**Statistics of complex eigenvalues in friction-induced vibration**

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**Abstract**

Self-excited vibrations appear in many mechanical systems with sliding contacts. There are several mechanisms whereby friction can cause the self-excited vibration to become unstable. Of these mechanisms, mode coupling is thought to be responsible for generating irritating high-frequency noise and vibration in brakes. Conventionally, in order to identify whether a system is stable or not, the complex eigenvalue analysis is performed. However, what has recently received much attention of researchers is the variability and uncertainty of input variables in the stability analysis of self-excited vibrations. For this purpose, a second-order perturbation method is extended and employed in the current study. The moments of the output distribution along with its joint moment generating function are used for quantifying the statistics of the complex eigenvalues. Moreover, the eigen-derivatives required for the perturbation method are presented in a way that they can deal with the asymmetry of the stiffness matrix and non-proportional damping. Since the eigen-derivatives of such systems are complex-valued numbers, it is mathematically more informative and convenient to derive the statistics of the eigenvalues in a complex form, without decomposing them into two real-valued real and imaginary parts. Then, the covariance and pseudo-covariance of the complex eigenvalues are used for determining the statistics of the real and imaginary parts. The reliability and robustness of the system in terms of stability can also be quantified by the approximated output distribution.

**Keywords:** friction-induced vibration, mode-coupling instability, complex eigenvalue analysis, variability and uncertainty, 2nd order perturbation method.

1. **Introduction**

Friction in mechanical systems can cause irritating noise and vibration. The sliding velocity, normal load, friction coefficient and stiffness of sliding components are some key factors making contributions to the excitation mechanisms. For example, in a simple “mass-on-belt” slider model, when the stiffness and relative velocity of the slider are fairly low, the difference between the static and kinetic friction can result in a repetitive motion with sticking and slipping phases. This motion is known as stick-slip [1]. However, instead of this discrete function, a number of smooth and non-smooth functions may be assigned to the relation between the friction coefficient and sliding velocity [2]. On this premise, Mills [3] developed a model for brake squeak instability. In this model, a decreasing function of the sliding velocity is considered for the transition from static to kinetic friction.

As brake systems experience several types of noises and vibrations, a number of frictional mechanisms have been proposed so far for the reasons behind these instabilities. By means of a lumped-mass brake model, North [4] showed even a constant friction coefficient can change stability of the system as a result of taking friction as a follower force. The idea of follower forces was followed by Ouyang et al. in [5]. Spurr [6] developed a theory in which the normal load was a function of the angle of inclination of a frictional slider in contact with a surface. It was illustrated that a reciprocating motion could be formed due to the locking (spragging) of the slider to the surface even if the friction coefficient remained constant. This mechanism is cited as sprag-slip in the literature. Kinkaid et al. [7] comprehensively reviewed several frictional mechanisms causing the instabilities of brake systems.

More recently, it is believed that mode-coupling is the main root cause of unstable oscillations in many mechanical systems with friction. Akay [8] cited a number of examples in which mode lock-in could generate self-exited vibrations. Perhaps the most challenging example is brake squeal due to its significant warranty cost to car manufacturers. Although there is no general consensus on its root cause, most researchers believe that the reason behind brake squeal is mode-coupling. Hoffmann et al. [9] demonstrated this mechanism in an undamped system with two degrees-of-freedom. It is shown that friction breaks the symmetry of the stiffness matrix since it relates the normal force to the tangential direction. Through increasing the friction coefficient, the imaginary parts of a pair of system eigenvalues get closer and at a unique point they merge. Simultaneously, the real parts of the system eigenvalues bifurcate. This unique point is called a bifurcation point. After the bifurcation point, one of the real parts of the eigenvalues becomes positive and causes the amplitude of vibration to grow exponentially.

The behaviour of friction-induced vibration about bifurcation points has studied by some researchers. Oestreich et al. [2] did a bifurcation analysis for stick-slip instability. Hetzler et al. [10] studied the bifurcation behaviour of a simple slider whose friction coefficient was a function of the sliding velocity. Sinou et al. [11] investigated the nonlinear behaviour of a system with sprag-slip instability using the centre manifold approach. The bifurcation behaviour of mode-coupling instability under the influence of damping was discussed in [12, 13]. It is worth mentioning that damping prevents the imaginary parts of eigenvalues from coalescing. Specifically, there is a region within which the imaginary parts of eigenvalues get close markedly and the real parts of eigenvalues start diverging. Thus, the same manner as the bifurcation is also observed when damping exists in the system.

