# Notes on Stability of Time-Delay Systems: Bounding Inequalities and Augmented Lyapunov-Krasovskii Functionals

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***Abstract*—The bounding inequalities and the Lyapunov-Krasovskii functionals (LKFs) are important for the stability analysis of time-delay systems. Much attention has been paid to develop tighter inequalities for improving stability criteria, while the contribution of the LKFs has not been considered when discussing the relationship between the tightness of inequalities and the conservatism of criteria. This note is concerned with this issue. Firstly, it is proved that, when a simple LKF is applied, the stability criteria obtained by the Wirtinger-based inequality and the Jensen inequality are equivalent although the Wirtinger-based inequality is tighter. It means that the tighter inequality does not always lead to a less conservative criterion. Secondly, it is found that a suitable augmented LKF with necessary integral vectors in its derivative is required to achieve the advantage of the Wirtinger-based inequality. Based on this observation, two delay-product-type terms are introduced into the LKF to establish new stability criteria. Finally, a numerical example is given to verify the equivalence statements and to show the benefit of the proposed criteria.**

***Index Terms*—Time-delay system, stability, bounding inequalities, augmented Lyapunov-Krasovskii functional**

## I. Introduction

Since the time delays arising in many systems may cause undesirable dynamics like performance degradation and even instability, the stability analysis of time-delay systems has become a hot topic in the past few decades [1], [2]. Many important techniques for this issue have been developed considering the following linear system with a time-varying delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-d(t)),\ t \geq 0 \\ x(t) = \phi(t), \quad t \in [-h, 0] \end{cases} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the state, $A$ and $A_d$ are the system matrices, the initial condition $\phi(t)$ is a continuously differentiable function, and $d(t)$ is the time-varying delay satisfying

$$0 \leq d(t) \leq h, \quad \mu_1 \leq \dot{d}(t) \leq \mu_2, \quad \forall t \geq 0 \tag{2}$$

Since the delays existing in the systems are usually time-varying, it is natural to investigate the stability problem in time-domain [3]. In this category, the Lyapunov-Krasovskii functional (LKF) method is a popular method [4]. The main problem of this method is that the stability criteria established bring more or less conservatism. Therefore, the important issue is how to reduce such conservatism.

The term, $\int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta$ with $R \geq 0$, is usually required for obtaining the delay-dependent criterion [5], [6]. Its derivative includes a single integral term, $\int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)ds$, which should be estimated for expressing the stability criteria in the form of tractable linear matrix inequalities (LMIs) [7]. Model transformations [8]–[10] and various free-weighting-matrix approach [11]–[14] are applied to estimate the integral term in the early research. An alternative type of method is to directly estimate the integral term via bounding inequalities, among which Jensen inequality (JI) [15] and Wirtinger-based inequality (WBI) [7] are frequently applied.

Since the LKF with augmented terms is found to be helpful to reduce the conservatism [16], various augmented LKFs have been developed by augmenting non-integral terms and/or integral terms using various state-related vectors, such as the delayed state vectors, $x(t-d(t))$, $x(t-h/2)$, and $x(t-h)$, the derivative of the state vector, $\dot{x}(t)$, the integral of the state vector, $\int_{t-d(t)}^{t} x(s)ds$, and $\int_{t-h}^{t-d(t)} x(s)ds$ *etc.* [4], [7], [13], [17]–[20], [37]. Compared with the simple form of LKF, the LKFs with augmented terms introduce several extra matrices, which provide more freedom for checking the feasibility of the LMI conditions in the criteria [21].

Most researches mainly aim at deriving less conservative criteria through the LKFs with more general form and/or the bounding inequalities with less estimation gap. It is predictable that more general LKFs can reduce the conservatism. How to choose the augmented vectors is an important problem. A few scholars have investigated theoretical comparisons among different techniques for estimating task [7], [22]–[26]. Based on those studies, it seems that the tighter inequality could lead to less conservative results. To the best of authors' knowledge, there is no work considering the contribution of the LKFs when discussing the relationship between the tightness of inequalities and the conservatism of criteria.

This note is concerned with the stability of system (1) and focuses on the link between the LKFs and two bounding inequalities during the analysis of the conservatism. Firstly, for the stability analysis of system (1) based on a simple LKF without any augmented term, it is proved that the criterion obtained by the WBI is equivalent to the one by the JI although the WBI is tighter. Secondly, by analyzing the LKF used in [7] and the difference of the WBI-based and the JI-

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based criteria, it is observed that the advantage of the WBI can be revealed by constructing a suitable augmented LKF, whose derivative includes some necessary integral vectors. Based on this observation, an augmented LKF with two delay-product-type terms is constructed and two stability criteria are established. Finally, a numerical example is given to verify the observation aforementioned and to show the benefit of the proposed criteria.

Throughout this note, the superscripts $T$ and $-1$ mean the transpose and the inverse of a matrix, respectively; $\mathcal{R}^n$ denotes the $n$-dimensional Euclidean space; $\|\cdot\|$ refers to the Euclidean vector norm; $P > 0$ $(\geq 0)$ means that $P$ is a real symmetric and positive-definite (semi-positive-definite) matrix; symmetric term in a symmetric matrix is denoted by $*$; and $\mathrm{Sym}\{Y\} = Y + Y^T$.

The following two lemmas are given for deriving the main results of this note.

