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Game theoretical approaches for pricing of Non-life insurance policies into a competitive market environment

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Abstract

Standard actuarial approaches for non-life insurance products suggest that the premium is divided into three main components: the actuarial price, the safety loading, and the loading for expenses. The number of product-specific policies from different companies has increased significantly, and strong market competition has boosted the demand for a competitive premium in global insurance market. Thus, the actuarial premium could eventually be altered by an insurer's marketing and management department regarding the competitive environment. Thus in this thesis, considering the competition in insurance market, game theoretical approaches are applied to investigate the influence of competition on general insurance pricing.

Firstly, a two-period deterministic N-player game is formulated to investigate the optimal pricing strategy by calculating the Nash equilibrium in an insurance market. Under that framework, each insurer is assumed to maximise its utility of wealth over the unit time interval. By analyzing the competition between each pair of insurers, the whole markets' competition is characterized through an aggregation. With the purpose of solving a game of N-players, the best-response potential game with non-linear aggregation is implemented. The existence of a Nash equilibrium is proved by finding a potential function of all insurers' payoff functions. A 12-player insurance game illustrates the theoretical findings under the framework in which the best-response selection premium strategies always provide the global maximum value of the corresponding payoff function.

Secondly, deterministic differential games are constructed with the purpose of studying the insurers' equilibrium premium in a competitive market. We apply an optimal

control theory to determine the open-loop Nash equilibrium premium strategies. In this direction, two models are formulated and studied. The market power of each insurance company is characterized by a price sensitive parameter, and the business volume is affected by the solvency ratio. Considering the average market premiums, the first model studies an exponential relation between premium strategies and volume of business. The other model initially characterizes the competition between any selected pair of insurers, then aggregates all the paired competitions in the market. Numerical examples illustrate the premium dynamics, and show that premium cycles may exist in equilibrium.

Thirdly, a multi-stage stochastic game will be constructed. Insurers are considered to be risk-averse, that is, insurers will to set risk-premiums on their products with the purpose of avoiding risk. Mean-variance Utility function will be adopted. The expenditures of insurance companies will be discussed separately as exposure related costs and non-exposure related costs. The expenditures of insurance companies will be discussed separately as exposure related costs and non-exposure-based costs. The exposure-based component is assumed to be stochastic.

Finally, summary of the conclusions complete the thesis.

Keywords: Insurance Market Competition; Non-life Insurance; Non-cooperative Game; Potential Game with Aggregation; Pure Nash Equilibrium; Price Cycles; Solvency Ratio

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Chapter 1

Introduction

A general insurance policy is an agreement between insurer and policyholder. Policyholder pays the premium while insurance company agrees to repay the policyholder for unpredictable losses within the contract time period. In the insurance world, the determination of an appropriate and attractive premium is always a highly challenging issue also because of the competition between different companies. Furthermore, the premium loading depends critically on the price that other insurers charge for comparable policies. Clapp [8] demonstrates using the seminal model by Rothschild and Stiglitz's ([51, 50]) that companies are able to use the quantity of insurance to compete for customers and, consequently, a bigger volume of business or a market share. By changing the level of indemnity while holding the premium rate constant (quantity competition), it is possible to induce customers to reveal their risk class.

On the other hand, insurance premiums contribute to the wealth while claims and other expenses counteract theoretically the growth of the insurer. Insurance pricing is a fundamental aspect that attracts the interest for both actuaries and academics. Standard actuarial approaches for non-life insurance products suggest that the premium is claimed to be divided into three main components: the *actuarial price*, the *safety loading* and the *loading for expenses*. The actuarial price is normally deduced according to different premium principles, such as the *Net Premium Principle*, the *Expected Value Premium Principle*, etc. (for overview of principles of premium pricing, see [49], [58]).

Classical approaches focus on determining the safety loading of each policy class proportional to the expected claim expenses or to the moment of it.

1.1 Motivation

However, in highly competitive environments which are dominated by a relatively small number of insurance companies (comparing with the magnitude and quantity of the finance-related firms), each insurer monitors, tries to predict reactions and takes advantages against the others, the actuarial premium might be altered eventually by the marketing and management department due to several reasons, such as the affordability of customers, the market conditions and the mutualization across the portfolio of customers to decrease risk. It is also proposed that the pricing cycles which can be found in different lines of insurance are affected by market competition, see [22]. These suggestions indicate that the insurance premium price should not be focused only on the risk perspectives. In order to study the competition between all insurers, a model is needed to be constructed to research on insurers' premium pricing interactions in a competitive insurance market.

1.2 Developments in Competitive Insurance Markets

This area of research has attracted the interest of many academics lately, but still there is little literature that has been done on how the competition might affect the insurance premiums and how the insurers will respond to changes in the levels of premiums being offered by the competitors.

Taylor [54] is the first from the actuarial community who mentioned that the competition is a key component for the insurance premium pricing, and he used the Australian market to extract some very useful remarks. From his study, the cyclical behaviour of the premium rates seems peculiar and raises several questions. For instance *"what the market is attempting to achieve by such pricing"* and *"what individual insurers* *are attempting to achieve in following the market"*. Analytically, the relation between the market's behaviour and the optimal response of an individual insurer was explored, whose objective is to maximize the expected present value of the wealth arising over a pre-defined finite time horizon period. He also assumed that the insurance products display a positive price-elasticity of demand. Thus, if the market as a whole begins underwriting at a loss, any attempt by a particular insurer to maintain profitability will result in a reduction of its volume of business. According to Taylor's results [54], the optimal strategies do not follow what someone might expect. Therefore, he stated that the optimal response depends upon various factors including:

- the predicted time which will elapse before a return of market rates into profitability,
- the price elasticity of demand for the insurance product under consideration, and
- the rate of return required on the capital supporting the insurance operation.

In his next paper, Taylor [55] noted that the optimum underwriting strategies might be substantially affected by the proper marginal expense rates which must be taken into account. It was first showed that the optimal strategy is not affected by the introduction of a component of fixed expenses, irrespective of the size of that component. However, the strategy is affected if the concomitant of the introduction of fixed expenses is the recognition of lower marginal expenses. It is possible to set limits on the effect of expenses on optimal underwriting strategy. The sharpness of these limits depends on:

- the extent of variation in marginal expense rates as demand varies;
- the price-elasticity of demand.

After almost two decades of silence, Emms and Haberman [23] extended significantly Taylor's ideas presented in [54, 55] considering the continuous form of his model and particularly, they assumed that the average premium is a positive random process with finite mean at time t and left the distribution for the mean claim size process unspecified. For two choices of the demand function, a smooth optimal control was calculated.

Later, Emms et al. [18] modelled market's average premium as a geometric Brownian motion. Consequently, the optimal strategy has two modes depending on the model parameters: a) either set an infinite premium and accumulate wealth from the existing customer base or b) set the premium at just above break-even in order to maximize market exposure whilst at the same time making a profit. Consequently, the optimal strategy for two particular approaches was investigated to adjust the premium. The first approach was based on a linear function of the market average premium, while the second one involved a linear combination of the break-even premium and the market average premium.

Simultaneously, Emms [25] determined the optimal strategy for an insurer which maximizes a particular objective over a fixed planning horizon and the premium by using a competitive demand model as well as the expected main claim size. Base on his approach, it is not enough that an insurer set a price to cover the claims if the rest of the market undercuts that price. Additionally, his demand law specified in such a way that the insurer's income and the exposure change was related to the market premium. Moreover, in [19] he studied the optimal premium pricing process into a competitive market with different types of constrains. Analytically, he calculated the premium strategy which maximises the objective of the insurer subject to a constrain on the control or constrains on the reserve that the insurer must hold. Premium restrictions lead to control constrains, while solvency requirements lead to state constrains. By assuming a deterministic control framework, the optimisation problem was solved by using elements from control parametrisation; see [57].

Finally, a simpler parametrisation was introduced by Emms [21], which represents the insurance market's response to an insurer adopting a pricing strategy determined via optimal control theory and claims are modelled using a lognormally distributed mean claim size rate. Analytically, a generalisation of the demand function which was mentioned in [19] had been considered which impacts significantly on the optimal premium strategy for an insurer.

Taylor ([54, 55]) and Emms et al. ([23, 18, 25, 19, 20, 24, 21]) studied fixed premium strategies and the sensitivity of the model to its parameters involved. In their approaches, the important parameters which determined the optimal strategies are the ratio of initial market average premium to break-even premium, the measure of the inverse elasticity of the demand function and the non-dimensional drift of the market average premium. In [41], a stochastic demand function was first introduced for the volume of business in a discrete-time set up extending further the previous ideas. Additionally, using also a linear discounted function for the wealth process of the company, a closed form (endogenous) formula was derived for the optimal premium strategy of the insurance company when it was expected to lose part of the market. Mathematically speaking, a maximization problem has been proposed for the wealth process of a company, which has been solved using stochastic dynamic programming. Thus, the optimal controller (i.e. the premium) has been defined endogenously by the market as the company struggles to increase its volume of business into a competitive environment with the same characteristics as in the previous literature.

In [42], the volume of business was formed to be a general stochastic demand function extending further the ides presented in [41] making the model more pragmatic and realistic. Thus, for the formulation of the volume of business, the company's reputation is also considered. According to [9], company's reputation has a strong influence on buying decisions or in other words, on the demand of the company's product. So, in that case the function for the volume of business emphasizes the ratio of the markets average to the company's premium, the past year experience, the company's reputation and a stochastic disturbance.

Very recently, in [43], the analytical solutions for some common special cases and a premium strategy concerning market's average premium is considered. In that paper the disturbance of the volume of business function was denoted by the set of all other stochastic variables that are considered to be relevant to the demand function (moreover, they are assumed to be independently distributed in time and Gaussian). What is more, the volume of business was modelled as a nonlinear function with respect to reserve, the premium, the noise and a quadratic performance criterion concerning the utility function to be implemented.

Except the idea of optimal pricing for general insurance, other approaches are also adopted to study the premium pricing. [56] constructs a simple but realistic insurance model to study the stability of premium rates, profitability, and the market concentration. The competitive premium is a maximum selection between several strategies. The exposure of the corresponding premium strategy of a selected insurer is calculated through the exposure exchange among all insurers.

1.3 Game-Theoretic Approaches

However, for all the models and approaches discussed in the previous subsection, a common assumption has been made that there exists a single insurer whose pricing strategy does not cause any reaction to the rest of the market's competitors. Thus, each insurance market participants' reaction can not be observed. The market price is independent of their own actions. In reality, this is not often the case. However, before we proceed further with the recently developed game theoretic approaches, it should be mentioned that Emms [21] released this assumption partially and he suggested a scenario where there is a leader in the market whose pricing is followed by other insurance companies. A simple parametrisation is introduced which represents the insurance market's response to an insurer's pricing strategy.

Game theoretical approaches have been introduced mostly lately in the premium pricing processes in non-life insurance products as they offer the opportunity to observe the pricing competition among the whole insurance market. In other words, the competition among insurers gives the pricing strategy of each market participants in a constructed insurance game, while one can only get a single insurer's pricing strategy through optimal control approach in previous studies. In this paper, as it will be discussed more extensively in the following subsection, a non-cooperative game model is considered in a competitive insurance market.

The use of game theory in actuarial science has a long history. The first attempts go back to Borch ([5, 6]) and Lemaire ([33, 34], who applied cooperative games to model insurer and reinsurer's risk transfer (for a review, see section 3.1 of [7]). Two models were applied in non-life insurance markets for non-cooperative games: a) the Bertrand oligopoly where insurers set premiums and b) the Cournot oligopoly where insurers choose volume of business. See [44, 48, 17] for the Bertrand model and also see [46, 45] for the Cournot model.

Emms [22] developed a model by applying a differential game-theoretic methodology. Under his framework, each insurer's price depends on other insurers' premium strategies. The whole insurance market is considered as two component: the sum of all insurers' exposure and the unallocated insurance exposure. Two price functions are investigated: the quotient price function and the difference price function. The calculation of the optimal pricing strategy requires the solution of multiple coupled optimization problems.In [22], he got the solution in a two-player game considering the competition as market average premium. What is more, two significant features of the model were also investigated in details in his approach: the effect of the limited total demand for policies and the uncertainty component for the determination of the break-even premium of an insurance policy.

Finally, very recently, Boonen [4] also proposed a way to optimally regulate bargaining for risk redistributions. Thus, he investigated the strategic interaction between two insurance companies who trade risk Over-the-Counter in one-period model. A Nash Equilibrium may exist in a game which the trading of risks occurs Over-The-Counter by restricting the strategy space a priori.

1.4 A New Approach: Potential Game with Aggregation

In our approach, with the purpose of solving the problem while the insurance market contains a large number of insurers, a two-stage insurance game is constructed which considers the competition in pairs. Thus, instead of just assuming that the competition is only through the market average premium, an aggregate game approach is formulated in order to further investigate the insurance market competition. In that direction, the Nobel-prize winning concept of *aggregative* game which was first proposed by Reinhard Selten in 1970 ([52]), who considered the aggregate as the sum of the players' strategies, is applied broadly in our approach. Thus, the derived strategy for all the insurers in the insurance market is presented as a single parameter, i.e. the aggregate. In more details, each insurers' utility (payoff) function is only depended on its own pricing premium strategy and the aggregate parameter.

Following also the suggestions by Taylor in [56], the market competition is measured through calculating an insurer's new volume of exposure by summing up all the policy flows during the competition between each other and the volume of exposure in a previous stage. An non-linear aggregate is obtained, which presents the strategies of all the insurers in the market. Moreover, a potential game approach is further investigated in order to prove the existence of Nash equilibrium in the insurance game. This approach also offers us an opportunity to simplify the problem of finding out Nash equilibrium by solving one single optimization problem, however, not detailed discussion will be provided here, as it is far beyond the scope of the present article.

Literature of potential games can trace back to Monderer and Shapley [38, 39], who created the also Nobel-prize winning concept of a *potential* game based on a congestion game. This proposed potential game technique does not only solve the congestion game itself, but also it can be regarded as an equilibrium refinement tool. Following this idea, the best-response potential games were introduced and characterized by Voorneveld in [60]. In his paper, it was proposed that for any best-response potential game, if the potential has a maximum over its domain, the best-response potential game has a Nash equilibrium.

Dubey et al. [16] were the first to embed the aggregate into potential games. Considering just a linear aggregation, they investigated a special type of best-response potential games which restrict the best-response selection that is continuously decreasing or increasing function. Then, it is proved that any game with linear aggregation and a decreasing or increasing continuous best-response selection, belongs to pseudo-potential games, which is pre-defined in their paper. By proving that any pseudo-potential game have a pure Nash equilibrium strategy, the existence of Nash equilibrium was obtained in this special class of potential game irrespective of whether strategy sets are convex or payoff functions quasi-concave.

The rest of paper is organized as follows. A two-period deterministic N-player game is formulated to investigate the optimal pricing strategy by calculating the Nash equilibrium in an insurance market in Chapter 2. Chapter 3 constructs deterministic differential games with the purpose of studying the insurers' equilibrium premium pricing in a competitive market. A multi-stage stochastic game is constructed in Chapter 4, considering the risk aversion of players. Chapter 5 is the conclusion which is the summary of this thesis.

Chapter 2

Potential Games with Aggregation in Non-cooperative General Insurance Markets

In the global insurance market, the number of product-specific policies from different companies has increased significantly, and strong market competition has boosted the demand for a competitive premium. Thus, in the present paper, by considering the competition between each pair of insurers, an N-player game is formulated to investigate the optimal pricing strategy by calculating the Nash equilibrium in an insurance market. Under that framework, each insurer is assumed to maximise its utility of wealth over the unit time interval. With the purpose of solving a game of N-players, the best-response potential game with non-linear aggregation is implemented. The existence of a Nash equilibrium is proved by finding a potential function of all insurers' payoff functions. A 12-player insurance game illustrates the theoretical findings under the framework in which the best-response selection premium strategies always provide the global maximum value of the corresponding payoff function.

Keywords: *Insurance Market Competition; Non-life Insurance; Potential Game with Aggregation; Pure Nash Equilibrium*

2.1 Introduction

2.1.1 Motivation

In the insurance world, determining an appropriate and attractive premium is always a highly challenging issue because of the competition among different companies. The premium loading depends critically on the price that the other insurers charge for comparable policies. [8] was able to demonstrate it using the seminal model by [51, 50]. Insurance pricing is a fundamental aspect that attracts the interest of both actuaries and academics. Standard actuarial approaches for non-life insurance products suggest that the premium is divided into three main components: the *actuarial price*, the *safety loading*, and the *loading for expenses*. The actuarial price is normally deduced according to different premium principles, such as the *Net Premium Principle*, the *Expected Value Premium Principle*, and others [49, 58]. Classical approaches focus on determining the safety loading of each policy class proportional to the expected claim expenses or to its moment.

However, in a highly competitive insurance environment which is dominated by a relatively small number of companies (compared with the banking sector and investment funds), each insurer monitors, attempts to predict reactions, and takes advantages against the others. Thus, the actuarial premium might eventually be altered by the marketing and management department for several reasons, such as the customer's affordability, the market conditions, and the mutualisation across the portfolio of customers to decrease risk. What is more, the pricing cycles, which are found in different lines of insurance, appear also to be affected by market competition [47, 36, 22]. These suggestions indicate that the insurance premium price should not focus only on the risk assessment. Consequently, to study the competition among insurers, a model needs to be formulated in order to investigate insurers' premium pricing interactions in the corresponding market.

For previous literatures research on optimal control theory, a common assumption was made that there exists a single insurer, whose pricing strategy does not cause any reaction to the rest of the market's competitors. Thus, for each participant in the insurance market, others reaction cannot be observed, and the premium remains eventually unaffected by their actions. In reality, this situation is not often the case.

Game theoretical approaches have been introduced mostly in the premium pricing processes of non-life insurance products. Competition among insurers reveals the pricing strategy of each market participant in a constructed insurance game, whereas one can only obtain a single insurer's pricing strategy through optimal control used in previous studies. However, in our approach, as it is discussed more extensively in the following subsection, a non-cooperative game model is designed for the insurance market implementing already well-defined parameters from the corresponding literature [54, 55, 25, 42].

2.1.2 A New Approach: Potential Game with Aggregation

In our approach, a two-stage non-life insurance game is constructed in a competitive market. Numerical solutions of Nash equilibria are obtained for a large number of insurers under the two-stage framework. Moreover, instead of simply parametrizing competition through comparison between single insurer's premium and the market average premium as it has been done so far in the relevant literature, an aggregate game approach is formulated to investigate further the insurance market competition. Different from [22], the existence of Nash equilibrium is proved under our framework.

The concept of *aggregative* game, which was first proposed by [52] by considering it as the sum of the players' strategies, is applied broadly in our approach. Thus, the derived strategy for all insurers in the insurance market is presented as a single parameter, i.e., the aggregate. In greater detail, each insurer's utility (payoff) function only depends on its own pricing premium strategy and the aggregate parameter.

Also following the suggestions by [56] and [22], market competition is measured by calculating an insurer's new volume of exposure and by summing up all of the policy flows during the competition between the insurers and the volume of exposure in a previous stage. A non-linear aggregate is obtained, which presents the strategies of all insurers in the market. Moreover, a potential game approach is further developed to prove the existence of a Nash equilibrium in the insurance game. This approach also gives us an opportunity to simplify the problem of determining the Nash equilibrium by solving a single optimisation problem.[1]

The literature on potential games can be traced back to [38, 39], who created the *potential* game concept on the basis of a congestion game. Their technique did not only solve the congestion game itself but also was regarded as an equilibrium refinement tool. Following their idea, the best-response potential games were introduced and characterised by [60]. His paper proposed that, for any best-response potential game, if the potential has a maximum over its domain, the best-response potential game has a Nash equilibrium.

[16] were the first to embed the aggregate into potential games. By considering just a linear aggregation, they investigated a special type of best-response potential game that restricts the best-response selection to a continuously decreasing or increasing function. Then, any game with linear aggregation and a decreasing or increasing continuous best-response selection is proved to belong to a pseudo-potential game, which is pre-defined in their paper. By proving that any pseudo-potential game has a pure Nash equilibrium strategy, the existence of a Nash equilibrium was obtained in this special class of potential games irrespective of whether strategy sets were convex or payoff functions were quasi-concave.

In this paper, for the first time according to our knowledge, these two game-theoretic techniques are successfully implemented to determine the premium strategy for modelling competition in a non-life insurance market. Thus, in greater detail, a *best-response potential* game with *non-linear* aggregation is constructed and discussed. Premiums per unit of exposure are regarded as the premium strategy, which makes our game to

[1]We won't discuss unnecessary technical details about how to introduce and solve numerically the single optimisation problem, as it is out of the scope of the present paper.

be suitable for different lines of product-specific policies. As a new side-effect result of our approach, when it is compared with the linear aggregation limitation in [16], we still prove the existence of a pure Nash equilibrium strategy when the aggregate is non-linear. This is novel result from a game-theoretic perspective. Furthermore, from the point of view of actuarial science, the pure Nash equilibrium existence of a constructed insurance game with a non-convex strategy set is obtained.[2] That is, insurers can avoid any premium range that is not preferred to price. We solve the insurance game with respect to two distinct insurance models by calculating the best-response equations system. The numerical result for a 12-player insurance game is presented under the assumption that the best-response selection premium strategies always give the global maximum value of the corresponding payoff function.

The remainder of this chapter is organised as follows. Section 2.2 introduces the formation of two insurance market competition models and constructs the game. In Section 2.3, the existence of a Nash equilibrium is proved using potential game techniques. Section 2.4 presents the simulation results of two models in a 12-insurer game.

2.2 Modelling Formulation and Preliminaries

2.2.1 Basic Notations and Assumptions

In this subsection, the necessary notation is provided and appropriate assumptions are introduced. Thus, in the next lines, the definition of key parameters is concentrated for a better understanding of the remaining paper:

[2] It is true that since we are able to extend the results of [16] for a non-linear aggregation, the concept of our model is possible to be used in other fields of economics. For this comment, we would like cordially to thank one of our reviewers who pointed this out to us. However, further discussion falls out of the scope of this paper, since various parameters from the relevant actuarial science literature are incorporated in the construction of our insurance model [54, 55, 25, 41, 42, 43].

N	Set of *insurers* in the insurance market, $N = \{1, \ldots, n\}$, $n \in \mathbb{N}$;
a_i	*Price sensitivity* (positive) parameter of insurer $i \in N$;
h_1, h_2	*Market presence limit factor*, which controls the amount of the flow of insurance policies attributable to the competition in the market;
p_i^1	*Premium value* (per unit of exposure) for insurer $i \in N$ at time $t = 1$;
$\mathbf{P}_i$	Set of *strategies* for insurer $i \in N$;
$\mathscr{P}$	Set of *joint strategies* for all insurers in the competitive market;
$\mathfrak{p}$	*Arbitrary profile* in $\mathscr{P}$;
$\mathfrak{p}_{-i}^1$	*Strategy profile* of other players at time $t = 1$, $\{P_1^1, \ldots, P_{i-1}^1, P_{i+1}^1, \ldots, P_n^1\}$;
q_i^1	*Exposure* (volume of business) for insurer $i \in N$ at time $t = 1$, which represents the number (quantity) of policies undertaken by $i \in N$;
Δq_i^1	*Marginal difference* of exposure volume for insurer $i \in N$ at time $t = 1$;
$\hat{q}_i^1$	*Actual (number of policies) volume of exposure* in the market coming to insurer $i \in N$ at time $t = 1$ from the unallocated exposure at time $t = 0$;
$\hat{\mathfrak{q}}_i^0$	*Given number of policies* in the market, which is intended to flow in or away from insurer $i \in N$ at time $t = 1$ from the unallocated exposure of time $t = 0$;

u_i^1 *Utility* of insurer i at time $t = 1$, which represents the net income of insurer $i \in N$ at time $t = 1$, depends on insurer i's premium and the aggregate of other players' strategies;

σ_i *Interacting function*, which represents the interaction between insurer i's payoff with the others in the market;

x_{-i}^1 Parameter indicating the *aggregation* of $\mathfrak{p}_{-i}^1$;

α_i *Cost ratio* of holding wealth of $i \in N$, generally higher than the risk-free rate, $\alpha_i \in (0, 1)$;

π_i^1 *Expected breakeven premium* (per unit of exposure) for insurer $i \in N$ at time $t = 1$, i.e., expectation of future claims plus other expenses. However, for purposes of simplicity, we skip the word "expected" when we refer to the breakeven premium in the remaining paper;

k_i *Breakeven ratio* for insurer $i \in N$, k_i is equal to π_i^1 divided by p_i^1;

θ^1 *Market stability factor*, which is used to describe the market's condition;

β_i *Best-response correspondences* for insurer i regarding all the other players' strategies;

R_i *Best-response correspondences* for insurer i regarding x_{-i}^1;

$\hat{r}_i$ *The maximal selections* of R_i;

Before we proceed further, the following general assumption is proposed.

Assumption 1: In the insurance market, for any insurer $i \in N$ at time $t = 1$,

- The breakeven premium (per unit of exposure) π_i^1 is assumed to be less than the corresponding premium p_i^1.
- Both π_i^1 and p_i^1 are positive quantities.

Entries of new insurers and insurance products are not taken into consideration. Insurers avoid to set premium under cost level [54, 55, 25, 41, 42, 43], and see the references therein. Thus, the case that $p_i^1 \leq \pi_i^1$ is not considered in this paper.

