PERRON IDENTITY FOR ARBITRARY BROKEN LINES.

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Abstract. In this paper we study the values of Markov-Davenport forms, which are specially normalized binary quadratic forms. We generalize the Perron identity for ordinary continued fractions for sails to the case of arbitrary broken lines.

Contents

Introduction

In this paper we give a geometric interpretation and generalization of the Perron identity relating minima of Markov-Davenport forms and their corresponding continued fractions, which says that for such a form f there exists a sequence (a_i) of positive integers such that

$$\min_{\Omega} \big|f\big| = \inf_{i\in\mathbb{Z}} \left(\frac{1}{a_i + [0; a_{i+1} : a_{i+2} : \ldots] + [0; a_{i-1} : a_{i-2} : \ldots]}\right),$$

where $\Omega = \mathbb{Z}^2 \setminus \{(0,0)\}$, and $[a_1 : \ldots : a_n]$ is the continued fraction of the sequence $(a_1, \ldots, a_n)$.

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The above formula was shown by A.Markov in his study of Markov-Davenport forms and the Markov spectrum below 3 in [6]. The statement holds for the whole Markov spectrum (see for example [7], and [1]). Our main result (Theorem 2.2) generalizes this formula to the case of arbitrary values of forms and the continued fractions of arbitrary broken lines. The theory of continued fractions for arbitrary broken lines was studied in papers [4, 2, 3] (see also [5]).

Organization of the paper. We start in Section 1 with necessary definitions and preliminary discussions. We also discuss the classical relation of the Markov spectrum and indefinite forms. In Section 2 we formulate and prove the main result of this paper, and give examples.

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1. Basic notions, definitions, and background

1.1. **Markov-Davenport forms.** Let $f(x, y)$ be a binary quadratic form with two distinct real linear factors

$$f(x, y) = (ax - by)(cx - dy),$$

for some real numbers a, b, c, and d. The discriminant of this form is

$$\Delta(f) = (ad - bc)^2.$$

Definition 1.1. The *Markov-Davenport* form $\hat{f}$ related to f is defined as

$$\hat{f} = \frac{f}{\sqrt{\Delta(f)}}.$$

Remark 1.2. In fact for any pair of distinct lines through the origin there exist precisely two Markov-Davenport forms, which differ by a sign.

1.2. **Background.** In this subsection we collect some classical facts which we generalise further in this paper. Let us recall the following general result. Let A be a doubly infinite sequence

$$\ldots, a_{-1}, a_0, a_1, \ldots, a_i, \ldots$$

of positive integers, and for each integer i define

$$\lambda_i(A) = a_i + [0; a_{i+1} : a_{i+2} : \ldots] + [0; a_{i-1} : a_{i-2} : \ldots],$$

where $[a_i; a_{i+1} : \ldots]$ is the regular continued fraction of the sequence $(a_i, a_{i+1}, \ldots)$. Define the value $M(A)$ to be

$$M(A) = \sup_{i \in \mathbb{Z}} \lambda_i(A).$$

The set of values $M(A)$ for all doubly infinite sequences of positive integers A is called the *Markov Spectrum*. If $M(A)<3$ then A is purely periodic (see [1]). In that case, the sequence A is related to an indefinite binary quadratic form $f(x,y)$, see for example [1]. Let $m(f)$ be the *Markov minimum* of f, defined by

$$m(f) = \min_{(x,y)\in\Omega} |f(x,y)|,$$

with $\Omega = \mathbb{Z}^2 \setminus (0,0)$, and let $\Delta(f)$ denote the discriminant of f. Then, from the paper [6] from Markov, we have the following result:

$$\frac{m(f)}{\sqrt{\Delta(f)}} = \inf_{i\in\mathbb{Z}} \left(\frac{1}{a_i + [0; a_{i+1} : a_{i+2} : \ldots] + [0; a_{i-1} : a_{i-2} : \ldots]} \right). \tag{1}$$

We generalise Equation (1), using notions of Integer Geometry, to describe a relationship between non-regular continued fractions and the value of the related Markov-Davenport form $f/\sqrt{\Delta(f)}$ at any point on the plane.

