field is in addition required to be the normal vector field of a family of hypersurfaces. The additional requirements needed if the space-time is stationary but not static are that the black hole is axisymmetric and invariant under simultanous reflection of time and and the angle around the symmetry axis.

[11]We refer to [26] for more details.

ensemble),

$$\delta E = T\delta\mathcal{S} - p\delta V + \mu\delta N , \qquad (3.20)$$

where $E$ is energy, $T$ is temperature, $\mathcal{S}$ is entropy, $p$ is pressure, $V$ is volume $\mu$ is chemical potential and $N$ is particle number. Given the relation between surface gravity and temperature, this suggests to interpret the area of the event horizon as the entropy of the black hole. This is surprising, since the entropy of normal thermodynamical systems is an extensive property, i.e., proportional to the volume rather than the surface area.

Like the zeroth law the first law can be derived for general covariant actions, under the same assumptions as for the zeroth law. Moreover, the mass, entropy, angular momentum and charge of the black hole are defined as surface charges, which are obtained by integrating a so-called Noether two-form over a closed spatial surface [69]. The Noether two-form is constructed out of Killing vector fields according to a certain algorithm. For the special case of Einstein gravity this definition reduces to the usual ones (i.e., the Komar or ADM constructions of mass and angular momentum, and the proportionality of entropy and horizon area).

Finally, let us turn to the second law of black hole mechanics, the Hawking area law. In contrast to the zeroth and first law, one does not assume the space-time to be stationary. Rather it can be time-dependent, and include processes such as the formation and fusion of black holes, as long as the time evolution is 'asymptotically predictable'[12]). The second law then states that the total area of event horizons is non-decreasing,

$$\delta A \geq 0 , \qquad (3.21)$$

which is obviously analogous to the second law of thermodynamics, which states the same for the entropy,

$$\delta\mathcal{S} \geq 0 . \qquad (3.22)$$

This reinforces the identification of area and entropy suggested by the first law. The second law has been derived using Einstein's field equation together with conditions on the energy-momentum tensor of matter (plus assuming 'predictability' of space-time). So far there is no general proof for the case of general covariant actions. However, examples have been studied, and the integrated Noether form is a good candidate for entropy in non-stationary space-times [40]. One interesting question is whether one should expect that the second law holds for all covariant actions. Since dynamical processes

---

[12]We refer to [33] for a precise definition and more details.

such as collision of black holes are admitted, the contents of the second law appears to be more sensitive to the details of the dynamics as the zeroth and first law. It is not clear whether all possible higher derivative actions give rise to 'sensible' physics which respects the second law. But one would certainly expect this to be true for string-effective actions, though this does not seem to have studied so far. Anyway, already the zeroth and first law provide compelling evidence for relating relating the surface gravity to the temperature and the area (integrated Noether two-form) to the entropy.

## 3.2   Quantum aspects of black holes

Let us now review the role of the Hawking effect [38] in making plausible the re-interpretation of geometrical as thermodynamic properites. This effect is derived by treating space-time geometry as a classical background, while matter is described by quantum field theory. In this framwork it has been shown that black holes emit thermal radiation, even if there is no ingoing radiation or matter. Moreover, the so-called Hawking temperature of this radiation is indeed proportional to the surface gravity:

$$T_{\text{Hawking}} = \frac{\kappa_S}{2\pi} \; .$$  (3.23)

In Einstein gravity the factor of proportionality between area and entropy is then fixed by the first law:

$$\mathcal{S} = \frac{A}{4}$$  (3.24)

(Newton's and Planck's constant and the speed of light serve as natural units, $G_N = \hbar = c = 1$). When using a covariant higher derivative action, the entropy is given by integrating the Noether two form $Q$ over the horizon $H$:

$$\mathcal{S} = 2\pi \oint_H Q \; .$$  (3.25)

It has been shown that the entropy can be expressed in terms of variational derivatives of the Lagrangian with respect to the Riemann tensor [39, 40]:

$$\mathcal{S} = 2\pi \oint_H \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma} \sqrt{h} \, d^2\Omega \; ,$$  (3.26)

where $\epsilon^{\mu\nu}$ is the normal bivector of $H$ (with a certain normalization), and $\sqrt{h} \, d^2\Omega$ is the induced volume element. If $\mathcal{L}$ is the Einstein-Hilbert Lagrangian, this formula reproduces the area law. If further terms containing

the Riemann tensor are present in $\mathcal{L}$ they induce explicit modifications of the area law.

Once the Hawking effect is taken into account, black holes can emit radiation, which implies that they loose mass and shrink, thus violating the second law of black hole mechanics. However, as soon as one takes the idea seriously that black holes carry entropy, one should consider the total entropy obtained by adding black hole entropy and the thermodynamical entropy of the exterior region. The generalized second law of thermodynamics, which states that the total entropy is non-decreasing, is expected to be valid in quantum gravity [41].

So far we have considered black hole entropy from what one might call the macroscopic or thermodynamical perspective. When dealing with many-constitutent systems one distinguishes two levels of description. The fundamental or microscopic level of description requires knowledge of the precise state of the system. For a classical gas this would require to specify the positions and momenta (and other quantities if internal excitations exist) of all atoms or molecules. At the thermodynamical or macroscopic level of description one only considers collective properties of the system, such as temperature, volume and pressure. Statistical mechanics asserts that these macroscopic quantities arise by 'coarse graining' microscopic quantities. E.g., temperature is the average energy per degree of freedom. Obviously many microstates will give rise to the same macrostate, where the latter is characterised by fixing only the macroscopic quantities. The so-called statistical or microscopic entropy measures how many different microstates give rise to the same macrosate. If $d(E, \dots)$ denotes the number of microstates corresponding to the macrostate with energy $E$, etc., then the corresponding microscopic entropy is

$$\mathcal{S}_{\mathrm{micro}} = \log d(E, \dots) \,. \tag{3.27}$$

In contrast the so-called macroscopic or thermodynamical entropy $\mathcal{S}_{\mathrm{macro}}$ is a purely macroscopic quantity, which can be characterized by its relation to other macroscopic quantities, such as temperature, free energy, etc. Both entropies are expected to be equal in the thermodynamcial limit, i.e., when the number of constituents goes to infinity.

The geometrical black hole entropy is analogous to the macroscopic entropy, because it has been defined through relations which only involve collective properties of the black hole, such as mass, charge and angular momentum. Any theory of quantum gravity is expected to provide a corresponding microscopic description of black holes, which in particular allows

one to identify its microstates. In particular the microscopic entropy should be equal to the macroscopic one, at least in the limit of large mass, which is analogous to the thermodynamical limit. This is widely regarded as a benchmark test for theories of quantum gravity.

## 3.3 Black holes and strings

In string theory four-dimensional black holes can be interpreted as arising from states in the full ten-dimensional string theory. These states might be string states, or winding states of higher-dimensional membranes (i.p. D-branes) [42]. One can test the expected relation between macroscopic and microscopic entropy by counting the ten-dimensional states which give rise to the same four-dimensional black hole. This comparison generically involves the variation of parameters such as the string coupling, and it is not a priori clear whether the number of states is preserved under this interpolation. But for a special subclass of states, the so-called supersymmetric states or BPS states, which we will review below, the interpolation is at least highly plausible. Moreover, both the macroscopic and the microscopic entropy can often be computed to high precision and it has been found that they match [9], even when subleading corrections are included [15]. In particular these tests are sensitive to the distinction between the area law and the generalized formula (3.26), and clearly show that string theory 'knows' about the modifications of the area law. In performing these prescision tests, special geometry plays a central role. It is the indispensible tool for constructing black hole solutions and extracting the macroscopic entropy from them. This will be the subject of the next section. We will not be able to cover the microscopic side of the story, i.e., the counting of microstates, in this article.

