Random Vibration of Systems with Singular Matrices

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Abstract

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In the area of stochastic engineering dynamics, a flourishing field of research has been connected to assessing the reliability of systems subjected to stochastic excitations. In particular, the development of analytical and numerical methodologies for the response statistics determination of multi-degree-of-freedom (MDOF) systems with potentially singular matrices are of high interest. These singular matrices can appear naturally in the systems governing equations of motion, for instance due to coupling of electro-mechanical equations in energy harvesting applications, or they are related to FEM modeling; they also appear due to a redundant degrees-of-freedom (DOF) modeling of the systems equation of motion. In the later case, for reasons pertaining to a less labor intensive formulation of the systems governing equations of motion, especially in case of large-scale MDOF systems, and/or from a computational efficiency perspective, the system governing equations of motion are derived by utilizing a redundant DOFs modeling. This results in equations of motion with singular mass, damping and stiffness matrices. Taking also into account that the classical state and frequency domain analysis methodologies for deriving the system stochastic response, have been developed ad hoc for the case of systems with non-singular matrices, the necessity for developing a framework for treating systems with singular matrices arises.

A novel Moore-Penrose (M-P) generalized matrix inverse based framework is developed for circumventing the difficulties arising from the redundant DOFs modeling of the systems governing equations of motion. The standard time and frequency domain analysis treatments have been extended to account for linear systems with singular matrices. A M-P based solution framework for the systems mean vector and covariance matrix is determined, first by solving the equations derived after the application of the standard state-variable formulation. By following a frequency domain analysis the corresponding mean vector and covariance matrices are derived. In the latter case, a M-P based expression is obtained for the system frequency response function (FRF) matrix, and subsequently utilizing the relationship that connects the impulse response function of the system excitation to the corresponding of its response, a M-P solution for the system response power spectrum is derived.

Next, the classical statistical linearization approximate methodology is generalized to account for nonlinear systems with singular matrices. Adopting a redundant DOFs modeling for the derivation of the systems governing equations of motion, and relying on the concept of the M-P generalized matrix inverse, the extended time and frequency domain analysis treatment are applied for deriving the response statistics of systems subjected to stochastic excitations. Working on the time domain, a family of optimal and response dependent equivalent linear matrices is derived. Extending a classical excitation-response relationship of the random vibration theory, and taking into account the aforementioned family of matrices, results in an iterative determination of the system response mean vector and covariance matrix. It is proved that setting the arbitrary element in the M-P solution for the equivalent linear matrices equal to zero yields a mean square error at least as low as the error corresponding to any non-zero value of the arbitrary element. The M-P based frequency domain analysis treatment also yields an iterative determination of the system response mean vector and covariance matrix. The generalization of a widely utilized formula that facilitates the application of statistical linearization is also given.

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To the sacred memory of my parents,
Christos and Eirini.

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Notations

The following notations and abbreviations are found throughout this thesis:

DOFs	Degrees-of-freedom
M-P	Moore-Penrose
FRF	Frequency response function matrix
SVD	Singular value decomposition
pdf	Probability density function
MND	Multivariate normal distribution
$\mathcal{C}^{m\times n}$	Class of the $m \times n$ complex matrices
$\mathbf{A}^{(j,k,\ldots,l)}$	$(j,k,\ldots,l)-$inverse of the $m \times n$ matrix $\mathbf{A}$
$\mathbf{A}^{+}$	The Moore-Penrose generalized inverse of the $m \times n$ matrix $\mathbf{A}$
$\mathbf{A}^{\mathrm{T}}$	Transpose of the $m \times n$ matrix $\mathbf{A}$
$\mathbf{A}^{*}$	Conjugate transpose of the $m \times n$ matrix $\mathbf{A}$
$\mathbb{E}[\,]$	Expectation operator
$\boldsymbol{q}$	n vector of the (generalized) displacements of a system
$\boldsymbol{Q}$	n vector containing the n (generalized) forces corresponding to $\boldsymbol{q}$
$\mathbf{M}$	$n \times n$ mass matrix
$\mathbf{C}$	$n \times n$ damping matrix
$\mathbf{K}$	$n \times n$ stiffness matrix
$\mathbf{h}(t)$	Impulse response matrix of an $n-$DOFs system
$\mathbf{w}_{\boldsymbol{\eta}}(t,\tau)$	Covariance matrix of the n vector $\boldsymbol{\eta}(t)$
$\boldsymbol{x}$	l vector of the displacements of a system - redundant DOFs modeling
$\boldsymbol{Q}_{\boldsymbol{x}}$	l vector of external forces corresponding to $\boldsymbol{x}$
$\mathbf{M}_{\boldsymbol{x}}$	$l \times l$ mass matrix - redundant DOFs modeling
$\mathbf{C}_{\boldsymbol{x}}$	$l \times l$ damping matrix - redundant DOFs modeling
$\mathbf{K}_{\boldsymbol{x}}$	$l \times l$ stiffness matrix - redundant DOFs modeling
$\boldsymbol{Q}^{c}_{\boldsymbol{x}}(t)$	Additional forces due to the presence of constraints
$\boldsymbol{w}$	l vector of virtual displacements due to the presence of constraints
$\bar{\mathbf{M}}_{\boldsymbol{x}}$	Augmented mass matrix - redundant DOFs modeling

$\bar{\mathbf{C}}_{\boldsymbol{x}}$	Augmented damping matrix - redundant DOFs modeling
$\bar{\mathbf{K}}_{\boldsymbol{x}}$	Augmented stiffness matrix - redundant DOFs modeling
$\mathbf{h}_{\boldsymbol{x}}(t)$	Generalized impulse response matrix - redundant DOFs modeling
$\otimes$	Kronecker product
$\lambda_i' s$	Eigenvalues of a matrix
$\sigma_i' s$	Singular values of a matrix
$\boldsymbol{\alpha}(\omega)$	FRF matrix of an $n-$DOFs system
$\mathbf{S}_{\boldsymbol{q}}(\omega)$	Power spectrum matrix of the system response ($n-$DOFs system)
$\mathbf{S}_{\boldsymbol{Q}}(\omega)$	Power spectrum matrix of the system excitation ($n-$DOFs system)
$\boldsymbol{\alpha}_{\boldsymbol{x}}(\omega)$	FRF matrix - redundant DOFs modeling
$\mathbf{S}_{\boldsymbol{x}}(\omega)$	Power spectrum matrix of the system response - redundant DOFs modeling
$\mathbf{S}_{\boldsymbol{Q}_{\boldsymbol{x}}}(\omega)$	Power spectrum matrix of the system excitation - redundant DOFs modeling
$\boldsymbol{\Phi}$	Nonlinear n vector function of the coordinate vector $\boldsymbol{q}$ and its derivatives
$\boldsymbol{\varepsilon}$	Error vector of the statistical linearization method
$\lVert\cdot\rVert_2$	Euclidean norm
$\hat{\boldsymbol{x}}$	$3n$ vector defined as $\hat{\boldsymbol{x}} = \begin{bmatrix}\boldsymbol{x} & \dot{\boldsymbol{x}} & \ddot{\boldsymbol{x}}\end{bmatrix}^{\mathrm{T}}$, where $\boldsymbol{x}$ an n vector
$\mathbf{M}_e$	Equivalent mass matrix - statistical linearization methodology
$\mathbf{C}_e$	Equivalent damping matrix - statistical linearization methodology
$\mathbf{K}_e$	Equivalent stiffness matrix - statistical linearization methodology
$\boldsymbol{\Phi}_{\boldsymbol{x}}$	Nonlinear vector - redundant DOFs modeling
$\bar{\mathbf{M}}_e$	Equivalent augmented mass matrix - statistical linearization - redundant DOFs
$\bar{\mathbf{C}}_e$	Equivalent augmented damping matrix - statistical linearization - redundant DOFs
$\bar{\mathbf{K}}_e$	Equivalent augmented stiffness matrix - statistical linearization - redundant DOFs
$\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}$	Augmented nonlinear vector - redundant DOFs modeling
$\boldsymbol{m}_{\boldsymbol{i*}}^{e\mathrm{T}}$	i^{th} row of $\mathbf{M}_e$ - statistical linearization methodology ($\bar{\mathbf{M}}_e$ - redundant DOFs)
$\boldsymbol{c}_{\boldsymbol{i*}}^{e\mathrm{T}}$	i^{th} row of $\mathbf{C}_e$ - statistical linearization methodology ($\bar{\mathbf{C}}_e$ - redundant DOFs)
$\boldsymbol{k}_{\boldsymbol{i*}}^{e\mathrm{T}}$	i^{th} row of $\mathbf{K}_e$ - statistical linearization methodology ($\bar{\mathbf{K}}_e$ - redundant DOFs)

”I am indebted to my father for living, but to my teacher for living well.”

Alexander the Great, 356-323 BC

Chapter 1

Introduction

1.1 Motivation and objectives

Random vibration analysis of dynamical systems, and particularly the dynamic analysis focusing on determining the response and reliability statistics of stochastically excited vibrating systems, has been a field of great interest and extensive study during the last decades. The engagement of Stochastic Calculus has, in general, been dictated by the inherent randomness in a wide range of complex and time-evolving natural phenomena, which has motivated the modeling and study of systems with stochastic parameters and initial/boundary conditions. Several random vibration methodologies have been developed over the past six decades, for quantifying the uncertain behavior of complex dynamical structural and mechanical systems, with varying degrees of success; see Refs [48, 61, 73, 60] for some indicative books, and Refs [72, 83, 57, 54] for some recently developed techniques such as the ones based on path integrals.

One of the major challenges in the direction of dynamic system analysis relates to the modeling of governing equations of motion for complex nonlinear systems. The formulation of the equations of motion of a multi-degree-of-freedom (MDOF) dynamic system is performed by utilizing the minimum number of (generalized) coordinates [58]. By doing so, the arising mass, damping and stiffness matrices of the system are not only non-singular but also symmetric and positive definite [80, 91, 16]. This characteristic actually facilitates the ensuing analysis, which via a number of solution techniques/methodologies (e.g. Ref. [91]) yields the derivation of closed forms for the system response statistics. In the field of multibody system dynamics, the smallest possible number of coordinates is also utilized for reasons including computational efficiency. Indeed, it can be argued that forming the multibody system equations of motion in terms of the independent degrees of freedom can ideally increase computational performance [40, 8, 27].

To elaborate further, depending on the complexity of the system under consideration, utilizing the minimum number of independent coordinates for formulating the equations of motion can be burdensome [93, 66]. Difficulties arise either in terms of the effort required for formulating the equations of motion, or in terms of the computation efficiency of the ensuing methodologies/techniques for the system stochastic response determination. For example, several approaches for generating the equations of motion [94], such as the ones relying on the computation of Lagrange multipliers [74, 85], require the application of constraints that are functionally independent. Verifying the above requirement is not a straightforward task, especially for large-scale complex systems. In case of large-scale multibody systems the disclosed difficulties in formulating the system governing equations of motion are magnified, as the complexity of the equations grows rapidly with increasing the number of constituent bodies and/or the number of degrees-of-freedom (DOFs) [85, 75, 92, 93, 66]. In fact, in many cases the choice of modeling utilizing the minimum number of DOFs/coordinates relates to excessive computational cost (e.g. [40, 8, 27, 28]). Employing the minimum number of DOFs can lead to limited flexibility regarding the form and nature of the constraints the system might be subjected to. Specifically, altering a constraint might require a complete remodeling of the multibody system. The degree of simplicity and the amount of effort required for deriving the governing equations of motion, especially for complex multibody systems, become key factors of the overall modeling approach in terms of assessing the performance of a methodology for obtaining the system equations of motion [92].

1.1.1 Utilization of redundant DOFs for Formulating the governing equations of motion of MDOF systems

Taking into account the challenges associated with formulating the governing equations of motion for MDOF systems, and working towards circumventing them, an alternative approach for deriving the system equations of motion is proposed. It can be argued that modeling by utilizing more than the minimum number of DOFs overcomes some of the barriers identified in Section 1.1, which are set by the standard approach, i.e. modeling with the minimum number of coordinates, and provides the modeler with enhanced flexibility [93].

Any complex multibody system can be decomposed into its constituent parts for each of which the equations of motion can be obtained readily [111, 110]. A number of constraint equations which, in essence, connect the individual subsystems, also arise [112]. The equations of motion of the decomposed subsystems along with the constraint equations can be used to form the composite system equations of motion in a less labor-intensive manner. However, due to the redundant coordinates modeling scheme, and particularly due to the fact that the extra utilized DOFs are not independent with each other, singular mass, damping and stiffness matrices can arise in the system equations of motion [59, 75, 114]. As a result, although it can be argued that in many cases (in particular when relatively complex systems are considered) the latter ”unconventional” modeling can be advantageous from a computational efficiency perspective (e.g. Refs [111], [66]), the presence of singular matrices renders inapplicable all the standard time and frequency domain techniques for determining the systems response statistics [91, 63].

It is deemed appropriate to mention that utilizing redundant coordinates is not the only reason for the appearance of singular matrices in the system equations of motion. Singularities may arise, for instance, in certain applications such as in the rotational motion of rigid bodies even if the minimum number of generalized coordinates are employed [114, 75, 113]. Besides the case where theoretically non-singular, but numerically ill-conditioned matrices may appear [51], singular matrices are naturally met in the formulation of the equations of motion of a certain class of smart structures. In this class of vibrating systems, the system mechanical equation of motion is coupled with the electrical equation yielding a differential-algebraic system of equations with a singular mass matrix [118, 51, 50]. A concise presentation of the topic is provided for completeness in Section 1.2. Note that some of the structural systems considered herein are related to the so-called descriptor systems described, in general, by a set of differential-algebraic equations; a more detailed presentation of the topic can be found in Refs [45], [77], [49], [46], [14], [19], [33], [62], [120], [32], [44].

1.2 Systems with singular matrices

In general, as also mentioned in Section 1.1, the mass, damping and stiffness matrices obtained by modeling the system governing equations of motion by utilizing the minimum number of coordinates are not only singular, but also symmetric and positive definite [16]. In the redundant DOFs modeling approach presented in Section 1.1.1, singular matrices arise in the systems governing equations of motion. Systems governing equations with singular matrices often appear in a wide range of applications in physical sciences and engineering. The significance of these problems has implied the development of several techniques for manipulating systems with singular matrices.

The definition of ill-conditioned algebraic systems of equations follows. A linear system of algebraic equations is called ill-conditioned, if the system coefficient matrix is ”almost” singular, i.e. its condition number is very large [10, 69]. Depending on the application that is examined, among the numerical treatments followed for efficiently manipulating ill-conditioned linear systems is to neglect the small terms of the ill-conditioned matrices by setting them equal to zero. This practice potentially yield singular matrices in the system governing equations [51]. Systems with large condition number are treated, in essence, the way systems with singular matrices are; although this is not always true (theoretically non-singular, but numerically ill-conditioned coefficient matrices). Extra care is required in manipulating such systems. The problem consists in that the obtained solution of the system under consideration is not robust, meaning that a small change in the system input can result tremendous change in its output. This, in turn, results inaccurate general solution to the original system [87]. Further steps regarding the regularization of the system are required, although it is generally agreed that the resulting regularized system will provide acceptable but not exact solutions [10, 11]. The vital significance of the regularization processes employed in solving ill-conditioned systems also becomes manifest from the fact that, as far as practical applications are concerned, the input of the system is obtained from experimental measurements, and as such, includes an inherent error due to the devices employed in recording the measurements. Considering the aforementioned, it is obvious that the solution of ill-conditioned systems can be problematic.

The necessity to efficiently manipulate ill-conditioned linear systems of algebraic equations is vital in diverse engineering applications. Some identical examples where such systems appear include the following. Ill-conditioned matrices arise when the analysis of large-scale constrained mechanical systems is considered, and particularly, when a generalized coordinates partitioning treatment aiming at an order reduction is followed [65]. Ill-conditioned systems appear when solving inverse problems. In general, such problems are ill-posed, and thus, the corresponding linear systems are ill-conditioned [89]. A first example is found when solving an inverse problem aiming at reconstructing the external loads acting on a structure [106]; this is significant considering the ability to derive external loads, e.g. earthquakes acting on a structure.

Ill-conditioned systems also appear when solving an inverse problem for identifying the external forces acting on structures [107]. Ill-posed systems of equations, which in turn yield ill-conditioned matrices, are related to inverse problems regarding the detection and identification of cracks in structures [87]. Ill-conditioned matrices are met in applications of motion simulation, which is either based on the kinematics or dynamics of biological as well as mechanical systems [64, 104]. Following closely Ref. [64], it is noted that in such cases, although the systems matrix of inertias cannot be singular, it can be extremely ill-conditioned, implying variate system accelerations. Ill-conditioned systems of equations also appear in power engineering applications, for example when load-flow problems are under consideration [105].

Regardless of the presence of ill-conditioned matrices in the linear system under consideration, another category of problems where singular matrices appear is that of energy harvesting, an emerging field of research with momentous real world applications. Energy harvesters can be described as devices of varying physical dimensions which utilize the ambient energy (usually arising from vibrating systems) for either powering themselves, or other connected to them devices. Considering the fast rising field of nanotechnology, which is included among the most impactful technologies over the next years, and particularly its applications aiming at the development of nano-sensors, the development of nano-scale energy harvesting devices becomes an emerging field of diverse potential applications. They can be utilized, for instance, for powering wireless sensor systems attached to engineering structures (e.g. buildings, bridges), which, in turn, monitor and transmit information regarding the current state of the structure. Regarding the presence of singular matrices in energy harvesting applications, they can naturally appear in the coupled governing equations of the electrical and mechanical system of harvesters, in problems such as powering low power electronic devices (e.g. wireless sensors) [5].

The appearance of singular matrices in the system equations of motion is also related to the dynamic analysis of a particular class of non viscously damped systems. In general, utilizing non-viscously damped models for describing the system damping forces constitutes a generalization of the standard viscous modeling approach. This is either dictated by reasons related to mathematical rigor of the utilized model, or/and by reasons related to modern composite materials analysis and design [117, 6, 67]. Indicatively, a general model for describing a non-viscously damped system is the so-called exponential damping model, in which the damping depends on the past history of motion via convolution integrals over kernel functions [117, 115]. For the analysis of linear systems with exponential damping, as well as linear systems with combined viscous and non-viscous (exponential) damping, a state space method and its extended version, respectively, can be utilized [115, 6]. The state space method is based on the introduction of a set of internal variables and for the particular case when the damping coefficient matrices of the system are rank deficient, singular matrices appear in the state space formulation. A more detailed presentation can be found in Refs [3, 2, 115, 6, 4].

Further applications where singular matrices naturally appear in the systems governing equations include the so-called smart structures, i.e. able to expand and regain their original position via a set of piezoelectrical elements attached to them (usually refer to beams). When formulating the governing equations of motion for such kind of vibrating systems, the system mechanical equation of motion is coupled with the electrical equation yielding a differential-algebraic system of equations with singular mass matrix; see Refs [118, 51, 9] for more details. Finally, some typical applications where systems with singular matrices appear, include dynamic system controlling [1], power electric systems analysis [19], and problems related to the hydrodynamics of planing ships [116].

1.3 Organization of the thesis

This thesis consists of five chapters followed by the list of published results and the cited publications list. A list of figures is included after the table of contents.

Chapter 1 comprises the introduction of the thesis where the motivation and objectives of the herein presented work are included. The problem of formulating the governing equations of motion for MDOF systems is discussed and a number of difficulties and limitation pertaining to the classical modeling analysis are presented. An unconventional approach based on utilizing redundant coordinates for formulating the equations of motion is provided, which in turn, addresses some of the aforementioned problems.

In chapter 2 a concise presentation of some mathematical tools utilized in the ensuing analysis is given. Taking into account the objective of this thesis, i.e. the treatment of singular matrix MDOF (linear and nonlinear) systems subjected to stochastic excitation, the concept of the generalized inverse of a singular matrix is analyzed. The analysis focuses on the Moore-Penrose (M-P) generalized inverse matrix, a potent mathematical tool utilized for addressing problems related to singular matrices. A number of properties and results regarding the application of M-P inverses, are provided. The concept of multivariate normal distribution is discussed. Taking into account the concept of the M-P generalized inverses, the particular case of multivariate normal distribution of random vectors with singular covariance matrices is also scrutinized.

Chapter 3 focuses on the stochastic response determination of MDOF systems with singular matrices. The system governing equations of motion are formulated by adopting the aforementioned, redundant DOFs, modeling yielding systems with singular mass, damping and stiffness matrices. As a result, the standard time and frequency domain analysis methodologies cannot be utilized for treating these unconventional systems. Exploiting the concept of M-P generalized matrix inverse for circumventing the associated to singular matrices difficulties, first a generalization of the standard time domain analysis methodology is proposed. As in the classical case, given a system subjected to stochastic excitation, the covariance matrix of the system response is derived by either solving a Lyapunov equation or by applying a complex modal analysis. In the latter case, it is noted that although impossible to derive a decoupled system of equations, an application of the singular value decomposition yields a solution for the system response statistics. Next, relying on the M-P inverse theory, the response statistics of singular matrix systems subjected to stochastic excitations are derived by applying a frequency domain analysis treatment. A M-P based frequency response function matrix is obtained, and subsequently, the system response power spectrum is determined by utilizing the standard relationship that connects the impulse response function of the system excitation to the corresponding function of its response. Pertinent numerical examples are included for validating the generalization of the proposed time and frequency domain analysis methodologies.

In chapter 4 the aforementioned solution framework is generalized to account for nonlinear systems. A generalization of the versatile statistical linearization approximate methodology to account for nonlinear systems with singular matrices is presented. Relying on the concept of the M-P generalized matrix inverse, a family of optimal and response dependent equivalent linear matrices is derived. This set of equations in conjunction with a linear system generalized excitation response relationship (depending on either a time or frequency domain analysis treatment is followed), leads to an iterative determination of the system response mean vector and covariance matrix. It is proved that setting the arbitrary element in the M-P solution for the equivalent linear matrices equal to zero yields a mean square error at least as small as the error corresponding to any non-zero value of the arbitrary element. The validity of the proposed techniques is demonstrated by pertinent numerical examples including several linear and nonlinear systems with singular matrices.

Finally, chapter 5 includes the concluding remarks and discusses potential future work based on the herein presented results.

Chapter 2

Mathematical prerequisites

2.1 Generalized matrix inverse theory

A common situation in many engineering applications is to seek for solutions to linear systems of algebraic equations, given by

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}, \tag{2.1}$$

where $\mathbf{A}$ is a rectangular $m \times n$ matrix, $\boldsymbol{x}$ an n vector and $\boldsymbol{b}$ is an m vector. In the particular case when $m = n$ and $\mathbf{A}$ is invertible, the system defined in Eq. (2.1) has a unique solution, given by

$$\boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{b}, \tag{2.2}$$

where $\mathbf{A}^{-1}$ is the inverse matrix of $\mathbf{A}$. On the contrary, in the general case when $\mathbf{A}$ is an arbitrary $m \times n$ matrix, deriving a solution to Eq. (2.1) is not a straightforward procedure. A solution can be found by utilizing the notion of the generalized inverse of a matrix.

It can be proved that for any $m \times n$ matrix $\mathbf{A}$, there exists a unique $n \times m$ matrix, denoted by $\mathbf{A}^{+}$, such that the following four identities

$$\mathbf{A}\mathbf{A}^{+}\mathbf{A} = \mathbf{A}, \tag{2.3}$$

$$\mathbf{A}^{+}\mathbf{A}\mathbf{A}^{+} = \mathbf{A}^{+}, \tag{2.4}$$

$$(\mathbf{A}\mathbf{A}^{+})^{*} = \mathbf{A}\mathbf{A}^{+}, \tag{2.5}$$

$$(\mathbf{A}^{+}\mathbf{A})^{*} = \mathbf{A}^{+}\mathbf{A}, \tag{2.6}$$

hold true; $\mathbf{A}^*$ denotes the conjugate transpose of the matrix $\mathbf{A}$. The matrix $\mathbf{A}^+$ is called the Moore-Penrose (M-P) generalized inverse of the $m \times n$ matrix $\mathbf{A}$, and the expressions given by Eqs. (2.3)-(2.6) represents the so-called M-P equations. If $\mathbf{A}$ is non-singular, its inverse, $\mathbf{A}^{-1}$, satisfies the M-P equations, and thus, $\mathbf{A}^+ = \mathbf{A}^{-1}$.

In general, several kind of generalized inverses can be defined for a given $m \times n$ matrix $\mathbf{A}$, and each one efficiently serves different purposes. The rationale behind the definition of the generalized inverse of $\mathbf{A}$ lies in which, or how many, of the four M-P equations are satisfied by the particular matrix. An $n \times m$ matrix $\mathbf{Z}$ which satisfies, for instance, Eq. (2.3) is called an $(1)-$inverse of $\mathbf{A}$, whereas if Eq. (2.3) and Eq. (2.5) are both satisfied, the matrix $\mathbf{Z}$ is referred to as the $(1,3)-$inverse of $\mathbf{A}$. Under the scope of the present thesis the utilization of generalized inverses is connected to solving linear systems of algebraic equations with singular matrices. More specifically, the solution of the algebraic system of Eq. (2.1), where either $\mathbf{A}$ is a singular $m \times m$, or an $m \times n$ matrix, is sought. In this particular case, the (1)-inverse of $\mathbf{A}$ is proved to be the most appropriate choice.

Definition 2.1. Let $\mathcal{C}^{m \times n}$ denotes the class of $m \times n$ complex matrices. For any $\mathbf{A} \in \mathcal{C}^{m \times n}$, let $\mathbf{A}^{j,k,\ldots,l} \in \mathcal{C}^{n \times m}$ denote a matrix such that the $(j), (k), \ldots, (l)$ of the M-P equations, Eqs. (2.3)-(2.6), hold true. Then, $\mathbf{A}^{(j,k,\ldots,l)}$ is called the $(j,k,\ldots,l)-$inverse of $\mathbf{A}$.

Theorem 2.2 ([13])**.** *Let* $\mathbf{A} \in \mathcal{C}^{m \times n}$*,* $\mathbf{B} \in \mathcal{C}^{p \times q}$ *and* $\mathbf{D} \in \mathcal{C}^{m \times q}$*. Then the matrix equation*

$$\mathbf{AXB} = \mathbf{D} \tag{2.7}$$

is consistent if, and only if for some $\mathbf{A}^{(1)}$*,* $\mathbf{B}^{(1)}$*,*

$$\mathbf{A}\mathbf{A}^{(1)}\mathbf{D}\mathbf{B}^{(1)}\mathbf{B} = \mathbf{D}, \tag{2.8}$$

in which case the general solution is

$$\mathbf{X} = \mathbf{A}^{(1)}\mathbf{D}\mathbf{B}^{(1)} + \mathbf{Y} - \mathbf{A}^{(1)}\mathbf{A}\mathbf{Y}\mathbf{B}\mathbf{B}^{(1)}, \tag{2.9}$$

for arbitrary $\mathbf{Y} \in \mathcal{C}^{n \times p}$*.*

In the particular case of algebraic systems described by Eq. (2.1), Theorem 2.2 takes the form given in the following corollary.

Corollary 2.3 ([13])**.** *Let* $\mathbf{A} \in \mathcal{C}^{m \times n}$ *and* $\boldsymbol{b}$ *be an* m *vector. Then, Eq. (2.1) is consistent if, and only if for some* $\mathbf{A}^{(1)}$*,*

$$\mathbf{A}\mathbf{A}^{(1)}\boldsymbol{b} = \boldsymbol{b}, \tag{2.10}$$

in which case the general solution of Eq. (2.1) is

$$\boldsymbol{x} = \mathbf{A}^{(1)}\boldsymbol{b} + (\mathbf{I} - \mathbf{A}^{(1)}\mathbf{A})\boldsymbol{y}, \tag{2.11}$$

where $\boldsymbol{y}$ *is an arbitrary* n *vector.*

If the M-P inverse of the matrix $\mathbf{A}$ is utilized instead of its $(1)-$inverse, ancillary identities arise (i.e. Eqs. (2.5)-(2.6)), which greatly facilitates the ensuing analysis. The M-P inverse of any $m \times n$ matrix $\mathbf{A}$, which can be determined by employing various techniques and methodologies such as a number of recursive formulae (e.g. [18], [47]), provides a tool for solving equations of the form defined in Eq. (2.1). For a singular square matrix $\mathbf{A}$, i.e. $det\mathbf{A} = 0$, utilizing the M-P inverse and taking into account Corollary 2.3, Eq. (2.1) yields

$$\boldsymbol{x} = \mathbf{A}^{+}\boldsymbol{b} + (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\boldsymbol{y}, \tag{2.12}$$

where $\boldsymbol{y}$ is an arbitrary n vector. A more detailed presentation of the topic can be found in Refs [13, 18, 12, 81].

2.1.1 Miscellaneous results

In this subsection, a number of useful for the ensuing analysis results are provided for completeness. Further, some properties/identities regarding the M-P generalized matrix inverse are provided in the form of corollary without its proof. The proofs of the identities are derived as a straightforward outcome of the M-P equations, Eqs. (2.3)-(2.6), and can be found in Refs [13, 18].

Suppose that $\mathbf{A}$ is an $m \times n$ matrix where, without loss of generality, $m < n$ and $rank(\mathbf{A}) < n$, i.e. $\mathbf{A}$ is not a full rank matrix. Then, $\mathbf{A}$ can be decomposed in the product of a full column rank matrix and a full row rank matrix.

Lemma 2.4 ([13])**.** *Let* $\mathbf{A} \in \mathcal{C}^{m\times n}$*, such that* $rank(\mathbf{A}) = r > 0$*. Then, there exist a set of matrices* $\mathbf{F} \in \mathcal{C}^{m\times r}$*, with* $rank(\mathbf{F}) = r$*, and* $\mathbf{G} \in \mathcal{C}^{r\times n}$*, with* $rank(\mathbf{G}) = r$*, such that*

$$\mathbf{A} = \mathbf{F}\mathbf{G}. \tag{2.13}$$

The factorization given in Eq. (2.13) is referred to as the full rank factorization of a matrix $\mathbf{A}$ and it will be proved of vital significance in the ensuing analysis, as it is connected to the following theorem for deriving an explicit formula for the M-P inverse of $\mathbf{A}$ [13, 84].

Theorem 2.5 ([13])**.** *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix, with* $rank(\mathbf{A}) = r > 0$*, and let* $\mathbf{A}$ *has the full rank factorization given in Eq. (2.13). Then, the* M-P *inverse of* $\mathbf{A}$ *is given by*

$$\mathbf{A}^{+} = \mathbf{G}^{*}(\mathbf{F}^{*}\mathbf{A}\mathbf{G}^{*})^{-1}\mathbf{F}^{*}. \tag{2.14}$$

A theorem that proves the existence of the M-P inverse, $\mathbf{A}^{+}$, of an $m \times n$ matrix $\mathbf{A}$, as well as some general properties of $\mathbf{A}^{+}$ follow.

