**A linear complementarity method for dynamic analysis of bridges under moving vehicles considering separation and surface roughness**

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# Abstract

In the present paper, a linear complementarity method for a vehicle-bridge dynamic system considering separation and random roughness is established. By introducing the linear complementarity relationship between the relative displacement of the wheels and the bridge at the contact points, the dynamic interaction problem of the vehicle-bridge coupled system is transformed into a standard linear complementarity problem, and two models with different connection relations between the wheels and the bridge are proposed. The presented models characterize the system with one unified formulation whether the wheels separate from the bridge or not, and the conventional trial-and-error iterative process in numerical simulation is avoided. In the numerical examples, the proposed method is verified by comparing it with the conventional method, and it is found that the velocity, the vehicle to bridge mass ratio and the road roughness have a significant influence on separation. By considering a vehicle model of three rigid bodies with four wheels and the randomness of the rail roughness in train-track-bridge system, the possibility of separation and the expectation of the maximum separation distance at different velocities are studied. The results show that it is very useful to carry out a stochastic analysis of the system and consider the influence of separation in vehicle and bridge design.

***Keywords****:* Vehicle-bridge system; Vibration; Separation; Linear complementarity problem; Random roughness

# 1. Introduction

The dynamic behaviour of beam structures, such as railway bridges, subject to moving loads has been investigated for over a century since Stocks [1] firstly brought this problem into attention. It is of great interest in many engineering applications, such as the design of bridges, railway tracks, cableways, etc., and a large number of papers related to this problem have been published to predict the dynamic responses of simple supporting structures under moving loads [2-8]. The early structural engineers found that under moving loads, structural dynamic deformations and stresses can be significantly higher than those caused by corresponding static loads [2,3]. Frýba [4] described the basic postulation of moving load problems and their analytical solutions. Olsson [5] discussed the assumption inherent in the moving force problem and solved it by the finite element method. In the moving force problem, the inertial force of the moving structure was neglected which would possibly miss some aspects of the physics involved and could not capture the interaction behaviour between the moving structure and the bridge during the travelling [7]. The moving mass problem was then suggested, which brings some improvements to the moving force model. The moving structure was considered a single mass [9,10] first, and an oscillator model was then considered which would be more appropriate for some applications [11-13]. More models with different degrees of complexity are used to represent realistic vehicles [14-16].

As the position of the moving structure changes with time, the coefficient matrices of the moving mass problem are time-dependent. There are two conventional ways to simulate the time-dependent equations of motion of vehicles-bridge systems. The first way assumes that the vehicle is in sliding contact with the bridge and solves the coupled equations of motion for the whole vehicle-bridge system through numerical integration in time domain, which requires a very small time step [17]. The other way is based on the uncoupled iteration method, in which each system (both the vehicles and the bridge) is solved separately and an iterative process in each time step is performed to find the equilibrium between the bridge and vehicle wheels. By using a proper estimate of the interaction forces, the accurate solution of the system can be obtained for a larger time step [17,18]. The paper adopts the second way.

All these papers mentioned above took no consideration of the possibility of separation between the moving object and the bridge, although separation was shown to be possible in theory and studied in some numerical simulations [19-25]. Lee [20] is perhaps the first researcher to study the separation between the moving mass and the supporting structure. In that case, the contact force was time-dependent and became zero under certain circumstances. The transition of the moving contact force from positive to zero was considered to be the onset of separation between the moving and supporting structures. It is found that the most important parameters influencing the separation are the sliding speed and the mass ratio between the moving mass and the beam [21]. Stăncioiu et al. [22,23] and Baez and Ouyang [24] studied separation and reattachment of an oscillator moving along a beam structure, and put forward a simplified method for computing the dynamic responses after the impact at reattachment. On the other hand, the road roughness of the bridge deck is a real physical phenomenon which almost exists in all bridges. It is found that road roughness magnifies the dynamic response due to the moving action of the mass, which would possibly be detrimental to the safety and serviceability of structures [26-28]. Furthermore, for a railway bridge, the rail roughness is a random process, which would increase the possibility of separation and make the responses harder to predict. Cheng et al. [25] studied separation considering the surface roughness of the bridge modelled as a continuous beam, and proposed an algorithm to account for the impact on the reestablishment of contact. It is found that separation often occurs in the vicinity of the crests of roughness, and the velocity has a significant influence on separation.