## 3.4 Black holes and supersymmetry

Before turning to the details, let us review the concepts of BPS states and BPS solitons.[13] Recall that the supercharges which generate supersymmetry transformations are spinors. If there is more than one such spinor, than the supersymmetry algebra admits central operators, which can be organised into a complex antisymmetric matrix. The skew eigenvalues $Z_{(i)}$ of this matrix are called the central charges. It can be shown that on any irreducible representation the mass is bounded from below by the absolute values of the

---

[13]In the following we use basic facts about supersymmetry algebras and their representations. See for example [10], Chapter II.

charges:

$$M \geq |Z_{(1)}| \geq |Z_{(2)}| \geq \cdots . \qquad (3.28)$$

Moreover, when the mass saturates one or several of these bounds, part of the supercharges operate trivially, and the corresponding multiplet is shorter than a generic massive multiplet. Such multiplets are called supersymmetric multiplets or BPS multiplets. When all inequalities are saturated, the resulting BPS multiplet is invariant under half of the supertransformations and has as many states as a massless multiplet. In the case of $N = 2$ supersymmetry considered in this article, the algebra has one single complex supercharge $Z$. Consequently, there are generic massive supermultiplets $M > |Z|$ and '$\frac{1}{2}$-BPS multiplets' with $M = |Z|$.

The concept of BPS state can be applied to solitons. By solitons we refer to solutions of the field equations which can be interpreted as particle-like objects. In particular, these solutions are required to have finite energy, and therefore must approach the ground state asymptotically. Since the energy localized in a small part of space-time, such 'lumps' can be thought of as 'extended particles'. One also requires that the solution is static (describing 'a massive particle in its rest frame') and free of naked singularities (we admit singularities covered by event horizons in order to include black holes). A soliton is then called supersymmetric or BPS, if it is invariant under part of the supersymmetry transformations. Let us denote the fields of the underlying action collectively by $\Phi$, the spinorial supersymmetry transformation parameters by $\epsilon$, the corresponding supersymmetry transformation by $\delta_\epsilon$ and the soliton solution by $\Phi_0$. Then a solution is BPS if there exists a choice of $\epsilon$ such that

$$(\delta_\epsilon \Phi)_{|\Phi_0} = 0 . \qquad (3.29)$$

Particular examples of BPS solitons are provided by supersymmetric black hole solutions of supergravity actions. In supergravity the supersymmetry transformation parameters depend on space-time, $\epsilon = \epsilon(x)$. Therefore the BPS condition implies the existence of a spinor field which generates a supertransformation under which the black hole solution is invariant. This is analogous to a Killing vector field, which generates a diffeomorphism under which the metric (and possibly other fields) are invariant. Therefore such spinor fields $\epsilon(x)$ are called Killing spinors (more accurately Killing spinor fields). The interested reader is referred to the monograph [43] for a detailed discussion of supersymmetric solutions.

# 4  Special geometry and black holes

## 4.1  Vector multiplets coupled to gravity

We are now in position to discuss BPS black hole solutions in $N = 2$ supergravity coupled to $n$ vector multiplets. This is the relevant part of the effective action for string compactifications preserving $N = 2$ supersymmetry. The general $N = 2$ vector multiplet action was constructed using the superconformal calculus [31].[14] The idea of this method is to start with a theory of $n+1$ rigidly supersymmetric vector multiplets and to impose that the theory is invariant under superconformal transformations. This implies that the prepotential has to be homogenous of degree 2 in addition to being holomorphic:

$$F(\lambda X^I) = \lambda^2 F(X^I) , \quad \lambda \in \mathbb{C}^* , \tag{4.30}$$

where now $I = 0, 1, \ldots, n$. Next one 'gauges' the superconformal transformation, that is one makes the Lagrangian locally superconformally invariant by introducing suitable connections. The new fields entering through this process are encoded in the so-called Weyl multiplet.[15] Finally, one imposes gauge conditions which reduce the local superconformal invariance to a local invariance under standard (Poincaré) supersymmetry. Through the gauge conditions some of the fields become functions of the others. In particular, only $n$ out of the $n+1$ complex scalars are independent. A convenient choice for the independent scalars is

$$z^A = \frac{X^A}{X^0} , \tag{4.31}$$

where $A = 1, \ldots, n$. This provides a set of special coordinates for the scalar manifold $M_{VM}$. In contrast, all $n + 1$ gauge fields remain independent. While one particular linear combination, the so-called graviphoton, belongs to the Poincaré supergravity multiplet, the other $n$ gauge fields sit in vector multiplets, together with the scalars $z^A$. The Weyl multiplet also provides physical degrees of freedom, namely the graviton and two gravitini.

From the underlying rigidly superconformal theory the supergravity theory inherits the invariance under symplectic rotations. For the gauge fields this is manifest, as $(F^I_{mn}, G_{I|mn})$ transforms as a vector under $Sp(2(n +$

---

[14]Further references on $N = 2$ vector multiplet Lagrangians and the superconformal calculus include [45, 46, 1, 47].

[15]One also needs to add a further 'compensating multiplet', which can be taken to be a hypermultiplet. We won't need to discuss this technical detail here. See for example [26] for more background material and references.

1), $\mathbb{R}$).[16] In the scalar sector $(X^I, F_I)$, where $F_I = \partial_I F$, also transforms as a vector, while the gravitational degrees of freedom are invariant. To maintain manifest symplectic invariance, it is advantagous to work with $(X^I, F_I)$ instead of $z^A$.

The underlying geometry can be described as follows [27, 28, 29]: the fields $X^I$ provide coordinates on the scalar manifold of the associated rigidly superconformal theory. This manifold has complex dimension $n + 1$, and can be immersed into $T^*\mathbb{C}^{n+1} \simeq \mathbb{C}^{2(n+1)}$ just as described in the previous section. The additional feature imposed by insisting on superconformal invariance is that the prepotential is homogenous of degree 2. Geometrically this implies that the resulting affine special Kähler manifold is a complex cone. The scalar manifold of the supergravity theory is parametrized by the scalars $z^A$ and has complex dimension $n$. It is obtained from the manifold of the rigidly superconformal theory by gauge-fixing the dilatation and $U(1)$ symmetry contained in the superconformal algebra. This amounts to taking the quotient of the complex cone with respect to the $\mathbb{C}^*$-action $X^I \to \lambda X^I$. Thus the scalar manifold $M_{VM}$ is the basis of the conical affine special Kähler manifold $C(M_{VM})$ of the rigid theory. For many purposes, including the study of black hole solutions, it is advantagous to work on $C(M_{VM})$ instead of $M_{VM}$. In particular, this allows to maintain manifest symplectic covariance, as we already noted. In physical terms this means that one can postpone the gauge-fixing of the dilatation and $U(1)$ transformations. The manifolds which can be obtained from conical affine special Kähler manifolds by a $\mathbb{C}^*$-quotient are called projective special Kähler manifolds. These are the target spaces of vector multiplets coupled to supergravity. All couplings in the Lagrangian and all relevant geometrical data of $M_{VM}$ are encoded in the prepotential. In particular, the affine special Kähler metric on $C(M_{VM})$ has Kähler potential

$$K_C(X^I, \overline{X}^I) = -i(X^I \overline{F}_I - F_I \overline{X}^I) , \qquad (4.32)$$

while the projective special Kähler metric on $M_{VM}$ has Kähler potential

$$K(z^A, \overline{z}^B) = -\log\left(-i(X^I \overline{F}_I - F_I \overline{X}^I)\right) , \qquad (4.33)$$

with corresponding metric

$$g_{a\overline{b}} = \frac{\partial^2 K(z^A, \overline{z}^B)}{\partial z^a \partial \overline{z}^b} . \qquad (4.34)$$

---

[16] The dual gauge fields $G_{I|mn}$ were introduced at the beginning of section 2.

18

In string theory the four-dimensional supergravity Lagrangians considered here are obtained by dimensional reduction of the ten-dimensional string theory on a compact six-dimensional manifold $X$ and restriction to the massless modes. Then the scalar manifold $M_{VM}$ is the moduli space of $X$. It turns out that the moduli spaces of Calabi-Yau threefolds provide natural realizations of special Kähler geometry [58]. Consider for instance the Calabi-Yau compactification of type-IIB string theory. In this case $M_{VM}$ is the moduli space of complex structures of $X$, the cone $M_{VM}$ is the moduli space of complex structures together with a choice of the holomorphic top-form, and $T^*\mathbb{C}^{n+1} \simeq \mathbb{C}^{2(n+1)}$ is $H^3(X, \mathbb{C})$, see [59].