Theorem 2.6 ([13])**.** *For any finite matrix* $\mathbf{A}$ *of complex elements,*

$$\mathbf{A}^{(1,4)}\mathbf{A}\mathbf{A}^{(1,3)} = \mathbf{A}^{+} \tag{2.15}$$

Corollary 2.7. *Let* $\mathbf{A} \in \mathcal{C}^{m\times n}$*,* $\mathbf{A}^{+}$ *denote its M-P inverse and* $\lambda \in \mathcal{C}$*. The following properties hold true.*

$$\begin{array}{ll}
(i) & (\mathbf{A}^{+})^{+} = \mathbf{A} \\
(ii) & (\mathbf{A}^{*})^{+} = (\mathbf{A}^{+})^{*} \\
(iii) & (\mathbf{A}^{\mathrm{T}})^{+} = (\mathbf{A}^{+})^{\mathrm{T}} \\
(iv) & \mathbf{A}^{+} = (\mathbf{A}^{*}\mathbf{A})^{+}\mathbf{A}^{*} = \mathbf{A}^{*}(\mathbf{A}\mathbf{A}^{*})^{+} \\
(v) & \lambda^{+} = \begin{cases} \frac{1}{\lambda} & , \textit{ if } \lambda \neq 0 \\ 0 & , \textit{ if } \lambda = 0 \end{cases} \\
(vi) & (\lambda\mathbf{A})^{+} = \lambda^{+}\mathbf{A}^{+} \\
(vii) & \textit{if } \mathbf{D} = diag(d_1, d_2, \ldots, d_n), \textit{ then, } \mathbf{D}^{+} = diag(d_1^{+}, d_2^{+}, \ldots, d_n^{+}) \\
(viii) & (\mathbf{U}\mathbf{A}\mathbf{V})^{+} = \mathbf{V}^{*}\mathbf{A}^{+}\mathbf{U}^{*} \textit{ where } \mathbf{U}, \mathbf{V} \textit{ unitary matrices} \\
(ix) & (\mathbf{A}\mathbf{B})^{+} = \mathbf{B}^{+}\mathbf{A}^{+}
\end{array}$$

2.2 Multivariate normal distribution

Assume that x is a random variable that follows the normal distribution, $x \sim N(\mu, \sigma^2)$, with mean and variance given by μ and σ^2, respectively. The probability density function (pdf) of x is given by [79]

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{(x-\mu)^2}{2\sigma^2}\right\},\ \infty < x < \infty. \tag{2.16}$$

Next, let $\boldsymbol{U} = (x_1, x_2, \ldots, x_p)$ be a random vector with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. The multivariate normal distribution (MND) is defined as a generalization of the (univariate) normal distribution with pdf given by Eq. (2.16). For the random vector $\boldsymbol{U}$ that follows the MND with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, namely $\boldsymbol{U} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, it can be proved that there is a $p \times r$ matrix $\mathbf{B}$ such that [88]

$$\boldsymbol{U} - \boldsymbol{\mu} = \mathbf{B}\boldsymbol{G} \;\text{ and }\; \mathbf{B}\mathbf{B}^{\mathrm{T}} = \boldsymbol{\Sigma}, \tag{2.17}$$

where $\boldsymbol{G} \sim N_m(\mathbf{0}, \mathbf{I})$ and $m = rank\boldsymbol{\Sigma} = p$; details on how $\mathbf{B}$ is determined are provided in Section 2.2.1. In this case, the $p-$dimension pdf of $\boldsymbol{U}$ is given by

$$\boldsymbol{p}(\boldsymbol{U}) = ((2\pi)^k|\boldsymbol{\Sigma}|)^{-\frac{1}{2}} \exp(-\frac{1}{2}(\boldsymbol{U} - \boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{U} - \boldsymbol{\mu})). \tag{2.18}$$

In case when $rank\boldsymbol{\Sigma} = k < p$, the matrix $\boldsymbol{\Sigma}$ is singular, and Eq. (2.18) cannot be employed in determining the $p-$dimension pdf of $\boldsymbol{U}$. To bypass this problem, let $\mathbb{M}(\boldsymbol{\Sigma})$ be the vector space spanned by the columns of the matrix $\boldsymbol{\Sigma}$ and $\mathbf{B}$ be a $p \times k$ matrix of orthonormal column vectors belonging to $\mathbb{M}(\boldsymbol{\Sigma})$. Further, let $\mathbf{N}$ be a $p \times (p-k)$ matrix of rank $p - k$ such that

$$\mathbf{N}^{\mathrm{T}}\boldsymbol{\Sigma} = \mathbf{0}. \tag{2.19}$$

Utilizing the transformation

$$\boldsymbol{X} = \mathbf{B}^{\mathrm{T}}\boldsymbol{U}, \;\; \boldsymbol{Z} = \mathbf{N}^{\mathrm{T}}\boldsymbol{U}, \tag{2.20}$$

the mean as well as the covariance matrix of the random vector $\boldsymbol{Z}$ are given by

$$\mathbb{E}(\boldsymbol{Z}) = \mathbf{N}^{\mathrm{T}}\boldsymbol{\mu} \tag{2.21}$$

and

$$cov(\boldsymbol{Z}, \boldsymbol{Z}) = \mathbf{N}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{N}, \tag{2.22}$$

respectively, so that

$$\boldsymbol{Z} = \mathbf{N}^{\mathrm{T}}\boldsymbol{\mu} \quad \text{with probability } 1. \tag{2.23}$$

The corresponding expectation and covariance matrix of $\boldsymbol{X}$, are

$$\mathbb{E}[\boldsymbol{X}] = \mathbf{B}^{\mathrm{T}}\boldsymbol{\mu} \tag{2.24}$$

and

$$cov(\boldsymbol{X}, \boldsymbol{X}) = \mathbf{B}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{B}, \tag{2.25}$$

respectively. Assuming that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the non-zero eigenvalues of $\mathbf{\Sigma}$, it can be proved that the determinant of the covariance matrix defined in Eq. (2.25) satisfies the relationship [103, 52]

$$|\mathbf{B}^T\mathbf{\Sigma}\mathbf{B}| = \lambda_1\lambda_2\ldots\lambda_k. \tag{2.26}$$

Combining Eq. (2.18) with Eq. (2.20) and manipulating yields [52]

$$\boldsymbol{p}(\boldsymbol{X}) = ((2\pi)^k|\mathbf{B}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{B}|)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\boldsymbol{X} - \mathbf{B}^{\mathrm{T}}\boldsymbol{\mu})^{\mathrm{T}}(\mathbf{B}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{B})^{-1}(\boldsymbol{U} - \mathbf{B}^{\mathrm{T}}\boldsymbol{\mu})\right\}. \tag{2.27}$$

Taking into account the transformation defined in Eq. (2.20), the distribution of $\boldsymbol{U}$ is determined via the corresponding distributions of the random vectors $\boldsymbol{Z}, \boldsymbol{X}$, i.e. utilizing Eq. (2.23) and Eq. (2.27), respectively. Denoting by α the expression included in the exponential of Eq. (2.27) and manipulating results

$$\begin{aligned}
\alpha &= -\frac{1}{2}(\mathbf{B}^{\mathrm{T}}\boldsymbol{U} - \mathbf{B}^{\mathrm{T}}\boldsymbol{\mu})^{\mathrm{T}}(\mathbf{B}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{B})^{-1}(\mathbf{B}^{\mathrm{T}}\boldsymbol{U} - \mathbf{B}^{\mathrm{T}}\boldsymbol{\mu}) \\
&= -\frac{1}{2}(\boldsymbol{U} - \boldsymbol{\mu})^{\mathrm{T}}\mathbf{\Sigma}^{(1)}(\boldsymbol{U} - \boldsymbol{\mu}),
\end{aligned} \tag{2.28}$$

where

$$\mathbf{\Sigma}^{(1)} = \mathbf{B}(\mathbf{B}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}} \tag{2.29}$$

denotes the $(1)-$inverse of the matrix $\mathbf{\Sigma}$. It is noted that Eq. (2.28) holds true for any particular choice of $(1)-$inverse matrix $\mathbf{\Sigma}^{(1)}$, and thus, taking into account that the set of $(1)-$inverse matrices includes the set of M-P matrix inverses as a subset [102, 88], Eq. (2.28) also holds true for the uniquely defined M-P inverse $\mathbf{\Sigma}^{+}$; for more details see Refs [103, 102]. Combining Eqs. (2.26) and (2.28) with Eq. (2.27), the pdf of $\boldsymbol{X}$ is written as

$$\boldsymbol{p}(\boldsymbol{X}) = ((2\pi)^k\lambda_1\lambda_2\ldots\lambda_n)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\boldsymbol{U} - \boldsymbol{\mu})^{\mathrm{T}}\mathbf{\Sigma}^{+}(\boldsymbol{U} - \boldsymbol{\mu})\right\}, \tag{2.30}$$

whereas combining Eqs. (2.20) and (2.23) yields

$$\mathbf{N}^{\mathrm{T}}\boldsymbol{U} = \mathbf{N}^{\mathrm{T}}\boldsymbol{\mu} \ \text{ with probability 1.} \tag{2.31}$$

The pdf of $\boldsymbol{U}$ is specified by Eq. (2.30) and Eq. (2.31), where the former is interpreted as the density on the hyperplane $\mathbf{N}^{\mathrm{T}}\boldsymbol{U} = \mathbf{N}^{\mathrm{T}}\boldsymbol{\mu}$. A more detailed presentation of the topic can be found in Refs [88, 102].

2.2.1 Further analysis of the results regarding singular matrices

For the determination of the matrix $\mathbf{B}$ of Eq. (2.20), the concept of full rank factorization of a matrix that is analyzed in Section 2.1.1, is employed. Assuming that the full rank factorization of the $p \times p$ matrix $\mathbf{\Sigma}$ is

$$\mathbf{\Sigma} = \mathbf{B}\mathbf{B}^{\mathrm{T}}, \tag{2.32}$$

where $\mathbf{B} \in \mathcal{C}^{p \times r}$ with $rank(\mathbf{B}) = r$ and $\mathbf{G} = \mathbf{B}^{\mathrm{T}}$, and taking into account Eq. (2.14), the M-P inverse of $\mathbf{\Sigma}$ takes the form

$$\mathbf{\Sigma}^{+} = \mathbf{B}(\mathbf{B}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}. \tag{2.33}$$

Combining Eq. (2.33) with Eq. (2.29), Eq. (2.30) is derived; $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of $\mathbf{\Sigma}$.

2.3 Summary

Chapter 2 comprises a brief introduction to the mathematical tools which are required for developing the forthcoming theoretical framework. Definitions and critical results pertaining the generalized matrix inverse theory, are presented. Particular emphasis is given to the Moore-Penrose generalized matrix inverse. A brief discussion on the multivariate normal distribution, and particularly the case of probability density functions with singular covariance matrices, is also included.

Chapter 3

Stochastic response of linear systems with singular matrices

3.1 Classical Approach

3.1.1 Time domain analysis

Following closely Ref. [91], the general form of the equations of motion of a lumped-parameter n degree-of-freedom ($n-$DOF) system is

$$\mathbf{M}\ddot{\boldsymbol{q}} + \mathbf{C}\dot{\boldsymbol{q}} + \mathbf{K}\boldsymbol{q} = \boldsymbol{Q}(t), \tag{3.1}$$

where $\mathbf{M}, \mathbf{C}, \mathbf{K}$ are symmetric $n \times n$ matrices, representing the mass, the damping and the stiffness of the system, respectively. The symbol $\boldsymbol{q}$ stands for an n vector containing the n (generalized) displacements of the system, and $\boldsymbol{Q}$ is an n vector containing the n (generalized) forces corresponding to $\boldsymbol{q}$.

The equations of motion for the $n-$DOF system of Eq. (3.1) can be cast into the state variable form by defining a $2n$ vector,

$$\boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix}. \tag{3.2}$$

Next, taking into account Eq. (3.2), Eq. (3.1) can be written, equivalently, in the form

$$\dot{\boldsymbol{z}} = \mathbf{G}\boldsymbol{z} + \boldsymbol{f}, \tag{3.3}$$

The results presented in this chapter are published in:

Fragkoulis et al. 2015, ASCE *J. Eng. Mech.*, Kougioumtzoglou et al. 2017, *J. Sound Vibr.*

where

$$\mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \tag{3.4}$$

and

$$\boldsymbol{f} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\boldsymbol{Q} \end{bmatrix}. \tag{3.5}$$

Note that in deriving Eqs. (3.4) and (3.5) it is assumed that the mass matrix $\mathbf{M}$ is non-singular, since systems with singular mass matrices are not common in a standard formulation of the system equations of motion in classical dynamics. In fact, when the minimum number of coordinates is utilized for formulating the equations of motion, the mass matrices are not only non-singular, but also symmetric and positive definite [80, 91].

The response of the system of Eq. (3.3) can be determined by utilizing the convolution integral

$$\boldsymbol{q}(t) = \int_0^t \mathbf{h}(\tau)\boldsymbol{Q}(t-\tau)d\tau, \tag{3.6}$$

where $\mathbf{h}(t)$ represents the impulse response matrix of the system, given by

$$\mathbf{h}(t) = \mathbf{b}(t)\mathbf{M}^{-1}. \tag{3.7}$$

In Eq. (3.7), $\mathbf{b}(t)$ is obtained by the relationship

$$\exp(\mathbf{G}t) = \begin{bmatrix} \mathbf{a}(t) & \mathbf{b}(t) \\ \mathbf{c}(t) & \mathbf{d}(t) \end{bmatrix}, \tag{3.8}$$

where all the sub-matrices are $n \times n$; see Ref. [91] for a more detailed presentation.

Statistical moments of the response of the linear MDOF system of Eq. (3.1) can be determined readily by direct manipulation of the state variable form of the equation of motion, i.e. Eq. (3.3). Denoting

$$\boldsymbol{m}_z = \mathbb{E}[\boldsymbol{z}(t)], \tag{3.9}$$

and taking expectations on Eq. (3.3), yields

$$\dot{\boldsymbol{m}}_z = \mathbf{G}\boldsymbol{m}_z + \boldsymbol{m}_f. \tag{3.10}$$

Eq. 3.10 can be solved to find $\boldsymbol{m}_z$ as a function of time. For a zero-mean excitation, the solution for $\boldsymbol{m}_z$ is given in the form

$$\boldsymbol{m}_z = \exp(\mathbf{G}t)\boldsymbol{m}_z(0), \tag{3.11}$$

where $\boldsymbol{m}_z \to 0$, as $t \to \infty$. Considering Eq. (3.3) and Eq. (3.10), the relationship

$$\dot{\boldsymbol{\lambda}} = \mathbf{G}\boldsymbol{\lambda} + \boldsymbol{\eta}(t), \tag{3.12}$$

is obtained, where the $2n$ vectors $\boldsymbol{\lambda}(t)$ and $\boldsymbol{\eta}(t)$ are given by

$$\boldsymbol{\lambda}(t) = \boldsymbol{z}(t) - \boldsymbol{m}_z(t) \tag{3.13}$$

and

$$\boldsymbol{\eta}(t) = \boldsymbol{f}(t) - \boldsymbol{m}_{\boldsymbol{f}}(t), \tag{3.14}$$

respectively. Taking into account the covariance matrix

$$\mathbf{V} = \mathbb{E}\left\{[\boldsymbol{z}(t) - \boldsymbol{m}_z(t)][\boldsymbol{z}(t) - \boldsymbol{m}_z(t)]^{\mathrm{T}}\right\}, \tag{3.15}$$

and considering Eqs. (3.12)-(3.15) yields

$$\dot{\mathbf{V}} = \mathbf{G}\mathbf{V} + \mathbf{V}\mathbf{G}^{\mathrm{T}} + \mathbf{S}, \tag{3.16}$$

where

$$\mathbf{S}(t) = \int_0^t \exp(\mathbf{G}(t-\tau))\left[\mathbf{w}_{\boldsymbol{\eta}}(t,\tau) + \mathbf{w}_{\boldsymbol{\eta}}^{\mathrm{T}}(t,\tau)\right] d\tau. \tag{3.17}$$

In Eq. (3.17), $\mathbf{w}_{\eta}(t,\tau)$ is the covariance matrix for the $2n$ vector $\boldsymbol{\eta}(t)$ defined in Eq. (3.15). For the specific case where the elements of $\boldsymbol{\eta}(t)$ are modeled as stationary white-noises, Eq. (3.17) implies

$$\mathbf{w}_{\boldsymbol{\eta}}(t,\tau) = \mathbf{D}\delta(t-\tau), \tag{3.18}$$

where $\mathbf{D}$ is a real, symmetric, non-negative matrix of constants. Substituting Eq. (3.18) into Eq. (3.17), Eq. (3.16) becomes

$$\dot{\mathbf{V}} = \mathbf{G}\mathbf{V} + \mathbf{V}\mathbf{G}^{\mathrm{T}} + \mathbf{D}. \tag{3.19}$$

3.2 A redundant coordinates modeling for formulating the systems governing equations of motion

An inherent assumption made for the mass, damping and stiffness matrices of any $n-$DOF system is that they all are non-singular (Section 1.1.1). Formulating the system equations of motion by utilizing a redundant coordinates modeling scheme, which can be advantageous in cases of complex multi-body systems, results singular mass, damping and stiffness matrices (Section 1.1.1). In such cases, the complex multi-body system can be decomposed into its constituent parts for each of which either the equations of motion are known, or they can be readily obtained. These equations can then be used to form the equations of motion of the overall composite system in a less labor-intensive manner [110, 112].

Considering that the governing equation of motion for the $n-$DOFs system of Eq. (3.1) (where $\boldsymbol{q}$ stands for an n vector containing the n (generalized) displacements of the system) is formed by utilizing more than the minimum coordinates, an $l-$DOF system of the form

$$\mathbf{M}_x\ddot{\boldsymbol{x}} + \mathbf{C}_x\dot{\boldsymbol{x}} + \mathbf{K}_x\boldsymbol{x} = \boldsymbol{Q}_x(t), \tag{3.20}$$

is derived. In Eq. (3.20), $\boldsymbol{x}$ is the l vector of the coordinates ($l \geqslant n$), $\boldsymbol{Q}_x$ is the l vector of external forces, and $\mathbf{M}_x, \mathbf{C}_x$ and $\mathbf{K}_x$ are the mass, damping and stiffness matrices, respectively, corresponding to the system of Eq. (3.20). Next, consider the case where the system of Eq. (3.20) is subjected to m constraints of the form

$$\mathbf{A}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\ddot{\boldsymbol{x}} = \boldsymbol{b}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t), \tag{3.21}$$

where $\mathbf{A}$ is an $m \times l$ matrix and $\boldsymbol{b}$ is an m vector.

In general, while the unconstrained system becomes constrained, additional forces arise to ensure that the constraints are satisfied [112, 109, 108, 53]. Eq. (3.20) becomes

$$\mathbf{M}_x\ddot{\boldsymbol{x}} + \mathbf{C}_x\dot{\boldsymbol{x}} + \mathbf{K}_x\boldsymbol{x} = \boldsymbol{Q}_x(t) + \boldsymbol{Q}_x^c(t), \tag{3.22}$$

where the l vector $\boldsymbol{Q}_x^c(t)$ denotes the additional aforementioned forces. The presence of constraints yields virtual displacements, described by the l vector $\boldsymbol{w}$, which is any non-zero vector satisfying the condition

$$\mathbf{A}\boldsymbol{w} = \mathbf{0}, \tag{3.23}$$

and at any instant of time t can be expressed as

$$\boldsymbol{w}^T \boldsymbol{Q}_x^c = \boldsymbol{w}^T \boldsymbol{N}. \tag{3.24}$$

The l vector $\boldsymbol{N}$ describes the nature of the non-ideal constraints and can be obtained by experimentation and/or observation. By employing the M-P inverse, $\mathbf{A}^+$, of the matrix $\mathbf{A}$, Eq. (3.23) is rewritten as

$$\boldsymbol{w} = (\mathbf{I} - \mathbf{A}^+\mathbf{A})\boldsymbol{y}, \tag{3.25}$$

or, equivalently,

$$\boldsymbol{w} = \tilde{\mathbf{A}}\boldsymbol{y}, \tag{3.26}$$

where

$$\tilde{\mathbf{A}} = \mathbf{I} - \mathbf{A}^+\mathbf{A} \tag{3.27}$$

and $\boldsymbol{y}$ is an arbitrary l vector. Substituting Eq. (3.25) into Eq. (3.24), yields

$$\tilde{\mathbf{A}}\boldsymbol{Q}_x^c = \tilde{\mathbf{A}}\boldsymbol{N}. \tag{3.28}$$

Pre-multiplying Eq. (3.22) by $\tilde{\mathbf{A}}$, and considering Eq. (3.28), the expression

$$\tilde{\mathbf{A}}\left\{\mathbf{M}_x\ddot{\boldsymbol{x}} + \mathbf{C}_x\dot{\boldsymbol{x}} + \mathbf{K}_x\boldsymbol{x}\right\} = \tilde{\mathbf{A}}(\boldsymbol{Q}_x + \boldsymbol{N}), \tag{3.29}$$

is obtained. Without loss of generality and for facilitating the ensuing analysis, the m vector $\boldsymbol{b}$ of the constrained Eq. (3.21), can be assumed to be of the form

$$\boldsymbol{b} = \boldsymbol{F} - \mathbf{E}\dot{\boldsymbol{x}} - \mathbf{L}\boldsymbol{x}. \tag{3.30}$$

Considering Eqs. (3.21) and (3.30) together with Eq. (3.29) implies

$$\bar{\mathbf{M}}_x\ddot{\boldsymbol{x}} = \begin{bmatrix} \tilde{\mathbf{A}}(\boldsymbol{Q}_x + \boldsymbol{N}) \\ \boldsymbol{F} \end{bmatrix} - \begin{bmatrix} \tilde{\mathbf{A}}\mathbf{C}_x\dot{\boldsymbol{x}} \\ \mathbf{E}\dot{\boldsymbol{x}} \end{bmatrix} - \begin{bmatrix} \tilde{\mathbf{A}}\mathbf{K}_x\boldsymbol{x} \\ \mathbf{L}\boldsymbol{x} \end{bmatrix} \tag{3.31}$$

or, equivalently

$$\bar{\mathbf{M}}_x\ddot{\boldsymbol{x}} = \begin{bmatrix} \tilde{\mathbf{A}}(\boldsymbol{Q}_x + \boldsymbol{N} + \boldsymbol{S}) \\ \boldsymbol{b} \end{bmatrix}, \tag{3.32}$$

where the m vector $\boldsymbol{b}$ and the $(m + l) \times l$ matrix $\bar{\mathbf{M}}_x$ are given by Eq. (3.30) and

$$\bar{\mathbf{M}}_x = \begin{bmatrix} \tilde{\mathbf{A}}\mathbf{M}_x \\ \mathbf{A} \end{bmatrix}, \tag{3.33}$$

respectively; the l vector $\boldsymbol{S}$ in Eq. (3.32) is given by

$$\boldsymbol{S} = -\mathbf{C}_x \dot{\boldsymbol{x}} - \mathbf{K}_x \boldsymbol{x}. \tag{3.34}$$

Employing Eq. (2.12), the M-P solution to Eq. (3.32) is

$$\ddot{\boldsymbol{x}} = \bar{\mathbf{M}}_x^+ \begin{bmatrix} \tilde{\mathbf{A}}(\boldsymbol{Q}_x + \boldsymbol{N} + \boldsymbol{S}) \\ \boldsymbol{b} \end{bmatrix} + (\mathbf{I} - \bar{\mathbf{M}}_x^+ \bar{\mathbf{M}}_x)\boldsymbol{y}, \tag{3.35}$$

where $\bar{\mathbf{M}}_x^+$ denotes the $l \times (l+m)$ M-P inverse matrix of $\bar{\mathbf{M}}_x$.

In order to simplify the M-P solution of Eq. (3.35), the following lemma is provided.

Lemma 3.1 ([113])**.** *Assume that $\bar{\mathbf{M}}_x$ is the $(m+l) \times l$ matrix defined in Eq. (3.33). Then, the relationship*

$$\bar{\mathbf{M}}_x^+ \begin{bmatrix} (\boldsymbol{Q}_x + \mathbf{A}^+ \boldsymbol{z}) + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix} = \bar{\mathbf{M}}_x^+ \begin{bmatrix} \boldsymbol{Q}_x + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix}, \tag{3.36}$$

holds true, for any m vector $\boldsymbol{z}$.

Proof. Taking into account Corollary (2.7), the expression

$$\bar{\mathbf{M}}_x^+ = (\bar{\mathbf{M}}_x^{\mathrm{T}} \bar{\mathbf{M}}_x)^+ \bar{\mathbf{M}}_x^{\mathrm{T}}, \tag{3.37}$$

holds true for the M-P inverse of the augmented mass matrix $\bar{\mathbf{M}}_x^+$ of Eq. (3.33).

Expanding the left hand side in Eq. (3.36) yields

$$\bar{\mathbf{M}}_x^+ \begin{bmatrix} (\boldsymbol{Q}_x + \mathbf{A}^+ \boldsymbol{z}) + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix} = (\bar{\mathbf{M}}_x^{\mathrm{T}} \bar{\mathbf{M}}_x)^+ \bar{\mathbf{M}}_x^{\mathrm{T}} \begin{bmatrix} (\boldsymbol{Q} + \boldsymbol{A}^+ \boldsymbol{z}) + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix}. \tag{3.38}$$

Considering the fourth M-P equation which is given by the expression $(\mathbf{A}^+\mathbf{A})^* = \mathbf{A}^+\mathbf{A}$ (see also Eq. (2.6)), Eq. (3.33) implies

$$\bar{\mathbf{M}}_x^{\mathrm{T}} = \begin{bmatrix} \mathbf{M}_x(\mathbf{I} - \mathbf{A}^+\mathbf{A}) & \mathbf{A}^{\mathrm{T}} \end{bmatrix}. \tag{3.39}$$

Combining Eq. (3.38) with Eq. (3.39) and manipulating, the expression

$$\bar{\mathbf{M}}_x^+ \begin{bmatrix} (\boldsymbol{Q}_x + \mathbf{A}^+ \boldsymbol{z}) + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix} = \bar{\mathbf{M}}_x^+ \begin{bmatrix} \boldsymbol{Q}_x + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix}, \tag{3.40}$$

arises, which proves Lemma 3.1. □

Taking into account Lemma 3.1 and assuming that the m vector $\boldsymbol{z}$ is given by

$$\boldsymbol{z} = -\mathrm{A}(\boldsymbol{Q}_x + \boldsymbol{N} + \boldsymbol{S}), \tag{3.41}$$

Eq. (3.36) yields

$$\bar{\mathrm{M}}_x^+ \begin{bmatrix} \tilde{\mathrm{A}}(\boldsymbol{Q}_x + \boldsymbol{N} + \boldsymbol{S}) \\ \boldsymbol{b} \end{bmatrix} = \bar{\mathrm{M}}_x^+ \begin{bmatrix} \boldsymbol{Q}_x + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix}. \tag{3.42}$$

Taking into consideration Eq. (3.42), Eq. (3.35) degenerates to the form

$$\ddot{\boldsymbol{x}} = \bar{\mathrm{M}}_x^+ \begin{bmatrix} \boldsymbol{Q}_x + \boldsymbol{N} + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix} + (\mathbf{I} - \bar{\mathrm{M}}_x^+ \bar{\mathrm{M}}_x)\boldsymbol{y}. \tag{3.43}$$

The preceding analysis is for the general case, where the constraints are considered to be non-ideal. Nevertheless, assuming in the ensuing analysis that the constraints are ideal, i.e. $\boldsymbol{N} = \mathbf{0}$, Eq. (3.43) becomes

$$\ddot{\boldsymbol{x}} = \bar{\mathrm{M}}_x^+ \begin{bmatrix} \boldsymbol{Q}_x + \boldsymbol{S} \\ \boldsymbol{b} \end{bmatrix} + (\mathbf{I} - \bar{\mathrm{M}}_x^+ \bar{\mathrm{M}}_x)\boldsymbol{y} \tag{3.44}$$

which, considering Eq. (3.34), can be written as

$$\ddot{\boldsymbol{x}} = \bar{\mathrm{M}}_x^+ \left[-\tilde{\mathrm{C}}_x \dot{\boldsymbol{x}} - \tilde{\mathrm{K}}_x \boldsymbol{x} + \tilde{\boldsymbol{Q}}_x\right] + (\mathbf{I} - \bar{\mathrm{M}}_x^+ \bar{\mathrm{M}}_x)\boldsymbol{y}. \tag{3.45}$$

In Eq. (3.45), the $(m+l) \times l$ matrices $\tilde{\mathrm{C}}_x$, $\tilde{\mathrm{K}}_x$ and the $(m+l)$ vector $\tilde{\boldsymbol{Q}}_x$ are given by

$$\tilde{\mathrm{C}}_x = \begin{bmatrix} \mathrm{C}_x \\ \mathrm{E} \end{bmatrix}, \tag{3.46}$$

$$\tilde{\mathrm{K}}_x = \begin{bmatrix} \mathrm{K}_x \\ \mathrm{L} \end{bmatrix} \tag{3.47}$$

and

$$\tilde{\boldsymbol{Q}}_x = \begin{bmatrix} \boldsymbol{Q}_x \\ \boldsymbol{F} \end{bmatrix}, \tag{3.48}$$

respectively.

It is noted that the simplified expression for the response acceleration given by Eq. (3.45) facilitates significantly (e.g. Ref. [41]) an efficient state variable formulation of the original equations of motion. Overall, the augmented system of equations can be concisely written in the alternative form

$$\bar{\mathrm{M}}_x \ddot{\boldsymbol{x}} + \bar{\mathrm{C}}_x \dot{\boldsymbol{x}} + \bar{\mathrm{K}}_x \boldsymbol{x} = \bar{\boldsymbol{Q}}_x(t) \tag{3.49}$$

where $\bar{\mathrm{M}}_x$, $\bar{\mathrm{C}}_x$ and $\bar{\mathrm{K}}_x$ denote the $(m+l) \times l$ augmented mass, damping and stiffness matrices and $\bar{\boldsymbol{Q}}_x$ denotes the $(m+l)$ augmented excitation vector. The augmented mass matrix is given by Eq. (3.33), whereas the augmented damping and stiffness matrices are given by

$$\bar{\mathrm{C}}_x = \begin{bmatrix} \tilde{\mathrm{A}}\mathrm{C}_x \\ \mathbf{E} \end{bmatrix} \tag{3.50}$$

and

$$\bar{\mathrm{K}}_x = \begin{bmatrix} \tilde{\mathrm{A}}\mathrm{K}_x \\ \mathbf{L} \end{bmatrix}, \tag{3.51}$$

respectively. The $(m+l)$ vector $\bar{\boldsymbol{Q}}_x$ is given by

$$\bar{\boldsymbol{Q}}_x = \begin{bmatrix} \tilde{\mathrm{A}}\boldsymbol{Q}_x \\ \boldsymbol{F} \end{bmatrix}. \tag{3.52}$$

3.3 Derivation of the response statistics for linear systems with singular matrices

3.3.1 Time domain analysis methodology

3.3.1.1 Moore-Penrose state variable formulation

In a similar manner as in the standard state-variable formulation included in Section 3.1.1, the augmented system of Eq. (3.49) can be cast into the state variable form by defining a $2l$ vector,

$$\boldsymbol{p}(t) = \begin{bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{bmatrix}. \tag{3.53}$$

Taking into account Eqs. (3.45) and (3.53), Eq. (3.49) becomes

$$\dot{\boldsymbol{p}} = \mathrm{G}_x \boldsymbol{p} + \boldsymbol{f}_x, \tag{3.54}$$

where the matrix of the system equations of motion state variable form is given by

$$\mathbf{G}_{\boldsymbol{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\bar{\mathbf{M}}_{\boldsymbol{x}}^{+}\bar{\mathbf{K}}_{\boldsymbol{x}} & -\bar{\mathbf{M}}_{\boldsymbol{x}}^{+}\bar{\mathbf{C}}_{\boldsymbol{x}} \end{bmatrix}, \tag{3.55}$$

and the corresponding excitation vector by

$$\boldsymbol{f}_{\boldsymbol{x}} = \begin{bmatrix} \mathbf{0} \\ \bar{\mathbf{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{Q}}_{\boldsymbol{x}} + (\mathbf{I} - \bar{\mathbf{M}}_{\boldsymbol{x}}^{+}\bar{\mathbf{M}}_{\boldsymbol{x}})\boldsymbol{y} \end{bmatrix}. \tag{3.56}$$

Setting $\boldsymbol{f}_{\boldsymbol{x}} = 0$, Eq. (3.54) becomes homogeneous and its general solution is given by

$$\boldsymbol{p}(t) = \exp(\mathbf{G}_{\boldsymbol{x}}t)\boldsymbol{p}(0), \tag{3.57}$$

where the $2l \times 2l$ matrix $\exp(\mathbf{G}_{\boldsymbol{x}}t)$ represents the transition matrix for the system.