Many researchers modelled the vehicle as an oscillator, which has only one wheel, thus only one equation of motion is required to demonstrate separation. But for a multi-wheel vehicle model, loss of contact for any wheel of the vehicle changes the time-dependent matrices by adding a DOF corresponding to the wheel separating from the bridge. Extra equations of motion are needed to describe all the possible combinations due to different wheels separating from the bridge, which makes it too complicated to deal with.

In this paper, a linear complementarity method [29,30] for vehicle-bridge system considering separation with road roughness is established. In the present simulation, a unified equation of motion is constructed to describe the system whether the wheels separate from the bridge or not. Therefore, the complex trial-and-error iterative processes in conventional numerical simulation are avoided (that is, the assumptions or the iterative process for determination of contact states is not required in the proposed method). The moving oscillator model with separation is firstly used to verify the present method by comparing it with the moving oscillator model with permanent contact. It is found that the road roughness significantly influences the separation region at various values of the velocity ratio and the vehicle to bridge mass ratio considered. The random track irregularity in railway and a multi-wheel vehicle are considered later. By implementing the Monte Carlo simulation, the possibility of separation and its influence on the dynamic responses are studied. Finally, the expectation of maximum distance of separation is determined, which is suggested as an indicator of the influence of separation.

# 2. Equations of motion of vehicle bridge systems and separation

As shown in Fig. 1, a simple beam model for a vehicle bridge is presented first. The vehicle is modelled as an oscillator with two degrees of freedom and . The sprung mass and the wheel are connected by a spring-damper, with stiffness and damping . The oscillator is moving along the bridge at a constant velocity .

Figure 1 in text

The vertical displacement of the sprung mass and the vertical displacement of the wheel are governed by a set of two equations of motion

where the overdot stands for the total derivative and is the interaction force at the wheel, is gravitational acceleration. Eq. can be written in matrix form

On the other hand, the equation of motion of a finite element model of bridge can be written as

where , and are the mass matrix, damping matrix and stiff matrix of the bridge respectively. is the load vector acting on the bridge, which can be obtained by

where is the influence matrix, which transforms the non-nodal load into equivalent nodal load, and can be written as

in which is the shape function vector of the bridge element in contact with the wheel,

where is the local coordinate of the contact point related to the wheel’s horizontal location which is changing with time, and

while is the length of the contact element, is the position transforming matrix of the element, consisting of 0 and 1. Similarly, the displacement of the bridge-wheel contact point can be obtained by

By applying the mode superposition method to the bridge, the displacement of the bridge can be expressed as

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where is the modal matrix of the bridge, and is the modal displacement vector for the bridge. Thus the equation of motion of the bridge can be written as

in which

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Conventionally, the wheel and the bridge are assumed to be in permanent contact, and hence the vertical displacement of the wheel is assumed to be identical to that of the bridge at the contact point. As a result, the displacement of the wheel can be expressed in terms of the bridge deflection at the contact point

where is the surface roughness at the bridge contact point, and

For the whole system, the displacements can be denoted by

where

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and is the identity matrix. Substituting Eq. into Eqs. and , and eliminating the dependent DOF of the wheel, accordingly yields the equations of motion of the system

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It can be noticed that coefficient matrices in Eq. are all time-dependent because the influence matrix changes with time.

In the system described above, the wheel and bridge are assumed to be in permanent contact. However it is possible that the moving vehicle could separate from the bridge during its horizontal travel. Considering separation, it is crucial to check whether or not the moving vehicle separates from the bridge, and it can be done by monitoring the values of the contact force. In this respect, the separation is considered to occur when these conditions are satisfied:

During the separation, the governing equations of motion of the vehicle and the bridge are now explicitly expressed as

where .