## 4.2   BPS black holes and the attractor mechanism

Let us then discuss BPS black hole solutions of $N = 2$ supergravity with $n$ vector multiplets. These are static, spherically symmetric solutions of the field equations, which are asymptotically flat, have regular event horizons, and possess 4 Killing spinors. Since the $N = 2$ superalgebra has 8 real supercharges, these are $\frac{1}{2}$-BPS solutions. Let us first have a look at pure four-dimensional $N = 2$ supergravity, i.e., we drop the vector multiplets, $n = 0$. The bosonic part of this theory is precisely the Einstein-Maxwell theory. In pure $N = 2$ supergravity, BPS solutions have been classified [60, 61, 62]. The number of linearly independent Killing spinor fields can be 8,4 or 0. This can be seen, for example, by investigating the integrability conditions of the Killing spinor equation.[17] Solutions with 8 Killing spinors are maximally supersymmetric and therefore considered as supersymmetric ground states. Examples are Minkowski space and $AdS^2 \times S^2$. Solutions with 4 Killing spinors are called $\frac{1}{2}$-BPS, because they are invariant under half as many supersymmetries as the ground state. They are solitonic realisations of states sitting in BPS representations. For static $\frac{1}{2}$-BPS solutions the space-time metric takes the form [60, 61]

$$ds^2 = -e^{-2f(\vec{x})}dt^2 + e^{2f(\vec{x})}d\vec{x}^2 \ , \tag{4.35}$$

where $\vec{x} = (x_1, x_2, x_3)$ are space-like coordinates and the function $f(\vec{x})$ must be such that $e^{f(\vec{x})}$ is a harmonic function with respect to $\vec{x}$. The solutions also have a non-trivial gauge field, which likewise can be expressed

---

[17]The classification of supersymmetric solutions has recently moved to the focus of interest. Readers who want to get an idea how the classification of supersymmetric solutions of four-dimensional $N = 2$ supergravity would work with 'modern', systematic methods can consult [63], where all supersymmetric solutions of minimal five-dimensional supergravity were constructed.

in terms of $e^{f(\vec{x})}$. This class of solutions of Einstein-Maxwell theory is known as the Majumdar-Papapetrou solutions [64, 65]. The only Majumdar-Papapetrou solutions without naked singularities are the multi-centered extremal Reissner-Nordstrom solutions, which describe static configurations of extremal black holes, see for example [66]. If one imposes in addition spherical symmetry, one arrives at the extremal Reissner-Nordstrom solution describing a single charged black hole. In this case the metric takes the form

$$ds^2 = -e^{-2f(r)}dt^2 + e^{2f(r)}(dr^2 + r^2 d\Omega^2) \,, \qquad (4.36)$$

where $r$ is a radial coordinate and $d\Omega^2$ is the line element on the unit two-sphere. The harmonic function takes the form

$$e^{f(r)} = 1 + \frac{q^2 + p^2}{r} \,, \qquad (4.37)$$

where $q, p$ are the electric and magnetic charge with respect to the graviphoton. The solution has two asymptotic regimes. In one limit, $r \to \infty$, it becomes asymptotically flat: $e^f \to 1$. In the other limit, $r \to 0$, which is the near-horizon limit, it takes the form

$$ds^2 = -\frac{r^2}{q^2 + p^2}dt^2 + \frac{q^2 + p^2}{r^2}dr^2 + (q^2 + p^2)d\Omega^2 \,. \qquad (4.38)$$

This is a standard form for the metric of $AdS^2 \times S^2$. The area of the two-sphere, which is the area of the event horizon of the black hole, is given by $A = 4\pi(q^2 + p^2)$. The two limiting solutions, flat Minkowski space-time and $AdS^2 \times S^2$ are among the fully supersymmetric solutions with 8 Killing spinors that we mentioned before. Thus, the extremal Reissner-Nordstrom black hole interpolates between two supersymmetric vacua [48]. This is a property familiar from two-dimensional kink solutions, and motivates the interpretation of supersymmetric black hole solutions as solitons, i.e., as particle-like collective excitations.

Let us now return to $N = 2$ supergravity with an arbitrary number $n$ of vector fields. We are interested in solutions which generalize the extremal Reissner-Nordstrom solution. Therefore we impose that the solution should be $\frac{1}{2}$-BPS, static, spherically symmetric, asymptotically flat, and that it should have a regular event horizon.[18] More general $\frac{1}{2}$-BPS solutions have been studied extensively in the literature, in particular in [49] and [16]. Recently, the classification of all $\frac{1}{2}$-BPS solutions was achieved in [50].

---

[18]This excludes both naked singularties and null singularities, where the horizon coincides with the singularity and has vanishing area.

BPS black holes in theories with $n$ vector multiplets depend on $n + 1$ gauge fields and on $n$ scalar fields. For any $\frac{1}{2}$-BPS solution, which is static and spherically symmetric, the metric can be brought to the form (4.37) [16]. The condition that the solution is static and spherically symmetric is understood in the strong sense, i.e., it also applies to the gauge fields and scalars. Thus gauge fields and scalars are functions of the radial coordinate $r$, only. Moreover the electric and magnetic fields are spherically symmetric, which implies that each field strength $F^I_{mn}(r)$ has only two independent components (see for example Appendix A of [26] for more details).

The electric and magnetic charges carried by the solution are defined through flux integrals of the field strength over asymptotic two-spheres:

$$(p^I, q_I) = \frac{1}{4\pi} \left( \oint F^I, \oint G_I \right) , \qquad (4.39)$$

where $F^I, G_I$ are the two-forms associated with the field strength $F^I_{mn}$ and their duals $G_{Imn}$. As a consequence, the charges transform as a vector under symplectic transformations. By contracting the charges with the scalars one obtains the symplectic function

$$Z = p^I F_I - q_I X^I . \qquad (4.40)$$

This field is often called the central charge, which is a bit misleading because $Z$ is a function of the fields $X^I$ and $F_I$ and therefore a function of the scalar fields $z^A$, which are space-time dependent.[19] Hence, in the backgrounds we consider, $Z$ is a function of the radial coordinate $r$. However, when evaluating this field in the asymptotically flat limit $r \to \infty$, it computes the electric and magnetic charge carried by the graviphoton, which combine into the complex central charge of the $N = 2$ algebra [67].

In particular, the mass of the black hole is given by

$$M = |Z|_\infty = M(p^I, q_I, z^A(\infty)) . \qquad (4.41)$$

Thus BPS black holes saturate the mass bound implied by the supersymmetry algebra. Note that the mass does not only depend on the charges, but also on the values of the scalars at infinity, which can be changed continuously.

---

[19]One can analyse BPS solutions without imposing the gauge conditions which fix the superconformal symmetry, and in fact it is advantagous to do so [15, 16]. Then the scalars are encoded in the fields $X^I(r)$, which are subject to gauge transformations. Once gauge conditions are imposed, one can express $Z(r)$ in terms of the physical scalar fields $z^A(r)$. See [26] for more details.

The other asymptotic regime is the event horizon. If the horizon is regular, then the solution must be fully supersymmetric in this limit [11]. Thus, while the bulk solution has 4 Killing spinors, both asymptotic limits have 8. In the near horizon limit, the metric (4.37) takes the form

$$ds^2 = -\frac{r^2}{|Z|^2_{\text{hor}}}dt^2 + \frac{|Z|^2_{\text{hor}}}{r^2}dr^2 + |Z|^2_{\text{hor}}d\Omega^2 \,, \qquad (4.42)$$

where $|Z|^2_{\text{hor}}$ is the value of $|Z|^2$ at the horizon. As in the extremal Reissner-Nordstrom solution, this is $AdS^2 \times S^2$. The area of the two-sphere, which is the area of the event horizon, is given by $A = 4\pi|Z|^2_{\text{hor}}$. Hence the Bekenstein Hawking entropy is

$$\mathcal{S}_{\text{macro}} = \frac{A}{4} = \pi|Z|^2_{\text{hor}} \,. \qquad (4.43)$$

A priori, $\mathcal{S}_{\text{macro}}$ depends on both the charges and the values of the scalars at the horizon, and one might expect that one can change the latter continuously. This would be incompatible with relating $\mathcal{S}_{\text{macro}}$ to a statistical entropy $\mathcal{S}_{\text{micro}}$ which counts states. But it turns out that the values of the scalar fields at the horizon are themselves determined in terms of the charges. Here, it is convenient to define $Y^I = \overline{Z}X^I$ and $F_I = F_I(Y) = \overline{Z}F_I(X)$.[20] In terms of these variables, the black hole attractor equations [11], which express the horizon values of the scalar fields in terms of the charges, take the following, symplectically covariant form:

$$\begin{pmatrix} Y^I - \overline{Y}^I \\ F_I - \overline{F}_I \end{pmatrix}_{\text{hor}} = i \begin{pmatrix} p^I \\ q_I \end{pmatrix} \,. \qquad (4.44)$$

