# A New Method of Updating Mass and Stiffness Matrices Simultaneously with No Spillover

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*Abstract:* A method for updating mass and stiffness matrices without spillover is presented, which requires the knowledge of only the few eigenpairs to be updated of the original undamped model. The finite element model updating problem with symmetric preserving and no spillover is formulated as a semi-definite programming problem, which can be efficiently solved by existing semi-definite programming algorithms. Numerical examples show that, using the presented updating method, the updated model accurately reproduces the “measured” modal data, while keeping the symmetry of mass and stiffness matrices and avoiding spillover.

*Keywords:* Finite element model updating, direct matrix updating, partially prescribed spectral information, without spurious modes

1. **INTRODUCTION**

Updating a finite element model to match measured modal data is important in design, construction and maintenance of engineering structures and mechanical systems. Due to the limitations of available computational methods to handle distributed parameter systems, the finite element method is generally used to discretize such systems to form finite element models. However, accurate geometric and material properties must be used in order for the finite element models to be predictable. Basically, finite element model updating (FEMU) is to incorporate the measured modal data into the finite element model to produce an improved finite element model with modal properties that closely match the experimental modal data. Then the updated model may be considered a better dynamic representation of the structure. Because of the importance of this problem, there now exists a relatively large quantity of works published in this field. Most of the works prior to 1995 are contained in the book by Friswell and Mottershead (1995), which conducted a comprehensive review on papers published before 1995. Some of the more recent results can be found in review articles (Datta, 2002; Dascotte, 2007; Datta, 2009; Mottershead et al., 2011) and references therein. The existing methods of FEMU can be classified as working in the frequency domain or the time domain. They have also been classified by other researchers into the following three classes: (i) Direct matrix updating methods, (ii) Iterative methods using modal data, and (iii) Methods using frequency response data, and some of them have been widely used and successfully applied to FEMU for a variety of structures.

Baruch, Berman and Nagy are the first advocates of the direct matrix updating methods. Assuming that the mass matrix was correct in his proposed method, Baruch (1978) used Lagrange multipliers to update the stiffness matrix by minimizing the discrepancy between the updated and analytical stiffness matrices. Berman(1979) introduced a formulation that modified the mass matrix and assumed that the measured modes were exact. Subsequently, Berman and Nagy(1983) combined the mass matrix adjustment procedure with the stiffness matrix adjustment procedure of Baruch (1978) to establish the so-called analytical model improvement (AMI) procedure. However, for the updated stiffness matrix, two additional constraint equations were included. Wei (1990)introduced an approach that could update the mass and stiffness matrices simultaneously using the measured eigenvector matrix as the reference. The effects due to mass and stiffness interaction were clearly determined from the corrected mass and stiffness matrices. Additionally, Kuo et al. (2006) recently proposed a direct method which seemed more efficient and reliable.

The iterative updating methods use the sensitivity of the parameters to update the model. They are usually posed as an optimization problem, and allow a number of model parameters to be systematically adjusted with respect to the measured modal data (natural frequencies and mode shapes) in order to minimize the objective function defined. The optimal solution is obtained using sensitivity-based optimization methods. Because of the nonlinear relationship between the vibration data and the physical parameters, an iterative optimization process is performed. There are a number of works that extensively discussed and demonstrated the results obtained from model updating using the optimization methods (Mottershead et al., 2011; Bakir et al., 2007; Charbel et al., 1993; Shahverdi et al., 2009). In a class of model updating methods using measured frequency response functions (FRF), the FE models are updated based on the fully damped response along a frequency axis and not on an estimated set of modal data. Most of the FRF-based model updating techniques relied on expression the system matrices as a polynomial of system parameters (Friswell et al., 1990; Imregun et al., 1995; Esfandiari et al., 2010). Arora (2011) conducted a detailed comparison of two approaches, i.e. a direct method, which used modal data, and an iterative method, which uses FRF data and was also a parameter-based method. They were evaluated with the objective that the frequency response functions (FRFs) obtained from updated FE models were able to predict the measured FRFs accurately. The updated results had shown that the iterative method gave 20% better matching of FRFs with the experimental data and also the predictions of the iterative method was better than the direct method beyond the considered frequency range. The FE model updating can also be performed with computational intelligence algorithms, such as neural networks, genetic algorithms, particle-swarm optimization, simulated annealing, and so on (Atalla et al., 1998; Marwala, 2010). It is worthwhile to note that Papadimitriou et al. (2000) presented a statistical methodology for optimally locating the sensors in structural model updating. The method could extract from the measured data the most information about the parameters of the model used to represent structural behavior.