2.2.2 Insurance Premium Pricing Model

For the proposed insurance model, every insurer must maximise its wealth. In this direction, a two-period framework: $t = 0, 1$ is investigated in a general insurance market. In line with the previous literature (see Section 2.1), the utility function u_i^1 that concerns insurer i with initial wealth u_i^0 is formulated as follows,

$$u_i^1 = -\alpha_i u_i^0 + (1 - \alpha_i)(p_i^1 - \pi_i^1)q_i^1. \tag{2.1}$$

For insurer i, p_i is the premium value per unit of exposure; q_i represents the holding exposure volume; π_i denotes the breakeven premium per unit of exposure, which includes risk premium and other expenses. $p_i,\ q_i,\ \pi_i$ are all positive and $\alpha_i \in (0, 1)$ is a given parameter that refers to the cost ratio of holding insurer i's wealth. As shown in Eq. (2.1), the net income of any insurer i is regarded as its utility u_i^1, and each insurer is assumed to receive the premium from policyholders at the beginning of time $t = 1$. We also assume that the insurance market contains $N = \{1, \dots, n\}$ insurers, and each insurer has perfect knowledge of its previous information. Moreover, $p_i^0, q_i^0, \pi_i^0, u_i^0$ are all known as constants at time $t = 1$. What is more, the value of q_i^1 implies competition in the market and must be determined analytically. An insurer's change in the number of policies is related to the deviation in the insurer's premium which is also connected to the market's premium level [13]. With the purpose of investigating exposure changes, marginal difference of exposure volume Δq_i^1 is defined in Eq. (2.2)

$$\Delta q_i^1 = q_i^1 - q_i^0. \tag{2.2}$$

We define the total market exposure $Q_m^1 > 0$ at time $t = 1$ as [22] did, which contains two components. The first part was related to the sum of the current exposure for each insurance company, i.e., $Q^1 = \sum_{i \in N} q_i^1 > 0$, and the second part had to do with the available (unallocated) exposure in the market, $\hat{Q}^1$, thus

$$Q_m^1 = Q^1 + \hat{Q}^1.$$

$\hat{Q}^1$ is allowed to be negative, and $\hat{Q}^1 \leq Q_m^1$. Policyholders may stop renewing policies at the end of time $t = 0$, and new clients may buy policies at the beginning of time $t = 1$ to become new policyholders. Consequently, Q^1 cannot be equal to Q^0, which causes the sum of all insurers' exposure change $\sum_{i \in N} \Delta q_i^1$ to take any value in $\mathbb{R}$. In our approach, instead of simply applying the demand function as it was the current trend (see the references in Section 2.1), the competition between any pair of insurers is now considered. Thus, additionally, the interaction between insurers' premiums needs to be formulated; consequently, Δq_i^1 is further analysed.

In the following two subsections, two distinct insurance models are introduced: a) the *simple exposure difference model I* (G_I), where $\sum_{i \in N} \Delta q_i^1$ might take any value in $\mathbb{R}$ and the available (unallocated) exposure of the insurance market $\hat{Q}^1$ is under consideration; b) the *advanced exposure difference model II* (G_{II}), which is used to further analyse policies for any insurer. Both models investigate the competition under the following assumption.

Let us define the transfer function ρ from insurer j to insurer i at time $t = 1$ as follows

$$\rho_{j \to i}^1 = 1 - \frac{p_j^0}{p_i^0} \frac{p_i^1}{p_j^1}. \tag{2.3}$$

The transfer function $\rho_{j \to i}^1$ in Eq. (2.3) describes that, for time $t = 1$, when the quotient of insurer i's premium and the previous premium $\frac{p_i^1}{p_i^0}$ is less than j's quotient $\frac{p_j^1}{p_j^0}$, insurer j's policies tend to flow to insurer i. The exposure of insurer i increases in the competition with j, whereas the exposure of j decreases. Policies flow in a reverse manner and $\frac{p_i^1}{p_i^0} > \frac{p_j^1}{p_j^0}$.

This assumption indicates that the preference of policyholders, i.e., when one insurer increases its premium and its competitor decreases its own premium, the insurer simultaneously decreases its attractiveness. When both insurers increase their premiums by different percentages, the insurer with the smaller increment becomes more attractive. Finally, in a similar manner, when both decrease their premiums, the insurer with the larger decrement becomes more attractive.

Insurer i gains exposure from the competition with insurer j when it offers a more attractive premium. However, policyholders sometimes choose an insurer's policies with higher premiums as the most preferable one because of a better reputation [42] (and the references therein). For this reason, the percentage changes in the premium are adapted in the transfer function rather than in the value of the premium itself. Note that the transfer function $\rho^1_{i\to j}$ can be either positive or negative. The policy amount of i is increased when $\rho^1_{i\to j} > 0$ and reduced when $\rho^1_{i\to j} < 0$.

By investigating the flow of policies between any pair of insurers, the entire insurance market competition can be evaluated by aggregating every competition among the different pairs of insurers. This topic is the focus of discussion in the following subsections.

Simple Exposure Difference Model I (G_I)

Let us consider that the competition in the insurance market is formulated as follows. First, the premium levels vary over time, which might even cause a change in the total number of policies in the market. Second, potential clients consider holding insurance policies when premiums decline. In contrast, the insurance market may lose clients if the market premium level is high.

In G_I, we assume that for any pair of insurers i and j, exposure q_i – which is related to gain or loss – is not equal to exposure q_j – which has to do with loss or gain – respectively. Thus, this assumption indicates that the available exposure joins or leaves the market because of competition between i and j. The expected exposure to flow by insurer i attributable to competition with j is given by

$$\begin{aligned} q^1_{j\to i} &= h_1 a_i \rho^1_{j\to i} q^0_i \\ &\neq -q^1_{i\to j}, \quad h_1 > 0. \end{aligned} \tag{2.4}$$

Here, we define a_i as the price sensitivity (positive) parameter of insurer i and h_1 as the market presence limit factor in G_I.

The exposure gain or loss from all other insurers to i is given by

$$\Delta q^1_i = \sum_{j\in N} q^1_{j\to i}. \tag{2.5}$$

Eqs. (2.4)–(2.5) are interpreted as follows. The strength (which is related to either gain or loss) of the exposure of insurer i attributable to the competition with j is demonstrated in Eq. (2.4). The premium p^1_j is modelled as being transferred to insurer i's premium by multiplying $\frac{p^0_i}{p^0_j}$ in $\rho^1_{i\to j}$ for the purpose of simultaneously comparing two insurers' premiums. Insurer i's price sensitivity parameter a_i is considered as information of insurer i for presenting the market power. Note that, regarding the transferred premium $\frac{p^0_i}{p^0_j}p^1_j$ as i's previous premium p^0_i, the item $a_i\rho^1_{j\to i}q^0_i$ is just the volume of business i's gain or loss when the price elasticity is a_i. In our case, the price elasticity of demand, a_i, is determined by imitating the concept of the [35] index, i.e., the leader in the insurance market which has the larger market power has lower price sensitivity and so on and so forth. In a competitive market, q^1_i depends not only on p^1_i but also on other insurers' premiums. Hence, instead of comparing the previous premium p^0_i, the transferred premium $\frac{p^0_i}{p^0_j}p^1_j$ is adopted to characterise the change in the volume of polices. In Eq. (2.4), h_1 is the market presence limit factor, which is used to limit the scale of the policies' flow amount. Because different stabilities exist in various insurance markets, h_1 can take different positive values.

The exposure difference Δq^1_i from the competition in the entire market is obtained by summing up all of the policies' gains or losses when competing with all insurers.

Note that $\sum_{i\in N}\Delta q_i^1$ is allowed not to be equal to zero. Regarding Eqs. (2.1)–(2.5), the utility function can be deduced.

We define $u^1_{G_I,i}$ (similar for $u^1_{G_{II},i}$, see Subsection 2.2.2) be the utility functions of insurer i at time $t=1$ in G_I (G_{II}).

Lemma 2.1. *For the simple exposure difference model I, the utility function $u^1_{G_I,i}$ of insurer i at time $t=1$ is given by*

$$\begin{aligned} u^1_{G_I,i} &= -\frac{a_i h_1 q_i^0(1-\alpha_i)}{p_i^0}(\sum_{j\in N}\frac{p_j^0}{p_j^1})(p_i^1)^2 \\ &+ (1-\alpha_i)[q_i^0 + nh_1a_iq_i^0 + \pi_i^1(\sum_{j\in N}\frac{p_j^0}{p_j^1})\frac{a_ih_1q_i^0}{p_i^0}]p_i^1 \\ &- \alpha_i u_i^0 - \pi_i^1(1-\alpha_i)[q_i^0 + na_ih_1q_i^0]. \end{aligned} \tag{2.6}$$

Proof. By combining Eqs. (2.2)–(2.5), we obtain the exposure of i considering that the competition occurred at time $t=1$.

$$\begin{aligned} q_i^1 &= q_i^0 + \Delta q_i^1 \\ &= q_i^0 + \sum_{j\in N} h_1 a_i \rho^1_{j\to i} q_i^0 \\ &= q_i^0 + \sum_{j\in N} h_1 a_i q_i^0 (1 - \frac{p_j^0}{p_i^0}\frac{p_i^1}{p_j^1}) \\ &= q_i^0 + nh_1a_iq_i^0 - \frac{a_ih_1q_i^0p_i^1}{p_i^0}\sum_{j\in N}\frac{p_j^0}{p_j^1}. \end{aligned}$$

By taking q_i^1 above into Eq. (2.1), we have that

$$
\begin{aligned}
u^1_{G_I,i} &= -\alpha_i u^0_i + (1-\alpha_i)(p^1_i - \pi^1_i)(q^0_i + nh_1 a_i q^0_i \\
&\quad -\frac{a_i h_1 q^0_i p^1_i}{p^0_i}\sum_{j\in N}\frac{p^0_j}{p^1_j}) \\
&= -\frac{a_i h_1 q^0_i(1-\alpha_i)}{p^0_i}(\sum_{j\in N}\frac{p^0_j}{p^1_j})(p^1_i)^2 \\
&+ (1-\alpha_i)[q^0_i + nh_1 a_i q^0_i + \pi^1_i(\sum_{j\in N}\frac{p^0_j}{p^1_j})\frac{a_i h_1 q^0_i}{p^0_i}]p^1_i \\
&- \alpha_i u^0_i - \pi^1_i(1-\alpha_i)[q^0_i + na_i h_1 q^0_i].
\end{aligned}
$$

□

Advanced Exposure Difference Model II (G_{II})

The modified exposure for insurer i can be further analysed. Different from G_I, in G_{II}, we concretely characterize the two components mentioned in Subsection 2.2.2, i.e., a) reallocated policies of the previous market Q^0, and b) policies from the (unallocated) exposure $\hat{Q}^1$.

Regarding the competition between any pair of insurers i and j, the number of exchange policies is characterised. The exposure gain or loss from i to j is obtained with respect to both insurers' premium strategy and market power. Given a positive market presence limit factor h_2, the strength of the flow of business between i and j is modelled as follows

$$
\begin{aligned}
q^1_{j\to i} &= h_2(a_i\rho^1_{j\to i}q^0_i - a_j\rho^1_{i\to j}q^0_j) \\
&= -q^1_{i\to j}, \quad h_2 > 0.
\end{aligned} \tag{2.7}
$$

As demonstrated in Eq. (2.7), both exposure i which tended to a gain or loss, $a_i\rho^1_{j\to i}q^0_i$, and exposure j which showed a potential loss or gain, $-a_j\rho^1_{i\to j}q^0_j$, represent the exchange strength from summing up the volume. The volume of the flow of exposure is further governed by h_2, which is defined as the positive market presence limit factor in G_{II}. Note that $\sum_{i \in N} \sum_{j \in N} q^1_{j \to i}$ equals to zero because of policies exchange between insurers in the component a). In the same way, for the b) component, the potential flow of policies, either attract or withdraw from the unallocated insurance market $\hat{Q}^1$, and it is modelled as $h_2 a_i (1 - \frac{p_i^1}{p_i^0}\theta^1) q_i^0$.

The flow of policies from the unallocated insurance market is modelled similarly to the concept of price elasticity: a comparison with previous premium price. Apart from the competition between pairs of insurers, they tend to lose policies to the available market when increasing their premiums and gain policies by lowering them. In addition, a positive market stability factor θ^1 is adopted to describe the market condition: $\theta^1 = 1$ indicates that the market faces a general condition; the insurance industry expands when $\theta^1 < 1$ because more policies tend to flow into the industry from the unallocated market; $\theta^1 > 1$, when the market faces a situation with challenges. Overall, the exposure gain or loss for i is given by

$$\Delta q_i^1 = \sum_{j \in N} q^1_{j \to i} + h_2 a_i (1 - \frac{p_i^1}{p_i^0}\theta^1) q_i^0, \quad \theta^1 > 0. \tag{2.8}$$

Following **Assumption 1**, $k_i \in (0,1)$. Then, the objective function for the G_{II} case can be deduced.

Lemma 2.2. *For the advanced exposure difference model II, the utility function $u^1_{G_{II},i}$ of insurer i at time $t = 1$ is given by*

$$\begin{aligned} u^1_{G_{II},i} &= -\frac{(1-k_i)(1-\alpha_i)h_2 a_i q_i^0}{p_i^0} (\sum_{j \in N} \frac{p_j^0}{p_j^1} + \theta^1)(p_i^1)^2 \\ &+ (1-k_i)(1-\alpha_i)(q_i^0 + (n+1)h_2 a_i q_i^0 - h_2 \sum_{j \in N} a_j q_j^0) p_i^1 \\ &+ (1-k_i)(1-\alpha_i) h_2 p_i^0 \sum_{j \in N} a_j q_j^0 \frac{p_j^1}{p_j^0} - \alpha_i u_i^0. \end{aligned} \tag{2.9}$$

Proof. Using Eqs. (2.7)–(2.8) instead, Lemma 2.2 can be showed similarly as Lemma 2.1. □

In the next Subsection, the construction of the game is presented and further discussed.

2.2.3 Game Construction

Normal Form Game

Let us define an N-insurer game, G, in a two-period framework: $t = 0, 1$. Each insurer i's strategy at time $t = 1$ is p_i^1, which stands for the action setting premium as the value of p_i^1, whereas $\mathbf{P}_i$ is the set of strategies. We use $\widetilde{P}_i^1$ to denote the equilibrium strategy for insurer i. Insurer i's payoff function is defined as $u_i^1 : \mathscr{P} \rightarrow \mathbb{R}$, where $\mathscr{P} \equiv \mathbf{P}_1 \times \cdots \times \mathbf{P}_N$ and $\mathfrak{p}$ is an arbitrary profile in $\mathscr{P}$. The notation $\mathfrak{p}_{-i}^1 \in \mathscr{P}_{-i}$ stands for $\{p_1^1, \ldots, p_{i-1}^1, p_{i+1}^1, \ldots, p_n^1\}$, which is used to represent the strategy profile of other players at time t. $(p_i^1, \mathfrak{p}_{-i}^1) \in \mathscr{P}$ decomposes a strategy profile in two parts, the insurer i's strategy and other insurers' components. Given this game in the insurance market, instead of calculating the optimal premium that maximises a single insurer's wealth, as was the case in the previous literature (see Section 2.1 for further details), the calculation of the Nash equilibrium is targeted.

Generally, from a game theory perspective, the Nash equilibrium is a prediction strategy that dictates the choices that each insurer is willing to make. Given the optimal strategy profile of other insurers, the market reaches a Nash equilibrium when no insurer can increase its total payoff by changing its strategy. The Nash equilibrium is defined through the best-response correspondences. In what it follows the next definitions should be stated.

Definition 2.3. [28] Define β_i by

$$\beta_i(\mathfrak{p}_{-i}^1) = \{p_i^1 \in \mathbf{P}_i : u_i^1(p_i^1, \mathfrak{p}_{-i}^1) \geq u_i^1(\acute{p}_i^1, \mathfrak{p}_{-i}^1), \forall \acute{p}_i^1 \in \mathbf{P}_i\}.$$

We call β_i the *best-response correspondences* for insurer i.

For any choice $\mathfrak{p}_{-i} \in \mathscr{P}_{-i}$ of others' strategies at time t, the set $\beta_i(\mathfrak{p}^1_{-i})$ of best replies of insurer i is given by

$$\beta_i(\mathfrak{p}^1_{-i}) = \operatorname*{argmax}_{p^1_i \in \mathbf{P}_i} u^1_i(p^1_i, \mathfrak{p}^1_{-i}).$$

Each player's predicted strategy must be a best response to the predicted strategies of the other players as the market reaches a Nash equilibrium.

Definition 2.4. [28] A strategy profile, $\widetilde{\mathfrak{p}}^1$, is a *Nash equilibrium* of the game (at time t) *if and only if* each player's strategy is a best response to the other players' strategies. That is

$$\widetilde{p}^1_i \in \beta(\widetilde{\mathfrak{p}}^1_{-i}), \quad \forall i \in N.$$

The best-response potential game technique is further considered, which is widely used to prove the existence of Nash equilibrium.

Definition 2.5. [60] A strategic game $\tilde{G} =< (\beta_i, \mathbf{P}_i)_{i\in N} >$ is a *best-response potential game* if there exists a function $f : \mathscr{P} \to \mathbb{R}$ such that

$$\forall i \in N, \forall \mathfrak{p}_{-i} \in \mathscr{P}_{-i} : \quad \beta_i(\mathfrak{p}_{-i}) = \operatorname*{argmax}_{p_i \in \mathbf{P}_i} f(p_i, \mathfrak{p}_{-i}).$$

The function f is called a best-response potential function of the game $\tilde{G}$.

The potential function f offers a new approach to determining the Nash equilibrium for the game $\tilde{G}$ by maximising f. Note that, f is a function, which depends on every insurer's strategy. If f has a maximum over $\mathscr{P}$, $\tilde{G}$ has a Nash equilibrium. A specific type of game, known as an *aggregate* game, is introduced to solve the Nash equilibrium for the N insurers' game.

Aggregate Games

With the additional requirement that each insurer's payoff is written as a function that depends only on its own strategy and an aggregate of the full strategy profile, a normal form game can be transformed into a game with aggregation. Formally, we have the following definition.

Definition 2.6. [37] An *aggregate game* in the insurance market, $G' =< (\mathbf{P}_i, u_i^1)_{i\in N}, g >$, is a normal form game with an extra condition that there exists an aggregate function, $g(\mathfrak{p}^1) : \mathscr{P} \longrightarrow \mathbf{M} \subseteq \mathbb{R}$, such that each player's payoff function can be further specialised to the aggregate form

$$\mathfrak{p}^1 \mapsto u_i^1(p_i^1, g(\mathfrak{p}^1)),$$

where $M^1 \in \mathbf{M}$, is called an aggregator of $\mathfrak{p}^1$.

The only requirement for a game to represent an aggregate game is that there exists an aggregate function [2]. To construct an insurance game with aggregation, a meaningful monotone aggregate function g is expected to be obtained. Here, the *Insurance Game I*, equipped with the objective function in the simple exposure difference model I, and the *Insurance Game II*, implemented with the objective function in the advanced exposure difference model II, are considered. Before we proceed further, the definitions of G_I and G_{II} are given as follows.

Definition 2.7. A game $G_I =< (\mathbf{P}_{G_I,i}, u^1_{G_I,i})_{i\in N} >$ has a finite set of players N, with compact, positive, pure strategy set $\mathbf{P}_{G_I,i}$ with respect to every i, whereas $u^1_{G_I,i}$ in Eq. (2.6) is the payoff function for i at time $t = 1$. This type of game is called *Insurance Game I*.

Similarly, Insurance Game II is defined as $G_{II} =< (\mathbf{P}_{G_{II},i}, u^1_{G_{II},i})_{i\in N} >$, with player set N, compact, positive, pure strategy set $\mathbf{P}_{G_{II},i}$ and payoff function $u^1_{G_{II},i}$ in Eq. (2.9).

2.3 Main Results

In this section, the theoretical results for models G_I and G_{II} are presented. However, before we proceed further with the existence of a Nash equilibrium, it is necessary to show that both G_I and G_{II} are aggregate games.

Lemma 2.8. *Based on the definition of payoff functions stated in the previous section, both G_I and G_{II} are aggregate games.*

Proof. Denote $M^1 = \sum_{j \in N} \frac{p_j^0}{p_j^1}$ as the aggregation of G_I game. Then, the payoff function in Eq. (2.6) turns out to be

$$\begin{aligned} u^1_{G_I,i} &= -\frac{a_i h_1 q_i^0 (1-\alpha_i)}{p_i^0} M^1 (p_i^1)^2 + (1-\alpha_i)[q_i^0 + n h_1 a_i q_i^0 + \pi_i^1 M^1 \frac{a_i h_1 q_i^0}{p_i^0}] p_i^1 \\ &- \alpha_i u_i^0 - \pi_i^1 (1-\alpha_i)[q_i^0 + n a_i h_1 q_i^0]. \end{aligned}$$

There exists an aggregate function $g(\mathfrak{p}^1) = \sum_{j \in N} \frac{p_j^0}{p_j^1}$ in G_I. For G_{II} game, we further denote $m^1 = \sum_{j \in N} a_j q_j^0 \frac{p_j^1}{p_j^0}$ as the other aggregation. Similarly, we obtain the payoff,

$$\begin{aligned} u^1_{G_{II},i} &= -\frac{(1-k_i)(1-\alpha_i) h_2 a_i q_i^0}{p_i^0} (m^1 + \theta^1)(p_i^1)^2 \\ &+ (1-k_i)(1-\alpha_i)(q_i^0 + (n+1) h_2 a_i q_i^0 - h_2 \sum_{j \in N} a_j q_j^0) p_i^1 \\ &+ (1-k_i)(1-\alpha_i) h_2 p_i^0 m^1 - \alpha_i u_i^0. \end{aligned}$$

Thus, the statement of the Lemma is derived. □

In aggregate games, for every player i, the other players in the competitive market are considered as a single player because their strategies aggregate through an interacting function $\sigma_i : \mathscr{P}_{-i} \to \mathbf{X}_{-i} \subseteq \mathbb{R}$. Intuitively, the other players influence i through the interaction function $\sigma_i(\mathfrak{p}^1_{-i})$. $\mathbf{X}_{-i} = \sigma_i(\mathscr{P}_{-i})$ is set to indicate the range of σ_i, whereas $x^1_{-i} = \sigma_i(\mathfrak{p}^1_{-i}) \in \mathbf{X}_{-i}$ for any t. With $x^1_{-i} = \sum_{j \neq i} \frac{p_j^0}{p_j^1}$, respectively, the G_I and G_{II} payoff functions are given as follows:

$$\begin{aligned} u^1_{G_I,i} &= -\frac{a_i h_1 q_i^0 (1-\alpha_i)}{p_i^0} x^1_{-i} (p_i^1)^2 \\ &+ (1-\alpha_i)[q_i^0 + (n-1) h_1 a_i q_i^0 + \pi_i^1 x^1_{-i} \frac{a_i h_1 q_i^0}{p_i^0}] p_i^1 \\ &- \alpha_i u_i^0 - \pi_i^1 (1-\alpha_i)[q_i^0 + (n-1) a_i h_1 q_i^0] \end{aligned}$$

and

$$\begin{aligned}
u^1_{G_{II},i} &= -\frac{(1-k_i)(1-\alpha_i)h_2 a_i q^0_i}{p^0_i}(x^1_{-i}+\theta^1)(p^1_i)^2 \\
&+ (1-k_i)(1-\alpha_i)(q^0_i + nh_2 a_i q^0_i - h_2\sum_{j\neq i} a_j q^0_j)p^1_i \\
&+ (1-k_i)(1-\alpha_i)h_2 p^0_i \sum_{j\neq i} a_j q^0_j \frac{p^1_j}{p^0_j} - \alpha_i u^0_i.
\end{aligned}$$

To generate Nash equilibrium premium strategies, $R_i : \mathbf{X}_{-i} \to 2^{\mathbf{P}_i}$, we need to define

$$R_i(x^1_{-i}) = \operatorname*{argmax}_{p^1_i\in\mathbf{P}} u^1_i(p^1_i, x^1_{-i}),$$

which coincides with $\beta_i(\mathfrak{p}^1_{-i})$. In other words, R_i describes how the interaction parameter $x^1_{-i} = \sigma_i(\mathfrak{p}^1_{-i})$ influences insurer i's best-response strategy.

In the case of G_I, we have

$$R_{G_I,i}(x^1_{-i}) = \operatorname*{argmax}_{p^1_i\in\mathbf{P}} u^1_{G_I,i}(p^1_i, x^1_{-i}). \tag{2.10}$$

$\hat{r}_{G_I,i}$ is defined as the maximal selections of $R_{G_I,i}(x^1_{-i})$, and for G_{II}, we have

$$R_{G_{II},i}(x^1_{-i}) = \operatorname*{argmax}_{p^1_i\in\mathbf{P}} u^1_{G_{II},i}(p^1_i, x^1_{-i}). \tag{2.11}$$

$\hat{r}_{G_{II},i}$ is defined as the maximal selections of $R_{G_{II},i}(x^1_{-i})$.

Before we prove that both G_I and G_{II} are best-response potential games, we need to recall first, Lemma 2.9 which is proposed by [30].