Eqs. and constitute the equations of motion during separation, and it is assumed that no discontinuity in the velocity or the deflection of the beam when the separation happens. This way, the initial conditions for these equations are given by the solution in Eq. at (shown in Fig. 2). The reattachment is considered to happen when the displacement of the wheel and the bridge at the same horizon location are equal to each other, as shown at . Since the wheel does not have an independent degree of freedom when it is in contact with the bridge, an assumption is made for reattachment: the beam displacement equals the vertical coordinate of the wheel after the reattachment and a very short-duration impact force is considered to act on the beam within () [22].

Figure 2 in text

The formulations after separation would become more complex if the vehicle has more than one wheel. Stăncioiu et al. [23] studied a two-wheel vehicle moving along a bridge considering separation, and 3 equations of motion are required due to different scenarios of separation of wheels from the bridge. Furthermore, for a vehicle model shown in Fig. 3, there are (where stands for combination) possible wheel separation scenarios, and hence 15 governing equations are needed to describe all the possible combinations, which are excessively complicated.

Figure 3 in text

# 3. Linear complementarity method for the rigid contact

Ever since an algebraic proof of the existence of equilibrium points for linear programming problems was given, leading to an efficient scheme for computing equilibrium points by Lemke and Howson [31], the complementary theory has been widely used in physics, mechanics, economics and operations research. In this paper, the relationship of the wheels and the bridge is established by linear complementarity method considering separation, in which the vehicle-bridge dynamic problem is transformed into a linear complementarity problem (LCP).

Take the moving oscillator model shown in Fig. 1 as an example. The equations of motion of the oscillator and the bridge have already been obtained, as shown as Eqs. and . By using a step-by-step integration method, the responses of the system can be expressed in terms of the interaction force at a certain time

where is the wheel location matrix, and . , , and are the recursive vectors at time generated by the step-by-step integration method. Take Newmark method as an example

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where , , , , , , are the parameters of Newmark method. In the numerical simulations, and are taken.

The displacement of the wheel at time can be obtained by

and the displacement of the bridge-wheel contact point at time can be obtained from Eq. , as

Substituting Eqs. and into Eqs. and , it can be obtained

in which

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It should be mentioned that the displacements of the wheel is always bigger than or equal to the displacement of the bridge at the contact point, and hence there exists the inequality relation

where is the road roughness at the contact point. It must be said that both the interaction force and the relative distance between the wheel and bridge must be nonnegative. So when a wheel separates from the bridge, the local interaction force becomes zero, meanwhile the distance is greater than zero; when the wheel and the bridge are in contact, the distance is zero while the interaction force is positive. These can be expressed mathematically as

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where “×” produces a new vector whose individual element is the product of the corresponding elements of the two vectors involved.

Substituting Eqs. and into and , it can be obtained

Eq. is a standard LCP equation, and the interaction force can be easily obtained by solving the LCP equation [30]. Then the responses of the system can be derived from Eqs. and .

The present method can be also used in the multi-wheel vehicle model as shown in Fig. 3. In this case, the equation of motion of the vehicle can be written as

where

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in which, is the displacement vector, is the vertical displacement, and is the rotation angle. Subscript denotes the DOF of the vehicle, including the vehicle body (denoted as subscript ) and the bogies (denoted as subscript ), and subscript denotes the DOF of the wheels. The mass matrices and the stiffness matrices of the vehicle in Eq. can be written as

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and

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The damping matrices are of the same form as the stiffness matrices, and

where is the interaction force acting on the -th wheel, , and are the mass of the vehicle body, bogie and the wheel respectively, and are rotational inertia of the vehicle body and the bogie respectively.

On the other hand, the equation of motion of the bridge can be written as Eq. . In this case, is a vector, which can be expressed as

and

in which is the shape function of the element where the -th wheel is located, and is the position matrix of that element. It can be obtained by applying Eq.