The name attractor equations refers to the behaviour of the scalar fields as functions of the space-time radial coordinate $r$. While the scalars can take arbitrary values at $r \to \infty$, they flow to fixed points, which are determined by the charges, for $r \to 0$. This fixed point behaviour follows when imposing that the event horizon is regular. Alternatively, one can show that to obtain a fully supersymmetric solution with geometry $AdS^2 \times S^2$ the scalars need to take the specific values dictated by the attractor equations [16]. This is due to the presence of non-vanishing gauge fields. The gauge fields in $AdS^2 \times S^2$ are covariantly constant, so that this can be viewed as an example of a flux compactification. In contrast, Minkowski space is also maximally supersymmetric, but the scalars can take arbitrary constant

---

[20] Note that $F_I$ is homogenous of degree 1.

values, because the gauge fields vanish. In type-II Calabi-Yau compactifications, the radial dependence of the scalar fields defines a flow on the moduli space, which starts at an arbitrary point and terminates at a fixed point corresponding to an 'attractor Calabi Yau.' Since the electric and magnetic charges $(p^I, q_I)$, which determine the fixed point, take discrete values, such attractor threefolds sit at very special points in the moduli space. This has been studied in detail in [51].

Using the fields $Y^I$ instead of $X^I$ to parametrize the scalars simplifies formulae and has the advantage that the $Y^I$ are invariant under the $U(1)$ transformations of the superconformal algebra. Note that

$$|Z|^2 = p^I F_I - q_I Y^I \ , \tag{4.45}$$

which is easily seen using the homogeneity properties of the prepotential. The diffeomorphism $X^I \to Y^I$ acts non-holomorphically on the cone $C(M_{VM})$, but operates trivially on its basis $M_{VM}$. Note in particular that

$$z^A = \frac{X^A}{X^0} = \frac{Y^A}{Y^0} \ . \tag{4.46}$$

## 4.3   The black hole variational principle

We now turn to the black hole variational principle, which was found in [14] and generalized in [13], motivated by the observations of [12]. First, we define two symplectic functions, the entropy function

$$\Sigma(Y^I, \overline{Y}^I, p^I, q_I) = \mathcal{F}(Y^I, \overline{Y}^I) - q_I(Y^I + \overline{Y}^I) + p^I(F_I + \overline{F}_I) \tag{4.47}$$

and the black hole free energy

$$\mathcal{F}(Y^I, \overline{Y}^I) = -i\left(\overline{Y}^I F_I - Y^I \overline{F}_I\right) \ . \tag{4.48}$$

The reason for our choice of terminology will become clear later. Now we impose that the entropy function is stationary, $\delta\Sigma = 0$, under variations of the scalar fields $Y^I \to Y^I + \delta Y^I$. Using that the prepotential is homogenous of degree two, it is easy to see that the conditions for $\Sigma$ being stationary are precisely the black hole attractor equations (4.44). Furthermore, at the

attractor point we find that[21]

$$\begin{aligned}
\mathcal{F}_{\text{attr}} &= -i\left(\overline{Y}^I F_I - Y^I \overline{F}_I\right)_{\text{attr}} = \left(q_I Y^I - p^I F_I\right)_{\text{attr}} \\
&= \left(q_I \overline{Y}^I - p^I \overline{F}_I\right)_{\text{attr}} = -|Z|^2_{\text{attr}}
\end{aligned} \tag{4.49}$$

and therefore

$$\Sigma_{\text{attr}} = |Z|^2_{\text{attr}} = \tfrac{1}{\pi} \mathcal{S}_{\text{macro}}(p^I, q_I) \ . \tag{4.50}$$

Here and in the following we use the label 'attr' (instead of 'hor' used previously) to indicate that quantities are evaluated at the attractor point determined by the electric and magnetic charges.

Thus, up to a constant factor, the entropy is obtained by evaluating the entropy function at its critical point. Moreover, a closer look at the variational principle shows us that, again up to a factor, the black hole entropy $\mathcal{S}_{\text{macro}}(p^I, q_I)$ is the Legendre transform of the free energy $\mathcal{F}(Y^I, \overline{Y}^I)$, where the latter is considered as a function of $x^I = \text{Re}(Y^I)$ and $y_I = \text{Re}(F_I)$. At this point the real variables discussed in the previous section become important again. Note that the change of variables $(Y^I, \overline{Y}^I) \to (x^I, y_I)$ is well defined provided that $\text{Im}(F_{IJ})$ is non-degenerate. This assumption will be satisfied in general, but breaks down in certain string theory applications, where one reaches the boundary of the moduli space.[22]

We are therefore led to rewrite the variational principle in terms of real variables. First, recall that the Hesse potential $H(x^I, y_I)$ is the Legendre transform of (two times) the imaginary part of the prepotential, see (2.7).[23] This Legendre transform replaces the independent variables $(x^I, u^I) = (\text{Re}(Y^I), \text{Im}(Y^I))$ by the independent variables $(x^I, y_I) = (\text{Re}(Y^I), \text{Re}(F_I))$ and therefore implements the change of variables $(Y^I, \overline{Y}^I) \to (x^I, y_I)$. Using (2.7) we find

$$H(x^I, y_I) = -\tfrac{i}{2}(\overline{Y}^I F_I - \overline{F}_I Y^I) = \tfrac{1}{2}\mathcal{F}(Y^I, \overline{Y}^I) \ . \tag{4.51}$$

Thus, up to a factor, the Hesse potential is the black hole free energy. We can now express the entropy function in terms of the real variables:

$$\Sigma(x^I, y_I, p^I, q_I) = 2H(x^I, y_I) - 2q_I x^I + 2p^I y_I \ . \tag{4.52}$$

---

[21] The relation $-i\left(\overline{Y}^I F_I - Y^I \overline{F}_I\right) = \left(q_I Y^I - p^I F_I\right)$ follows from the definitions of $Z$ and $Y^I$ together with the homogenity of the prepotential (once the dilatational symmetry of the fields $X^I$ has been gauge fixed). Therefore it holds irrespective of whether the scalar fields take their attractor values or not.

[22] See for example [13] for a discussion of some of the implications.

[23] Note that this is the Hesse potential of the affine special Kähler metric on $C(M_{VM})$. The projective special Kähler metric on $M_{VM}$ is obtained by the $\mathbb{C}^*$-quotient.

If we impose that $\Sigma$ is stationary with respect to variations of $x^I$ and $y_I$, we get the black hole attractor equations in real variables:

$$\frac{\partial H}{\partial x^I} = q_I \ , \quad \frac{\partial H}{\partial y_I} = -p^I \ . \qquad (4.53)$$

Plugging this back into the entropy function we obtain

$$\mathcal{S}_{\mathrm{macro}} = 2\pi \left( H - x^I \frac{\partial H}{\partial x^I} - y_I \frac{\partial H}{\partial y_I} \right)_{\mathrm{attr}} \ . \qquad (4.54)$$

Thus, up to a factor, the black hole entropy is the Legendre transform of the Hesse potential. This is an intriguing observation, because it relates the black hole entropy, which is a space-time quantity, directly to the special geometry encoding the scalar dynamics. In string theory compactifications this relates the geometry of four-dimensional space-time to the geometry of the compact internal space $X$. The Hesse potential appears to be closely related to the action functional underlying the geometry of $X$ in Hitchin's approach to manifolds with special holonomy [52, 53, 54].

