ITINERARIES FOR INVERSE LIMITS OF TENT MAPS: A BACKWARD VIEW

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Abstract. Previously published admissibility conditions for an element of $\{0,1\}^{\mathbb{Z}}$ to be the itinerary of a point of the inverse limit of a tent map are expressed in terms of forward orbits. We give necessary and sufficient conditions in terms of backward orbits, which is more natural for inverse limits. These backward admissibility conditions are not symmetric versions of the forward ones: in particular, the maximum backward itinerary which can be realised by a tent map mode locks on intervals of kneading sequences.

1. Introduction

Inverse limits of tent maps have been much investigated, not only because of their intrinsic interest as topological spaces, but also because they are closely related to other topics in dynamical systems such as hyperbolic attractors and Hénon maps. A recent highlight is the proof by Barge, Bruin, and Štimac of the Ingram Conjecture [2], which states that the inverse limits of distinct tent maps are non-homeomorphic.

Kneading theory is widely used in the study of the dynamics of unimodal maps, and has been extended to and applied in the context of inverse limits of tent maps by several authors (e.g. [3, 4]). A key starting point for the application of such symbolic techniques is understanding the *admissibility conditions* under which a sequence of symbols is realised as the itinerary of a point of the inverse limit. In previous works, such admissibility conditions have been adapted from those for kneading theory of unimodal maps, and as such are based on the forward itineraries of points. This is somewhat unnatural in the context of inverse limits, where the main focus is on backward orbits.

In this paper we develop admissibility conditions for inverse limits which are based on backward itineraries. One might naïvely expect these conditions to be symmetric versions of the forward ones but, with the exception of certain special cases (tent maps of *irrational* or *rational endpoint* type), this is not the case. The essential content of the forward conditions is that every forward sequence must be less than or equal to the kneading sequence of the tent map f, in the unimodal order. For the backward conditions, the kneading sequence is replaced by two sequences, so that backward sequences are bounded by a stepped line. Moreover, these two sequences mode-lock on intervals in parameter space — what changes as the parameter varies within such an interval is the location of the step between the two sequences.

In Section 2 we review the forward admissibility conditions. This theory is well established, but we make some minor modifications which enable us to give admissibility conditions which are strictly necessary and sufficient (Lemmas 3 and 5), which seem not to have appeared before. The basis of the backward admissibility conditions is the stratification of the space of unimodal maps by *height*, a

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number in $[0, 1/2]$ which is associated to each unimodal map [8]. This theory is reviewed in Section 3.1, before the main results are stated and proved. Theorem 14 gives necessary and sufficient backward admissibility conditions in the symmetric case; Theorem 16 is the analogous result in the non-symmetric case; and Theorem 17 provides a striking illustration of the asymmetry of forward and backward itineraries: the maximum backward itinerary which can be realised by a tent map mode locks on intervals of kneading sequences.

2. Forward admissibility

2.1. **Basic definitions.** Throughout the paper, $I = [a, b]$ is a compact interval and $f\colon I \to I$ is a tent map of slope $\lambda \in (\sqrt{2}, 2)$: that is, there is some $c \in (a, b)$ such that f has constant slope λ on $[a, c]$ and constant slope $-\lambda$ on $[c, b]$. Moreover, we assume that I is the dynamical interval (or core) of f, so that $f(c) = b$ and $f(b) = a$.

Let $\{0, 1\}^{\mathbb{N}}$ and $\{0, 1\}^{\mathbb{Z}}$ denote the spaces of semi-infinite and bi-infinite sequences over $\{0, 1\}$, with their natural product topologies. We denote elements of the former with lower-case letters, and of the latter with upper-case letters. We write $\sigma\colon \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ and $\sigma\colon \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ for the corresponding shift maps. If $S \in \{0, 1\}^{\mathbb{Z}}$, we denote by $\overrightarrow{S}$ and $\overleftarrow{S}$ the elements of $\{0, 1\}^{\mathbb{N}}$ defined by $\overrightarrow{S}_r = S_r$ and $\overleftarrow{S}_r = S_{-1-r}$ for $r \geq 0$: therefore $\overrightarrow{\sigma^r(S)} = S_r S_{r+1} \dots$ and $\overleftarrow{\sigma^r(S)} = S_{r-1} S_{r-2} \dots$ for each $r \in \mathbb{Z}$. We say that S *does not end* 0^∞ (respectively *does not start* 0^∞) if infinitely many of the entries of $\overrightarrow{S}$ (respectively $\overleftarrow{S}$) are 1.

If $n \geq 1$ then a *word of length* n is an element of $\{0, 1\}^n$. We say that a word W is *even* (respectively *odd*) if it contains an even (respectively odd) number of 1s. If $s \in \{0, 1\}^{\mathbb{N}}$ and W is a word of length n, then we write $s = W \dots$ to mean that $s_i = W_i$ for $0 \leq i \leq n - 1$.

We denote by $\preceq$ the *unimodal order* on $\{0, 1\}^{\mathbb{N}}$ (also known as the *parity lexicographical order*), which is defined as follows: if s and t are distinct elements of $\{0, 1\}^{\mathbb{N}}$, then $s \prec t$ if and only if the word $s_0 \dots s_r$ is even, where $r \geq 0$ is least with $s_r \neq t_r$. An element s of $\{0, 1\}^{\mathbb{N}}$ is said to be *shift-maximal* if $\sigma^r(s) \preceq s$ for all $r \geq 0$.

There are several different approaches to assigning itineraries in $\{0, 1\}^{\mathbb{N}}$ to points of I under the action of f. Those differences which are not cosmetic are concerned with the straightforward but vexed question of how to code the critical point c, and therefore affect the itineraries of only countably many points. One may introduce a third symbol C; make an arbitrary choice of 0 or 1 as the code of the critical point; allow either of these symbols, leading to multiple itineraries for certain points – an approach whose ramifications are compounded when the critical point is periodic; or take limits à la Milnor-Thurston [9]. The approach which we adopt here is to code c with a choice of 0 or 1 which depends on f in the case where c is periodic; and to allow either code for c if c is not periodic. This convention, as well as being ideal for our results, has the added benefit – quite independent of the main results of the paper – of leading to admissibility conditions for itineraries which are strictly necessary and sufficient (Lemma 3), at least in the case of tent maps or other unimodal maps which admit no homtervals (i.e. for which distinct points have distinct itineraries).