*Lemma 1:* (Jensen inequality [15] and Wirtinger-based inequality [7]) For given symmetric positive definite matrix $R \in \mathcal{R}^{n\times n}$, scalars $a$ and $b$ satisfying $a < b$, and vector $\omega : [a, b] \mapsto \mathcal{R}^n$ such that the integrations concerned are well defined, the following inequalities hold

$$(b-a)\int_a^b \omega^T(s)R\omega(s)ds \geq \chi_1^T R\chi_1 \tag{3}$$

$$(b-a)\int_a^b \omega^T(s)R\omega(s)ds \geq \chi_1^T R\chi_1 + 3\chi_2^T R\chi_2 \tag{4}$$

where $\chi_1 = \int_a^b \omega(s)ds$ and $\chi_2 = \chi_1 - \frac{2}{b-a}\int_a^b\int_a^s \omega(u)duds$.

*Lemma 2:* ( [7], [27]) For given vectors $\beta_1$ and $\beta_2$, scalar $\alpha$ in the interval $(0, 1)$, symmetric positive definite matrix $R \in \mathcal{R}^{n\times n}$ and any matrix $X \in \mathcal{R}^{n\times n}$ satisfying $\begin{bmatrix} R & X \\ * & R\end{bmatrix} \geq 0$, the following inequality holds

$$\frac{1}{\alpha}\beta_1^T R\beta_1 + \frac{1}{1-\alpha}\beta_2^T R\beta_2 \geq \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}^T \begin{bmatrix} R & X \\ * & R\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \tag{5}$$

*Proof:* The above lemma can be obtained from Lemma 3 of [7] by setting $\beta_i = W_i\xi, i = 1, 2$ [28].

## II. Stability criteria by a simple LKF

This section develops two stability criteria through a simple LKF and proves the equivalence of them.

*Theorem 1:* For given $h$, $\mu_1$, and $\mu_2$, system (1) is asymptotically stable, if one of the following conditions holds:

C1: there exist symmetric $n\times n$ matrices $P > 0$, $Q \geq 0$, $R \geq 0$, $Z \geq 0$, and any $n \times n$ matrix $S$, such that the following holds for $\dot{d}(t) \in \{\mu_1, \mu_2\}$:

$$\Phi_1 = \begin{bmatrix} R & S \\ * & R\end{bmatrix} \geq 0, \qquad \Phi_2 = \Xi_1 - \Xi_2 < 0 \tag{6}$$

C2: there exist symmetric $n\times n$ matrices $P > 0$, $Q \geq 0$, $R \geq 0$, $Z \geq 0$, and any $2n \times 2n$ matrix $U$, such that the following holds for $\dot{d}(t) \in \{\mu_1, \mu_2\}$:

$$\Psi_1 = \begin{bmatrix} \tilde{R} & U \\ * & \tilde{R}\end{bmatrix} \geq 0, \qquad \Psi_2 = \Xi_3 - \Xi_4 < 0 \tag{7}$$

Box I: Notations used in Theorem 1

$$\Xi_1 = \mathrm{Sym}\{\bar{e}_1^T P\bar{e}_s\} + \bar{e}_1^T(Q+Z)\bar{e}_1 - (1-\dot{d}(t))\bar{e}_2^T Q\bar{e}_2 - \bar{e}_3^T Z\bar{e}_3 + h^2\bar{e}_s^T R\bar{e}_s$$
$$\Xi_2 = \begin{bmatrix}\bar{e}_1-\bar{e}_2\\ \bar{e}_2-\bar{e}_3\end{bmatrix}^T\begin{bmatrix} R & S \\ * & R\end{bmatrix}\begin{bmatrix}\bar{e}_1-\bar{e}_2\\ \bar{e}_2-\bar{e}_3\end{bmatrix}$$
$$\Xi_3 = \mathrm{Sym}\{e_1^T Pe_s\} + e_1^T(Q+Z)e_1 - (1-\dot{d}(t))e_2^T Qe_2 - e_3^T Ze_3 + h^2 e_s^T Re_s$$
$$\Xi_4 = \begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_4 \\ e_2 - e_3 \\ e_2 + e_3 - 2e_5\end{bmatrix}^T \begin{bmatrix}\tilde{R} & U \\ * & \tilde{R}\end{bmatrix}\begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_4 \\ e_2 - e_3 \\ e_2 + e_3 - 2e_5\end{bmatrix}, \ \tilde{R} = \mathrm{diag}\{R, 3R\}$$
$$\bar{e}_s = [A, A_d, 0],\ \bar{e}_i = [0_{n\times(i-1)n}, I_{n\times n}, 0_{n\times(3-i)n}], i = 1, 2, 3$$
$$e_s = [A, A_d, 0, 0, 0],\ e_i = [0_{n\times(i-1)n}, I_{n\times n}, 0_{n\times(5-i)n}], i = 1, 2, \cdots, 5$$

where the related notations are defined in Box I. Moreover, the above two conditions are equavilant to each other, i.e., there exists a set of feasible solutions of (6) if and only if there exists a set of feasible solutions of (7) [Note that it is also true for the case that the delay change rates are unavailable].

*Proof:* Construct an LKF without any augmented term as follows

$$V(t) = x^T(t)Px(t) + \int_{t-d(t)}^t x^T(s)Qx(s)ds + \int_{t-h}^t x^T(s)Zx(s)ds + h\int_{-h}^0\int_{t+\theta}^t \dot{x}^T(s)R\dot{x}(s)dsd\theta \tag{8}$$

where $P > 0, Q \geq 0, Z \geq 0, R \geq 0$. It can be found that it satisfies $V(t) \geq \epsilon||x(t)||^2$ for a sufficiently small $\epsilon > 0$.