We can also relate the black hole free energy to another quantity of special geometry. In terms of complex variables we observe that

$$\mathcal{F}(Y^I, \overline{Y}^I) = K_C(Y^I, \overline{Y}^I) := i(\overline{Y}^I F_I - \overline{F}_I Y^I) \ . \qquad (4.55)$$

Comparing to (4.32) it appears that we should interpet $K_C(Y^I, \overline{Y}^I)$ as the Kähler potential of an affine special Kähler metric on $C(M_{VM})$. Since the diffeomorphism $X^I \to Y^I$ is non-holomorphic, this is not the same special Kähler structure as with (4.32). However, we already noted that the diffeomorphism acts trivially on $M_{VM}$, see (4.46). Moreover it is easy to see that when taking the quotient with respect to the $\mathbb{C}^*$-action $Y^I \to \lambda Y^I$, then the resulting projective special Kähler metric with Kähler potential $K(Y^I, \overline{Y}^I) = -\log K_C(Y^I, \overline{Y}^I)$ is the same as the one derived from (4.33), because the two Kähler potentials differ only by a Kähler transformation. It appears that in the context of black hole solutions the affine special Kähler metric associated with the rescaled scalars $Y^I$ is of more direct importance than the one based on the $X^I$. The same remark applies to the Hesse potential, which depends on the real coordinates associated to $Y^I$. Note that the scalars $Y^I$ do not only encode the values of the Calabi-Yau moduli $z^A$ via (4.46) but also, via (4.45) the size of the two-sphere in the black-hole space-time.[24] While variations of the moduli correspond to variations along the

---

[24]This is not only true at the horizon but throughout the whole black hole solution.

basis of the cone $C(M)$, variations of the radius of the two-sphere correspond to motions along the radial direction of the cone.

Note that it is more natural to identify the free energy with the Hesse potential than the Kähler potential. The first reason is that the various Legendre transforms involve the real and not the complex coordinates. The second reason is that, as we will discuss below, we need to generalize the supergravity Lagrangian in order to take into account certain corrections appearing in string theory. We will see that this works naturally by introducing a generalized Hesse potential.

Before turning to this subject, we also remark that the terms in the entropy function (4.47) which are linear in the charges, and which induce the Legendre transform, have yet another interpretation in terms of supersymmetric field theory. Namely, the symplectic function

$$W = q_I Y^I - p^I F_I \qquad (4.56)$$

has the form of an $N = 2$ superpotential. The four-dimensional supergravity Lagrangian we are studying does not have a superpotential. However, the near-horizon solution has the form $AdS^2 \times S^2$ and carries non-vanishing, covariantly constant gauge fields. The dimensional reduction on $S^2$ is a flux compactification, with fluxes parametrized by $(p^I, q_I)$, and the resulting two-dimensional theory will possess a superpotential. This also provides an alternative interpretation of the attractor mechanism, as the resulting scalar potential will lift the degeneracy of the moduli.

## 4.4 Quantum corrections to black holes solutions and entropy

So far we only considered supergravity Lagrangians which contain terms with at most two derivatives. The effective Lagrangians derived from string theory also contain higher derivative terms, which modify the dynamics at short distances. These terms describe interactions between the massless states which are mediated by massive string states. While the effective Lagrangian does not contain the massive string states explicitly, it is still possible to describe their impact on the dynamics of the massless states.

In $N = 2$ supergravity a particular class of higher derivative terms can be taken into account by giving the prepotential an explicit dependence on an additional complex variable $\Upsilon$, which is proportional to the lowest component of the Weyl multiplet [57, 68]. The resulting function $F(Y^I, \Upsilon)$ is required to be holomorphic in all its variables, and to be (graded) homoge-

nous of degree two:[25]

$$F(\lambda Y^I, \lambda^2 \Upsilon) = \lambda^2 F(Y^I, \Upsilon) \ . \tag{4.57}$$

Assuming that it is analytic at $\Upsilon = 0$ one can expand it as

$$F(Y^I, \Upsilon) = \sum_{g=0}^{\infty} F^{(g)}(Y^I) \Upsilon^g \ . \tag{4.58}$$

Then $F^{(0)}(Y^I)$ is the prepotential, while the functions $F^{(g)}(Y^I)$ with $g > 0$ appear in the Lagrangian as the coefficients of various higher-derivative terms. These include in particular terms quadratic in the space-time curvature, and therefore one often loosely refers to the higher derivative tems as $R^2$-terms.

In type-II Calabi Yau compactifications the functions $F^{(g)}(Y^I)$ can be computed using (one of) the topologically twisted version(s) of the theory [56]. They are related to the partition functions $Z_{\text{top}}^{(g)}$ of the topologically twisted string on a world sheet with genus $g$ by $F^{(g)} = \log Z_{\text{top}}^{(g)}$. Therefore they are called the (genus-$g$) topological free energies.

It was shown in [15, 16] that the black hole attractor mechanism can be generalized to the case of Lagrangians based on a general function $F(Y^I, \Upsilon)$. The attractor equations still take the form (4.44), but the prepotential is replaced by the full function $F(Y^I, \Upsilon)$. The additional variable $\Upsilon$ takes the value $\Upsilon = -64$ at the horizon. The evaluation of the generalized entropy formula (3.26) for $N = 2$ supergravity gives [15]:

$$\mathcal{S}_{\text{macro}}(q^I, p_I) = \pi \left( |Z|^2 + 4\text{Im}(\Upsilon F_\Upsilon) \right)_{\text{attr}} \ , \tag{4.59}$$

where $F_\Upsilon = \partial_\Upsilon F$.[26] Note that symplectic covariance is manifest, as the entropy is the sum of two symplectic functions. While the first term corresponds to the area law, the second term is an explicit modification which depends on the coefficients $F^{(g)}$, $g > 0$, of the higher derivative terms.

It was shown in [13] that the variational principle generalizes to the case with $R^2$-terms. The black hole free energy $\mathcal{F}$ is now proportional to a generalized Hesse potential $H(x^I, y_I, \Upsilon, \overline{\Upsilon})$, which in turn is proportional to the Legendre transform of the imaginary part of the function $F(Y^I, \Upsilon)$:

$$H(x^I, y_I, \Upsilon, \overline{\Upsilon}) = 2\text{Im}F(x^I + iu^I, \Upsilon) - 2y_I u^I \ .$$

---

[25]Since we are interested in black hole solutions, we use rescaled fields $Y^I, \Upsilon$.

[26]At the attractor point, $\Upsilon$ takes the value $\Upsilon = -64$.

In terms of complex fields $Y^I$ this becomes

$$
\begin{aligned}
H(x^I, y_I, \Upsilon, \overline{\Upsilon}) &= -\tfrac{i}{2}(\overline{Y}^I F_I - \overline{F}_I Y^I) - i(\Upsilon F_\Upsilon - \overline{\Upsilon F}_{\overline{\Upsilon}}) \quad (4.60)\\
&= \tfrac{1}{2}\mathcal{F}(Y^I, \overline{Y}^I, \Upsilon, \overline{\Upsilon}) .
\end{aligned}
$$

The entropy function (4.52), the attractor equations (4.53) and the formula for the entropy (4.54), which now includes correction terms to the area law, remain the same, except that one uses the generalized Hesse potential. From (4.60) it is obvious that the black hole free energy naturally corresponds to a generalized Hesse potential (defined by the Legendre transform of the prepotential) and not to a 'generalized Kähler potential', which would only give rise to the first term on the right hand side of (4.60).

There is a second class of correction terms in string-effective supergravity Lagrangians. Quantum corrections involving the massless fields lead to modifications which correspond to adding non-holomorphic terms to the function $F(Y^I, \Upsilon)$. The necessity of such non-holomorphic terms can be seen by observing that otherwise the invariance of the full string theory under T-duality and S-duality is not captured by the effective field theory. In particular, one can show that the black hole entropy can only be T- and S-duality invariant if non-holomorphic corrections are taken into account [55].[27] From the point of view of string theory the presence of these terms is related to a holomorphic anomaly [56, 57].

As the holomorphic $R^2$-corrections, the non-holomorphic corrections can be incorporated into the black hole attractor equations and the black hole variational principle [55, 13]. The non-holomorphic terms are encoded in a function $\Omega(Y^I, \overline{Y}^I, \Upsilon, \overline{\Upsilon})$, which is real valued and homogenous of degree two. To incorporate non-holomorphic terms into the variational principle one has to define the generalized Hesse potential as the Legendre transform of $2\mathrm{Im}F + 2\Omega$:

$$
H(x^I, \hat{y}_I, \Upsilon, \overline{\Upsilon}) = 2\mathrm{Im}F(x^I + iu^I, \Upsilon, \overline{\Upsilon}) + 2\Omega(x^I, u^I, \Upsilon, \overline{\Upsilon}) - 2\hat{y}_I u^I , \quad (4.61)
$$

where $\hat{y}_I = y_I + i(\Omega_I - \Omega_{\overline{I}})$ and $\Omega_I = \frac{\partial \Omega}{\partial Y^I}$ and $\Omega_{\overline{I}} = \frac{\partial \Omega}{\partial \overline{Y}^I}$. Up to these modifications, the attractor equations, the entropy function, and the entropy remain as in (4.53), (4.52) and (4.54). Also note from (4.61) that if $\Omega$ is harmonic, it can be absorbed into $\mathrm{Im}F$, because it then is the imaginary

---

[27]We are referring to compactifications with exact T- and S-duality symmetry. These are mostly compactifications with $N = 4$ supersymmetry, which, however, can be studied in the $N = 2$ framework. We refer to [55, 70, 13] for details.

part of holomorphic function. Thus, the non-holomorphic modifications of the prepotentail correspond to non-harmonic functions $\Omega$.