Suppose first that c is a periodic point of f, of period n, and define $\varepsilon(f) = 0$ (respectively $\varepsilon(f) = 1$) if an even (respectively odd) number of the points $\{f^r(c) : 1 \leq r < n\}$ lie in $(c, b]$. Then define the *itinerary* $j(x) \in \{0, 1\}^{\mathbb{N}}$ of $x \in I$ by

$$j(x)_r = \begin{cases} 0 & \text{if } f^r(x) \in [a, c), \\ 1 & \text{if } f^r(x) \in (c, b], \\ \varepsilon(f) & \text{if } f^r(x) = c \end{cases} \qquad \text{for each } r \in \mathbb{N}.$$

Define the *kneading sequence* $\kappa(f) \in \{0,1\}^{\mathbb{N}}$ of f by $\kappa(f) = j(b)$. By construction, $\kappa(f) = (W\varepsilon(f))^\infty$, where $W\varepsilon(f)$ is an even word of length n.

In the case where c is not a periodic point of f, we say that $s \in \{0,1\}^{\mathbb{N}}$ *is an itinerary of* $x \in I$ if $f^r(x) \in [a,c]$ whenever $s_r = 0$, and $f^r(x) \in [c,b]$ whenever $s_r = 1$. Therefore each $x \in I$ has a unique itinerary unless $c \in \operatorname{orb}(x,f) = \{f^r(x) : r \geq 0\}$, in which case it has exactly two itineraries.

Define the kneading sequence $\kappa(f) \in \{0,1\}^{\mathbb{N}}$ of f to be the itinerary of b (which is unique since $b = f(c)$ and c is not periodic). Therefore if $f^r(x) = c$ for some $r \geq 0$, then the two itineraries of x are $s_0 \ldots s_{r-1} {}^0_1 \kappa(f)$ for some $s_0, \ldots, s_{r-1} \in \{0,1\}$.

Remark 1. If s is an itinerary for $x \in I$, then $\sigma^r(s)$ is an itinerary for $f^r(x)$ for each $r \geq 0$, regardless of whether or not c is a periodic point. It is standard (see for example [5, 7]) that the unimodal order on itineraries reflects the usual order on the interval I. Since f is uniformly expanding on each of its two branches, distinct points $x, y \in I$ cannot share a common itinerary. If s and t are itineraries of x and y, we therefore have that $x < y \implies s \prec t$; while if $s \prec t$, then either $x < y$, or s and t are the two itineraries of $x = y$ in the case where c is not periodic.

Let
$$j_f = \{s \in \{0,1\}^{\mathbb{N}} : s \text{ is an itinerary of some } x \in I\}.$$
An element of j_f is said to be *admissible (for f)*.

The *inverse limit* $\widehat{I}$ of $f \colon I \to I$ is defined by
$$\widehat{I} = \{\mathbf{x} \in I^{\mathbb{Z}} : f(x_r) = x_{r+1} \text{ for all } r \in \mathbb{Z}\},$$
topologized as a subspace of the product $I^{\mathbb{Z}}$. This definition differs from the standard one, in which only indices $r \leq 0$ are considered, but is homeomorphic to it, since x_0 determines x_r for all $r > 0$, and is more convenient for our purposes. Let $\widehat{f} \colon \widehat{I} \to \widehat{I}$ be the shift map defined by $\widehat{f}(\mathbf{x})_r = x_{r+1}$ for all r, a homeomorphism which is called the *natural extension* of f. The projection $\pi_0 \colon \widehat{I} \to I$ defined by $\mathbf{x} \mapsto x_0$ is a semi-conjugacy from $\widehat{f}$ to f.

We define itineraries of elements of $\widehat{I}$, lying in $\{0,1\}^{\mathbb{Z}}$, in the same way as itineraries of points of I under f: they provide symbolic representations of the points of $\widehat{I}$ which are not directly related to the dynamics of $\widehat{f}$. Thus if c is a periodic point of f, then each $\mathbf{x} \in \widehat{I}$ has a unique itinerary $J(\mathbf{x})$ defined by
$$J(\mathbf{x})_r = \begin{cases} 0 & \text{if } x_r \in [a,c), \\ 1 & \text{if } x_r \in (c,b], \\ \varepsilon(f) & \text{if } x_r = c \end{cases} \qquad \text{for each } r \in \mathbb{Z}.$$
On the other hand, if c is not a periodic point of f, then we say that $S \in \{0,1\}^{\mathbb{Z}}$ is an itinerary of $\mathbf{x} \in \widehat{I}$ if $x_r \in [a,c]$ whenever $S_r = 0$, and $x_r \in [c,b]$ whenever $S_r = 1$. Therefore $\mathbf{x}$ has a unique itinerary if $x_r \neq c$ for all r; and has exactly two itineraries if $x_r = c$ for some r, which are $\ldots S_{r-2}\, S_{r-1} {}^0_1 \kappa(f)$. Note that if S is an itinerary for $\mathbf{x} \in \widehat{I}$ and $r \in \mathbb{Z}$, then $\overrightarrow{\sigma^r(S)}$ is an itinerary for $x_r \in I$ under f.

Let
$$J_f = \{S \in \{0,1\}^{\mathbb{Z}} : S \text{ is an itinerary of some } \mathbf{x} \in \widehat{I}\}.$$
An element of J_f is said to be *admissible (for $\widehat{I}$)*. If S is admissible, then it is the itinerary of only one $\mathbf{x} \in \widehat{I}$, since each x_r is determined by its itinerary $\overrightarrow{\sigma^r(S)}$. The map $g \colon J_f \to \widehat{I}$ which sends each itinerary to the corresponding element of $\widehat{I}$ is a semiconjugacy (at most two-to-one) between the

subshift $\sigma\colon J_f \to J_f$ and the natural extension $\widehat{f}\colon \widehat{I} \to \widehat{I}$ of f. For this reason we refer to $\sigma\colon J_f \to J_f$ as the *symbolic natural extension* of f.

Remark 2. The condition that $\lambda > \sqrt{2}$ is equivalent to the tent map f's not being renormalizable; which is equivalent in turn to the condition $\kappa(f) \succ 101^\infty$.

The condition that $\lambda < 2$ is equivalent to $\kappa(f) \prec 10^\infty$. We exclude the case $\lambda = 2$ to avoid having to treat it separately in lemma and theorem statements: since every element of $\{0,1\}^{\mathbb{N}}$ (respectively $\{0,1\}^{\mathbb{Z}}$) is admissible for f (respectively $\widehat{I}$) when $\lambda = 2$, there is no loss in so doing.

2.2. **Admissibility conditions.** The following result, which gives conditions under which an element of $\{0,1\}^{\mathbb{N}}$ is admissible for f, is well known. We nevertheless provide a proof (following those of [5] and [7]), since it is a key result in the paper and our definition of itineraries is slightly non-standard.