Based on the solution of the homogeneous equation, the response to a non-zero forcing, $\boldsymbol{f}_{\boldsymbol{x}}$, is given by

$$\boldsymbol{p(t)} = \exp(\mathbf{G}_{\boldsymbol{x}}t)\boldsymbol{p}(0) + \int_0^t \exp[\mathbf{G}_{\boldsymbol{x}}(t-\tau)]\boldsymbol{f}_{\boldsymbol{x}}(\tau)d\tau, \tag{3.58}$$

which, under the assumption that $\boldsymbol{p}(0) = 0$, becomes

$$\boldsymbol{p}(t) = \int_0^t \exp(\mathbf{G}_{\boldsymbol{x}}\tau)\boldsymbol{f}_{\boldsymbol{x}}(t-\tau)d\tau. \tag{3.59}$$

Eq. (3.59) is a convolution integral between the input $\boldsymbol{f}_{\boldsymbol{x}}(t)$ and the output $\boldsymbol{p}(t)$. Defining

$$\exp(\mathbf{G}_{\boldsymbol{x}}t) = \begin{bmatrix} \mathbf{a}_{\boldsymbol{x}}(t) & \mathbf{b}_{\boldsymbol{x}}(t) \\ \mathbf{c}_{\boldsymbol{x}}(t) & \mathbf{d}_{\boldsymbol{x}}(t) \end{bmatrix}, \tag{3.60}$$

Eq. (3.59) yields

$$\begin{bmatrix} \boldsymbol{x}(t) \\ \dot{\boldsymbol{x}}(t) \end{bmatrix} = \int_0^t \begin{bmatrix} \mathbf{a}_{\boldsymbol{x}}(\tau) & \mathbf{b}_{\boldsymbol{x}}(\tau) \\ \mathbf{c}_{\boldsymbol{x}}(\tau) & \mathbf{d}_{\boldsymbol{x}}(\tau) \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{H}(t-\tau) \end{bmatrix} d\tau, \tag{3.61}$$

where

$$\boldsymbol{H}(t-\tau) = \bar{\mathbf{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{Q}}_{\boldsymbol{x}}(t-\tau) + (\mathbf{I} - \bar{\mathbf{M}}_{\boldsymbol{x}}^{+}\bar{\mathbf{M}}_{\boldsymbol{x}})\boldsymbol{y}(t-\tau). \tag{3.62}$$

Taking into account Eqs. (3.61) and (3.62) yields

$$\boldsymbol{x}(t) = \int_0^t \mathbf{b}_{\boldsymbol{x}}(\tau)\boldsymbol{H}(t-\tau)d\tau. \tag{3.63}$$

It is deemed appropriate to make two remarks. As noted also in Ref. [112] the expression of Eq. (3.45) for the response acceleration vector is not unique. As shown in Eq. (3.63), the response displacement vector $\boldsymbol{x}(t)$ is not unique, as well. Due to the fact that, in general, for systems of the form of Eq. (3.1) a unique response displacement/acceleration vector is experimentally observed, it is reasonable to apply conditions so that the system response is uniquely defined. As shown in Ref. [112], in case that the $(m+l)\times l$ matrix $\bar{\mathbf{M}}_x$ has full rank, i.e. $rank(\bar{\mathbf{M}}_x)=l$, yields

$$\bar{\mathbf{M}}_x^+ = (\bar{\mathbf{M}}_x^{\mathrm{T}}\bar{\mathbf{M}}_x)^{-1}\bar{\mathbf{M}}_x^{\mathrm{T}}, \tag{3.64}$$

so that

$$(\mathbf{I}-\bar{\mathbf{M}}_x^+\bar{\mathbf{M}}_x)=\mathbf{0}. \tag{3.65}$$

Eq. (3.63) can be equivalently written as

$$\boldsymbol{x}(t)=\int_0^t \mathbf{h}_x(\tau)\bar{\boldsymbol{Q}}_x(t-\tau)d\tau, \tag{3.66}$$

where

$$\mathbf{h}_x(t)=\mathbf{b}_x(t)\bar{\mathbf{M}}_x^+ \tag{3.67}$$

can be considered as the uniquely defined "generalized" impulse response matrix. Considering next the expression derived in case $\bar{\mathbf{M}}_x$ has full rank, ie. Eq. (3.65), Eq. (3.49) and Eq. (3.56) become

$$\ddot{\boldsymbol{x}}=\bar{\mathbf{M}}_x^+(-\bar{\mathbf{C}}_x\dot{\boldsymbol{x}}-\bar{\mathbf{K}}_x\boldsymbol{x}+\bar{\boldsymbol{Q}}_x) \tag{3.68}$$

and

$$\boldsymbol{f}_x=\begin{bmatrix}\mathbf{0}\\ \bar{\mathbf{M}}_x^+\bar{\boldsymbol{Q}}_x\end{bmatrix}, \tag{3.69}$$

respectively.

For the determination of the system response statistical moments, in a similar manner as in Section 3.1.1, taking expectations on Eq. (3.54) yields an equation for the system response mean vector in the form

$$\dot{\boldsymbol{m}}_x=\mathbf{G}_x\boldsymbol{m}_x+\boldsymbol{m}_{f_x}. \tag{3.70}$$

The corresponding equation for the system response covariance matrix becomes

$$\dot{\mathbf{V}}_x=\mathbf{G}_x\mathbf{V}_x+\mathbf{V}_x\mathbf{G}_x^{\mathrm{T}}+\mathbf{S}_x, \tag{3.71}$$

where

$$\mathbf{S}_{\boldsymbol{x}} = \int_0^t \exp(\mathbf{G}_{\boldsymbol{x}}(t-\tau)) \left[\mathbf{w}_{\boldsymbol{\eta}_{\boldsymbol{x}}}(t,\tau) + \mathbf{w}^{\mathrm{T}}_{\boldsymbol{\eta}_{\boldsymbol{x}}}(t,\tau)\right] d\tau. \tag{3.72}$$

For the case where the elements of $\boldsymbol{\eta}_{\boldsymbol{x}}$ are regarded to be stationary white noises, Eq. (3.71) becomes

$$\dot{\mathbf{V}}_{\boldsymbol{x}} = \mathbf{G}_{\boldsymbol{x}}\mathbf{V}_{\boldsymbol{x}} + \mathbf{V}_{\boldsymbol{x}}\mathbf{G}^{\mathrm{T}}_{\boldsymbol{x}} + \mathbf{D}_{\boldsymbol{x}}, \tag{3.73}$$

where $\mathbf{D}_{\boldsymbol{x}}$ is a real, symmetric, non-negative matrix of constants and $\mathbf{G}_{\boldsymbol{x}}$, $\boldsymbol{f}_{\boldsymbol{x}}$ are given by Eq. (3.55) and Eq. (3.69), respectively. Focusing on the system stationary response, i.e.

$$\dot{\mathbf{V}}_{\boldsymbol{x}} = \mathbf{0}, \tag{3.74}$$

Eq. (3.73) becomes

$$\mathbf{G}_{\boldsymbol{x}}\mathbf{V}_{\boldsymbol{x}} + \mathbf{V}_{\boldsymbol{x}}\mathbf{G}^{\mathrm{T}}_{\boldsymbol{x}} + \mathbf{D}_{\boldsymbol{x}} = \mathbf{0}. \tag{3.75}$$

Eq. (3.75) is a Lyapunov equation which is a special case of the Sylvester equation of the form

$$\mathbf{AX} + \mathbf{XB} + \mathbf{Q} = \mathbf{0}. \tag{3.76}$$

The Sylvester Eq. (3.76) has a unique solution if and only if the matrices $\mathbf{A}$ and $-\mathbf{B}$ have no common eigenvalues [15]. A sufficient condition for Eq. (3.75) to have a unique solution is none of the eigenvalues of the matrix $\mathbf{G}_{\boldsymbol{x}}$ are equal to zero or, equivalently, the rows of $\mathbf{G}_{\boldsymbol{x}}$ are linearly independent with each other (for more details see Ref. [22]). However, due to the fact that more than the minimum number of coordinates are used for the system modeling, most likely the rows of $\mathbf{G}_{\boldsymbol{x}}$ will not be linearly independent; thus, a special treatment is needed for solving Eq. (3.75).

Eq. (3.75) can be rewritten as

$$(\mathbf{I}_{2l} \otimes \mathbf{G}_{\boldsymbol{x}} + \mathbf{G}_{\boldsymbol{x}} \otimes \mathbf{I}_{2l}) vec\mathbf{V}_{\boldsymbol{x}} = vec(-\mathbf{D}_{\boldsymbol{x}}), \tag{3.77}$$

where $vec\mathbf{V}_{\boldsymbol{x}}$ and $vec(-\mathbf{D}_{\boldsymbol{x}})$ are $(2l)^2$ vectors formed by stacking all columns of $\mathbf{V}_{\boldsymbol{x}}$ and $-\mathbf{D}_{\boldsymbol{x}}$ respectively, on top of one another; also, $\mathbf{I}_{2l}\otimes\mathbf{G}_{\boldsymbol{x}}$ and $\mathbf{G}_{\boldsymbol{x}}\otimes\mathbf{I}_{2l}$ denote the Kronecker products of the pairs of matrices $\mathbf{I}_{2l}, \mathbf{G}_{\boldsymbol{x}}$ and $\mathbf{G}_{\boldsymbol{x}}, \mathbf{I}_{2l}$, respectively. Equivalently, Eq. (3.77) is expressed in the form

$$\mathbf{W}\boldsymbol{v} = \boldsymbol{d}, \tag{3.78}$$

where $\mathbf{W}$ is a $(2l)^2 \times (2l)^2$ matrix, and the vectors $\boldsymbol{v}$ and $\boldsymbol{d}$ are given by

$$\boldsymbol{v} = vec(\mathbf{V}_{\boldsymbol{x}}) \tag{3.79}$$

and

$$\boldsymbol{d} = vec(-\mathbf{D}_{\boldsymbol{x}}), \tag{3.80}$$

respectively. Involving Eq. (2.12) for the M-P inverse of a matrix, the general solution to Eq. (3.78) is

$$\boldsymbol{v} = \mathbf{W}^{+}\boldsymbol{d} + (\mathbf{I}_{(2l)^2} - \mathbf{W}^{+}\mathbf{W})\boldsymbol{y}, \tag{3.81}$$

where $\boldsymbol{y}$ is an arbitrary $(2l)^2$ vector.

3.3.1.2 Moore-Penrose state variable formulation – a numerical example

As a numerical example of the M-P state variable formulation that is proposed in Section 3.3.1.1, consider the system of two rigid masses m_1 and m_2 in Figure 3.1. The masses move horizontally as a result of an applied random force $Q_2(t)$. Let the mass m_1 be connected to the foundation by a linear spring and a linear damper with coefficients k_1 and c_1, respectively. A mass m_2 is connected to m_1 by a linear spring and a linear damper with coefficients k_2 and c_2, respectively. $Q_2(t)$ is a white-noise process with a correlation function $w_{Q_2(t)} = 2\pi S_0 \delta(\tau)$, where S_0 is the (constant) power spectrum value for $Q_2(t)$. Note in passing that the white noise excitation assumption is introduced in this example only to simplify the related calculations. It is emphasized that the herein developed framework can readily handle excitations with arbitrary autocorrelation function forms; see Eq. (3.71) and Eq. (3.72). q_1, q_2 are the generalized displacements, shown in Figure 3.1.

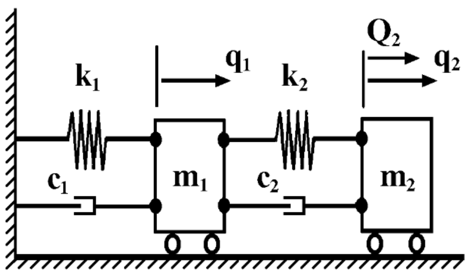

FIGURE 3.1: A two degree-of-freedom linear structural system under stochastic excitation.

The equations of motion governing the system in Figure 3.1 can be written in the matrix form of Eq. (3.1), where the matrices $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ are given by (see also Ref. [91])

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}. \tag{3.82}$$

The coordinates and system excitation vectors are given by

$$\boldsymbol{x} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{3.83}$$

and

$$\boldsymbol{Q} = \begin{bmatrix} 0 \\ Q_2(t) \end{bmatrix}, \tag{3.84}$$

respectively. Eq. (3.19) is formed; thus, obtaining a system of algebraic equations to be solved for the 16 unknowns of matrix $\mathbf{V}$ (in reality there are 10 unknowns due to symmetry). Focusing on the stationary system response, i.e. $\dot{\mathbf{V}} = \mathbf{0}$, and considering the parameters values $m_1 = m_2 = m = 1, c_1 = c_2 = c = 0.1, k_1 = k_2 = k = 1$ and $S_0 = 10^{-3}$, numerical solution of the Lyapunov Eq. (3.19) yields

$$\mathbf{V} = \begin{bmatrix} 0.0438 & 0.0690 & 0.0000 & -0.0012 \\ 0.0690 & 0.1132 & 0.0012 & 0.0000 \\ 0.0000 & 0.0012 & 0.0188 & 0.0251 \\ -0.0012 & 0.0000 & 0.0251 & 0.0441 \end{bmatrix}. \tag{3.85}$$

Consider next the system of two masses m_1 and m_2 of the system in Figure 3.1 modeled as a multi-body one, and consisting of two separate subsystems as shown in Figure 3.2; see also Ref. [112]. The two sub-systems are related based on the constraint

$$x_2 = x_1 + d, \tag{3.86}$$

where d is the length of mass m_1. The "unconstrained" equations of motion are derived by treating the three coordinates ($\bar{x}_1, x_2$ and $\bar{x}_3$) as independent with each other. The equation of motion of the composite system is derived by including the constraint of Eq. (3.86), or, equivalently written as

$$x_2 = \bar{x}_1 + l_{10} + d, \tag{3.87}$$

where l_{10} is the unstretched length of the spring k_1.

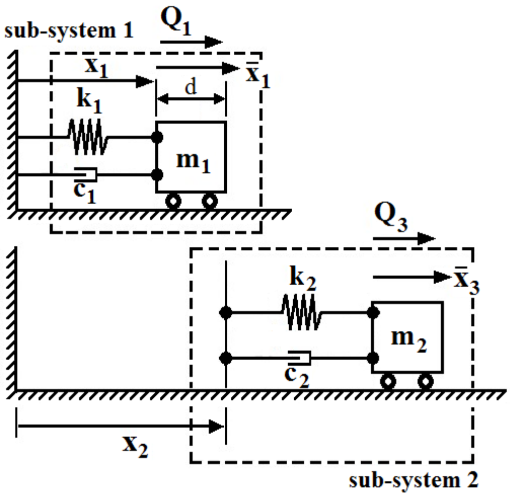

FIGURE 3.2: A three degree-of-freedom linear structural system under stochastic excitation.

The total kinetic and potential energies for the two sub-systems are

$$T = \frac{1}{2}m_1\dot{\bar{x}}_1^2 + \frac{1}{2}m_2(\dot{x}_2 + \dot{\bar{x}}_3)^2 \tag{3.88}$$

and

$$V = \frac{1}{2}k_1\bar{x}_1^2 + \frac{1}{2}k_2\bar{x}_3^2, \tag{3.89}$$

respectively. By forming the Lagrangian function

$$\boldsymbol{L}(\bar{x}_1, x_2, \bar{x}_3, \dot{\bar{x}}_1, \dot{x}_2, \dot{\bar{x}}_3) = \boldsymbol{T} - \boldsymbol{V}, \tag{3.90}$$

and utilizing the Euler-Lagrange equations [68], yields

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{L}}{\partial \dot{x}_i}\right) - \frac{\partial \boldsymbol{L}}{\partial x_i} - \frac{\partial \boldsymbol{F}}{\partial \dot{x}_i} = \boldsymbol{Q}, \tag{3.91}$$

where

$$\boldsymbol{F}(\bar{x}_i, \dot{\bar{x}}_i) = -\frac{1}{2}c_i\dot{\bar{x}}_i^2 \tag{3.92}$$

is the damping force ($i = 1, 3$) and $\boldsymbol{Q}$ the external excitation. Manipulating Eq. (3.91) yields

$$m_1\ddot{\bar{x}}_1 + c_1\dot{\bar{x}}_1 + k_1\bar{x}_1 = 0, \tag{3.93}$$

$$m_2\ddot{x}_2 + m_2\ddot{\bar{x}}_3 = Q_3, \tag{3.94}$$

$$m_2\ddot{x}_2 + m_2\ddot{\bar{x}}_3 + c_2\dot{\bar{x}}_3 + k_2\bar{x}_3 = Q_3, \tag{3.95}$$

where

$$\bar{x}_1 = x_1 - l_{10} \tag{3.96}$$

and

$$\bar{x}_3 = x_3 - l_{20}. \tag{3.97}$$

In Eqs. (3.96)-(3.97), l_{10}, l_{20} denote the unstretched lengths of the springs k_1 and k_2.

The matrix form for the equations of motion takes the form given in Eq. (3.20), where

$$\mathbf{M}_{\boldsymbol{x}} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & m_2 \\ 0 & m_2 & m_2 \end{bmatrix}, \quad \mathbf{C}_{\boldsymbol{x}} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_2 \end{bmatrix}, \quad \mathbf{K}_{\boldsymbol{x}} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_2 \end{bmatrix}, \tag{3.98}$$

$$\boldsymbol{Q}_{\boldsymbol{x}} = \begin{bmatrix} 0 \\ Q_3 \\ Q_3 \end{bmatrix} \tag{3.99}$$

and

$$\boldsymbol{x} = \begin{bmatrix} \bar{x}_1 \\ x_2 \\ \bar{x}_3 \end{bmatrix}. \tag{3.100}$$

Differentiating the constraint of Eq. (3.87) the two sub-systems are subject to, yields

$$\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\bar{x}}_1 \\ \ddot{x}_2 \\ \ddot{\bar{x}}_3 \end{bmatrix} = 0. \tag{3.101}$$

The $m \times l$ matrix $\mathbf{A}$ and the m vector $\boldsymbol{b}$ in Eq. (3.21) take the form

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \tag{3.102}$$

and

$$b = 0, \tag{3.103}$$

respectively. As in the previous example, assume that $m_1 = m_2 = m = 1, c_1 = c_2 = c = 0.1, k_1 = k_2 = k = 1$ and Q_3 is a white noise excitation with power spectrum amplitude $S_0 = 10^{-3}$. Note that $rank(\bar{\mathbf{M}}_{\boldsymbol{x}}) = 3$, i.e. the 4×3 matrix $\bar{\mathbf{M}}_{\boldsymbol{x}}$ has full rank.

Hence, Eq. (3.33) becomes

$$\bar{\mathbf{M}}_{x} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \tag{3.104}$$

whereas Eqs. (3.50)-(3.52) become

$$\bar{\mathbf{C}}_{x} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\mathbf{K}}_{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\boldsymbol{Q}}_{x} = \begin{bmatrix} 0.5w(t) \\ 0.5w(t) \\ w(t) \\ 0 \end{bmatrix}. \tag{3.105}$$

The M-P inverse of the matrix $\bar{\mathbf{M}}_{x}$ is

$$\bar{\mathbf{M}}_{x}^{+} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 2 & 1 \end{bmatrix}. \tag{3.106}$$

Substituting Eq. (3.106) into Eq. (3.55) yields

$$\mathbf{G}_{x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -0.1 & 0 & 0.1 \\ -1 & 0 & 1 & -0.1 & 0 & 0.1 \\ 1 & 0 & -2 & 0.1 & 0 & -0.2 \end{bmatrix}. \tag{3.107}$$

Focusing on the stationary system response, i.e. $\dot{\mathbf{V}}_{x} = \mathbf{0}$, and considering that Q_3 is a white-noise excitation, the matrix $\mathbf{D}_{x}$ in Eq. (3.75), takes the form

$$\mathbf{D}_{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\pi 10^{-3} \end{bmatrix}, \tag{3.108}$$

whereas the Lyapunov Eq. (3.75) becomes

$$\mathbf{G}_{\boldsymbol{x}}\mathbf{V}_{\boldsymbol{x}} + \mathbf{V}_{\boldsymbol{x}}\mathbf{G}_{\boldsymbol{x}}^{\mathrm{T}} = -\mathbf{D}_{\boldsymbol{x}}. \tag{3.109}$$

Note that due to the fact that not all rows of $\mathbf{G}_{\boldsymbol{x}}$ are linearly independent with each other (compare the fourth and the fifth rows), the solution of Eq. (3.109) is not unique. Following the Kronecker product approach described in Eqs. (3.77)-(3.81) yields

$$\mathbf{V}_{\boldsymbol{x}} = \begin{bmatrix} 0.0438 & 0.0438 & 0.0252 & 0 & 0 & -0.0012 \\ 0.0438 & y_1 & 0.0252 & 0 & 0 & -0.0012 \\ 0.0252 & 0.0252 & 0.0190 & 0.0012 & 0.0012 & 0 \\ 0 & 0 & 0.0012 & 0.0188 & 0.0188 & 0.0063 \\ 0 & 0 & 0.0012 & 0.0188 & 0.0188 & 0.0063 \\ -0.0012 & -0.0012 & 0 & 0.0063 & 0.0063 & 0.0127 \end{bmatrix}. \tag{3.110}$$

Note also that almost all the elements of the matrix $(\mathbf{I}_{(2l)^2} - \mathbf{W}^{+}\mathbf{W})$ in Eq. (3.81) are zero. Interestingly, the only non-zero one is the element in the diagonal in the position (2,2) corresponding to the additional auxiliary DOF x_2. The only element of $\mathbf{V}_{\boldsymbol{x}}$ affected is the element $\mathbf{V}_{\boldsymbol{x}}(2,2) = \mathbb{E}(x_2^2)$. Hence, the presence of the arbitrary vector $\boldsymbol{y}$ does not affect, in essence, the calculated $\mathbf{V}_{\boldsymbol{x}}$.

Comparing Eqs. (3.85) and (3.110), the variance $\mathbb{E}[q_1^2]$ as well as the variance $\mathbb{E}[\dot{q}_1^2]$ obtained in the first example, coincide with the respective ones in the second one, i.e $\mathbb{E}[\bar{x}_1^2]$ and $\mathbb{E}[\dot{\bar{x}}_1^2]$. Taking expectations in the equation that connects the two reference systems, that is

$$\bar{x_3} = q_2 - q_1, \tag{3.111}$$

and utilizing Eq. (3.85) yields

$$\begin{aligned} \mathbb{E}[\bar{x}_3^2] &= \mathbb{E}[q_2^2] + \mathbb{E}[q_1^2] - 2\mathbb{E}[q_1 q_2] \\ &= 0.0190 \end{aligned} \tag{3.112}$$

and

$$\begin{aligned} \mathbb{E}[\dot{\bar{x}}_3^2] &= \mathbb{E}[\dot{q}_2^2] + \mathbb{E}[\dot{q}_1^2] - 2\mathbb{E}[\dot{q}_1 \dot{q}_2] \\ &= 0.0127, \end{aligned} \tag{3.113}$$

which are indeed in agreement with the corresponding values in Eq. (3.110). It can be readily verified that the rest of the elements of the matrix given by Eq. (3.110) are also in agreement with the respective ones of Eq. (3.85).

3.3.1.3 Complex modal analysis

In the standard formulation of the linear random vibration theory, computing the "complex modal matrix" whose columns are the eigenvectors, or "complex modes" of matrix $\mathbf{G}$ of Eq. (3.4) facilitates not only the efficient evaluation of $\exp(\mathbf{G}t)$ in Eq. (3.8), and thus, of the system impulse response matrix of Eq. (3.7), but also plays an instrumental role in decoupling the original coupled system of equations (Eq. (3.1)); see for example Refs [91, 38, 60, 16]. In this section it is shown that a similar treatment of the system of Eq. (3.49) does not yield in general a decoupling of the equations of motion. However, a singular value decomposition treatment of matrix $\mathbf{G}_{\boldsymbol{x}}$ of Eq. (3.55) facilitates the efficient computation of the system response statistics.

Let $\lambda_1, \lambda_2, \ldots, \lambda_{2l}$ be the eigenvalues of the $2l \times 2l$ matrix $\mathbf{G}_{\boldsymbol{x}}$ given by Eq. (3.55), so that the first r of them are non zero and the remaining $2l - r$ are equal to zero. The eigen-decomposition of $\mathbf{G}_{\boldsymbol{x}}$ yields

$$\mathbf{G}_{\boldsymbol{x}}\mathbf{T} = \mathbf{T}\boldsymbol{\eta}_{\boldsymbol{x}}, \tag{3.114}$$

where $\boldsymbol{\eta}_{\boldsymbol{x}}$ is the diagonal matrix given by

$$\boldsymbol{\eta}_{\boldsymbol{x}} = diag(\lambda_1, \lambda_2, \ldots, \lambda_r, 0, 0, \ldots, 0), \tag{3.115}$$

and

$$\mathbf{T} = [\boldsymbol{v}_1\ \boldsymbol{v}_2 \ldots \boldsymbol{v}_{2l}] \tag{3.116}$$

is the $2l$ "complex modal matrix" formed by the eigenvectors of $\mathbf{G}_{\boldsymbol{x}}$. Due to the presence of zero eigenvalues, the eigenvectors are not linearly independent, which means that the matrix $\mathbf{T}$ is singular. The singular value decomposition (SVD) of $\mathbf{G}_{\boldsymbol{x}}$, yields

$$\mathbf{G}_{\boldsymbol{x}} = \mathbf{U}\boldsymbol{\eta}_{\boldsymbol{x}}\boldsymbol{\Psi}^*, \tag{3.117}$$

where the matrix $\boldsymbol{\eta}_{\boldsymbol{x}}$ is $2l \times 2l$ diagonal of the form

$$\boldsymbol{\eta}_{\boldsymbol{x}} = diag(\sigma_1, \sigma_2, \ldots, \sigma_r, 0, 0, \ldots, 0). \tag{3.118}$$

In Eq. (3.118),

$$\sigma_j = \sqrt{\lambda_j}, \tag{3.119}$$

for $j = 1, 2, \ldots, 2l$, denote the singular values of the matrix $\mathbf{G}_{\boldsymbol{x}}$. The $2l \times 2l$ matrix $\boldsymbol{\Psi} = \left[\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_{2l}\right]$ is unitary, i.e. $\boldsymbol{\Psi}\boldsymbol{\Psi}^* = \boldsymbol{\Psi}^*\boldsymbol{\Psi} = \mathbf{I}$, where "$*$" denotes the conjugate transpose and $\boldsymbol{\psi}_j$ is an eigenvector corresponding to each singular value σ_j for $j =$

$1, 2, \ldots, 2l$. $\mathbf{U} = \left[\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{2l}\right]$ is a $2l \times 2l$ unitary matrix, i.e. $\mathbf{UU}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}$, and each one of the $2l-$vectors $\boldsymbol{u}_j$ is equal to

$$\boldsymbol{u}_j = \frac{\mathbf{G}_x \boldsymbol{\psi}_j}{\sigma_j}, \tag{3.120}$$

for $j = 1, 2, \ldots, 2l$.

To determine the impulse response matrix, $\mathbf{h}_x(t)$, of Eq. (3.67) the matrix $\exp(\mathbf{G}_x t)$ of Eq. (3.60) has to be evaluated first.

The transformation

$$\boldsymbol{p} = \boldsymbol{\Psi} \boldsymbol{z}_x, \tag{3.121}$$

is used and the state variable form of Eq. (3.54) becomes

$$\dot{\boldsymbol{z}}_x = \boldsymbol{\Psi}^* \mathbf{G}_x \boldsymbol{\Psi} \boldsymbol{z}_x + \boldsymbol{\Psi}^* \boldsymbol{f}_x. \tag{3.122}$$

Taking into consideration Eq. (3.117), Eq. (3.122) can be rewritten as

$$\dot{\boldsymbol{z}}_x = \boldsymbol{\Psi}^* \mathbf{U} \boldsymbol{\eta}_x \boldsymbol{z}_x + \boldsymbol{g}_x, \tag{3.123}$$

where

$$\boldsymbol{g}_x = \boldsymbol{\Psi}^* \boldsymbol{f}_x. \tag{3.124}$$

It is critical to note that due to the form of Eq. (3.123) the equations of motion cannot be decoupled. Unlike a standard complex modal analysis (e.g. Ref. [91]) utilizing the minimum number of degrees of freedom, the formulation herein yields a matrix $\mathbf{G}_x$ with some of its eigenvalues being zero. As a result, not all the eigenvectors forming the "complex modal matrix" $\mathbf{T}$ are linearly independent with each other; thus, leading to inability to perform a standard eigenvalue decomposition of $\mathbf{G}_x$ (see Eq. (3.114)). In other words, the matrix $\boldsymbol{\Psi}^*\mathbf{U}$ in Eq. (3.123) cannot be a unitary matrix rendering the system of coupled equations of Eq. (3.123) an uncoupled one. Overall, in contrast to a standard analysis/modeling where a complex modal analysis yields an uncoupled system of equations, this is not possible when utilizing more than the minimum number of degrees-of-freedom. Nevertheless, it is shown in the ensuing analysis that relying on an SVD of the matrix $\mathbf{G}_x$ greatly facilitates the numerical computation of the system response statistics.

Proceeding with the analysis, Eq. (3.54) has been cast into Eq. (3.123), which has the general solution

$$\boldsymbol{z}_{\boldsymbol{x}}(t) = \exp(\boldsymbol{\Psi}^{*}\mathbf{U}\boldsymbol{\eta}_{\boldsymbol{x}}t)\boldsymbol{z}_{\boldsymbol{x}}(0) + \int_{0}^{t} \exp(\boldsymbol{\Psi}^{*}\mathbf{U}\boldsymbol{\eta}_{\boldsymbol{x}}(t-\tau))\boldsymbol{g}_{\boldsymbol{x}}(\tau)d\tau. \tag{3.125}$$

Under the assumption that the system is initially at rest, Eq. (3.125) becomes

$$\boldsymbol{z}_{\boldsymbol{x}}(t) = \int_{0}^{t} \mathbf{H}_{\boldsymbol{x}}(s)\boldsymbol{g}_{\boldsymbol{x}}(t-s)ds, \tag{3.126}$$

where $\mathbf{H}_{\boldsymbol{x}}(t)$ is given by

$$\mathbf{H}_{\boldsymbol{x}}(t) = \exp(\boldsymbol{\Psi}^{*}\mathbf{U}\boldsymbol{\eta}_{\boldsymbol{x}}t). \tag{3.127}$$

Once $\boldsymbol{z}_{\boldsymbol{x}}$ is computed, the $2l$ vector $\boldsymbol{p}$ can be determined by using the transformation given by Eq. (3.121).

Taking expectation on Eq. (3.123), then taking into account Eq. (3.124) and considering the stationary response, i.e. $\dot{\boldsymbol{m}}_{\boldsymbol{z}_{\boldsymbol{x}}} = \mathbf{0}$, the equation

$$\boldsymbol{\eta}_{\boldsymbol{x}}\boldsymbol{m}_{\boldsymbol{z}_{\boldsymbol{x}}} = -\mathbf{U}^{*}\boldsymbol{m}_{\boldsymbol{f}_{\boldsymbol{x}}}, \tag{3.128}$$

arises. Taking into account Eq. (2.12), Eq. (3.128) has the general solution

$$\boldsymbol{m}_{\boldsymbol{z}_{\boldsymbol{x}}} = -\boldsymbol{\eta}_{\boldsymbol{x}}^{+}\mathbf{U}^{*}\boldsymbol{m}_{\boldsymbol{f}_{\boldsymbol{x}}} + (\mathbf{I}_{2l} - \boldsymbol{\eta}_{\boldsymbol{x}}^{+}\boldsymbol{\eta}_{\boldsymbol{x}})\boldsymbol{y}. \tag{3.129}$$

In Eq. (3.129), $\boldsymbol{\eta}_{\boldsymbol{x}}^{+}$ is the M-P inverse of $\boldsymbol{\eta}_{\boldsymbol{x}}$ and $\boldsymbol{y}$ is an arbitrary $2l$ vector. Also, using Eq. (3.121), the expression

$$\boldsymbol{m}_{\boldsymbol{p}} = -\boldsymbol{\Psi}\boldsymbol{\eta}_{\boldsymbol{x}}^{+}\mathbf{U}^{*}\boldsymbol{m}_{\boldsymbol{f}_{\boldsymbol{x}}} + \boldsymbol{\Psi}(\mathbf{I}_{2l} - \boldsymbol{\eta}_{\boldsymbol{x}}^{+}\boldsymbol{\eta}_{\boldsymbol{x}})\boldsymbol{y}, \tag{3.130}$$

is obtained, where $\mathbf{U}$, $\boldsymbol{\Psi}$ are the SVD unitary matrices. Regarding the determination of the Moore-Penrose inverse of the $2l \times 2l$ matrix $\boldsymbol{\eta}_{\boldsymbol{x}}$, this is given by

$$\sigma_j = \begin{cases} \sigma_j^{-1} & , \text{ if } \sigma_j \neq 0 \\ 0 & , \text{ if } \sigma_j = 0 \end{cases} ; \tag{3.131}$$

see Corollary 2.7.

The covariance matrices of the transformed state vector $\boldsymbol{z}_{\boldsymbol{x}}$ and the $2l$ vector $\boldsymbol{g}_{\boldsymbol{x}}$ given by Eq. (3.126) and Eq. (3.124), respectively, can be easily related as follows.

Defining the covariance matrix of $\boldsymbol{z_x}$ as

$$\mathbf{w}_{\boldsymbol{z_x}}(\tau) = \mathbb{E}\left[(\boldsymbol{z_x}(\tau) - \boldsymbol{m}_{\boldsymbol{z_x}})(\boldsymbol{z_x}(t+\tau) - \boldsymbol{m}_{\boldsymbol{z_x}})^*\right] \tag{3.132}$$

and the covariance matrix of $\boldsymbol{g_x}$ as

$$\mathbf{w}_{\boldsymbol{g_x}}(\tau) = \mathbb{E}\left[(\boldsymbol{g_x}(\tau) - \boldsymbol{m}_{\boldsymbol{g_x}})(\boldsymbol{g_x}(t+\tau) - \boldsymbol{m}_{\boldsymbol{g_x}})^*\right], \tag{3.133}$$

and considering Eq. (3.126), the covariance input-output relationship is given by

$$\mathbf{w}_{\boldsymbol{z_x}}(\tau) = \int_0^\infty \int_0^\infty \mathbf{H}_{\boldsymbol{x}}(s_1)\mathbf{w}_{\boldsymbol{g_x}}(\tau + s_1 - s_2)\mathbf{H}_{\boldsymbol{x}}^*(s_2)\, ds_1 ds_2, \tag{3.134}$$

where $\mathbf{H}_{\boldsymbol{x}}(t)$ is given by Eq. (3.127).