Lemma 2.9. *The game $<(\beta_i, \boldsymbol{P}_i)_{i\in N}>$ is a best-response potential game if and only if there exists a real-valued function, $f :\to \mathbb{R}$, such that:*

$$\tilde{\mathfrak{p}}^1 \succeq \mathfrak{p}^1 \Rightarrow f(\tilde{\mathfrak{p}}^1) \geq f(\mathfrak{p}^1) \tag{2.12}$$

and

$$\tilde{\mathfrak{p}}^1 \succ \mathfrak{p}^1 \Rightarrow f(\tilde{\mathfrak{p}}^1) > f(\mathfrak{p}^1), \tag{2.13}$$

where the previous two binary relations are defined as:

$$\begin{aligned}
\tilde{\mathfrak{p}}^1 \succeq \mathfrak{p}^1 \quad &\Leftrightarrow \quad \exists i \in N, \quad s.t.\ [\tilde{\mathfrak{p}}^1_{-i} = \mathfrak{p}^1_{-i}, \textit{and} \quad \tilde{\mathfrak{p}}^1_i \in R_i(x^1_{-i})] \\
\tilde{\mathfrak{p}}^1 \succ \mathfrak{p}^1 \quad &\Leftrightarrow \quad [\tilde{\mathfrak{p}}^1 \succeq \mathfrak{p}^1, \quad \textit{and} \quad \mathfrak{p}^1_{-i} \notin R_i(x^1_{-i})]
\end{aligned}$$

The next lemma is useful for the main result of our paper. Its proof is rather technical, and for better understanding, we present it using intermediate steps.

Lemma 2.10. *Both G_I and G_{II} are best-response potential games.*

Proof. Initially, G_I is considered.

- **Step 1: State the best-response potential function.**

  - **Convex hull of $\mathbf{X}_{-i}$.**

    In the case that $\mathbf{P}_i$ is *not* convex, $\mathbf{X}_{-i}$ is *not* convex as well. Denote Σ_{-i} as the convex hull of $\mathbf{X}_{-i}$, which is obviously compact.

    For G_I, $R_{G_I,i}$ is the best-response correspondences to x^1_{-i} of i. We extend $R_{G_I,i}$ in a piecewise linear fashion to $\Phi_{G_I,i}$, defined on the domain Σ_{-i}. $\Phi_{G_I,i}$ coincides with $R_{G_I,i}$ on $\mathbf{X}_{-i}$. For any $s \in \Sigma_{-i} \setminus \mathbf{X}_{-i}$ define

    $$\Phi_{G_I,i}(s) = \frac{z-s}{z-y} R_{G_I,i}(y) + \frac{s-y}{z-y} R_{G_I,i}(z),$$

    with $y = \max\{v \in \mathbf{X}_{-i} | v \leq s\}$ and $z = \min\{v \in \mathbf{X}_{-i} | v \geq s\}$.

  - **For any insurer i, linearly enhance the best response domain to be the same as its strategy domain.**

    Let $\hat{\mathbf{P}}_{G_I,i}$ denote the range of player i's best response map, and the set be

$\{p_i^1 \in \mathbf{P}_{G_I,i} : p_i^1 \in R_i(\sigma_i(\mathfrak{p}_{-i}^1))\} \subseteq \mathbf{P}_{G_I,i}$. Denote $\phi_{G_I,i}^1$ as the selections of $\Phi_{G_I,i}$, which is continuous on Σ_{-i}. We further define a mapping $O_i(\phi_{G_I,i}^1)$, which linearly enhances the domain $\hat{\mathbf{P}}_{G_I,i}$ to $\mathbf{P}_{G_I,i}$. In addition, $r_{G_I,i}^1$ is defined as the selection of $O_i(\phi_{G_I,i}^1)$. In other words,

$$\forall i, \exists \hat{x}_{-i}^1 \quad s.t. \quad p_i^1 \in O_i(\Phi_{G_I,i}(\hat{x}_{-i}^1)).$$

Let $\perp_i^1 = \min_{\mathfrak{p}_{-i}^1 \in \mathscr{P}_{-i}^1} \sigma_i(\mathfrak{p}_{-i}^1)$, $\top_i^1 = \max_{\mathfrak{p}_{-i}^1 \in \mathscr{P}_{-i}^1} \sigma_i(\mathfrak{p}_{-i}^1)$, and extend each $r_{G_I,i}^1$ to $[\perp_i^1, \top_i^1]$ along the line with [31].

- **We state that the following Eq. (2.14) is the best-response potential function of G_I.**

$$f(p_i^1, \mathfrak{p}_{-i}^1) = \sum_i \left[p_i^0 \int_{\perp_i^1}^{\top_i^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\} d\tau - \frac{p_i^0}{p_i^1} \perp_i \right] + \sum_{i<j} \frac{p_i^0 p_j^0}{p_i^1 p_j^1}. \tag{2.14}$$

- **Step 2: Prove that Eqs. (2.12) and (2.13) are true.**

  - **Prove that each of the correspondences $R_{G_I,i} : \mathbf{X}_{-i} \to 2^{\mathbf{P}_i}$ is a strictly decreasing selection; that is, for every R_i, all $x_{-i}^1 \in \mathbf{X}_{-i}$ such that $R_i(\bar{x}_{-i}^1) > R_i(x_{-i}^1)$ whenever $\bar{x}_{-i}^1 \leq x_{-i}^1$.**

    The statement is satisfied as long as the conditions of Topkis' Theorem (see [59] for details) are satisfied, i.e. each $\mathbf{P}_i$ is a lattice, every $u_{G_I,i}(p_i^1, x_{-i}^1)$ supermodular in p_i^1,and has strictly decreasing differences in p_i^1 and x_{-i}^1.

    Since p_i^1 is one-dimensional for all i, the first two of these requirements are satisfied: $\mathbf{P}_i$ is a lattice for all i; every $u_{G_I,i}$ supermodular in p_i^1. In addition, because u_i^1 is twice differentiable, $u_{G_I,i}(p_i^1, x_{-i}^1)$ has strictly decreasing differences in p_i^1 and x_{-i}^1 if and only if $\partial^2 u_{G_I,i}(p_i^1, x_{-i}^1)/\partial p_i^1 \partial x_{-i}^1 < 0$. In an insurance game G_I, we have

$$\begin{aligned}\partial^2 u_{G_I,i}(p_i^1, x_{-i}^1)/\partial x_{-i}^1 \partial p_i^1 &= \partial\{-2\frac{(1-\alpha_i)h_1 a_i q_i^0}{p_i^0} x_{-i}^1 p_i^1 + (1-\alpha_i)[q_i^0 \\ &+ (n-1)h_1 a_i q_i^0 + \pi_i^1 x_{-i}^1 \frac{a_i h_1 q_i^0}{p_i^0}]\}/\partial x_{-i}^1 \\ &= \frac{a_i h_1 q_i^0 (1-\alpha_i)}{p_i^0}(\pi_i^1 - 2p_i^1) < 0.\end{aligned}$$

According to the assumption that for any $i,\ t$, $\pi_i^1 < p_i^1$, the above item is negative. Hence, $u_i^1(p_i^1, x_{-i}^1)$ has strictly decreasing differences in p_i^1 and x_{-i}^1. Because $O_i(\phi_{G_I,i}^1)$ enhance the domain $\hat{\mathbf{P}}_{G_I,i}$ linearly, $r_{G_I,i}^1$ coincides with $\phi_{G_I,i}^1$. One can deduce that if $\hat{x}_{-i}^1 > x_{-i}^1$, we have $p_i^1 < \tilde{p}_i^1$ and vice versa.

– **The comparison between $f(\tilde{p}_i^1, \mathfrak{p}_{-i}^1)$ and $f(p_i^1, \mathfrak{p}_{-i}^1)$.**

With equilibrium premium $\tilde{p}_i^1$ of i in $\tilde{\mathfrak{p}}^1$, the difference between $f(\tilde{\mathfrak{p}}^1)$ and $f(\mathfrak{p}^1)$ is demonstrated as

$$\begin{aligned}& f(\tilde{p}_i^1, \mathfrak{p}_{-i}^1) - f(p_i^1, \mathfrak{p}_{-i}^1) \\ = & \sum_{i\in N}\left[\int_{\perp_i^1}^{\top_i^1} p_i^0 \cdot \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau\right] - \\ & \sum_{i\in N}\left[\int_{\perp_i^1}^{\top_i^1} p_i^0 \cdot \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau\right] - \sum_{i\in N}\left[\frac{p_i^0}{\tilde{p}_i^1}\cdot \perp_i^1\right] + \sum_{i\in N}\left[\frac{p_i^0}{p_i^1}\cdot \perp_i^1\right] \\ & + \left[\frac{p_i^0}{\tilde{p}_i^1} - \frac{p_i^0}{p_i^1}\right]\cdot \sum_{j\neq i}\frac{p_j^0}{p_j^1}\end{aligned}$$

$$
\begin{aligned}
&= \int_{\perp_i^1}^{\top_i^1} p_i^0 \cdot \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau - \int_{\perp_i^1}^{\top_i^1} p_i^0 \cdot \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau \\
&\quad -\frac{p_i^0}{\tilde{p}_i^1} \cdot \perp_i^1 + \frac{p_i^0}{p_i^1} \cdot \perp_i^1 + \frac{p_i^0}{\tilde{p}_i^1} \cdot x_{-i}^1 - \frac{p_i^0}{p_i^1} \cdot x_{-i}^1 \\
&= p_i^0 \Bigg[\int_{\perp_i^1}^{\top_i^1} \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau - \int_{\perp_i^1}^{\top_i^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau \\
&\quad - \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{\tilde{p}_i^1} d\tau + \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{p_i^1} d\tau \Bigg].
\end{aligned}
$$

When $\hat{x}_{-i}^1 > x_{-i}^1$,

$$
\begin{aligned}
& f(\tilde{P}_i^1, \mathfrak{p}_{-i}^1) - f(p_i^1, \mathfrak{p}_{-i}^1) \\
&= \int_{\perp_i^1}^{x_{-i}^1} \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau + \int_{x_{-i}^1}^{\hat{x}_{-i}^1} \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau \\
&+ \int_{\hat{x}_{-i}^1}^{\top_i^1} \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau - \int_{\perp_i^1}^{x_{-i}^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau \\
&- \int_{x_{-i}^1}^{\hat{x}_{-i}^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau - \int_{\hat{x}_{-i}^1}^{\top_i^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau \\
&- \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{\tilde{p}_i^1} d\tau + \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{p_i^1} d\tau \\
&= \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{\tilde{p}_i^1} d\tau + \int_{x_{-i}^1}^{\hat{x}_{-i}^1} -\frac{1}{r_{G_I,i}^1(\tau)} d\tau + \int_{\hat{x}_{-i}^1}^{\top_i^1} -\frac{1}{r_{G_I,i}^1(\tau)} d\tau \\
&\qquad - \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{p_i^1} d\tau - \int_{x_{-i}^1}^{\hat{x}_{-i}^1} -\frac{1}{p_i^1} d\tau - \int_{\hat{x}_{-i}^1}^{\top_i^1} -\frac{1}{r_{G_I,i}^1(\tau)} d\tau \\
&\qquad - \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{\tilde{p}_i^1} d\tau + \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{p_i^1} d\tau \\
&\qquad = \int_{x_{-i}^1}^{\hat{x}_{-i}^1} [\frac{1}{p_i^1} - \frac{1}{r_{G_I,i}^1(\tau)}] d\tau > 0.
\end{aligned}
$$

When $\hat{x}_{-i}^1 < x_{-i}^1$,

$$
\begin{aligned}
&f(\tilde{P}_i^1, \mathfrak{p}_{-i}^1) - f(p_i^1, \mathfrak{p}_{-i}^1) \\
&= \int_{\perp_i^1}^{\hat{x}_{-i}^1} \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau + \int_{\hat{x}_{-i}^1}^{x_{-i}^1} \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau \\
&+ \int_{x_{-i}^1}^{\top_i^1} \min\{-\frac{1}{\tilde{p}_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau - \int_{\perp_i^1}^{\hat{x}_{-i}^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau \\
&- \int_{\hat{x}_{-i}^1}^{x_{-i}^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau - \int_{x_{-i}^1}^{\top_i^1} \min\{-\frac{1}{p_i^1}, -\frac{1}{r_{G_I,i}^1(\tau)}\}d\tau \\
&- \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{\tilde{p}_i^1}d\tau + \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{p_i^1}d\tau \\
&= \int_{\perp_i^1}^{\hat{x}_{-i}^1} -\frac{1}{\tilde{p}_i^1}d\tau + \int_{\hat{x}_{-i}^1}^{x_{-i}^1} -\frac{1}{r_{G_I,i}^1(\tau)}d\tau + \int_{x_{-i}^1}^{\top_i^1} -\frac{1}{r_{G_I,i}^1(\tau)}d\tau \\
&- \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{p_i^1}d\tau - \int_{x_{-i}^1}^{\hat{x}_{-i}^1} -\frac{1}{p_i^1}d\tau - \int_{\hat{x}_{-i}^1}^{\top_i^1} -\frac{1}{r_{G_I,i}^1(\tau)}d\tau \\
&- \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{\tilde{p}_i^1}d\tau + \int_{\perp_i^1}^{x_{-i}^1} -\frac{1}{p_i^1}d\tau \\
&= \int_{\hat{x}_{-i}^1}^{x_{-i}^1} [\frac{1}{r_{G_I,i}^1(\tau)} - \frac{1}{p_i^1}]d\tau > 0.
\end{aligned}
$$

It is obvious that if $\hat{x}_{-i}^1 = x_{-i}^1$, this item equals zero. In this case, $p_i^1, \tilde{p}_i^1 \in R_{G_I,i}(\sigma_i(\mathfrak{p}_{-i}^1))$ (i.e. if Eq. (2.12) holds but not Eq. (2.13)), $f(\tilde{P}_i^1, \mathfrak{p}_{-i}^1) - f(p_i^1, \mathfrak{p}_{-i}^1) = 0$. Eq. (2.12) is proved to be true in an insurance game G_I. If not, Eq. (2.13) is proved.

- **Step 3: Conclusion**

  We conclude that when $(p_i^1, \mathfrak{p}_{-i}^1), (\tilde{p}_i^1, \mathfrak{p}_{-i}^1) \in \mathbf{P}_i$,$(\tilde{p}_i^1, \mathfrak{p}_{-i}^1) \succeq (\succ)(p_i^1, \mathfrak{p}_{-i}^1) \Rightarrow f(\tilde{p}_i^1, \mathfrak{p}_{-i}^1) - f(p_i^1, \mathfrak{p}_{-i}^1) \geq (>)0$, with respect to Lemma 2.10. An insurance game G_1 is the best-response potential game, whereas f is the best-response potential function.

Similarly, in G_{II},

$$\begin{aligned}\partial^2 u^1_{G_{II},i}(p^1_i, x^1_{-i})/\partial x^1_{-i}\partial p^1_i &= \partial\{-2\frac{(1-\alpha_i)(1-k_i)h_2 a_i q^0_i}{p^0_i}(x^1_{-i}+\theta^1)p^1_i \\ &\quad +(1-k_i)(1-\alpha_i)(q^0_i + nh_2 a_i q^0_i - h_2\sum_{j\neq i} a_j q^0_j)\}/\partial x^1_{-i} \\ &= -2\frac{(1-\alpha_i)(1-k_i)h_2 a_i q^0_i}{p^0_i}p^1_i < 0.\end{aligned}$$

We also obtain that $u^1_{G_{II},i}(p^1_i, x^1_{-i})$ has strictly decreasing differences in p^1_i and x^1_{-i}. By replacing $r^1_{G_I,i}$ by $r^1_{G_{II},i}$ in f from Eq. (2.14), one obtains the best-response potential function of $u^1_{G_{II},i}$ in G_{II}. □

Following the discussion so far, one can deduce the useful Theorem, which is the main theoretical result of our paper.

Theorem 2.11. *The Nash equilibrium at time $t = 1$ in both G_I and G_{II} exists.*

Proof. In G_I, let us suppose that

$$\tilde{\mathfrak{p}}^1 \in \operatorname{argmax} f(p^1_i, \mathfrak{p}^1_{-i}).$$

Such a $\tilde{\mathfrak{p}}^1$ exists because $\mathbf{P}_i$ is compact for any i and f is continuous. If $\tilde{\mathfrak{p}}^1$ is not a Nash equilibrium of G_1, then $f(c^1_i, \tilde{\mathfrak{p}}^1_{-i}) > f(\tilde{\mathfrak{p}}^1)$ for some $c^1_i \in \mathbf{P}_i$, contradicting that $\tilde{\mathfrak{p}}^1$ maximises f. Hence, the Nash equilibrium exists in G_I. Similarly, it can be shown that the Nash equilibrium exists in G_{II}.

□

2.4 Numerical Example

In this section, a numerical example with 12 major non-life insurance companies based on the number of contracts (i.e., volume of business) they have in their portfolios is proposed to illustrate the main modelling characteristics and theoretical findings of our paper. A scenario which investigates insurers with different market power is considered by consisting of a market leading insurer with $796,139$ contracts, nine almost equal insurers with around $300,000$ contracts and two followers with only around $200,000$ contracts.[3] Referring to the premium values at time $t = 0$, the pricing strategy for the entire market of insurers is derived by finding the Nash equilibrium premiums at time $t = 1$. The impact of different parameters involved in the process to the equilibrium premiums is also analysed. To generate results that are comparable to those existing in the literature of actuarial science and for simplicity in our calculations, convex premium strategy sets are considered in the numerical example.[4]

Insurance Companies i	Premium p_i^0	Number of Contracts q_i^0	Price sensitivity parameter a_i
1	€269.09	298,269	2.0
2	€282.07	303,673	2.0
3	€377.06	282,224	2.0
4	€371.52	304,609	2.0
5	€281.56	295,769	2.0
6	**€377.83**	**796,139**	**1.9**
7	€257.88	298,304	2.0
8	**€366.99**	**200,135**	**2.1**
9	**€347.58**	**211,314**	**2.1**
10	€351.18	299,690	2.0
11	€364.11	299,995	2.0
12	€291.22	319,453	2.0

TABLE 2.1: 12 insurance companies are considered from the Greek insurance market in 2010. Premium values and number of contracts are based on data from *the Hellenic Association of Insurance Companies*. Price sensitivity parameter for every insurer demonstrated in the table is used as a benchmark.

Data is used from the Greek market, as it was presented in [41, 43]. Thus, the premium prices are calculated in Euros. Let us assume that the number of contracts at time $t = 0$ is demonstrated in Table 2.1. With respect to t, this dataset is adopted for a 12-player game because the insurers' premium prices and exposure in the previous

[3] We don't have here any intention to develop any type of Stackelberg leadership model. However, the Greek insurance market might be considered as an ideal case for this model. Thus, it will be considered as a future work.

[4] We recall that the theoretical results did not assume any type of convexity.

period are used. With an intention to describe insurance companies' market power, the price sensitivity parameter, a_i, for all insurers i is characterised further.

The standard values of price sensitivity parameter are set up in Table 2.1, and they can be used as a benchmark. As it was already demonstrated, insurer 6 is considered to be the market leader with the lowest price sensitivity parameter $a_6 = 1.9$, because it occupies significant greater market weight compared with other insurers. Correspondingly, insurers 8 and 9 are regarded as market followers, which have price sensitivity parameters of value 2.1. All of the others insurers' price sensitivity parameter take the value of 2.0 in our insurance game.[5]

The diversity of the price sensitivity parameter for the insurers obviously affect the equilibrium premium profiles. Different values of a_i are investigated through a simulation. However, for any i, a_i^1 are restricted in $[1.5, 2.5]$. Using the previously demonstrated market data, the Nash equilibrium premium profiles are calculated for both G_I and G_{II}.

2.4.1 Insurance Game I Simulation Results

In Insurance Game I, G_I , the Nash equilibrium premium profiles are calculated with respect to the market's data at time $t = 0$; see Table 2.1. Table 2.2 sets up also ad hoc the main parameters. Note that for any insurer i in G_1, the breakeven premium π_i^1 is not assumed to be proportional to p_i^1. The percentage between π_i^1 and p_i^1 is used to describe the cost structure of i.

Number of market participants	n	12
Market presence limit factor	h_1	0.09
The breakeven ratio of every insurer	k_i	0.5

TABLE 2.2: Environmental parameter values in G_I

From Eq. (2.6), the second order condition of payoff is negative for each insurer i in G_I. Hence, when the stationary point is in the domain $\mathbf{P}_{G_I,i}$, i's payoff is maximized.

[5]The values for a_i have been considered ad hoc based on the concept of [35] index. Unfortunately, we don't have access to more detailed data, and some of the model parameters are rather artificial. This is common in the corresponding literature [25, 24]

What is more, one can find out the Nash equilibrium profiles by implementing the following algorithm:

Step 1: For each insurer i, set the first order condition of its payoff function equal to zero as the maximum selection(s). From Eq. (2.10),

$$\hat{r}_{G_I,i} : \frac{1 + (n-1)h_1 a_i + \pi_i^1 x_{-i}^1 \frac{a_i h_1}{p_i^0}}{2h_1 a_i x_{-i}^1} p_i^0 = 0.$$

Step 2: Solve the system of $r_{G_I,i}^1$.

Step 3: Select the profile(s) corresponding to each insurer's premium located in $\mathbf{P}_{G_I,i}$, which is (are) the Nash equilibrium premium profile(s).

Be aware that when the derived values are located outside of $\mathbf{P}_{G_I,i}$, then these are not the equilibrium premiums, as the edges of the premium domain reach a maximum instead. Furthermore, it indicates that the Nash equilibrium still exists even though the calculated premium profile have not located inside of $\mathbf{P}_{G_I,i}$. However, this case won't be analysed further here.

Let us now characterize the premium strategies set $\mathbf{P}_{G_I,i}$. For each insurer i, the premiums are restricted to take values between €180 and €800 during any period, i.e., $p_i^1 \in [€180, €800]$. In addition the other parameters are restrained, i.e., the market presence limit factor $h_1 \in [0.07, 0.11]$ and the breakeven premium $\pi_i^1 \in [30\% p_i^1, 70\% p_i^1]$, for any i, t. Numerical results for the system of equations $r_{G_I,i}^1$ are generated using m-file "fsolve". It should be mentioned that the Nash equilibrium premium profile might not be unique. However, among these results we chose the first positive premium profile which located in $\mathbf{P}_{G_I,i}$.[6] This result is illustrated in Figure 2.1. Figure 2.2 shows the corresponding number of contracts from insurers 1 to 12.

The ratio between insurers' equilibrium premiums at time $t = 1$ is correlative to the previous premium ratio in Figure 2.1. Note that the market leader insurer 6 tends to increase its premium, which leads to a reduction of its policy numbers in Figure 2.2.

[6] Among all the possible positive profiles, we pick up the smallest one based on the iterative algorithm of the Matlab, m-file "fsolve".

Larger market power offers insurer 6 the advantage in competition, which allows it to increase its premium until equilibrium for seeking higher profit.

Figure 2.3 demonstrates the effect of the increasing parameter π_6^1 in G_I. In Figure 2.3, adjustment for a single insurer's breakeven premium ratio is investigated. The market leader, insurer 6, is modelled to increase π_6^1 from 30% to 70% of p_i^1, whereas all other insurers keep the ratio at 50%. The increase in the breakeven premium ratio of insurer 6 is observed to cause not only an increase in its equilibrium premium but also a slight incremental increase in other insurers' premiums.

Price sensitivity parameter, a_i, strongly affects the equilibrium premium of each insurer i. The effects of modifying a_i with regard to the market leading insurer 6 and the market follower 8 are illustrated in Figures 2.4 and 2.5, and all other parameters remain the same as before. Figure 2.4 shows that the two players' equilibrium premiums decrease as the price sensitivity parameter decreases. In Figure 2.5, the number of contracts is observed to increase as a_i increases for both insurers 6 and 8. In addition, in both Figures 2.4 and 2.5, the slope of insurer 6 is obviously larger than that of insurer 8, indicating that parameter a_i is more sensitive with respect to the market leader than the market follower.

The values of parameters a_6 and h_1 strongly affect the equilibrium premium at time $t = 1$. We give an example of insurer 6 about the sensitivity with respect to these two parameters in Figure 2.6.

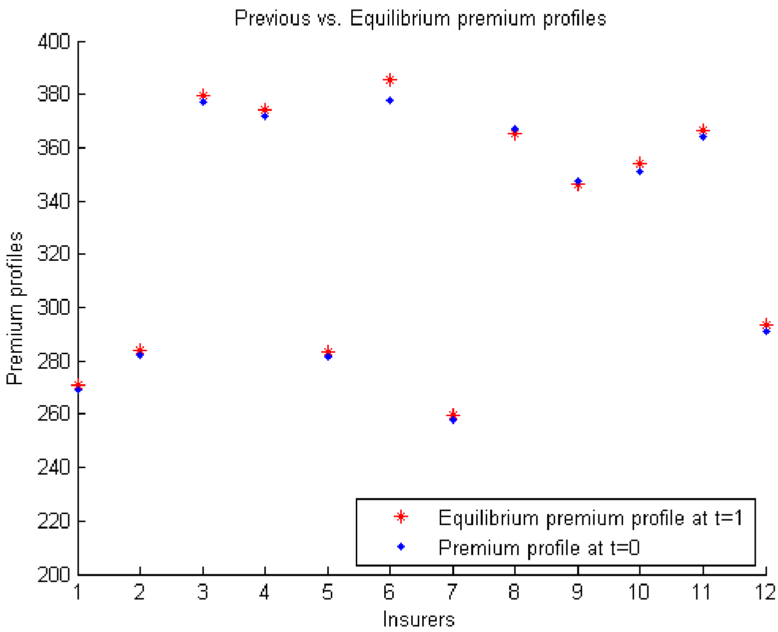

FIGURE 2.1: Previous (at time $t = 0$) vs. equilibrium (at time $t = 1$) premium profiles in G_I. The red solid line is the equilibrium premium profile at time $t = 1$ with respect to 12 insurers, which is on the x-axis. Premium values are given on the y-axis. The blue dash line represents the previous premium profile given in the Table 2.1.