Lemma 3 (Admissibility conditions for f)**.** *Write $\kappa(f) = \kappa$. Let $s \in \{0,1\}^{\mathbb{N}}$. Then $s \in j_f$ if and only if the following three conditions hold:*

(a) $\sigma^r(s) \preceq \kappa$ for all $r \geq 0$;
(b) $\sigma(\kappa) \preceq s$; and
(c) if c is periodic and $\sigma^r(s) = \kappa$ for some $r > 0$, then $s_{r-1} = \varepsilon(f)$.

Proof. Let s be an itinerary of $x \in I$. Since a and b have unique itineraries $\sigma(\kappa)$ and κ, s must satisfy (a) and (b) by Remark 1. Moreover, if c is periodic and $r > 0$, then $\sigma^r(s) = \kappa \implies f^r(x) = b \implies f^{r-1}(x) = c \implies s_{r-1} = \varepsilon(f)$, so that (c) holds too.

For the converse, let $s \in \{0,1\}^{\mathbb{N}}$ satisfy (a), (b), and (c). We shall show that s is an itinerary of some $x \in I$. We can suppose that $\sigma(\kappa) \prec s \prec \kappa$, since otherwise s is the itinerary of either a or b.

Suppose first that c is not periodic. Define

$$\begin{aligned} L &= \{x \in I : \text{ all itineraries } t \text{ of } x \text{ have } t \prec s\}, \\ R &= \{x \in I : \text{ all itineraries } t \text{ of } x \text{ have } t \succ s\}. \end{aligned}$$

Then $a \in L$ and $b \in R$. We shall show that L and R are open in I, so that there is some $x \notin L \cup R$. It is impossible for such a point x to have one itinerary smaller than s and one larger than s, since if s lies strictly between $t_0 \dots t_{r-1}0\kappa$ and $t_0 \dots t_{r-1}1\kappa$ then $\sigma^{r+1}(s) \succ \kappa$, contradicting (a). Therefore s is an itinerary of x, as required.

Let $x \in L$. We need to show that if $y > x$ is sufficiently close to x, then any itinerary of y is smaller than s. If $c \notin \operatorname{orb}(x, f)$ then this is obvious. If $c = f^r(x)$ for some (unique) $r \geq 0$ then x has itineraries $t_0 \dots t_{r-1}{}^0_1\kappa$. We can suppose that $s_i = t_i$ for $0 \leq i \leq r-1$, since otherwise the result is obvious. If $t_0 \dots t_{r-1}$ is even then, since both of the itineraries of x are smaller than s, we have $s = t_0 \dots t_{r-1}1u$ for some $u \in \{0,1\}^{\mathbb{N}}$ with $u \prec \kappa$. Let $j \geq 0$ be least with $u_j \neq \kappa_j$. Pick $z > x$ sufficiently close to x that if $y \in (x, z)$ then $f^i(y) \neq c$ for $0 \leq i \leq r + j + 2$. Then any $y \in (x, z)$ has all itineraries of the form $t_0 \dots t_{r-1}1\kappa_0 \dots \kappa_j \dots$, and the result follows. The argument is analogous if $t_0 \dots t_{r-1}$ is odd.

The proof that R is open in I is similar.

Now suppose that c is periodic of period n. Write $\varepsilon = \varepsilon(f)$ and $\overline{\varepsilon} = 1 - \varepsilon(f)$. Let $W \in \{0,1\}^{n-1}$ be such that $\kappa = (W\varepsilon)^\infty$. Define

$$\begin{aligned} L &= \{x \in I : j(x) \prec s\}, \\ R &= \{x \in I : j(x) \succ s\}. \end{aligned}$$

Since $a \in L$ and $b \in R$, it suffices to show that L and R are open in I.

Let $x \in L$. We need to show that, for all $y > x$ sufficiently close to x, we have $j(y) \prec s$. If $c \notin \operatorname{orb}(x, f)$ then this is obvious, so we suppose that there is some least $r \geq 0$ with $f^r(x) = c$. Therefore

$$t := j(x) = t_0 \, \ldots \, t_{r-1} \, \varepsilon \, (W\varepsilon)^\infty.$$

We can suppose that $s_i = t_i$ for $0 \leq i \leq r-1$, since otherwise the result is obvious. We distinguish two cases:

Case A: $t_0 \, \ldots \, t_{r-1} \, \varepsilon$ is an even word. Since $t \prec s$ and $\sigma^{r+1}(s) \preceq (W\varepsilon)^\infty$ by (a) we have that $s = t_0 \, \ldots \, t_{r-1} \, \overline{\varepsilon} \, u$ for some $u \in \{0,1\}^{\mathbb{N}}$, which satisfies $u \prec (W\varepsilon)^\infty$ by (c). Write $u = (W\varepsilon)^k \, v$ for $k \geq 0$ as large as possible, so that $v \in \{0,1\}^{\mathbb{N}}$ satisfies $v \prec (W\varepsilon)^\infty$ and does not have $W\varepsilon$ as an initial subword. In particular, $v \prec w$ for any $w \in \{0,1\}^{\mathbb{N}}$ which does start with $W\varepsilon$. We have $s = t_0 \, \ldots \, t_{r-1} \, \overline{\varepsilon} \, (W\varepsilon)^k \, v$.

Pick $z > x$ sufficiently close to x that if $y \in (x, z)$ then $f^i(y) \neq c$ for $0 \leq i \leq (k+1)n + r + 1$. Then $j(y) = t_0 \, \ldots \, t_{r-1} \, \overline{\varepsilon} \, (W\varepsilon)^{k+1} \, \ldots$ for all $y \in (x, z)$. This is because $f^r(y)$ is less than c (respectively greater than c) if $t_0 \, \ldots \, t_{r-1}$ is odd (respectively even); and $f^{r+1}(y) < b$. Since $t_0 \, \ldots \, t_{r-1} \, \overline{\varepsilon} \, (W\varepsilon)^k$ is an odd word, it follows that $j(y) \prec s$ for all such y, as required.

Case B: $t_0 \, \ldots \, t_{r-1} \, \varepsilon$ is an odd word. Since $t \prec s$ we have $s = t_0 \, \ldots \, t_{r-1} \, \varepsilon \, u$ for some $u \in \{0,1\}^{\mathbb{N}}$ with $u \prec (W\varepsilon)^\infty$ (this last statement again since $t \prec s$). Write $u = (W\varepsilon)^k \, v$ for $k \geq 0$ as large as possible: then $s = t_0 \, \ldots \, t_{r-1} \, \varepsilon \, (W\varepsilon)^k \, v$, where $v \prec (W\varepsilon)^\infty$ does not have $W\varepsilon$ as an initial subword.