In terms of the complex variables the attractor equation are

$$\begin{pmatrix} Y^I - \overline{Y}^I \\ F_I + 2i\Omega_I - \overline{F}_I + 2i\Omega_{\overline{I}} \end{pmatrix} = i \begin{pmatrix} p^I \\ q_I \end{pmatrix} . \tag{4.62}$$

The modified expressions for the free energy and the entropy function can be found in [13].

At this point it is not quite clear what the $R^2$-corrections and the non-holomorphic corrections mean in terms of special geometry. Since they correspond to higher derivative terms in the Lagrangian, they do not give rise to modifications of the metric on the scalar manifold, which, by definition, is the coefficient of the scalar two-derivative term.[28] It would be very interesting to extend the framework of special geometry such that the functions $F^{(g)}$ get an intrinsic geometrical meaning.

## 4.5   Black hole partition functions and the topological string

Let us now discuss how the black hole variational principle is related to black hole partition functions and the topological string. We start by relating the variational principle described in the last sections to the variational principle used in [12]. One can start from the generalized Hesse potential and perform partial Legendre transforms by imposing only part of the attractor equations. If this subset of fields is properly chosen one obtains a reduced variational principle, which yields the remaining attractor equations, and, by further extremisation, the black hole entropy. Specifically, one can solve the magnetic attractor equations $Y^I - \overline{Y}^I = ip^I$ by setting[29]

$$Y^I = \tfrac{1}{2}(\phi^I + ip^I) . \tag{4.63}$$

Plugging this back, the new, reduced entropy function is

$$\Sigma(p^I, \phi^I, q_I) = \mathcal{F}_E(p^I, \phi^I, \Upsilon, \overline{\Upsilon}) - q_I \phi^I , \tag{4.64}$$

where[30]

$$\mathcal{F}_E(p^I, \phi^I, \Upsilon, \overline{\Upsilon}) = 4 \left( \mathrm{Im}F(Y^I, \Upsilon) + \Omega(Y^I, \overline{Y}^I, \Upsilon, \overline{\Upsilon}) \right)_{\mathrm{mgn}} \tag{4.65}$$

---

[28]See however [72], where such an interpretation was proposed.

[29]Obviously, $\phi^I = 2x^I$. We use $\phi^I$ to be consistent with the notation used in [13]. The conventions of [12] are slightly different.

[30]We suppressed the dependence of $\Sigma$ on $\Upsilon$, but indicated it for $\mathcal{F}_E$ in order to make explicit that we included the higher derivative corrections.

Here the label 'mgn' indicates that the magnetic attractor equations have been imposed, i.e., $Y^I = \frac{1}{2}(\phi^I + ip^I)$. Both $\mathcal{F}(Y^I, \overline{Y}^I, \Upsilon, \overline{\Upsilon}) = 2H(x^I, \hat{y}_I, \Upsilon, \overline{\Upsilon})$ and $\mathcal{F}_E(p^I, \phi^I, \Upsilon, \overline{\Upsilon})$ are interpreted as free energies, which, however, refer to different statistical ensembles. In the microcanonical ensemble the electric and magnetic charges are kept fixed, while they fluctuate around a mean value in the canoncial ensemble. The transition between these two ensemble is made by changing the independent variables, i.e., one eliminates the electric and magnetic charges $q_I, p^I$ in favour of the corresponding chemical potentials, which are the electrostatic and magnetostatic potentials $\phi^I, \chi_I$.[31] By virtue of the equations of motion the potentials coincide, up to a factor, with the real coordinates on $C(M_{VM})$: $\phi^I = 2x^I$, $\chi_I = 2\hat{y}_I$. In black hole thermodynamics the electrostatic and magnetostatic potentials are evaluated at the horizon. Note that both sets of thermodynamical variables correspond to different real symplectic coordinates on $C(M_{VM})$: the charges to the imaginary part, the potentials to the real part of the symplectic vector $(Y^I, F_I)$.

As an intermediate step, one can go to the mixed ensemble, where the magnetic charges are kept fixed, while the electric charges fluctuate. Then the independent variables are $p^I$ and $\phi^I$. This indicates that $\mathcal{F}$ is the free energy with respect to the canonical ensemble, while $\mathcal{F}_E$ is the free energy with respect to the mixed ensemble.

If one imposes that $\Sigma(p^I, \phi^I, q_I)$ is stationary with respect to variations of $\phi^I$, then one obtains the electric attractor equations $(F_I - 2i\Omega_I) - (\overline{F}_I + 2i\Omega_{\overline{I}}) = iq_I$ (4.62). Plugging these back one sees that at the stationary point $\Sigma_{\text{attr}} = \frac{1}{\pi}\mathcal{S}_{\text{macro}}(p^I, q_I)$ and that the macroscopic entropy is the partial Legendre transform of the free energy $\mathcal{F}_E(p^I, \phi^I, \Upsilon, \overline{\Upsilon})$.

Actually, the black hole free energy introduced in [12] includes the contribution from holomorphic higher derivative terms, but not the non-holomorphic corrections. Let us denote this quantity by $\mathcal{F}_{\text{OSV}}(p^I, \phi^I, \Upsilon)$. It is proportional to the imaginary part of the generalized holomorphic prepotential $F(Y^I, \Upsilon)$. If the model under consideration has been obtained by compactification of type-II string theory on a Calabi-Yau threefold, then the prepotential is in turn proportional to the so-called topological free energy $F_{\text{top}}$, which is the logarithm of the all-genus partition function of the topological type-II string, $Z_{\text{top}} = e^{F_{\text{top}}}$. In our conventions the precise relation

---

[31] Since the charges play the roles of particle number in non-relativistic thermodynamics, it might appear more logical to call the 'microcanonical' ensemble canonical, and the 'canonical' ensemble grand canonical. However, we follow the terminology established in the recent literature on the OSV conjecture.

between the free energies is

$$\pi \mathcal{F}_{\text{OSV}} = 4\pi \text{Im} F = 2\text{Re} F_{\text{top}} \ . \tag{4.66}$$

Therefore the free energy $\mathcal{F}_{\text{OSV}}$ is related to the topological partition function by [12]

$$e^{\pi \mathcal{F}_{\text{OSV}}(p,\phi,\Upsilon)} = |Z_{\text{top}}|^2 \ . \tag{4.67}$$

This supports the idea to take the interpretation of $\mathcal{F}_{\text{OSV}}(p, \phi, \Upsilon)$ as the free energy of the black hole seriously. Then it should be related to the partition function of the black hole with respect to the mixed ensemble, which is defined by

$$Z_{\text{mixed}}(p, \phi) = \sum_q d(p, q) e^{q\phi} \ , \tag{4.68}$$

where $d(p, q)$ is the number of BPS microstates with charges $p^I, q_J$, and $q\phi := q_I \phi^I$. This relation is a formal discrete Laplace transform which relates the microscopic partition function, i.e., the state degeneracy, to the mixed partition function. The standard relation between free energy and partition function would imply that $Z_{\text{mixed}} = e^{\pi \mathcal{F}_{\text{OSV}}}$. However, from our discussion of the black hole variational principle and of the role of non-holomorphic corrections it appears to be natural to contemplate including non-holomorphic terms, thus replacing $\mathcal{F}_{\text{OSV}}$ by $\mathcal{F}_E$.[32] Thus we should leave open the option that there are subleading corrections to the relation between the black hole partition function and the topological string partition function. The weak version of the OSV conjecture [12] is:

$$Z_{\text{mixed}}(p, \phi) \approx e^{\pi \mathcal{F}_{\text{OSV}}(p,\phi)} = |Z_{\text{top}}(p, \phi)|^2 \ , \tag{4.69}$$

where $\approx$ means equality in the limit of large charges, which is the semiclassical limit. Evidence for this form of the conjecture will be given below. We will also see that the conjecture needs to be modified as soon as subleading corrections are included.