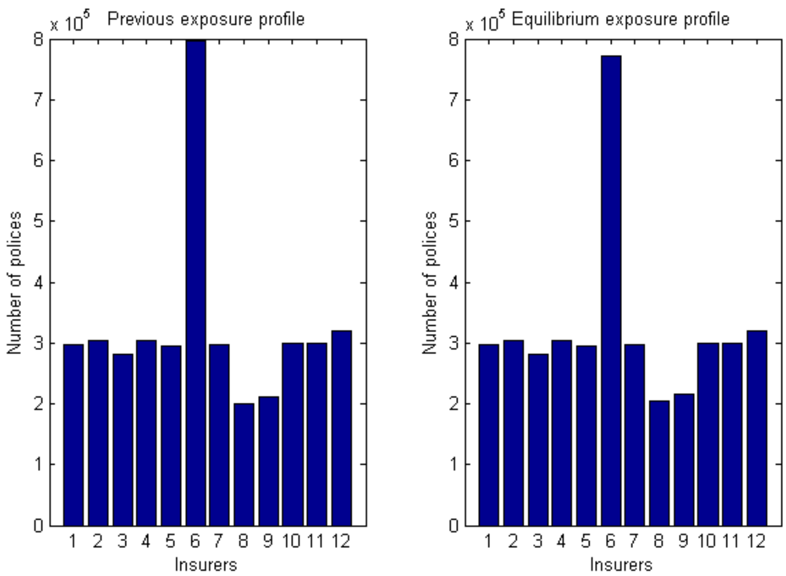

FIGURE 2.2: Previous (at time $t = 0$) vs. equilibrium (at time $t = 1$) number of policies in G_I. The left figure illustrates the number of contracts with respect to 12 insurers at time $t = 0$, which are given in Table 2.1. The right figure shows the equilibrium number of contracts at time $t = 1$.

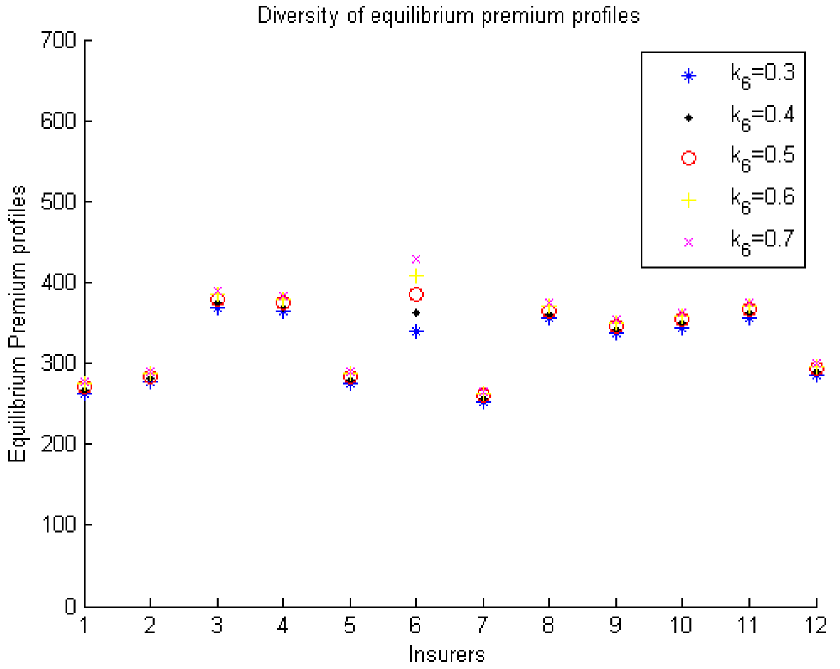

FIGURE 2.3: Diversity of equilibrium premium profiles with different π_6^1 in G_I. The market leader 6's breakeven premium ratio is investigated, which takes values from 30% to 70%. The corresponding 5 different equilibrium premium profiles are given.

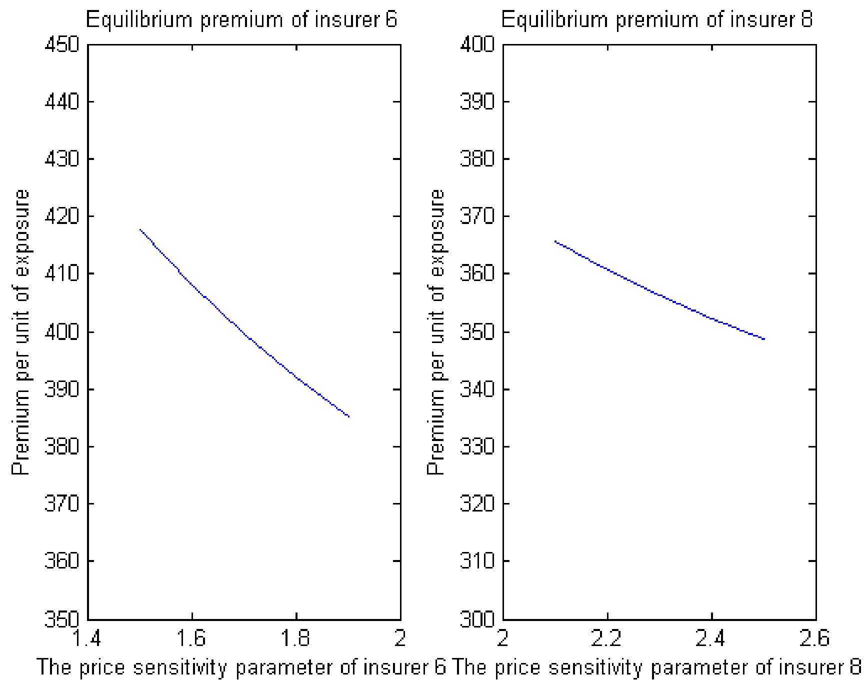

FIGURE 2.4: Equilibrium premium sensitivity test of a_6 and a_8 in G_I.

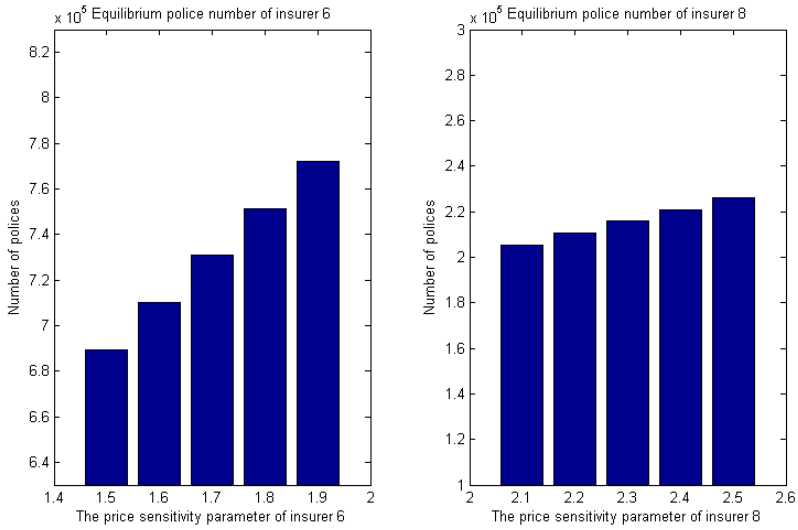

FIGURE 2.5: Equilibrium number of policies sensitivity test of a_6 and a_8 in G_I.

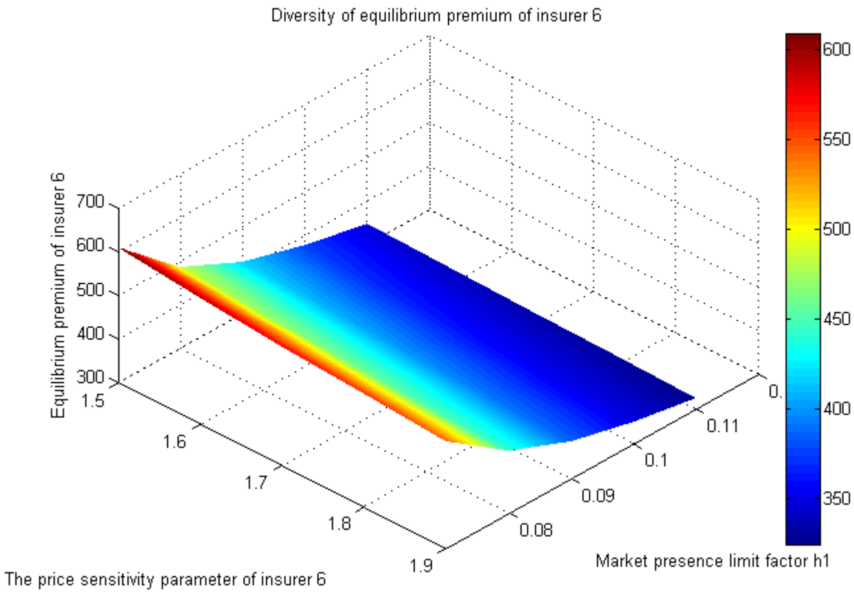

FIGURE 2.6: Diversity of insurer 6's equilibrium premium in G_I. Different equilibrium premium values are given, with respect to different a_6 and h_1.

2.4.2 Insurance Game II Simulation Results

Using Table 2.1, and the same parameters reported in Table 2.3, the Nash equilibrium premium profiles in G_{II} are calculated. From Eq. (2.9), the second order condition of payoff is negative for each insurer i in G_{II}. Similarly, we use the algorithm which is presented for the case G_I by assuming that $\hat{r}^1_{G_{II},i}$ is defined by

$$\hat{r}^1_{G_{II},i} : \frac{q_i^0 + nh_2 a_i q_i^0 - h_2 \sum_{j\neq i} a_j q_j^0}{2h_2 a_i q_i^0 (x^1_{-i} + \theta^1)} p_i^0 = 0,$$

where $\hat{r}^1_{G_{II},i}$ is the maximal selection of $R_{G_{II},i}(x^1_{-i})$, see Eq. (2.11), for i at time $t = 1$ in G_{II}.

Note that the breakeven ratio k_i does not affect the best-reply selection in G_{II}. If the calculated premium for each insurer is located in $\mathbf{P}_{G_I,i}$, the Nash equilibrium is unique in G_{II}, since the equation of $\hat{r}^1_{G_{II},i}$ is a linear one.

Number of market participants	n	12
Market presence limit factor	h_2	0.0205
Market stability factor	θ^1	1

TABLE 2.3: Environmental parameter values in G_{II}

In G_{II}, for each insurer i, the premiums are retained between €180 and €900 during any period, i.e., $p_i^1 \in [€150, €900]$. The other parameters are also restricted, such as the market presence limit factor $h_2 \in [0.0203, 0.0207]$ and the market stability factor $\theta^1 \in [0.8, 1.2]$ for any t. Figures 2.7 and 2.8, respectively, show the equilibrium premium profile and number of contracts from insurers 1 to 12.

In Figure 2.7, similar to G_I, market leader insurer 6 tends to increase its premium until equilibrium. As exposure flows between insurers are enhanced, the ratio between insurers' equilibrium premium in G_{II} significantly diverge from the previous. Compared with G_I, the market leader has a greater advantage in the competition, which generates a larger reduction in the policy numbers than in Figure 2.2. Market followers 8 and 9 reduce their premiums significantly to increase their exposure. As demonstrated in Figure 2.8, the equilibrium number of policies of insurers 8 and 9 approximately reach the other insurer's level, excluding the market leader insurer 6.

With the other parameters unaffected, the impacts of modifying a_i in G_2 with regard to the market leading insurer 6 and the market follower 8 are illustrated in Figures 2.9

and 2.10. Similarly as G_I, Figures 2.9 and 2.10 indicate that both players' equilibrium premiums in G_{II} decrease and the number of contracts increases as the price sensitivity parameter a_i decreases. In addition, we also conclude that the parameter a_i with respect to the market leader is more sensitive than the market follower in G_{II}. Comparing with Figures 2.4 and 2.5 in G_I that a_6 is more sensitive than a_8 in G_{II} is also noteworthy.

A new parameter, market stability factor θ^1, significantly affect the equilibrium premium profile in G_{II}. Figure 2.11 illustrates the diversity of the equilibrium premium profiles with a varying market stability factor θ^1 from 0.8 to 1.2. As θ^1 represents the whole market's business condition, it is reasonable to expect the equilibrium profile entirely moves up or down with different θ^1.

Similarly as in G_I, we test the sensitivity of a_6 and h_2 for G_{II} in Figure 2.12. As we can observe, h_2 is much more sensitive than h_1, an tiny increase of just 10^{-4} in h_2 causes a compelling decrease in equilibrium premium for insurer 6.

Overall, we observe that insurers with larger market power take advantage in the competition, and they tend to increase their premium to reach equilibrium. On the other hand, insurers with less market power tend to decrease their premium requesting a bigger volume of exposure. The price sensitivity parameter, a_i, is quite sensitive. The market presence limit factor h_1, h_2, and the market stability factor θ^1 have an impact on the market equilibrium levels, which control the exposure of volume flow among the insurers and the exposures volume flow into or away from the insurance market, respectively. Different with G_I, a breakeven premium for i appears not to affect the insurer's equilibrium premium in G_{II}.

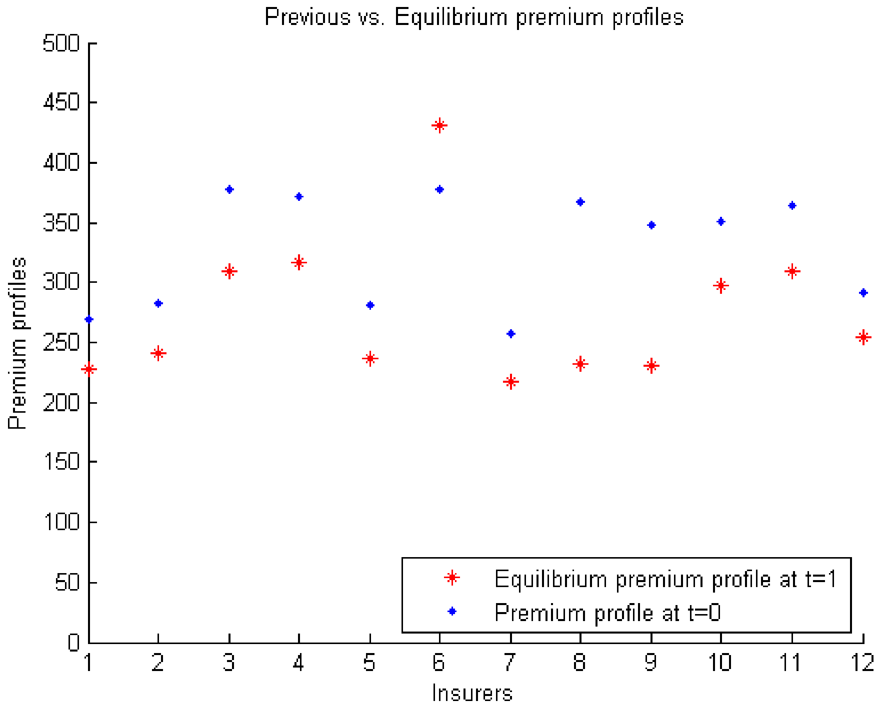

FIGURE 2.7: Previous (at time $t = 0$) vs. equilibrium (at time $t = 1$) premium profiles in G_{II}. Similar with Figure 2.1.

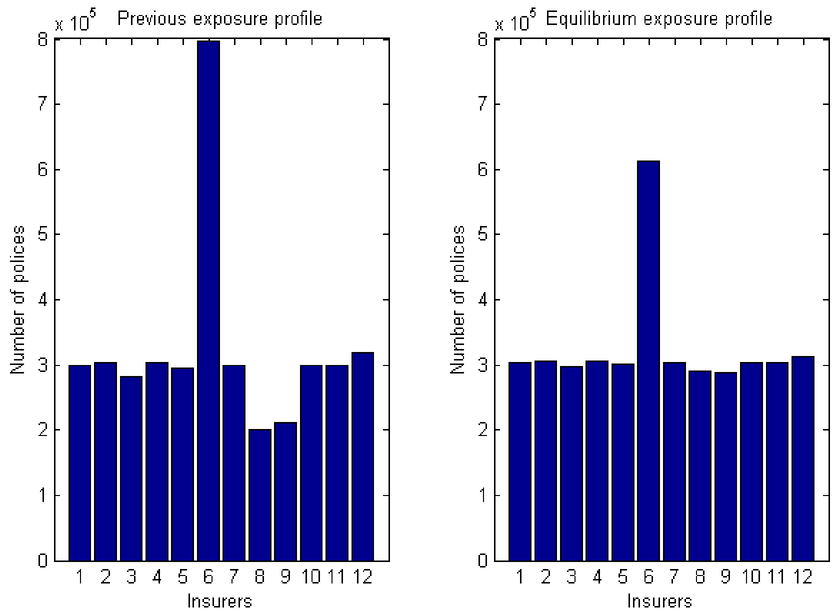

FIGURE 2.8: Previous (at time $t = 0$) vs. equilibrium (at time $t = 1$) number of policies in G_{II}. Similar with Figure 2.2.

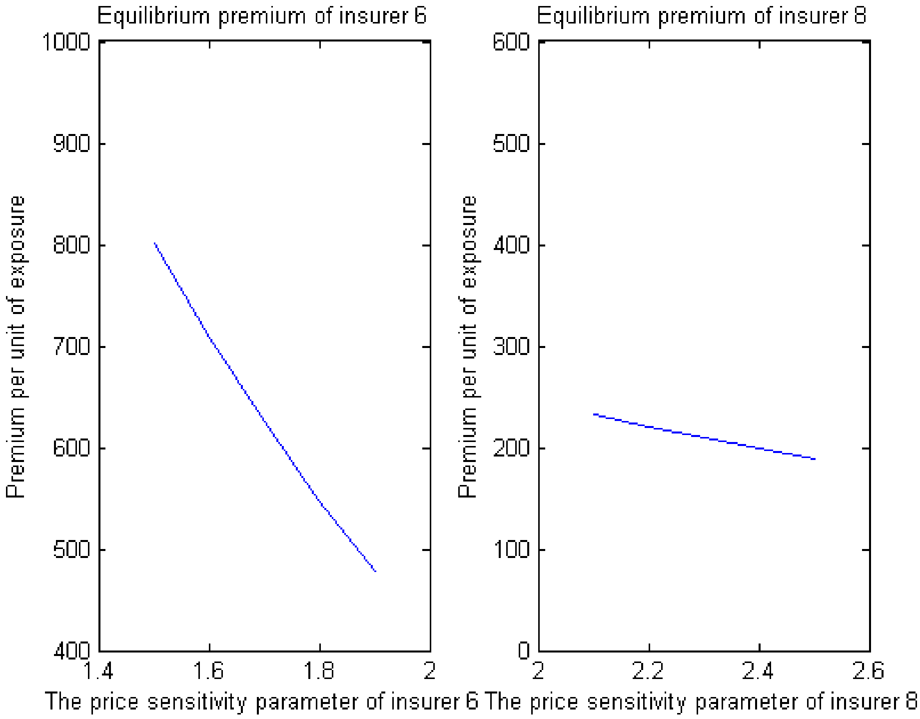

FIGURE 2.9: Equilibrium premium sensitivity test of a_6 and a_8 in G_{II}.

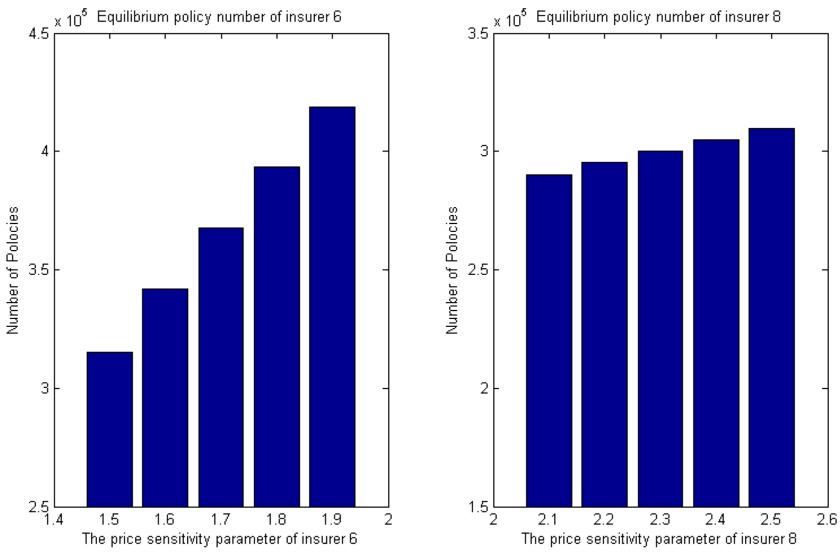

FIGURE 2.10: Equilibrium number of policies sensitivity test of a_6 and a_8 in G_{II}.

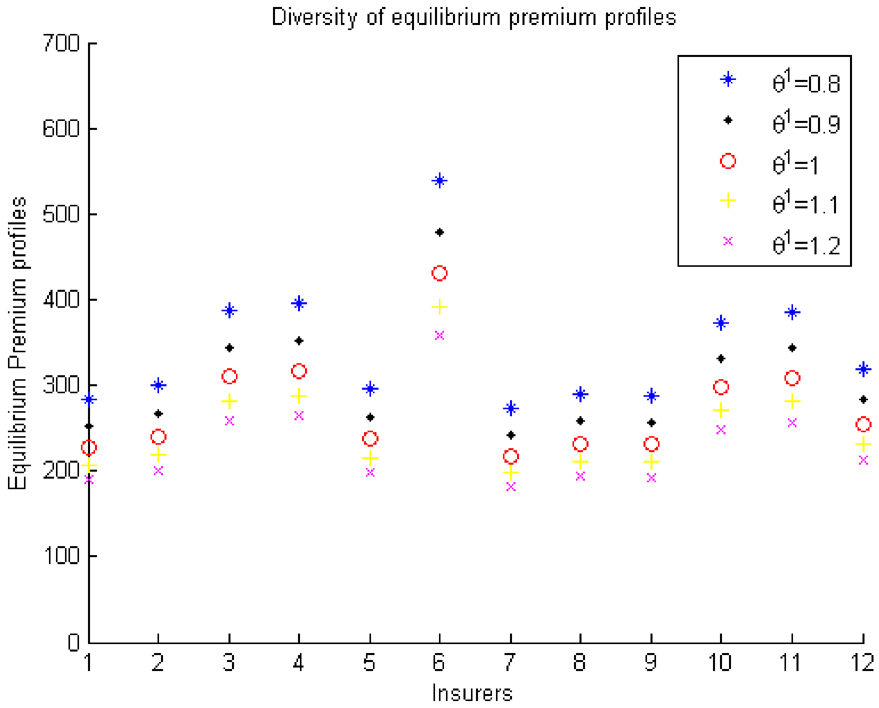

FIGURE 2.11: Diversity of equilibrium premium profiles with different θ^1 in G_{II}. The market stability factor θ^1 is investigated which takes values from 0.8 to 1.2. The corresponding 5 different equilibrium premium profiles are given.

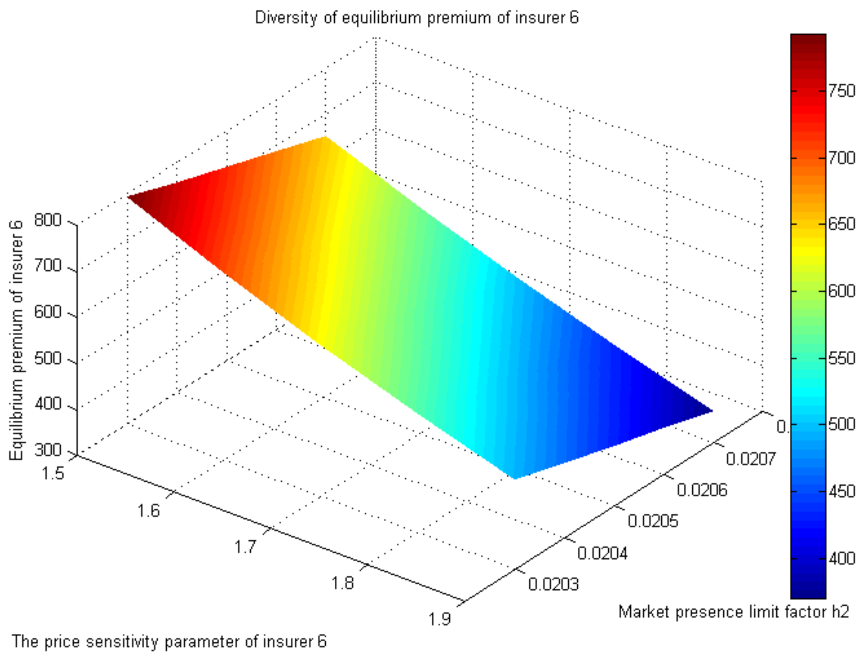

FIGURE 2.12: Diversity of insurer 6's equilibrium premium in G_{II}. Different equilibrium premium values are given, with respect to different a_6 and h_2.