Based on the mode-coupling mechanism, the amplitude of vibration grows without bound in a linear system. However, what is evident in practice is that the existence of nonlinearities in a structure may impede this growth after a number of cycles. Subsequently, the reciprocating motion of the system becomes periodic and its trajectories tend to a closed curve known as Limit Cycle. Depending on the central aim of a study, size of the problem and accessible computational facilities, a linear or nonlinear approach may be taken toward friction-induced vibration. In [14], Ouyang et al. discussed the pros and cons of the linear and nonlinear approaches. Although the computational efficiency of the linear stability analysis, i.e. the complex eigenvalue analysis, is appreciated, it is emphasized that the nonlinear approaches can be more informative. Unfortunately, the nonlinear approaches are mostly performed by numerical integrations which are computationally very expensive. Therefore, some researchers used different simplification and linearisation techniques in order to include the contributions of nonlinearities in the linearised system [11, 15, 16 and 17].

In addition to the effects of nonlinearities, two inseparable parts of any friction-induced problem are variability and uncertainty [18]. In the literature, structural uncertainties are divided into two categories: reducible and irreducible uncertainties [19]. Reducible uncertainties, also simply known as uncertainty, are mostly due to the lack of analysts’ knowledge. Friction, contact, thermal effects, humidity, wear and loading distributions are typical examples, and each one introduces a degree of uncertainty to the problem. If more realistic models of these factors are incorporated in the problem, more reliable results can subsequently be obtained. On the other hand, irreducible uncertainties, also called variability, originate in the deviation of material properties from the nominal values, imperfection of component geometries and dissimilarity of assemblies. As this group of uncertainties are inherent in the manufacturing processes, it is referred to as irreducible uncertainty.

For the brake squeal problem, Tison et al. [20] demonstrated that the prediction of unstable modes would be improved if the variability of the pad-to-disc contact interface was modelled by a random field. Sarrouy et al. [21] used polynomial chaos expansions for performing the complex eigenvalue analysis in a stochastic way. Oberst and Lai [22] took an experimental approach for the statistical analysis of squeal instability. Culla and Massi [23] employed Monte Carlo simulation for quantifying the probability of squeal occurrence in a simplified brake model.

In order to incorporate the influence of variability and uncertainty in friction-induced instabilities, probabilistic, non-probabilistic, or mixed uncertainty techniques may be employed. The advantage of probabilistic approaches is that they are usually more informative in terms of the statistics of outputs. However, their major drawback is that they can be computationally expensive. For example, Monte Carlo simulation is known as the most practical, yet simplest, technique of uncertainty analysis [24]. In practice, nevertheless, it is not always feasible to collect a large number of deterministic results which are required by this approach. Therefore, researchers have been encouraged to use more efficient uncertainty techniques such as the perturbation methods and polynomial chaos expansions.

The technique to be employed in this paper is the 2nd order perturbation method. Adhikari and Friswell [25] applied this method for random matrix eigenvalue problems. As there was no sliding contact with friction in their examples, the stiffness matrix remained symmetric. Moreover, the effect of non-proportional damping was not discussed in [25]. Consequently, the distributions of the imaginary parts of eigenvalues, i.e. frequencies, were the point of interest in their work. However, this paper focuses on friction-induced vibration with non-proportional damping. Friction makes the stiffness matrix asymmetric and introducing non-proportional damping means that the problem should be solved in the state-space. What is important here is to obtain the probabilistic distributions of the real parts of eigenvalues for evaluating the system stability. For this purpose, a few expressions are derived for the statistics of the complex eigenvalues. The first moment of the output distribution is used for the mean value and the joint moment generating function is employed for the calculation of the variance/covariance of the scatter of the complex eigenvalues. The statistics of the real and imaginary parts is then calculated by the use of the derived expressions.

1. **Stability analysis of friction-induced vibration with non-proportional damping**

In general, there are two numerical approaches toward friction-induced problems: the complex eigenvalue analysis and the transient analysis. As the complex eigenvalue analysis deals with linear systems, it can predict the onset of instability. In fact, this approach finds eigenvalues of a system in the complex form, i.e. where and are the real and the imaginary parts of the th eigenvalue, and then assess the system stability by the sign of the real part. Based on the mode-coupling mechanism, one or more real parts of the eigenvalues being positive indicate instability. In contrast with viscous damping causing the amplitude of oscillations to decay exponentially (negative real parts), a positive real part enlarges the amplitude of oscillations without bound.