As far as the determination of the elements of $\mathbf{H}_{\boldsymbol{x}}(t)$ is concerned, the Cayley-Hamilton theorem can be employed yielding [82]

$$\begin{aligned} \mathbf{H}_{\boldsymbol{x}}(t) &= \exp(\boldsymbol{\Psi}^*\mathbf{U}\boldsymbol{\eta_x}t) \\ &= \sum_{k=0}^{r-1} \alpha_k (\boldsymbol{\Psi}^*\mathbf{U}\boldsymbol{\eta_x})^k. \end{aligned} \tag{3.135}$$

The coefficients $\alpha_k,\ k = 1, 2, \ldots, r-1$ can be found by solving the following system of linear equations

$$\exp(\lambda_i) = \sum_{k=0}^{r-1} \alpha_k \lambda_i^k, \tag{3.136}$$

where $i = 1, 2, \ldots, r$ and λ_i are the eigenvalues of the matrix $\boldsymbol{\Psi}^*\mathbf{U}\boldsymbol{\eta_x}$. Using the obtained formula for the determination of the elements of $\mathbf{H}_{\boldsymbol{x}}(t)$, the elements of the covariance matrix $\mathbf{w}_{\boldsymbol{z_x}}$, can be determined.

After determining the covariance matrix $\mathbf{w}_{\boldsymbol{z_x}}$, by utilizing the transformation given by Eq. (3.121), the covariance matrix $\mathbf{w}_{\boldsymbol{p}}$ can be determined as well. In this regard,

$$\mathbf{w}_{\boldsymbol{p}}(\tau) = \boldsymbol{\Psi}\mathbf{w}_{\boldsymbol{z_x}}(\tau)\boldsymbol{\Psi}^*. \tag{3.137}$$

Similarly, using Eq. (3.124), the matrices of $\boldsymbol{g_x}(t)$ and $\boldsymbol{f_x}(t)$ are related via the formula

$$\mathbf{w}_{\boldsymbol{g_x}}(\tau) = \boldsymbol{\Psi}^*\mathbf{w}_{\boldsymbol{f_x}}(\tau)\boldsymbol{\Psi}. \tag{3.138}$$

Assuming that $\boldsymbol{f_x}$ is a zero-mean white noise vector process (and thus, the response mean vector processes in Eqs. (3.129)-(3.130) yield zero values) with correlation function

$$\mathbf{w}_{\boldsymbol{f_x}}(t,\tau) = \mathbf{D}_{\boldsymbol{x}}\delta(t-\tau), \tag{3.139}$$

where $\mathbf{D}_{\boldsymbol{x}}$ is a real, symmetric, non-negative matrix of constants, the covariance matrix of $\boldsymbol{g_x}$ is given by

$$\mathbf{w}_{\boldsymbol{g_x}}(\tau) = \boldsymbol{\Psi}^*\mathbf{D}_{\boldsymbol{x}}\boldsymbol{\Psi}\delta(t-\tau). \tag{3.140}$$

Note that $\mathbf{H}_{\boldsymbol{x}}(t)$ can be determined by other alternative more elegant methods than by using the Cayley-Hamilton theorem (see Ref. [29, 70, 23]). In this regard, setting

$$\mathbf{R} = \boldsymbol{\Psi}^*\mathbf{U}\boldsymbol{\eta_x}, \tag{3.141}$$

the determination of the impulse response function is equivalent to the determination of the matrix $\exp(\mathbf{R}t)$, which can be determined as a finite polynomial in $\mathbf{R}$, with analytic functions of t as coefficients. Once the eigenvalues of $\mathbf{R}$ are known, i.e. $\mu_1, \mu_2, \ldots, \mu_s$, it might be more convenient to express $\exp(\mathbf{R}t)$ in terms of polynomials in $(\mathbf{R} - \mu_i\mathbf{I})$. In the following analysis, the equations systems arising for determining the coefficient functions, are proven to be triangular, and thus, can be readily solved; see Ref. [23] for more details.

As a first step for the determination of $\exp(\mathbf{R}t)$, assume that the matrix $\mathbf{R}$ is in its Jordan form, $\mu_1, \mu_2, \ldots, \mu_s$ are its s distinct eigenvalues, and $m_i, i = 1, 2, \ldots, s$ is the algebraic multiplicity of each eigenvalue μ_i. Assume that

$$M_{\mathbf{R}}(x) = \prod_{j=1}^{s}(x-\mu_j)^{m_j}, \tag{3.142}$$

is the minimal polynomial of $\mathbf{R}$ (i.e. the monic polynomial $\boldsymbol{P}$ of least degree such that $\boldsymbol{P}(\mathbf{R}) = \mathbf{0}$). For $r \geq 0, s \geq 1$ and $1 \leq k \leq s$, let

$$H_s(r) = \left\{(a_1, a_2, \ldots, a_s) \in \mathbb{N}^s : a_i \geq 0 \text{ and } \sum_{i=1}^{s} a_i = r\right\} \tag{3.143}$$

and

$$H_s^{(k)}(r) = \left\{(a_1, a_2, \ldots, a_s) \in H_s(r) : a_k = 0\right\}. \tag{3.144}$$

The exponential of matrix $\mathbf{R}t$ is equal to

$$\exp(\mathbf{R}t) = \sum_{k=1}^{s}\left[\sum_{r=0}^{m_k-1} f_{k,r}(t)(\mathbf{R}-\mu_k\mathbf{I})^r\right]\prod_{j=1,j\neq k}^{s}(\mathbf{R}-\mu_j\mathbf{I})^{m_j}, \tag{3.145}$$

where the coefficient functions $f_{k,0}(t), f_{k,1}(t), \ldots, f_{k,m_k-1}(t)$ satisfy the equation

$$\sum_{r=0}^{i} f_{k,r}(t) \sum_{a \in H_s^{(k)}(i-r)} \prod_{j=1, j \neq k}^{s} \binom{m_j}{a_j} (\mu_k - \mu_j)^{m_j - a_j} = \exp(\mu_k t) \frac{t^i}{i!}, \tag{3.146}$$

for $i = 0, 1, \ldots, m_k - 1$ and each $k = 1, 2, \ldots, s$. It is noted that in case when the matrix $\mathbf{R}$ defined in Eq. (3.141) has district eigenvalues, the system given by Eq. (3.146) is especially simple. Nevertheless, the herein presented framework is also valid when the matrix $\mathbf{R}$ has repeated eigenvalues [23].

3.3.1.4 Complex modal analysis – a numerical example

Consider the multi-body system presented as an example in the Moore-Penrose state-variable formulation Section 3.3.1.2, where the matrix $\mathbf{G}_{\boldsymbol{x}}$ is given by Eq. (3.107). The system consists of two separate subsystems of masses m_1 and m_2, respectively, related based on the constraint given by Eq. (3.86).

Determining the SVD of $\mathbf{G}_{\boldsymbol{x}}$, the unitary matrices $\mathbf{U}$, $\boldsymbol{\Psi}$, as well as the diagonal matrix of the singular values $\boldsymbol{\eta}_{\boldsymbol{x}}$, are found to be equal to

$$\mathbf{U} = \begin{bmatrix} -0.0214 & 0.8204 & 0 & -0.5692 & -0.0507 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.0309 & 0.5685 & 0 & 0.8213 & -0.0351 & 0 \\ 0.4695 & -0.0326 & 0 & -0.0177 & -0.5274 & 0.7071 \\ 0.4695 & -0.0326 & 0 & -0.0177 & -0.5274 & -0.7071 \\ -0.7468 & -0.0410 & 0 & 0.0281 & -0.6632 & 0 \end{bmatrix}, \tag{3.147}$$

$$\boldsymbol{\Psi} = \begin{bmatrix} -0.5660 & 0.0242 & 0 & 0.0638 & 0.8216 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0.8168 & 0.0168 & 0 & -0.0921 & 0.5693 & 0 \\ -0.0638 & 0.8216 & 0 & -0.5660 & -0.0242 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.0921 & 0.5693 & 0 & 0.8168 & -0.0168 & 0 \end{bmatrix} \tag{3.148}$$

and

$$\boldsymbol{\eta}_x = \begin{bmatrix} 2.9784 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0015 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9944 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4768 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.149}$$

respectively.

Utilizing Eq. (3.141), the matrix $\mathbf{R}$ is given by

$$\mathbf{R} = \begin{bmatrix} -0.1827 & -0.0017 & 0 & 0.9912 & -0.0131 & 0 \\ -0.1176 & -0.0208 & 0 & 0.0015 & -0.3875 & 0 \\ 1.3984 & -0.0326 & 0 & -0.0176 & -0.2515 & 0 \\ -2.6207 & -0.0150 & 0 & -0.0785 & -0.1159 & 0 \\ 0.0035 & 1.0006 & 0 & 0 & -0.0180 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \tag{3.150}$$

which has the following six eigenvalues; that is,

$$\begin{aligned} \lambda_1 &= -0.1309 + 1.6127i, \\ \lambda_2 &= -0.1309 - 1.6127i, \\ \lambda_3 &= -0.0191 + 0.6177i, \\ \lambda_4 &= -0.0191 - 0.6177i, \\ \lambda_5 &= 0, \;\; \lambda_6 \;=\; 0 \end{aligned} \tag{3.151}$$

and its minimal polynomial has the form

$$M_{\mathbf{R}}(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)(x - \lambda_4)x^2. \tag{3.152}$$

Taking into consideration the analysis provided in Eqs. (3.143)-(3.146), the matrix exponential $\exp(\mathbf{R}t)$ is decomposed in the form

$$\exp(\mathbf{R}t) = \mathbf{p}_1 \exp(\lambda_1 t) + \mathbf{p}_2 \exp(\lambda_2 t) + \mathbf{p}_3 \exp(\lambda_3 t) + \mathbf{p}_4 \exp(\lambda_4 t) + \mathbf{p}_5 t + \mathbf{p}_6, \tag{3.153}$$

where the coefficients $\mathbf{p}_i$ are given by

$$\mathbf{p}_1 = \frac{\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4}{a_{12}a_{13}a_{14}\lambda_1^2}\mathbf{R}^2, \tag{3.154}$$

$$\mathbf{p}_2 = \frac{\mathbf{b}_1\mathbf{b}_3\mathbf{b}_4}{a_{21}a_{23}a_{24}\lambda_2^2}\mathbf{R}^2, \tag{3.155}$$

$$\mathbf{p}_3 = \frac{\mathbf{b}_1\mathbf{b}_2\mathbf{b}_4}{a_{31}a_{32}a_{34}\lambda_3^2}\mathbf{R}^2, \tag{3.156}$$

$$\mathbf{p}_4 = \frac{\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3}{a_{41}a_{42}a_{43}\lambda_4^2}\mathbf{R}^2, \tag{3.157}$$

$$\mathbf{p}_5 = \frac{\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4}{a_{51}a_{52}a_{53}a_{54}}\mathbf{R} \tag{3.158}$$

and

$$\mathbf{p}_6 = \left\{\mathbf{I} - \frac{(a_{52}a_{53}a_{54} + a_{51}a_{53}a_{54} + a_{51}a_{52}a_{54} + a_{51}a_{52}a_{53})}{a_{51}a_{52}a_{53}a_{54}}\mathbf{R}\right\}. \tag{3.159}$$

The expressions a_{ij} for $i, j = 1, 2, \ldots, 5$ and $\mathbf{b}_k$ for $k = 1, 2, 3, 4$ are defined, in turn, by

$$a_{ij} = \lambda_i - \lambda_j, \tag{3.160}$$

for $i, j = 1, 2, \ldots, 5$, and

$$\mathbf{b}_k = \mathbf{R} - \lambda_k \mathbf{I}, \tag{3.161}$$

for $k = 1, 2, 3, 4$.

$\mathbf{H}_{\boldsymbol{x}}(t)$ (see Eq. (3.127)) is conveniently expressed in terms of the eigenvalues of $\mathbf{R}$ and, thus, it can be easily determined. Following closely the example presented in the Moore-Penrose state-variable formulation (Section 3.3.1.2), assuming that the excitation $Q_2(t)$ is modeled as a white noise process, and employing Eqs. (3.134) and (3.140), the covariance matrix $\mathbf{w}_{z_x}$, can be determined in an efficient manner.

Using the decomposition of $\mathbf{H}_{\boldsymbol{x}}(t)$ obtained in Eq. (3.153), the double integral of Eq. (3.134), can be decomposed, and simplified in the form

$$\begin{aligned}\mathbf{H}_{\boldsymbol{x}}\mathbf{w}_{\boldsymbol{g_x}}\mathbf{H}_{\boldsymbol{x}}^{*} &= \sum_{i=1}^{4} e^{\lambda_i t}\left\{\sum_{j=1}^{4} e^{\bar{\lambda}_j s}\mathbf{p}_i\boldsymbol{\Psi}^{*}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{\Psi}\mathbf{p}_j^{*} + s\mathbf{p}_i\boldsymbol{\Psi}^{*}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{\Psi}\mathbf{p}_5^{*} + \mathbf{p}_i\boldsymbol{\Psi}^{*}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{\Psi}\mathbf{p}_6^{*}\right\}\delta \\ &+ \sum_{i=1}^{2} t^{2-i}\left\{\sum_{j=1}^{4} e^{\bar{\lambda}_j s}\mathbf{p}_{4+i}\boldsymbol{\Psi}^{*}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{\Psi}\mathbf{p}_j^{*} + s\mathbf{p}_{4+i}\boldsymbol{\Psi}^{*}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{\Psi}\mathbf{p}_5^{*} + \mathbf{p}_{4+i}\boldsymbol{\Psi}^{*}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{\Psi}\mathbf{p}_6^{*}\right\}\delta,\end{aligned} \tag{3.162}$$

where $\boldsymbol{\Psi}$ is the SVD unitary matrix. The matrices $\mathbf{p}_r, r = 1, 2, \ldots, 6$ are given by Eqs. (3.154)-(3.159) and the matrix $\mathbf{D}_{\boldsymbol{x}}$ by

$$\mathbf{D}_{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\pi 10^{-3} \end{bmatrix}, \tag{3.163}$$

respectively; see also Eq. (3.108).

Evaluating the matrices $\mathbf{p}_i(\boldsymbol{\Psi}^{*}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{\Psi})\mathbf{p}_j^{*},\ i, j = 1, 2, \ldots, 6$, it is noted that

$$\mathbf{p}_i(\boldsymbol{\Psi}^{*}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{\Psi})\mathbf{p}_5^{*} = \mathbf{p}_i(\boldsymbol{\Psi}^{*}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{\Psi})\mathbf{p}_6^{*} = 0, \tag{3.164}$$

for $i = 1, 2, 3, 4$, and

$$\mathbf{p}_5(\boldsymbol{\Psi}^{*}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{\Psi})\mathbf{p}_j^{*} = \mathbf{p}_6(\boldsymbol{\Psi}^{*}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{\Psi})\mathbf{p}_j^{*} = 0, \tag{3.165}$$

for $j = 1, 2, \ldots, 6$. Taking into consideration Eq. (3.164) and Eq. (3.165), Eq. (3.162) takes a much simpler form, which being substituted in Eq. (3.134), yields

$$\mathbf{w}_{\boldsymbol{z_x}} = \sum_{i=1}^{4}\sum_{j=1}^{4}\mathbf{I}_{i,j}, \tag{3.166}$$

where

$$\mathbf{I}_{i,j}(\tau) = \int_0^{\infty}\int_0^{\infty} e^{\lambda_i t}e^{\bar{\lambda}_j s}\mathbf{p}_i(\boldsymbol{\Psi}^{*}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{\Psi})\mathbf{p}_j^{*}\delta(\tau + t - s)dtds \tag{3.167}$$

or, equivalently,

$$\mathbf{I}_{i,j}(\tau) = -\mathbf{p}_i(\boldsymbol{\Psi}^{*}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{\Psi})\mathbf{p}_j^{*}\frac{e^{\bar{\lambda}_j\tau}}{\lambda_i + \bar{\lambda}_j}\ , \tag{3.168}$$

for $i, j = 1, 2, 3, 4$. Eq. (3.166) yields

$$\mathbf{w}_{\mathbf{z_x}}(\tau) = -\sum_{i=1}^{4}\sum_{j=1}^{4} \frac{e^{\bar{\lambda}_j \tau}}{\lambda_i + \bar{\lambda}_j} \mathbf{p}_i(\mathbf{\Psi}^* \mathbf{D}_x \mathbf{\Psi})\mathbf{p}_j^* \tag{3.169}$$

and for $\tau = 0$, Eq. (3.169) becomes

$$\begin{aligned} \mathbf{w}_{\mathbf{z_x}}(0) &= -\sum_{i=1}^{4}\sum_{j=1}^{4} \frac{\mathbf{p}_i(\mathbf{\Psi}^* \mathbf{D}_x \mathbf{\Psi})\mathbf{p}_j^*}{\lambda_i + \bar{\lambda}_j} \\ &= \begin{bmatrix} 0.0035 & 0.0011 & 0.0004 & 0.0006 & -0.0029 & 0.0043 \\ 0.0011 & 0.0227 & 0.0190 & -0.0008 & 0.0011 & -0.0008 \\ 0.0004 & 0.0190 & 0.0188 & -0.0056 & 0.0001 & 0 \\ 0.0006 & -0.0008 & -0.0056 & 0.0086 & -0.0009 & 0.0005 \\ -0.0029 & 0.0011 & 0.0001 & -0.0009 & 0.0593 & -0.0504 \\ 0.0043 & -0.0008 & 0 & 0.0005 & -0.0504 & 0.0438 \end{bmatrix}. \end{aligned} \tag{3.170}$$

Using Eq. (3.137), the covariance matrix of $\mathbf{p}$ becomes

$$\mathbf{w}_{\mathbf{p}}(0) = \begin{bmatrix} 0.0438 & 0.0438 & 0.0252 & 0 & 0 & -0.0012 \\ 0.0438 & 0.0438 & 0.0252 & 0 & 0 & -0.0012 \\ 0.0252 & 0.0252 & 0.0190 & 0.0012 & 0.0012 & 0 \\ 0 & 0 & 0.0012 & 0.0188 & 0.0188 & 0.0063 \\ 0 & 0 & 0.0012 & 0.0188 & 0.0188 & 0.0063 \\ -0.0012 & -0.0012 & 0 & 0.0063 & 0.0063 & 0.0127 \end{bmatrix}, \tag{3.171}$$

which is in total agreement with the respective results determined via the solution of the Lyapunov equation (see Eq. (3.110)). It is deemed necessary to mention that in contrast to the matrix calculated in Eq. (3.110), the covariance matrix obtained by Eq. (3.171) does not have any arbitrary elements y. This is due to the fact that the solution of Eq. (3.109), was based on the M-P inverse Eq. (2.12), whereas no such concept was invoked for determining Eq. (3.171) via a complex modal analysis treatment. Thus, it can be argued that there is additional merit in utilizing a "generalized" complex modal analysis for treating systems with singular matrices.

3.3.2 Frequency domain analysis methodology

In this section, the response of linear systems with singular matrices subject to stochastic excitation is determined by applying a frequency domain analysis treatment. The herein developed frequency domain response analysis methodology can be construed as an alternative to the Moore-Penrose time domain technique developed in Section 3.3.1; see also Ref. [41].

3.3.2.1 Standard linear systems with non-singular matrices

Some elements of the frequency domain stochastic response analysis of systems with standard non-singular matrices are provided in the following for completeness. The statistics of the system response, $\boldsymbol{q}(t)$, to an external excitation, $\boldsymbol{Q}(t)$, are determined in the frequency domain by utilizing input-output relationships, involving the frequency response function (FRF) matrix $\boldsymbol{\alpha}(\omega)$ [91]. Specifically, consider the equations of motion of an $n-$DOF linear system given by Eq. (3.1). Utilizing generalized coordinates for formulating the system equations of motion yields mass, damping and stiffness matrices that are not only non-singular, but also symmetric and positive definite. To determine the system FRF matrix $\boldsymbol{\alpha}(\omega)$, consider an excitation of the form

$$\boldsymbol{Q}(t) = \boldsymbol{Q}_0 \exp(i\omega t), \tag{3.172}$$

where ω denotes the frequency and $\boldsymbol{Q}_0$ is an amplitude vector. Considering the response displacement vector to be of the form

$$\boldsymbol{q}(t) = \boldsymbol{\alpha}(\omega)\boldsymbol{Q}(t), \tag{3.173}$$

where $\boldsymbol{\alpha}(\omega)$ is the $n \times n$ FRF matrix and substituting Eqs. (3.172)-(3.173) into Eq. (3.1) yields

$$\boldsymbol{\alpha}(\omega) = \mathbf{R}^{-1}, \tag{3.174}$$

where

$$\mathbf{R} = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}. \tag{3.175}$$

A spectral excitation-response (input-output) relationship can be determined by utilizing the FRF matrix of Eq. (3.174) in the form

$$\mathbf{S}_{\boldsymbol{q}}(\omega) = \boldsymbol{\alpha}(\omega)\mathbf{S}_{\boldsymbol{Q}}(\omega)\boldsymbol{\alpha}^{\mathrm{T}*}(\omega), \tag{3.176}$$

where $\mathbf{S}_{\boldsymbol{q}}(\omega)$ and $\mathbf{S}_{\boldsymbol{Q}}(\omega)$ are the system response and excitation power spectrum matrices, respectively, $\boldsymbol{Q}(t)$ represents an arbitrary stationary stochastic vector process, and $\boldsymbol{\alpha}^{\mathrm{T}*}(\omega)$ denotes the conjugate transpose of $\boldsymbol{\alpha}(\omega)$; see Ref. [91] for a more detailed presentation. System response second-order statistics can be readily determined based on

Eq. (3.176). For instance, utilizing Eq. (3.176) the response displacement and velocity moments $\mathbb{E}\left[q_i^2(t)\right]$ and $\mathbb{E}\left[\dot{q}_i^2(t)\right]$ are given by

$$\mathbb{E}\left[q_i^2(t)\right] = \int_{-\infty}^{\infty} \mathrm{S}_{q_i q_i}(\omega)\mathrm{d}\omega \tag{3.177}$$

and

$$\mathbb{E}\left[\dot{q}_i^2(t)\right] = \int_{-\infty}^{\infty} \omega^2 \mathrm{S}_{q_i q_i}(\omega)\mathrm{d}\omega, \tag{3.178}$$

respectively.

3.3.2.2 Linear systems with singular matrices

As described in Section 3.2, it can be argued that there are cases where utilizing more than the minimum number (redundant) DOFs for formulating the equations of motion of a complex dynamical system can be advantageous, especially from a computational efficiency perspective; see Refs [41, 93] for a detailed discussion. Following the redundant coordinates modeling scheme presented in Section 3.2, the $n-$DOF system of Eq. (3.1) can be alternatively described by Eq. (3.49), where the augmented mass, damping and stiffness matrices for the system are defined in Eq. (3.33) and Eqs. (3.50)-(3.51), respectively. Note, however, that due to the utilization of additional/redundant DOFs, the augmented mass, damping and stiffness matrices are singular.

Focusing on the frequency domain, the problem of determining the FRF matrix of a system with singular mass, damping and stiffness matrices is considered. The system of Eq. (3.49) is excited by a harmonic force of the form defined in Eq. (3.172). The system response is given by

$$\boldsymbol{x}(t) = \boldsymbol{\alpha}_{\boldsymbol{x}}(\omega)\bar{\boldsymbol{Q}}_{\boldsymbol{x}}(t), \tag{3.179}$$

where $\boldsymbol{\alpha}_{\boldsymbol{x}}(\omega)$ is the $l \times (m+l)$ FRF matrix. Eq. (3.179) is differentiated twice with respect to time and the obtained expressions, along with Eq. (3.179), are substituted in Eq. (3.49) yielding

$$\mathbf{R}_{\boldsymbol{x}}\boldsymbol{\alpha}_{\boldsymbol{x}}(\omega) = \mathbf{I}. \tag{3.180}$$

In Eq. (3.180) the $(m+l) \times l$ matrix $\mathbf{R}_{\boldsymbol{x}}$ is given by

$$\mathbf{R}_{\boldsymbol{x}} = -\omega^2\bar{\mathbf{M}}_{\boldsymbol{x}} + i\omega\bar{\mathbf{C}}_{\boldsymbol{x}} + \bar{\mathbf{K}}_{\boldsymbol{x}}. \tag{3.181}$$

The M-P inverse of the matrix $\mathbf{R}_{\boldsymbol{x}}$ is employed for solving Eq. (3.180). Specifically, utilizing Eq. (2.12), the FRF matrix takes the form

$$\boldsymbol{\alpha}_{\boldsymbol{x}}(\omega) = \mathbf{R}_{\boldsymbol{x}}^{+} + (\mathbf{I} - \mathbf{R}_{\boldsymbol{x}}^{+}\mathbf{R}_{\boldsymbol{x}})\mathbf{Y}, \tag{3.182}$$

where $\mathbf{R}_{\boldsymbol{x}}^{+}$ is the $l \times (m+l)$ M-P inverse of $\mathbf{R}_{\boldsymbol{x}}$ and $\mathbf{Y}$ is an arbitrary $l \times (m+l)$ matrix.

It is noted that the presence of the arbitrary matrix $\mathbf{Y}$ on the right hand side of Eq. (3.182) yields a non-unique solution for the FRF matrix. Nevertheless, depending on the rank of $\mathbf{R}_{\boldsymbol{x}}$, a uniquely defined FRF matrix can be derived. Specifically, taking into account the full rank factorization of a matrix (see Lemma 2.4), it is readily seen that if $\mathbf{R}_{\boldsymbol{x}}$ has full rank, its M-P inverse takes the form

$$\mathbf{R}_{\boldsymbol{x}}^{+} = (\mathbf{R}_{\boldsymbol{x}}^{*}\mathbf{R}_{\boldsymbol{x}})^{-1}\mathbf{R}_{\boldsymbol{x}}^{*} \tag{3.183}$$

(see Eq. (2.14)) and taking into account Eq. (3.183), the expression

$$\mathbf{I} - \mathbf{R}_{\boldsymbol{x}}^{+}\mathbf{R}_{\boldsymbol{x}} = \mathbf{0}, \tag{3.184}$$

holds true. Combining Eq. (3.184) with Eq. (3.182), the FRF matrix is uniquely defined as

$$\boldsymbol{\alpha}_{\boldsymbol{x}}(\omega) = \mathbf{R}_{\boldsymbol{x}}^{+}. \tag{3.185}$$

Following Ref. [91] the standard spectral excitation-response relationship of Eq. (3.176) is generalized and given in the form

$$\mathbf{S}_{\boldsymbol{x}}(\omega) = \boldsymbol{\alpha}_{\boldsymbol{x}}(\omega)\mathbf{S}_{\bar{\boldsymbol{Q}}_{\boldsymbol{x}}}(\omega)\boldsymbol{\alpha}_{\boldsymbol{x}}^{\mathrm{T}*}(\omega), \tag{3.186}$$

where $\mathbf{S}_{\boldsymbol{x}}(\omega)$ and $\mathbf{S}_{\bar{\boldsymbol{Q}}_{\boldsymbol{x}}}(\omega)$ are the system response and excitation power spectrum matrices, respectively. System response second-order statistics can be readily determined based on Eq. (3.186). For instance, utilizing Eq. (3.186) the response displacement and velocity moments $\mathbb{E}\left[x_i^2(t)\right]$ and $\mathbb{E}\left[\dot{x}_i^2(t)\right]$ are given as

$$\mathbb{E}\left[x_i^2(t)\right] = \int_{-\infty}^{\infty} \mathrm{S}_{x_i x_i}(\omega)\mathrm{d}\omega \tag{3.187}$$

and

$$\mathbb{E}\left[\dot{x}_i^2(t)\right] = \int_{-\infty}^{\infty} \omega^2 \mathrm{S}_{x_i x_i}(\omega)\mathrm{d}\omega, \tag{3.188}$$

respectively.

It is deemed appropriate to note the evaluation of the FRF matrix $\boldsymbol{\alpha}_{\boldsymbol{x}}(\omega)$ of Eq.

(3.185) can be simplified in many cases by circumventing the computation of the M-P inverse of $\mathbf{R}_x$ of Eq. (3.181). In the context of generalizing the classical modal analysis treatment to account for systems with singular matrices, it was shown recently in Ref. [78] that the problem of determining the natural frequencies of the augmented system given by Eq. (3.49) is related to solving an eigenvalue problem for the $l \times l$ matrix $\bar{\mathbf{M}}_x^+ \bar{\mathbf{K}}_x$ and determining the $l \times l$ modal matrix, $\bar{\mathbf{\Psi}}$. Considering the transformation

$$\boldsymbol{x} = \bar{\mathbf{\Psi}} \boldsymbol{p}, \tag{3.189}$$

the system governing equation of motion Eq. (3.49) becomes

$$\mathbf{L}\ddot{\boldsymbol{p}} + \mathbf{D}\dot{\boldsymbol{p}} + \mathbf{N}\boldsymbol{p} = \boldsymbol{P}. \tag{3.190}$$

In Eq. (3.190), $\mathbf{L}, \mathbf{N}$ denote the $l \times l$ diagonal matrices given by

$$\mathbf{L} = \bar{\mathbf{\Psi}}^{-1} \bar{\mathbf{M}}_x^+ \bar{\mathbf{M}}_x \bar{\mathbf{\Psi}} \tag{3.191}$$

and

$$\mathbf{N} = \bar{\mathbf{\Psi}}^{-1} \bar{\mathbf{M}}_x^+ \bar{\mathbf{K}}_x \bar{\mathbf{\Psi}}, \tag{3.192}$$

respectively, whereas the l vector $\boldsymbol{P}$ has the form

$$\boldsymbol{P} = \bar{\mathbf{\Psi}}^{-1} \bar{\mathbf{M}}_x^+ \bar{\boldsymbol{Q}}_x. \tag{3.193}$$

The $l \times l$ matrix $\boldsymbol{D}$ is given by

$$\mathbf{D} = \bar{\mathbf{\Psi}}^{-1} \bar{\mathbf{M}}_x^+ \bar{\mathbf{C}}_x \bar{\mathbf{\Psi}} \tag{3.194}$$

and, in general, is not a diagonal matrix; see Ref. [78] for a more detailed presentation. Nevertheless, in many cases, and based on a reasonable assumption of light damping (e.g. Refs [91, 24]), a satisfactory approximation can be obtained by neglecting the off-diagonal elements of $\mathbf{D}$; thus, yielding a diagonal $\mathbf{D}$ matrix. In this regard, the FRF matrix of the system of Eq. (3.190) is given by

$$\mathbf{\Lambda}(\omega) = \mathbf{R}_{\mathbf{\Lambda}}^{-1}, \tag{3.195}$$

where

$$\mathbf{R}_{\mathbf{\Lambda}} = -\omega^2 \mathbf{L} + i\omega \mathbf{D} + \mathbf{N}. \tag{3.196}$$

Considering Eqs. (3.189) and (3.193), as well as the relation

$$\boldsymbol{p} = \boldsymbol{\Lambda}(\omega)\boldsymbol{P}, \tag{3.197}$$

leads to

$$\boldsymbol{x} = \bar{\boldsymbol{\Psi}}\boldsymbol{\Lambda}(\omega)\bar{\boldsymbol{\Psi}}^{-1}\bar{\mathrm{M}}_{\boldsymbol{x}}^{+}\bar{\boldsymbol{Q}}, \tag{3.198}$$

which, combined with Eq. (3.179), yields

$$\boldsymbol{\alpha}_{\boldsymbol{x}}(\omega) = \bar{\boldsymbol{\Psi}}\boldsymbol{\Lambda}(\omega)\bar{\boldsymbol{\Psi}}^{-1}\bar{\mathrm{M}}_{\boldsymbol{x}}^{+}. \tag{3.199}$$

The FRF matrix obtained in Eq. (3.199) can be further simplified if taken into account that the FRF matrix $\boldsymbol{\Lambda}(\omega)$ is diagonal. Eq. (3.199) yields

$$\boldsymbol{\alpha}_{\boldsymbol{x}}(\omega) = \left\{\sum_{k=1}^{l} \boldsymbol{x}^{(k)}\boldsymbol{y}^{(k)}\alpha_k^{'}(\omega)\right\}\bar{\mathrm{M}}_{\boldsymbol{x}}^{+}, \tag{3.200}$$

where $\boldsymbol{x}^{(k)}, \boldsymbol{y}^{(k)}, k = 1, 2, \ldots, l$ correspond to the $k-th$ column of the modal matrix $\bar{\boldsymbol{\Psi}}$ and to the $k-th$ row of $\bar{\boldsymbol{\Psi}}^{-1}$, respectively; $\alpha_k^{'}(\omega), k = 1, 2, \ldots, l$ is the $k-th$ diagonal element of the matrix $\boldsymbol{\Lambda}(\omega)$.

Eq. (3.200) is a rather useful series expression for $\boldsymbol{\alpha}_{\boldsymbol{x}}(\omega)$, which circumvents the potentially cumbersome numerical evaluation of the M-P inverse indicated in Eq. (3.185). Also, in many applications, the series may be truncated to only the first few terms, with little loss of accuracy [91].