In spite of these developments mentioned above, there are still some unsolved issues with the FEMU problem. For example, these existing methods may reproduce the given set of measured data while keeping symmetry of updated matrices; however, they cannot guarantee that extra, spurious modes are not introduced into the frequency range of interest after updating (Friswell and Mottershead, 1995). When updating a model, it is desirable to match only the measured modal data without tampering with the other unmeasured modal data in the original model. Such an updating method is known to have no spillover. In the last few years, Carvalho and Datta, and their collaborators (2007), Mao and Dai (2012) respectively developed two novel methods to solve the problem with only the stiffness matrix updated, and with both the stiffness and mass matrices updated, for the undamped model. However, the updating method proposed in Mao and Dai (2012) cannot reproduce the measured data accurately for a numerical example in their paper. Additionally, Chu et al. (2007, 2008) presented a complete theory on when model updating of damped systems with no spillover is possible.

The purpose of this paper is to propose a new method of updating mass and stiffness matrices simultaneously avoiding spillover and preserving symmetry. A matrix equation is formulated for updating mass and stiffness matrices, which requires the knowledge of only the few eigenpairs to be updated of the original undamped model. Setting the residual norm of this matrix equation as the objective function, the FEMU problem is posed as an optimization problem, which is solved by semi-definite programming (SDP) techniques. The presented updating method is shown to reproduce the measured data accurately for those numerical examples in Mao and Dai (2012), while avoiding spillover. The remainder of this paper is organized as follows. A necessary and sufficient condition for no spillover updating is established in Section 2. The outline of SDP techniques and some relevant software packages are discussed, and then the FEMU problem is formulated as SDP problems in Section 3. Three numerical examples are presented to illustrate the efficiency of the proposed method in Section 4. Finally, some conclusions are drawn in Section 5.

**2. A NECESSARY AND SUFFICIENT CONDITION FOR NO SPILLOVER UPDATING**

Consider an *n*-degree-of-freedom undamped vibration system that is modelled by the following set of second-order ordinary differential equations:

 (1)

where is displacement vector, are mass and stiffness matrices, respectively. In general,  is symmetric and positive definite, and is symmetric and positive semi-definite, denoted by, , where is the transpose operation. For the sake of convenience, the model is simply denoted by.

It is well known that if is a fundamental solution of Eq. (1), then the natural frequency and the mode shape vectormust satisfy the following generalized eigenvalue equation:

  (2)

where is the square of the *i*th natural frequency, called the *i*th eigenvalue, andis the corresponding *i*th mode shape, called the *i*th eigenvector. Eq. (2) can be written in a compact representation as follows:

 (3)

where and be eigenvalue and eigenvector matrices of the analytical model. Letandbe a set of *p* eigenvalues and eigenvectors measured from an experimental structure. Mathematically, the model updating problem may be formulated as follows (Mao and Dai, 2012):

*Problem FEMU:* Given an analytical modeland a set of its associated eigenpairs with, and another set of measured eigenpairs  from an experimental or a real-life structure have been obtained, update the analytical model to of the same structure such that:

1) ≥ 0.

2) The subset is replaced by as *p* eigenpairs of the updated model.

3) The remaining (unknown) *n*-*p* eigenpairs of the updated model stay the same as those of the original model.

For convenience, the following partitions and notation are used:

, , , ,

,, and , ,

and assume that the eigenvector matrix *X* of the analytical modelsatisfies the normalization condition. Moreover, the following assumptions are made in this paper:

(A1)  (an empty set);

(A2) ;

(A3) All the above-mentioned eigenvalues are distinct.

The FEMU problem, as stated above, concerns finding symmetric corrective matricesand  such that the following eigen-matrix equations of the updated model hold simultaneously:

  (4)

 (5)

A necessary and sufficient condition will be firstly established forandto satisfy Eq. (4). Before that, two lemmas have to be introduced below.

*Lemma 2.1* (Sylvester’s law of nullity) (Abadir et al., 2005). For any two matrices, and, if, then, $r(∙)$ denotes the rank of a matrix.

*Lemma 2.2*. If the rank of a matrix A is zero, i.e., *r* (*A*) = 0, then matrix A is a null matrix, i.e., *A*=0.