However, in reality, the amplitude of vibration often grows to some level and then tends to a steady state motion which is known as Limit Cycle Oscillation (LCO). In general, the only way can safely deal with any form of nonlinearity is numerical integrations. Unfortunately, numerical integrations can become computationally so expensive that it is impractical to apply to large-scale finite element models. As a result, industry usually relies on the results of the complex eigenvalue analysis and attempts to design a system in the way that it remains stable under various circumstances.

The equations of motion for a system with a sliding contact and non-proportional damping can be written as

|  |  |
| --- | --- |
|  | (1) |

where ,and are the mass, damping and asymmetric stiffness matrices, respectively. In order to perform the complex eigenvalue analysis, two aspects of the above equation should be considered. First, the stiffness matrix is not symmetric; the left and right eigenvectors are not the same. Secondly, damping of the system can be either proportional or non-proportional. As a result, the state-space equations of motion should be used in place of the second-order equations. It is worth mentioning that one may employ the second-order equations of motion directly [26], but it is mathematically more convenient to use the state-space form:

|  |  |
| --- | --- |
|  | (2) |

where and.

To derive the eigenvalues and eigenvectors of equation (2), the complex eigenvalue analysis is carried out. In other words, where represents the eigenvalues of the system. The solution consists of pairs of eigenvalues which are complex conjugates. Since the stiffness matrix is not symmetric, the associated eigenvectors are derived in the following way:

|  |  |
| --- | --- |
|  | (3) |

where and are the right and left eigenvectors, respectively. Later on, the normalized eigenvectors will be used in the equations. The normalization means [27]:

|  |  |
| --- | --- |
| . | (4) |

Up to this point, no uncertainties are considered in input variables. In fact, the above procedure is the general deterministic approach toward friction-induced vibration. Now what is important to know is how variability and uncertainty of the input variables can be propagated to the output space by means of an efficient statistical approach. For this purpose, the second-order perturbation method is extended in this paper.

1. **Uncertainty analysis via perturbation method**

In the literature, structural uncertainties are classified into two groups: aleatory and epistemic [19]. Yet it is not always possible to make a distinction between them. Aleatory uncertainty, also called irreducible uncertainty or variability, refers to the random nature in the system properties, which originates from manufacturing processes. Variations of material properties, component geometries and assemblies are the most significant examples of this type of uncertainty. Epistemic uncertainty, known as reducible uncertainty, is mainly due to the lack of information. This type of uncertainty is called reducible since future advances and/or investigations can provide new insights into the problems which are not fully understood yet.

In spite of the extensive investigations that have been conducted so far, the predictions of the behaviour of friction, contact and wear have remained very complicated. In addition to the tribological interactions, the degree of uncertainty in a friction-induced vibration problem is increased by the thermal effects, humidity and diverse loading cases. Variability also imposes a considerable level of uncertainty to the problem. For example, in brake systems, the properties of friction materials may vary from one brake to another significantly. Ignoring the effect of variability in the complex eigenvalue analysis then causes underestimation or overestimation of the number of unstable modes. Therefore, in order to make more reliable predictions of unstable modes, it is necessary to perform uncertainty analysis.

Uncertainty propagation may be carried out via probabilistic, non-probabilistic or mixed uncertainty techniques. Briefly speaking, non-probabilistic approaches are not as informative as probabilistic ones. If one is interested in finding the ranges within which the output variables fluctuate, non-probabilistic approaches will be the right choice. Of course, these predictions can also be made by a fuzzy method. This method evaluates fuzzy membership functions of the outputs, which are more informative than crisp intervals [28]. However, in most applications, it is well worth quantifying the probability distributions of the results in order to evaluate the reliability and robustness of a system.

The most fundamental, yet practical probabilistic method of uncertainty analysis is Monte Carlo simulation. The conventional Monte Carlo simulation spreads millions of samples over the design space and collects the results of the sampled points by means of a mapping function. In fact, the mapping function is the relation between the output and input variables. Then, the statistics of the outputs are quantified by the use of the output scatter. In the case of mode-coupling instability, the mapping function is the complex eigenvalue analysis. Note there is no explicit expression for the relation between the real or imaginary parts of eigenvalues and the input variables unless a system consists of a very limited number of degrees-of-freedom. The complex eigenvalues and eigenvectors of every sampled point should be calculated in order to obtain the scatter of output space. In this study, the conventional Monte Carlo simulation is used for the validation of the proposed method. What is obvious though is that the computational workload of the Monte Carlo simulation is massive. Specifically, in large-scale finite element models, even running a single complex eigenvalue analysis of a detailed brake model may take one or two days. It is thus impractical to run millions of such simulations.