The displacements of the wheels at time can be obtained as

where is the wheel location matrix, and , in which is a 4×4 identity matrix. The deflections of the bridge-wheel contact points are

Substituting Eqs. and into Eqs. and , yields

, , and can be similarly obtained from Eqs. and ,

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It should be noticed that for each wheel, the relationship in Eqs. and are still valid, and hence there exist the inequality relations below

where is the road roughness at the -th wheel-bridge contact point, is the interaction force and is the relative distance between the -th wheel and bridge. For all the wheels, it yields

 *,*

Substituting Eqs. and into Eq. , it can be obtained

Eq. is also a standard LCP equation, and the interaction force can be easily obtained by solving the LCP equation. It is worthy noticing that only one equation is used whether the wheels separate from the bridge or not, and no trial-and-error iterative process is needed, which is a major advantage over Stăncioiu et al. [22]. It should be noted that, the impact effect is not considered in the present rigid contact LCP model when the wheel gets reconnected with the bridge. The proposed method considered the vehicle and the bridge as uncoupled subsystems, and the motion of the subsystems is governed by the equations of motion of their own. Thus, when the wheel and the bridge reestablish contact again, a constant contact force is assumed in that small time-step.

# 4. Linear complementarity method for the elastic contact

For rigid contact in the vehicle bridge model, the solution of the equation cannot easily converge sometimes. Therefore, an elastic contact model is used to make the dynamic contact condition more flexible to improve the numerical treatment of the dynamic contact condition.

The elastic contact model assumes each of the wheels is connected to the bridge by a linear and tensionless spring (with stiffness ), thus the interaction force between the -th wheel and the bridge shown in Fig. 3 can be written as

where is the original compression of the *i*-th spring due to gravity, and is the compression variable, defined by

As a result, the interaction force vector can be expressed as

where

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By substituting Eq. into Eqs. and , the equation of motion of the system can be obtained as

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By using a relaxation factor for the -th wheel, the linear complementarity condition can be obtained as

Thus for all the wheels, it gives

where . Eq. is also an LCP equation.

The present algorithm can be implemented by the following 5 steps:

1) Generate the recursion functions and by using a step-by-step integration method for Eq. ;

where **.**

2) Obtain the relationship of the displacement vector of the wheels and the bridge at the contact points;

3) Substitute Eq. into Eq. , gives

4) Acquire the interaction force by solving the LCP of Eq. ;

5) Obtain the responses of the system by substituting the interaction force back into the step-by-step integration algorithm.

# 5. Numerical examples

## 5.1 Responses of moving oscillator model

The moving oscillator model shown in Fig. 1 is considered in this section, and the bridge is modelled as a simply supported beam with a single-span, which may have a harmonically varying surface roughness. In accordance with notations used in Reference [20], six dimensionless parameters are used, as shown in Table 1.

Table 1 in text

where is the first bending natural frequency of the bridge, is the density, is the cross section area, is the length of the bridge. , and the oscillator is assumed to move at a constant speed *V*. It is noted that the only stiffness ratio is required in the elastic contact model.

Generally, the following specific parameters are considered: , , , , and . The bridge is modelled with 20 finite elements, and the modal-damping ratio is set to be 0.02. In the presentation of results in this section, the response magnification factor is used, which is defined as

where is the response, and is the corresponding static response, which is defined as follows: if is the displacement, if is the acceleration, if represents force.

### 5.1.1 Simply supported bridge without roughness under a moving oscillator

A perfectly smooth bridge is firstly analysed in this sub-section, and the full time histories of the response factors for the system with different contact models are shown in Fig. 4. The dimensionless time (position of vehicle ) is used as the horizontal axis. Three different models used here are: the LCP rigid contact model, the LCP elastic model and the traditional permanent contact model.

Figure 4 in text

As shown in Fig. 4, the responses of the system calculated respectively by rigid contact and elastic contact model of the LCP method, are almost of the same value, and some differences can be seen compared with those of the traditional permanent contact model. The imposition of the connection in the permanent contact model would reduce the responses of the system, which could be observed in these figures, especially in Fig. 4c and Fig. 4d. It can be also found that separation occurs when the oscillator is going to exit the bridge, and the contact force becomes zero in the LCP models (both rigid contact and elastic contact models), whereas it becomes negative in the permanent contact model because of the neglecting of separation.

### 5.1.2 Simply supported bridge with harmonic roughness under an oscillator

In this sub-section, the bridge is assumed to have a harmonically varying surface roughness represented by [25]

where and are the amplitude and wave length of the surface irregularity, respectively. They are assumed to be , . is the initial phase.