With assumptions (A1) to (A3) and the above two lemmas, Theorem 2.1 can be stated below.

*Theorem 2.1* (No spillover updating). The real symmetric corrective matricesandsatisfy Eq. (4), *if and only if* andsatisfy the following matrix Eq. (6a) or (6b).

  (6a)

  (6b)

*Proof:* Necessity:

 Using the normalization of the eigenvector matrix *X* of the analytical model, it is easy to show thatand. Partition of the left-hand side of the previous matrix equation, that is,, leads to

  (7)

As, one can get,and  . So it follows that

  (8)

From Eqs. (7) and (8), one can derive

  (9)

  (10)

It follows from Eqs. (9) - (10) that

 (11)

Moreover, subtracting the eigen-matrix equationof from Eq. (4) yields

 (12)

When Eq. (12) is used in Eq. (11), Eq. (6a) results (end of proof of the necessity).

In addition, multiplying Eq. (6a) on the right by, the matrix equation (6b) is obviously obtained.

Sufficiency: Suppose matrix equation (6a) holds. It follows from Eq. (11) that. Letand. Because all modes ofare orthogonal to one another, that is, *X*2 contains *n*-*p* orthogonal modes, obviously. Lemma 2.1 dictates that. Therefore one can conclude that. It can immediately be inferred from lemma 2.2 that the matrix formulation (12) is valid. Then it is easy to verify that

 

This completes the proof. □

Note that andsatisfying Eq. (6a) or Eq. (6b) ensure that the eigenvalues  and corresponding eigenvectors of the original analytical model  are not affected by updating. Now, letandbe in a specific form as follows:

, (13)

where, are real symmetric matrices to be determined. Substituting (13) into Eq. (6b) and Eq. (5), it is easy to show that the following two matrix equations can be obtained.

 (14)

 (15)

Here it is worthwhile to note that (i) determination of andthat satisfy Eqs. (4) and (5) so as to solve the FEMU problem is now equivalently transformed to solve Eqs. (14) and (15) to determine and; (ii) Eq. (14) works exclusively with only a small number of finite element frequencies and mode shapes that need to be updated in the analytical model, in contrast to Eq. (4) that involves a large number of the unmeasured frequencies and mode shapes of .

In what follows the procedure to determineand, the FEMU problem is posed as an SDP problem based on Eqs. (14) and (15).

**3. FORMULATIN THE FEMU PROBLEM AS AN SDP PROBLEM**

The SDP is an extension of linear programming with a linear objective function in a sense that in addition to linear constraints, the additional constraint that the matrix be positive semi-definite (note that this additional constraint cannot be expressed as a linear one) must be imposed, that is (Helmberg, 2002),



  (16)

where *X* is a variable matrix, *C*, *A*1, …, *Am* are given symmetric matrices, and is the trace operator. The SDP unifies several convex optimization problems (e.g., linear and quadratic programming), and many primal-dual interior-point methods for linear and quadratic programming have been naturally extended to solve SDPs (Nesterov et al., 1994), where the number of arithmetic operations required by the algorithms is bounded by a polynomial of the problem size. This extension thus admits theoretically efficient solution procedures based on iterating interior points that either follow the central path or decrease a potential function. As a result, SDPs are not much harder to solve than ordinary linear and quadratic programming problems. In recent years, SDP has received increasing attention for its various fields of applications, such as nonlinear and time-varying system analysis, controller synthesis, optimal statistical model designs, and structural optimization (Todd, 2001; Boyd et al., 2004; Ohsaki et al., 1999), due to its versatility to model and solve problems arising in many areas. Examples of converting these problems into the standard primal problem (16) or its dual can be found in Todd (2001) and Boyd et al. (2004). More recently, Lin et al. (2010) described some applications of SDP techniques to quadratic inverse eigenvalue problems (QIEPs). They claimed that unlike other numerical methods, the SDP approach presented a unified, efficient, and tractable scheme for solving QIEPs.