Fortunately, the perturbation method can overcome the issue of computational costs. In contrast with the Monte Carlo simulation, this method approximates the statistics of the outputs in one run. In this method, the outputs, i.e. the complex eigenvalues, are assumed to be a function of the random input variables. Then, this function is expanded about a point of the input distributions. This point must have the most contribution to the probability distribution. Depending on the orders retained in the perturbation series, the method is called the first- or second-order perturbation. Due to the smallness of higher order terms, they are usually neglected in practice.

If the joint probability distribution of the input variables is Gaussian, the expansion of the complex eigenvalues about the mean value can be written in a quadratic form as [25]:

|  |  |
| --- | --- |
|  | (5) |

where represents th random input variable, is the gradient vector and is the Hessian matrix.

The eigen-derivatives which are reported in [25] cannot cope with the problem under the current study due to the asymmetry of the stiffness matrix and the existence of non-proportional damping. Plaut and Huseyin [29] derived the expressions of eigenvalue and eigenvector derivatives in non-self-adjoint systems. These derivatives can be re-written for the eigenvalues and eigenvectors of equation (2) as:

|  |  |
| --- | --- |
|  | (6) |

where .

Moreover,

|  |  |
| --- | --- |
|  | (7) |

The quadratic form of the complex eigenvalues which are derived in equation (5) cannot directly be used since is a random variable. In other words, this equation is not deterministic. However, the statistics of the complex eigenvalues can be obtained by the use of this quadratic form. For the convenience of readers, the theoretical background of this approach is briefly discussed in the following section.

1. **Statistics of complex eigenvalues**

The main objective of this section is to predict the statistics of the complex eigenvalues without decomposing them into the real and imaginary parts. As the eigen-derivatives expressed in (6) and (7) produce complex-valued numbers, the moments and cumulants of the output distributions are also derived in the complex form.

* 1. *Derivation of the mean value*

In the theory of statistics, the mean value of a random variable is its first moment about the origin. Considering the fact that the input variables are real-valued while is a complex function of the inputs, the mean value or the first moment of the th eigenvalue can be calculated by:

|  |  |
| --- | --- |
|  | (8) |

where and represent the expected value of and the joint probability density function of,respectively. Vector consists of *m* real-valued random input variables. Any joint probability density function can be substituted in place of . However, in many cases but the multivariate Gaussian distribution, it is not feasible to derive a closed-form expression for . In [30], the joint probability density function of an *m*-dimensional Gaussian distribution is given in a quadratic form:

|  |  |
| --- | --- |
|  | (9) |

where is the covariance matrix of the input variables .

It is more convenient to transform the variable of integration (8) to :

|  |  |
| --- | --- |
|  | (10) |

Due to the fact that the covariance matrix is positive definite, it can be decomposed as , where is a non-singular lower triangle matrix [31]. By the use of the Jacobian matrix, the variable of integration in (10) is transformed to :

|  |  |
| --- | --- |
|  | (11) |

Since , the multiple integral given in (11) turns into the product of *m* single variable integrals. Indeed, that is one of the advantages of using the theory of quadratic forms. Considering and , equation (11) becomes:

|  |  |
| --- | --- |
|  | (12) |

The symmetry of the matrix brings about the same left and right eigenvectors. If and respectively represent the diagonal matrix of the eigenvalues and the matrix of the normalized eigenvectors of , pre- and post-multiplying this matrix by and results in:

|  |  |
| --- | --- |
|  | (13) |

where . As that matrix is diagonal, . Therefore, the mean value of the th complex eigenvalues can explicitly be determined as:

|  |  |
| --- | --- |
|  | (14) |

The eigenvalues of are equivalent to those of [31]. Furthermore, the summation of eigenvalues of any matrix is evidently equal to the trace of that matrix. Therefore, equation (14) can be re-written as

|  |  |
| --- | --- |
|  | (15) |

It is worth mentioning that the expression derived for the mean value of the complex eigenvalues has the same form as the one reported in [25] for a symmetric system without damping. However, equation (15) produces a complex number whose real part is the mean value of the real parts of the th complex eigenvalues and whose imaginary part is the mean value of the imaginary parts.