Figure 5 in text

As shown in Fig. 5, the harmonic surface roughness has significant influences on the responses of the system. It can be observed that the responses of elastic contact model with an initial phase are very close to those of the rigid contact model with the initial phase , and obviously different from those of the elastic contact model with an initial phase zero. The reason is that the elastic deformation of the contact spring delays the responses excited by the roughness of the contact surface, and the delayed phase is equal to . This phenomenon is similar to the half-wave loss in physics. It can be inferred that there would be no delay when the stiffness of the contact surface tends to infinity, which suggests that the rigid contact model is a special case of the elastic contact. In reality, the stiffness of the contact surface () can be reasonably determined by Hertzian theory of elastic deformation [32].

### 5.1.3 Separate region at various velocity ratios and vehicle to bridge mass ratios

The separation region with respect to velocity ratio and vehicle to bridge mass ratio considering a smooth bridge surface or a harmonic surface roughness is shown in Fig. 6a and b respectively. The white region indicates that no separation occurs in the whole time duration, while the grey region indicates that there is at least one event of wheel separation from the bridge. As shown in these figures, the road roughness has crucial influences on the separation region, and no separation exists at certain values, if the vehicle to bridge mass ratio or the velocity of the oscillator is small.

Figure 6 in text

## 5.2 Train-bridge system subject to random track irregularity

As the interaction between the train and the bridge due to random tack irregularity in a railway bridge is an issue of great concern in many countries, a multi-wheel vehicle-bridge model subject to random track irregularity as shown in Fig. 3 is considered in this section. The relationship between the wheels and the bridge are assumed to be in elastic contact, and the LCP method proposed in Section 4 is implemented. The vehicle parameters are listed in Table 2, and a simply supported beam is considered with the parameters: N/m2, kg/m, m and the modal-damping ratio is set to be 0.02, and the critical speed is 103.93 m/s. The rail roughness is generated by trigonometric function method based on the American rail irregularity power spectrum densities [33], and in the Monte Carlo simulation, 1000 samples of rail roughness are used.

Table 2 in text

### 5.2.1 Responses due to random roughness

The responses of the train-bridge system considering both the random track irregularity and separation are shown in Fig. 7. In the figure, the grey curves are time history responses considering a single irregularity sample based on American class 5 rail irregularity power spectrum density, the solid lines are the expectation of the responses, and the 3 rule is used to estimate the maximum responses, shown as dashed dot lines.

Figure 7 in text

The 3-rule is a generic term of some statistical hypothesis tests whose statistics are known as normal. The upper and lower limits are considered to be , where represents the standard deviation (SD), and is the expectation operator. An observation is considered an outlier if its least-squares residual exceeds three times of its SD. As shown in Fig. 7, most of the responses are in the envelope lines based on the rule. It can be considered appropriate to use the rule to estimate the maximum responses of the vehicle bridge-system.

### 5.2.2 Possibility of separation due to random roughness

The possibility of separation due to random tack roughness for all wheels is shown in Fig. 9. 1000 samples of track roughness based on the American Class 5 track irregularity PSD are taken into account.

Figure 8 in text

As shown in Fig. 8, the possibility of separation of different wheels is not the same, and generally the possibility of separation of the front wheels at each bogie is larger than that of the rear wheels. On the other hand, the velocity is also very important to the possibility of separation, and in the discussed cases, the faster the vehicle travels, the bigger possibility of separation occurs. The possibility of separation can also be considered an indicator of the safety of the vehicle-bridge system.

### 5.2.3 Expectation of maximum distance between wheels and bridge due to separation

Fig. 9 shows the expectations of the maximum separation distance between the wheels and bridge at different track irregularities. As shown in these figures, the better surface condition of the bridge is, the smaller separation distance will result, and in some circumstances no separation could ever occur if the roughness condition is good enough (as Class 6 roughness). On the other hand, velocity is another important factor to the maximum separation distance. Generally, the expectation of maximum distance mostly increases with the velocity, but not for all the wheels (e.g. Fig. 9d). It can be found that the moment acting in between the vehicle body and the bogies is also important to the separation distance, since the rotation of the vehicle body and the bogies has a different influence on different wheels. In the present investigation, it is assumed that there is no discontinuity in the velocity or the deflection of the beam when the separation happens, and the responses do not change much compared with the permanent contact model. However, the separation between the wheels and the bridge could be detrimental to the safety and serviceability of the vehicle and/or the bridge, especially for high-speed trains. As a result, it would be necessary to consider the possibility of the separation in vehicle and bridge design, and the expectation of the maximum separation distance could be used to estimate the safety and serviceability of the vehicle during travels. The knowledge of the train speed that leads to separation should be very useful in train design and operations.