Chapter 3

Non-Cooperative Dynamic Games for General Insurance Markets

In the insurance industry, the number of product-specific policies from different companies has increased significantly. The strong market competition has boosted the demand for a competitive premium. In actuarial science, scant literature still exists on how competition actually affects the calculation and the pricing cycles of company's premiums. In this paper, we model premium dynamics via a differential game, and study the insurers' equilibrium premium pricing in a competitive market. We apply an optimal control theory methodology to determine the open-loop Nash equilibrium premium strategies. In this direction, two models are formulated and studied. The market power of each insurance company is characterized by a price sensitive parameter, and the business volume is affected by the solvency ratio. Considering the average market premiums, the first model studies an exponential relation between premium strategies and volume of business. The other model initially characterize the competition between any selected pair of insurers, then aggregates all the paired competitions in the market. Numerical examples illustrate the premium dynamics, and show that premium cycles may exist in equilibrium.

Keywords: Insurance Market Competition; Price Cycles; Non-cooperative Game; Solvency Ratio

3.1 Introduction

3.1.1 Motivation

This paper constructs two models for determining the price of general policies in competitive, non-cooperative, insurance markets. In the corresponding literature, there is little available research on how insurance premiums are modelled in competitive markets and how they respond to changes offered by competitors [12, 25, 22]. Despite the fact that in many lines of insurance the presence of underlying cycles is clearly observed, a mathematical formulation, modelling and analysis of those underwriting strategies have been a constant endeavour to better understand the behaviour of insurance markets [10, 47, 15, 13, 61, 11, 32, 56, 36, 22].

Since the competition is getting higher among insurance companies, and in several markets worldwide, the domination by a relatively few companies appears often in the determination of insurance premium prices, a fair, but also a commercially attractive premium is not any more a simple risk assessment exercise, but a highly challenging decision. Consequently, the demand of a mathematical model is more essential than ever to investigate the connectivity among the competitors in the corresponding markets and to understand the formulation of pricing cycles.

3.1.2 New approach: Generalized finite-time differential game models

In this paper, generalized finite-time differential games with finite number of players are constructed. The formulation allows to investigate the mechanism for the pricing cycles by solving NE premium profiles. When the market reaches a NE, no insurer can increase its payoff by modifying its strategy (over the time) given by the optimal strategy profile of other insurers. As in [22], the optimal control theory methodology is incorporated.

Moreover, under a continuous-time framework, for any time unit, the number of new contracts is modelled considering competition, while the loss of exposure due to policy termination is assumed to be proportional to the current volume of exposure. In this direction, two competition-related models are proposed, studied and compared: (1) *Model I* adopts the exponential demand function of [54, 55, 22] considering the market average premium; while, (2) *Model II* is formulated based on the aggregate game in [62].

Analytically, the price sensitivity parameter, which has been proposed in [62], is implemented as a market power parameter. The solvency ratio is the capital per unit of premium. Solvency ratio is taken into consideration in the competition between each pair of insurers, as it is observable by the policyholders. In [56], it is stated that the management department will adjust its actuarial premium price with respect to the current solvency ratio. Considering historical data, when the capital amount is relatively high compared with actuarial premium value, insurance companies prefer to increase its premium value. The reason is that the insurers are more confident to pay the claims under this condition. In the present paper, we implement the concept of solvency ratio in the competition. Differently from [56], we develop an optimization problem where the solvency ratio is embedded in formulating an insurance game. In particular, we assume that if the (observable) solvency ratio is high, the number of new contracts sold will be affected less by other insurers' premium strategies . Interestingly, pricing cycles are observed in the numerical example of Model II, even without the consideration of any stochastic parameter.

The remainder of this paper is organized as follows. Section 3.2 introduces the construction of the two-insurance market competition models. In Section 3.3, the optimization problem is formulated for the two models, and the Hamiltonians are presented. Section 3.4 presents a numerical example for each of those two models.

3.2 Model Construction

3.2.1 Baseline model

Let $\mathbf{N} = \{1, \ldots, n\}$ be the finite set of insurers in the market. For a given time period $[0, T]$, we assume that every insurer $i \in \mathbf{N}$ aims to maximize the *net present value* of its terminal wealth. At every point in time $t \in [0, T]$, every insurer $i \in \mathbf{N}$ makes a decision to set the premium $p_i(t)$ (per unit of exposure). The decisions of all insurers in the market lead to the state variable $\theta_i(t) = (k_i(t), q_i(t))$, where $k_i(t) > 0$ is the capital (per unit of exposure) of insurer i, and $q_i(t) > 0$ is the volume of exposure of insurer i at time t, which represents the number of policies. Denote $P_i(t) = \{p_i(t') : t' \leq t\}$ and $\Theta_i(t) = \{\theta_i(t') : t' \leq t\}$. Moreover, we write $M_{-i}(t) = \{M_j(t) : j \in \mathbf{N} \backslash \{i\}\}$ for any function M, $\dot{k}_i(t) = \frac{d}{dt}k_i(t)$ and $\dot{q}_i(t) = \frac{d}{dt}q_i(t)$.

In line with [23], we assume that there is a fixed length τ of insurance policies, and all new and existing policyholders are required to pay the current premium rate $p_i(t)$. We illustrate in Figure 3.1 how the underwriting polices affect the exposure volume of the insurer.

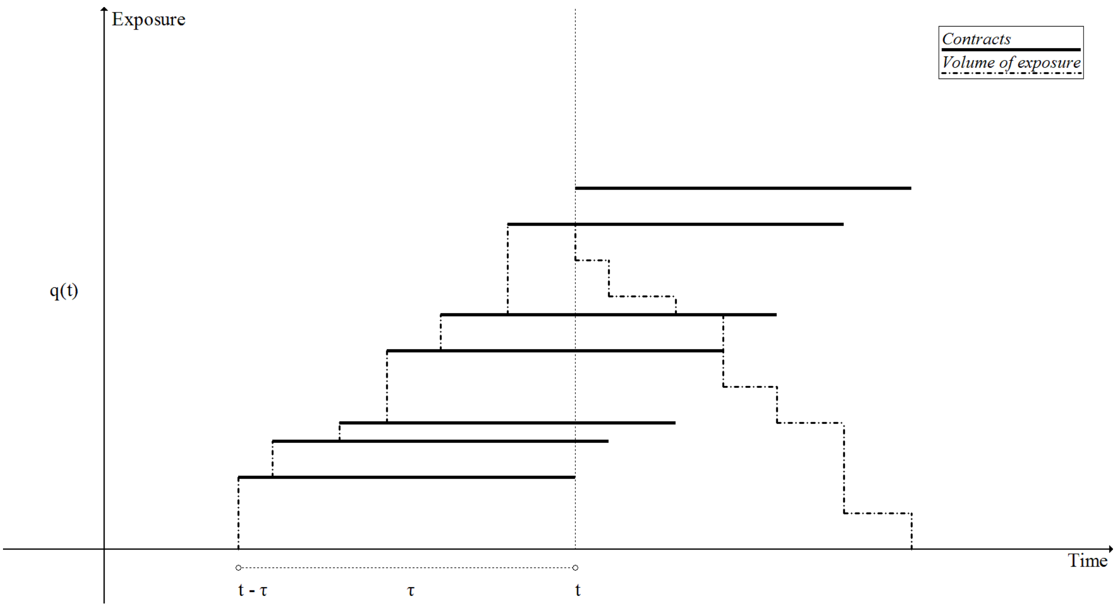

FIGURE 3.1: The dashed line is the volume of exposure $q(t)$ with respect to time t, while thick lines denote the duration of policies with the same start date.

The change in exposure at any time t can be split up into the one gained due to the generation of new contracts and the other one lost due to policy termination. In order to proceed further, in the next paragraphs we identify the necessary details of those two effects.

In order to use conventional control theory, we follow [23] by assuming that the loss due to policy termination is proportional to $\tau^{-1}\, q_i(t)$ for any insurer i. Then, the state equation of exposure for insurer i is given by

$$\dot{q}_i(t) = m_i(t)q_i(t) - \tau^{-1}\, q_i(t), \tag{3.1}$$

where $m_i(t)$ is the marginal number of new policies sold at time t per unit of exposure. The value of $m_i(t)$ may depend on $(p_i(t), \theta_i(t), p_{-i}(t), \theta_{-i}(t))$. As illustrated in Figure 3.1, the bottom line indicates the group of exposure (policies) expired at time t, which is $\tau^{-1}\, q_i(t)$; the top line represents the group of new policies $m_i(t)q_i(t)$; and the rest of lines stand for the holding policies at time t.

Define $I_i(t, t+\Delta t)$ as the premium income of insurer i in period $[t, t+\Delta t)$ and $C_i(t, t+\Delta t)$ as the cost of holding capital. Here, we assume that the premiums are paid at the beginning of each contract and all insurance policies have a fixed length τ, and so $I_i(t, t+\Delta t)$ is the premium income of the new contracts generated. For a small time period Δt after t, we have

$$I_i(t, t+\Delta t) = p_i(t)\, m_i(t)\, q_i(t)\, \Delta t + o(\Delta t^2).$$

Define $\pi_i > 0$ as the constant break-even premium (per unit of exposure) for insurer i, and $\beta_i \in (0, 1)$ as the depreciation of capital for insurer i. The break-even premium π_i is the deterministic insurance claim that needs to be paid per unit of exposure (see, e.g., [22]). The cost of holding capital $C_i(t, t+\Delta t)$ during the period $[t, t+\Delta t)$ is given by

$$C_i(t, t+\Delta t) = \beta_i\, K_i(t)\, \Delta t + o(\Delta t^2).$$

Moreover, the insurer needs to pay $\pi_i \, q_i(t) \, \Delta t + o(\Delta t^2)$ for insurance claims during the period $[t, t + \Delta t)$. The total capital difference for insurer i between time t and $t + \Delta t$ is given by

$$\begin{aligned} \Delta K_i(t) &= I_i(t, t+\Delta t) - C_i(t, t+\Delta t) - \pi_i \, q_i(t) \, \Delta t + o(\Delta t^2) \\ &= (p_i(t) \, m_i(t) - \pi_i - \beta_i \, k_i(t)) \; q_i(t)\Delta t + o(\Delta t^2). \end{aligned}$$

The volume of insurer's exposure is also modified considering the entry of new business and the expiration of existing policies. The difference of capital per exposure in period $[t, \Delta t)$ equals to

$$\begin{aligned} \Delta k_i(t) &= k_i(t+\Delta t) - k_i(t) \\ &= \frac{K_i(t+\Delta t)}{q_i(t+\Delta t)} - k_i(t) \\ &= \frac{K_i(t) + \Delta K_i(t)}{q_i(t) + \dot{q}_i(t)\Delta t} - k_i(t) + o(\Delta t^2) \\ &= \frac{k_i(t) + (p_i(t) \, m_i(t) - \pi_i - \beta_i \, k_i(t)) \; \Delta t}{1 + (m_i(t) - \tau^{-1}) \; \Delta t} - k_i(t) + o(\Delta t^2) \\ &= \big(p_i(t) \, m_i(t) - \pi_i - k_i(t) \; \big(\beta_i + m_i(t) - \tau^{-1}\big)\big) \; \Delta t + o(\Delta t^2), \end{aligned}$$

by using a Taylor series expansion. Therefore, the state equation of capital per exposure for insurer i is given by

$$\dot{k}_i(t) \;=\; p_i(t) \, m_i(t) - \pi_i - k_i(t) \; \big(\beta_i + m_i(t) - \tau^{-1}\big) \, . \tag{3.2}$$

In line with [22], we propose the following time-separable utility function for insurer i:

$$u_i(P_i(T); \Theta_i(T)) = \int_0^T e^{-\zeta t} F_i(p_i(t); \theta_i(t)) dt. \tag{3.3}$$

Here, $\zeta \in (0, 1)$ is the discount factor, and

$$F_i(p_i(t); \theta_i(t)) = (p_i(t)m_i(t) - \pi_i - \beta_i k_i(t)) q_i(t). \tag{3.4}$$

A set of control functions $t \mapsto (p_1^*(t), p_2^*(t), \dots, p_n^*(t))$ is a NE for the game within the class of open-loop strategies if the following holds. For any insurer i, the control $p_i^*(\cdot)$ provides a solution to the optimal control problem:

$$\textit{maximize} \quad u_i(P_i(T); \Theta_i(T)), \tag{3.5}$$

over the set of controllers, $P_i(T)$, where the set of controllers of other insurers, $P_{-i}(t)$, is feasible, and the system has dynamics: $\{p_1(0), ..., p_n(0)\}$ and $\{\theta_1(0), ..., \theta_n(0)\}$ given, and

$$\begin{aligned} \dot{k}_i(t) &= p_i(t)\, m_i(t) - \pi_i - k_i(t)\, \left(\beta_i + m_i(t) - \tau^{-1}\right), \\ \dot{q}_i(t) &= m_i(t) q_i(t) - \tau^{-1}\, q_i(t), \end{aligned}$$

for all $i \in \mathbf{N}$. Here we assume k_i and q_i always exist for all i and t.

The rate of generating business m_i is affected by market competition. We propose two models in Sections 3.3.1 and 3.3.2 that are defined as *Model I* and *Model II*, respectively. Inspired by [54, 55] and [22], Model I investigates exponential relations between exposure volume and premium competition. Market average premium is also considered. On the other hand, Model II characterizes exposure volume regarding the aggregation of competition among all the pairs of insurers. Furthermore, the price elasticity function concept is adopted to investigate the exposure volume change in Model II, as an extension of [62].

3.2.2 Model I Formulation

Model I adopts the exponential demand function proposed in [54, 55] and [22] for modelling the competition between any pair of insurers. Let us define the function ρ_i of insurer i at time t as follows,

$$\rho_i(t) = -(p_i(t) - \bar{p}_{-i}(t)),$$

where $\bar{p}_{-i}(t)$ is the average premium of all the other insurers in insurance market except i. When $\rho_i(t)$ is positive, insurer i's premium $p_i(t)$ is less than $\bar{p}_{-i}(t)$; then, we assume that insurer i tends to gain exposure from the rest of insurance market. Policies flow in a reverse manner when ρ_i is negative. We model the rate of selling new policies $m_i(t)$ for insurer i at time t as

$$m_i(t) = \tau^{-1}\, h_1\, r_i\, e^{b_i\, \rho_i(t) - \frac{p_i(t)}{k_i(t)}}, \tag{3.6}$$

where $b_i > 0$ is the price sensitivity parameter of insurer $i \in \mathbf{N}$, $h_1 > 0$ is a market presence limit factor, and $r_i > 0$ is a benchmark parameter of insurer i.

In line with the exponential demand function in [54, 55], we initially model m as $\tau^{-1}\, e^{b_i \rho_i(t)}$. When, $p_i(t) < \bar{p}_{-i}(t)$, $\tau^{-1}\, e^{b_i \rho_i(t)}$ is larger than τ^{-1}. We further augment this effect with an influence of the solvency ratio on competition, which is a new concept in our paper.

We study the solvency ratio, and its impact on the premium pricing strategy of insurer i. [56] assumed that the management department will adjust the actuarial premium price by comparing the insurer's current solvency ratio and a benchmark solvency ratio. Inspired by [56], we model the solvency ratio as $\frac{k_i(t)}{p_i(t)}$. We further modify the exponential component to $e^{b_i \rho_i(t) + ln(r_i) - \frac{p_i^t}{k_i^t}}$, where r_i is a positive benchmark solvency ratio for insurer i. When solvency ratio $\frac{k_i(t)}{p_i(t)}$ increases, the rate of selling new policies m_i^t increases, which describes insurers with larger solvency ratio can obtain more policies.

3.2.3 Model II Formulation

With the price sensitive parameter h_2 proposed by [62], Model II initially specifies the flow of policies between any pair of insurers. The entire insurance market competition can be evaluated by aggregating among the different pairs of insurers. For any insurer j, let us define the transfer function $\rho_{j\to i}(t)$ from insurer j to insurer i at time t as follows

$$\rho_{j\to i}(t) = 1 - \frac{p_i(t)}{p_j(t)}.$$

The transfer function $\rho_{j\to i}(t)$ describes the key assumption: for time t, when insurer i's premium is less than insurer j's premium, insurer j's policies tend to flow to insurer i. Policies flow in a reverse direction when $p_i(t) > p_j(t)$. We assume that the exposure flow from insurer j to insurer i is given by

$$q_{j\to i}(t) \quad = \quad h_2\, a_i\, \rho_{j\to i}(t)\, \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\, q_i(t), \tag{3.7}$$

where $a_i > 0$ is the price sensitivity parameter of insurer i, $h_2 > 0$ is a market presence limit factor, and $r_i > 0$ is a benchmark parameter of insurer i. Typically, we have $q_{j\to i}(t) \neq -q_{i\to j}(t)$.

The exposure changes over time follow from the competition in the entire market. It is obtained by summing up all the bilateral policies' gains or losses. The aggregate exposure gain or loss for insurer i is then given by

$$\dot{q}_i(t) = \sum_{j\in\mathbf{N}, j\neq i} q_{j\to i}(t). \tag{3.8}$$

We allow that $\sum_{i\in\mathbf{N}} \dot{q}_i(t)$ is not equal to zero, since potential customers may enter (leave) the insurance market when the premiums are low (high).

Substituting (3.7) in (3.8) yields that the rate of generation of new policies for insurer i at time t in Model II is given by

$$m_i(t) = \tau^{-1} + h_2\, a_i\, \frac{1}{r_i}\, e^{\frac{p_i(t)}{k_i(t)}} \sum_{j\in\mathbf{N}, j\neq i} \left(1 - \frac{p_i(t)}{p_j(t)}\right). \tag{3.9}$$

3.3 Theoretical Results

As it was discussed in Section 3.2, the NE pricing strategy for the ith insurer follows from a maximization problem over the set of feasible premium strategies given the feasible pricing strategies of the other insurer [14]. Thus, in this section, the corresponding Hamiltonians and related results for Models I and II are presented.

3.3.1 Optimisation Problem for Model I

From (3.4)-(3.6), we derive

$$F_i(p_i(t);\theta_i(t)) = \left(p_i(t)\,\tau^{-1}\,h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}} - \pi_i - \beta_i\,k_i(t)\right)\,q_i(t). \quad (3.10)$$

We obtain the following dynamics for the state variables of insurer i:

$$\begin{aligned}\dot{k}_i(t) &= p_i(t)\,\tau^{-1}\,h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}} - \pi_i \\ &\quad -k_i(t)\,\left(\beta_i + \tau^{-1}\,h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}} - \tau^{-1}\right), \end{aligned} \quad (3.11)$$

and

$$\dot{q}_i(t) = \left(h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}} - 1\right)\tau^{-1}q_i(t). \quad (3.12)$$

With the objective function and state equations, the Hamiltonian for the ith insurer is given by[1]

$$\begin{aligned} H_i &= e^{-\zeta t}\left(p_i(t)\,\tau^{-1}\,h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}}-\pi_i-\beta_i\,k_i(t)\right)\,q_i(t) \\ &\quad +\sum_{j\in\mathbf{N}}\mu_{ij}(t)\left[p_j(t)\,\tau^{-1}\,h_1\,r_j\,e^{b_j\,(\bar{p}_{-j}(t)-p_j(t))-\frac{p_j(t)}{k_j(t)}}-\pi_j\right. \\ &\quad \left.-k_j(t)\left(\beta_j+\tau^{-1}\,h_1\,r_j\,e^{b_j\,(\bar{p}_{-j}(t)-p_j(t))-\frac{p_j(t)}{k_j(t)}}-\tau^{-1}\right)\right] \\ &\quad +\sum_{j\in\mathbf{N}}\lambda_{ij}(t)\left(h_1\,r_j\,e^{b_j\,(\bar{p}_{-j}(t)-p_j(t))-\frac{p_j(t)}{k_j(t)}}-1\right)\tau^{-1}q_j(t). \end{aligned} \tag{3.13}$$

For any $j \in \mathbf{N}$, the adjoint equations are given by

$$\frac{d\lambda_{ij}(t)}{dt} = -\frac{\partial H_i}{\partial q_j(t)},\ \ \lambda_{ij}(T)=0, \tag{3.14}$$

$$\frac{d\mu_{ij}(t)}{dt} = -\frac{\partial H_i}{\partial k_j(t)},\ \ \mu_{ij}(T)=0. \tag{3.15}$$

Lemma 3.1. *For any j and t, it holds that $\lambda_{ij}(t)=0$ and $\mu_{ij}(t)=0$ for all $j\neq i$.*

Proof. For $j \neq i$, it holds that

$$\frac{d\lambda_{ij}(t)}{dt} = -\lambda_{ij}(t)\left(h_1\,r_j\,e^{b_j\,(\bar{p}_{-j}(t)-p_j(t))-\frac{p_j(t)}{k_j(t)}}-1\right)\tau^{-1}. \tag{3.16}$$

If $\lambda_{ij}(t)\neq 0$, let $A=\left(h_1\,r_j\,e^{b_j\,(\bar{p}_{-j}(t)-p_j(t))-\frac{p_j(t)}{k_j(t)}}-1\right)\tau^{-1}$. When $A\neq 0$, we have

$$|\frac{d\lambda_{ij}(t)}{A\cdot\lambda_{ij}(t)}| = dt \Leftrightarrow \int|\frac{d\lambda_{ij}(t)}{A\cdot\lambda_{ij}(t)}| = \int dt \Leftrightarrow \ln|\lambda_{ij}(t)| = |A\cdot t|+c$$

and, hence, we get $|\lambda_{ij}(t)| = e^{|A*t|+c}$. With $\lambda_{ij}(T)=0$, it is a contradiction. When $A=0$, we have $\lambda_{ij}(t)=0$ by construction. Hence $\lambda_{ij}(t)=0$, when $j\neq i$. We can prove that $\mu_{ij}(t)=0$ for all $j\neq i$ in a similar way. □

[1]With slight abuse of notation, we do not explicitly write that the Hamiltonian depends on $(p_i(t),\theta_i(t),p_{-i}(t),\theta_{-i}(t))$.

Using Lemma 3.1, the Hamiltonian in (3.13) simplifies to

$$\begin{aligned}
H_i \;=\;& e^{-\zeta t}\left(p_i(t)\,\tau^{-1}\,h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}}-\pi_i-\beta_i\,k_i(t)\right)q_i(t)\\
&+\mu_{ii}(t)\left[p_i(t)\,\tau^{-1}\,h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}}-\pi_i\right.\\
&\left.-k_i(t)\left(\beta_i+\tau^{-1}\,h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}}-\tau^{-1}\right)\right]\\
&+\lambda_{ii}(t)\left(h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}}-1\right)\tau^{-1}q_i(t).
\end{aligned} \tag{3.17}$$

From the adjoint equations, we have

$$\begin{aligned}
\frac{d\lambda_{ii}(t)}{dt} \;=\;& -\frac{\partial H_i}{\partial q_i(t)}\\
=\;& -e^{-\zeta t}\left(p_i(t)\,\tau^{-1}\,h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}}-\pi_i-\beta_i\,k_i(t)\right)\\
&-\lambda_{ii}(t)\left(h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}}-1\right)\tau^{-1},
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
\frac{d\mu_{ii}(t)}{dt} \;=\;& -\frac{\partial H_i}{\partial k_i(t)}\\
=\;& -e^{-\zeta\,t}\,q_i(t)\left(p_i(t)^2\,k_i(t)^{-2}\,\tau^{-1}\,h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}}-\beta_i\right)\\
&-\mu_{ii}(t)\left[-\beta_i-\tau^{-1}\,h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}}+\tau^{-1}\right.\\
&\left.+\left(p_i(t)-k_i(t)\right)\,p_i(t)\,k_i(t)^{-2}\,\tau^{-1}\,h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}}\right]\\
&-\lambda_{ii}(t)\,p_i(t)\,k_i(t)^{-2}\,\tau^{-1}\,h_1\,r_i\,e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}}\,q_i(t).
\end{aligned} \tag{3.19}$$

The first-order conditions of the Hamiltonian, defined in ((3.17)), are given by

$$\begin{aligned}\frac{\partial H_i}{\partial p_i(t)} &= e^{-\zeta\, t}\, q_i(t)\, \left(p_i(t)\, \left(-b_i - k_i(t)^{-1}\right) + 1\right) \tau^{-1}\, h_1\, r_i\, e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}} \\ &+\mu_{ii}(t) \Bigg[(p_i(t)-k_i(t))\, \left(-b_i - k_i(t)^{-1}\right) \tau^{-1}\, h_1\, r_i\, e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}} \\ &+\tau^{-1}\, h_1\, r_i\, e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}} \Bigg] \\ &+\lambda_{ii}(t)\, q_i(t)\, \left(-b_i - k_i(t)^{-1}\right) \tau^{-1}\, h_1\, r_i\, e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}}. \end{aligned} \tag{3.20}$$

which must equal zero for all $t \in [0, T]$ and $i \in \mathbb{N}$.

