**Static output feedback for partial eigenstructure assignment of undamped vibration systems**

**Jiafan Zhanga,\*[[1]](#footnote-1), Jianping Yea , Huajiang Ouyangb**

aSchool of Mechanical Engineering, Wuhan Polytechnic University, Wuhan 430023, People’s Republic of China

bSchool of Engineering, University of Liverpool, UK

**Abstract**: A novel method for partial eigenstructure assignment of undamped vibration systems using acceleration and displacement output feedback is presented in this paper. It is based on modifications of mass and stiffness that preserve partial eigenstructure. A numerical algorithm for determining the required control gain matrices of acceleration and displacement output feedback, which assign the desired eigenstructure, is developed. This algorithm is easy to implement, and works directly on the second-order system model. More importantly, the algorithm allows the output matrix and the input matrix to be specified beforehand and also leads naturally to a small norm solution of the gain matrices. Finally, some numerical results are presented to demonstrate the effectiveness and accuracy of the proposed algorithm.

**Keywords**: Partial eigenstructure assignment; Static output feedback; Undamped vibration system; Acceleration and displacement feedback

**1. Introduction**

Active vibration control techniques of engineering structures have been extensively studied during the past three decades. The dynamic response of a vibrating system can be altered by changing the system's natural frequencies and mode shapes, namely, its modal characteristics, which is also referred to as the eigenstructure (i.e. the eigenvalues and eigenvectors). Thus the eigenvalue or eigenstructure reallocation or assignment is a common control strategy in active vibration suppression.

Eigenvalue assignment and eigenstructure assignmentworking directly on second-order dynamic system models has attracted much attention over the last ten years, partly because of the demands in general control and vibration control applications in engineering, and partly because of the advantage of those peculiar properties afforded by the second-order system models. Another issue being taken into account in the area of research is that, in real applications, it is needed to change only a few undesirable eigenvalues or undesirable part of eigenstructure which are purposefully assigned to desired values, and it is desirable to keep all other eigenpairs unchanged. This problem is called *partial eigenvalue or eigenstructure assignment*. Some major effort can be seen from the literature to tackle this problem, for example, in [1-14] on damped and undamped second-order vibration systems.

All these above approaches solved the problem by *full-state feedback*, but in most practical situations the full states are not directly available. From a practical standpoint, a more attractive procedure would be one which is based upon feeding back only the measured variables, i.e., *static* *output feedback* (SOF). For eigenvalue or eigenstructure assignment of first-order state-space system models via SOF, a great deal of research exists, and numerical algorithms and some readily verifiable necessary or sufficient conditions for determining solvability have been proposed. Many results, however, are mainly theoretical in nature and there are no good numerical algorithms available in many cases when a specific system is known to be solvable. It is believed that the solution techniques that work well on small-sized systems may be doomed as the system size increases [15]. Someone suggests that every effort should be made to exploit the particular structure of a given SOF problem. The starting points for further information about SOF are the survey papers [15, 16], as well as the more recent papers [17-19]. As for those working directly on second-order system models via SOF, few results can be seen from the literature. Lin and Wang proposed a solution to the partial eigenvalue assignment problem for the second-order damped vibration system models by SOF [20]. They considered the elements of the output matrix and the input matrix as design variables as well, and explained the research problem for this setting in [20] as follows: *For the usual partial eigenvalue assignment problem by output feedback*, *the input and output matrices are in general fixed*. *However*, *it seems very difficult to relocate unwanted eigenvalues to desired values while keeping all wanted eigenstructure unchanged with fixing input and output matrices*. *To our knowledge*, *there is no result in this direction*. In addition, they set the input matrix to be the transpose of the output matrix, namely, the collocated actuator and sensor configuration.