As in Case A, if $y > x$ is close enough to x then $j(y) = t_0 \, \ldots \, t_{r-1} \, \varepsilon \, (W\varepsilon)^{k+1} \ldots$. This is because $f^r(y)$ is less than c (respectively greater than c) if $t_0 \, \ldots \, t_{r-1}$ is odd (respectively even). Since $t_0 \, \ldots \, t_{r-1} \, \varepsilon \, (W\varepsilon)^k$ is an odd word, the result follows.

The proof that R is open in I is analogous: in this case, the argument when $t_0 \, \ldots \, t_{r-1} \, \varepsilon$ is even is similar to case B, while the argument when $t_0 \, \ldots \, t_{r-1} \, \varepsilon$ is odd is similar to case A. $\square$

The following straightforward lemma enables us to convert these conditions into admissibility conditions for $\widehat{I}$.

Lemma 4. *Let $S \in \{0,1\}^{\mathbb{Z}}$. Then $S \in J_f$ if and only if $\overrightarrow{\sigma^r(S)} \in j_f$ for all $r \in \mathbb{Z}$.*

Proof. If $S \in J_f$, then S is an itinerary of some $\mathbf{x} \in \widehat{I}$. For each $r \in \mathbb{Z}$, $\overrightarrow{\sigma^r(S)}$ is an itinerary of $x_r \in I$, and hence lies in j_f.

Conversely, suppose that $\overrightarrow{\sigma^r(S)} \in j_f$ for all $r \in \mathbb{Z}$. Then for each $r \in \mathbb{Z}$ there is a unique $x_r \in I$ which has $\overrightarrow{\sigma^r(S)}$ as an itinerary. Now if $s \in \{0,1\}^{\mathbb{N}}$ is an itinerary for $x \in I$, then $\sigma(s)$ is an itinerary for $f(x)$: hence $f(x_r) = x_{r+1}$ for each r. Therefore $\mathbf{x} = (x_r)$ is an element of $\widehat{I}$ with itinerary S. $\square$

Lemma 5 (Forward admissibility conditions for $\widehat{I}$)**.** *Write $\kappa(f) = \kappa$. Let $S \in \{0,1\}^{\mathbb{Z}}$. Then $S \in J_f$ if and only if the following three conditions hold:*

(A) $\overrightarrow{\sigma^r(S)} \preceq \kappa$ for all $r \in \mathbb{Z}$;
(B) S does not start 0^∞; and
(C) If c is periodic and $\overrightarrow{\sigma^r(S)} = \kappa$ for some $r \in \mathbb{Z}$, then $S_{r-1} = \varepsilon(f)$.

Proof. Suppose that conditions (A), (B), and (C) hold. Let $r \in \mathbb{Z}$. By Lemma 4, we need to show that $\overrightarrow{\sigma^r(S)}$ satisfies the conditions of Lemma 3. Conditions (a) and (c) of Lemma 3 are immediate from (A) and (C). For (b), suppose for a contradiction that $\overrightarrow{\sigma^r(S)} \prec \sigma(\kappa)$. By (B), there is some greatest $i < r$ with $S_i = 1$. Then $\overrightarrow{\sigma^i(S)} = 10^{r-i-1}\overrightarrow{\sigma^r(S)} \succ \kappa$, which contradicts (A).

For the converse, suppose that $\overrightarrow{\sigma^r(S)}$ satisfies the conditions of Lemma 3 for each $r \in \mathbb{Z}$. We need to show that S satisfies conditions (A), (B), and (C). Conditions (A) and (C) are immediate from (a) and (c) of Lemma 3. For (B), suppose for a contradiction that there is some $R \in \mathbb{Z}$ such that $S_r = 0$ for all $r \le R$. Since $\kappa \prec 10^\infty$, there is some $k > 0$ with $\kappa = 10^k1\dots$. Then $\overrightarrow{\sigma^{R-k}(S)} = 0^{k+1}\dots \prec \sigma(\kappa)$, contradicting (b) of Lemma 3. $\square$

3. Backward admissibility

3.1. **Height.** In order to establish backward admissibility conditions we will use the *height function* $q\colon \{0,1\}^{\mathbb{N}} \to [0,1/2]$ introduced in [8]. Here we recall the definition of this function, and state those of its properties which we will use.

Convention 6. All rationals m/n will be assumed to be written in lowest terms.

Let $q \in (0,1/2]$. We associate to q a sequence $(k_i(q))_{i\ge1}$ in $\mathbb{N}$ as follows. Let L_q be the straight line $y = qx$ in $\mathbb{R}^2$. For each $i \ge 1$, define $k_i(q)$ to be two less than the number of vertical lines $x =$ integer which L_q intersects for $y \in [i-1, i]$.

If $q = m/n$ is rational, then define the word $c_q \in \{0,1\}^{n+1}$ by

$$c_q = 10^{k_1(q)}110^{k_2(q)}11\dots110^{k_m(q)}1.$$

On the other hand, if q is irrational, then let $c_q = 10^{k_1(q)}110^{k_2(q)}11\dots \in \{0,1\}^{\mathbb{N}}$. (These sequences c_q are closely related to Sturmian sequences of slope q, which can also be defined as cutting sequences of L_q, see for example [1].)

Example 7. Figure 1 shows the line $L_{5/17}$ for $x \in [0,17]$. The numbers of intersections with vertical coordinate lines for $y \in [i-1,i]$ are 4, 3, 4, 3, and 4 for $i = 1$, $i = 2$, $i = 3$, $i = 4$, and $i = 5$ respectively. Hence $k_1(5/17) = k_3(5/17) = k_5(5/17) = 2$, while $k_2(5/17) = k_4(5/17) = 1$. Therefore $c_{5/17} = 100110110011011001$, a word of length 18.

More generally, if $q = m/n$ then the word c_q is evidently palindromic, and contains $n-2m+1$ zeroes divided 'as even-handedly as possible' into m (possibly empty) subwords, separated by 11. For example, for each $n \ge 2$ we have $c_{1/n} = 10^{n-1}1$; $c_{2/(2n+1)} = 10^{n-1}110^{n-1}1$; $c_{3/(3n+1)} = 10^{n-1}110^{n-2}110^{n-1}1$; and $c_{3/(3n+2)} = 10^{n-1}110^{n-1}110^{n-1}1$.