1.3. **LLS sequences for broken lines.** In this subsection we recall the definition of LLS sequence in [5, p. 138], and formally set the expression for the cross product of two-dimensional vectors. We work in the oriented two-dimensional plane with the origin $O = (0,0)$.

First let

$$f_1(x,y) = ax - by, \text{ and}$$
$$f_2(x,y) = cx - dy,$$

for real numbers a, b, c, and d, and let $\Delta(f) = (ad - bc)^2$. We set

$$f(x,y) = \frac{f_1(x,y)f_2(x,y)}{\sqrt{\Delta(f)}}.$$

We start with the following general definition.

Definition 1.3. Consider the Markov-Davenport form f. A broken line $A_0 \ldots A_n$ is an *f-broken line* if the following conditions hold:

- $f_1(A_0) = 0$;
- $f_2(A_n) = 0$;
- all edges of the broken line are of positive length;
- for every $k = 1, \ldots, n$ the line $A_{k-1}A_k$ does not pass through the origin.

Let us define the oriented Euclidean area via formal cross product expression for two-dimensional vectors.

Definition 1.4. Let $X = (x_1, x_2)$, $Y = (y_1, y_2)$, $V = (v_1, v_2)$, and $W = (w_1, w_2)$. Set

$$|XY \times VW| = (x_1 - y_1)(v_2 - w_2) - (v_1 - w_1)(x_2 - y_2).$$

We call this expression *the oriented Euclidean area* for the parallelogram spanned by XY and VW.

Remark 1.5. The oriented Euclidean area for the parallelogram spanned by XY and VW may be negative.

Definition 1.6. Given an f-broken line $A_0 \ldots A_n$ define

$$a_{2k} = |OA_k \times OA_{k+1}|, \quad k = 0, \ldots, n;$$
$$a_{2k-1} = \frac{|A_kA_{k-1} \times A_kA_{k+1}|}{a_{2k-2}a_{2k}}, \quad k = 1, \ldots, n.$$

The sequence $(a_0, \ldots, a_{2n})$ is called the *LLS sequence* for the broken line. This sequence encodes the integer angles and integer lengths of the broken line, as can be found in [5]. The expression $[a_0; \ldots : a_{2n}]$ is said to be the *continued fraction for the broken line* $A_0 \ldots A_n$. Note that the values a_i may be negative, but not 0.

In the proofs in Section 2 we need the following result:

Theorem 1.7. ([5, Corollary 11.11, p. 144]**.)** *Consider a broken line $A_0 \ldots A_n$ that has the LLS sequence $(a_0, \ldots, a_{2n})$, with $A_0 = (1, 0)$, $A_1 = (1, a_0)$, and $A_n = (x, y)$. Let*

$$\alpha = [a_0; a_1 : \ldots : a_{2n}]$$

be the corresponding continued fraction for this broken line. Then

$$\frac{y}{x} = \alpha.$$

For the case of an infinite value for $\alpha = [a_0; a_1 : \ldots : a_{2n}]$,

$$\frac{x}{y} = 0.$$

2. Theorem on values of Markov-Davenport forms

2.1. **Formulation.** Let us introduce a useful choice of orientation of the linear factors of Markov-Davenport forms.