Calculating the derivative of the LKF along the solution of system (1) yields

$$\dot{V}(t) = \xi_1^T(t)\Xi_1\xi_1(t) - h\int_{t-h}^t \dot{x}^T(s)R\dot{x}(s)ds \tag{9}$$
$$= \xi_2^T(t)\Xi_3\xi_2(t) - h\int_{t-h}^t \dot{x}^T(s)R\dot{x}(s)ds \tag{10}$$

where

$$\xi_1(t) = [x^T(t), x^T(t-d(t)), x^T(t-h)]^T$$
$$\xi_2(t) = [\xi_1^T(t), v_1^T(t), v_2^T(t)]^T$$
$$v_1(t) = \int_{t-d(t)}^t \frac{x(s)}{d(t)}ds, \quad v_2(t) = \int_{t-h}^{t-d(t)} \frac{x(s)}{h-d(t)}ds$$

*(1) Proof of C1*: when $\Phi_1 \geq 0$ holds for any matrix $S$, using JI (3) and (5) to estimate the integral term in (9) yields [27]

$$h\int_{t-h}^t \dot{x}^T(s)R\dot{x}(s)ds \geq \frac{h}{d(t)}v_3^T(t)Rv_3(t) + \frac{h}{h-d(t)}v_4^T(t)Rv_4(t) \geq \xi_1^T(t)\Xi_2\xi_1(t) \tag{11}$$

where $v_3(t) = x(t) - x(t - d(t))$ and $v_4(t) = x(t - d(t)) - x(t - h)$. Then,

$$\dot{V}(t) \leq \xi_1^T(t)(\Xi_1 - \Xi_2)\xi_1(t) = \xi_1^T(t)\Phi_2\xi_1(t) \tag{12}$$

Thus, $\Phi_2 < 0$ leads to $\dot{V}(t) \leq -\epsilon_2||x(t)||^2$ for a sufficiently small scalar $\epsilon_2 > 0$, which ensures the asymptotical stability of system (1).

*(2) Proof of C2*: when $\Psi_1 \geq 0$ holds for any matrix $U$, using WBI (4) and (5) to estimate the integral term in (10)

yields [7]

$$h\int_{t-h}^{t}\dot{x}^T(s)R\dot{x}(s)ds \geq \frac{h}{d(t)}\begin{bmatrix}v_3(t)\\v_5(t)\end{bmatrix}^T\begin{bmatrix}R & 0\\0 & 3R\end{bmatrix}\begin{bmatrix}v_3(t)\\v_5(t)\end{bmatrix} + \frac{h}{h-d(t)}\begin{bmatrix}v_4(t)\\v_6(t)\end{bmatrix}^T\begin{bmatrix}R & 0\\0 & 3R\end{bmatrix}\begin{bmatrix}v_4(t)\\v_6(t)\end{bmatrix}$$

$$\geq \xi_2^T(t)\Xi_4\xi_2(t) \quad (13)$$

where $v_5(t) = x(t) + x(t-d(t)) - 2v_1(t)$ and $v_6(t) = x(t-d(t)) + x(t-h) - 2v_2(t)$. Then

$$\dot{V}(t) \leq \xi_2^T(t)(\Xi_3 - \Xi_4)\xi_2(t) = \xi_2^T(t)\Psi_2\xi_2(t) \quad (14)$$

Thus, $\Psi_2 < 0$ leads to $\dot{V}(t) \leq -\epsilon||x(t)||^2$ for a sufficiently small scalar $\epsilon > 0$, which ensures the asymptotical stability of system (1).

*(3) Proof of equivalence.* At first, an equivalent condition of $\Psi_2 < 0$ is obtained. Setting $U = \begin{bmatrix}S & S_2\\S_3 & S_4\end{bmatrix}$ in $\Psi_2$ and carrying out simple calculations yield:

$$\Psi_2 = \begin{bmatrix}I & 0 & 0 & 0 & 0\\0 & I & 0 & 0 & 0\\0 & 0 & I & 0 & 0\\I & I & 0 & -2I & 0\\0 & I & I & 0 & -2I\end{bmatrix}^T \Psi_3 \begin{bmatrix}I & 0 & 0 & 0 & 0\\0 & I & 0 & 0 & 0\\0 & 0 & I & 0 & 0\\I & I & 0 & -2I & 0\\0 & I & I & 0 & -2I\end{bmatrix} \quad (15)$$

where $\Psi_3 = \begin{bmatrix}\Xi_1 - \Xi_2 & \Pi_1\\ * & \Pi_2\end{bmatrix}$, $\Pi_1 = -\begin{bmatrix}\bar{e}_1 - \bar{e}_2\\ \bar{e}_2 - \bar{e}_3\end{bmatrix}^T\begin{bmatrix}0 & S_2\\S_3^T & 0\end{bmatrix}$, and $\Pi_2 = -\begin{bmatrix}3R & S_4\\ * & 3R\end{bmatrix}$. Thus, $\Psi_2 < 0 \iff \Psi_3 < 0$.