* 1. *Derivation of the covariance*

What is known as the variance or covariance of a complex random variable is a real-valued number which is calculated by

|  |  |
| --- | --- |
|  | (16) |

where “\*” denotes conjugate of a complex number and “H” indicates Hermitian, i.e. conjugate transpose, of a complex matrix. Before starting to derive the expression of the covariance, it is important to re-consider the central aim of this study, which is the evaluation of the system stability under the influence of structural uncertainties. In other words, the distribution of the real parts of the complex eigenvalues is the main point of interest in this study. If and represent the variance of the real and imaginary parts, and also and stand for the covariance between the real and imaginary parts, equation (16) can be expressed with these terms as:

|  |  |
| --- | --- |
|  | (17) |

due to the fact that .

Obviously, equation (17) is not sufficient to calculate the variance of the real and imaginary parts. However, what can resolve this issue is the use of pseudo-covariance:

|  |  |
| --- | --- |
|  | (18) |

Similar to the covariance, the pseudo-covariance can also be related to the variances of the real and imaginary parts:

|  |  |
| --- | --- |
|  | (19) |

As seen, in contrast with the covariance, is a complex number and it provides a way to determine the variance of the real and imaginary parts along with equation (17).

Now, in order to derive an expression for the covariance of the complex eigenvalues, the joint moment generating function is used. In fact, the moment generating function is an alternative way of calculating the moments of a random variable [32]. The joint moment generating function of and can be written as

|  |  |
| --- | --- |
|  | (20) |

Using the definition of the expected value results in:

|  |  |
| --- | --- |
|  | (21) |

This integral is, in fact, the generalized version of Gaussian integral to an *m*-dimensional domain. Like the procedure for the mean value, the solution of this integral can also be found:

|  |  |
| --- | --- |
|  | (22) |

where is a symmetric matrix.

 According to (22), the joint moment generating function can explicitly be determined:

|  |  |
| --- | --- |
|  | (23) |

In the theory of statistics, it is quite common to use the cumulant generating function rather than the moment generating function. The cumulant generating function is the natural logarithm of the moment generating function and the second cumulant of a distribution is the variance. Hence, the covariance of the complex eigenvalues is obtained via

After differentiating and equating and to zero, the covariance of the complex eigenvalues can be expressed as

|  |  |
| --- | --- |
|  | (24) |

Similarly, the pseudo-covariance is obtained by

|  |  |
| --- | --- |
|  | (25) |

The pseudo-covariance is the same with the variance of eigenvalues given in [25], yet the difference here is that it produces complex-valued numbers. As discussed earlier, expressions (24) and (25) will be employed together for the calculation of and .

1. **A lumped mass model**

In the literature, friction-induced vibration is often investigated in systems with limited degrees-of-freedom. However, Butlin and Woodhouse [33] discovered that very low-order models may not represent the problem well. It was shown that more reliable results could be achieved when enough degrees-of-freedom were included in a reduced model and the minimum number of degrees-of-freedom of a reduced brake model was four. Therefore, the model under this study consists of four degrees-of-freedom. Ouyang [34] employed this model in order to demonstrate the technique of pole-assignment for stabilizing friction-induced vibrations. This model is shown in figure 1. Here, the only difference is that there is a contact damping at the slider-belt interface in addition to the grounded damping in [34]. This lumped mass model may be imagined as a simplified brake system in which the slider serves as the disc and acts as the brake pad with the stiffness of , and in different directions. The angle of inclination for is . The contact stiffness and damping at the slider-belt interface are and , respectively. Masses and may be considered as representing the abutment and the pad backplate. The stiffness of the abutment is and the backplate is . Mass has one degree-of-freedom along direction, is free to move in both and directions and can only oscillate in y-direction. The normal force is and the friction force is . It is assumed that the belt velocity is constant and no stick-slip motion is experienced. In other words, the static and kinetic friction coefficients are the same.



Figure 1. A lumped mass model

The matrices of mass, damping and stiffness corresponding to the vector are:

|  |  |
| --- | --- |
| ; ; |  (26) |

where (), (), , and . The matrices given in (26) reveal how the friction force breaks the symmetry of stiffness and damping matrices by relating the normal force to the tangential one.