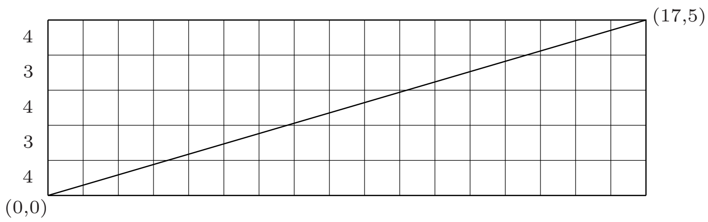

Figure 1. $c_{5/17} = 100110110011011001$

The following statement, which is Lemma 2.7 of [8], is essential for the definition of height.

Lemma 8. *The function* $(0,1/2] \cap \mathbb{Q} \to \{0,1\}^{\mathbb{N}}$ *defined by* $q \mapsto (c_q0)^\infty$ *is strictly decreasing with respect to the unimodal order on* $\{0,1\}^{\mathbb{N}}$. $\square$

We now define the *height* $q(s) \in [0, 1/2]$ of $s \in \{0,1\}^{\mathbb{N}}$ by

$$q(s) = \inf\left(\{q \in (0, 1/2] \cap \mathbb{Q} \,:\, (c_q 0)^\infty \prec s\} \cup \{1/2\}\right).$$

By Lemma 8, the height function $q\colon \{0,1\}^{\mathbb{N}} \to [0,1/2]$ is decreasing with respect to the unimodal order on $\{0,1\}^{\mathbb{N}}$ and the usual order on $[0,1/2]$.

In order to state the properties of height which we require, we need some additional notation. For each rational $q = m/n \in (0, 1/2)$, let $w_q \in \{0,1\}^{n-1}$ be defined by $(w_q)_i = (c_q)_i$ for $0 \le i \le n-2$; and let $\widehat{w}_q$ be the reverse of w_q, so that $(\widehat{w}_q)_i = (w_q)_{n-2-i}$ for $0 \le i \le n-2$. We therefore have $c_q = w_q 01$ for each q. Since c_q is an even word, w_q and $\widehat{w}_q$ are odd words.

Define $\mathrm{lhe}(q)$ and $\mathrm{rhe}(q)$ to be the shift-maximal elements of $\{0,1\}^{\mathbb{N}}$ given by

$$\mathrm{lhe}(q) = (w_q 1)^\infty \qquad \text{and} \qquad \mathrm{rhe}(q) = c_q(1\widehat{w}_q)^\infty.$$

The first three statements of the following lemma characterize those elements of $\{0,1\}^{\mathbb{N}}$ which have given height. Most significant, from the point of view of this paper, is that irrational heights are realised by single elements of $\{0,1\}^{\mathbb{N}}$, while rational heights q other than 0 are realised on intervals in $\{0,1\}^{\mathbb{N}}$ with left and right hand endpoints $\mathrm{lhe}(q)$ and $\mathrm{rhe}(q)$. Note (cf. Remark 2) that, by (a), the condition $\lambda \in (\sqrt{2}, 2)$ is equivalent to $q(\kappa(f)) \in (0, 1/2)$.

Lemma 9 (Properties of height)**.**

(a) Let $s \in \{0,1\}^{\mathbb{N}}$. Then $q(s) = 0$ if and only if $s = 10^\infty$; and $q(s) = 1/2$ if and only if $s \preceq 101^\infty$.
(b) Let $s \in \{0,1\}^{\mathbb{N}}$ and $q \in (0,1/2)$ be rational. Then $q(s) = q$ if and only if $\mathrm{lhe}(q) \preceq s \preceq \mathrm{rhe}(q)$.
(c) Let $s \in \{0,1\}^{\mathbb{N}}$ and $q \in (0,1/2)$ be irrational. Then $q(s) = q$ if and only if $s = c_q$.
(d) Let $q = m/n \in (0,1/2)$, and $1 \le r \le m$. Then the word $10^{k_r(q)+1}110^{k_{r+1}(q)}11\dots110^{k_m(q)}1$ disagrees with c_q within the shorter of their lengths, and is greater than it in the unimodal order.
(e) Let $q \in (0,1/2)$ be rational, and let $s = c_q \dots \in \{0,1\}^{\mathbb{N}}$. Then $q(s) \le q$.
(f) Let $s \in \{0,1\}^{\mathbb{N}}$ with $q = q(s) \in (0,1/2)$ rational. Then either $s = \mathrm{lhe}(q)$, or there is some $k \ge 0$ and $t \in \{0,1\}^{\mathbb{N}}$ such that $s = (w_q 1)^k\, t$, and either $t = c_q \dots$ or $q(t) < q$.
(g) Let κ be the kneading sequence of a tent map, with $q = q(\kappa)$ rational. Then either $\kappa = \mathrm{lhe}(q)$, or $\kappa = c_q \dots$.

Proof. For (a), the characterization of height 0 is immediate from the definition of height and the fact that $c_{1/n} = 10^{n-1}1$; and the characterization of height 1/2 is Lemma 3.3 of [8]. (b) is Lemma 3.4 of [8], and (c) follows from the definition of height and that L_q does not pass through any integer lattice points other than $(0,0)$. (d) is Lemma 63 of [6]. For (e), we need only observe that if $s = c_q \dots$ then $s \succ \mathrm{lhe}(q)$, and use (b).

For (f), if $s \ne \mathrm{lhe}(q)$ then $s \succ \mathrm{lhe}(q)$. Let $k \ge 0$ be greatest such that s starts with the word $(w_q 1)^k$: then $s = (w_q 1)^k\, t$, where $t \succ \mathrm{lhe}(q)$ does not start with $w_q 1$. Since $t \succ \mathrm{lhe}(q)$ we have $q(t) \le q$. If $q(t) = q$ then, by (b) (and recalling that $c_q = w_q 01$) we have $(w_q 1)^\infty \prec t \preceq w_q 01(1\widehat{w}_q)^\infty$. Therefore, since t does not start with $w_q 1$, it must start with $w_q 0$; moreover, since $w_q 0$ is an odd word, t must start with $w_q 01 = c_q$, or it would be greater than $\mathrm{rhe}(q)$. This also proves (g), using the observation that if $s = \kappa$ is a kneading sequence then, by Lemma 3, we must have $k = 0$, since otherwise we would have $\sigma^{nk}(\kappa) \succ \kappa$. ☐

Example 10. Let $q = 1/3$, so that $c_q = 1001$, $w_q = 10$, and $\widehat{w}_q = 01$. Then $\mathrm{lhe}(q) = (101)^\infty$ and $\mathrm{rhe}(q) = 1001(101)^\infty = 10(011)^\infty$. Therefore $q(s) = 1/3$ if and only if $(101)^\infty \preceq s \preceq 10(011)^\infty$.