The first step is to prove that $C1 \implies C2$, i.e., *if the matrices $(P, Q, R, Z, S)$ are any feasible solutions of (6), then the matrices $(P, Q, R, Z, U = \begin{bmatrix}S & 0\\0 & 0\end{bmatrix})$ must be the feasible solutions of (7).*

- If $(R, S)$ are feasible solutions of $\Phi_1 \geq 0$, then

$$\left.\begin{matrix}\Phi_1 \geq 0\\R \geq 0\end{matrix}\right\} \implies \begin{bmatrix}\Phi_1 & 0\\ * & \begin{bmatrix}3R & 0\\0 & 3R\end{bmatrix}\end{bmatrix} \geq 0 \implies \begin{bmatrix}\begin{bmatrix}R & 0\\0 & 3R\end{bmatrix} & \begin{bmatrix}S & 0\\0 & 0\end{bmatrix}\\ * & \begin{bmatrix}R & 0\\0 & 3R\end{bmatrix}\end{bmatrix} \geq 0$$

  Thus, $(R, U = \begin{bmatrix}S & 0\\0 & 0\end{bmatrix})$ are the solutions of $\Psi_1 \geq 0$.
- If $(P, Q, R, Z, S)$ are feasible solutions of $\Phi_2 < 0$, then

$$\left.\begin{matrix}\Phi_2 < 0\\R \geq 0\end{matrix}\right\} \implies \begin{bmatrix}\Xi_1 - \Xi_2 & \begin{bmatrix}\bar{e}_1 - \bar{e}_2\\ \bar{e}_2 - \bar{e}_3\end{bmatrix}^T\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\\ * & -\begin{bmatrix}3R & 0\\ * & 3R\end{bmatrix}\end{bmatrix} < 0$$

  Thus, the matrices $(P, Q, R, Z, U = \begin{bmatrix}S & 0\\0 & 0\end{bmatrix})$ are feasible solutions of $\Psi_3 < 0$ and also the feasible solutions of $\Psi_2 < 0$ due to $\Psi_2 < 0 \iff \Psi_3 < 0$.

The second step is to prove that $C2 \implies C1$, i.e., *if the matrices $(P, Q, R, Z, U = \begin{bmatrix}S & S_2\\S_3 & S_4\end{bmatrix})$ are the feasible solutions of (7), then the matrices $(P, Q, R, Z, S)$ must be the feasible solutions of (6).*

- If $(R, U = \begin{bmatrix}S & S_2\\S_3 & S_4\end{bmatrix})$ are feasible solutions of $\Psi_1 \geq 0$, then

$$\Psi_1 \geq 0 \implies \begin{bmatrix}R & S & 0 & S_2\\ * & R & S_3^T & 0\\ * & * & 3R & S_4\\ * & * & * & 3R\end{bmatrix} \geq 0 \implies \Phi_1 = \begin{bmatrix}R & S\\ * & R\end{bmatrix} \geq 0$$

  Thus, $(R, S)$ must be the feasible solutions of $\Phi_1 \geq 0$.
- If $(P, Q, R, Z, U = \begin{bmatrix}S & S_2\\S_3 & S_4\end{bmatrix})$ are solutions of $\Psi_2 < 0$, then they are also the solutions of $\Psi_3 < 0$, thus

$$\Psi_3 = \begin{bmatrix}\Xi_1 - \Xi_2 & \Pi_1\\ * & \Pi_2\end{bmatrix} < 0 \implies \Phi_2 = \Xi_1 - \Xi_2 < 0$$

  Thus, the matrices $(P, Q, R, Z, S)$ must be the feasible solutions of $\Phi_2 < 0$.

By combining the above two steps, the equivalence of two conditions in Theorem 1 is proved. From the proof procedure, it can be found that the aformentioned discussion is still true when setting $Q = 0$, which means that two conditions are still equivalent when the delay change rates are unavailable. ■

Theorem 1 shows that, based on the same LKF, (8), the WBI-based and the JI-based criteria are equivalent. That is, although the extra term of the WBI ($3\chi_2^T R\chi_2$ in (4)) successfully reduces the estimation gap of the JI, it does not provide extra freedom to reduce the conservatism. Then, the following problem arises:

- *Why cannot the additionally positive term, $3\chi_2^T R\chi_2$, in (4) provide freedom to increase the feasibility of the WBI-based criteria ?*

This problem can be explained from the locations of additional terms introduced by the $3\chi_2^T R\chi_2$. By setting $\xi_3(t) = [v_5^T(t), v_6^T(t)]^T$, the following is true

$$\begin{aligned}\dot{V}(t) &\leq \xi_2^T(t)\Xi_3\xi_2(t) - \xi_2^T(t)\Xi_4\xi_2(t)\\ &= \xi_1^T(t)\Xi_1\xi_1(t) - \begin{bmatrix}\xi_1(t)\\ \xi_3(t)\end{bmatrix}^T\begin{bmatrix}\Xi_2 & -\Pi_1\\ * & -\Pi_2\end{bmatrix}\begin{bmatrix}\xi_1(t)\\ \xi_3(t)\end{bmatrix}\\ &= \begin{bmatrix}\xi_1(t)\\ \xi_3(t)\end{bmatrix}^T\begin{bmatrix}\Xi_1 - \Xi_2 & \Pi_1\\ * & \Pi_2\end{bmatrix}\begin{bmatrix}\xi_1(t)\\ \xi_3(t)\end{bmatrix}\end{aligned} \quad (16)$$

The extra terms in $\Xi_4$ introduced by the $3\chi_2^T R\chi_2$ is moved to terms $\Pi_1$ and $\Pi_1$ in $\Psi_3$. It is found that $\Psi_3 < 0$ requires $\Xi_1 - \Xi_2 < 0$. Thus, $\Phi_2 = \Xi_1 - \Xi_2 < 0$ is a necessary condition of $\Psi_2 < 0$ (due to $\Psi_2 < 0 \iff \Psi_3 < 0$). That is, the extra terms $\Pi_i, i = 1, 2$ from $3\chi_2^T R\chi_2$ in $\Psi_2 < 0$ do not relax the constraint of $\Phi_2 < 0$ and do not reduce the conservatism.

*Remark 1:* Many new integral inequalities, such as auxiliary function-based inequality [34] and Bessel-Legendre inequality [35], are developed by introducing more extra terms to reduce the estimation gap of the JI. It is predictable that those inequalities are also no contribution to reduce the conservatism if only a simple LKF without augmented terms is applied.