In this paper we attack the partial eigenstructure assignment problem by SOF for second-order undamped vibration system models. The main contribution of this paper consists of the following: (1) The input matrix and the output matrix here can be prescribed and chosen in a simple form, and the collocated actuator and sensor configuration is not necessary. Two measured variables, the acceleration and displacement, are used and correspondingly there are two output matrices, respectively. (2) The proposed algorithm only needs those few eigenpairs to be assigned and the analytical mass and stiffness matrices of the original vibration system, and also leads naturally to a small norm solution of the output feedback gain matrices.

The work here is based on our recent article [14] where we obtained a partial eigenstructure modification formulation. In [14] a necessary and sufficient condition was proposed for the incremental mass and stiffness matrices that modify some eigenvalues or eigenpairs while keeping other eigenpairs unchanged, and an efficient numerical algorithm was suggested for partial eigenstructure assignment of undamped vibration systems using acceleration and displacement state feedback. In what follows, the partial eigenstructure modification formulation is presented, and the problem involved, some notations and assumptions are described in Section 2. A partial eigenstructure assignment algorithm is proposed to determine the acceleration and displacement output feedback gain matrices in Section 3. In Section 4, some numerical results are provided to demonstrate the effectiveness of the proposed method.

**2.** **The problem description**

**2.1 A partial eigenstructure modification formulation**

Consider an *n*-degree-of-freedom undamped vibration system that is modelled by the following set of second-order ordinary differential equations:

(1)

where is the displacement vector, is the vector of external forces, and , and are constant mass and stiffness matrices, respectively. In general, is symmetric and positive definite, and is symmetric and positive semi-definite, i.e*.*, .

It is well known that if is a fundamental solution of (1), then the natural frequency and the mode shape vector must satisfy the following generalized eigenvalue equation:

(2)

where the *i*th eigenvalue is the square of the *i*th natural frequency , and is the corresponding *i*th eigenvector. Eq. (2) can be written in a compact representation as follows:

(3)

where and make up the complete eigenstructure of the system (1), and satisfies the mass-normalised condition .

Suppose that the system described by (1) is modified by the incremental mass and stiffness matrices and . Then the motion of the modified system is governed by

(4)

and it satisfies the following eigen-matrix equation:

(5)

where and are the complete eigenstructure of the modified system (4).

In [14] a necessary and sufficient condition was proposed for the incremental mass and stiffness matrices that modify some eigenvalues or eigenpairs while keeping other eigenpairs unchanged, which is crucial to address the partial eigenstructure assignment problem by SOF in this paper and thus is shown in the following:

(6)

where and are submatrices of and , and are composed of eigenvalues and eigenvectors to be modified in the system (1), respectively. It implies that, if and satisfies (6), the following eigen-matrix equation then holds:

(7)

where and are submatrices of and , and are composed of unchanged eigenvalues and eigenvectors of the system (1). Here and . Eq. (7) means that and is also the eigenpairs of the modified system.

**2.2 Partial eigenstructure assignment of undamped vibration systems by SOF**

When active control forces is exerted on a undamped vibration system, Eq. (1) now becomes

(8)

where is known as the control input matrix, and without loss of generality, is assumed to have a full column rank, that is, . The control force vector is a time-dependent real vector. For the SOF, could take the following particular form:

(9)

where and are displacement and acceleration output feedback gain matrices, respectively. , are output vectors which represent measured displacements and accelerations, respectively. They have the following form:

and (10)

where , are displacement and acceleration output matrices, respectively, and are generally assumed to be full row rank. Substituting (9) and (10) into (8) yields the closed-loop system (11) of (1) as follows.

(11)

Let and denote the closed-loop eigenvalue and eigenvector matrices, respectively, then the following eigen-matrix equation of the closed-loop system (11) holds:

(12)

*Now* *the partial eigenstructure assignment problem by SOF is to find output feedback gain matrices*  *and* *such that eigenpairs* *and* *of* (1) *is replaced by eigenpairs* *and* *of the closed-loop system* (11), *while the remaining* *eigenpairs of* (11) *are also eigenpairs of the system* (1), *i.e.* *and* .