 There are now many software packages for solving SDP problems of a certain size, and three commonly used ones among them are SDPA (Yamashita et al.,2010), SDPT3 (Tütüncü et al. 2003), and SeDuMi (Sturm, 1999). There are also a number of interfaces that facilitate the use of SDP software. One of them, called YALMIP (Lofberg, 2004), is used to implement numerical examples in this paper. YALMIP is a free MATLAB-based toolbox that serves as a convenient interface for multiple external optimization solvers, and supports a large number of optimization classes, such as linear, quadratic, second order cone, semi-definite, mixed integer conic, geometric, local and global polynomial, multiparametric, bilevel and robust programming. The YALMIP commands unify and facilitate the different formats in SDP software. When applied to QIEPs, it makes the description of various structural constraints, such as positive definiteness, nonnegativity, sparsity patterns, and prescribed entries of the coefficient matrices, extremely simple and offers via the well established SDP theory and algorithms a reliable and conclusive answer within the specified numerical tolerance (Lin et al., 2010).

 The FEMU problem in this paper may be formulated as an SDP problem in two forms.

*Formulation 1:* Min



where and  are some preselected weight factors, andis Frobenius norm. Note that *J* is a convex but nonlinear function inand. This objective function can be easily rewritten as a second-order Lorentz cone programming problem via the YALMIP commands.

  *Formulation 2:* Min

Subject to 

 Substituting and, obtained by the SDP optimization computation above, into Eq. (13) for symmetric corrective matrices and, then one could optimally accomplish model updating with no spillover and symmetry preserved.

**4. NUMERICAL EXAMPLES**

Three numerical examples are given to show the application of the above-established method for solving the FEMU problem. All codes are run in MATLAB 7.11 on a personal computer with 2.4 GHz CPU and 2.0 GB physical memory.

*Example* 4.1 (*Example* 4.1 in Mao and Dai (2012)). Consider the original analytical model as follows:



 

Taking *p*=3, the computed modal data are, and

 

The ‘measured’ modal data are taken as, and

 

 From *Formulation* 1 of the SDP optimization with weight factors, the updated mass and stiffness matrices are given as follows:

 

 

It is easy to calculate, . Another indicator showing the accuracy of no spillover is.

 From *Formulation* 2 of the SDP optimization, the updated mass and stiffness matrices are given as follows:

 

 

It is easy to calculate,,.

*Remark:* The results show that the measured modal data are accurately matched, and the updated model has no spillover using two SDP formulations, which have similar accuracy to that in Mao and Dai (2012).

*Example* 4.2 (*Example* 4.2 in Mao and Dai (2012)). The matrices *M*a and *K*a of the original model are

 

 

Taking *p*=2, the computed modal data are, and

 

The ‘measured’ modal data are taken as, and

 

 From *Formulation* 1 of the SDP optimization with weight factors, the updated mass and stiffness matrices are given as follows:

 

 

It is easy to calculate, .

From *Formulation* 1 with weight factors, the updated mass and stiffness matrices are not listed for the sake of saving space. It is found, in this situation, that,, and the match with the measured modal data is improved further.

From *Formulation* 2, the updated mass and stiffness matrices are given as follows:

 

 

It is easy to calculate, .

*Remark*: The results show that the updated model matches the measured modal data accurately, and has no spillover using two SDP formulations. However, for the same example in Mao and Dai (2012), their method can update the model to avoid spillover, but cannot reproduce the measured data accurately.

*Example* 4.3. Here matrices *M*a and *K*a are

 

 Taking *p*=2, the first two eigenpairs andof are taken as the computed modal data. The measured modal data are taken as, and, where matrix *T* is taken as

 

 From *Formulation* 1 with weight factors, it is easy to calculate, ; for weight factors, , ; for weight factors, , . The computational times using the SDP algorithm are respectively 4.56, 6.79, and 7.92 seconds.

From *Formulation* 2, it is easy to calculate,. The computational time using the SDP algorithm in this case is 8.55 seconds.

It should be pointed out that most existing SDP algorithms can only efficiently solve problems with up to a thousand linear constraints and matrices of a thousand or so in dimension, because they fail to exploit the sparsity of the underlying problem. Additionally, the algorithms fail to maintain physical connectivity between the elements of the mass and stiffness matrices. How to maintain the physical feasibility of the updated matrices is a subject for further investigation.

**5. CONCLUSIONS**

A direct mass and stiffness matrices updating method is presented in this paper. Based on a matrix equation that the corrective mass and stiffness matrices should satisfy for no spillover updating, the finite element model updating problem is formulated as a semi-definite programming (SDP) optimization problem, which can be solved using the well established SDP algorithms and an efficient and tractable scheme. The updated mass and stiffness matrices are naturally symmetric, and are also positive definite or positive semi-definite, due to the characteristics of the SDP algorithm itself. Numerical experiments indicate that the presented method produces accurate results.

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