The second-order condition of the Hamiltonian is given by

$$\begin{aligned}\frac{\partial^2 H_i}{\partial p_i(t)^2} &= e^{-\zeta\, t}\, q_i(t)\, \left(2\, \left(-b_i - k_i(t)^{-1}\right) + \left(b_i + k_i(t)^{-1}\right)^2\right) \tau^{-1}\, h_1\, r_i\, e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}} \\ &+\mu_{ii}(t) \Bigg[2\, \left(-b_i - k_i(t)^{-1}\right) \tau^{-1}\, h_1\, r_i\, e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}} \\ &+ \left(p_i^t - k_i^t\right)\, \left(b_i + k_i(t)^{-1}\right)^2 \tau^{-1}\, h_1\, r_i\, e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}} \Bigg] \\ &+\lambda_{ii}(t)\, q_i(t)\, \left(b_i + k_i(t)^{-1}\right)^2 \tau^{-1}\, h_1\, r_i\, e^{b_i\,(\bar{p}_{-i}(t)-p_i(t))-\frac{p_i(t)}{k_i(t)}}. \end{aligned} \tag{3.21}$$

It is well-known in optimal control theory that the solution of the first-order conditions is a NE when the second-order conditions of Hamiltonians are non-positive for all $t \in [0, T]$ and $i \in \mathbb{N}$.

3.3.2 Optimisation Problem for Model II

In Model II, we derive

$$F_i(p_i(t); \theta_i(t)) = \left(p_i(t)\, \left(\tau^{-1} + \sum_{j \in \mathbf{N}, j \neq i} h_2\, a_i\, \left(1 - \frac{p_i(t)}{p_j(t)}\right)\, \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \right) - \pi_i - \beta_i\, k_i(t) \right) q_i(t), \tag{3.22}$$

Analogous to (3.11)-(3.12), we derive the following state equations

$$\begin{aligned}\dot{k}_i(t) &= p_i(t)\left(\tau^{-1} + \sum_{j\in\mathbf{N}, j\neq i} h_2\, a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\right) \\ &\quad -\pi_i - k_i(t)\left(\beta_i + \sum_{j\in\mathbf{N}, j\neq i} h_2\, a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\right), \qquad (3.23)\\ \dot{q}_i(t) &= \sum_{j\in\mathbf{N}, j\neq i} h_2\, a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\, q_i(t). \qquad (3.24)\end{aligned}$$

With the objective function and state equations, the Hamiltonian for insurer i is given by

$$\begin{aligned}H_i &= e^{-\zeta t}\left(p_i(t)\left(\tau^{-1} + \sum_{j\in\mathbf{N}, j\neq i} h_2\, a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\right) - \pi_i - \beta_i\, k_i(t)\right) q_i(t) \\ &\quad + \sum_{j\in\mathbf{N}} \mu_{ij}(t)\left[p_j(t)\left(\tau^{-1} + \sum_{\ell\in\mathbf{N}, \ell\neq j} h_2\, a_j \left(1 - \frac{p_j(t)}{p_\ell(t)}\right) \frac{e^{\frac{p_j(t)}{k_j(t)}}}{r_j}\right)\right. \\ &\quad \left. -\pi_j - k_j(t)\left(\beta_j + \sum_{\ell\in\mathbf{N}, \ell\neq j} h_2\, a_j \left(1 - \frac{p_j(t)}{p_\ell(t)}\right) \frac{e^{\frac{p_j(t)}{k_j(t)}}}{r_j}\right)\right] \\ &\quad + \sum_{j\in\mathbf{N}} \lambda_{ij}\left(\sum_{\ell\in\mathbf{N}, \ell\neq j} h_2\, a_j \left(1 - \frac{p_j(t)}{p_\ell(t)}\right) \frac{e^{\frac{p_j(t)}{k_j(t)}}}{r_j}\, q_j(t)\right). \qquad (3.25)\end{aligned}$$

For any $j \in \mathbf{N}$, the adjoint equations are given by

$$\begin{aligned}\frac{d\lambda_{ij}(t)}{dt} &= -\frac{\partial H_i}{\partial q_j(t)}, \lambda_{ij}(T) = 0, \qquad (3.26)\\ \frac{d\mu_{ij}(t)}{dt} &= -\frac{\partial H_i}{\partial k_j(t)}, \mu_{ij}(T) = 0. \qquad (3.27)\end{aligned}$$

Lemma 3.2. *For any i and t, it holds that $\lambda_{ij}(t) = 0$ and $\mu_{ij}(t) = 0$ for all $j \neq i$.*

Proof. The proof is similar to the proof of Lemma 3.1, and so it is omitted. □

Due to Lemma 3.2, the Hamiltonian in ((3.25)) simplifies to

$$\begin{aligned}
H_i &= e^{-\zeta t}\left(p_i(t)\left(\tau^{-1}+\sum_{j\in\mathbf{N},j\neq i} h_2\,a_i\left(1-\frac{p_i(t)}{p_j(t)}\right)\frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\right)-\pi_i-\beta_i\,k_i(t)\right)q_i(t)\\
&+\mu_{ii}(t)\left[p_i(t)\left(\tau^{-1}+\sum_{j\in\mathbf{N},j\neq i} h_2\,a_i\left(1-\frac{p_i(t)}{p_j(t)}\right)\frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\right)\right.\\
&\left.-\pi_i-k_i(t)\left(\beta_i+\sum_{j\in\mathbf{N},j\neq i} h_2\,a_i\left(1-\frac{p_i(t)}{p_j(t)}\right)\frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\right)\right]\\
&+\lambda_{ii}\left(\sum_{j\in\mathbf{N},j\neq i} h_2\,a_i\left(1-\frac{p_i(t)}{p_j(t)}\right)\frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\,q_i(t)\right),
\end{aligned}\tag{3.28}$$

From the adjoint equations, we get

$$\begin{aligned}
\frac{d\mu_{ii}(t)}{dt} &= -\frac{\partial H_i}{\partial k_i(t)}\\
&= -e^{-\zeta,t}\,q_i(t),\left[p_i(t)\left(\sum_{j\in\mathbf{N},j\neq i} h_2\,a_i\left(1-\frac{p_i(t)}{p_j(t)}\right)\frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\left(-(k_i(t))^2\,p_i(t)\right)\right)\right.\\
&\left.-\sum_{j\in\mathbf{N},j\neq i} h_2\,a_i\frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\left(-\frac{1}{p_j(t)}+\frac{1}{k_i(t)}\left(1-\frac{p_i(t)}{p_j(t)}\right)\right)\right]\\
&-\mu_{ii}(t)\left[(p_i(t)-k_i(t))\left(\sum_{j\in\mathbf{N},j\neq i} h_2\,a_i\left(1-\frac{p_i(t)}{p_j(t)}\right)\frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\left(-(k_i(t))^2\,p_i(t)\right)\right)-\beta_i\right.\\
&\left.-\sum_{j\in\mathbf{N},j\neq i} h_2\,a_i\left(1-\frac{p_i(t)}{p_j(t)}\right)\frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\right]\\
&-\lambda_{ii}\,q_i(t)\left(\sum_{j\in\mathbf{N},j\neq i} h_2\,a_i\left(1-\frac{p_i(t)}{p_j(t)}\right)\frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i}\left(-(k_i(t))^2\,p_i(t)\right)\right),
\end{aligned}\tag{3.29}$$

and

$$\begin{aligned}
\frac{d\lambda_{ii}(t)}{dt} &= -\frac{\partial H_i}{\partial q_i(t)} \\
&= -e^{-\zeta t} \left(p_i(t) \left(\tau^{-1} + \sum_{j\in\mathbf{N}, j\neq i} h_2\, a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \right) - \pi_i - \beta_i\, k_i(t) \right) \\
&\quad -\lambda_{ii} \left(\sum_{j\in\mathbf{N}, j\neq i} h_2\, a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \right). \qquad (3.30)
\end{aligned}$$

The first-order conditions of the Hamiltonian in ((3.28)) are given by

$$\begin{aligned}
\frac{\partial H_i}{\partial p_i(t)} &= e^{-\zeta\, t}\, q_i(t) \Bigg[\tau^{-1} + \left(\sum_{j\in\mathbf{N}, j\neq i} h_2\, a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \right) \\
&\quad + p_i(t) \left(\sum_{j\in\mathbf{N}, j\neq i} h_2\, a_i \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-\frac{1}{p_j(t)} + \frac{1}{k_i(t)} \left(1 - \frac{p_i(t)}{p_j(t)}\right)\right)\right) \Bigg] \\
&\quad + \mu_{ii}(t) \Bigg[\tau^{-1} + \left(\sum_{j\in\mathbf{N}, j\neq i} h_2\, a_i \left(1 - \frac{p_i(t)}{p_j(t)}\right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \right) \\
&\quad - (p_i(t) - k_i(t)) \left(\sum_{j\in\mathbf{N}, j\neq i} h_2\, a_i \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-\frac{1}{p_j(t)} + \frac{1}{k_i(t)} \left(1 - \frac{p_i(t)}{p_j(t)}\right)\right)\right) \Bigg] \\
&\quad + \lambda_{ii}(t)\, q_i(t) \left(\sum_{j\in\mathbf{N}, j\neq i} h_2\, a_i \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-\frac{1}{p_j(t)} + \frac{1}{k_i(t)} \left(1 - \frac{p_i(t)}{p_j(t)}\right)\right)\right) (3.31)
\end{aligned}$$

which must equal zero for all $t \in [0, T]$ and $i \in \mathbb{N}$.

The second-order condition of the Hamiltonian in ((3.28)) is given by

$$
\begin{aligned}
\frac{\partial^2 H_i}{\partial r_i(t)^2} = \; & e^{-\zeta\, t}\, q_i(t) \Bigg[2 \left(\sum_{j\in \mathbf{N}, j\neq i} h_2\, a_i\, \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-\frac{1}{p_j(t)} + \frac{1}{k_i(t)} \left(1 - \frac{p_i(t)}{p_j(t)} \right) \right) \right) \\
& + p_i(t) \left(\sum_{j\in \mathbf{N}, j\neq i} h_2\, a_i\, \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(\frac{-2k_i(t) + p_j(t) - p_i(t)}{k_i(t)^2\, p_j(t)} \right) \right) \Bigg] \\
& + \mu_{ii}(t) \Bigg[2 \left(\sum_{j\in \mathbf{N}, j\neq i} h_2\, a_i\, \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-\frac{1}{p_j(t)} + \frac{1}{k_i(t)} \left(1 - \frac{p_i(t)}{p_j(t)} \right) \right) \right) \\
& + (p_i(t) - k_i(t)) \left(\sum_{j\in \mathbf{N}, j\neq i} h_2\, a_i\, \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(\frac{-2k_i(t) + p_j(t) - p_i(t)}{k_i(t)^2\, p_j(t)} \right) \right) \Bigg] \\
& + \lambda_{ii}(t)\, q_i(t) \left(\sum_{j\in \mathbf{N}, j\neq i} h_2\, a_i \left(1 - \frac{p_i(t)}{p_j(t)} \right) \frac{e^{\frac{p_i(t)}{k_i(t)}}}{r_i} \left(-(k_i(t))^2\, p_i(t) \right) \right)
\end{aligned}
\tag{3.32}
$$

which must be non-positive for all $t \in [0, T]$ and $i \in \mathbb{N}$.

3.4 Numerical Application

In this section, the numerical application for both two models is introduced in a competitive insurance market. Instead of calculating the NE with respect to many players, more attention is paid to the formation of premium pricing cycles regarding market competition.

3.4.1 Model I

A two players' insurance game is constructed for Model I, with insurers 1 and 2. Then, we have ten variables ($p_1, p_2, k_1, k_2, q_1, q_2, \mu_{11}, \mu_{22}, \lambda_{11}, \lambda_{22}$), two first-order conditions of the Hamiltonian for the two players, and eight ODEs ($\dot{k_1}$, $\dot{k_2}$, $\dot{q_1}$, $\dot{q_2}$, $\dot{\mu_{11}}$, $\dot{\mu_{22}}$, $\dot{\lambda_{11}}$, $\dot{\lambda_{22}}$).

We observe that when (3.20) equals to zero, the two players' premium is not correlated, thus $\dot{p_1}$ can be obtained by differentiating the corresponding solution. Similarly, $\dot{p_2}$ can also be calculated. Under these circumstances, we will have 10 variables and 10

ODEs regarding each variable. Considering the initial conditions and terminal conditions, a Bounded Value Problem (BVP) is formulated, which can be solved using Matlab Programming.

Algorithm of Calculating Equilibrium Pricing Strategy for Model I

In this section, the main steps of the algorithm are presented, whereas the appendix provides the details of the algorithm.

Step 1: Calculate p_1 when the first-order condition of the Hamiltonian for player 1 is satisfied.

Step 2: Differentiate the p_1 obtained in Step 1 with respect to time t to obtain $\dot{p}_1$.

Step 3: Repeat steps 1 and 2 with respect to insurer 2 to calculate $\dot{p}_2$.

Step 4: Right now we have ten variables, $p_1, p_2, k_1, k_2, q_1, q_2, \mu_{11}, \mu_{22}, \lambda_{11}, \lambda_{22}$ and the corresponding ordinary differential equations. This is a BVP with six conditions from the initial information of both players and four terminal conditions from (3.14) and (3.15), which can be solved by ”bvp45” in Matlab.

Step 5: Test whether the second-order condition of Hamiltonian for both two players are negative during the whole time interval $[0, T]$. If yes, accept the result; If no, reject.

Algorithmic Steps using Matlab Programming for Model I

The steps 1 to 3 are presenting using Matlab:

Matlab - Step 1:
$\%$ Type in (3.20) with respect to insurer 1, denoted as *firstorderH1* . 1: x1=solve(*firstorderH1*==0, p_1) .

Matlab - Step 2:
$\%$ Create symbolic variables with respect to t; 1: $odex1$=diff$(x1(t), t)$; $\%$ $odex1$ includes *diff* $(k_1(t), t)$, *diff* $(q_1(t), t)$, *diff* $(\mu_{11}(t), t)$, *diff* $(\lambda_{11}(t), t)$. (3.11), (3.12), (3.19), and (3.18) provide all the above differential equations. $\%$ Substitute *diff* $(k_1(t), t)$, *diff* $(q_1(t), t)$, *diff* $(\mu_{11}(t), t)$, *diff* $(\lambda_{11}(t), t)$ in $odex1$. $\dot{p}_1$ is obtained.

Matlab - Step 3:

init=bvpinit(linspace(0,3,1000),@bc_init);
sol=bvp4c(@rhs_bvp, @bc_bvp,init);
t=linspace(0,3,1000);
BS=deval(sol,t);
plot(t,BS(1,:));

function rhs=rhs_bvp(t,y);
rhs=[$\dot{p}_1$; $\dot{p}_2$;$\dot{k}_1$;$\dot{k}_2$;$\dot{q}_1$;$\dot{q}_2$;$\dot{\mu}_{11}$;$\dot{\mu}_{22}$;$\dot{\lambda}_{11}$;$\dot{\lambda}_{22}$]

function bc=$bc\_bvp(yl, yr)$
bc=[yl(1)-0.88; yl(2)-1.05; yl(3)-0.6; yl(4)-1; yr(5)-5225; yr(6)-13700; yr(7); yr(8); yr(9); yr(10)];

% @bc_init is the guess.

Numerical Example of Model I

Here, we illustrate an insurance game considering a period of three years. The scenery in this section is modelled to investigate the competition among two candidates: one player represents a large market power insurer, while the other is regarded as a relatively weaker insurer.

Table 3.1 demonstrates the parameter values of our insurance game. Table 3.2 illustrates the initial information, including the initial premium, volume of exposure and the capital per exposure regarding both two insurers.An insurance company with a greater market power has a larger price sensitivity parameter b. The break-even premium of both insurers are assumed to be constant during the whole period.

Number of market participants	n	2
Market presence limit factor	h_1	1
Break-even premium of insurer 1	π_1	0.6
Break-even premium of insurer 2	π_2	0.609
Price sensitivity parameter of insurer 1	b_1	0.2
Price sensitivity parameter of insurer 2	b_2	0.28
Standard solvency ratio factor of insurer 1	r_1	3.3
Standard solvency ratio factor of insurer 2	r_2	2.2
Expected rate of return of insurer 1	β_1	0.03
Expected rate of return of insurer 2	β_2	0.03
time valuing of money	ζ	0.02

TABLE 3.1: Parameter values for Model I

Initial premium of insurer 1	$p_1(0)$	0.88
Initial premium of insurer 2	$p_2(0)$	1.05
Initial exposure volume of insurer 1	$q_1(0)$	5225
Initial exposure volume of insurer 2	$q_2(0)$	13700
Initial capital per exposure of insurer 1	$k_1(0)$	0.6
Initial capital per exposure of insurer 2	$k_2(0$	1

TABLE 3.2: Initial information of insurer 1 and insurer 2

The existence of a NE can be proved by finding out one under the game set up from Tables 3.1 and 3.2. With the algorithm introduced in Section 3.4.1, a premium profile of both insurers is calculated, which is presented in Figure 3.2, with negative second-order Hamiltonian profiles observed for both insurers Figure 3.5. Since the second-order conditions of the Hamiltonian are satisfied, the premium profile that follows from the first-order conditions constitutes a NE. Figures 3.3 and 3.4 describe the exposure volume and capital per exposure accordingly.

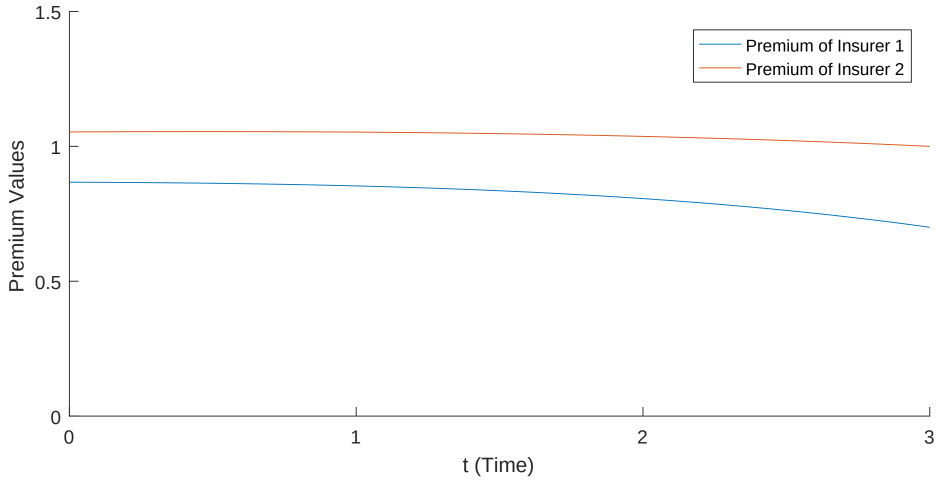

FIGURE 3.2: Equilibrium premium profiles of both insurers over three years time in Model I. The blue line represents insurer 1's premium profile and the red line shows insurer 2's premium profile. Premium values are given on y-axis while the corresponding time is on x-axis.

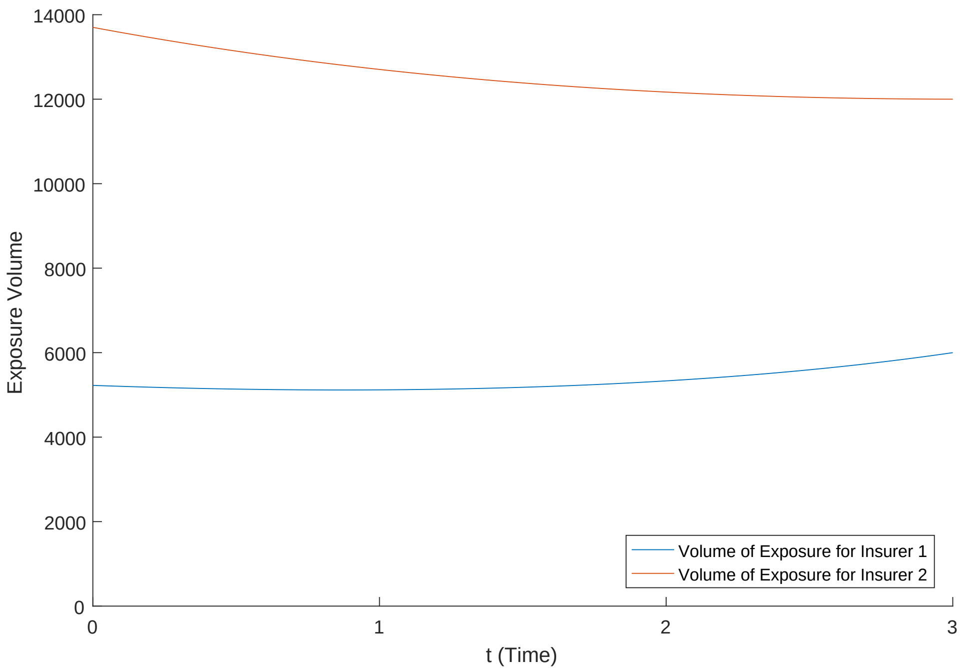

FIGURE 3.3: Volume of exposure profiles regarding both insurers over three years time in Model I. The blue line represents insurer 1's volume of exposure profile and the red line shows insurer 2's. Volume of exposure are given on y-axis while the corresponding time is on x-axis.

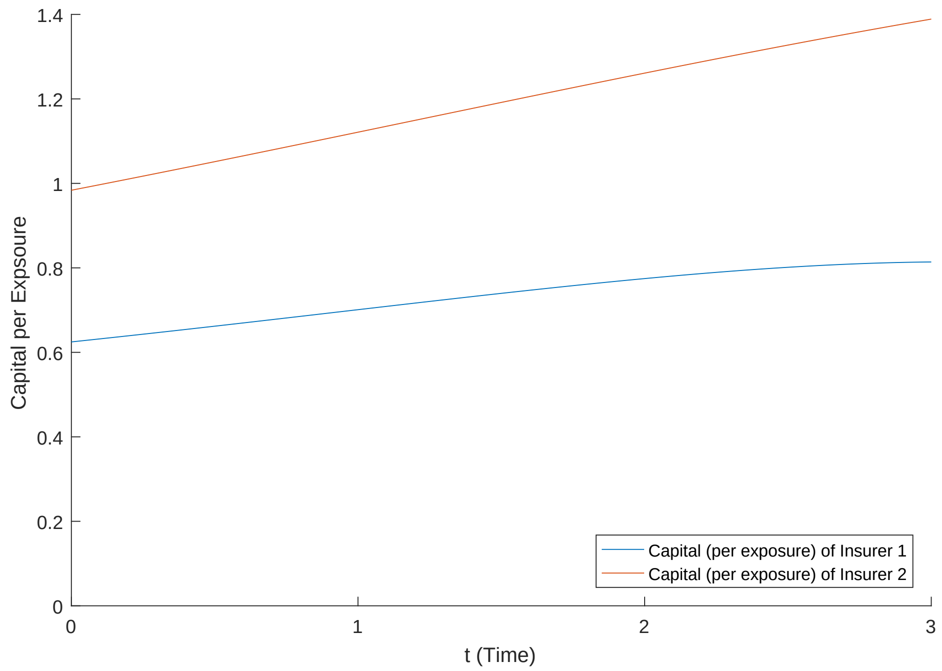

FIGURE 3.4: Exposure per capital profiles for both insurers over three years time in Model I. The blue line represents insurer 1's capital profiles (per exposure) and the red line shows insurer 2's. Capital are given on y-axis while the corresponding time is on x-axis.

Although there is a slightly decrement of value of premium, the larger market power of insurer 2 yields that the equilibrium premiums of insurer 2 keep in a relatively high level through the whole time horizon. Insurer 1 adopts a relatively low pricing level with the purpose of absorbing more policies. Insurer 2 slightly lower its capital (per exposure) through the competition while insurer's capital (per exposure) inappreciably increased. No pricing cycles appear in the equilibrium of Model I. The equilibrium strategies of the two insurers keep stable over the 3-year period. Different sets of parameters are tested for Model I, and none of the results illustrate any cycles. An obvious reason is that there is no correlation between two players' premium while the first-order conditions of Hamiltonian are satisfied. Similar findings can be confirmed in [22].

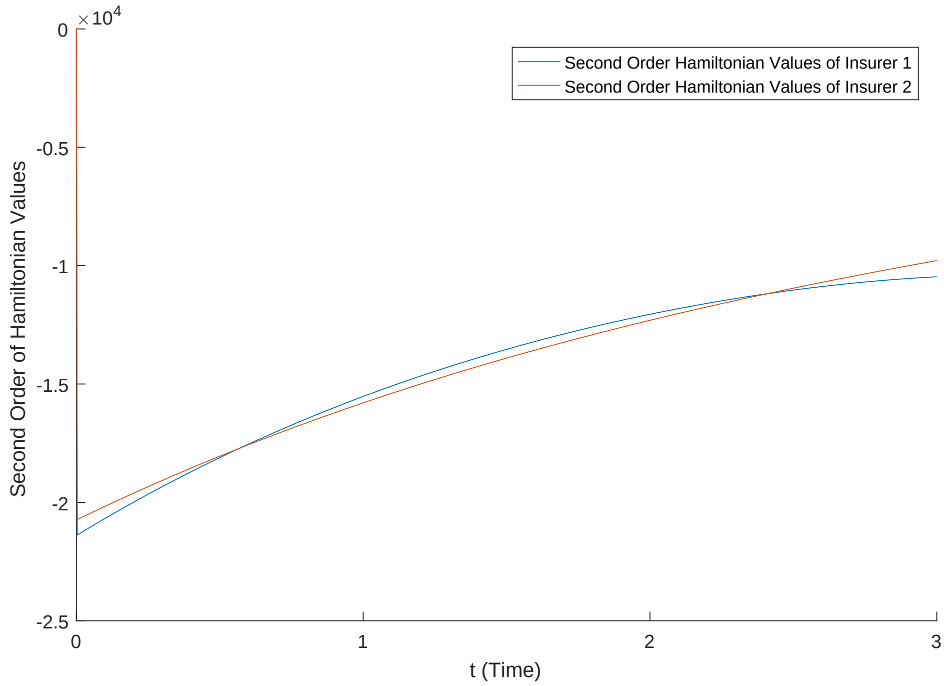

FIGURE 3.5: Second-order conditions of the Hamiltonians for both insurers in Model I. The blue one refers to insurer 1 while the red represent insurer 2. Both insures' second-order conditions keep negative through the whole time horizon.