Additionally, throughout this paper, we also assume that (a) , for , and is a finite value. (b) or , . (c) The system (8) with (9), (10) is controllable and observable.

**3. SOF assignment algorithm**

Taking and from the comparison of (5) and (12), and substituting them into (6) gives

(13)

since is of full column rank. Obviously, if and satisfy (13), then the remaining eigenpairs of the closed-loop system (11) remain the same as eigenpairs and of the original system (1). On the other hand, and must also implement some given eigenstructure assignment. By definition,

(14)

Now Eqs. (13) and (14) are two key matrix equations used to solve our problem.

In what follows, we firstly present the general solutions of (13) for and . Eq.(13) is rewritten as

(15)

Let the rank , and use QR decomposition (or singular value decompositions), one gets

(16)

where is an orthogonal matrix with and , and is an upper triangular and nonsingular matrix. Let

*=* (17)

where , . Thus, we obtain the general solutions from (15), (16) and (17) (see [14] for details) as follows.

(18)

where is to be determined below.

Secondly, the solutions of (14) for and are sought below. Eq. (14) is rewritten as

(19)

Substituting the QR decomposition of , i.e. , where is an orthogonal matrix with and , and is an upper triangular and nonsingular matrix, into (19) gives

that is

(20)

(21)

As is known, for the given assigned eigenvalues of (11), not any given corresponding eigenvectors can be assigned to (11) such that there exist and satisfying (14) or (19). It should be pointed out, from (21), that “achievable” eigenvectors corresponding to for the eigenstructure assignment here must belong to the right null space of matrix, . In order to ensure the accuracy of partial eigenstructure assignment, it is necessary to ‘condition’ the given . For the details of the conditioning algorithm refer to [14]. The conditioned eigenvectors corresponding to are denoted by , which will replace in the following calculation of the given eigenstructure assignment of (19).

Now the solution of (20) can be sought. Substituting the general solutions (18) into (20) gives

(22)

Eq. (22) is the matrix equation of the form , where , and are given matrices of appropriate dimensions and matrix needs to be determined. The necessary and sufficient condition for the existence of solutions on this type of matrix equation is [21], where the superscript denotes the Moore–Penrose inverse of a matrix. In particular, the solvability of (22) for is related to , and with given assigned eigenvalues and the control input matrix *B*. If the rank ( is the number of eigenvalues to be assigned), then Eq. (22) for has solutions. This condition means , i.e. . In the event that this condition is not satisfied, different *B*and output matrices *Cd*, *Ca*must be selected. The choices are plentiful and hence the solvability is not believed to be a problem. Supposing here that Eq. (22) has solutions, then a unique minimal norm solution of (22) for matrix is obtained as follows [21]:

(23)

Substituting the obtained back into (18), then output feedback gain matrices and are eventually determined, which could solve the problem of partial eigenstructure assignment by SOF in this paper.

**4. Numerical examples**

To demonstrate the performance of the present algorithm, two numerical examples are analysed in this section, using MATLAB 7.11.

**Example 4.1**[14]In this example, *n* = 6, *m* = *r* = *p* = 3, and



, 

The open-loop eigenvalues are

. This means,

. Let .

The original eigenvector matrix , and the assigned and are listed in Table 1.

Table 1. Eigenvector matrices , and

Two different configurations of output matrices and the corresponding assignment results are presented in the following:

(a) , i.e..

The displacement and acceleration output-feedback gain matrices and their *F*- norms are shown in Table 2.

Table 2. Output feedback gain matrices and and their norms

The *F*- norms of the closed-loop eigen-matrix equations are

(b) , , .

The displacement and acceleration output-feedback gain matrices and their *F*- norms are shown in in Table 3.

Table 3. Output feedback gain matrices and and their norms

The *F*- norms of the closed-loop eigen-matrix equations are

Note that for the configuration, the resulting assigned frequencies and modes are quite accurate too.