3.3.2.3 Frequency domain analysis of linear systems with singular matrices – a numerical example

As a numerical example, the 3-DOF linear system of rigid masses shown in Figure 3.3, is considered. The first mass m_1 is attached to the foundation by a linear spring and a linear damper with coefficients k_1 and c_1, respectively. It is also connected to the other two masses m_2 and m_3 by two linear springs with coefficients k_2 and k_4. The mass m_2 is connected to the third mass by a linear spring with coefficient k_3 and a linear damper with damping coefficient c_2. The system is excited by a stochastic force $Q_3(t)$ applied on mass m_3 and modeled as a white-noise process with a correlation function $w_{Q_3}(t) = 2\pi S_0\delta(t)$. The value S_0 stands for the (constant) power spectrum value of $Q_3(t)$. The generalized displacements of the masses m_1, m_2 and m_3 due to the applied force, are denoted by q_1, q_2 and q_3, respectively.

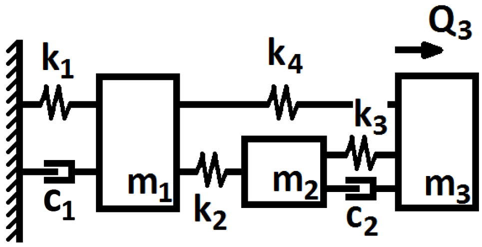

FIGURE 3.3: A three degree-of-freedom linear system under stochastic excitation.

Following a standard Newtonian, or Lagrangian approach [58], the linear system equations of motion have the form given by Eq. (3.1), where the 3×3 mass, damping and stiffness matrices are given by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \tag{3.201}$$

$$\mathbf{C} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & -c_2 \\ 0 & -c_2 & c_2 \end{bmatrix} \tag{3.202}$$

and

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 + k_4 & -k_2 & -k_4 \\ -k_2 & k_2 + k_3 & -k_3 \\ -k_4 & -k_3 & k_3 + k_4 \end{bmatrix}, \tag{3.203}$$

respectively. The displacement vector is given as

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \tag{3.204}$$

whereas the excitation vector $\boldsymbol{Q}(t)$ is given by

$$\boldsymbol{Q} = \begin{bmatrix} 0 \\ 0 \\ Q_3 \end{bmatrix}. \tag{3.205}$$

The parameters values in this example are $m_1 = m_3 = 2, m_2 = 1, c_1 = c_2 = 0.1$ and $k_1 = k_2 = k_3 = k_4 = 1$, and $S_0 = 10^{-3}$. Considering next Eqs. (3.176)-(3.177), the stationary covariance matrix of the system response displacement is given by

$$\mathbf{V}_{\boldsymbol{q}} = \begin{bmatrix} 0.0493 & 0.0623 & 0.0644 \\ 0.0623 & 0.0805 & 0.0846 \\ 0.0644 & 0.0846 & 0.0916 \end{bmatrix}, \tag{3.206}$$

whereas the stationary covariance matrix of the system response velocity is

$$\mathbf{V}_{\dot{\boldsymbol{q}}} = \begin{bmatrix} 0.0106 & 0.0110 & 0.0086 \\ 0.0110 & 0.0142 & 0.0131 \\ 0.0086 & 0.0131 & 0.0170 \end{bmatrix}. \tag{3.207}$$

To demonstrate the herein developed frequency domain based methodology for systems with singular matrices, the system shown in Figure 3.3 is decomposed into several separate systems, which are treated independently. As it is seen in Figure 3.4, the number of modeling coordinates used for deriving the system equations of motion is increased by two. The coordinates vector of the redundant DOFs system becomes

$$\boldsymbol{x} = \begin{bmatrix} \bar{x}_1 \\ x_2 \\ \bar{x}_3 \\ x_4 \\ \bar{x}_5 \end{bmatrix}, \tag{3.208}$$

where $\bar{x}_1, \bar{x}_3$ and $\bar{x}_5$ correspond to the displacements of the masses m_1, m_2 and m_3 and the coordinates x_2, x_4 correspond to the additional DOFs.

The sub-systems are related via two constraint equations, namely

$$x_1 + d = x_2 \tag{3.209}$$

and

$$x_2 + x_3 + d = x_4, \tag{3.210}$$

where d is the physical length of the masses (same for m_1, m_2 and m_3). The constraint equations can also be written as

$$\bar{x}_1 + l_{1,0} + d = x_2 \tag{3.211}$$

and

$$x_2 + \bar{x}_3 + l_{3,0} + d = x_4, \tag{3.212}$$

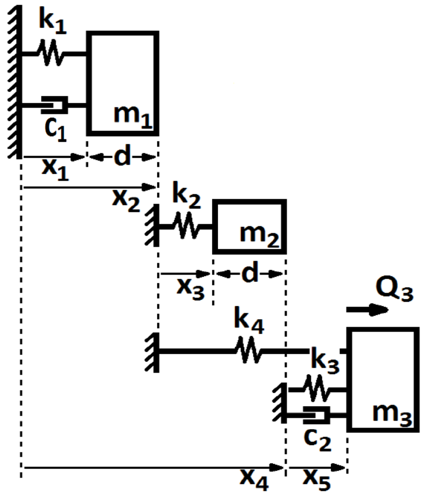

FIGURE 3.4: A three degree-of-freedom linear system under stochastic excitation utilizing redundant coordinates.

where $l_{1,0}$ is the unstretched length of the mass m_1, and $l_{3,0}$ is the unstretched length of m_3.

To derive the system equations of motion, the total kinetic energy of the system is given by

$$T = \frac{1}{2}m_1\dot{\bar{x}}_1^2 + \frac{1}{2}m_2(\dot{x}_2 + \dot{\bar{x}}_3)^2 + \frac{1}{2}m_3(\dot{x}_4 + \dot{\bar{x}}_5)^2 \tag{3.213}$$

and the total potential energy by

$$V = \frac{1}{2}k_1\bar{x}_1^2 + \frac{1}{2}k_2\bar{x}_3^2 + \frac{1}{2}k_3\bar{x}_5^2 + \frac{1}{2}k_4(-x_2 + x_4 + \bar{x}_5)^2. \tag{3.214}$$

The standard variational formulation [58] involving the Lagrangian function

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = T - V \tag{3.215}$$

leads to the Euler-Lagrange equations, and thus, to the system equations of motion of the form of Eq. (3.49). The mass, damping and stiffness matrices become

$$\mathbf{M}_x = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & m_2 & 0 & 0 \\ 0 & m_2 & m_2 & 0 & 0 \\ 0 & 0 & 0 & m_3 & m_3 \\ 0 & 0 & 0 & m_3 & m_3 \end{bmatrix}, \tag{3.216}$$

$$\mathbf{C}_x = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_2 \end{bmatrix} \tag{3.217}$$

and

$$\mathbf{K}_x = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & k_4 & 0 & -k_4 & -k_4 \\ 0 & 0 & k_2 & 0 & 0 \\ 0 & -k_4 & 0 & k_4 & k_4 \\ 0 & -k_4 & 0 & k_4 & k_3 + k_4 \end{bmatrix}, \tag{3.218}$$

respectively. Differentiating twice with respect to time Eqs. (3.211)-(3.212), the 2×5 matrix $\mathbf{A}$ defined in Eq. (3.21) takes the form

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix}, \tag{3.219}$$

whereas the 2 vector $\boldsymbol{b}$ becomes

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.220}$$

Next, the 7×5 augmented mass, damping and stiffness matrices given by Eq. (3.33) and Eqs. (3.50)-(3.51), are determined. Taking into account Eq. (3.21) and Eq. (3.30), and substituting the parameters values yield

$$\bar{\mathbf{M}}_x = \begin{bmatrix} 0.8 & 0.2 & 0.2 & 0.4 & 0.4 \\ 0.8 & 0.2 & 0.2 & 0.4 & 0.4 \\ -0.4 & 0.4 & 0.4 & 0.8 & 0.8 \\ 0.4 & 0.6 & 0.6 & 1.2 & 1.2 \\ 0 & 0 & 0 & 2 & 2 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix}, \tag{3.221}$$

$$\bar{\mathbf{C}}_x = \begin{bmatrix} 0.04 & 0 & 0 & 0 & 0 \\ 0.04 & 0 & 0 & 0 & 0 \\ -0.02 & 0 & 0 & 0 & 0 \\ 0.02 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.222}$$

and

$$\bar{\mathbf{K}}_{\boldsymbol{x}} = \begin{bmatrix} 0.4 & 0.2 & -0.2 & -0.2 & -0.2 \\ 0.4 & 0.2 & -0.2 & -0.2 & -0.2 \\ -0.2 & -0.6 & 0.6 & 0.6 & 0.6 \\ 0.2 & -0.4 & 0.4 & 0.4 & 0.4 \\ 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.223}$$

Employing Eq. (3.52), the augmented excitation vector is given by

$$\bar{\boldsymbol{Q}}_{\boldsymbol{x}} = \begin{bmatrix} 0.2Q_3 \\ 0.2Q_3 \\ 0.4Q_3 \\ 0.6Q_3 \\ Q_3 \\ 0 \\ 0 \end{bmatrix}. \tag{3.224}$$

To determine the system response statistics via the herein developed frequency domain methodology, the 7×5 matrix $\mathbf{R}_{\boldsymbol{x}}$ is obtained via Eq. (3.181). Utilizing Eq. (3.185) the FRF matrix $\boldsymbol{\alpha}_{\boldsymbol{x}}(\omega)$ is determined. It is noted that Eq. (3.185) is utilized instead of Eq. (3.182) as the 7×5 matrix $\mathbf{R}_{\boldsymbol{x}}$ has full rank, i.e. $rank(\mathbf{R}_{\boldsymbol{x}}) = 5$, and thus, the FRF matrix is uniquely defined. Combining Eq. (3.186) with Eq. (3.187), the covariance matrix of the system response displacement is given by

$$\mathbf{V}_{\bar{\boldsymbol{x}}} = \begin{bmatrix} 0.0493 & 0.0493 & 0.0130 & 0.0623 & 0.0021 \\ 0.0493 & 0.0493 & 0.0130 & 0.0623 & 0.0021 \\ 0.0130 & 0.0130 & 0.0052 & 0.0182 & 0.0019 \\ 0.0623 & 0.0623 & 0.0182 & 0.0805 & 0.0040 \\ 0.0021 & 0.0021 & 0.0019 & 0.0040 & 0.0030 \end{bmatrix} \tag{3.225}$$

and combining Eq. (3.186) with Eq. (3.188), the covariance matrix of the system response velocity is determined to be

$$\mathbf{V}_{\dot{\bar{\boldsymbol{x}}}} = \begin{bmatrix} 0.0106 & 0.0106 & 0.0004 & 0.0110 & -0.0024 \\ 0.0106 & 0.0106 & 0.0004 & 0.0110 & -0.0024 \\ 0.0004 & 0.0004 & 0.0028 & 0.0032 & 0.0013 \\ 0.0110 & 0.0110 & 0.0032 & 0.0142 & -0.0011 \\ -0.0024 & -0.0024 & 0.0013 & -0.0011 & 0.0050 \end{bmatrix}. \tag{3.226}$$

For the comparison of the results obtained by the standard and the herein proposed methodology, the matrices given by Eqs. (3.225)-(3.226) are compared to those given by Eqs. (3.206)-(3.207). Indicatively, it is seen that the variances $\mathbb{E}[q_1^2]$ and $\mathbb{E}[\dot{q}_1^2]$ coincide with their counterparts, i.e. $\mathbb{E}[\bar{x}_1^2]$ and $\mathbb{E}[\dot{\bar{x}}_1^2]$. Considering the equations that connect the reference systems depicted in Figure 3.4, i.e.

$$\bar{x}_3 = q_2 - q_1 \tag{3.227}$$

and

$$\bar{x}_5 = q_3 - q_2, \tag{3.228}$$

yields

$$\begin{aligned} \mathbb{E}[\bar{x}_3^2] &= \mathbb{E}[q_1^2] + \mathbb{E}[q_2^2] - 2\mathbb{E}[q_1 q_2] \\ &= 0.0052 \end{aligned} \tag{3.229}$$

and

$$\begin{aligned} \mathbb{E}[\bar{x}_5^2] &= \mathbb{E}[q_2^2] + \mathbb{E}[q_3^2] - 2\mathbb{E}[q_2 q_3] \\ &= 0.0030. \end{aligned} \tag{3.230}$$

The variances computed in Eqs. (3.229)-(3.230) are equal to the corresponding ones in positions $(3, 3)$ and $(5, 5)$ of matrix $\mathbf{V}_{\bar{x}}$. The same agreement for the response velocity variances can be readily verified by comparing Eq. (3.207) with Eq. (3.226).

As noted in Section 3.2, the FRF matrix $\boldsymbol{\alpha}_{\boldsymbol{x}}(\omega)$ can be alternatively determined without computing the M-P inverse of the matrix $\mathbf{R}_{\boldsymbol{x}}$ in Eq. (3.181). Instead, a generalized modal analysis approach can be employed. Following closely Ref. [78], the modal matrix for the system in Figure 3.4 is computed as

$$\bar{\boldsymbol{\Psi}} = \begin{bmatrix} 0.1740 & 0.4880 & 0.5151 & 0.0000 & -0.0000 \\ 0.1740 & 0.4880 & 0.5151 & -0.1305 & 0.8116 \\ -0.6401 & -0.3004 & 0.1517 & 0.3601 & 0.1281 \\ -0.4661 & 0.1877 & 0.6668 & -0.8507 & 0.5554 \\ 0.5590 & -0.6310 & 0.0413 & 0.3601 & 0.1281 \end{bmatrix}. \tag{3.231}$$

Utilizing the transformation of Eq. (3.189) and taking into account Eq. (3.191), Eq. (3.192) and Eq. (3.194), the system equation of motion of Eq. (3.190) arises. Also, the 5×5

diagonal FRF matrix of Eq. (3.195) becomes

$$\mathbf{\Lambda}(\omega) \approx \begin{bmatrix} \frac{1}{-\omega^2+0.1162i\omega+2.5726} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{-\omega^2+0.0703i\omega+1.7620} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{-\omega^2+0.0135i\omega+0.1655} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{-\omega^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{-\omega^2} \end{bmatrix}. \tag{3.232}$$

Combining Eq. (3.199), or Eq. (3.200), with Eqs. (3.231)-(3.232), the FRF matrix is determined, and thus, the covariance matrix of the system response displacement is given by

$$\mathbf{V}_x = \begin{bmatrix} 0.0492 & 0.0492 & 0.0132 & 0.0624 & 0.0020 \\ 0.0492 & 0.0492 & 0.0132 & 0.0624 & 0.0020 \\ 0.0132 & 0.0132 & 0.0050 & 0.0182 & 0.0019 \\ 0.0624 & 0.0624 & 0.0182 & 0.0806 & 0.0039 \\ 0.0020 & 0.0020 & 0.0019 & 0.0039 & 0.0032 \end{bmatrix}, \tag{3.233}$$

which is in agreement with Eq. (3.225) obtained via utilizing the M-P inverse of $\mathbf{R}_x$.

3.4 Summary

In chapter 3, adopting a redundant coordinates modeling for deriving the systems governing equations of motion and utilizing the M-P generalized matrix inverse of a singular matrix, a time as well as a frequency domain methodology is developed for stochastic response determination of MDOF linear structural systems with singular matrices. Following the time domain methodology, a complex modal analysis treatment, in conjunction with an SVD of the system transition matrix, is developed for deriving the linear system response statistics. The difference with the standard complex modal analysis framework - minimum DOFs modeling - is that a decoupling of the equations of motion cannot be achieved by utilizing a redundant DOFs modeling approach. For the frequency domain methodology, a M-P FRF is determined for linear systems with singular matrices. A series expansion of the M-P FRF that serves as an alternative for circumventing the potentially cumbersome numerical evaluation of the M-P inverse is also presented. The theoretical framework is validated by pertinent numerical examples. The obtained results are in total agreement with these derived by following a standard analysis (in time and frequency domain), where the minimum number of coordinates is utilized in deriving the systems governing equations of motion.

Chapter 4

Stochastic response of nonlinear systems with singular matrices

4.1 Nonlinear systems stochastic response determination

The study of nonlinear engineering systems subjected to stochastic excitation has flourished during the last decades, yielding the development of research fields such as that of nonlinear stochastic dynamics. This upward trend is driven by the fact that most physical phenomena such as wind, earthquakes, ocean waves, explosive vibration etc., apart from their nonlinear nature, enclose an inherent uncertainty efficiently described by Stochastic Calculus. The necessity of first understanding the impact of the aforementioned phenomena in everyday life, and then managing that impact in terms of designing systems/structures that are capable of manipulating, or resisting to that impact, yielded the development of several analytical and numerical methodologies/techniques, which formulate the stochastic behavior of such systems.

Diverse mathematical tools have been developed for encountering the critical problem of deriving the nonlinear system response statistics, yielding several approximate and analytical approaches [72]. Some of the most frequently employed follow. The most versatile among the available methodologies for efficiently deriving the nonlinear system response statistics is, undoubtedly, the application of the Monte Carlo simulation (MCS) method [86]. The reason for this wide spread of the MCS method is the small number of nonlinear problems retaining an exact solution. However, the excessive computational cost related to its application, especially in case of large-scale complex multibody systems, renders the MCS solution scheme a burdensome option.

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Therefore, a number of approximate methodologies have been developed throughout the years as alternative techniques to MCS method. Among them, another well established technique for determining the (MDOF) nonlinear system response statistics, capable of addressing several random vibration problems, is the stochastic averaging method [90, 121]. The recently derived analytical techniques based on the utilization of the Wiener path integral framework [21], constitute one more approach for determining the nonlinear (MDOF) system response statistics [56]. Relying on the concept of the most probable trajectory, an approximate expression for the probability density function of nonlinear system response is derived. The method is extended to account for linear and nonlinear oscillators endowed with fractional derivatives elements [30]. Finally, a widely accepted technique for approximately deriving the nonlinear system response statistics, is the statistical linearization methodology. Among all methods quoted herein, the statistical linearization methodology is the most straightforward in its implementation and readily applied to multibody system modeling for a wide range of nonlinearities. This last method is concisely presented in Section 4.2; a more detailed presentation can be found in Ref. [91].

4.2 Statistical linearization approximate methodology

In Chapter 3, the standard time and frequency domain methodology techniques of the random vibration theory for the stochastic response determination of linear systems (e.g. Refs. [91], [60], [63]), are generalized to account for the case of linear systems with singular matrices. As also noted in Section 4.1, most mathematical models derive from applications involve nonlinear systems. Therefore, the analysis and results obtained in Chapter 3, are extended to account for nonlinear systems with singular matrices. In this context, a generalized statistical linearization approximate methodology, which plays an instrumental role in the ensuing analysis (Chapter 4), is also proposed [41, 55]. A brief introduction to the classical statistical linearization methodology follows.

4.2.1 Classical method

Statistical linearization has been one of the most versatile approximate methodologies for determining the stochastic response of nonlinear structural and mechanical systems efficiently [91, 98, 20, 25, 26, 99, 100, 71]. The main objective of the methodology relates to the replacement of the original nonlinear system with an ”equivalent linear” one by appropriately minimizing the error vector corresponding to the difference between the two systems. The rationale behind this procedure is that closed form analytical expressions exist for the response statistics of a linear system, and thus, the response statistics of the equivalent linear system can be used as an approximation for the response statistics of the original nonlinear system. According to the standard implementation of the methodology, the minimization criterion relates typically to the mean square error, whereas the Gaussian assumption for the system response probability density functions is commonly adopted [91]. Note, that although more sophisticated implementations of the statistical linearization that relax the aforementioned assumptions and/or employ various other minimization criteria exist [98], these versions typically lack versatility. One of the reasons for the wide utilization of the standard statistical linearization methodology has been, undoubtedly, its versatility in addressing a wide range of nonlinear behaviors. In particular, the Gaussian response assumption in conjunction with the mean square error minimization criterion facilitates the derivation of closed form expressions for the equivalent linear elements (e.g. stiffness, damping coefficients, etc.) as functions of the response statistics.

Taking into consideration the general form of the equation of motion of a lumped-parameter n degree of freedom system given by Eq. (3.1), the general form of the equation of motion of a nonlinear MDOF vibratory system is given by

$$\mathbf{M}\ddot{\boldsymbol{q}} + \mathbf{C}\dot{\boldsymbol{q}} + \mathbf{K}\boldsymbol{q} + \boldsymbol{\Phi}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \boldsymbol{Q}(t). \tag{4.1}$$

In Eq. (4.1), $\mathbf{M}, \mathbf{C}, \mathbf{K}$ denote the $n \times n$ mass, damping and stiffness matrices of the system, $\boldsymbol{\Phi}$ is a nonlinear n vector function of the generalized coordinate vector $\boldsymbol{q}$ and its derivatives, and $\boldsymbol{Q}$ denotes the n vector system excitation.

In general, the presence of the nonlinear term $\boldsymbol{\Phi}$ in Eq. (4.1) hinders the derivation of an exact solution for the system response. An exact solution can only be derived under an assumption including a small number of DOFs, n, and also considering restrictions for the nonlinear term, $\boldsymbol{\Phi}$ as well as the system response vector, $\boldsymbol{q}$ [91]. An approximate solution for the system response statistics is sought. As aforementioned, the idea of the method lies in the replacement of the non-linear term of Eq. (4.1) with auxiliary linear ones (for which the exact analytic formula for the solution is known) yielding the formulation of the so-called equivalent linear system. A concise description of the method follows; see Ref. [91] for a more detailed presentation.

The equivalent linear system is firstly defined. The nonlinear vector $\mathbf{\Phi}$ in Eq. (4.1) is replaced by equivalent mass, damping and stiffness matrices yielding

$$(\mathbf{M} + \mathbf{M}_e)\ddot{\boldsymbol{q}} + (\mathbf{C} + \mathbf{C}_e)\dot{\boldsymbol{q}} + (\mathbf{K} + \mathbf{K}_e)\boldsymbol{q} = \boldsymbol{Q}(t), \tag{4.2}$$

where $\mathbf{M}_e, \mathbf{C}_e$ and $\mathbf{C}_e$ denote the $n \times n$ equivalent matrices. The difference between the original and the equivalent linear system is formed, defining the error vector

$$\boldsymbol{\varepsilon} = \mathbf{\Phi}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) - \mathbf{M}_e\ddot{\boldsymbol{q}} - \mathbf{C}_e\dot{\boldsymbol{q}} - \mathbf{K}_e\boldsymbol{q}. \tag{4.3}$$

The objective of the method is to minimize ε in some statistical sense, for every $\boldsymbol{q}$ belonging to a certain class of functions of the independent variable t [91, 101]. Among several criteria, the mean square minimization criterion

$$\mathbb{E}\left[\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\varepsilon}\right] = \text{minimum}, \tag{4.4}$$

is employed. In Eq. (4.4), the Euclidean norm of the error $\boldsymbol{\varepsilon}$, i.e. $\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\varepsilon} = \|\boldsymbol{\varepsilon}\|_2^2$, is utilized. The solution of Eq. (4.4) corresponds to determining the elements of $\mathbf{M}_e, \mathbf{C}_e$ and $\mathbf{K}_e$ or, equivalently the equivalent linear system of Eq. (4.2), which in turn, constitutes an efficient approximation of the original nonlinear system of Eq. (4.1).

4.2.2 Minimization criteria

The mean square minimization criterion utilized in deriving the elements of the equivalent linear system of Eq. (4.2), is the most widely used criterion in the literature. The reason for this happening is that the criterion is straightforward in its implementation and can confront a wide range of nonlinearities [91]. Nevertheless, several different criteria have been developed throughout the last decades. Some of them are immediately derived from the classical minimization criterion and are summarized as follows. For example, by slightly alternating the classical case and particularly by requiring that the difference between the original nonlinear system and its linear correspondent is orthogonal to the system response, a novel minimization criterion arises [35]. A detailed circumstantial review of the available criteria is found in Refs [96, 97]; further reviews are included in Refs [35, 26].

Following closely Ref. [96], a general discretization of the minimization criteria include, among others, the so-called energy criteria, the spectral density criteria and the probability density criteria. In the first category, i.e. the energy based minimization criteria, either the system potential energy or its the energy dissipation function is minimized. The mean square difference is formed, for instance, in terms of the original nonlinear system potential energy and the corresponding of its equivalent linear counterpart [34, 119]. Alternatively, the mean square value of the nonlinear system potential energy is set equal to the corresponding energy of the linear counterpart [36]. A similar rationale is followed if the system energy dissipation function is chosen instead of its potential energy [37]. Many generalizations of these two criteria also exist; see Ref. [96] for more details. As for the spectral density criteria, they are related to applying the linearization technique in frequency domain. The original nonlinear system is replaced by an equivalent linear one, that is described by random coefficients with a known probability density function. These random coefficients are, in turn, described by a conditional probability expression and for their derivation, this expression is set equal to a corresponding conditional probability of the original nonlinear system [17, 96]. The probability density criteria are based on determining the coefficients of the equivalent linear system by employing a probabilistic metric in the probability density space criterion [96, 95].

4.2.3 Modeling the nonlinear systems governing equations of motion with redundant coordinates

The redundant coordinates modeling presented in Section 3.2 can be followed for deriving the governing equations of motion of nonlinear systems. In this section a concise presentation of the technique in case of nonlinear systems is given for completeness.

Following a redundant DOFs modeling scheme, the $n-$DOF nonlinear system of Eq. (4.1) is construed as a collection of sub-systems modeled separately, yielding an overall $l-$DOF system ($l \geq n$) with governing equations of motion given by

$$\mathbf{M}_{\boldsymbol{x}}\ddot{\boldsymbol{x}} + \mathbf{C}_{\boldsymbol{x}}\dot{\boldsymbol{x}} + \mathbf{K}_{\boldsymbol{x}}\boldsymbol{x} + \boldsymbol{\Phi}_{\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \boldsymbol{Q}_{\boldsymbol{x}}(t), \tag{4.5}$$

where $\boldsymbol{x}$ stands for an $l-$DOF vector of coordinates ($l \geq n$), $\boldsymbol{Q}_{\boldsymbol{x}}$ is the l vector of the external forces and $\mathbf{M}_{\boldsymbol{x}}, \mathbf{C}_{\boldsymbol{x}}$ and $\mathbf{K}_{\boldsymbol{x}}$ are the $l \times l$ mass, damping and stiffness matrices, respectively. The augmented nonlinear vector for the $l-$DOF system is given by the l vector $\boldsymbol{\Phi}_{\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}})$.

As in the case of linear systems, additional constraint equations which are given by Eq. (3.21) and connect the aforementioned subsystems, arise. The constraint equations imply a number of additional forces, $\boldsymbol{Q}_{\boldsymbol{x}}^{\mathrm{c}}(t)$, and thus, Eq. (4.5) is transformed into

$$\mathbf{M}_x\ddot{\boldsymbol{x}} + \mathbf{C}_x\dot{\boldsymbol{x}} + \mathbf{K}_x\boldsymbol{x} + \mathbf{\Phi}_x(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) = \boldsymbol{Q}_{\boldsymbol{x}}(t) + \boldsymbol{Q}_{\boldsymbol{x}}^{c}(t), \tag{4.6}$$

Moreover, virtual displacements appear due to the additional forces $\boldsymbol{Q}_{\boldsymbol{x}}^{\mathrm{c}}(t)$; these displacements satisfy the condition given in Eq. (3.23) and at any instant of time t can be expressed as in Eq. (3.24). Taking into account that in the case of nonlinear systems, the $(m+l)$ vector $\boldsymbol{S}$ becomes

$$\boldsymbol{S} = -\mathbf{\Phi}_x - \mathbf{C}_x\dot{\boldsymbol{x}} - \mathbf{K}_x\boldsymbol{x}, \tag{4.7}$$

and following the arguments stated in Section 3.2 (Eqs. (3.26)-(3.45)), the corresponding to Eq. (3.44) expression takes the form

$$\ddot{\boldsymbol{x}} = \bar{\mathbf{M}}_x^{+}\left[-\tilde{\mathbf{C}}_x\dot{\boldsymbol{x}} - \tilde{\mathbf{K}}_x\boldsymbol{x} - \tilde{\mathbf{\Phi}}_x + \tilde{\boldsymbol{Q}}_{\boldsymbol{x}}\right] + (\mathbf{I} - \bar{\mathbf{M}}_x^{+}\bar{\mathbf{M}}_x)\boldsymbol{y}. \tag{4.8}$$

In Eq. (4.8), the $(m+l)\times l$ matrices $\tilde{\mathbf{C}}_x$, $\tilde{\mathbf{K}}_x$ and the $(m+l)$ vector $\tilde{\boldsymbol{Q}}_{\boldsymbol{x}}$ are identical with those defined in Section 3.2 (see Eqs. (3.46)-(3.47)), whereas the $(m+l)$ vector $\tilde{\mathbf{\Phi}}_x$ is given by

$$\tilde{\mathbf{\Phi}}_x = \begin{bmatrix}\mathbf{\Phi}_x \\ \mathbf{0}\end{bmatrix}; \tag{4.9}$$

also, $\bar{\mathbf{M}}_x^{+}$ denotes the M-P inverse of the matrix $\bar{\mathbf{M}}_x$ that is defined in Eq. (3.33).

The augmented equations of motion of nonlinear systems, derived via a redundant coordinates modeling scheme are given by

$$\bar{\mathbf{M}}_x\ddot{\boldsymbol{x}} + \bar{\mathbf{C}}_x\dot{\boldsymbol{x}} + \bar{\mathbf{K}}_x\boldsymbol{x} + \bar{\mathbf{\Phi}}_x(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) = \bar{\boldsymbol{Q}}_{\boldsymbol{x}}(t), \tag{4.10}$$

where the $(m+l)\times l$ augmented mass, damping and stiffness matrices, $\bar{\mathbf{M}}_x$, $\bar{\mathbf{C}}_x$ and $\bar{\mathbf{K}}_x$, are defined in Eqs. (3.33), (3.50) and Eq. (3.51), respectively. The $(m+l)$ vector $\bar{\boldsymbol{Q}}_{\boldsymbol{x}}$ is defined in Eq. (3.52), whereas the $(m+l)$ nonlinear vector $\bar{\mathbf{\Phi}}_x$ is given by

$$\bar{\mathbf{\Phi}}_x = \begin{bmatrix}\tilde{\mathbf{A}}\mathbf{\Phi}_x \\ \mathbf{0}\end{bmatrix}; \tag{4.11}$$

$\tilde{\mathbf{A}}$ denotes the $l\times l$ matrix of Eq. (3.27). A more detailed presentation on the construction of the equations of motion for a nonlinear system with singular matrices can be found in Ref. [42].

4.3 Generalized statistical linearization approximate methodology

The statistical linearization approximate methodology is generalized to account for the nonlinear system with singular matrices of Eq. (4.10). To account for singular matrices, a generalization of a formula [91, 7, 39] based on a Gaussian response assumption and related to the expectation of the derivatives of the nonlinear function $\bar{\mathbf{\Phi}}_{\boldsymbol{x}}$ is proved [55, 43].

Following closely Ref. [91], an equivalent linear system is sought in the form

$$(\bar{\mathbf{M}}_{\boldsymbol{x}} + \bar{\mathbf{M}}_e)\ddot{\boldsymbol{x}} + (\bar{\mathbf{C}}_{\boldsymbol{x}} + \bar{\mathbf{C}}_e)\dot{\boldsymbol{x}} + (\bar{\mathbf{K}}_{\boldsymbol{x}} + \bar{\mathbf{K}}_e)\boldsymbol{x} = \bar{\boldsymbol{Q}}_{\boldsymbol{x}}(t), \tag{4.12}$$

where $\bar{\mathbf{M}}_e, \bar{\mathbf{C}}_e$ and $\bar{\mathbf{K}}_e$ denote the equivalent linear $(m+l) \times l$ mass, damping and stiffness matrices, respectively, to account for the nonlinearity of the original system.