Before talking about the uncertainty analysis of the system shown in figure 1, it is well worth looking at the results of the deterministic approaches. As discussed earlier, the complex eigenvalue analysis is the typical deterministic approach for determining stability of a system. For the mentioned set of input parameters, the complex eigenvalue analysis is performed with four different friction coefficients. The results are shown in table 1. Although each input parameter makes a contribution to the system stability, the friction coefficient usually plays the most significant role in low order models. As seen in table 1, the real part of the first eigenvalue becomes positive when is larger than . The same behaviour is observed for the forth eigenvalue if .

Table 1. The complex eigenvalues of the system with three different friction coefficients ()

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Eigenvalue | First pair | Second pair | Third pair | Forth pair |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |

In order to understand the bifurcation behaviour of the system versus the friction coefficient, the real and imaginary parts of all eigenvalues are shown in figure 2. As seen, the existence of damping prevents the imaginary parts of eigenvalues from coalescing, yet they get close markedly.



Figure 2. The bifurcation behaviour of the system under study

If the variations of other input parameters are also considered, it will be observed that the real parts of the eigenvalues are nonlinear functions of the other input parameters as well as the friction coefficient. As a case in point, Sinou and Jézéquel [13] emphasized that damping makes the bifurcation behaviour of friction-induced problems very complicated. Then, it is important to figure out how the system instability is influenced under the variations of input variables.

1. **The effect of variability and uncertainty**

Imagine the system shown in figure 1 represents a very simplified brake model. Many experimental studies have shown that the level of variability and uncertainty of friction material is very high. Not only the variability of material properties ( and), but also the friction coefficient (), contact stiffness () and contact damping () are major sources of uncertainties in this problem. When the friction coefficient is 0.4, according to table 1, all eigenvalues of the system are stable. It would be very useful to find out whether the system would remain stable subjected to uncertainty. For this purpose, it is assumed that the deviation of and from their nominal values is 5 percent and the deviation of and is 10 percent. As the friction coefficient can change significantly in a brake application, 15 percent variation is allocated to . Now it is aimed to evaluate the statistics of the real and imaginary parts of the eigenvalues via the second-order perturbation method.

Moreover, the conventional Monte Carlo simulation is run for the verification of the results produced by the perturbation method. One-million samples are made via a multivariate Gaussian random generator. The mean value and variance of the real and imaginary parts are calculated separately and compared with the outcomes of the expressions given in section 4. Table 2 to table 5 list the results of the perturbation method and Monte Carlo simulation.

Table 2. The mean of the real parts of the complex eigenvalues

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | Real part of the 1st pair | Real part of the 2nd pair | Real part of the 3rd pair | Real part of the 4th pair |
| Perturbation |  |  |  |  |
| Monte Carlo  |  |  |  |  |

Table 3. The mean of the imaginary parts of the complex eigenvalues

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | Imaginary part of the 1st pair | Imaginary part of the 2nd pair | Imaginary part of the 3rd pair | Imaginary part of the 4th pair |
| Perturbation | 8.7516 | 12.1715 | 16.7552 | 19.8448 |
| Monte Carlo | 8.7515 | 12.1716 | 16.7553 | 19.8450 |

Table 4. The variance of the real parts of the complex eigenvalues

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | Real part of the 1st pair | Real part of the 2nd pair | Real part of the 3rd pair | Real part of the 4th pair |
| Perturbation |  |  |  |  |
| Monte Carlo  |  |  |  |  |

Table 5. The variance of the imaginary parts of the complex eigenvalues

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | Imaginary part of the 1st pair | Imaginary part of the 2nd pair | Imaginary part of the 3rd pair | Imaginary part of the 4th pair |
| Perturbation |  |  |  |  |
| Monte Carlo  |  |  |  |  |

According to the results, the error in the prediction of the mean values is practically zero. The maximum error in the approximations of variances is 15 percent. However, due to the smallness of the variances, the error seems fairly large.

The distinct advantage of the 2nd-order perturbation method is that it produces the results in just one run while one-million simulations have been used for determining the same results via Monte Carlo simulation. In actual applications, running Monte Carlo simulation for large models is almost impossible due to the required time for the computation of the results. Instead, the second-order perturbation method suggests an efficient way of uncertainty analysis with remarkable accuracy in the prediction of results.