**Example 4.2** In this example, *n* = 3, *m* = *r* = *p* = 2, and

, , 

The open-loop eigenvalues and eigenvectors are

, 

This means,  and .

Let , and the output matrices

, .

For four groups of eigenvectors , the corresponding gain matrices and are presented in Table 4.

Table 4. Various eigenvector matrices and the corresponding gain matrices

The results in Table 4 show that the partial eigenstructure are assigned successfully using our algorithm, and it is obvious that different choices of eventually affect the resulting feedback gain matrices. From our experience gained from calculating some numerical examples, in all situations the solvability of (22) is satisfied after the assigned eigenvectors are conditioned according to Eq. (21).

It should be pointed out that the closed-loop mass matrix would be nonsingular (when is nonsingular) if the partial eigenstructure or partial finite eigenvalues are assigned successfully using our algorithm.

**5. Conclusions**

For undamped vibration systems, we propose a method to solve the partial eigenstructure assignment problem by static output feedback control. The proposed algorithm can accurately assign prescribed eigenpairs while keeping other unassigned eigenpairs unchanged for this particular second-order model, which mainly involves numerically stable matrix computations, such as QR decomposition (or singular value decomposition). More importantly, the successful assignment can be achieved with some predetermined input and output matrices. It should be noted, however, that the proposed algorithm may present difficult computational problems if the mass matrix is nearly singular, since it involves the inverse computation of the mass matrix.

**References**

[1] D.J. Inman, A. Kress, Eigenstructure assignment using inverse eigenvalue methods, Journal of Guidance, Control, and Dynamics. 18 (1995) 625–627.

[2] B.N. Datta, S. Elhay, Y.M. Ram, Orthogonality and partial pole assignment for the symmetric definite quadratic pencil, Linear Algebra Applications. 257 (1997) 29–48.

[3] Y.M. Ram, S. Elhay, Pole assignment in vibratory systems by multi-input control, Journal of Sound and Vibration. 230 (2000) 309–321.

[4] B.N. Datta, S. Elhay, Y.M. Ram, D.R. Sarkissian, Partial eigenstructure assignment for the quadratic pencil, Journal of Sound and Vibration. 230 (2000) 101–110.

[5] E.K. Chu, Pole assignment for second-order systems, Mechanical Systems and Signal Processing. 16 (2002) 39–59.

[6] J.F. Zhang, Partial pole assignment for general vibration systems by multi-input state feedback (in Chinese), Journal of Vibration and Shock. 20 (2001) 45–47.

[7] S.F. Xu, J. Qian, Orthogonal basis selection method for robust partial eigenvalue assignment problem in second-order control systems, Journal of Sound and Vibration. 317 (2008) 1–19.

[8] J. Qian, S.F. Xu, Robust partial eigenvalue assignment problem for the second-order system, Journal of Sound and Vibration. 282 (2005) 937–948.

[9] B.N. Datta, W.W. Lin, J.N. Wang, Robust partial pole assignment for vibrating structures with aerodynamic effect, IEEE Transactions on Automatic Control. 51 (2006) 1979–1984.

[10] S. Brahma, B.N. Datta, An optimization approach for minimum norm and robust partial quadratic eigenvalue assignment problems for vibrating structures, Journal of Sound and Vibration. 324 (2009) 471–489.

[11] Z.J. Bai, B.N. Datta, J.W. Wang, Robust and minimum norm partial quadratic eigenvalue assignment in vibrating systems: a new optimization approach, Mechanical Systems and Signal Processing. 24 (2010) 766–783.

[12] Y.F. Cai, J. Qian, S.F. Xu, The formulation and numerical method for partial quadratic eigenvalue assignment problems, Numerical Linear Algebra with Applications .184 (2011) 637–652.

[13] J.F. Zhang, H. Ouyang, Y. L. Zhang, J.P. Ye, Partial quadratic eigenvalue assignment in vibrating systems using acceleration and velocity feedback. Inverse Problems in Science and Engineering. 23(3) (2015) 479–97.