The above discussion shows that other treatments are required to show the advantage of the tighter WBI. In [7], the WBI-based criteria are found less conservative than the JI-based ones. By comparing the proof of Theorem 1 and Theorem 7 of [7], the key difference is that the LKF used in

[7] includes an augmented non-integral term, which activates the advantage of the WBI. Then another problem arises:

- *If any type of the augmented LKF or what special form of augmented LKF can reveal the advantage of the WBI?*

*Remark 2:* The following statements can be summarized for the above problem.

1) when the cross terms introduced by the augmented terms do not bring any $v_i(t), i = 1, 2$ related terms, the WBI-based criteria would not provide less conservatism than the JI-based ones; and
2) when the cross terms introduced by the augmented terms have the link between the $v_i(t), i = 1, 2$ and other vectors, the WBI-based criteria may be less conservative than the JI-based ones.

On the one hand, for an augmented term (for example, augmented non-integral terms including $x(t-d(t))$, $x(t-h/2)$, and/or $x(t-h)$), whose derivative will not lead to any cross term with information of $v_i(t), i = 1, 2$, the following could be obtained via the JI and the WBI, respectively:

- The JI-based estimation:

$$\dot{V}_{ji}(t) \le \eta^T(t)\Theta\eta(t) - \xi_1^T(t)\Xi_2\xi_1(t) = \eta^T(t)\Theta_1\eta(t)$$

- The WBI-based estimation:

$$\begin{aligned}\dot{V}_{wbi}(t) &\le \eta^T(t)\Theta\eta(t) - \begin{bmatrix}\xi_1(t)\\ \xi_3(t)\end{bmatrix}^T \begin{bmatrix}\Xi_2 & -\Pi_1\\ * & -\Pi_2\end{bmatrix}\begin{bmatrix}\xi_1(t)\\ \xi_3(t)\end{bmatrix}\\ &= \begin{bmatrix}\eta(t)\\ \xi_3(t)\end{bmatrix}^T \begin{bmatrix}\Theta_1 & \Pi_1\\ * & \Pi_2\end{bmatrix}\begin{bmatrix}\eta(t)\\ \xi_3(t)\end{bmatrix} \qquad (17)\end{aligned}$$

where $\eta^T(t)\Theta\eta(t)$ includes all parts of the derivative except $h\int_{t-h}^{t}\dot{x}^T(s)R\dot{x}(s)ds$, and $\eta(t) = \begin{bmatrix}\xi_1(t)\\ \eta_0(t)\end{bmatrix}$, $\Theta_1 = \Theta + \begin{bmatrix}\Xi_2 & 0\\ 0 & 0\end{bmatrix}$ with $\eta_0(t)$ being the additional vectors caused by the augmented vectors of the LKF (for example, $\dot{x}(t-d(t))$, $x(t-h/2)$, $\dot{x}(t-h/2)$, $\dot{x}(t-h)$).

It can be found that system stability requires the condition of $\Theta_1 < 0$ based on the JI-based method. Although the WBI introduces extra terms, $\Pi_i, i = 1, 2$, the stability of system still requires the condition of $\Theta_1 < 0$ based on the WBI-based method. Therefore, the WBI-based criteria developed through this type of augmented LKF cannot provide less conservatism.

On the other hand, the augmented LKF used in [7] satisfies statement 2) of Remark 2 and the results in [7] verify this statement. Specifically, by constructing an augmented non-integral term ($\tilde{x}^T(t)P\tilde{x}(t)$ of [7]), some additional cross terms arising in its derivative are introduced into the obtained LMI conditions, and $\Psi_3 = \begin{bmatrix}\Xi_1 - \Xi_2 & \Pi_1\\ * & \Pi_2\end{bmatrix} < 0$ is renewed by a relaxed one, $\tilde{\Psi}_3 = \begin{bmatrix}\Xi_1 - \Xi_2 + \Gamma_1 & \Pi_1 + \Gamma_2\\ * & \Pi_2\end{bmatrix} < 0$, where $\Gamma_i, i = 1, 2$ are the combinations of matrices introduced by augmented vectors in non-integral term of the LKF, and $\Gamma_2$ links the $v_i(t), i = 1, 2$ and other vectors. Thus, $\Pi_i, i = 1, 2$ obtained by the extra term of the WBI ($3\chi_2^T R\chi_2$) becomes helpful to reduce the conservatism.

## III. Criteria via a delay-product-type LKF

Two delay-product-type terms satisfying statement 2) of Remark 2 are developed to construct a new LKF, shown as follows

$$\begin{aligned}\bar{V}(t) &= \zeta_1^T(t)\bar{P}\zeta_1(t) + d(t)\zeta_2^T(t)P_1\zeta_2(t) + [h-d(t)]\zeta_3^T(t)P_2\zeta_3(t)\\ &\quad + \int_{t-d(t)}^{t}\zeta_4^T(s)\bar{Q}\zeta_4(s)ds + \int_{t-h}^{t-d(t)}x^T(s)Zx(s)ds\\ &\quad + h\int_{-h}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)R\dot{x}(s)dsd\theta \qquad (18)\end{aligned}$$

where

$$\zeta_1(t) = \left[x^T(t), x^T(t-d(t)), \textstyle\int_{t-d(t)}^{t}x^T(s)ds, \int_{t-h}^{t-d(t)}x^T(s)ds\right]^T$$
$$\zeta_2(t) = \left[x^T(t),\ x^T(t-d(t)),\ v_1^T(t)\right]^T$$
$$\zeta_3(t) = \left[x^T(t),\ x^T(t-d(t)),\ v_2^T(t)\right]^T$$
$$\zeta_4(s) = \left[x^T(s),\ \dot{x}^T(s)\right]^T$$

By using (4) and (5) to estimate the derivative of the LKF, the following theorem is obtained.