By a formal Laplace transform we can equivalently formulate this conjecture as a prediction of the state degeneracy in terms of the free energy, by

$$d(p, q) \approx \int d\phi e^{\pi [\mathcal{F}_{\text{OSV}} - q\phi]} \ . \tag{4.70}$$

---

[32]This makes sense microscopically, because the non-holomorphic corrections to the supergravity effective action are related to the holomorphic anomaly of the topological string [56, 57]. The role of the holomorphic anomaly for the OSV conjecture has also been investigated in [71].

Here $d\phi = \prod_I d\phi^I$, and the $\phi^I$ are taken to be complex and integrated along a contour encircling the origin. The relation (4.70) is intriguing, as it relates the black hole microstates directly to the topological string partition function. Note that a saddle point evalation of the integral gives

$$d(p,q) \approx e^{\mathcal{S}_{\mathrm{macro}}(p,q)} \; , \tag{4.71}$$

because at the critical point of the integrand we have $\pi[\mathcal{F}_E - q_I\phi^I]_{\mathrm{attr}} = \mathcal{S}_{\mathrm{macro}}(p,q)$. Thus the microscopic entropy $\mathcal{S}_{\mathrm{micro}}(p,q) = \log d(p,q)$ and the macroscopic entropy $\mathcal{S}_{\mathrm{macro}}(p,q)$ argree to leading order in the semiclassical limit.[33]

There are several problems which indicate that the proposal (4.70) must be modified. The number of states $d(p,q)$ should certainly be invariant under stringy symmetries such as S-duality and T-duality. In the context of compactifications with $N \geq 2$ supersymmetry, where duality symmetries are realized as symplectic transformation, this also means that $d(p,q)$ should be a symplectic function. However, in the approach of [12] the electric and magnetic charges are treated differently, so that there is no manifest symplectic covariance. A related issue is how to take into account non-holomorphic corrections. While [12] is based on the holomorphic function $F(Y^I, \Upsilon)$, it is clear that non-holomorphic terms have to enter one way or another, because they are needed to make $d(p,q)$ duality invariant. A concrete proposal for modifying (4.70) was made in [13]. It is based on the free energy $\mathcal{F} = 2H$, i.e., on the generalized Hesse potential, instead of $\mathcal{F}_{\mathrm{OSV}}$. This allows one to treat electric and magnetic charges on equal footing and to keep symplectic covariance manifest.

The covariant version of (4.69) is

$$e^{\pi\mathcal{F}(\phi,\chi)} = e^{2\pi H(x,\hat{y})} \approx Z_{\mathrm{can}}(\phi,\chi) = \sum_{p,q} d(p,q)e^{\pi(q\phi-p\chi)} \; , \tag{4.72}$$

where $\phi^I = 2x^I$ and $\chi_I = 2\hat{y}_I$ are the electrostatic and the magnetostatic potentials, respectively, and $Z_{\mathrm{can}}(\phi,\chi)$ is the partition function of the black hole with respect to the canonical ensemble. By a formal Laplace transform we can reformulate the conjecture (4.72) as a prediction of the state degeneracy:

$$d(p,q) \approx \int dx d\hat{y} e^{\pi\Sigma(x,\hat{y},p,q)} \; . \tag{4.73}$$

---

[33] $\mathcal{S}_{\mathrm{macro}}$ and $\mathcal{S}_{\mathrm{micro}}$ are expected to be different, once subleading terms are taken into account, because they refer to different statisticle ensembles.

In absence of $R^2$- and non-holomorphic corrections, the measure $dxdy = \prod_{I,J} dx^I dy_J$ is proportional to the top power of the symplectic form $dx^I \wedge dy_I$ on $C(M_{VM})$ and therefore is symplectically invariant. In the presence of $R^2$- and non-holomorphic corrections, $dx d\hat{y}$ is the appropriate generalization. Since $\Sigma$ is a symplectic function, we have found a manifestly symplectically covariant version of (4.70).

As before, the variational principle ensures that in saddle point approximation we have $d(p,q) \approx \exp(\mathcal{S}_{\text{macro}})$, as $\mathcal{S}_{\text{macro}}$ is the Legendre transform of the Hesse potential and hence the saddle point value of $\pi\Sigma$. In order to compare (4.73) to (4.70), we can rewrite (4.73) in terms of the complex variables and perform the integral over $\text{Im}Y^I$ in saddle point approximation, i.e., we perform a Gaussian integration with respect to the subspace where the magnetic attractor equations are satisfied. The result is [13]

$$d(p,q) \approx \int d\phi \sqrt{\Delta^-(p,\phi)} e^{\pi[\mathcal{F}_E - q\phi]} \qquad (4.74)$$

and modifies (4.70) in two ways: first, in contrast to [12] we have included non-holomorphic terms into the free energy $\mathcal{F}_E$; second, the integral contains a measure factor $\Delta^-(p,\phi)$, whose explicit form can be found in [13]. The measure factor is needed in order to be consistent with symplectic covariance.

The proposals (4.70) and (4.73) can be tested by comparing the black hole entropy to the microscopic state degeneracy. There are some cases where these are either known exactly, or where at least subleading contributions are accessible. While this chapter is far from being closed, there seems to be agreement by now that (4.70) needs to be modified by a measure factor [73, 72, 13]. In particular, the measure factors extracted from the evaluation of exact dyonic state degeneracies in $N = 4$ compactifications [74] are consistent, at the semiclassical level, with the proposal (4.73) [13]. Detailed investigations of microscopical $N = 2$ partition functions have clarified the origin of the asymptotic holomorphic factorization of the black hole partition function, $Z_{\text{mixed}} \approx |Z_{\text{top}}|^2$: it results from simultanous contributions of branes and anti-branes to the state degeneracy [75, 76, 77, 78]. Recently, the refined analysis of [79] has identified a microscopic measure factor, which agrees with the one found in [72, 13] in the semiclassical limit.

## Acknowledgments

talk given at the 'Bernardfest' in Utrecht, and I would like to thank the organisers for the opportunity to speak on the occasion of Bernard's anniversary. My special thanks goes to Vicente Cortés for inviting me to contribute this article to the Handbook on Pseudo-Riemannian Geometry and Supersymmetry. Furthermore, I would like to thank the referee for suggesting to make this article accessible to a larger readership by including an introduction to black holes in general and to supersymmetric black hole solutions in particular. This material has been incluced as a separate section (Section 3). The extended version of this article was written during a stay as Senior Research Fellow at the Erwin Schrödinger for Mathematical Physics in Vienna.

# References

[1] B. de Wit and A. Van Proeyen, Nucl. Phys. B 245 (1984) 89.

[2] N. Seiberg and E. Witten, Nucl. Phys. B 426 (1994) 19, hep-th/9407087.

[3] S. Kachru and C. Vafa, Nucl. Phys. B 450 (1995) 69, hep-th/9505105.

[4] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, Phys. Lett. B 361 (1995) 59, hep-th/9505162.

[5] V. Cortés, C. Mayer, T. Mohaupt and F. Saueressig, JHEP 03 (2004) 028, hep-th/0312001.

[6] V. Cortés, C. Mayer, T. Mohaupt and F. Saueressig, JHEP 06 (2005) 025, hep-th/0503094.

[7] S. Cecotti, S. Ferrara and L. Girardello, Int. J. Mod. Phys. A 4 (1989) 2475.

[8] S. Ferrara and S. Sabharwal, Nucl. Phys. B 332 (1990) 317.

[9] A. Strominger and C. Vafa, Phys. Lett. B 379 (1996) 99, hep-th/9601029.

[10] J. Wess and J. Bagger, Supersymmetry and Supergravity, Princeton UP, 1992.

[11] S. Ferrara, R. Kallosh and A. Strominger, Phys. Rev. D 52 (1995) 5412, hep-th/9508072. A. Strominger, Phys. Lett. B 383 (1996) 39, hep-th/9602111. S. Ferrara and R. Kallosh, Phys. Rev. D 54 (1996) 1514, hep-th/9602136. S. Ferrara and R. Kallosh, Phys. Rev. D 54 (1996) 1525, hep-th/9603090.