[14] J.F. Zhang, H. Ouyang, J. Yang, Partial eigenstructure assignment for undamped vibration systems using acceleration and displacement feedback, Journal of Sound and Vibration. 333 (1)(2014) 1–12.

[15] V. L. Syrmos, C. T. Abdallah, P. Dorato, and K. Grigoriadis, Static output feeback – a survey, Automatica. 33(2)( 1997) 25–137.

[16] J. Rosenthal, J. C. Willems, Open problems in the area of pole placement. in: Open problems in mathematical systems and control theory, V. D. Blondel, E. D. Sontag, M. Vidyasager, and J. C. Willems, Eds. Springer Verlag, 1998, pp.181–191.

[17] M. Fu, Pole placement via static output feedback is NP-hard, IEEE Transactions on Automatic Control. 49(5)( 2004) 855–857.

[18] K. Yang, R. Orsi, Static Output Feedback Pole Placement via a Trust Region Approach, IEEE Transactions on Automatic Control. 52(11)( 2007)2146 –2150.

[19] A. X. Wang, U. Konigorski, On Linear solutions of the output feedback pole assignment problem, IEEE Transactions on Automatic Control. 58(9)(2013) 2354– 2359.

[20] W.W. Lin, J.N. Wang, Partial pole assignment for the quadratic pencil by output feedback control with feedback designs, Numerical Linear Algebra with Applications 12(10)( 2005) 967–979.

[21] G.H. Golub, C.F. Van Loan, Matrix Computations (Third Edition), Johns Hopkins University Press, Baltimore, 1996.

Table 1. Eigenvector matrices , and

|  |
| --- |
|  |
| 1.0000 1.0000 1.0000  **0.7039 0.4525 0.3001**  -0.1529 -0.5317 -0.8832 **-1.0000 -1.0000 -1.0000**  0.5469 -0.4235 -0.6561 **-0.0819 0.2105 0.0565**  -0.1454 -0.3288 0.1410  **0.7738 0.4879 -0.3469**  0.1655 -0.5899 0.7440 **-0.1773 -0.1195 0.0361**  -0.1005 0.1960 0.1847 **-0.7552 0.7418 -0.1234** |

|  |
| --- |
|  |

1.0000 1.0000 1.0000 **1.0000 1.0000 1.0000**

-0.0152 -0.1317 -0.3832 **-0.0312 -0.2149 -0.7661**

0.6469 -0.3235 -0.5561 **0.6878 -0.2187 -0.7466**

-0.2454 -0.4288 0.2410 **-0.1563 -0.4360 0.0829**

0.2655 -0.3899 0.5440 **0.2342 -0.6176 0.8050**

-0.2005 0.2960 0.2847 **-0.1103 0.2460 0.3105**

|  |
| --- |
|  |

Table 2. Output feedback gain matrices and and their norms

|  |
| --- |
|  |
| -0.3094 0.5211 3.0889  0.0927 0.3437 3.7486 8.0040  -3.6947 2.8966 4.2351 |
|  |
|  |
| -0.0455 0.2027 0.0084  -0.0505 0.2321 0.0102 0.5010  -0.0960 0.3767 0.0115 |

Table 3. Output feedback gain matrices and and their norms

|  |
| --- |
|  |
| -0.2900 -0.0146 -0.2132  0.1131 -0.2679 -0.0325 4.5489  -3.6465 1.8884 -1.9011 |
|  |
|  |
| 0.0522 0.0069 -0.0396  0.0604 0.0084 -0.0457 0.1544  0.0926 0.0093 -0.0709 |

Table 4. Various eigenvector matrices and the corresponding gain matrices

|  |
| --- |
| solvability of (22) for  (Y/N) |
| Y  Y  Y  Y |

1. The corresponding author, 0086-27-85617871/0086-27-85617871, jfz@whpu.edu.cn. [↑](#footnote-ref-1)