*Theorem 2:* For given $h$, $\mu_1$, and $\mu_2$, system (1) is asymptotically stable, if the one of the following conditions holds

C1: there exist $4n \times 4n$ matrix $\bar{P} > 0$, $3n \times 3n$ matrices $P_1 > 0$ and $P_2 > 0$, $2n \times 2n$ matrix $\bar{Q} > 0$, $n \times n$ matrices $Z > 0$ and $R > 0$, and any $2n \times 2n$ matrix $U$, such that the following holds:

$$\Omega_1 = \begin{bmatrix}\tilde{R} & U\\ * & \tilde{R}\end{bmatrix} \ge 0, \quad \Omega_2 = \Upsilon_1 + \Upsilon_1^T + \Upsilon_2 - \bar{\Xi}_4 < 0 \qquad (19)$$

C2: there exist symmetric $4n \times 4n$ matrix $\bar{P}$, symmetric $3n \times 3n$ matrices $P_1$ and $P_2$, symmetric $2n \times 2n$ matrix $\bar{Q} > 0$, symmetric $n \times n$ matrices $Z > 0$ and $R > 0$, $2n \times 2n$ matrix $U$, and $n \times n$ matrix $X$, such that $\Omega_1 \ge 0$, $\Omega_2 < 0$ and the following holds:

$$\Omega_3 \ge 0, \quad \Omega_4 > 0 \qquad (20)$$

where the related notations are defined in Box II.

*Proof:* The derivative of $P_1$- and $P_2$-dependent delay-product-type terms can be obtained as

$$\begin{aligned}&\frac{d}{dt}\left\{d(t)\zeta_2^T(t)P_1\zeta_2(t) + [h-d(t)]\zeta_3^T(t)P_2\zeta_3(t)\right\}\\ &= \dot{d}(t)\zeta_2^T(t)P_1\zeta_2(t) - \dot{d}(t)\zeta_3^T(t)P_2\zeta_3(t)\\ &\quad + 2d(t)\zeta_2^T(t)P_1\begin{bmatrix}\dot{x}(t)\\ (1-\dot{d}(t))\dot{x}(t-d(t))\\ \frac{x(t)-(1-\dot{d}(t))x(t-d(t))-\dot{d}(t)v_1(t)}{d(t)}\end{bmatrix}\\ &\quad + 2(h-d(t))\zeta_3^T(t)P_2\begin{bmatrix}\dot{x}(t)\\ (1-\dot{d}(t))\dot{x}(t-d(t))\\ \frac{(1-\dot{d}(t))x(t-d(t))-x(t-h)+\dot{d}(t)v_2(t)}{h-d(t)}\end{bmatrix}\end{aligned}$$

Then, calculating the derivative of other terms in (18) and using $\Omega_1 \ge 0$, (4), and (5) to estimate the $R$-dependent single integral term yield

$$\begin{aligned}\dot{\bar{V}}(t) &= \xi_4^T(t)\left(\Upsilon_1 + \Upsilon_1^T + \Upsilon_2\right)\xi_4(t) - h\int_{t-h}^{t}\dot{x}^T(s)R\dot{x}(s)ds\\ &\le \xi_4^T(t)\Omega_2\xi_4(t) \qquad (21)\end{aligned}$$

Box II: Notations used in Theorem 2

$$\Upsilon_1 = \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ d(t)\tilde{e}_4 \\ h_d(t)\tilde{e}_5 \end{bmatrix}^T \bar{P} \begin{bmatrix} \tilde{e}_s \\ (1-\dot{d}(t))\tilde{e}_6 \\ \tilde{e}_1-(1-\dot{d}(t))\tilde{e}_2 \\ (1-\dot{d}(t))\tilde{e}_2-\tilde{e}_3 \end{bmatrix} + \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_4 \end{bmatrix}^T P_1 \begin{bmatrix} d(t)\tilde{e}_s \\ d(t)(1-\dot{d}(t))\tilde{e}_6 \\ \tilde{e}_1-(1-\dot{d}(t))\tilde{e}_2-\dot{d}(t)\tilde{e}_4 \end{bmatrix} + \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_5 \end{bmatrix}^T P_2 \begin{bmatrix} (h-d(t))\tilde{e}_s \\ (h-d(t))(1-\dot{d}(t))\tilde{e}_6 \\ (1-\dot{d}(t))\tilde{e}_2-\tilde{e}_3+\dot{d}(t)\tilde{e}_5 \end{bmatrix}$$

$$\Upsilon_2 = \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_s \end{bmatrix}^T \bar{Q} \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_s \end{bmatrix} - (1-\dot{d}(t)) \begin{bmatrix} \tilde{e}_2 \\ \tilde{e}_6 \end{bmatrix}^T \left(\bar{Q} - \begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} \tilde{e}_2 \\ \tilde{e}_6 \end{bmatrix} - \tilde{e}_3^T Z \tilde{e}_3 + h^2 \tilde{e}_s^T R \tilde{e}_s + \dot{d}(t) \left\{ \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_4 \end{bmatrix}^T P_1 \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_4 \end{bmatrix} - \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_5 \end{bmatrix}^T P_2 \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_5 \end{bmatrix} \right\}; \ \Omega_3 = \begin{bmatrix} EP_1E^T & X \\ * & EP_2E^T+Z \end{bmatrix}$$