The error vector, $\boldsymbol{\varepsilon}$, between the nonlinear and the equivalent linear system is defined as

$$\boldsymbol{\varepsilon} = \bar{\mathbf{\Phi}}_{\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) - \bar{\mathbf{M}}_e\ddot{\boldsymbol{x}} - \bar{\mathbf{C}}_e\dot{\boldsymbol{x}} - \bar{\mathbf{K}}_e\boldsymbol{x}. \tag{4.13}$$

The mean square error is chosen to be minimized (see Eq. (4.4)) for determining the equivalent linear matrices. This yields the equations

$$\frac{\partial}{\partial m_{ij}}\mathbb{E}[\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\varepsilon}] = 0, \tag{4.14}$$

$$\frac{\partial}{\partial c_{ij}}\mathbb{E}[\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\varepsilon}] = 0 \tag{4.15}$$

and

$$\frac{\partial}{\partial k_{ij}}\mathbb{E}[\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\varepsilon}] = 0, \tag{4.16}$$

where m_{ij}^e, c_{ij}^e and k_{ij}^e are the (i, j) elements of the matrices $\bar{\mathbf{M}}_e, \bar{\mathbf{C}}_e$ and $\bar{\mathbf{K}}_e$, respectively. Combining Eq. (4.13) with Eq. (4.4), the minimization criterion can be equivalently written as

$$\sum_{i=1}^{m+l} D_i^2 = \text{minimum}, \tag{4.17}$$

where

$$D_i^2 = \mathbb{E}\left\{\left[\bar{\Phi}_{i,\boldsymbol{x}} - \sum_{j=1}^{l}(m_{ij}^e\ddot{x}_j + c_{ij}^e\dot{x}_j + k_{ij}^e x_j)\right]^2\right\}, \tag{4.18}$$

for $i = 1, 2, \ldots, (m + l)$ and

$$\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}} = \left[\bar{\Phi}_{i,\boldsymbol{x}}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}})\right]^{\mathrm{T}}, \tag{4.19}$$

for $i = 1, 2, \ldots, (m+l)$. Due to the form of the expression in Eq. (4.17), the minimization criterion can be equivalently written as

$$D_i^2 = \text{minimum}, \tag{4.20}$$

for $i = 1, 2, \ldots, (m + l)$. Minimizing Eq. (4.20) yields the equations

$$\mathbb{E}\left[\bar{\Phi}_{i,\boldsymbol{x}}\hat{\boldsymbol{x}}\right] = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}] \begin{bmatrix} \boldsymbol{k}_{\boldsymbol{i}*}^{e\mathrm{T}} \\ \boldsymbol{c}_{\boldsymbol{i}*}^{e\mathrm{T}} \\ \boldsymbol{m}_{\boldsymbol{i}*}^{e\mathrm{T}} \end{bmatrix}, \tag{4.21}$$

for $i = 1, 2, \ldots, (m + l)$. The $3l$ vector $\hat{\boldsymbol{x}}$ of Eq. (4.21) is defined as

$$\hat{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{x}} \end{bmatrix} \tag{4.22}$$

and $\boldsymbol{m}_{\boldsymbol{i}*}^{e\mathrm{T}}$, $\boldsymbol{c}_{\boldsymbol{i}*}^{e\mathrm{T}}$ and $\boldsymbol{k}_{\boldsymbol{i}*}^{e\mathrm{T}}$ correspond to the i^{th} row of $\bar{\mathbf{M}}_e$, $\bar{\mathbf{C}}_e$ and $\bar{\mathbf{K}}_e$, respectively.

The determination of the equivalent linear elements in Eq. (4.21) requires the inversion of $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$. The question arises whether this $3l \times 3l$ matrix $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$ which appears in Eq. (4.21), is singular or not. As proved in Ref. [101], a necessary and sufficient condition for $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$ to be singular is that at least one of the components of the $3l$ vector $\hat{\boldsymbol{x}}$ defined in Eq. (4.22), can be expressed as a linear combination of the remaining components. Note that in the formulation herein it is assumed a priori that the elements of the coordinates vector $\boldsymbol{x}$ are not independent with each other as more than the minimum coordinates are utilized in forming the equations of motion. It is anticipated that some of the elements of $\hat{\boldsymbol{x}}$ are linearly dependent. The matrix $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$ in Eq. (4.21) is singular.

In order to simplify further Eq. (4.21) the following proposition is introduced, which can be construed as a generalization of the theorem proved in Ref. [7].

Proposition 4.1 ([55]). *Let the $3l$ vector $\hat{\boldsymbol{x}}$ be a zero mean jointly Gaussian random vector and $\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}} : \mathbb{R}^{3l} \to \mathbb{R}^{3l}$ be a smooth multivariate function. Then, the expression*

$$\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}\mathbb{E}[\bar{\Phi}_{i,\boldsymbol{x}}\hat{\boldsymbol{x}}] = \mathbb{E}[\nabla\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}(\hat{\boldsymbol{x}})], \tag{4.23}$$

where $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}$ *denotes the* M-P *inverse matrix of* $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$*, holds true.*

Proof. Taking into account the definition of the expected value and assuming that the joint Gaussian pdf of $\hat{\boldsymbol{x}}$ is denoted by $\boldsymbol{p}(\hat{\boldsymbol{x}})$, the expression

$$\mathbb{E}[\nabla\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}(\hat{\boldsymbol{x}})] = \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\nabla\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}(\hat{\boldsymbol{x}})\boldsymbol{p}(\hat{\boldsymbol{x}})d\hat{\boldsymbol{x}}^{\mathrm{T}}, \tag{4.24}$$

holds true. As noted in Section 4.3, the matrix $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$ is singular. Considering a multivariate Gaussian distribution with a singular covariance matrix $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$ [52, 88, 102], the pdf of $\hat{\boldsymbol{x}}$ is given by

$$\boldsymbol{p}(\hat{\boldsymbol{x}}) = ((2\pi)^{k}|\mathbf{B}^{\mathrm{T}}\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]\mathbf{B}|)^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}\hat{\boldsymbol{x}}^{\mathrm{T}}\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}\hat{\boldsymbol{x}}\right\}. \tag{4.25}$$

In Eq. (4.25), the M-P inverse of the matrix $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$ has the form

$$\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+} = \mathbf{B}(\mathbf{B}^{\mathrm{T}}\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}, \tag{4.26}$$

and $\mathbf{B}$ satisfies the relationship

$$|\mathbf{B}^{\mathrm{T}}\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]\mathbf{B}| = \lambda_1\lambda_2\dots\lambda_\rho, \tag{4.27}$$

where $\lambda_i,\ i = 1, 2, \dots, \rho$ denote the non-zero eigenvalues of the singular matrix $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$ [52].

Following closely Ref. [7], the right hand side of Eq. (4.24) is integrated by parts yielding

$$\mathbb{E}[\nabla\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}(\hat{\boldsymbol{x}})] = \boldsymbol{r} - \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\bar{\Phi}_{i,\boldsymbol{x}}\nabla\boldsymbol{p}(\hat{\boldsymbol{x}})d\hat{\boldsymbol{x}}^{\mathrm{T}}, \tag{4.28}$$

where $\boldsymbol{r}$ is a $3l$ vector with

$$r_i = \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\left\{\bar{\Phi}_{i,\boldsymbol{x}}\boldsymbol{p}(\hat{\boldsymbol{x}})|_{x_i=-\infty}^{x_i=+\infty}\right\}\prod_{\substack{j=1\\ i\neq j}}^{3l}dx_j,\ i = 1, 2, \dots, 3l. \tag{4.29}$$

Without loss of generality, the quantity in the brackets in Eq. (4.29) is assumed next to be zero at $x_i = \pm\infty$. This is further substantiated by the form of $\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}$ ordinarily met in practice; see also Ref. [7]. Eq. (4.28) becomes

$$\mathbb{E}[\nabla\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}(\hat{\boldsymbol{x}})] = -\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\bar{\Phi}_{i,\boldsymbol{x}}\nabla\boldsymbol{p}(\hat{\boldsymbol{x}})d\hat{\boldsymbol{x}}^{\mathrm{T}}. \tag{4.30}$$

Note that applying the nabla operator to Eq. (4.25) yields

$$\nabla \boldsymbol{p}(\hat{\boldsymbol{x}}) = -\frac{1}{2}\boldsymbol{p}(\hat{\boldsymbol{x}})\nabla\left(\hat{\boldsymbol{x}}^{\mathrm{T}}\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}\hat{\boldsymbol{x}}\right) \tag{4.31}$$

and noticing that $\nabla\left(\hat{\boldsymbol{x}}^{\mathrm{T}}\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}\hat{\boldsymbol{x}}\right) = 2\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}\hat{\boldsymbol{x}}$, Eq. (4.31) becomes, equivalently,

$$\nabla \boldsymbol{p}(\hat{\boldsymbol{x}}) = -\boldsymbol{p}(\hat{\boldsymbol{x}})\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}\hat{\boldsymbol{x}}. \tag{4.32}$$

Considering Eq. (4.32), Eq. (4.30) takes the form

$$\mathbb{E}[\nabla\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}(\hat{\boldsymbol{x}})] = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\bar{\Phi}_{i,\boldsymbol{x}}\hat{\boldsymbol{x}}\boldsymbol{p}(\hat{\boldsymbol{x}})d\hat{\boldsymbol{x}}^{\mathrm{T}} \tag{4.33}$$

or, equivalently,

$$\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}\mathbb{E}[\bar{\Phi}_{i,\boldsymbol{x}}\hat{\boldsymbol{x}}] = \mathbb{E}[\nabla\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}(\hat{\boldsymbol{x}})], \tag{4.34}$$

which proves Eq. (4.23). □

For the singular matrix $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$ of Eq. (4.21), the M-P inverse matrix on the left hand side of Eq. (4.23) is also singular, and thus, taking into account Eqs. (2.3)-(2.6) and Eq. (2.12), Eq. (4.23) has the following M-P type solution

$$\mathbb{E}[\bar{\Phi}_{i,\boldsymbol{x}}\hat{\boldsymbol{x}}] = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]\mathbb{E}[\nabla\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}(\hat{\boldsymbol{x}})] + \left\{\mathbf{I} - \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}\right\}\boldsymbol{w}, \tag{4.35}$$

where $\boldsymbol{w}$ is an arbitrary $3l$ vector.

For $\boldsymbol{w} = \mathbf{0}$ a particular solution for $\mathbb{E}[\bar{\Phi}_{i,\boldsymbol{x}}\hat{\boldsymbol{x}}]$ is obtained in the form

$$\mathbb{E}[\bar{\Phi}_{i,\boldsymbol{x}}\hat{\boldsymbol{x}}] = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]\mathbb{E}[\nabla\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}(\hat{\boldsymbol{x}})]. \tag{4.36}$$

Eq. (4.36) was utilized in Ref. [42], and can be construed as a direct generalization of the standard relationship for non-singular matrices [91, 7]. The step of arbitrarily choosing the solution of Eq. (4.36) for $\mathbb{E}[\bar{\Phi}_{i,\boldsymbol{x}}\hat{\boldsymbol{x}}]$ corresponding to $\boldsymbol{w} = \mathbf{0}$ can be circumvented by directly treating Eq. (4.23). Eqs. (4.21) and (4.23) are pre-multiplied by $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}$ and $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$, respectively, yielding

$$\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]\begin{bmatrix}\boldsymbol{k}_{\boldsymbol{i}*}^{e\mathrm{T}}\\\boldsymbol{c}_{\boldsymbol{i}*}^{e\mathrm{T}}\\\boldsymbol{m}_{\boldsymbol{i}*}^{e\mathrm{T}}\end{bmatrix}=\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]\mathbb{E}\begin{bmatrix}\dfrac{\partial\bar{\Phi}_{i,x}}{\partial\boldsymbol{x}}\\\dfrac{\partial\bar{\Phi}_{i,x}}{\partial\dot{\boldsymbol{x}}}\\\dfrac{\partial\bar{\Phi}_{i,x}}{\partial\ddot{\boldsymbol{x}}}\end{bmatrix}, \tag{4.37}$$

for $i=1,2,\ldots,(m+l)$.

Likewise for the determination of the equivalent linear elements in Eq. (4.37) the inversion of $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$ is required, and since $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$ is singular, its M-P inverse is employed for determining an expression for the elements m_{ij}^e, c_{ij}^e and k_{ij}^e of the equivalent linear augmented matrices. Considering Eq. (2.12), Eq. (4.37) is written in the form

$$\begin{bmatrix}\boldsymbol{k}_{\boldsymbol{i}*}^{e\mathrm{T}}\\\boldsymbol{c}_{\boldsymbol{i}*}^{e\mathrm{T}}\\\boldsymbol{m}_{\boldsymbol{i}*}^{e\mathrm{T}}\end{bmatrix}=\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]\mathbb{E}\begin{bmatrix}\dfrac{\partial\bar{\Phi}_{i,x}}{\partial\boldsymbol{x}}\\\dfrac{\partial\bar{\Phi}_{i,x}}{\partial\dot{\boldsymbol{x}}}\\\dfrac{\partial\bar{\Phi}_{i,x}}{\partial\ddot{\boldsymbol{x}}}\end{bmatrix}+\boldsymbol{g}(\boldsymbol{y}), \tag{4.38}$$

for $i=1,2,\ldots,(m+l)$, where the $3l$ vector

$$\boldsymbol{g}(\boldsymbol{y})=(\mathbf{I}-\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}])\boldsymbol{y}, \tag{4.39}$$

is the arbitrary part of the M-P inverse based general solution of Eq. (4.37). It is deemed important to note that when the minimum number of coordinates, n, is utilized, $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$ is a non-singular matrix yielding

$$\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}=\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{-1}. \tag{4.40}$$

In that case, the vector $\hat{\boldsymbol{x}}$ of Eq. (4.22) becomes

$$\hat{\boldsymbol{x}}=\begin{bmatrix}\boldsymbol{q} & \dot{\boldsymbol{q}} & \ddot{\boldsymbol{q}}\end{bmatrix}^{\mathrm{T}} \tag{4.41}$$

and combining Eq. (4.39) with Eq. (4.40), Eq. (4.38) takes the well-established form

$$\begin{bmatrix}\boldsymbol{k}_{\boldsymbol{i}*}^{e\mathrm{T}}\\\boldsymbol{c}_{\boldsymbol{i}*}^{e\mathrm{T}}\\\boldsymbol{m}_{\boldsymbol{i}*}^{e\mathrm{T}}\end{bmatrix}=\mathbb{E}\begin{bmatrix}\dfrac{\partial\Phi_{i,q}}{\partial\boldsymbol{q}}\\\dfrac{\partial\Phi_{i,q}}{\partial\dot{\boldsymbol{q}}}\\\dfrac{\partial\Phi_{i,q}}{\partial\ddot{\boldsymbol{q}}}\end{bmatrix}, \tag{4.42}$$

for $i = 1, 2, \ldots, n$. Eq. (4.42) represents the celebrated expressions for determining the elements of the equivalent linear mass, damping and stiffness matrices in the standard formulation of the statistical linearization methodology [91]. When formulating the system equations of motion by employing additional degrees-of-freedom, $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$ is singular and the generalized version of Eq. (4.42) (i.e. Eq. (4.38)) needs to be considered. Regarding Eq. (4.38), it can be readily seen that a critical step for the practical implementation of the generalized statistical linearization methodology is the choice of the arbitrary element $\boldsymbol{y}$. It is proved in the following proposition that the solution obtained by setting the arbitrary element $\boldsymbol{y}$ equal to zero is not only intuitively the simplest but it is also at least as good (in the sense of minimizing the mean square error, where the error $\boldsymbol{\varepsilon}$ is defined in Eq. (4.13)) as any other solution obtained by selecting an arbitrary non-zero value for $\boldsymbol{y}$. Setting $\boldsymbol{y} = \mathbf{0}$, Eq. (4.38) becomes

$$\begin{bmatrix} \boldsymbol{k}_{\boldsymbol{i*}}^{e\mathrm{T}} \\ \boldsymbol{c}_{\boldsymbol{i*}}^{e\mathrm{T}} \\ \boldsymbol{m}_{\boldsymbol{i*}}^{e\mathrm{T}} \end{bmatrix} = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]\mathbb{E}\begin{bmatrix} \dfrac{\partial\bar{\Phi}_{i,x}}{\partial\boldsymbol{x}} \\ \dfrac{\partial\bar{\Phi}_{i,x}}{\partial\dot{\boldsymbol{x}}} \\ \dfrac{\partial\bar{\Phi}_{i,x}}{\partial\ddot{\boldsymbol{x}}} \end{bmatrix}, \tag{4.43}$$

for $i = 1, 2, \ldots, (m + l)$.

Assume next that $(m_{ij}^e, c_{ij}^e, k_{ij}^e)$ is the set of parameters arising from solving Eq. (4.43) and corresponding to the equivalent matrices $\bar{\mathbf{M}}_e$, $\bar{\mathbf{C}}_e$ and $\bar{\mathbf{K}}_e$. Selecting an arbitrary vector $\boldsymbol{y} = \boldsymbol{y}_0 \neq \mathbf{0}$ in Eq. (4.39), a different set of parameters, $(m_{ij}^{'e}, c_{ij}^{'e}, k_{ij}^{'e})$, corresponding to matrices $\bar{\mathbf{M}}_e^{'}$, $\bar{\mathbf{C}}_e^{'}$, $\bar{\mathbf{K}}_e^{'}$, is obtained via Eq. (4.38); see also Ref. [101].

Proposition 4.2 ([42]). *Let* $D_i^2(m_{ij}^e, c_{ij}^e, k_{ij}^e)$ *and* $D_i^2(m_{ij}^{'e}, c_{ij}^{'e}, k_{ij}^{'e})$ *denote the errors as defined in Eq. (4.18) corresponding to the parameters values* $m_{ij}^e, c_{ij}^e, k_{ij}^e$ *and* $m_{ij}^{'e}, c_{ij}^{'e}, k_{ij}^{'e}$, *respectively. Then,*

$$D_i^2(m_{ij}^e, c_{ij}^e, k_{ij}^e) \leq D_i^2(m_{ij}^{'e}, c_{ij}^{'e}, k_{ij}^{'e}), \tag{4.44}$$

for $i = 1, 2, \ldots, (m + l)$ *and* $j = 1, 2, \ldots, l$.

Proof. It is seen from Eq. (4.18) that the quantity $D_i^2(m_{ij}^e, c_{ij}^e, k_{ij}^e)$ is a quadratic polynomial with respect to the parameters m_{ij}^e, c_{ij}^e and k_{ij}^e. Therefore, its mixed partial derivatives concerning $m_{ij}^e, c_{ij}^e, k_{ij}^e$ of order higher that two vanish. Taking into account

Eq. (4.38), the two sets of parameters are connected via the expressions

$$\begin{aligned} m_{ij}^{\prime e} &= m_{ij}^{e} + g_{m,i}(y_0), && (4.45)\\ c_{ij}^{\prime e} &= c_{ij}^{e} + g_{c,i}(y_0), && (4.46)\\ k_{ij}^{\prime e} &= k_{ij}^{e} + g_{k,i}(y_0), && (4.47) \end{aligned}$$

where the terms $g_{m,i}(y_0), g_{c,i}(y_0), g_{k,i}(y_0), i = 1, 2, \ldots, m+l$, represent the arbitrary elements as defined in Eq. (4.39). Considering a Taylor's expansion around $(m_{ij}^e, c_{ij}^e, k_{ij}^e)$, yields

$$\begin{aligned} D_i^2(m_{ij}^{\prime e}, c_{ij}^{\prime e}, k_{ij}^{\prime e}) &= D_i^2(m_{ij}^e, c_{ij}^e, k_{ij}^e) \\ &+ \sum_{j=1}^{l}\left(\frac{\partial D_i^2}{\partial m_{ij}^e} g_{m,i}(y_0) + \frac{\partial D_i^2}{\partial c_{ij}^e} g_{c,i}(y_0) + \frac{\partial D_i^2}{\partial k_{ij}^e} g_{k,i}(y_0)\right) \\ &+ \frac{1}{2}\mathbb{E}\left\{\left[\sum_{j=1}^{l}(g_{m,i}(y_0)\ddot{x}_j + g_{c,i}(y_0)\dot{x}_j + g_{k,i}(y_0)x_j)\right]^2\right\}, \end{aligned} \tag{4.48}$$

for $i = 1, 2, \ldots, m+l$, where the terms $g_{m,i}(y_0)$, $g_{c,i}(y_0)$ and $g_{k,i}(y_0)$ denote the distance between the two sets of parameters.

Taking into account Eqs. (4.14)-(4.16), the necessary conditions for minimizing Eq. (4.20) are

$$\begin{aligned} \frac{\partial D_i^2}{\partial m_{ij}^e} &= 0, && (4.49)\\ \frac{\partial D_i^2}{\partial c_{ij}^e} &= 0 && (4.50) \end{aligned}$$

and

$$\frac{\partial D_i^2}{\partial k_{ij}^e} = 0. \tag{4.51}$$

Utilizing Eqs. (4.49)-(4.51), the first sum on the right hand side of Eq. (4.48) is zero and Eq. (4.48) takes the form

$$D_i^2(m_{ij}^{\prime e}, c_{ij}^{\prime e}, k_{ij}^{\prime e}) = D_i^2(m_{ij}^e, c_{ij}^e, k_{ij}^e) + \frac{1}{2}\mathbb{E}\left\{J_i^2\right\}, \tag{4.52}$$

for $i = 1, 2, \ldots, m + l$, where

$$J_i = \sum_{j=1}^{l}(g_{m,i}(y_0)\ddot{x}_j + g_{c,i}(y_0)\dot{x}_j + g_{k,i}(y_0)x_j). \tag{4.53}$$

Taking into account that $\mathbb{E}\left\{J_i^2\right\} \geq 0$, for all $i = 1, 2, \dots, m + l$, Eq. (4.52) proves the argument stated in Eq. (4.44). $\square$

Based on Eq. (4.44), utilizing Eq. (4.43) yields equivalent linear elements corresponding to an error that is at least as small (in a mean square sense) as any other obtained by utilizing a non-zero $\boldsymbol{y}$ vector in Eq. (4.38).

It is noted that comparing the standard Eq. (4.42) with its generalized counterpart Eq. (4.43) the equivalent linear matrices in Eq. (4.43) have typically a more complex structure than their counterparts in Eq. (4.42). Due to the fact that in Eq. (4.43) the product $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]^{+}\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$ does not yield a unitary matrix, the equivalent linear matrices are anticipated to have many more non-zero components than in the case of utilizing Eq. (4.42). This observation is further highlighted in the numerical example section. Additionally, the determination of the equivalent linear matrices in Eq. (4.43) requires the knowledge of the response covariance matrix $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$. An additional system of equations is needed that relates the two sets of unknowns, i.e. the response covariance matrix and the equivalent linear elements. Focusing on the linearized system of Eq. (4.12), generalized excitation-response relationships can be employed. As seen in Sections 3.3.1 and 3.3.2, the standard state-variable formulation, the complex modal analysis as well as the classical frequency domain analysis were generalized for treating systems with singular matrices and determining the augmented system response covariance matrix [41, 55]. These approaches are included in the following subsections.

4.3.1 State variable formulation and analysis

The first step of the state variable formulation consists in deriving the state variable form of the system equations of motion. Considering the M-P inverse of the $\bar{\mathbf{M}}_x + \bar{\mathbf{M}}_e$ matrix, the augmented equivalent linear system of Eq. (4.12) can be cast in the corresponding of Eq. (3.54) form, i.e.

$$\dot{\boldsymbol{p}} = \mathbf{G}_x \boldsymbol{p} + \boldsymbol{f}_x, \tag{4.54}$$

where the state vector $\boldsymbol{p}$ is given by Eq. (3.53) and the $2l \times 2l$ matrix $\mathbf{G}_x$ (see also Eq. (3.55)) is given by

$$\mathbf{G}_x = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -(\bar{\mathbf{M}}_x + \bar{\mathbf{M}}_e)^{+}(\bar{\mathbf{K}}_x + \bar{\mathbf{K}}_e) & -(\bar{\mathbf{M}}_x + \bar{\mathbf{M}}_e)^{+}(\bar{\mathbf{C}}_x + \bar{\mathbf{C}}_e) \end{bmatrix}. \tag{4.55}$$

The $2l$ vector $\boldsymbol{f_x}$ (see also Eq. (3.56)) is given by

$$\boldsymbol{f_x} = \begin{bmatrix} \mathbf{0} \\ (\bar{\mathbf{M}}_{\boldsymbol{x}} + \bar{\mathbf{M}}_e)^+ \bar{\boldsymbol{Q}}_{\boldsymbol{x}} \end{bmatrix}. \tag{4.56}$$

For zero initial conditions (i.e. $\boldsymbol{p}(0) = \mathbf{0}$), the solution of the state variable form of the augmented system equation of motion is

$$\boldsymbol{p}(t) = \int_0^t \exp(\mathbf{G}_{\boldsymbol{x}}\tau) \boldsymbol{f_x}(t-\tau) d\tau, \tag{4.57}$$

where the $2l \times 2l$ transition matrix $\exp(\mathbf{G}_{\boldsymbol{x}}t)$ has a block form which is similar to that of Eq. (3.60). Taking into account Eq. (4.57) and the block form of the transition matrix, the response displacement vector, $\boldsymbol{x}$, is

$$\boldsymbol{x}(t) = \int_0^t \mathbf{h}_{\boldsymbol{x}}(\tau) \bar{\boldsymbol{Q}}_{\boldsymbol{x}}(t-\tau) d\tau, \tag{4.58}$$

where

$$\mathbf{h}_{\boldsymbol{x}}(t) = \mathbf{b}_{\boldsymbol{x}}(t)(\bar{\mathbf{M}}_{\boldsymbol{x}} + \bar{\mathbf{M}}_e)^+ \tag{4.59}$$

can be construed as the uniquely defined ”*generalized*” impulse response matrix; see also the arguments followed in deriving Eqs. (3.58)-(3.63).

In deriving Eq. (4.58) arguments for neglecting the arbitrary term associated with the M-P inverse of the $\bar{\mathbf{M}}_{\boldsymbol{x}} + \bar{\mathbf{M}}_e$ matrix have been employed. These relate to uniquely defining a response acceleration vector (see also Eq. (4.8)) as suggested by experimental observations [112]; a detailed discussion of the topic can be found in Section 3.3.1.1.

Similar to the linear case described in Section 3.3.1.1, manipulating Eq. (4.54) and taking expectations yields the equation for the system response covariance matrix in the following, corresponding to Eq. (3.71), form

$$\dot{\mathbf{V}}_{\boldsymbol{x}} = \mathbf{G}_{\boldsymbol{x}} \mathbf{V}_{\boldsymbol{x}} + \mathbf{V}_{\boldsymbol{x}} \mathbf{G}_{\boldsymbol{x}}^{\mathrm{T}} + \mathbf{S}_{\boldsymbol{x}}, \tag{4.60}$$

where the state matrix $\mathbf{G}_{\boldsymbol{x}}$ is in the present case given by Eq. (4.55). Following similar arguments to those stated in Section 3.3.1.1, and focusing on stationary white noise excitation and stationary system response, a Lyapunov equation for determining the covariance matrix of the system response is obtained (see Eq. (3.75)). As highlighted in Ref. [41], the Lyapunov equation does not have a unique solution due to the form of the augmented matrix $\mathbf{G}_{\boldsymbol{x}}$ of Eq. (4.55). Nevertheless, recasting it in a form that utilizes the

Kronecker product, it has been shown that a solution for the response covariance matrix can be provided; see Eqs. (3.77)-(3.81) for more details.

4.3.2 Complex modal analysis

Focusing on a complex modal analysis treatment, due to the form of the matrix $\mathbf{G}_{\boldsymbol{x}}$ of Eq. (4.55), its eigenvectors that correspond to its zero eigenvalues are linearly dependent. A standard eigen-decomposition analysis cannot be performed as is the case for modeling using generalized coordinates. The singular value decomposition (SVD) method can be applied for $\mathbf{G}_{\boldsymbol{x}}$ yielding a corresponding to Eq. (3.117) expression, i.e.

$$\mathbf{G}_{\boldsymbol{x}} = \mathbf{U}\boldsymbol{\eta}_{\boldsymbol{x}}\boldsymbol{\Psi}^{*}. \tag{4.61}$$

As in Section 3.3.1.3, the diagonal $2l \times 2l$ matrix $\boldsymbol{\eta}_{\boldsymbol{x}}$ is given by

$$\boldsymbol{\eta}_{\boldsymbol{x}} = diag(\sigma_1, \sigma_2, \dots, \sigma_r, 0, 0, \dots, 0). \tag{4.62}$$

where the singular values $\sigma_j, j = 1, 2, \dots, 2l$ of the matrix $\mathbf{G}_{\boldsymbol{x}}$ are given by Eq. (3.119), whereas the $2l \times 2l$ matrices $\mathbf{U} = \left[\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_{2l}\right]$ and $\boldsymbol{\Psi} = \left[\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \dots, \boldsymbol{\psi}_{2l}\right]$ are unitary; $\boldsymbol{\psi}_j$ is the eigenvector corresponding to the singular value σ_j $(j = 1, 2, \dots, 2l)$ whereas $\boldsymbol{u}_j$ is equal to $\boldsymbol{u}_j = \dfrac{\mathbf{G}_{\boldsymbol{x}}\boldsymbol{\psi}_j}{\sigma_j}$.

Utilizing the SVD of Eq. (4.61), Eq. (4.54) can be alternatively written as

$$\dot{\boldsymbol{z}}_{\boldsymbol{x}} = \boldsymbol{\Psi}^{*}\mathbf{U}\boldsymbol{\eta}_{\boldsymbol{x}}\boldsymbol{z}_{\boldsymbol{x}} + \boldsymbol{g}_{\boldsymbol{x}}, \tag{4.63}$$

where $\boldsymbol{g}_{\boldsymbol{x}}$ and $\boldsymbol{p}$ are given by Eq. (3.124) and Eq. (3.121), respectively. Eq. (4.57) becomes

$$\boldsymbol{z}_{\boldsymbol{x}}(t) = \int_0^t \mathbf{H}_{\boldsymbol{x}}(s)\boldsymbol{g}_{\boldsymbol{x}}(t-s)ds, \tag{4.64}$$

where

$$\mathbf{H}_{\boldsymbol{x}}(t) = \exp(\boldsymbol{\Psi}^{*}\mathbf{U}\boldsymbol{\eta}_{\boldsymbol{x}}t). \tag{4.65}$$

As pointed out in Section 3.3.1.3 (see also Ref. [41]), a complex modal analysis does not result in uncoupling the coupled system of Eq. (4.63). The product $\boldsymbol{\Psi}^{*}\mathbf{U}$ does not yield a unitary matrix as in the case of utilizing the minimum number of coordinates, and thus, $\mathbf{H}_{\boldsymbol{x}}(t)$ is not diagonal. Nevertheless, relying on a SVD of matrix

$\mathbf{G}_{\boldsymbol{x}}$ facilitates significantly the numerical computation of the system response statistics. Considering Eq. (4.64) and manipulating yields the covariance matrix $\mathbf{w}_{\boldsymbol{z_x}}$ of the response vector $\boldsymbol{z_x}$ in the form

$$\mathbf{w}_{\boldsymbol{z_x}}(\tau) = \int_0^{\infty} \int_0^{\infty} \mathbf{H}_{\boldsymbol{x}}(s_1)\mathbf{w}_{\boldsymbol{g_x}}(\tau + s_1 - s_2)\mathbf{H}^*_{\boldsymbol{x}}(s_2)\, ds_1 ds_2. \tag{4.66}$$

The relationship $\boldsymbol{p} = \boldsymbol{\Psi z_x}$ (see also Eq. (3.121)) can be used for determining the covariance matrix of the response vector $\boldsymbol{p}$ in the form

$$\mathbf{w}_{\boldsymbol{p}}(\tau) = \boldsymbol{\Psi}\mathbf{w}_{\boldsymbol{z_x}}(\tau)\boldsymbol{\Psi}^*; \tag{4.67}$$

a more detailed presentation of how Eqs. (4.66)-(4.67) are derived can be found in Section 3.3.1.3.

4.3.3 Frequency domain analysis

Consider that a redundant coordinates modeling formulation is followed for deriving the governing equations of motion of an $n-$DOF nonlinear system. The augmented form of the nonlinear system equations of motion are given by Eq. (4.10), and the augmented nonlinear vector $\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}$ is given by Eq. (4.11); see Section 4.2.3 for more details. Following the generalized version of the statistical linearization methodology presented in Section 4.3, an equivalent to Eq. (4.10) linear system is sought in the form given by Eq. (4.12).

Comparing Eqs. (3.49) and (4.12), the FRF matrix of the equivalent linear system of Eq. (4.12) is given by

$$\boldsymbol{\alpha}_{\mathrm{e}}(\omega) = \mathbf{R}_e^+, \tag{4.68}$$

where $\mathbf{R}_e^+$ denotes the M-P inverse of the matrix

$$\mathbf{R}_e = -\omega^2(\bar{\mathbf{M}}_{\boldsymbol{x}} + \bar{\mathbf{M}}_e) + i\omega(\bar{\mathbf{C}}_{\boldsymbol{x}} + \bar{\mathbf{C}}_e) + (\bar{\mathbf{K}}_{\boldsymbol{x}} + \bar{\mathbf{K}}_e). \tag{4.69}$$

Without loss of generality, it has been assumed in Eq. (4.68) that the $\mathbf{R}_e$ matrix has full rank. In a different case, the generalized expression of the FRF matrix arising from solving Eq. (3.180), i.e. Eq. (3.182), should be considered. The response statistics are determined via applying Eqs. (3.187)-(3.188). The generalized statistical linearization methodology presented in Section 4.3 is followed.

As in the application of the time domain analysis, determining the equivalent linear matrices in Eq. (4.37) requires knowledge of the response covariance matrix $\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}]$. The herein derived frequency domain input-output Eq. (3.186) is utilized as the additional set of equations required for relating the covariance matrix and the equivalent linear matrices. The developed generalized statistical linearization methodology can be construed as the frequency domain alternative to the time-domain methodology presented in Section 4.3.1 (see also Ref. [42]).

4.3.4 Mechanization of the generalized statistical linearization methodology

Depending on either the generalized statistical linearization scheme presented in Section 4.3 is used in conjunction with a time or frequency domain analysis treatment, a coupled nonlinear system has to be solved for determining the system response covariance matrix as well as the elements of the equivalent linear matrices $\bar{\mathbf{M}}_e$, $\bar{\mathbf{C}}_e$ and $\bar{\mathbf{K}}_e$. In both time and frequency domain approaches, Eq. (4.43) constitutes the first equation of the coupled nonlinear system.