We will say that f is of *irrational type* if $q(\kappa(f))$ is irrational; that it is of *rational (left hand or right hand) endpoint type* if $\kappa(f)$ is equal to $\mathrm{lhe}(q)$ or $\mathrm{rhe}(q)$ respectively for some rational q; and that it is of *rational interior type* otherwise.

The following result is the essential fact which makes it possible to relate heights of forward sequences to heights of backward sequences. The real content of the lemma is the final sentence — the infimal height forward is equal to the infimal height backward, for any bi-infinite sequence S which does not start or end 0^∞.

Lemma 11. *Let $S \in \{0,1\}^{\mathbb{Z}}$.*

(a) If S does not end 0^∞ then $\inf_{r\in\mathbb{Z}} q(\overleftarrow{\sigma^r(S)}) \le \inf_{r\in\mathbb{Z}} q(\overrightarrow{\sigma^r(S)})$.

(b) If S does not start 0^∞ then $\inf_{r\in\mathbb{Z}} q(\overrightarrow{\sigma^r(S)}) \le \inf_{r\in\mathbb{Z}} q(\overleftarrow{\sigma^r(S)})$.

In particular, if S neither starts nor ends 0^∞, then $\inf_{r\in\mathbb{Z}} q(\overrightarrow{\sigma^r(S)}) = \inf_{r\in\mathbb{Z}} q(\overleftarrow{\sigma^r(S)})$.

Proof. To prove (a), suppose for a contradiction that S does not end 0^∞ and that $\inf_{r\in\mathbb{Z}} q(\overrightarrow{\sigma^r(S)}) < \inf_{r\in\mathbb{Z}} q(\overleftarrow{\sigma^r(S)})$. Let $q = m/n$ be a rational with $\inf_{r\in\mathbb{Z}} q(\overrightarrow{\sigma^r(S)}) < q < \inf_{r\in\mathbb{Z}} q(\overleftarrow{\sigma^r(S)})$. Then there is some $k \in \mathbb{Z}$ with $q(\overrightarrow{\sigma^k(S)}) < q$: replacing S with one of its shifts, we can assume without loss of generality that $k = 0$, so that $q(\overrightarrow{S}) < q$. We will show that there is some $r \in \mathbb{Z}$ with $q(\overleftarrow{\sigma^r(S)}) \le q$, which will be the required contradiction to $q < \inf_{r\in\mathbb{Z}} q(\overleftarrow{\sigma^r(S)})$.

Since $q(\overrightarrow{S}) < q$, Lemma 9(b) gives that $\overrightarrow{S} \succ \mathrm{rhe}(q) = c_q(1\widehat{w}_q)^\infty$. If $\overrightarrow{S} = c_q\ldots$, then, since c_q is palindromic, we have $\overleftarrow{\sigma^{n+1}(S)} = c_q\ldots$, and hence $q(\overleftarrow{\sigma^{n+1}(S)}) \le q$ by Lemma 9(e). We therefore suppose that c_q is not an initial subword of $\overrightarrow{S}$, so that there is some i with $1 \le i \le m$ and some $\ell \ge 1$ with $\overrightarrow{S} = 10^{k_1}110^{k_2}11\ldots110^{k_{i-1}}110^{k_i+\ell}1\ldots$ (we write $k_j = k_j(q)$ and use the fact that S does not end 0^∞ to get the final 1). Writing r for the length of this initial subword of $\overrightarrow{S}$, we have

$$\begin{aligned}
\overleftarrow{\sigma^r(S)} &= 10^{k_i+\ell}110^{k_{i-1}}11\ldots110^{k_2}110^{k_1}1\ldots \\
&\succeq 10^{k_i+1}110^{k_{i-1}}11\ldots110^{k_2}110^{k_1}1\ldots \\
&= 10^{k_{m+1-i}+1}110^{k_{m+2-i}}11\ldots110^{k_{m-1}}110^{k_m}1\ldots \qquad \text{(since } c_q \text{ is palindromic)} \\
&\succ (c_q0)^\infty \qquad \text{(by Lemma 9(d))},
\end{aligned}$$

so that $q(\overleftarrow{\sigma^r(S)}) \le q$ by the definition of height, as required.

Statement (b) follows by applying (a) to the reverse of S. $\square$

Remark 12. It is possible for one of the infima to be a minimum, and the other not to be attained. Consider, for example, the sequence S with $\overrightarrow{S} = (101)^\infty = \mathrm{lhe}(1/3)$, and $\overleftarrow{S} = 1^\infty$. Then $q(\overrightarrow{S}) = 1/3$, but $q(\overleftarrow{\sigma^r(S)}) > 1/3$ for all $r \in \mathbb{Z}$.

3.2. **Backward admissibility conditions.** In this section we will state and prove 'backward' admissibility conditions: that is, admissibility conditions which are expressed in terms of $\overleftarrow{\sigma^r(S)}$ rather than $\overrightarrow{\sigma^r(S)}$. We do this first in the symmetric case (Theorem 14), where they are analogous to the 'forward' conditions of Lemma 5; and then in the non-symmetric case (Theorem 16), where they take a quite different form. We start with a lemma which will be the main part of the proof of necessity for both theorems.

Lemma 13. *Write $\kappa = \kappa(f)$ and $q = q(\kappa)$. Let $S \in J_f$.*

(a) If q is irrational, then $\overleftarrow{S} \preceq \kappa$.

(b) If $q = m/n$ is rational, then $\overleftarrow{S} \preceq \mathrm{rhe}(q)$. Moreover, if either $\kappa = \mathrm{lhe}(q)$ or $\overrightarrow{S} \succ \sigma^{n+1}(\kappa)$, then $\overleftarrow{S} \preceq \mathrm{lhe}(q)$.

Proof. By Lemma 5, S does not start 0^∞.

(a) Suppose for a contradiction that q is irrational and that $\overleftarrow{S} \succ \kappa$. Then $q(\overleftarrow{S}) < q$, since, by Lemma 9(c), κ is the unique element of $\{0,1\}^{\mathbb{N}}$ with height q. By Lemma 11(b) there is some r with $q(\overrightarrow{\sigma^r(S)}) < q$, so that $\overrightarrow{\sigma^r(S)} \succ \kappa$, again by Lemma 9(c). This contradicts Lemma 5.

(b) Now let $q = m/n$ be rational. If $\overleftarrow{S} \succ \mathrm{rhe}(q)$ then $q(\overleftarrow{S}) < q$, and we get a contradiction to Lemma 5 as in (a). We will therefore suppose that $\overleftarrow{S} \preceq \mathrm{rhe}(q)$ in the remainder of the proof.