$$\bar{\Xi}_4 = \begin{bmatrix} \tilde{e}_1-\tilde{e}_2 \\ \tilde{e}_1+\tilde{e}_2-2\tilde{e}_4 \\ \tilde{e}_2-\tilde{e}_3 \\ \tilde{e}_2+\tilde{e}_3-2\tilde{e}_5 \end{bmatrix}^T \begin{bmatrix} \tilde{R} & U \\ * & \tilde{R} \end{bmatrix} \begin{bmatrix} \tilde{e}_1-\tilde{e}_2 \\ \tilde{e}_1+\tilde{e}_2-2\tilde{e}_4 \\ \tilde{e}_2-\tilde{e}_3 \\ \tilde{e}_2+\tilde{e}_3-2\tilde{e}_5 \end{bmatrix}; \quad \Omega_4 = P + \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_0 \end{bmatrix}^T [d(t)P_1+(h-d(t))P_2] \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_0 \end{bmatrix} + Sym\left\{ \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_0 \end{bmatrix}^T \left( P_1 \begin{bmatrix} \hat{e}_0 \\ \hat{e}_0 \\ \hat{e}_3 \end{bmatrix} + P_2 \begin{bmatrix} \hat{e}_0 \\ \hat{e}_0 \\ \hat{e}_4 \end{bmatrix} \right) \right\} + \frac{\Omega_3}{h}$$

$\tilde{e}_s = [A, A_d, 0, 0, 0, 0]$, $\tilde{e}_i = [0_{n\times(i-1)n}, I_{n\times n}, 0_{n\times(6-i)n}], i = 1, 2, \cdots, 6$; $\hat{e}_0 = [0_{n\times 4n}]$, $\hat{e}_i = [0_{n\times(i-1)n}, I_{n\times n}, 0_{n\times(4-i)n}], i = 1, 2, \cdots, 4$

where $\xi_4(t) = [\xi_2^T(t), \dot{x}^T(t-d(t))]^T$.

Thus, $\Omega_2 < 0$ leads to $\dot{\bar{V}}(t) \le -\epsilon||x(t)||^2$ for a sufficiently small scalar $\epsilon > 0$.

*(1) Proof of C1:* The condition ($\bar{P} > 0$, $P_1 > 0$, $P_2 > 0$, $\bar{Q} > 0$, $Z > 0$, $R > 0$) leads to $V(t) \ge \epsilon||x(t)||^2$ for a sufficiently small $\epsilon > 0$. Therefore, system (1) is asymptotically stable if C1 is satisfied.

*(2) Proof of C2:* Inspired by the idea of [31], a relaxed condition for ensuring the positive definite of LKF is obtained. Based on $Z > 0$ and JI (3), the $Z$-dependent term can be estimated as

$$\int_{t-h}^{t-d(t)} x^T(s)Zx(s)ds \ge \frac{[(h-d(t))v_2(t)]^T Z[(h-d(t))v_2(t)]}{(h-d(t))}$$

The $P_1$- and $P_2$-dependent terms can be rewritten as

$$\begin{aligned} & d(t)\zeta_2^T(t)P_1\zeta_2(t) + [h-d(t)]\zeta_3^T(t)P_2\zeta_3(t) \\ = & \begin{bmatrix} x(t) \\ x(t-d(t)) \\ 0 \end{bmatrix}^T [d(t)P_1 + (h-d(t))P_2] \begin{bmatrix} x(t) \\ x(t-d(t)) \\ 0 \end{bmatrix} \\ & +2 \begin{bmatrix} x(t) \\ x(t-d(t)) \\ 0 \end{bmatrix}^T \left\{ P_1 \begin{bmatrix} 0 \\ 0 \\ w_1(t) \end{bmatrix} + P_2 \begin{bmatrix} 0 \\ 0 \\ w_2(t) \end{bmatrix} \right\} \\ & + \frac{[w_1(t)]^T EP_1E^T[w_1(t)]}{d(t)} + \frac{[w_2(t)]^T EP_2E^T[w_2(t)]}{h-d(t)} \end{aligned}$$

where $w_1(t) = d(t)v_1(t)$, $w_2(t) = (h-d(t))v_2(t)$, and $E = [0 \ \ 0 \ \ I]$. Based on $\Omega_3 > 0$ and (5), the following holds

$$\begin{aligned} & \frac{[w_1(t)]^T EP_1E^T[w_1(t)]}{d(t)} + \frac{[w_2(t)]^T (EP_2E^T+Z)[w_2(t)]}{h-d(t)} \\ \ge & \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}^T \frac{\Omega_3}{h} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \end{aligned} \tag{22}$$

Therefore, LKF (18) can be estimated as

$$\begin{aligned} \bar{V}(t) \ge & \xi_5^T(t)\Omega_4\xi_5(t) + \int_{t-d(t)}^{t} \zeta_4^T(s)\bar{Q}\zeta_4(s)ds \\ & + h\int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta \end{aligned} \tag{23}$$

where $\xi_5(t) = [x^T(t), \ x^T(t-d(t)), \ w_1(t), \ w_2(t)]^T$. Thus, the condition ($\Omega_4 > 0$, $\bar{Q} > 0$, $R > 0$) leads to $V(t) \ge \epsilon||x(t)||^2$ for a sufficiently small $\epsilon > 0$. Therefore, system (1) is asymptotically stable if C2 is satisfied. ■

*Remark 3:* Similar to the discussion in Remark 3 of [17], conditions of Theorem 2 can be guaranteed for all time-varying delay satisfying (2) if they are satisfied for $(d(t), \dot{d}(t)) \in [0, h] \times [-\mu_1, \mu_2]$. Since the condition ($\bar{P} > 0$, $P_1 > 0$, $P_2 > 0$) of C1 is relaxed to be $\Omega_3 > 0$ and $\Omega_4 > 0$, C2 of Theorem 2 is less conservative.