Definition 2.1. Let

$$f(x, y) = (ax - by)(cx - dy),$$

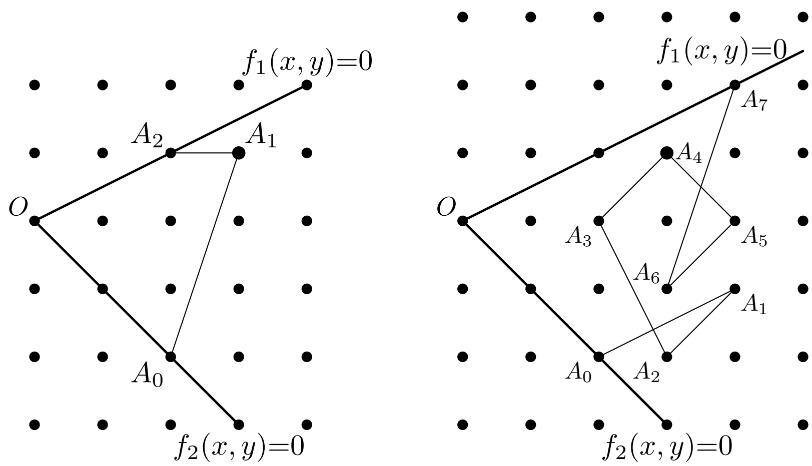

FIGURE 1. Well oriented f-broken lines.

and let $A_0 \dots A_n$ be an f-broken line, with $A_0 = (x_0, y_0)$, $A_n = (x_n, y_n)$. Then we say that $A_0 \dots A_n$ is a *well oriented* f-broken line if

$$ax_0 - by_0 = 0$$
$$cx_n - dy_n = 0$$

when $ad - bc > 0$, or

$$ax_n - by_n = 0$$
$$cx_0 - dy_0 = 0$$

when $ad - bc < 0$.

We state the main result of this paper.

Theorem 2.2. *Let $\hat{f}$ be a Markov-Davenport form related to f as in Definition 1.1 and let $L = A_0 \dots A_{n+m}$ be a well oriented f-broken line (here n and m are arbitrary positive integers). Let*

$$(a_0, a_1, \dots, a_{2n+2m})$$

be the LLS sequence of L, and let $k = m + n + 1$. Then

$$\hat{f}(A_k) = \frac{1}{a_{2k-1} + [0; a_{2k-2} : \dots : a_0] + [0; a_{2k} : \dots : a_{2n+2m}]}. \tag{2}$$

2.2. **Examples.** Before proving the theorem we illustrate its statement with the following examples. Let

$$f(x, y) = (x + y)(x - 2y).$$

(i) Let $A_0A_1A_2$ be an f-broken line, with $A_0{=}(2,1)$, $A_1{=}(3,1)$, and $A_2{=}(2,-2)$. Figure 1 (Left) shows the solutions to $f(x,y)=0$, and the broken line $A_0A_1A_2$.

We have that

$$\begin{aligned}\frac{f(A_1)}{\sqrt{\Delta(f)}} &= \frac{f(3,1)}{\sqrt{9}}\\ &= \frac{4}{3}\\ &= \frac{1}{-\frac{3}{8}+\frac{1}{8}+\frac{1}{1}}\\ &= \frac{1}{-3/8+[0;8]+[0;1]}\\ &= \frac{1}{a_1+[0;a_0]+[0;a_2]},\end{aligned}$$

where (a_0,a_1,a_2) is the LLS sequence of the broken line A.

(ii) Let $A = A_0 \ldots A_7$ be the broken line with vertices at integer points as in Figure 1 (Right), and again we consider $(3,1)$, which is now A_4. Then $f(x,y)=f_1(x,y)f_2(x,y)$ is given by

$$f(x,y)=(x-2y)(x+y),$$

the discriminant is $\Delta(f)=3$, and

$$\begin{aligned}\frac{f(A_4)}{\sqrt{\Delta(f)}} &= \frac{f(3,1)}{\sqrt{9}}\\ &= \frac{4}{3}\\ &= \frac{1}{-\frac{1}{4}+\frac{1}{4}+\frac{3}{4}}\\ &= \frac{1}{-\frac{1}{4}+L_1+L_2},\end{aligned}$$

where

$$L_1 = \left[0;2:\frac{3}{8}:4:-\frac{3}{20}:-5:-\frac{1}{30}:6\right], \text{ and}$$

$$L_2 = \left[0;-4:\frac{1}{8}:-4:-\frac{1}{20}:10\right]$$

are the continued fractions of the LLS sequences of the broken lines $A_0 \dots A_4$ and $A_4 \dots A_7$ respectively.