Figure 9 in text

# 6. Conclusions

By introducing the linear complementarity relations linking the relative displacement and the interaction force between the wheel and the bridge at the contact points, a new model considering separation for a vehicle-bridge system is established. It transforms the dynamic interaction problem of the vehicle-bridge system into a standard linear complementarity problem. Therefore the complex trial-and-error iterative process in conventional numerical methods is avoided, and the iterative process for the determination of contact states is not required. The presented methodology is verified in the numerical simulations and it is very effective in dealing with multiple contact points. It is found that the vehicle to bridge mass ratio, the velocity of the vehicle and the road roughness are the main factors influencing the separation of the system. A Monte Carlo simulation is implemented by considering random rail roughness and separation of a train-bridge system. Numerical results show that it is necessary to carry out a stochastic analysis of the train-bridge system since the track irregularity is a random process, and separation should be considered in bridge and vehicle design since it can be detrimental to the safety and serviceability of the vehicle on the bridge and the bridge itself.

It should be noted that, the separation described in the present paper is not likely to happen in current train operations. However, this is not impossible for high-speed trains. As found from numerical simulations, as the train speed increases, the tendency of separation increases. So this research looks at a potential scientific problem that could be a serious one in near future. Additionally, the method developed can be used in other moving-load problems than train-track interaction.

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# Table captions

**Table 1** Dimensionless parameters of the system

**Table 2** Parameters of the vehicle

# Tables

**Table 1** Dimensionless parameters of the system

|  |  |
| --- | --- |
| Parameters | Definition |
| velocity ratio |  |
| unsprung to sprung mass ratio |  |
| vehicle to bridge mass ratio |  |
| bridge to vehicle frequency ratio |  |
| vehicle damping ratio |  |
| stiffness ratio |  |

**Table 2** Parameters of the vehicle

|  |  |  |
| --- | --- | --- |
| Item | Unit | Data |
| Mass of vehicle body () | kg | 31994 |
| Roll mass moment of body () | kg m2 | 2100000 |
| Mass of bogie () | kg | 3333 |
| Roll mass moment of bogie () | kg m2 | 3200 |
| Mass of wheel-pair () | kg | 1650 |
| Vertical stiffness of first suspension system () | kN/m | 4720 |
| Vertical stiffness of 2nd suspension system () | kN/m | 160 |
| Vertical damping of first suspension system () | kN s/m | 160 |
| Vertical damping of 2nd suspension system () | kN s/m | 200 |
| Half-distance of two bogies () | m | 8.75 |
| Half-distance of two wheel-pairs () | m | 1.25 |

# Figure captions

**Fig. 1.** A simple vehicle-bridge coupled system model

**Fig. 2.** Time history for a succession of contact-separation-reattachment

**Fig. 3.** Multi-wheel vehicle and bridge system

**Fig. 4.** Response magnification factors of system without roughness: (a) displacement of oscillator, (b) displacement of wheel, (c) displacement of bridge middle point and (d) interaction force.

**Fig. 5.** Response magnification factors of system considering harmonic irregularity: (a) displacement of oscillator, (b) displacement of wheel, (c) displacement of bridge middle point and (d) interaction force.

**Fig. 6.** Separate region due to velocity ratio and vehicle to bridge mass ratio: (a) with perfect smooth surface and (b) with harmonic roughness.

**Fig. 7.** Time history responses of system: (a) displacement of vehicle, (b) acceleration of vehicle, (c) displacement of bridge middle point and (d) acceleration of bridge middle point.

**Fig. 8.** The possibility of separation due to velocity.

**Fig. 9.** Expectation of Maximum distance between wheel and bridge due to random roughness: (a) 1st wheel, (b) 2nd wheel, (c) 3rd wheel and (d) 4th wheel.