[12] H. Ooguri, A. Strominger and C. Vafa, Phys. Rev. D 70 (2004) 106007, hep-th/0405146.

[13] G. L. Cardoso, B. de Wit, J. Käppeli and T. Mohaupt, hep-th/0601108.

[14] K. Behrndt, G. L. Cardoso, B. de Wit, R. Kallosh, D. Lüst and T. Mohaupt, Nucl. Phys. B 488 (1997) 236, hep-th/9610105.

[15] G. L. Cardoso, B. de Wit and T. Mohaupt, Phys. Lett. B 451 (1999) 309, hep-th/9812082.

[16] G. L. Cardoso, B. de Wit, J. Käppeli and T. Mohaupt, JHEP 0012 (2000) 019, hep-th/0009234.

[17] K. Behrndt, I. Gaida, D. Lüst, S. Mahapatra and T. Mohaupt, Nucl. Phys. B 508 (1997) 659, hep-th/9706096.

[18] G. Neugebauer and D. Kramer, Ann. der Physik (Leipzig) 24 (1969) 62.

[19] P. Breitenlohner, D. Maison and G. W. Gibbons, Commun. Math. Phys. 120 (1988) 295.

[20] G. Clement and D. V. Gal'tsov, Phys. Rev. D 54 (1996) 6136, hep-th/9607043.
D. V. Gal'tsov and O. A. Rytschkov, Phys. Rev. D 58 (1998) 122001, hep-th/9801160.

[21] K. S. Stelle, hep-th/9803116.

[22] M. Gunaydin, A. Neitzke, B. Pioline and A. Waldron, hep-th/0512296.

[23] M. Rocek, C. Vafa and S. Vandoren, math.DG/0603048.

[24] M. Rocek, C. Vafa and S. Vandoren, JHEP 02 (2006) 062, hep-th/0512206.

[25] S. Ivanov, I. Minchev and S. Zamkovoy, math.DG/0511657.

[26] T. Mohaupt, Fortsch. Phys. 49 (2001) 3, hep-th/0007195.

[27] D. S. Freed, Commun. Math. Phys. 203 (1999) 31, hep-th/9712042.

[28] N. J. Hitchin, Asian. J. Math. 3 (1999) 77, math.dg/9901069.

[29] D. V. Alekseevsky, V. Cortés and C. Devchand, J. Geom. Phys. 42 (2002) 85, math.dg/9910091.

[30] G. W. Gibbons, M. B. Green and M. J. Perry, Phys. Lett. B 370 (1996) 37, hep-th/9511080.

[31] B. de Wit, P. G. Lauwers and A. Van Proeyen, Nucl. Phys. B 255 (1985) 569.

[32] J. De Jaegher, B. de Wit, B. Kleijn and S. Vandoren, Nucl. Phys. B 514 (1998) 553, hep-th/9707262.

[33] R. Wald, General Relativity, University of Chicago Press, 1984.

[34] R. Wald, Quantum Field Theory and Curved Spacetime and Black Hole Thermodynamics, University of Chicago Press, 1994.

[35] V. Frolov and I. Novikov, Black Hole Physcis: Basic Concepts and New Developments, Springer, 1998.

[36] J. M. Bardeen, B. Carter and S. W. Hawking, Comm. Math. Phys. 31 (1973) 161.

[37] L. Racz and R. Wald, Class. Quant. Grav. 9 (1992) 2643. L. Racz and R. Wald, gr-qc/9507055.

[38] S. W. Hawking, Comm. Math. Phys. 43 (1974) 199.

[39] T. Jacobson, G. Kang and R. Myers, Phys. Rev. D 49 (1994) 6587, gr-qc/9312023.

[40] V. Iyer and R. Wald, Phys. Rev. D 50 (1994) 846, gr-qc/9403028.

[41] J.D. Bekenstein, Phys. Rev. D7 (1973) 2333.

[42] L. Susskind, Some speculations about black hole entropy in string theory, in: C. Teitelboim (ed.), The black hole, p. 118, hep-th/9309145.
G. T. Horowitz and J. Polchinski, Phys. Rev. D 55 (1997) 6189, hep-th/9612146.

[43] T. Ortin, Gravity and Strings, Cambridge University Press, 2004.

[44] V. Cortés, A holomorphic representation formula for parabolic hyperspheres, in: B. Opozda, U. Simon and M. Wiehe (eds.), PDEs, Submanifolds and Affine Differential Geometry, Banach Center Publications, Warsaw (2000), math.dg/0107037.

[45] B. de Wit, J. van Holten and A Van Proyen, Nucl. Phys. B 167 (1980) 186. (E.: ibid. B 172 (1980) 543).

[46] B. de Wit, P. G Lauwers and A. van Proeyen, Nucl. Phys. B 184 (1981) 77. (E.: ibid B 222 (1983) 516).

[47] E. Cremmer, C. Kounnas, A. Van Proeyen, J. P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, Nucl. Phys. B 250 (1985) 385.

[48] G. W. Gibbons, *Supersymmetric Soliton States in Extended Supergravity Theories*. In P. Breitenlohner and H. P Dürr (eds.), Unified Theories of Elementary Particles, Springer, 1982.

[49] K. Behrndt, D. Lüst and W. A. Sabra, Nucl. Phys B 510 (1998) 264, hep-th/9705169.

[50] P. Meessen and T. Ortin, hep-th/0603099.

[51] G. Moore, hep-th/9807087.

[52] N. Hitchin, math.dg/0010054.

[53] V. Pestun and E. Witten, Lett. Math. Phys. 74 (2005) 21, hep-th/0503083.

[54] V. Pestun, hep-th/0512189.

[55] G. L. Cardoso, B. de Wit and T. Mohaupt, Nucl. Phys. B 567 (2000) 87, hep-th/9906094.

[56] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Comm. Math. Phys. 165 (1993) 311, hep-th/9309140.

[57] I. Antoniadis, E. Gava, N. S. Narain and T. R. Taylor, Nucl. Phys. B 413 (1994) 162, hep-th/9307158.

[58] A. Strominger, Commun. Math. Phys. 133 (1990) 163.

[59] V. Cortés, Trans. Amer. Math. Soc. 350 (1998) 19.

[60] G. W. Gibbons and C. M. Hull, Phys. Lett. B 106 (1982) 190.

[61] K. P. Tod, Phys. Lett. B 121 (1983) 241.

[62] J. Kowalski-Glikman, Phys. Lett. B 150 (1985) 125.

[63] J.P. Gauntlett, J.B. Gutowski, C. Hull, S. Pakis and H.S. Reall, Class. Quant. Grav. 20 (2003) 4587, hep-th/0209114.

[64] S.D. Majumdar, Phys. Rev. 72 (1947) 390.

[65] A. Papapetrou, Proc. Roy. Irish Acad. A51 (1947) 191.

[66] S. Chandrasekhar, The Mathematical Theory of Black Holes, Oxford Science Publishers, 1992.

[67] C. Teitelboim, Phys. Lett. B 69 (1977) 240.

[68] B. de Wit, Nucl. Phys. Proc. Suppl. 49 (1996) 191, hep-th/9602060.

[69] R. Wald, Phys. Rev. D 48 (1993) 3427, gr-qc/9307038.

[70] G. L. Cardoso, B. de Wit, J. Käppeli and T. Mohaupt, JHEP 12 (2004) 075, hep-th/0412287.

[71] E. Verlinde, hep-th/0412139.

[72] D. Shih and X. Yin, hep-th/0508174.

[73] A. Dabholkar, F. Denef, G. W Moore and B. Pioline, JHEP 08 (2005) 021, hep-th/0502157, JHEP 10 (2005) 096, hep-th/0507014.

[74] D. P. Jatkar and A. Sen, JHEP 04 (2006) 018, hep-th/0510147.

[75] D. Gaiotto, A. Strominger and X. Yin, hep-th/0602046.

[76] P. Kraus and F. Larsen, hep-th/0607138.

[77] C. Beasley, D. Gaiotto, M. Gucia, L. Huang, A. Strominger and X. Yin, hep-th/0608021.

[78] J. de Boer, M.C.N. Cheng, R. Dijkgraaf, J. Manshot and E. Verlinde, JHEP 11 (2006) 024, hep-th/0608059.

[79] F. Denef and G. Moore, hep-th/0702146.