Remark 2.3. We see from these examples that when calculating the value of the form $\hat{f}$ at the point A_k, the choice of the well oriented f-broken line does not affect the value of the right hand side of the Perron identity, so long as it contains the point A_k as a vertex.

2.3. **Proofs.** We first give a Lemma which will be ueful in the proof of Theorem 2.2.

Lemma 2.4. Let $\hat{f}$ be a Markov-Davenport form related to f as in Definition 1.1, with

$$f(x, y) = (ax - by)(cx - dy),$$

and a, c not both zero. Let $B_0B_1B_2$ be an oriented f-broken line, and let

$$(b_0, b_1, b_2)$$

be the LLS sequence of $B_0B_1B_2$. Then

$$\frac{1}{b_1 + [0; b_0] + [0; b_2]} = \frac{(ax_k - by_k)(cx_k - dy_k)}{(ad - bc)}.$$

Proof. From Definition 1.6 we have that the LLS sequence for the broken line is

$$\begin{aligned} b_0 &= |OB_0 \times OB_1|, \\ b_2 &= |OB_1 \times OB_2|, \\ b_1 &= \frac{|B_1B_0 \times B_1B_2|}{b_0b_2}. \end{aligned}$$

We calculate these values, with the points $B_0 = (\beta_0, \gamma_0)$, $B_2 = (\beta_2, \gamma_2)$, $B_1 = (x_k, y_k)$, and O being the origin. Then

$$\begin{aligned} b_0 &= \begin{vmatrix} \beta_0 & \gamma_0 \\ x_k & y_k \end{vmatrix} = \beta_0 y_k - x_k \gamma_0, \\ b_2 &= \begin{vmatrix} x_k & y_k \\ \beta_2 & \gamma_2 \end{vmatrix} = x_k \gamma_2 - y_k \beta_2, \\ b_1 &= \begin{vmatrix} x_k - \beta_0 & y_k - \gamma_0 \\ x_k - \beta_2 & y_k - \gamma_2 \end{vmatrix} = \beta_0\gamma_2 - \beta_2\gamma_0 - \beta_0 y_k + x_k\gamma_0 - x_k\gamma_2 + y_k\beta_2. \end{aligned}$$

So

$$b_0 + b_1 + b_2 = \beta_0\gamma_2 - \beta_2\gamma_0,$$

and we get

$$\begin{aligned}
\frac{1}{b_1+[0;b_0]+[0;b_2]} &= \frac{1}{\dfrac{b_1}{b_2b_0}+\dfrac{1}{b_2}+\dfrac{1}{b_0}}\\
&= \frac{b_2b_0}{b_1+b_2+b_0}\\
&= \frac{(\beta_0y_k-x_k\gamma_0)(x_k\gamma_2-y_k\beta_2)}{\beta_0\gamma_2-\beta_2\gamma_0}.
\end{aligned}$$

By the definition of B_0 and B_2 we have that $f_1(\beta_0,\gamma_0)=0$ and $f_2(\beta_2,\gamma_2)=0$. Now we rearrange these two equations to get

$$\begin{aligned}
\beta_0 &= \frac{b}{a}\gamma_0,\\
\beta_2 &= \frac{d}{c}\gamma_2,
\end{aligned}$$

and substitute them into the expression

$$\frac{(\beta_0y_k-x_k\gamma_0)(x_k\gamma_2-y_k\beta_2)}{\beta_0\gamma_2-\beta_2\gamma_0}$$

to get

$$\begin{aligned}
\frac{1}{b_1+[0;b_0]+[0;b_2]} &= \frac{(\beta_0y_k-x_k\gamma_0)(x_k\gamma_2-y_k\beta_2)}{\beta_0\gamma_2-\beta_2\gamma_0}\\
&= \frac{\left(\dfrac{b}{a}\gamma_0y_k-x_k\gamma_0\right)\left(x_k\gamma_2-\dfrac{d}{c}\gamma_2y_k\right)}{\dfrac{b}{a}\gamma_0\gamma_2-\dfrac{d}{c}\gamma_0\gamma_2}\\
&= \frac{\gamma_0\gamma_2(by_k-ax_k)(cx_k-dy_k)}{\gamma_0\gamma_2(bc-ad)}\\
&= \frac{(ax_k-by_k)(cx_k-dy_k)}{(ad-bc)}.
\end{aligned}$$