3.4.2 Model II

Regarding now Model II, like in Section 3.4.1, we consider a two players' game, with insurers 1 and 2. Similarly as before, we have ten variables (p_1, p_2, k_1, k_2, q_1, q_2, μ_{11}, μ_{22}, λ_{11}, λ_{22}), two first-order conditions of the Hamiltonian for the two players, and eight ODEs ($\dot{k_1}$, $\dot{k_2}$, $\dot{q_1}$, $\dot{q_2}$, $\dot{\mu_{11}}$, $\dot{\mu_{22}}$, $\dot{\lambda_{11}}$, $\dot{\lambda_{22}}$).

We can eliminate two variables p_2 and λ_{22}, and transfer $\dot{\lambda_{22}}$ to the differential equation of p_1. Under these circumstances, we will have eight variables and eight ODEs regarding each variable. The backward integration considers the standard Mean Value Theorem, which is adopted in this section in order to solve the BVP.

Algorithm of Calculating Equilibrium Pricing Strategy for Model II

In this section, the main steps of the algorithm are presented, whereas the appendix provides the details of the algorithm.

Step 1: Calculate p_2 when the first-order condition of the Hamiltonian for player 1 is satisfied.

Step 2: Get an expression of λ_{22} from the first-order condition of insurer 2's Hamiltonian, with p_2 excluded.

Step 3: Differentiate the expression of λ_{22} with respect to time t. Generate an ordinary differential equation of p_1.

Step 4: Apply a backward iteration of the system with the first-order conditions of Hamiltonian for insurer 1 and 2. Terminal values of 10 variables are required to be used as inputs. From (3.26) and (3.27), it follows that $\mu_{11}(T) = \mu_{22}(T) = \lambda_{11}(T) = \lambda_{22}(T) = 0$. For the other 6 variables, $p_1(T)$, $p_2(T)$, $k_1(T)$, $k_2(T)$, $q_1(T)$, $q_2(T)$ need to satisfy (3.31) in order to be used as inputs.

Since $\mu_{11}(T) = \mu_{22}(T) = \lambda_{11}(T) = \lambda_{22}(T) = 0$, (3.31) does not depend on $q_1(T)$ and $q_2(T)$ at time T. We use the Matlab solver 'fsolve' to provide $p_1(T)$, $p_2(T)$ when $k_1(T)$, $k_2(T)$ are fixed, via (3.31).

Then, $q_1(T)$ and $q_2(T)$ will be guessed. Terminal values of the 10 variables are used as inputs in the backward iteration.

Step 5: Stop until the initial value of p_1, p_2, k_1, k_2, q_1 and q_2 from backward iteration equals to the initial data value. Otherwise, we adjust the guess of $k_1(T)$, $k_2(T)$, $q_1(T)$ and $q_2(T)$.

Step 6: From the previous step, we collect the terminal values that yield the correct initial values. We check whether the second-order conditions of the Hamiltonians of both players are negative during the whole time interval $[0, T]$. If yes, accept the equilibrium; If no, reject.

Remark 3.3. A game with more players can be also investigated with a similar algorithm. More loops are required to calculate the equilibrium pricing strategy.

Algorithmic Steps using Matlab Programming for Model II

The steps 1 to 6 are presenting using Matlab:

Matlab - Step 1:
% Type in (3.31) with respect to insurer 1, denoted as *FirstorderH1* . 1: x2=solve(*FirstorderH1*==0, p_2) .

Matlab - Step 2:
% Type in (3.31) with respect to insurer 2, denoted as *FirstorderH2* . 1: *FirstorderH2_fh*=matlabFunction(*FirstorderH2*) ; 2: *FirstorderH2_new*=*FirstorderH2*_fh(a_2,r_2,h_2,τ^{-1},ζ,t,p_1,**x2**,k_2,q_2,μ_{22},λ_{22}) ; 3: x10=solve(*FirstorderH2_new*==0,λ_{22}) .

Matlab - Step 3:

% Create symbolic variables with respect to t;
1: $dx10$=diff$(x10(t), t)$;
% $dx10$ includes *diff* $(p_1(t), t)$, *diff* $(k_1(t), t)$, *diff* $(k_2(t), t)$, *diff* $(q_1(t), t)$, *diff* $(q_2(t), t)$, *diff* $(\mu_{11}(t), t)$, *diff* $(\mu_{22}(t), t)$, *diff* $(\lambda_{11}(t), t)$. (3.24), (3.25), (3.29), and (3.30) provide all the above differential equations, except *diff* $(p_1(t), t)$.
% Similar as step 2, substitute *diff* $(k_1(t), t)$, *diff* $(k_2(t), t)$, *diff* $(q_1(t), t)$, *diff* $(q_2(t), t)$, *diff* $(\mu_{11}(t), t)$, *diff* $(\mu_{22}(t), t)$, *diff* $(\lambda_{11}(t), t)$ with the corresponding differential equations using *'matlabFunction'*. A new function is generated including *diff* $(p_1(t), t)$, which is denoted as *x_10*;
2: *x_10* - *diff* $(\lambda_{22}(t), t)$ == 0.
% (3.30) provides *diff* $(\lambda_{22}(t), t)$, an equation of *diff* $(p_1(t), t)$ is obtained.

Matlab - Step 4:

1: x0=[0.000001 0.000001];
2: p=fsolve(@premium,x0);
3: function F = premium(u);
% Substitute $p_1(N)$ as $u(1)$, $p_2(N)$ as $u(2)$ in $\frac{\partial H_1}{\partial p_1}$ and $\frac{\partial H_2}{\partial p_2}$.
4: F = [$\frac{\partial H_1}{\partial p_1}$; $\frac{\partial H_2}{\partial p_2}$] ;
5: end .

6: for i=N:-1:1 ;
7: t = (i-1) * T/(N-1) ;
8: % 8 ODE systems with 8 variables:
$k_1(i-1) = -dt * \textit{diff}\,(k_1(t), t) + k_1(i)$;
$k_2(i-1) = -dt * \textit{diff}\,(k_2(t), t) + k_2(i)$;
$q_1(i-1) = -dt * \textit{diff}\,(q_1(t), t) + q_1(i)$;
$q_2(i-1) = -dt * \textit{diff}\,(q_2(t), t) + q_2(i)$;
$\mu_{11}(i-1) = -dt * \textit{diff}\,(\mu_{11}(t), t) + \mu_{11}(i)$;
$\mu_{22}(i-1) = -dt * \textit{diff}\,(\mu_{22}(t), t) + \mu_{22}(i)$;
$\lambda_{11}(i-1) = -dt * \textit{diff}\,(\lambda_{11}(t), t) + \lambda_{11}(i)$;
syms v
% Substitute *diff* $(p_1(t), t)$ with $\frac{p_1(i)-v}{dt}$.
% Replace p_2, λ_{22} with $x2$,$x10$ correspondingly in *diff* $(\lambda_{22}(t), t)$.
$p_1(i-1) = vpasolve(\textit{x\_10} - \textit{diff}\,(\lambda_{22}(t), t) == 0)$;
9: end.

Matlab - Step 5:

```
1: k1=linspace(1.6,2,9) ;
2: x0=[0.000001 0.000001];
3: Y=zeros(length(k1),2);
4: for z=1:length(k1) ;
5:     fun= @(x)premium(x,k1(z)) ;
6:     Y(z,:)=fsolve(fun,x0);
7: end

8: for z=1:length(k1)
9: p1(N,z) = Y(:,1); ...
10: for i=N:-1:1
11: W = [0,1];
     ...
12: if (p1(1,z) > 0.885) && (p1(1,z) < 0.895) && (p2(1,z) > 1.535) && (p2(1,z) < 1.545)
     ...
13: W(index,1)=[k1(z),1];
14: index = index +1;
15: end
16: end
17: end
```

Numerical Example of Model II

An 3-years time insurance game regarding Model II is introduced in this section. The scenery in this section also investigates the competition among a large market power insurer and a relatively weaker insurer.

Table 3.3 demonstrates the parameter values in the insurance game. Table 3.4 illustrates the initial information of the two players.

According to [62], referring to Lerner Index, insurance company with greater market power would have a lower price sensitivity parameter a. Considering a monopoly insurance market, insurer will not lose any policies while increasing its premium value, that is, its price sensitivity parameter $a = 0$. The break-even premium of both insurers are assumed to be constants during the 3 years.

Number of market participants	n	2
Market presence limit factor	h_2	0.58
Break-even premium of insurer 1	π_1	0.6
Break-even premium of insurer 2	π_2	0.61
Price sensitivity parameter of insurer 1	a_1	2
Price sensitivity parameter of insurer 2	a_2	1.5
Standard solvency ratio factor of insurer 1	r_1	2
Standard solvency ratio factor of insurer 2	r_2	3.5
Expected rate of return of insurer 1	β_1	0.03
Expected rate of return of insurer 2	β_2	0.03
Discount factor	ζ	0.02

TABLE 3.3: Parameter values for Model II

Initial premium of insurer 1	$p_1(0)$	0.89
Initial premium of insurer 2	$p_2(0)$	1.54
Initial exposure volume of insurer 1	$q_1(0)$	3240
Initial exposure volume of insurer 2	$q_2(0)$	5240
Initial capital per exposure of insurer 1	$k_1(0)$	1.28
Initial capital per exposure of insurer 2	$k_2(0$	1.96

TABLE 3.4: Initial information of insurer 1 and insurer 2

With the algorithm introduced in Section 3.4.2, a NE premium profile for the two insurers is calculated, which is presented in Figure 3.6. Similarly as for Model I, verification that it is a NE follows from the negative second-order Hamiltonian profiles for both insurers Figure 3.10. Figures 3.7 and 3.8 describe the exposure volume and capital per exposure, accordingly. Pricing cycles are observed in the whole time period. Figure 3.6 supports the opinion in previous literatures [10, 15, 61, 11, 32, 36] that pricing cycles in insurance markets are caused by market competition. Although the premiums between the two insurers are not proportional, the shape of premium cycle profiles is similar. Figure 3.6 suggests that insurer 1 follows insurer 2's pricing strategy. The premium of insurer 1 even falls below the break-even premium level from the 3rd month to the 6th month in order to keep competitive and attract more policies. The two insurers' total capital, displayed in Figure 3.9, remains stable for the first two years. Due to the increment of premium, both insurers gain massive capital in the third year, which is particularly true for insurer 1.

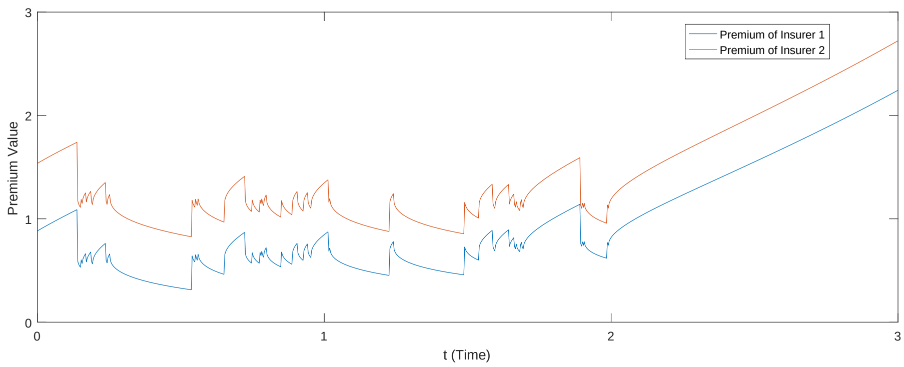

FIGURE 3.6: Equilibrium premium profiles of both insurers over three years time in Model II. The blue line represents insurer 1's premium profile and the red line shows insurer 2's premium profile. Premium values are given on y-axis while the corresponding time is on x-axis.

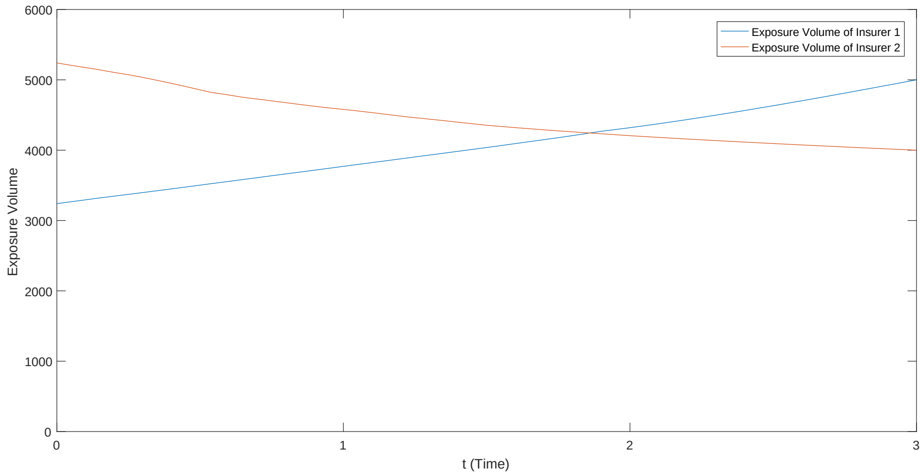

FIGURE 3.7: Volume of exposure profiles regarding both insurers over three years time in Model II. The blue line represents insurer 1's volume of exposure profile and the red line shows insurer 2's. Volume of exposure are given on y-axis while the corresponding time is on x-axis.

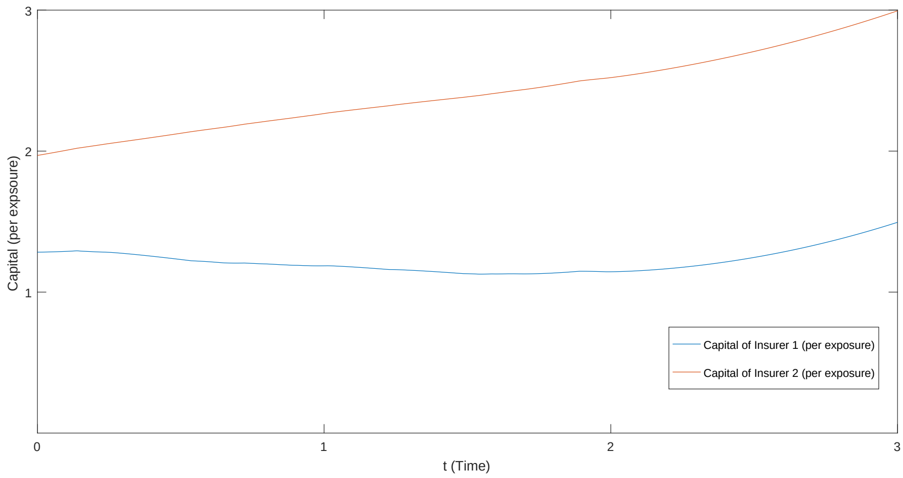

FIGURE 3.8: Exposure (per capital) profiles regarding both insurers over three years time in Model II. The blue line represents insurer 1's capital profiles (per exposure) and the red line shows insurer 2's. Capital are given on y-axis while the corresponding time is on x-axis.

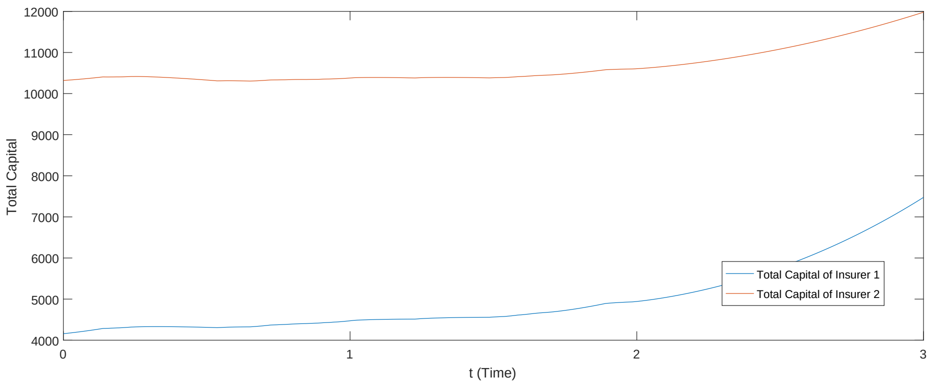

FIGURE 3.9: Total capital profiles regarding both insurers over three years time in Model II. The blue line represents insurer 1's capital profiles and the red line shows insurer 2's. Capital are given on y-axis while the corresponding time is on x-axis.

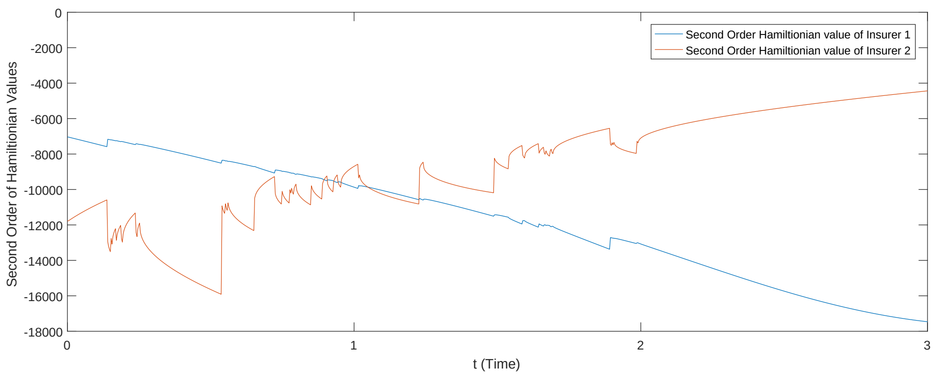

FIGURE 3.10: Second-order conditions of the Hamiltonians for both insurers in Model II. The blue one refers to insurer 1 while the red represent insurer 2. Both insures' second-order conditions keep negative through the whole time horizon.

Pricing cycles in Model II are appeared to be related with profit margins. When the break-even premium is relatively low regarding insurers' premium, insurers intend to compete with their rivals and more pricing cycles are appeared. Figure 3.11 demonstrates the equilibrium premium profiles while break-even premium $\pi_1 = 0.8, \pi_1 = 0.85$ and other parameters remain the same in Tables 3.3 and 3.4. Figure 3.12 shows the equilibrium premium profiles while break-even premium $\pi_1 = 1.2, \pi_1 = 1.1$. As illustrated in Figure 3.11, premium cycles are founded, but with less amounts; while both insurers' equilibrium premium slightly increased over the time horizon smoothly.

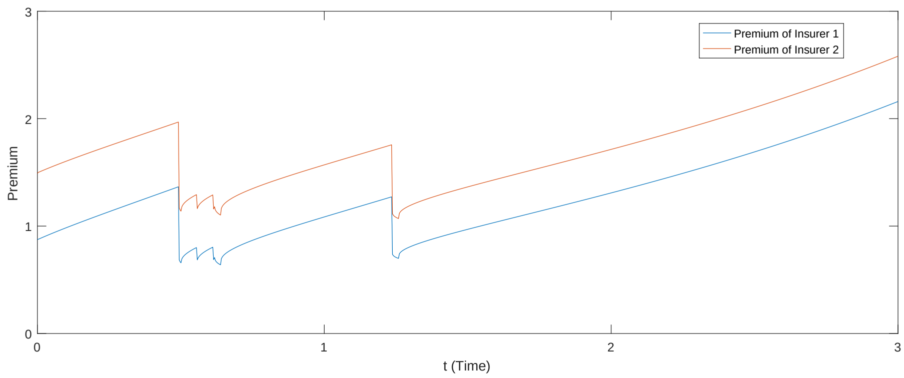

FIGURE 3.11: Equilibrium premium profiles of both insurers over three years time in Model II, while $\pi_1 = 0.8, \pi_1 = 0.85$. The blue line represents insurer 1's premium profile and the red line shows insurer 2's premium profile. Premium values are given on y-axis while the corresponding time is on x-axis.

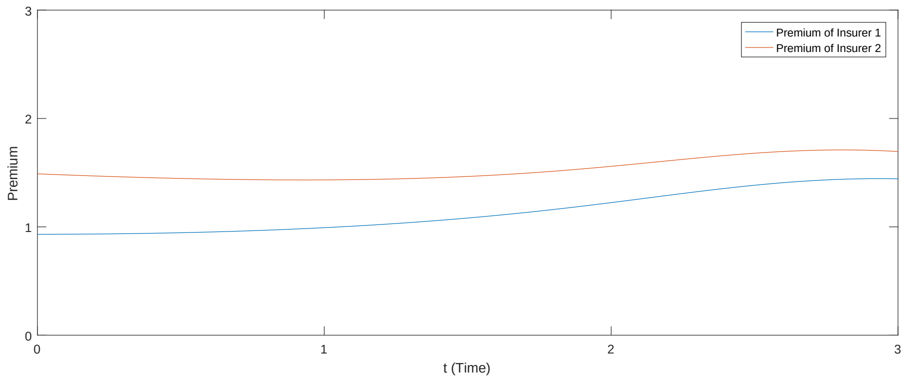

FIGURE 3.12: Equilibrium premium profiles of both insurers over three years time in Model II, while $\pi_1 = 1.2, \pi_1 = 1.1$. The blue line represents insurer 1's premium profile and the red line shows insurer 2's premium profile. Premium values are given on y-axis while the corresponding time is on x-axis.

Chapter 4

Multi-stage Stochastic General Insurance Games with Risk Aversion Players

4.1 Motivation

In general insurance market, the claims of policies are always the focused part to be calculated since the claim of each policy is unpredictable. The uncertainty of repaid claims differs insurance product from other products. In the previous chapters, the constant break-even premium framework is expected to be further analyzed. The expenditures of insurance companies will be discussed separately as exposure related costs and non-exposure-based costs. In this chapter, the exposure-based component is assumed to be stochastic.

A multi-stage stochastic game will be constructed. Insurers are considered to be risk-averse, that is, insurance companies would like to set risk-premiums on their products with the purpose of avoiding risk. Constant absolute risk-aversion (CARA) function will be adopted in this chapter for each player. The multi-stage insurance game will be solved by finding out the subgame perfect Nash equilibrium. A numerical example will be given to explore the effect of risk aversion on premium pricing strategy with stochastic claims .

4.2 Baseline Model

Considering an insurance market with **N**=1,...,n insurers, this section investigate non-cooperative games over **M**=1,...,m stages for insurance market. Each insurer is a player and assumed to maximize its expected utility of net income for every single stage. That is, the preferences of insurance companies are explicitly myopic. [1]

Different from other products, the claim of insurance policies are not fixed. Regarding the uncertain claim, we explicitly separate the "break-even premium" (which is mentioned in both Chapter 2 and Chapter 3) into two components: exposure-based π and non-exposure-based $\hat{\pi}$. For $i \in$ **N** and $s \in$ **M**, we propose $\hat{\pi}$ is a constant and

$$\pi_i^s \sim N\left(\mu_i^s, \sigma_i^{s2}\right) \tag{4.1}$$

$\hat{\pi}$ are the costs which are not related with policies, such as labor costs, operating costs, building expenses, etc. π investigates the costs based on holding the exposures, such as claims, etc. The costs per exposure are modelled by summing up all the risks of expenditure and then divided into each exposure. Note that for any insurer i at different stage s, μ_i^s and σ_i^s could be different due to market conditions. Insurance companies usually predict the values based on previous stages' information and market situations.

In line with Chapter 2, the net income I_i^s that concerns insurer i and stage s is formulated as follows,

$$I_i^s = -\alpha_i w_i^{s-1} + (1-\alpha_i)(p_i^s - \pi_i^s)q_i^s - \hat{\pi}_i^s.$$

[1]This chapter aims to investigate the influence of stochastic claims on premium pricing. In addition, the dynamic programming has been considered in previous chapter. Hence we assume that insurers are myopic. Relevant assumptions are used in many papers of game theory [53, 27, 38, 29, 26].

For insurer i, p_i^s is the premium value per unit of exposure at stage s; q_i^s represents the holding exposure volume at stage s; w_i^s is the holding wealth at stage s. $p_i^s,\ q_i^s,\ \pi_i^s$, $\hat{\pi}_i^s$ are all positive and $\alpha_i \in (0,1)$ is a given parameter that refers to the cost ratio of holding insurer i's wealth. Each insurer is assumed to receive the premium from policyholders at the beginning of s and $p_i^s,\ q_i^s,\ \pi_i^s, \hat{\pi}_i^s$ remain the same at stage s, i.e. time in interval $[s, s+1)$. It is also assumed that each insurer has perfect knowledge of its previous information, which is regarded as constants at stage s. A decision of premium p_i^s will affect the volume of exposure q_i^s for insurer i due to the competition with other insurers. The volume changes of q among stages cause further differences to the net income of insurers and the market competition henceforth.