*Remark 4:* The $\bar{P}$-dependent augmented term in (18) is similar to the one in [17], and the $P_1$- and $P_2$-dependent delay-product-type terms in (18) are developed based on the similar terms for discrete-time systems [29]. Several terms in the derivative of them, $\Upsilon_1$ and $\Upsilon_2$, satisfy statement 2) of Remark 2 and are helpful to reduce the conservatism.

*Remark 5:* If the $R$-dependent single integral term in $\dot{\bar{V}}(t)$ is estimated by using the JI, then $\bar{\Xi}_4$ in $\Omega_2$ becomes $\bar{\Xi}_2 = \begin{bmatrix} \tilde{e}_1-\tilde{e}_2 \\ \tilde{e}_2-\tilde{e}_3 \end{bmatrix}^T \begin{bmatrix} R & S \\ * & R \end{bmatrix} \begin{bmatrix} \tilde{e}_1-\tilde{e}_2 \\ \tilde{e}_2-\tilde{e}_3 \end{bmatrix}$. Then, the quadratic term related to $v_1(t)$ in $\Omega_2$ becomes to be $-v_1^T(t)\left[\dot{d}(t)EP_1E^T\right]v_1(t)$, which is positive for $\dot{d}(t) = \mu_1$. Thus, one cannot find any feasible solutions of the condition, $\Omega_2 < 0$.

## IV. A NUMERICAL EXAMPLE

A numerical example is given to verify the main results proposed in this note.

*Example 1:* Consider system (1) with the parameters

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \tag{24}$$

The maximal admissible delay upper bounds (MADUPs) with respect to various $\mu$ calculated by the proposed criteria, together with the ones reported in literature, are listed in Table I, where Th. and Co. indicate Theorem and Corollary, respectively. The number of decision variable (NoV) is also given to compare the computation complexity in the table, where $NoV_{[32]} = 2[(1+n_\phi)^2 + (6+2n_\phi)(7+2n_\phi)+2]n^2 + (3+n_\phi)n > 90n^2 + 3n$ with $n$ being the order of system matrix and $n_\phi > 0$ being the order of the filter system.

The MADUPs calculated by the JI-based criterion (Th.1.C1) and the WBI-based criterion (Th.1.C2) are identical, which verifies the statement given in Theorem 1. The results also show the less conservatism of Theorem 2.C2 in comparison with Theorem 2.C1. Moreover, the less conservatism of Theorem 2.C2 compared with the ones in literature can be found from the table. Although the NoV of Theorem 2.C2 is bigger than that of the criteria [7], [13], [26], [30], [33], the MADUPs provided by Theorem 2.C2 are larger. Compared with the criteria in [17], [32], [36], Theorem 2.C2 obtains less conservative MADUPs but requires less computation complexity.

TABLE I
MADUPS FOR VARIOUS $\mu = -\mu_1 = \mu_2$.

| Methods | $\mu = -\mu_1 = \mu_2$ | | | | NoVs |
|---|---|---|---|---|---|
| | 0.1 | 0.2 | 0.5 | 0.8 | |
| Th.1 [13] | 3.605 | 3.039 | 2.043 | 1.492 | $11.5n^2 + 2.5n$ |
| Th.1 [33] | 3.611 | 3.047 | 2.072 | 1.590 | $7n^2 + 4n$ |
| Co.3 [36] | 3.669 | 3.163 | 2.337 | 1.934 | $31.5n^2 + 7.5n$ |
| Th.7 [7] | 4.703 | 3.834 | 2.420 | 2.137 | $10n^2 + 3n$ |
| Th.1 [30] | 4.753 | | 2.429 | 2.183 | $24n^2 + 4n$ |
| Th.2.C2 [26] | 4.714 | | 2.608 | 2.375 | $23n^2 + 4n$ |
| Th.3 [32] | 4.794 | 3.995 | 2.682 | 1.957 | $NoV_{[32]}$ |
| Th.1 [17] | 4.788 | 4.060 | 3.055 | 2.615 | $65n^2 + 11n$ |
| Th.1.C1 | 3.658 | 3.163 | 2.337 | 1.934 | $3n^2 + 2n$ |
| Th.1.C2 | 3.658 | 3.163 | 2.337 | 1.934 | $6n^2 + 2n$ |
| Th.2.C1 | 4.809 | 4.089 | 3.062 | 2.638 | $24n^2 + 7n$ |
| Th.2.C2 | 4.809 | 4.091 | 3.109 | 2.710 | $25n^2 + 7n$ |

## V. CONCLUSIONS

This note has investigated the stability analysis of systems with time-varying delay. It has discussed the contribution of LKFs in the analysis of the relationship between the tightness of inequalities and the conservatism of criteria. On the one hand, the equivalence of the stability criterion obtained by the WBI and the one by the JI, all based on a class of simple LKF, has been proved theoretically. It means that a tighter inequality does not always lead to a better criterion. On the other hand, it has been also found that the advantage of the WBI compared with the JI can be revealed by combining it with a suitable augmented LKF. Based on this observation, the delay-product-type terms has been applied to construct the novel LKF, under which two stability criteria have been established. Finally, the equivalence of WBI- and JI-based criteria and the benefit of the proposed criteria have been demonstrated through a numerical example.

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