■

Proof of Theorem 2.2. Let $f(x,y)$ be the form

$$f(x,y)=f_1(x,y)f_2(x,y),$$

where

$$\begin{aligned}
f_1(x,y) &= (ax-by), \text{ and}\\
f_2(x,y) &= (cx-dy).
\end{aligned}$$

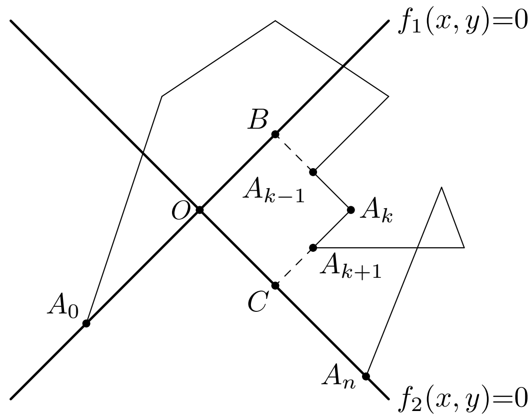

FIGURE 2. The two well oriented f-broken lines $A_0 \dots A_n$ and BA_kC.

We assume that a and c are not both zero. If they are both zero, then the solutions to $f_1 = 0$ and $f_2 = 0$ coincide. Without loss of generality we assume that

$$\begin{aligned} f_1(A_0) &= 0, \text{ and} \\ f_2(A_n) &= 0. \end{aligned}$$

Since L is a well oriented f-broken line, this fixes the discriminant $\Delta(f) = ad - bd$ to be positive.

Let the points $B = (\beta_1, \beta_2)$ and $C = (\gamma_1, \gamma_2)$ be the intersections of the line segments OA_0 and A_kA_{k-1}, and OA_n and A_kA_{k+1} respectively, as in Figure 2. Then by Theorem 1.7, the continued fractions of the broken lines BA_k and $A_0 \dots A_k$ coincide, as do the continued fractions of the broken lines A_kC and $A_k \dots A_n$. The angles at A_k are the same for both broken lines BA_kC and $A_0 \dots A_n$, and so the corresponding elements of the LLS sequences also equal.

This allows us to replace the two broken lines $A_0 \dots A_k$ and $A_k \dots A_n$ with the broken lines CA_k and A_kB respectively. Let the coordinates of A_k be (x_k, y_k).

The discriminant is $\Delta(f) = (ad - bc)^2$, so the value of the Markov-Davenport form at A_k is

$$\frac{f(x_k, y_k)}{|ad - bc|} = \frac{(ax_k - by_k)(cx_k - dy_k)}{|ad - bc|}.$$

Let the LLS sequence for the broken line BA_kC be

$$(b_0, b_1, b_2).$$

We have from Lemma 2.4 that

$$\frac{1}{b_1+[0;b_0]+[0;b_2]} = \frac{(ax_k-by_k)(cx_k-dy_k)}{(ad-bc)}.$$

Finally we know that $ad-bc>0$, and so

$$\begin{aligned}\hat{f}(A_k) &= \frac{f(x_k,y_k)}{|ad-bc|}\\ &= \frac{1}{b_1+[0;b_0]+[0;b_2]}\\ &= \frac{1}{a_{2k-1}+[0;a_{2k-2}:\ldots:a_0]+[0;a_{2k}:\ldots:a_{2n+2m}]}.\end{aligned}$$

■

Remark 2.5. If we had chosen the non oriented broken line CA_nB, our value would be

$$\frac{1}{b_1+[0;b_0]+[0;b_2]} = -\frac{(ax_k-by_k)(cx_k-dy_k)}{(ad-bc)}.$$

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