Based on a modeling utilizing more than the minimum number of DOFs, in conjunction with a time domain analysis treatment, Eqs. (4.43) and (4.60) (or, alternatively, Eq. (4.43) and Eqs. (4.66)-(4.67) if a complex modal analysis is employed) constitute a coupled nonlinear system of equations to be solved for determining the system response covariance matrix and the equivalent linear elements. Utilizing a frequency domain approach, the equivalent linear matrices $\bar{\mathbf{M}}_e$, $\bar{\mathbf{C}}_e$ and $\bar{\mathbf{K}}_e$ as well as the system response covariance matrix are determined by solving the coupled nonlinear system comprised of Eqs. (3.187)-(3.188) and Eq. (4.43).

For the solution of the coupled system, any appropriate standard numerical optimization scheme can be applied [76]. The following iterative procedure can be utilized as an alternative straightforward approach.

The first step consists of selecting initial values for the equivalent linear matrices, $\bar{\mathbf{M}}_e$, $\bar{\mathbf{C}}_e$ and $\bar{\mathbf{K}}_e$, which are set equal to null matrices. Following the selection of an appropriate convergence criterion, the following two steps are repeated successively.

- If a time domain analysis treatment is followed, the system response covariance matrix is determined via Eq. (4.60) (or, alternatively via Eqs. (4.66)-(4.67)). In a similar manner, the corresponding covariance matrix for the frequency domain analysis is computed via Eq. (3.187) and Eq. (3.188).

- Combining Eq. (4.43) with the system response covariance matrix obtained in the previous step (depending on either a time or frequency domain analysis treatment is followed), updated values for the equivalent linear matrices are calculated.

The iterative method stops when convergence is attained.

4.4 Numerical Examples

4.4.1 Generalized statistical linearization in conjunction with a time domain analysis treatment

As a numerical example the system of two rigid masses m_1 and m_2 shown in Figure 4.1 is considered. It is assumed that the mass m_1 is connected to the ground by a nonlinear spring of the linear-plus-cubic type and by a linear damper with coefficient c_1. A mass m_2 is connected to m_1 by a linear spring and a linear damper with coefficients k_2 and c_2, respectively. The applied random force $Q_2(t)$ is modeled as a white-noise process with a correlation function $w_{Q_2}(t) = 2\pi S_0 \delta(t)$, where S_0 is the (constant) power spectrum value of $Q_2(t)$. q_1, q_2 are the generalized displacements, as shown in Figure 4.1.

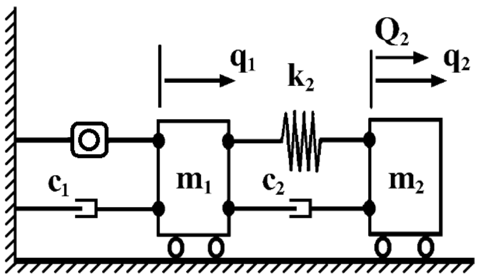

FIGURE 4.1: A two degree-of-freedom nonlinear structural system under stochastic excitation.

Utilizing generalized coordinates the equations of motion governing the system in Figure 4.1 can be written in the matrix form of Eq. (4.1), where the matrices

$\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ are given by (see also Ref. [91])

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} (c_1 + c_2) & -c_2 \\ -c_2 & c_2 \end{bmatrix} \tag{4.70}$$

and

$$\mathbf{K} = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \tag{4.71}$$

respectively; the coordinate vector $\boldsymbol{q}$ and the excitation vector $\boldsymbol{Q}(t)$ are given by

$$\boldsymbol{q} \;=\; \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{4.72}$$

and

$$\boldsymbol{Q} \;=\; \begin{bmatrix} 0 \\ Q_2(t) \end{bmatrix}, \tag{4.73}$$

whereas the nonlinear function $\boldsymbol{\Phi}$ takes the form

$$\boldsymbol{\Phi}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \begin{bmatrix} \varepsilon_1 k_1 q_1^3 \\ 0 \end{bmatrix}. \tag{4.74}$$

Taking into account the fact that the minimum number of DOFs are utilized in modeling the system equations of motion, i.e. $\mathbb{E}[\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}^{\mathrm{T}}]^{+} = \mathbb{E}[\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}^{\mathrm{T}}]^{-1}$, Eq. (4.42) yields

$$\mathbf{K}_e = \begin{bmatrix} 3\varepsilon_1 k_1 \sigma_{q_1}^2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.75}$$

Focusing on the stationary system response, i.e. $\dot{\mathbf{V}} = \mathbf{0}$, a standard statistical linearization procedure in conjunction with a complex modal analysis treatment (e.g. Ref. [91]) for the values $m_1 = m_2 = m = 1, c_1 = c_2 = c = 0.1, k_1 = k_2 = k = 1$ and $S_0 = 10^{-3}$, is applied. Regarding the numerical implementation, convergence based on the criterion

$$\left| \frac{\mathbf{K}_e^{j+1} - \mathbf{K}_e^{j}}{\mathbf{K}_e^{j}} \right| > 10^{-5}, \tag{4.76}$$

where the j index denotes the $j - th$ iteration, is satisfied after eight iterations. The initial value $\mathbf{K}_e^0$ has been set equal to zero. By applying a complex modal analysis treatment, the eigenvalues of the system after the last iteration are

$$\lambda_1 = -0.1308 - 1.6389i \quad , \quad \lambda_2 = -0.1308 + 1.6389i,$$
$$\lambda_3 = -0.0192 - 0.6422i \quad , \quad \lambda_4 = -0.0192 + 0.6422i, \tag{4.77}$$

whereas the corresponding eigenvectors are

$$\begin{aligned} \boldsymbol{v}_1^T &= \begin{bmatrix} -0.0357 - 0.4466i & 0.0188 + 0.2626i & 0.7366 & -0.4328 - 0.0036i \end{bmatrix}, \\ \boldsymbol{v}_2^T &= \begin{bmatrix} -0.0357 + 0.4466i & 0.0188 - 0.2626i & 0.7366 & -0.4328 + 0.0036i \end{bmatrix}, \\ \boldsymbol{v}_3^T &= \begin{bmatrix} -0.4260 - 0.0014i & -0.7255 & 0.0090 - 0.2736i & 0.0139 - 0.4659i \end{bmatrix}, \\ \boldsymbol{v}_4^T &= \begin{bmatrix} -0.4260 + 0.0014i & -0.7255 & 0.0090 + 0.2736i & 0.0139 + 0.4659i \end{bmatrix}. \end{aligned} \tag{4.78}$$

The obtained covariance matrix of the system response is given by

$$\mathbf{V} = \begin{bmatrix} 0.0386 & 0.0639 & 0 & -0.0010 \\ 0.0639 & 0.1102 & 0.0010 & 0 \\ 0 & 0.0010 & 0.0178 & 0.0252 \\ -0.0010 & 0 & 0.0252 & 0.0462 \end{bmatrix}. \tag{4.79}$$

Consider next that the system of two masses m_1 and m_2 depicted in Figure 4.1 is modeled as a multi-body one, consisting of two separate subsystems as shown in Figure 4.2; see also Ref. [41]. The two subsystems are related based on the constraint

$$x_2 = x_1 + d, \tag{4.80}$$

where d is the length of mass m_1. The "*unconstrained*" equations of motion are derived by treating the three coordinates ($\bar{x}_1, x_2$ and $\bar{x}_3$) as independent with each other. The equation of motion of the composite system is derived by including the constraint defined in Eq. (4.80), or, equivalently

$$x_2 = \bar{x}_1 + l_{1,0} + d, \tag{4.81}$$

where $l_{1,0}$ is the unstretched length of the spring k_1. Based on a Lagrangian formulation of the equations of motion, Eq. (4.5) is formed [41]. The 3×3 matrices $\mathbf{M}_{\boldsymbol{x}}, \mathbf{C}_{\boldsymbol{x}}$ and

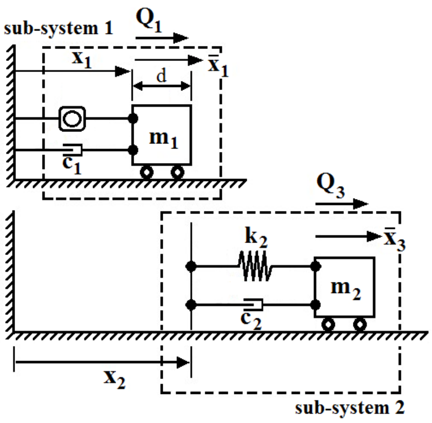

FIGURE 4.2: A three degree-of-freedom nonlinear structural system under stochastic excitation.

$\mathbf{K}_x$ become

$$\mathbf{M}_x = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & m_2 \\ 0 & m_2 & m_2 \end{bmatrix}, \quad \mathbf{C}_x = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_2 \end{bmatrix}, \quad \mathbf{K}_x = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_2 \end{bmatrix}, \tag{4.82}$$

whereas the 3 vectors $\boldsymbol{\Phi}_x$ and $\boldsymbol{Q}_x$ become

$$\boldsymbol{\Phi}_x(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \begin{bmatrix} \varepsilon_1 k_1 \bar{x}_1^3 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{Q}_x = \begin{bmatrix} 0 \\ Q_3 \\ Q_3 \end{bmatrix}. \tag{4.83}$$

The coordinates vector is given by

$$\boldsymbol{x} = \begin{bmatrix} \bar{x}_1 \\ x_2 \\ \bar{x}_3 \end{bmatrix} \tag{4.84}$$

where the variables $\bar{x}_1$ and $\bar{x}_3$ are defined as

$$\bar{x}_1 = x_1 - l_{1,0} \tag{4.85}$$

and

$$\bar{x}_3 = x_3 - l_{2,0}, \tag{4.86}$$

respectively. In Eq. (4.85), $l_{2,0}$ is the unstretched length of the spring k_2. Differentiating the constraint of Eq. (4.81), the two sub-systems are subject to, yields

$$\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\bar{x}}_1 \\ \ddot{x}_2 \\ \ddot{\bar{x}}_3 \end{bmatrix} = 0. \tag{4.87}$$

The matrix $\mathbf{A}$ and the vector $\boldsymbol{b}$ of Eq. (3.21) take the form

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \tag{4.88}$$

and

$$b = 0, \tag{4.89}$$

respectively. Utilizing Eqs. (3.49), (4.82), (4.83) and (4.88), the new augmented equation of motion can be determined. Taking into account Eq. (3.33) and Eqs. (3.51)-(3.52), the linear equivalent augmented mass, damping and stiffness matrices are given by

$$\bar{\mathbf{M}}_{\boldsymbol{x}} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \tag{4.90}$$

and

$$\bar{\mathbf{C}}_{\boldsymbol{x}} = \begin{bmatrix} 0.05 & 0 & 0 \\ 0.05 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\mathbf{K}}_{\boldsymbol{x}} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.91}$$

respectively. The vectors $\bar{\boldsymbol{Q}}_{\boldsymbol{x}}$ and $\bar{\boldsymbol{\Phi}}_{\boldsymbol{x}}$ are given by

$$\bar{\boldsymbol{Q}}_{\boldsymbol{x}} = \begin{bmatrix} 0.5w(t) \\ 0.5w(t) \\ w(t) \\ 0 \end{bmatrix}, \quad \bar{\boldsymbol{\Phi}}_{\boldsymbol{x}} = \begin{bmatrix} 0.5\varepsilon_1 k_1 \bar{x}_1^3 \\ 0.5\varepsilon_1 k_1 \bar{x}_1^3 \\ 0 \\ 0 \end{bmatrix}. \tag{4.92}$$

Applying Eq. (4.43) for determining the equivalent linear stiffness matrix $\bar{\mathbf{K}}_e$ yields

$$\boldsymbol{k}_{\mathbf{1*}}^{eT} = \begin{bmatrix} \boldsymbol{r}(1,1) & \boldsymbol{r}(1,2) & \boldsymbol{r}(1,3) \\ \boldsymbol{r}(2,1) & \boldsymbol{r}(2,2) & \boldsymbol{r}(2,3) \\ \boldsymbol{r}(3,1) & \boldsymbol{r}(3,2) & \boldsymbol{r}(3,3) \end{bmatrix} \begin{bmatrix} \frac{3}{2}\varepsilon_1 k_1 \sigma_{\bar{x}_1}^2 \\ 0 \\ 0 \end{bmatrix}, \tag{4.93}$$

$$\boldsymbol{k}_{\mathbf{2*}}^{eT} = \begin{bmatrix} \boldsymbol{r}(1,1) & \boldsymbol{r}(1,2) & \boldsymbol{r}(1,3) \\ \boldsymbol{r}(2,1) & \boldsymbol{r}(2,2) & \boldsymbol{r}(2,3) \\ \boldsymbol{r}(3,1) & \boldsymbol{r}(3,2) & \boldsymbol{r}(3,3) \end{bmatrix} \begin{bmatrix} \frac{3}{2}\varepsilon_1 k_1 \sigma_{\bar{x}_1}^2 \\ 0 \\ 0 \end{bmatrix}, \tag{4.94}$$

$$\boldsymbol{k}_{\mathbf{3*}}^{eT} = \begin{bmatrix} \boldsymbol{r}(1,1) & \boldsymbol{r}(1,2) & \boldsymbol{r}(1,3) \\ \boldsymbol{r}(2,1) & \boldsymbol{r}(2,2) & \boldsymbol{r}(2,3) \\ \boldsymbol{r}(3,1) & \boldsymbol{r}(3,2) & \boldsymbol{r}(3,3) \end{bmatrix} \mathbb{E} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}, \tag{4.95}$$

and

$$\boldsymbol{k}_{\mathbf{4*}}^{eT} = \begin{bmatrix} \boldsymbol{r}(1,1) & \boldsymbol{r}(1,2) & \boldsymbol{r}(1,3) \\ \boldsymbol{r}(2,1) & \boldsymbol{r}(2,2) & \boldsymbol{r}(2,3) \\ \boldsymbol{r}(3,1) & \boldsymbol{r}(3,2) & \boldsymbol{r}(3,3) \end{bmatrix} \mathbb{E} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}, \tag{4.96}$$

where $\boldsymbol{r}(i,j), i,j = 1,2,\ldots,9$ denotes the element of the matrix

$$\mathbf{r} = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T]^{+}\mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T], \tag{4.97}$$

in position (i,j). Considering Eq. (4.93), the equivalent linear matrix $\bar{\mathbf{K}}_e$ can be concisely written as

$$\bar{\mathbf{K}}_e = \frac{3}{2}\varepsilon_1 k_1 \sigma_{\bar{x}_1}^2 \begin{bmatrix} \boldsymbol{r}(1,1) & \boldsymbol{r}(2,1) & \boldsymbol{r}(3,1) \\ \boldsymbol{r}(1,1) & \boldsymbol{r}(2,1) & \boldsymbol{r}(3,1) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.98}$$

Comparing Eqs. (4.75) and (4.98) it is noted that although the general form of the equivalent linear stiffness matrices is similar, the equivalent linear matrix of Eq. (4.98) has more non-zero elements. This is due to the presence of matrix $\mathbf{r}$ which, unlike the generalized coordinates modeling case, is not unitary. Employing Eq. (4.55), the matrix $\mathbf{G}_{\boldsymbol{x}}$ takes the form

$$\mathbf{G}_{\boldsymbol{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\bar{\mathbf{M}}_{\boldsymbol{x}}^{+}(\bar{\mathbf{K}}_{\boldsymbol{x}} + \bar{\mathbf{K}}_e) & -\bar{\mathbf{M}}_{\boldsymbol{x}}^{+}\bar{\mathbf{C}}_{\boldsymbol{x}} \end{bmatrix}, \tag{4.99}$$

where the M-P inverse of $\bar{\mathbf{M}}_{\boldsymbol{x}}$ is found by Eq. (3.33) to be equal to

$$\bar{\mathbf{M}}_{\boldsymbol{x}}^{+} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 2 & 1 \end{bmatrix}. \tag{4.100}$$

As in the case of the covariance matrix obtained in Eq. (4.79) for the $2-$DOF system, a complex modal analysis treatment is utilized for deriving the covariance matrix of the system response. To be consistent with the previously obtained result, the convergence criterion and error are the same as those utilized for deriving Eq. (4.79). Convergence is reached after eight iterations. Employing Eqs. (4.61)-(4.65), the eigenvalues of the matrix $\boldsymbol{\Psi}^{*}\mathbf{U}\boldsymbol{\eta}_{\boldsymbol{x}}$, where $\boldsymbol{\Psi}, \mathbf{U}, \boldsymbol{\eta}_{\boldsymbol{x}}$ are defined in Eq. (4.61), after the last iteration are

$$\lambda_1 = -0.1308 - 1.6389i, \;\; \lambda_2 = -0.1308 + 1.6389i, \tag{4.101}$$

$$\lambda_3 = -0.0192 - 0.6422i, \;\; \lambda_4 = -0.0192 + 0.6422i \tag{4.102}$$

and

$$\lambda_5 = \lambda_6 = 0, \tag{4.103}$$

whereas the corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} -0.0145 - 0.4629i \\ -0.0432 - 0.0020i \\ 0.4009 + 0.0278i \\ 0.7540 \\ 0.0051 - 0.0227i \\ -0.0343 + 0.2281i \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -0.0145 + 0.4629i \\ -0.0432 + 0.0020i \\ 0.4009 - 0.0278i \\ 0.7540 \\ 0.0051 + 0.0227i \\ -0.0343 - 0.2281i \end{bmatrix}, \tag{4.104}$$

$$\boldsymbol{v}_3 = \begin{bmatrix} -0.0308 + 0.0028i \\ 0.0006 - 0.4181i \\ -0.0177 - 0.3418i \\ -0.0060 - 0.0025i \\ 0.6740 \\ 0.5027 - 0.0111i \end{bmatrix}, \boldsymbol{v}_4 = \begin{bmatrix} -0.0308 - 0.0028i \\ 0.0006 + 0.4181i \\ -0.0177 + 0.3418i \\ -0.0060 + 0.0025i \\ 0.6740 \\ 0.5027 + 0.0111i \end{bmatrix} \tag{4.105}$$

and

$$\boldsymbol{v}_5^{\mathrm{T}} = \boldsymbol{v}_6^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.106}$$

After determining the eigenvalues and eigenvectors of the matrix $\mathbf{\Psi}^*\mathbf{U}\boldsymbol{\eta}_{\boldsymbol{x}}$, Eq. (4.66) evaluated at $\tau = 0$ takes the form

$$\mathbf{w}_{\boldsymbol{z}_{\boldsymbol{x}}}(0) = -\sum_{i=1}^{4}\sum_{j=1}^{4}\frac{\mathbf{p}_i(\mathbf{\Psi}^*\mathbf{D}_{\boldsymbol{x}}\mathbf{\Psi})\mathbf{p}_j^*}{\lambda_i + \bar{\lambda}_j}, \tag{4.107}$$

where $\lambda_i, i = 1, 2, 3, 4$ are given by Eq. (4.101) and $\mathbf{D}_{\boldsymbol{x}}$ is a real, symmetric, non-negative matrix of constants given by

$$\mathbf{D}_{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\pi 10^{-3} \end{bmatrix}. \tag{4.108}$$

In Eq. (4.107), the expressions $\mathbf{p}_i, i = 1, 2, 3, 4$ denote 6×6 matrices defined in terms of the matrix $\mathbf{\Psi}^*\mathbf{U}\boldsymbol{\eta}_{\boldsymbol{x}}$, as well as the eigenvalues calculated in Eq. (4.101). For example, $\mathbf{p}_1$ is defined as (see Ref. [41] for more details)

$$\mathbf{p}_1 = \frac{(\mathbf{\Psi}^*\mathbf{U}\boldsymbol{\eta}_{\boldsymbol{x}} - \lambda_2\mathbf{I})(\mathbf{\Psi}^*\mathbf{U}\boldsymbol{\eta}_{\boldsymbol{x}} - \lambda_3\mathbf{I})(\mathbf{\Psi}^*\mathbf{U}\boldsymbol{\eta}_{\boldsymbol{x}} - \lambda_4\mathbf{I})(\mathbf{\Psi}^*\mathbf{U}\boldsymbol{\eta}_{\boldsymbol{x}})^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)\lambda_1^2}. \tag{4.109}$$

Employing Eq. (4.67), the covariance matrix of the system response is given by

$$\mathbf{w}_{\boldsymbol{p}}(0) = \begin{bmatrix} 0.0386 & 0.0386 & 0.0253 & 0 & 0 & -0.0010 \\ 0.0386 & 0.0386 & 0.0253 & 0 & 0 & -0.0010 \\ 0.0253 & 0.0253 & 0.0210 & 0.0010 & 0.0010 & 0 \\ 0 & 0 & 0.0010 & 0.0178 & 0.0178 & 0.0074 \\ 0 & 0 & 0.0010 & 0.0178 & 0.0178 & 0.0074 \\ -0.0010 & -0.0010 & 0 & 0.0074 & 0.0074 & 0.0136 \end{bmatrix}. \tag{4.110}$$

Comparing Eqs. (4.79) and (4.110), the variance $\mathbb{E}[q_1^2]$ as well as $\mathbb{E}[\dot{q}_1^2]$ obtained for the system examined in Figure 4.1, coincide with the respective ones for the system of Figure 4.2, i.e $\mathbb{E}[\bar{x}_1^2]$ and $\mathbb{E}[\dot{\bar{x}}_1^2]$. Taking expectations in the equation that connects the two reference systems, that is $\bar{x}_3 = q_2 - q_1$, and utilizing Eq. (4.79) yields

$$\begin{aligned} \mathbb{E}[\bar{x}_3^2] &= \mathbb{E}[q_2^2] + \mathbb{E}[q_1^2] - 2\mathbb{E}[q_1 q_2] \\ &= 0.0210 \end{aligned} \tag{4.111}$$

and

$$\begin{aligned}\mathbb{E}[\dot{x}_3^2] &= \mathbb{E}[\dot{q}_2^2] + \mathbb{E}[\dot{q}_1^2] - 2\mathbb{E}[\dot{q}_1\dot{q}_2] \\ &= 0.0136,\end{aligned} \tag{4.112}$$

which are indeed in agreement with the corresponding values in Eq. (4.110). It can be readily verified that the rest of the elements of the matrix given in Eq. (4.110) are also in agreement with the respective ones of Eq. (4.79). Alternatively, the response covariance matrix $\mathbf{V}_x$ can be obtained by utilizing a state variable formulation in conjunction with the Lyapunov equation of Eq. (3.75); see Ref. [41] for more details.

4.4.2 Generalized statistical linearization in conjunction with a frequency domain analysis treatment

4.4.2.1 2-DOF nonlinear system with singular matrices

The $2-$DOF nonlinear system examined in Section 4.4.1 by following a time domain analysis, is examined herein by applying a frequency domain analysis treatment. Assume that the mass m_1 is connected to the foundation by a nonlinear spring of the linear-plus-cubic type and by a linear damper with coefficient c_1. The mass m_2 is connected to m_1 by a linear spring and a linear damper with coefficients k_2 and c_2, respectively. The system is excited by a random force $Q_2(t)$ which is modeled as a white-noise process with a correlation function $w_{Q_2}(t) = 2\pi S_0 \delta(t)$, where S_0 is the (constant) power spectrum value of $Q_2(t)$. The generalized displacements are given by q_1 and q_2 (see Figure 4.1).

As in case of the system depicted in Figure 4.1, the system equations of motion are written in the matrix form of Eq. (4.1), where the matrices $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ are given by Eq. (4.70), whereas the coordinate vector $\boldsymbol{q}$ and the excitation vector $\boldsymbol{Q}$ are defined in Eqs. (4.72) and (4.73), respectively. The nonlinear function $\boldsymbol{\Phi}$ takes the form given by Eq. (4.74), and utilizing Eq. (4.42), the equivalent linear stiffness matrix has the form of Eq. (4.75).

The standard statistical linearization procedure is applied. The parameters values used are $m_1 = m_2 = m = 1, c_1 = c_2 = c = 0.1, k_1 = k_2 = k = 1$, and $S_0 = 10^{-3}$. The value of the power spectrum for the excitation is $S_0 = 10^{-3}$. As in previous case, convergence is attained after eight iterations, subject to the criterion defined in Eq. (4.76) for the same initial value, i.e. $\mathbf{K}_e^0 = \mathbf{0}$. At the end of the iterative solution procedure, the covariance matrix of the system response displacement is determined as

$$\mathbf{V}_{\boldsymbol{q}} = \begin{bmatrix} 0.0386 & 0.0639 \\ 0.0639 & 0.1102 \end{bmatrix}, \tag{4.113}$$

whereas the covariance matrix of the system response velocity is

$$\mathbf{V}_{\dot{\boldsymbol{q}}} = \begin{bmatrix} 0.0178 & 0.0252 \\ 0.0252 & 0.0458 \end{bmatrix}. \tag{4.114}$$

Utilizing a redundant coordinates modeling scheme, the three coordinates $\bar{x}_1, x_2$ and $\bar{x}_3$ shown in Figure 4.2 are considered. The constraint equation $x_2 = x_1 + d$ (see Eq. (4.80)), with d being the length of mass m_1, serves to connect the two subsystems of mass m_1 and mass m_2 (Figure 4.2). Differentiating Eq. (4.80) twice with respect to time, the constraint equation is written in the matrix form of Eq. (3.21), i.e. $\mathbf{A}\ddot{\boldsymbol{x}} = \boldsymbol{b}$, where $\mathbf{A}$ and $\boldsymbol{b}$ are, in turn, given by Eq. (4.88) and Eq. (4.89), respectively. The augmented mass, damping and stiffness matrices of the system can be determined (see Eqs. (4.90)-(4.91)). The augmented excitation vector and the nonlinear vector of the system are given by Eq. (4.92). Note in passing that the variable $\bar{x}_1$ in Eq. (4.92) corresponds to the displacement of the first mass and is defined as $\bar{x}_1 = x_1 - l_{1,0}$, where $l_{1,0}$ is the unstretched length of the spring k_1.

The generalized statistical linearization methodology is applied. Following the algorithm described in Section 4.3.4, Eq. (4.43) is utilized for determining the equivalent linear stiffness matrix, $\bar{\mathbf{K}}_e$, yielding the same expression as in Eq. (4.98). Once again, it is noted that due to the presence of the non-unitary matrix $\boldsymbol{r}$ of Eq. (4.97), the augmented matrix $\bar{\mathbf{K}}_e$ has more non-zero elements than the equivalent stiffness matrix $\mathbf{K}_e$ (compare Eqs. (4.75) and (4.98)). The same convergence criterion as the one employed in deriving the covariance matrix of system response displacement and velocity, i.e. Eq. (4.113) and Eq. (4.114), is used, whereas convergence is reached after eight iterations. Noticing that in this case the 4×3 matrix $\mathbf{R}_e$, has full rank, and thus, Eq. (3.185) is used for determining the FRF matrix $\boldsymbol{\alpha}_{\boldsymbol{x}}(\omega)$, the covariance matrix of the system response displacement is determined to be

$$\mathbf{V}_{\bar{\boldsymbol{x}}} = \begin{bmatrix} 0.0386 & 0.0386 & 0.0253 \\ 0.0386 & 0.0386 & 0.0253 \\ 0.0253 & 0.0253 & 0.0210 \end{bmatrix}, \tag{4.115}$$

whereas the system response velocity covariance matrix is computed as

$$\mathbf{V}_{\dot{\bar{x}}} = \begin{bmatrix} 0.0178 & 0.0178 & 0.0074 \\ 0.0178 & 0.0178 & 0.0074 \\ 0.0074 & 0.0074 & 0.0132 \end{bmatrix}. \tag{4.116}$$

Comparing the results, it is seen that the variance $\mathbb{E}[q_1^2]$ in Eq. (4.113) coincide with the variance $\mathbb{E}[\bar{x}_1^2]$ in Eq. (4.115). Similarly, the variances $\mathbb{E}[\dot{q}_1^2]$ and $\mathbb{E}[\dot{\bar{x}}_1^2]$ in Eq. (4.114) and Eq. (4.116), coincide with each other. Taking into account the expression $\bar{x}_3 = q_2 - q_1$ that relates the two reference systems yields

$$\begin{aligned} \mathbb{E}[\bar{x}_3^2] &= \mathbb{E}[q_2^2] + \mathbb{E}[q_1^2] - 2\mathbb{E}[q_1 q_2] \\ &= 0.0210 \end{aligned} \tag{4.117}$$

and

$$\begin{aligned} \mathbb{E}[\dot{\bar{x}}_3^2] &= \mathbb{E}[\dot{q}_2^2] + \mathbb{E}[\dot{q}_1^2] - 2\mathbb{E}[\dot{q}_1 \dot{q}_2] \\ &= 0.0132, \end{aligned} \tag{4.118}$$

which agree with the corresponding values in Eqs. (4.115)-(4.116).

It should be noted that the herein obtained results are in total agreement with the ones obtained when the problem is solved by following an alternative time-domain methodology presented in Section 4.4.1; see also Refs. [41, 42].

4.4.2.2 3-DOF nonlinear system with singular matrices

In this example, nonlinearities are considered in the system studied in Section 3.3.2.3. As seen in Figure 4.3, it is assumed that the damping force connecting mass m_1 with the foundation is given by $c_1\dot{q}_1(1 + \epsilon\,|\dot{q}_1|)$.

The system mass, damping and stiffness matrices, as well as the system coordinates and the vector of the excitation force are given by Eqs. (3.201)-(3.203) and Eqs. (3.204)-(3.205), respectively. The nonlinear vector $\mathbf{\Phi}$ of Eq. (4.1) takes the form

$$\mathbf{\Phi} = \begin{bmatrix} \epsilon_1 c_1 \dot{q}_1\,|\dot{q}_1| \\ 0 \\ 0 \end{bmatrix}. \tag{4.119}$$

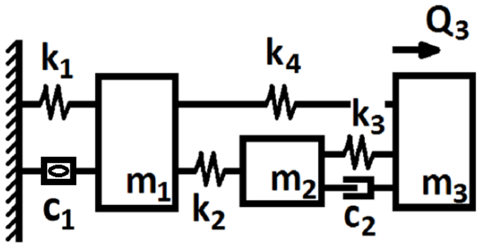

FIGURE 4.3: A three degree-of-freedom nonlinear system under stochastic excitation.

Following the standard statistical linearization approach [91], the equivalent linear damping matrix of the system becomes

$$\mathbf{C}_e = \frac{4\epsilon_1 c_1}{\sqrt{2\pi}}\sqrt{\mathbb{E}[\dot{q}_1^2]}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{4.120}$$

Regarding the numerical implementation of the iterative solution scheme, the convergence criterion is given by

$$\left|\frac{\mathbf{C}_e^{j+1} - \mathbf{C}_e^{j}}{\mathbf{C}_e^{j}}\right| > 10^{-5}, \tag{4.121}$$

where j denotes the $j-th$ iteration and $\mathbf{C}_e^0$ is set equal to zero. After eight iterations, the covariance matrices of the system response displacement and velocity are given by

$$\mathbf{V}_{\boldsymbol{q}} = \begin{bmatrix}0.0379 & 0.0477 & 0.0491\\ 0.0477 & 0.0616 & 0.0646\\ 0.0491 & 0.0646 & 0.0702\end{bmatrix} \tag{4.122}$$

and

$$\mathbf{V}_{\dot{\boldsymbol{q}}} = \begin{bmatrix}0.0084 & 0.0085 & 0.0063\\ 0.0085 & 0.0110 & 0.0099\\ 0.0063 & 0.0099 & 0.0133\end{bmatrix}, \tag{4.123}$$

respectively.

Utilizing the redundant coordinates modeling, the system presented in Figure 4.3 is decomposed as seen in Figure 4.4 and the augmented nonlinear vector of Eq.

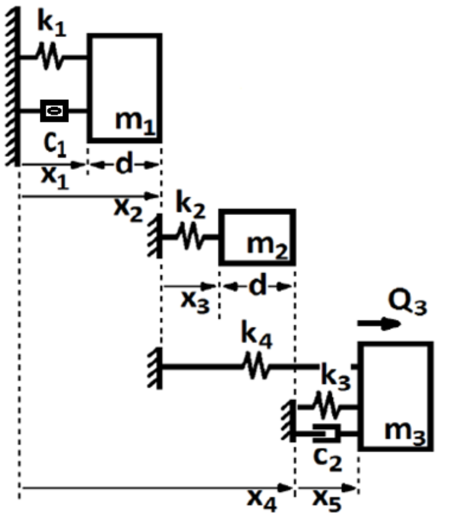

FIGURE 4.4: A three degree-of-freedom nonlinear system under stochastic excitation utilizing redundant coordinates.