* 1. *Profile of the output distribution*

Generally speaking, when the mapping function is nonlinear, the projection of a multivariate Gaussian variable to the output space is no longer Gaussian. Depending on how nonlinear a function is, the output distribution can be skewed or its kurtosis may be different from the normal distribution. In the theory of statistics, the skewness and kurtosis are the shape descriptors for real-valued random variables. Although it is doable to quantify the higher moments of a complex random variable, the interpretation and relation of these moments to its real and imaginary parts are not very straightforward. Therefore, the approximation of output distributions with the Gaussian probability density function should not perfectly represent the results. However, it is useful to portray the output profile via Gaussian probability density function in order to have a better understanding of the distributions of the results. In the literature, it is also common to use the normal/Gaussian probability density function for this purpose [25].

A bivariate Gaussian distribution is considered for the real and imaginary parts of each eigenvalue. This assumption helps to plot the elliptical contours of the scatter of the results. The ellipse equation is [30]:

|  |  |
| --- | --- |
|  | (27) |

where and are the mean value of the real and imaginary parts of the th eigenvalue. and are the associated standard deviations. Parameter is the correlation between the real and imaginary parts, i.e. , which is calculated by (17) and (19). Moreover, the least-squares fit of the real and imaginary parts of the complex eigenvalues which are produced by Monte Carlo simulation is determined. The results are shown in figure 3.



Figure 3. Contours of the real and imaginary of the eigenvalues

In figure 3, the predicted elliptical contours of the first and second eigenvalues are well-matched with the least-squares fits. However, a noticeable discrepancy is found for the third and forth eigenvalues. The reason behind is, in fact, the skewness and kurtosis of the outputs. To illustrate this point, the distributions of the real parts of the eigenvalues are plotted in figure 4. The histograms show the results of the Monte Carlo simulation, while the red circle-dash lines indicate the approximated distributions via Gaussian probability density function. Likewise, the distributions of the imaginary parts are shown in figure 5.



Figure 4. The distribution of real parts of the eigenvalues

As seen, the distributions of both real and imaginary parts of the first and second eigenvalue are nearly Gaussian. Consequently, the least-squares fits and the predicted elliptical contours are well-matched in figure 3. To support this statement, the skewness and kurtosis of the real and imaginary parts are listed in tables 6 and 7. It is worth mentioning that the kurtosis of the normal distribution is equal to three, but in the theory of statistics it is also common to use excess kurtosis, which is defined as kurtosis minus three. However, in this study, the excess kurtosis is not used.

In contrast with the first and second modes, the real parts of the third and forth eigenvalues are negatively and positively skewed, respectively (table 6). Looking at the skewness and kurtosis of the imaginary parts, it can be seen that all of them are almost Gaussian. The reason is that the imaginary parts are not much influenced by the friction coefficient.



Figure 5. The distributions of the imaginary parts of the eigenvalues

Table 6. The skewness of the real and imaginary parts

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Eigenvalue | The 1st pair | The 2nd pair | The 3rd pair | The 4th pair |
| Real parts | -0.030 | -0.278 | -0.965 | 0.706 |
| Imaginary parts | 0.049 | -0.290 | -0.014 | -0.017 |

Table 7. The kurtosis of the real and imaginary parts

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Eigenvalue | The 1st pair | The 2nd pair | The 3rd pair | The 4th pair |
| Real parts | 3.208 | 3.055 | 8.391 | 6.119 |
| Imaginary parts | 3.005 | 3.200 | 3.024 | 2.989 |

The second important conclusion is that although all of the eigenvalues of the system were stable before the uncertainty analysis, the variation of the input parameters causes the first eigenvalue of the system to exceed the stable region and become positive for some samples.

Once again, it is imagined that the system represents a simplified brake model. There is no doubt that at the end of the production, well designed and manufactured brakes are still likely to experience friction-induced instability during services because of inherent uncertainty and variability in them. The probability distributions of outputs then help to evaluate how robust and reliable a design is in terms of noise and vibration. Due to this fact, uncertainty analysis of brake systems has received intensive attention of car manufacturers recently.