To prove the 'moreover' statement, consider first the case where $\kappa = \mathrm{lhe}(q)$, and assume for a contradiction that $\overleftarrow{S} \succ \mathrm{lhe}(q) = (w_q1)^\infty$. By Lemma 9(f), we can write $\overleftarrow{S} = (w_q1)^k t$, where $k \geq 0$, and either $q(t) < q$ or $t = c_q \ldots$. If $q(t) < q$ then we get a contradiction as in (a). On the other hand, if $t = c_q \ldots$ then $\overrightarrow{\sigma^{-(k+1)n-1}(S)} = c_q \ldots \succ \kappa$ (since c_q is palindromic), contradicting Lemma 5.

It remains to show that if $\mathrm{lhe}(q) \prec \kappa \preceq \mathrm{rhe}(q)$ and $\sigma^{n+1}(\kappa) \prec \overrightarrow{S} \preceq \mathrm{rhe}(q)$, then $\overleftarrow{S} \preceq \mathrm{lhe}(q)$. Using Lemma 9(g), we write $\kappa = c_q u$, where $u = \sigma^{n+1}(\kappa)$. Suppose for a contradiction that $\overleftarrow{S} \succ \mathrm{lhe}(q)$. In particular, $q(\overleftarrow{S}) = q$ since $\overleftarrow{S} \preceq \mathrm{rhe}(q)$.

By Lemma 9(f) we have $\overleftarrow{S} = (w_q1)^k t$ for some $k \geq 0$, where either $q(t) < q$ or $t = c_q \ldots$. If $q(t) < q$ then we get a contradiction to Lemma 5 as in (a), so we can assume that $t = c_q \ldots$. Then $\overrightarrow{\sigma^{-(k+1)n-1}(S)} = c_q(1\widehat{w}_q)^k \overrightarrow{S} \succ c_q(1\widehat{w}_q)^k u$, since $\overrightarrow{S} \succ u$ by assumption. By Lemma 5 we therefore have $c_q(1\widehat{w}_q)^k u \prec \kappa = c_q u$, so that $k > 0$ and $(1\widehat{w}_q)^k u \prec u$. However this inequality gives $u \succ (1\widehat{w}_q)^\infty$, so that $\kappa = c_q u \succ c_q(1\widehat{w}_q)^\infty = \mathrm{rhe}(q)$, which is the required contradiction.

$\square$

Theorem 14 (Backward admissibility conditions for $\widehat{I}$: symmetric case)**.** *Suppose that f is either of irrational type, or of rational endpoint type. Write $\kappa(f) = \kappa$. Let $S \in \{0,1\}^{\mathbb{Z}}$. Then $S \in J_f$ if and only if the following three conditions hold:*

(a) $\overleftarrow{\sigma^r(S)} \preceq \kappa$ for all $r \in \mathbb{Z}$;

(b) S does not end 0^∞; and

(c) If f is of left hand endpoint type and $\overrightarrow{\sigma^r(S)} = \kappa$ for some $r \in \mathbb{Z}$, then $S_{r-1} = 1$.

Proof. Suppose that $S \in J_f$. Then $\sigma^r(S) \in J_f$ for all $r \in \mathbb{Z}$, and so (a) is a consequence of Lemma 13. If (b) did not hold then (since some $S_r = 1$ by Lemma 5(B)), there would be an r with $\overrightarrow{\sigma^r(S)} = 10^\infty$, contradicting Lemma 5(A). Condition (c) is immediate from Lemma 5(C).

For the converse, suppose that $S \notin J_f$, so that one of conditions (A), (B), and (C) of Lemma 5 fails. We must show that one of conditions (a), (b), and (c) is false. We write $q = q(\kappa)$.

If (A) fails, then let $r \in \mathbb{Z}$ with $\overrightarrow{\sigma^r(S)} \succ \kappa$. Suppose first that $q(\overrightarrow{\sigma^r(S)}) < q$ (which must necessarily be the case if q is irrational or q is rational and $\kappa = \mathrm{rhe}(q)$). By Lemma 11(a), either (b) is false or there is some i with $q(\overleftarrow{\sigma^i(S)}) < q$, so that $\overleftarrow{\sigma^i(S)} \succ \kappa$, and (a) is false. It remains to consider the case where $q = m/n$, $\kappa = \mathrm{lhe}(q)$ and $q(\overrightarrow{\sigma^r(S)}) = q$. By Lemma 9(f), we have $\overrightarrow{\sigma^r(S)} = (w_q1)^k t$, where either

$q(t) < q$ or $t = c_q \dots$. If $q(t) < q$ then, by Lemma 11(a), either (a) or (b) is false; while if $t = c_q \dots$ then $\overleftarrow{\sigma^{r+(k+1)n+1}(S)} = c_q \dots \succ \kappa$, and (a) is false.

If (B) fails, then either $S_r = 0$ for all r, in which case (b) is false; or there is some r with $\overleftarrow{\sigma^r(S)} = 10^\infty$, in which case (a) is false. Clearly if (C) fails then (c) is false. $\square$

Remark 15. It is immediate from Lemma 5 and Theorem 14 that if f is of irrational type, or if $\kappa(f) = \mathrm{rhe}(m/n)$ for some m/n, then the *reversing* function $\rho\colon \{0,1\}^\mathbb{Z} \to \{0,1\}^\mathbb{Z}$ defined by $\rho(S)_r = S_{-r}$ restricts to a homeomorphism $J_f \to J_f$, which conjugates the symbolic natural extension $\sigma\colon J_f \to J_f$ to its inverse. On the other hand, if $\kappa(f) = \mathrm{lhe}(m/n)$, then ρ does not restrict to a self-homeomorphism of J_f. For example, if $\kappa(f) = \mathrm{lhe}(1/3) = (101)^\infty$, then the sequences with $\overleftarrow{S} = (101)^\infty$ and $\overrightarrow{S} = {}^0_1 1^\infty$ are both admissible (and hence correspond to different points of $\widehat{I}$); while the sequence with $\overrightarrow{S} = (101)^\infty$ and $\overleftarrow{S} = 01^\infty$ is not admissible: and, even if our conventions were changed so that it were, it would represent the same point as the sequence with $\overleftarrow{S} = 1^\infty$.