(4.11) becomes

$$\bar{\mathbf{\Phi}}_{\boldsymbol{x}} = \begin{bmatrix} 0.4\epsilon_1 c_1 \dot{\bar{x}}_1 \left|\dot{\bar{x}}_1\right| \\ 0.4\epsilon_1 c_1 \dot{\bar{x}}_1 \left|\dot{\bar{x}}_1\right| \\ -0.2\epsilon_1 c_1 \dot{\bar{x}}_1 \left|\dot{\bar{x}}_1\right| \\ 0.2\epsilon_1 c_1 \dot{\bar{x}}_1 \left|\dot{\bar{x}}_1\right| \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.124}$$

The equivalent damping matrix $\mathbf{C}_e$ is obtained by applying the generalized statistical linearization methodology; that is, Eq. (4.43) yields

$$\bar{\mathbf{C}}_e = \frac{0.8\epsilon_1 c_1}{\sqrt{2\pi}} \sqrt{\mathbb{E}[\dot{\bar{x}}_1^2]} \begin{bmatrix} 2\boldsymbol{r}(6,6) & 2\boldsymbol{r}(7,6) & 2\boldsymbol{r}(8,6) & 2\boldsymbol{r}(9,6) & 2\boldsymbol{r}(10,6) \\ 2\boldsymbol{r}(6,6) & 2\boldsymbol{r}(7,6) & 2\boldsymbol{r}(8,6) & 2\boldsymbol{r}(9,6) & 2\boldsymbol{r}(10,6) \\ -\boldsymbol{r}(6,6) & -\boldsymbol{r}(7,6) & -\boldsymbol{r}(8,6) & -\boldsymbol{r}(9,6) & -\boldsymbol{r}(10,6) \\ \boldsymbol{r}(6,6) & \boldsymbol{r}(7,6) & \boldsymbol{r}(8,6) & \boldsymbol{r}(9,6) & \boldsymbol{r}(10,6) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.125}$$

whereas the iterative solution procedure using the convergence criterion defined in

Eq. (4.121) (i.e. the same criterion as in Eqs. (4.122)-(4.123)), yields the response covariance matrices

$$\mathbf{V}_{\bar{x}} = \begin{bmatrix} 0.0379 & 0.0379 & 0.0098 & 0.0477 & 0.0014 \\ 0.0379 & 0.0379 & 0.0098 & 0.0477 & 0.0014 \\ 0.0098 & 0.0098 & 0.0041 & 0.0139 & 0.0016 \\ 0.0477 & 0.0477 & 0.0139 & 0.0616 & 0.0029 \\ 0.0014 & 0.0014 & 0.0016 & 0.0029 & 0.0027 \end{bmatrix} \tag{4.126}$$

and

$$\mathbf{V}_{\dot{\bar{x}}} = \begin{bmatrix} 0.0084 & 0.0084 & 0.0001 & 0.0085 & -0.0022 \\ 0.0084 & 0.0084 & 0.0001 & 0.0085 & -0.0022 \\ 0.0001 & 0.0001 & 0.0025 & 0.0026 & 0.0010 \\ 0.0085 & 0.0085 & 0.0026 & 0.0110 & -0.0012 \\ -0.0022 & -0.0022 & 0.0010 & -0.0012 & 0.0046 \end{bmatrix}. \tag{4.127}$$

Taking into account the equations that connect the reference systems, i.e. Eqs. (3.227)-(3.228), it can be readily verified that the covariance matrices in Eqs. (4.122)-(4.123) are in total agreement with the corresponding covariance matrices in Eqs. (4.126)-(4.127).

4.5 Summary

In chapter 4 the standard statistical linearization approximate methodology for determining the stochastic response of nonlinear dynamic systems is generalized to account for systems with singular matrices. Based on the concept of the M-P inverse of a singular matrix and utilizing theoretical results pertaining to random vectors that follow the multivariate normal distribution with singular covariance matrix, the general expression of the random vibration theory that relates the stochastic system excitation to its response is derived. The generalized version of the statistical linearization methodology is applied in conjunction with a time as well as a frequency domain analysis treatment. A coupled system of nonlinear algebraic equations yielding the system response mean vector and covariance matrix, is constructed and solved. The employment of the M-P generalized matrix inverse implies a family of solutions for determining the equivalent mass, damping and stiffness matrices of the linearized system. A proposition for deriving a unique solution out of the family of solutions is also given along with its proof. The theoretically obtained results for systems with singular matrices are validated by pertinent numerical examples and are compared with the results derived by utilizing the standard statistical linearization methodology.

Chapter 5

Concluding remarks and future research

5.1 Conclusion

This chapter consists of the main conclusions of the thesis, including remarks regarding the techniques/methodologies utilized, the numerical implementations executed as well as the results obtained. Also, potential directions for future research related to the derived results are identified.

Chapter 1 comprises an introduction to the thesis, including the motivation and objectives this research work is intended to meet. The case of systems with governing equations of motion having singular mass, damping and stiffness matrices has been discussed. Application examples of such systems, including, but not limited to, the utilization of additional/redundant coordinates in formulating the equations of motion, ill-conditioned systems and modeling the equations of motion of smart materials has been presented. The organization of forthcoming results has been included in a separate section.

In Chapter 2, some critical mathematical results for the herein presented work has been provided. The main elements of the generalized matrix inverse theory has been presented. Particular attention has been given in analyzing the concept of the Moore-Penrose (M-P) generalized matrix inverse, which prevails in the herein presented work. Several pertain to the forthcoming analysis results and properties of the M-P matrix inverses has been included. Some results regarding the multivariate normal distribution, and particularly the case of probability density functions with singular covariance matrices, has been given. The latter is decisively connected to the proof of a proposition required in generalizing the standard statistical linearization approximate methodology.

In Chapter 3, certain concepts and relationships of the linear random vibration theory have been modified and generalized to account for structural systems with singular matrices. Adopting a redundant (generalized) coordinates modeling for deriving the systems governing equations of motion, singular mass, damping and stiffness matrices appeared. Relying on the M-P inverse of a singular matrix, the standard time domain analysis methodology has been extended to account for systems with singular matrices. By applying a state variable formulation as well as a complex modal analysis treatment, the mean vector and covariance matrix of linear systems with singular matrices have been determined. It has been shown that applying a complex modal analysis treatment, unlike the standard system modeling case, does not lead to decoupling of the equations of motion. Nevertheless, relying on a singular value decomposition of the system transition matrix facilitates significantly the efficient computation of the system response statistics. The standard frequency domain analysis methodology has been also extended to account for linear systems with singular matrices. Aiming at the determination of the linear system response power spectrum, a M-P frequency response function (FRF) has been determined. A series expansion for the M-P FRF which circumvents the potentially cumbersome numerical evaluation of the M-P inverse, has been presented as well. Validation of the theoretically obtained results has been provided by pertinent examples of $2-$ and $3-$DOFs systems, referring to the time and frequency domain approaches, respectively.

Chapter 4 comprises a generalization of the standard statistical linearization approximate methodology for determining the stochastic response of nonlinear dynamic systems to account for systems with singular matrices. Relying on the M-P generalized matrix inverse, and taking into account the form of the probability density function for a random vector following the multivariate normal distribution with singular covariance matrix, a extension of a general expression of the random vibration theory relating the stochastic system excitation to its response has been derived. A family of optimal and response dependent equivalent linear matrices has been obtained, and after combining them with the aforementioned result, a coupled system of nonlinear algebraic equations, has been constructed and solved, yielding the derivation of the system response mean vector and covariance matrix. It has been noted that the utilization of the M-P generalized matrix inverse yields an expression for determining the statistical linearization equivalent linear matrices which includes an arbitrary part. This implies a family of solutions for the equivalent linear matrices of the method. A proposition has been proved showing that the solution obtained by setting the arbitrary part in the M-P expression equal to zero is at least as good (in a mean square error minimization sense) as any other solution corresponding to a non-zero value. This proof greatly facilitates the practical implementation of the technique as it promotes the utilization of the intuitively simplest solution among a family of possible solutions. Validation of the proposed methodology has been done by pertinent examples including $2-$ and $3-$DOFs nonlinear systems.

The results presented in this thesis propose a novel theoretical framework for deriving the stochastic response of linear and nonlinear systems with singular matrices. These singularities are potentially caused by adopting an unconventional formulation of the systems equations of motion that is based on a redundant coordinates modeling. Relying on the concept of M-P generalized matrix inverse, the barriers set by the presence of singular matrices in the systems governing equations of motion are circumvented.

5.2 Future research

A potential future extension of the present research work lies in the direction of stochastic dynamics with fractional derivatives aiming at the still challenging problem of assessing the reliability analysis of multi degree-of-freedom systems endowed with fractional derivative terms (FMDOF). Apart from the fact that Fractional Calculus has proved itself in recent years as an essential and potent mathematical modeling tool (e.g. applications in viscoelasticity, control theory, biophysics), the merit of potential extension stems from the lack of methods/techniques for manipulating FMDOF systems. A recently proposed state variable formulation for the stochastic response derivation of linear SDOF systems with fractional derivative elements [31], can be potentially extended to SDOF systems with singular matrices. An extension to FMDOF systems with singular matrices may also be considered as future work.

Further future work may be connected to introducing uncertainties in the herein proposed framework. The study of MDOF systems with stochastic coefficient, singular matrices, appears interesting, and to the best of the author's knowledge has not yet been addressed in the literature. The herein developed state variable framework for solving equations with singular matrices can be utilized in engineering applications where coupled systems of governing equations yield singular matrices.

List of publications

Journal papers

- V. C. Fragkoulis, I. A. Kougioumtzoglou, and A. A. Pantelous. Linear random vibration of structural systems with singular matrices, ASCE *Journal of Engineering Mechanics*, 142(2):04015081-11, 2015.
- V. C. Fragkoulis, I. A. Kougioumtzoglou, and A. A. Pantelous. Statistical linearization of nonlinear structural systems with singular matrices, ASCE *Journal of Engineering Mechanics*, 142(9):04016063-11, 2016.
- I. A. Kougioumtzoglou, V. C. Fragkoulis, A. A. Pantelous, and A. Pirrotta. Random vibration of linear and nonlinear structural systems with singular matrices: A frequency domain approach, *Journal of Sound and Vibration*, 404:84-101, 2017.

Conference papers

- V. C. Fragkoulis, I. A. Kougioumtzoglou, and A. A. Pantelous, Random vibration of linear systems with singular mass matrices, In *Proceedings of the 7th International Conference on Computational Stochastic Mechanics (CSM 7)*, Santorini, Greece, 15-18 June, 2014, G. Deodatis, P. D. Spanos (Eds.), Research Publishing, ISBN: 978-981-09-5348-5, p. 277-285.
- V. C. Fragkoulis, I. A. Kougioumtzoglou, A. A. Pantelous, and A. Pirrotta, Higher order matrix differential equations with singular coefficient matrices, In *Proceedings of the 12th International Conference of Numerical Analysis and Applied Mathematics (ICNAAM 2014)*, Rhodes, Greece, 22-28 September, 2014, American Institute of Physics (AIP) Conf. Proc. 1648, 340002-1340002-4; doi: 10.1063/1.4912578.

- V. C. Fragkoulis, I. A. Kougioumtzoglou, and A. A. Pantelous, Statistical linearization of nonlinear systems with singular mass matrices, In *Proceedings of the Engineering Mechanics Institute Conference (EMI 2015)*, Stanford University, June 16-19, 2015.
- V. C. Fragkoulis, I. A. Kougioumtzoglou, and A. A. Pantelous, Stochastic response determination of linear and nonlinear dynamical systems with singular matrices, In *Proceedings of the 11th HSTAM International Congress on Mechanics (HSTAM 2016)*, Athens, Greece, 27-30 May, 2016, ISBN: 978-618-82609-1-7, editors, H. G. Georgiades and V. K. Koumousis.
- V. C. Fragkoulis, I. A. Kougioumtzoglou, A. A. Pantelous, and A. Pirrotta, A frequency domain methodology for determining the stochastic response of systems with singular matrices, In *Proceedings of the Engineering Mechanics Institute Conference (EMI 2017)*, UC San Diego, June 4-7, 2017.
- V. C. Fragkoulis, I. A. Kougioumtzoglou, A. A. Pantelous, and A. Pirrotta, A Moore-Penrose frequency domain approach for stochastic response determination of structural systems with singular matrices, In *Proceedings of the 12th International Conference On Structural Safety And Reliability (ICOSSAR 2017)*, 6-10 August, 2017, TU Wien, Vienna, Austria.

Bibliography

[1] T. H. S. Abdelaziz. Eigenstructure assignment by displacement–acceleration feedback for second-order systems. *Journal of Dynamic Systems, Measurement, and Control*, 138(6):064502, 2016.

[2] S. Adhikari. Dynamics of nonviscously damped linear systems. *Journal of Engineering Mechanics*, 128(3):328–339, 2002.

[3] S. Adhikari. *Structural dynamic analysis with generalized damping models: analysis*. John Wiley & Sons, 2013.

[4] S. Adhikari and M. I. Friswell. Calculation of eigensolution derivatives for non-viscously damped systems. *AIAA journal*, 44(8):1799–1806, 2006.

[5] S. Adhikari, M. I. Friswell, and D. J. Inman. Piezoelectric energy harvesting from broadband random vibrations. *Smart Materials and Structures*, 18(11):115005, 2009.

[6] S. Adhikari and N. Wagner. Direct time-domain integration method for exponentially damped linear systems. *Computers & structures*, 82(29):2453–2461, 2004.

[7] T. S. Atalik and S. Utku. Stochastic linearization of multi-degree-of-freedom non-linear systems. *Earthquake Engineering & Structural Dynamics*, 4(4):411–420, 1976.

[8] D.-S. Bae and E. J. Haug. A recursive formulation for constrained mechanical system dynamics: Part ii. closed loop systems. *Journal of Structural Mechanics*, 15(4):481–506, 1987.

[9] A. V. Balakrishnan. Vibrating systems with singular mass-inertia matrices. 1996.

[10] R. Bellman, R. Kalaba, and J. Lockett. Dynamic programming and ill-conditioned linear systems. *Journal of Mathematical Analysis and Applications*, 10(1):206–215, 1965.

[11] R. Bellman, R. Kalaba, and J. Lockett. Dynamic programming and ill-conditioned linear systemsii. *Journal of Mathematical Analysis and Applications*, 12(3):393–400, 1965.

[12] A. Ben-Israel. Generalized inverses of matrices: a perspective of the work of penrose. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 100, pages 407–425. Cambridge University Press, 1986.

[13] A. Ben-Israel and T. N. E. Greville. *Generalized inverses: theory and applications*, volume 15. Springer Science & Business Media, 2003.

[14] S. P. Bhat and D. S. Bernstein. Second-order systems with singular mass matrix and an extension of guyan reduction. *SIAM Journal on Matrix Analysis and Applications*, 17(3):649–657, 1996.

[15] R. Bhatia and P. Rosenthal. How and why to solve the operator equation ax- xb= y. *Bulletin of the London Mathematical Society*, 29(1):1–21, 1997.

[16] R. E. D. Bishop, S. Michaelson, and G. M. L. Gladwell. *The matrix analysis of vibration*. Cambridge University Press, 1979.

[17] R. Bouc. The power spectral density of response for a strongly non-linear random oscillator. *Journal of Sound and Vibration*, 175(3):317–331, 1994.

[18] S. L. Campbell and C. D. Meyer. *Generalized inverses of linear transformations*. SIAM, 2009.

[19] S. L. Campbell and N. J. Rose. A second order singular linear system arising in electric power systems analysis. *International Journal of Systems Science*, 13(1):101–108, 1982.

[20] T. K. Caughey. Equivalent linearization techniques. *The Journal of the Acoustical Society of America*, 35(11):1706–1711, 1963.

[21] M. Chaichian and A. Demichiev. *Path Integrals in Physics: Volume I Stochastic Processes and Quantum Mechanics*, volume 1. Institute of Physics Publishing, Bristol and Philadelphia, 2001.

[22] C.-T. Chen. *Linear system theory and design*. Oxford University Press, Inc., 1995.

[23] H.-W. Cheng and S. S.-T. Yau. More explicit formulas for the matrix exponential. *Linear algebra and its applications*, 262:131–163, 1997.

[24] R. W. Clough and J. Penzien. *Dynamics of Structures, 2nd edition*. McGraw-Hill, New York, 1993.

[25] S. H. Crandall. Is stochastic equivalent linearization a subtly flawed procedure? *Probabilistic Engineering Mechanics*, 16(2):169–176, 2001.

[26] S. H. Crandall. A half-century of stochastic equivalent linearization. *Structural Control and Health Monitoring*, 13(1):27–40, 2006.

[27] J. H. Critchley and K. S. Anderson. A generalized recursive coordinate reduction method for multibody system dynamics. *International Journal for Multiscale Computational Engineering*, 1(2&3), 2003.

[28] D. de Falco, E. Pennestrì, and L. Vita. Investigation of the influence of pseudoinverse matrix calculations on multibody dynamics simulations by means of the udwadia-kalaba formulation. *Journal of Aerospace Engineering*, 22(4):365–372, 2009.

[29] N. Del Buono and L. Lopez. A survey on methods for computing matrix exponentials in numerical schemes for odes. *Computational Science ICCS 2003*, pages 673–673, 2003.

[30] A. Di Matteo, I. A. Kougioumtzoglou, A. Pirrotta, P. D. Spanos, and M. Di Paola. Stochastic response determination of nonlinear oscillators with fractional derivatives elements via the wiener path integral. *Probabilistic Engineering Mechanics*, 38:127–135, 2014.

[31] M. Di Paola, F. P. Pinnola, and P. D. Spanos. Analysis of multi-degree-of-freedom systems with fractional derivative elements of rational order. In *Fractional Differentiation and Its Applications (ICFDA), 2014 International Conference on*, pages 1–6. IEEE, 2014.

[32] G.-R. Duan and L. Huang. Robust pole assignment in descriptor second-order dynamical systems. *Acta Automatica Sinica*, 33(8):888–892, 2007.

[33] Y. Dumont, D. Goeleven, and M. Rochdi. Reduction of second order unilateral singular systems. applications in mechanics. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 81(4):219–245, 2001.

[34] I. Elishakoff. Method of stochastic linearization revised and improved. In *Computational Stochastic Mechanics*, pages 101–111. Springer, 1991.

[35] I. Elishakoff and L. Andriamasy. The tale of stochastic linearization technique: Over half a century of progress. In *Nondeterministic Mechanics*, pages 115–189. Springer, 2012.

[36] I. Elishakoff and R. Zhang. Comparison of new energy-based versions of the stochastic linearization technique. In *IUTAM Symposium*, pages 201–211, 1992.

[37] I. Elishakoff and X. Zhang. An appraisal of different stochastic linearization techniques. *Journal of Sound and Vibration*, 153(2):370–375, 1992.

[38] T. Fang and Z.-N. Wang. Complex modal analysis of random vibrations. *AIAA journal*, 24(2):342–344, 1986.

[39] L. Faravelli, F. Casciati, and M. P. Singh. Stochastic equivalent linearization algorithms and their applicability to hysteretic systems. *Meccanica*, 23(2):107–112, 1988.

[40] R. Featherstone. *Robot Dynamics Algorithms*. Kluwer, New York, 1987.

[41] V. C. Fragkoulis, I. A. Kougioumtzoglou, and A. A. Pantelous. Linear random vibration of structural systems with singular matrices. *Journal of Engineering Mechanics*, 142(2):04015081, 2015.

[42] V. C. Fragkoulis, I. A. Kougioumtzoglou, and A. A. Pantelous. Statistical linearization of nonlinear structural systems with singular matrices. *Journal of Engineering Mechanics*, 142(9):04016063, 2016.

[43] V. C. Fragkoulis, I. A. Kougioumtzoglou, A. A. Pantelous, and A. Pirrotta. A moore-penrose frequency domain approach for stochastic response determination of structural systems with singular matrices. In *Proceedings of the 12th International Conference on Structural Safety & Reliability, August 6-10, Vienna, Austria*.

[44] V. C. Fragkoulis, I. A. Kougioumtzoglou, A. A. Pantelous, and A. Pirrotta. Higher order matrix differential equations with singular coefficient matrices. In *AIP Conference Proceedings*, volume 1648, page 340002. AIP Publishing, 2015.

[45] B. Gashi and A. A. Pantelous. Linear backward stochastic differential equations of descriptor type: regular systems. *Stochastic Analysis and Applications*, 31(1):142–166, 2013.

[46] B. Gashi and A. A. Pantelous. Linear backward stochastic differential systems of descriptor type with structure and applications to engineering. *Probabilistic Engineering Mechanics*, 40:1–11, 2015.

[47] T. N. E. Greville. Some applications of the pseudoinverse of a matrix. *SIAM review*, 2(1):15–22, 1960.

[48] M. Grigoriu. *Stochastic calculus: applications in science and engineering*. Springer Science & Business Media, 2013.

[49] G. I. Kalogeropoulos, A. D. Karageorgos, and A. A. Pantelous. On the solution of higher order linear homogeneous complex σ–α descriptor matrix differential systems of apostol–kolodner type. *Journal of the Franklin Institute*, 351(3):1756–1777, 2014.

[50] T. Kamada, T. Fujita, T. Hatayama, T. Arikabe, N. Murai, S. Aizawa, and K. Tohyama. Active vibration control of frame structures with smart structures using piezoelectric actuators (vibration control by control of bending moments of columns). *Smart Materials and Structures*, 6(4):448, 1997.

[51] D. T. Kawano, M. Morzfeld, and F. Ma. The decoupling of second-order linear systems with a singular mass matrix. *Journal of Sound and Vibration*, 332(25):6829–6846, 2013.

[52] C. G. Khatri. Some results for the singular normal multivariate regression models. *Sankhyā: The Indian Journal of Statistics, Series A*, pages 267–280, 1968.

[53] P. B. Koganti and F. E. Udwadia. Unified approach to modeling and control of rigid multibody systems. *Journal of Guidance, Control, and Dynamics*, 2016.

[54] I. A. Kougioumtzoglou, A. Di Matteo, P. D. Spanos, A. Pirrotta, and M. Di Paola. An efficient wiener path integral technique formulation for stochastic response determination of nonlinear mdof systems. *Journal of Applied Mechanics*, 82(10):101005, 2015.

[55] I. A. Kougioumtzoglou, V. C. Fragkoulis, A. A. Pantelous, and A. Pirrotta. Random vibration of linear and nonlinear structural systems with singular matrices: A frequency domain approach. *Journal of Sound and Vibration*, 404:84–101, 2017.

[56] I. A. Kougioumtzoglou and P. D. Spanos. An analytical wiener path integral technique for non-stationary response determination of nonlinear oscillators. *Probabilistic Engineering Mechanics*, 28:125–131, 2012.

[57] I. A. Kougioumtzoglou and P. D. Spanos. Nonstationary stochastic response determination of nonlinear systems: A wiener path integral formalism. *Journal of Engineering Mechanics*, 140(9):04014064, 2014.

[58] C. Lanczos. *The variational principles of mechanics*. Courier Corporation, 2012.

[59] A. Laulusa and O. A. Bauchau. Review of classical approaches for constraint enforcement in multibody systems. *Journal of Computational and Nonlinear Dynamics*, 3(1):011004, 2008.

[60] J. Li and J. Chen. *Stochastic dynamics of structures*. John Wiley & Sons, 2009.

[61] Y.-K. Lin. *Probabilistic theory of structural dynamics*. Krieger Publishing Company, 1976.

[62] P. Losse and V. Mehrmann. Controllability and observability of second order descriptor systems. *SIAM journal on control and optimization*, 47(3):1351–1379, 2008.

[63] L. D. Lutes and S. Sarkani. *Random vibrations: analysis of structural and mechanical systems*. Butterworth-Heinemann, 2004.

[64] A. A. Maciejewski. Dealing with the ill-conditioned equations of motion for articulated figures. *IEEE Computer Graphics and Applications*, 10(3):63–71, 1990.

[65] N. K. Mani, E. J. Haug, and K. E. Atkinson. Application of singular value decomposition for analysis of mechanical system dynamics. *Journal of Mechanisms, Transmissions, and Automation in Design*, 107(1):82–87, 1985.

[66] L. Mariti, N. P. Belfiore, E. Pennestrì, and P. P. Valentini. Comparison of solution strategies for multibody dynamics equations. *International Journal for Numerical Methods in Engineering*, 88(7):637–656, 2011.

[67] D. J. McTavish and P. C. Hughes. Modeling of linear viscoelastic space structures. *Journal of Vibration and Acoustics*, 115(1):103–110, 1993.

[68] T. Mestdag, W. Sarlet, and M. Crampin. Second-order dynamical systems of lagrangian type with dissipation. *Differential Geometry and its Applications*, 29:S156–S163, 2011.

[69] C. D. Meyer. *Matrix analysis and applied linear algebra*, volume 2. Siam, 2000.

[70] C. Moler and C. Van Loan. Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. *SIAM review*, 45(1):3–49, 2003.

[71] A. Naess, F. Galeazzi, and M. Dogliani. Extreme response predictions of nonlinear compliant offshore structures by stochastic linearization. *Applied ocean research*, 14(2):71–81, 1992.

[72] A. Naess and J. M. Johnsen. Response statistics of nonlinear, compliant offshore structures by the path integral solution method. *Probabilistic Engineering Mechanics*, 8(2):91–106, 1993.

[73] D. E. Newland. *An introduction to random vibrations, spectral & wavelet analysis*. Courier Corporation, 2012.

[74] P. E. Nikravesh. *Computer-aided analysis of mechanical systems*. Prentice-Hall, Inc., 1988.

[75] P. E. Nikravesh, R. A. Wehage, and O. K. Kwon. Euler parameters in computational kinematics and dynamics. part 1. *Journal of Mechanisms, Transmissions, and Automation in Design*, 107(3):358–365, 1985.

[76] J. Nocedal and S. Wright. *Numerical Optimization, 2nd Edition*. Springer, New York, 2006.

[77] A. A. Pantelous, A. D. Karageorgos, and G. I. Kalogeropoulos. A new approach for second-order linear matrix descriptor differential equations of apostol–kolodner type. *Mathematical Methods in the Applied Sciences*, 37(2):257–264, 2014.

[78] A. A. Pantelous and A. Pirrotta. Modal analysis of multi-degrees-of-freedom systems with singular matrices: Analytical dynamics approach. *Journal of Engineering Mechanics*, 143(6):06017005, 2017.

[79] A. Papoulis and S. U. Pillai. *Probability, random variables, and stochastic processes*. Tata McGraw-Hill Education, 2002.

[80] L. A. Pars. *A treatise on analytical dynamics*. CT: Oxbow Press, Woodridge, 1979.

[81] R. Penrose. A generalized inverse for matrices. In *Mathematical proceedings of the Cambridge philosophical society*, volume 51, pages 406–413. Cambridge University Press, 1955.

[82] A. J. Pettofrezzo. *Matrices and transformations*. Courier Corporation, 1978.

[83] A. Pirrotta and R. Santoro. Probabilistic response of nonlinear systems under combined normal and poisson white noise via path integral method. *Probabilistic Engineering Mechanics*, 26(1):26–32, 2011.

[84] R. Piziak and P. L. Odell. Full rank factorization of matrices. *Mathematics magazine*, 72(3):193–201, 1999.

[85] S. Pradhan, V. J. Modi, and A. K. Misra. Order n formulation for flexible multibody systems in tree topology: Lagrangian approach. *Journal of guidance, control, and dynamics*, 20(4):665–672, 1997.

[86] C. Proppe, H. J. Pradlwarter, and G. I. Schuëller. Equivalent linearization and monte carlo simulation in stochastic dynamics. *Probabilistic Engineering Mechanics*, 18(1):1–15, 2003.

[87] D. Rabinovich, D. Givoli, and S. Vigdergauz. Xfem-based crack detection scheme using a genetic algorithm. *International Journal for Numerical Methods in Engineering*, 71(9):1051–1080, 2007.

[88] C. R. Rao. *Linear statistical inference and its applications*, volume 2. Wiley New York, 1973.

[89] L. Reichel and G. Rodriguez. Old and new parameter choice rules for discrete ill-posed problems. *Numerical Algorithms*, pages 1–23, 2013.

[90] J. B. Roberts and P. D. Spanos. Stochastic averaging: an approximate method of solving random vibration problems. *International Journal of Non-Linear Mechanics*, 21(2):111–134, 1986.

[91] J. B. Roberts and P. D. Spanos. *Random Vibration and Statistical Linearization*. Dover Publications, New York, 2003.

[92] W. O. Schiehlen. Dynamics of complex multibody systems. *Solid Mechanics Archives*, 9:159–195, 1984.

[93] A. D. Schutte and F. E. Udwadia. New approach to the modeling of complex multibody dynamical systems. *Journal of Applied Mechanics*, 78(2):021018, 2011.

[94] A. A. Shabana. *Dynamics of multibody systems*. Cambridge university press, 2013.

[95] L. Socha. Statistical and equivalent linearization techniques with probability density criteria. *Journal of Theoretical and Applied Mechanics*, 37(2):369–382, 1999.

[96] L. Socha. Linearization in analysis of nonlinear stochastic systems: recent resultspart i: theory. *Applied Mechanics Reviews*, 58(3):178–205, 2005.

[97] L. Socha. Linearization in analysis of nonlinear stochastic systems, recent resultspart ii: applications. *Applied Mechanics Reviews*, 58(5):303–315, 2005.

[98] L. Socha. *Linearization methods for stochastic dynamic systems*, volume 730. Springer Science & Business Media, 2007.

[99] P. D. Spanos. Stochastic linearization in structural dynamics. *Appl. Mech. Rev*, 34(1):1–8, 1981.

[100] P. T. D. Spanos. Formulation of stochastic linearization for symmetric or asymmetric mdof nonlinear systems. *Journal of Applied Mechanics*, 47(1):209–211, 1980.

[101] P.-T. D. Spanos and W. D. Iwan. On the existence and uniqueness of solutions generated by equivalent linearization. *International Journal of Non-linear mechanics*, 13(2):71–78, 1978.

[102] M. S. Srivastava and C. G. Khatri. *An Introduction to Multivariate Statistics*. Lect. Notes Phys. 730. North-Holland, New York, 1979.

[103] M. S. Srivastava and D. von Rosen. Regression models with unknown singular covariance matrix. *Linear algebra and its applications*, 354(1-3):255–273, 2002.

[104] D. Terzopoulos and A. Witkin. Physically based models with rigid and deformable components. *IEEE Computer Graphics and Applications*, 8(6):41–51, 1988.

[105] S. C. Tripathy, G. D. Prasad, O. P. Malik, and G. S. Hope. Load-flow solutions for ill-conditioned power systems by a newton-like method. *IEEE Transactions on Power apparatus and Systems*, (10):3648–3657, 1982.

[106] E. Turco. Load distribution modelling for pin-jointed trusses by an inverse approach. *Computer Methods in Applied Mechanics and Engineering*, 165(1-4):291–306, 1998.

[107] E. Turco. A strategy to identify exciting forces acting on structures. *International Journal for Numerical Methods in Engineering*, 64(11):1483–1508, 2005.

[108] F. E. Udwadia and R. E. Kalaba. A new perspective on constrained motion. *Proceedings: Mathematical and Physical Sciences*, 439(1906):407–410, 1992.

[109] F. E. Udwadia and R. E. Kalaba. On motion. *Journal of the Franklin Institute*, 330(3):571–577, 1993.

[110] F. E. Udwadia and R. E. Kalaba. Explicit equations of motion for mechanical systems with nonideal constraints. *Journal of Applied Mechanics*, 68(3):462–467, 2001.

[111] F. E. Udwadia and R. E. Kalaba. *Analytical dynamics: a new approach*. Cambridge University Press, 2007.

[112] F. E. Udwadia and P. Phohomsiri. Explicit equations of motion for constrained mechanical systems with singular mass matrices and applications to multi-body dynamics. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 462, pages 2097–2117. The Royal Society, 2006.

[113] F. E. Udwadia and A. D. Schutte. Equations of motion for general constrained systems in lagrangian mechanics. *Acta mechanica*, 213(1):111–129, 2010.

[114] F. E. Udwadia and T. Wanichanon. On general nonlinear constrained mechanical systems. *Numerical Algebra, Control and Optimization*, 3(3):425–443, 2013.

[115] N. Wagner and S. Adhikari. Symmetric state-space method for a class of nonviscously damped systems. *AIAA journal*, 41(5):951–956, 2003.

[116] X. Wang and A. H. Day. Numerical instability in linearized planing problems. *International journal for numerical methods in engineering*, 70(7):840–875, 2007.

[117] J. Woodhouse. Linear damping models for structural vibration. *Journal of Sound and Vibration*, 215(3):547–569, 1998.

[118] S. X. Xu and T. S. Koko. Finite element analysis and design of actively controlled piezoelectric smart structures. *Finite elements in analysis and design*, 40(3):241–262, 2004.

[119] X. Zhang, I. Elishakoff, and R. Zhang. A stochastic linearization technique based on minimum mean square deviation of potential energies. In *Stochastic Structural Dynamics 1*, pages 327–338. Springer, 1991.

[120] X. Zhang and X. Liu. Output regulation for matrix second order singular systems via measurement output feedback. *Journal of the Franklin Institute*, 349(6):2124–2135, 2012.

[121] W. Q. Zhu. Recent developments and applications of the stochastic averaging method in random vibration. *Applied Mechanics Reviews*, 49(10S):S72–S80, 1996.