Referring to (4.1), one can deduce that

$$I_i^s \sim N\left(\mu_{Ii}^s, \sigma_{Ii}^s\right).$$

Where

$$\begin{aligned} \mu_{Ii}^s &= -\alpha_i w_i^{s-1} + (1-\alpha_i)(p_i^s - \mu_i^s)q_i^s - \hat{\pi}_i^s, \\ \sigma_{Ii}^s &= (1-\alpha_i)^2\, q_i^{s2}\, \sigma_i^{s2}. \end{aligned}$$

Other than previous chapters, insurers are regarded as risk averse, i.e, risk premium will be applied in pricing insurance policies. Furthermore, Constant absolute risk-aversion (CARA) function is adopted. That is,

$$U(I_i^s) = -e^{-\lambda_i^s I_i^s}, \qquad \lambda_i^s > 0.$$

λ_i^s is denoted as risk averse parameter. Since I_i^s is normally distributed, according to [3], the objective of insurer i to maximize the expected utility $\mathbb{E}[U(I_i^s)]$ of the net income at stage s is equivalent to

$$
\begin{aligned}
Max_{p_i^s} \qquad u_i^s &= \mu_{Ii}^s - \frac{1}{2}\,\lambda_i^s\,\sigma_{Ii}^s \\
&= -\alpha_i w_i^{s-1} + (1-\alpha_i)(p_i^s - \mu_i^s)q_i^s - \hat{\pi}_i^s \\
&\quad -\frac{1}{2}\,\lambda_i^s\,(1-\alpha_i)^2\,q_i^{s2}\,\sigma_i^{s2}.
\end{aligned} \tag{4.2}
$$

u_i^s is the payoff of insurer i at stage s.

Moreover note that the Arrow-Pratt index of absolute risk aversion is given by

$$
-\frac{U''(I_i^s)}{U'(I_i^s)} = \lambda_i^s.
$$

This means that, at stage s, the larger λ_i^s, the more risk averse insurer i is.

Similar as previous chapters, the value of q_i^s needed to be further analyzed which implies the competition of insurance market.

4.3 Market Competition

This section investigates the quotient price function in [22], which considers the market average premium (exclude insurer itself) as an aggregate. The exposure volume is modelled through the comparison insurer's premium strategy with market average premium at current stage,

$$
q_i^s = \left(h - a_i\,\frac{p_i^s}{\bar{p}_{-i}^s}\right)^+ q_i^{s-1}
$$

Where h is the market presence limit factor, which controls the amount of exposure insurers could gain attributable to the competition. a_i is price sensitivity parameter of insurer i [2]. Both h and a_i are positive. $\bar{p}_{-i}^s$ is the market average premium exclude insurer i at stage s. One can deduce the following lemma from Eq. (4.2).

[2]Price sensitivity parameter a_i plays a similar role regarding previous chapters, which is not further investigated here.

Lemma 4.1. *For the above proposed quotient price function, the payoff u_i^s of insurer i at stage s is given by*

$$\begin{aligned}
u_i^s &= -\alpha_i w_i^{s-1} + (1-\alpha_i)\left(p_i^s - \mu_i^s\right)\left(h - a_i \frac{p_i^s}{\bar{p}_{-i}^s}\right) q_i^{s-1} - \hat{\pi}_i^s \\
&= -\frac{1}{\bar{p}_{-i}^s}\,(1-\alpha_i)\; a_i\, q_i^{s-1}\left(1 + \frac{1}{2}\,\sigma_i^{s2}\,\lambda_i^s\,(1-\alpha_i)\,\frac{a_i}{\bar{p}_{-i}^s}\, q_i^{s-1}\right) p_i^{s2} \\
&\quad + (1-\alpha_i)\; q_i^{s-1}\left(h + \frac{a_i\,\mu_i^s}{\bar{p}_{-i}^s} + \lambda_i\, h\,(1-\alpha_i)\,\frac{a_i}{\bar{p}_{-i}^s}\,\sigma_i^{s2}\, q_i^{s-1}\right) p_i^s \\
&\quad -\alpha_i\, w_i^{s-1} - \hat{\pi}_i^s - (1-\alpha_i)\, q_i^{s-1}\,\mu_i^s\, h - \frac{1}{2}\,\lambda_i^s\,(1-\alpha_i)^2\, h^2\,\sigma_i^{s2}\, q_i^{s-1^2}. \qquad (4.3)
\end{aligned}$$

4.4 Game Construction

Let us define an **N**-insurer game, G, in a **M**-stage framework: for a stage $s \in$ **M**, the number of insurer is n. Each insurer i's strategy at stage s is P_i^s, which stands for the action setting premium as the value of p_i^s, whereas $\mathbf{P}_i$ is the set of strategies. $\mathfrak{P}_i \equiv P_i^1 \times \cdots \times P_i^m$ is the strategy profile for i over all stages. We use $\widetilde{P}_i^s$ to denote the equilibrium strategy for insurer i at stage s and $\widetilde{\mathfrak{P}}_i$ to denote the equilibrium strategy profile over all stages. Insurer i's payoff function is defined as $u_i^s : \mathscr{P}^s \to \mathbb{R}$, where $\mathscr{P}^s \equiv \mathbf{P}_1^s \times \cdots \times \mathbf{P}_N^s$ and $\mathfrak{p}$ is an arbitrary profile in $\mathscr{P}$. The notation $\mathfrak{p}_{-i}^s \in \mathscr{P}_{-i}$ stands for $\{P_1^s, \ldots, P_{i-1}^s, P_{i+1}^s, \ldots, P_{N^s}^s\}$, which is used to represent the strategy profile of other players at time t. $(P_i^s, \mathfrak{P}_{-i}^s) \in \mathscr{P}^s$ decomposes a strategy profile in two parts, the insurer i's strategy and other insurers' components. Given this game in the insurance market, instead of calculating the optimal premium that maximises a single insurer's wealth, the calculation of the subgame perfect Nash equilibrium is targeted. Note that every subgame perfect equilibrium is a Nash equilibrium.

Here we give the definition of a multi-stage stochastic insurance game with risk aversion players with respect to the model above.

Definition 4.2. A game $G =< (\mathbf{P}_i, u_i^s)_{i\in\mathbf{N}, s\in\mathbf{M}} >$ has a finite set of players **N**, a finite set of stages **M**, with compact, convex, positive strategy set $\mathbf{P}_i$ with respect to every i, whereas u_i^s is the payoff function for i at stage s. This type of game is called *Multi-stage Stochastic Insurance Game with Risk Aversion Players*.

Moreover, the definition of Subgame Perfect Nash Equilibrium is given as follows.

Definition 4.3 (*Subgame Perfect Nash Equilibrium*)**.** [40] A strategy profile $\widetilde{\mathfrak{P}}$ is a Subgame Perfect Nash Equilibirum (SPE) in game G if for any subgame G' of G, $\widetilde{\mathfrak{P}}_{|G'}$ is a Nash Equilibrium of G' .

4.5 Main Results

Lemma 4.4. *Based on the payoff functions stated in the Eq.* (4.3)*, G is an aggregate game.*

Proof. Denote $g = \bar{p}_{-i}^s$ as the aggregate of $\mathbf{G}_I$ game. Then, the payoff function turns out to be

$$\begin{aligned} u_i^s &= -\alpha_i w_i^{s-1} + (1-\alpha_i)\,(p_i^s - \mu_i^s)\left(h - a_i\,\frac{p_i^s}{g}\right)\,q_i^{s-1} - \hat{\pi}_i^s \\ &= -\frac{1}{g}\,(1-\alpha_i)\;a_i\,q_i^{s-1}\,\left(1 + \frac{1}{2}\,\sigma_i^{s2}\,\lambda_i^s\;(1-\alpha_i)\;\frac{a_i}{g}\,q_i^{s-1}\right)\,p_i^{s2} \\ &\quad + (1-\alpha_i)\;q_i^{s-1}\left(h + \frac{a_i\,\mu_i^s}{g} + \lambda_i\,h\;(1-\alpha_i)\;\frac{a_i}{g}\,\sigma_i^{s2}\,q_i^{s-1}\right)\,p_i^s \\ &\quad -\alpha_i\,w_i^{s-1} - \hat{\pi}_i^s - (1-\alpha_i)\,q_i^{s-1}\,\mu_i^s\,h - \frac{1}{2}\,\lambda_i^s\,(1-\alpha_i)^2\,h^2\,\sigma_i^{s2}\,q_i^{s-1\,2}. \end{aligned}$$

There exists an aggregate function in G, where the payoff function is only depend on insurer i's strategy P_i^s and g. Thus, the statement of the Lemma is derived. □

Lemma 4.5. *The Nash equilibrium at any stage s in G exists.*

Proof. $\mathbf{P}_i$, is a compact, convex strategy set, and u_i^s is a concave function of p_i^s. The Nash equilibrium at any stage s exists according to fixed point theorem [1].

□

Theorem 4.6. *Subgame Perfect Nash Equilibrium (SPE) in game G exists.*

Proof. For each stage of game G, it is a subgame G' of G with imperfect information. From Lemma 4.5, start from the last stage m by backward induction and at each stage there exist a equilibrium premium strategy. If SPE does not exist in game G, it is a contradiction. □

Lemma 4.7. *Considering the game G with the lower bound $\perp_{\boldsymbol{P}_i}$ and up bound $\top_{\boldsymbol{P}_i}$ for $\boldsymbol{P}_i$. If $\tilde{p}_i^s \in [\perp_{\boldsymbol{P}_i}, \top_{\boldsymbol{P}_i}]$, $\tilde{p}_i^s$ is best-response correspondence for insurer i. While*

$$\tilde{p}_i^s = \frac{\left(h + \frac{a_i\, \mu_i^s}{\bar{p}_{-i}^s} + \lambda_i\, h\, (1-\alpha_i)\, \frac{a_i}{\bar{p}_{-i}^s}\, \sigma_i^{s2}\, q_i^{s-1}\right)}{2\, a_i \left(1 + \frac{1}{2}\, \sigma_i^{s2}\, \lambda_i^s\, (1-\alpha_i)\, \frac{a_i}{\bar{p}_{-i}^s}\, q_i^{s-1}\right)}\, \bar{p}_{-i}^s \tag{4.4}$$

Proof. $\tilde{p}_i^s$ is the solution while the first order derivatives of $u_i^s = 0$. It is shown below that the first order derivatives of u_i^s is negative.

$$\frac{\partial^2 u_i^s}{\partial p_i^{s2}} = -\frac{2}{\bar{p}_{-i}^s}\, (1-\alpha_i)\, a_i\, q_i^{s-1} \left(1 + \frac{1}{2}\, \sigma_i^{s2}\, \lambda_i^s\, (1-\alpha_i)\, \frac{a_i}{\bar{p}_{-i}^s}\, q_i^{s-1}\right) < 0$$

When $\tilde{p}_i^s \in [\perp_{\mathbf{P}_i}, \top_{\mathbf{P}_i}]$, $\tilde{p}_i^s$ is a global maximum. Thus, the statement of the Lemma is derived. □

4.6 Numerical Example

In this section, a numerical example with 5 non-life insurance companies based on the number of contracts (i.e., volume of business) they have in their portfolios is proposed to illustrate the main modelling characteristics and theoretical findings of this chapter regarding 5 stages. Referring to the information at previous $s-1$, the pricing strategy for the entire market of insurers is derived by finding the Nash equilibrium premiums at stage s. The impact of different parameters involved in the process to the equilibrium premiums is also analyzed. [3]

[3]Finding out a pure strategy SPE is the approach to solve the game G in the section of application. The uniqueness of Nash Equilibrium at each stage is not further investigated in this chapter.

It is considered here that all insurance companies price policies at any positive real numbers, i.e. $P_i^s \in (0, +\infty)$. This means that, if $\tilde{p}_i^s$ is positive, it is the equilibrium premium for insurer i at stage s. Regarding stage s, one can find out the Nash equilibrium profiles by implementing the following algorithm in Matlab:

Step 1: Type in Eq.(4.4) for all i .

Step 2: Solve the system of $\tilde{p}_i^s$ using code 'fsolve' starting at a positive point.

Parameters	Insurer 1	Insurer 2	Insurer 3	Insurer 4	Insurer 5
q_i^0	1000	2000	3000	2000	500
w_i^0	40000	40000	100000	40000	10000
h	3	3	3	3	3
α_i	0.03	0.03	0.03	0.03	0.03
a_i	2.2	2	1.6	2	2.5
$\hat{\pi}_i^s, \forall s \in \mathbf{M}$	15000	20000	30000	20000	8000
$\lambda_i^s, \forall s \in \mathbf{M}$	0.25	0.1	0.2	0.25	0.1
$\mu_i^s, \forall s \in \mathbf{M}$	100	100	100	100	100
$\sigma_i^s, \forall s \in \mathbf{M}$	0.2	0.2	0.2	0.2	0.2

TABLE 4.1: Benchmark Parameters

The standard values of all parameters are set up in Table 4.1, and they can be used as a benchmark. A scenario is investigated with a market leader insurer 3, two equivalent size insurers 2 $\&$ 4 and market followers 1 $\&$ 5. The price sensitivity parameter a_i which indicates the market power of insurers is assumed to be the same over all stages. The less a_i indicates larger market power since insurer receives more policies while competing with other insurers. The only difference between insurer 2 $\&$ 4 is that they have different degrees of risk inverse as $\lambda_2^s = 0.1$ and $\lambda_4^s = 0.25$. As mentioned in previous section, the larger λ_i^s , the more risk averse insurer i is. This means, insurer 4 would like to set a larger risk premium to avoid risk.

Stages s	Insurer 1	Insurer 2	Insurer 3	Insurer 4	Insurer 5
s=1	204.3472	216.3396	255.7188	220.8273	185.5081
s=2	203.9184	216.0736	255.6116	220.1621	185.2177
s=3	203.4642	215.7725	255.3761	219.4802	184.9129
s=4	203.0978	215.5162	255.2366	218.9803	184.6662
s=5	202.8270	215.3555	255.2158	218.5868	184.6028

TABLE 4.2: Equilibrium Premium Values $\tilde{p}_i^s$

Stages s	Insurer 1	Insurer 2	Insurer 3	Insurer 4	Insurer 5
s=1	955	2009	3080	1893	462
s=2	912	2018	3131	1792	433
s=3	874	2009	3184	1721	402
s=4	838	1999	3238	1653	373
s=5	801	2007	3285	1584	405

TABLE 4.3: Volume of Exposure q_i^s

Stages s	Insurer 1	Insurer 2	Insurer 3	Insurer 4	Insurer 5
s=1	50476	54879	130050	50999	14239
s=2	45525	55450	139840	38781	11308
s=3	40962	54348	144560	27500	8713
s=4	37460	53261	152520	19967	6317
s=5	33922	52313	159360	12304	8990

TABLE 4.4: Wealth w_i^s

Table 4.2 illustrates the calculated SPE profiles with the benchmark data from Table 4.1, the corresponding exposure volume and wealth are demonstrated in Table 4.3 and Table 4.4 respectively. Given by the benchmark price sensitivity parameters, the whole market intends to decrease premiums over the 5 stages. Market leader insurer 3 gains policies from other four insurers step by step and it is the only insurer who has a monotone increasing wealth. [4] Since insurer 2 and insurer 4 have different risk averse degree, the performance of these two players are significantly different. As expected, insurer 4 price its premium higher than insurer 2 at each stage of game in order to cover a higher risk premium. Regarding the same μ_i^s and σ_i^s over all 5 stages, insurer 2 accumulate more wealth.

However, in realistic, μ_i^s and σ_i^s could be different between insurers at different stages due to market conditions. Insurance companies usually predict the values based on previous stages' information and market situations.

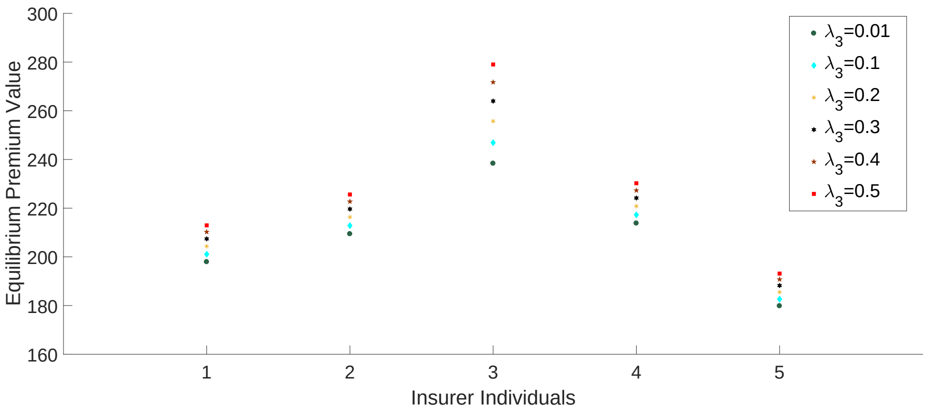

FIGURE 4.1: Diversity of equilibrium premium profiles with different λ_3. Regarding the first stage game with other parameters remain the same in Table 4.1, insurer 3's risk averse parameter is investigated, which takes values from 0.01 to 0.5. The corresponding 6 different equilibrium premium profiles are given in the legend.

Regarding the first stage of game, Figure 4.1 investigates the equilibrium premium profiles' changes when the market leader insurer 3 change its degree of risk averse. The other parameters remain the same in Table 4.1. As illustrated, its equilibrium premium value increases as λ_3^1 is larger and the whole market follows its pricing strategy. The smaller scale of insurer, the change of premium is less. The similar situation happens with different μ_3^1 and σ_3^1, which is shown in Figure 4.2 and Figure 4.3 respectively.

[4]Market competition situation could be different with different price sensitivity parameters other than the benchmark one.

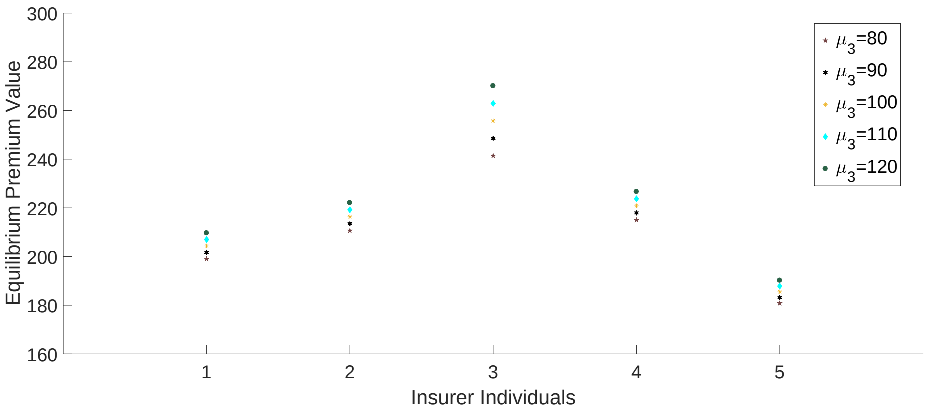

FIGURE 4.2: Diversity of equilibrium premium profiles with different μ_3. Regarding the first stage game with other parameters remain the same in Table 4.1, μ_3 is investigated, which takes values from 80 to 120. The corresponding 5 different equilibrium premium profiles are given in the legend.

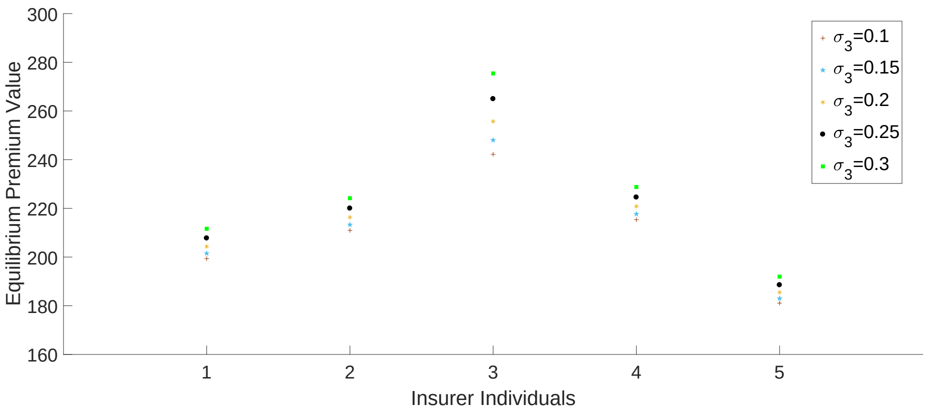

FIGURE 4.3: Diversity of equilibrium premium profiles with different σ_3. Regarding the first stage game with other parameters remain the same in Table 4.1, σ_3 is investigated, which takes values from 0.1 to 0.3. The corresponding 5 different equilibrium premium profiles are given in the legend.

Differ from the benchmark parameters in Table 4.1, μ_i^s and σ_i^s keeps changing in real insurance markets. Considering the random choice of $\mu_i^s \in [70, 120]$ and $\sigma_i^s \in [0.01, 0.4]$, the equilibrium premium profiles of all 5 insurers over 5 stages are given in Figure 4.4. Pricing cycles can be observed.

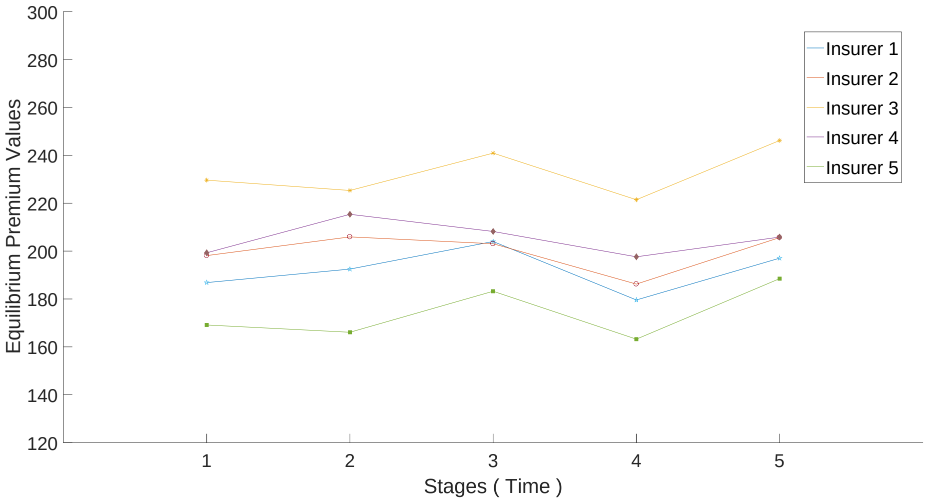

FIGURE 4.4: Equilibrium Premium Profiles of 5 insurers over 5 stages.

Chapter 5

Conclusion

This thesis investigates game-theoretical approaches to pricing general insurance premiums in competitive non-corporative market environments. Both deterministic stochastic Games were constructed under different assumptions with the purpose of pricing equilibrium premium by solve the games.

Chapter 2 models two-stage non-cooperative games in an insurance market to investigate how the competition impacts the pricing process of non-life insurance products. Insurers compete to maximise their payoffs in a second stage by adjusting premium pricing strategies, which leads to diversity of the volume of exposure. We further characterise one insurer's second-stage modified volume of exposure in a way that sums up the exposure flows in or out during competitions with other insurers. The modified second volumes of exposure in any two insurers' competition are characterised by transferring one insurer's second stage premium to the other's first-stage premium and modelling the changing volume through a definition of price elasticity. Two models are discussed in detail regarding the modified volume of exposure: simple exposure difference model I (G_I) and advanced exposure difference model II (G_{II}). Using payoffs in these two models, two N-player games are constructed with non-linear aggregate and positive, compact but not necessarily convex, premium strategy sets. A potential game with an aggregation technique is applied: we prove the existence of a pure Nash equilibrium of these two games by determining the potential functions. Both games' pure

Nash equilibriums can be solved by calculating the best-response equation systems. The numerical results for 12-player insurance games are presented under the framework that the best-response selection premium strategies always provide the global maximum value of the corresponding payoff function.

Chapter 3 models a generalized finite-time differential game in an insurance market to study how the competition impacts the pricing process of non-life insurance products. An optimal control theory approach is applied to determine premiums in the open-loop Nash Equilibrium. Two models are proposed. The first one (Model I) adopts the exponential demand function proposed by [54, 55] and [22], and the second one (Model II) is formulated based on the aggregate exposure proposed by [62]. The motivation behind the consideration and implementation of models I and II is related to compare the existing directions in the corresponding insurance literature. Numerical examples illustrate the premium dynamics, and show that premium cycles do exist in equilibrium for the Model II.

Chapter 4 constructs a multi-stage stochastic game with risk aversion players. The expenditures of insurance companies are discussed separately as exposure related costs and non-exposure related costs. The exposure related break-even premium is modelled as stochastic. A numerical example of 5 players during 5stages is shown to analyze the effect of insurers' risk averse on premium pricing.

To conclude, other than classic approaches of general insurance premium pricing, this thesis studies the competition in insurance markets based on different game structures. Limited by the difficulty of adapting real data from market, the ad hoc parameters are selected to obtain reasonable equilibrium premium values in numerical examples, including pricing sensitivity parameters a_i, market presence limit factor h, etc. Further researches may be continued with two directions: adapting new concepts from game theory to investigate the premium pricing strategy in general insurance market; or, improve the exist models with characterizing the algorithm of parameters' selection.

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