* 1. *Reliability and robustness*

In general, a reliability analysis is performed for evaluating the robustness of a design. In fact, the reliability analysis quantifies the failure probability of the design. The term “failure” is widely used in engineering for various undesired states of a system. In this study, failure means that one of the system eigenvalues becomes unstable, i.e. the sign of the associated real part turns positive. For brake analysts, it is very important to know what percentage of a brake design will ‘fail’ either at the end of production or due to the aging effects. Then, the structural modifications which are usually done to reduce brake noises can be carried out for decreasing the likelihood of unstable vibration. The failure probability is obtained as:

|  |  |
| --- | --- |
|  | (28) |

Among the four modes of the system shown in figure 1, only the first one exceeds the stable region under the influence of the uncertainties. In figure 4, the green triangles highlight the border of the unstable region. By the use of the Monte Carlo simulation results, the percentage of failure for the first mode is 35.7. If the output distribution is approximated by the Gaussian density function, the predicted probability of failure is calculated by

|  |  |
| --- | --- |
|  | (29) |

where and . Accordingly, the approximated failure probability of the first mode is 35.2 percent. This percentage is calculated by the estimated mean value and standard deviation via the perturbation method.

Moreover, one may say that the failure probability is quite large for the system under this study. In fact, the design point is deliberately set slightly below the critical friction coefficient in order to demonstrate how bad a design can be if the variability and uncertainty of the inputs are not considered carefully.

1. **Discussion about the application of an uncertainty analysis**

Discussions with brake engineers reveal that sometimes there is confusion about the difference between the sensitivity analysis used for structural modifications of a brake and the uncertainty analysis of a brake design. It is worthwhile to discuss this issue here.

What is common at the design stage of brake systems is to modify the structures so that no unstable mode is observed in the results of the complex eigenvalue analysis. In order to come up with an idea of how these modifications should be made, a sensitivity analysis is usually carried out for finding the input variables having more influences on the unstable modes. For example, a brake analyst may realize that a 40 percent increase in the disc’s elastic modulus results in a quieter design. However, this figure should not be confused with the uncertainty of the system. As mentioned in the introduction section, uncertainties can be due to the lack of knowledge. If it is already known that the mean value of a design variable should be increased by 40 percent, it does not mean that the uncertainty of this variable is 40 percent. Even if a modification leads to a quieter design, a quieter brake is not guaranteed because of uncertainty and therefore uncertainty analysis is still needed for the new design point. Conventionally, when the eigenvalues of a system like the one under this study are stable at the design point by the deterministic approach, it is assumed as an acceptable design. However, the variations of the input variables can result in a noticeable probability of failure. Therefore, the mean values of the input variables must be modified in the way that the failure probability decreases to an acceptable level.

As a consequence of this confusion, sometimes brake engineers believe that uncertainty propagation techniques lose their accuracy for large variations of input parameters. The fact is that these methods are developed for a reasonable range of variations. Since large variations of input variables affect the system properties significantly, they should not be taken as uncertainties.

1. **Conclusions**

Mode-coupling instability may occur in many mechanical systems with sliding contacts. According to this mechanism, the real parts of the complex eigenvalues of a system become positive when certain system parameters vary and this indicates unstable oscillations. The common deterministic approach is to perform the complex eigenvalue analysis in order to evaluate stability of the system. However, the complex eigenvalue analysis cannot deal with the variability and uncertainty of input variables, which are inherent in any systems with frictional contacts. Uncertainty quantification of the outputs is then vital for the reliable stability analysis of such problems.

The statistics of the complex eigenvalues are studied in this paper with the application to friction-induced vibration problems. In order to deal with non-proportional damping and the asymmetry of the stiffness matrix, the state-space equations are used. The second-order perturbation method is then extended for incorporating the variability and uncertainty of input variables. A few expressions for calculating the mean value, covariance and pseudo-covariance of the complex eigenvalues are derived. The mean value of the eigenvalues is, in fact, a complex number which provides the mean values of the real and the mean value of the imaginary parts. The covariance and pseudo-covariance of the complex eigenvalues include information of the variance of the real parts, the variance of imaginary parts and the covariance of the real and imaginary parts. The correlation between the real and imaginary parts of the eigenvalues can also be obtained by these statistical measures.

The most important outcome of this study is that the distributions of the real and imaginary parts of the complex eigenvalues can efficiently be approximated without decomposing them into two real-valued numbers (real and imaginary parts). In this way, the results are produced in just one-run, while Monte Carlo simulation requires a large number of samples to find the statistical measures of the outputs. Monte Carlo simulation is, nevertheless, more informative since the skewness and kurtosis of the real and imaginary parts can also be evaluated.

This study also allows the likelihood of unstable vibration to be predicted and then provides a very useful tool for design.

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