Theorem 16 (Backward admissibility conditions for $\widehat{I}$: non-symmetric case)**.** *Suppose that f is of rational interior type. Write $\kappa(f) = \kappa$ and $q = m/n = q(\kappa)$. Let $S \in \{0,1\}^\mathbb{Z}$. Then $S \in J_f$ if and only if the following four conditions hold:*

(a) $\overleftarrow{\sigma^r(S)} \preceq \mathrm{rhe}(q)$ *for all* $r \in \mathbb{Z}$*;*
(b) $\overleftarrow{\sigma^r(S)} \preceq \mathrm{lhe}(q)$ *for all* $r \in \mathbb{Z}$ *for which* $\overrightarrow{\sigma^r(S)} \succ \sigma^{n+1}(\kappa)$*;*
(c) S *does not end* 0^∞*; and*
(d) If c *is periodic and* $\overrightarrow{\sigma^r(S)} = \kappa$ *for some* $r \in \mathbb{Z}$*, then* $S_{r-1} = \varepsilon(f)$*.*

Proof. The argument that if $S \in J_f$ then conditions (a) – (d) hold proceeds in the same way as in the proof of Theorem 14, using Lemmas 5 and 13.

Suppose, then, that $S \notin J_f$, so that one of conditions (A), (B), and (C) of Lemma 5 fails. As in the proof of Theorem 14, if (B) fails then either (c) or (a) is false, and if (C) fails then (d) is false. To complete the proof, we show that if (A) fails then one of (a), (b), or (c) is false. We therefore assume that there is some $r \in \mathbb{Z}$ with $\overrightarrow{\sigma^r(S)} \succ \kappa$. Since f is of interior type, we have $\kappa = c_q \dots$ by Lemma 9(g).

If $q(\overrightarrow{\sigma^r(S)}) < q$, then by Lemma 11(a), either (c) is false, or there is some i with $q(\overleftarrow{\sigma^i(S)}) < q$, so that $\overleftarrow{\sigma^i(S)} \succ \mathrm{rhe}(q)$, and (a) is false.

If $q(\overrightarrow{\sigma^r(S)}) = q$, then $\overrightarrow{\sigma^r(S)} = c_q\, t$ for some $t \in \{0,1\}^\mathbb{N}$. Write $\kappa = c_q\, u$, where $u \in \{0,1\}^\mathbb{N}$. Since $\overrightarrow{\sigma^r(S)} \succ \kappa$ we have $\overrightarrow{\sigma^{r+n+1}(S)} = t \succ u = \sigma^{n+1}(\kappa)$. On the other hand we have $\overleftarrow{\sigma^{r+n+1}(S)} = c_q \dots \succ \mathrm{lhe}(q)$: so condition (b) is false. $\square$

At this stage it is conceivable that the conditions of Theorem 16 are in fact a symmetric version of those of Lemma 5, expressed in a different way. Our final result establishes that this is not the case, by showing that the maximum backward itinerary which can be realised by a tent map with given kneading sequence mode locks on rational height intervals. This contrasts with the maximum admissible forward itinerary, which is the kneading sequence itself.

Theorem 17 (Mode-locking of maximum backward itinerary)**.** *Suppose that f is of rational interior type, with $q(\kappa(f)) = q \in \mathbb{Q}$. Then $s = \mathrm{rhe}(q)$ is the greatest element of $\{0,1\}^\mathbb{N}$ with the property that there is some $S \in J_f$ with $\overleftarrow{S} = s$.*

Proof. It is immediate from Theorem 16 that $\overleftarrow{S} \preceq \mathrm{rhe}(q)$ for all $S \in J_f$. It is therefore only necessary to exhibit an element S of J_f with $\overleftarrow{\sigma^r(S)} = \mathrm{rhe}(q)$ for some $r \in \mathbb{Z}$.

Let $S \in \{0,1\}^{\mathbb{Z}}$ be given by $\overrightarrow{S} = (w_q 0)^\infty$ and $\overleftarrow{S} = (1\widehat{w}_q)^\infty$, so that

$$\overleftarrow{\sigma^{n+1}(S)} = 10\widehat{w}_q(1\widehat{w}_q)^\infty = c_q(1\widehat{w}_q)^\infty = \mathrm{rhe}(q),$$

where $q = m/n$, using $c_q = w_q 01 = 10\widehat{w}_q$ (as it is palindromic). We will show that $\overrightarrow{\sigma^r(S)} \prec \kappa(f)$ for all $r \in \mathbb{Z}$, so that $S \in J_f$ by Lemma 5.

The sequence $(w_q 0)^\infty$ is shift-maximal, since it is the saddle-node pair of $(w_q 1)^\infty = \mathrm{lhe}(q)$. Moreover, there do not exist shift-maximal sequences s with $(w_q 1)^\infty \prec s \prec (w_q 0)^\infty$. For such a sequence s would necessarily have initial subword w_q. If $s = w_q 1\ldots$, let $k \geq 1$ be greatest such that $s = (w_q 1)^k t$ for some $t \in \{0,1\}^{\mathbb{N}}$. Then $t \succ (w_q 1)^\infty$, as $s \succ (w_q 1)^\infty$, and since $w_q 1$ is not an initial subword of t we have $t \succ s$, contradicting the shift-maximality of s. On the other hand, if $s = w_q 0t$ for some $t \in \{0,1\}^{\mathbb{N}}$, then $t \succ (w_q 0)^\infty$, as $s \prec (w_q 0)^\infty$ and $w_q 0$ is odd, and so $t \succ s$, again contradicting shift-maximality.

Since $(w_q 0)^\infty$ is not the kneading sequence of a tent map (its minimal repeating word being odd) and $\kappa(f) \succ \mathrm{lhe}(q)$ is shift-maximal, we have $\kappa(f) \succ (w_q 0)^\infty$. Hence, for any $r \geq 0$, we have $\overrightarrow{\sigma^r(S)} = \sigma^r((w_q 0)^\infty) \preceq (w_q 0)^\infty \prec \kappa(f)$, establishing the result in the case $r \geq 0$.

For the case $r < 0$, write $k_i = k_i(q)$ for $1 \leq i \leq m$, so that we have $w_q = 10^{k_1}1^2 0^{k_2} 1^2 \ldots 1^2 0^{k_{m-1}} 1^2 0^{k_m - 1}$. Let $r < 0$. If $\overrightarrow{\sigma^r(S)}$ does not have initial subword 10, then clearly $\overrightarrow{\sigma^r(S)} \prec \kappa(f)$. If it does have initial subword 10, then there is some $1 \leq i \leq m$ such that

$$\overrightarrow{\sigma^r(S)} = 10^{k_i}1^2 0^{k_{i+1}} 1^2 \ldots 1^2 0^{k_m - 1} 1 \ldots \prec 10^{k_i} 1^2 0^{k_{i+1}} 1^2 \ldots 1^2 0^{k_m - 1} 0\, (w_q 0)^\infty,$$

which is a shift of the shift-maximal sequence $(w_q 0)^\infty$. Therefore $\overrightarrow{\sigma^r(S)} \prec (w_q 0)^\infty \prec \kappa(f)$ as required. □

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