Ruin Probabilities in Classical Risk Models with Gamma Claims

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Abstract

In this paper we provide three equivalent expressions for ruin probabilities in a Cramér-Lundberg model with gamma distributed claims. The results are solutions of integro-differential equations, derived by means of (inverse) Laplace transforms. All the three formulas have infinite series forms, two involving Mittag-Leffler functions and the third one involving moments of the claims distribution. This last result applies to any other claim size distributions that exhibits finite moments.

1 Introduction

Deriving the ruin probability is a central topic in risk theory literature. Starting from the basic collective insurance risk model introduced by Cramér and Lundberg at the beginning of last century (Lundberg, 1903, 1926; Cramér, 1930), researchers are still analyzing concrete instances of it or amending some of its features to make it more practical. The classical Cramér-Lundberg is a compound Poisson model, accounting for claims (losses) arriving independently at exponential times, random in size, but independent and identical distributed.

One direction of research considers altering the assumptions of independence or memory loss of claim arrivals, thus analyzing ruin probabilities in renewal models (Andersen, 1957) or models with various dependence structures (Albrecher and Boxma, 2004, 2005; Constantinescu et al., 2013). Considering a gamma aggregate claims process, Dufresne et al. (1991) derived bounds for the ruin probabilities. Adding financial considerations to the model, such as returns in investments, see e.g. Paulsen (1998); Frolova et al. (2002); Kalashnikov and Norberg (2002); Paulsen (2008); Albrecher et al. (2012); Ramsden and Papaioannou (2017), interest rate models, see e.g. Cai and Dickson (2004) or perturbations in premium cash-flow, see e.g. Temnov (2014), asymptotics of ruin probabilities have been derived. Lévy risk models were considered and first passege and exit times were derived via fluctuation theory and scale functions, see e.g. Furrer et al. (1997); Furrer (1998); Yang and Zhang (2001); Avram et al. (2002); Kyprianou (2006); Palmowski and Pistorius (2009); Hubalek and Kyprianou (2011).

However, the direction that captured the most attention over the last hundred years involves ruin results for particular claims' distributions. Numerous approximations (Beekman, 1969; De Vylder, 1978; Kingman, 1962; Bloomfield and Cox, 1972) and asymptotic results have been derived (Klüppelberg et al., 2004; Palmowski and Pistorius, 2009), especially for heavy-tailed claims (Ramsay, 2003). However, ever since the explicit form of ruin probability in the case of exponential claims sizes was established (Cramér, 1930), searching for explicit formulas for other (light-tailed) distributions becomes a frequent direction of research.

This paper falls into this latter category, exploring the classical ruin model with gamma distributed claims, extending and generalizing earlier results of Thorin (1973). Among the first distributions considered in risk theory literature are the integer shaped gammas, or the so-called Erlang distributions. These have rational Laplace transforms and at the same time are phase-type distributed, a class dense in the class of continuous distributions. The ruin probability has closed form expressions for classical risk models with phase-type claims, see e.g. Asmussen and Albrecher (2010) or rational Laplace transform distributions, see e.g. Albrecher et al. (2010). In this paper we go beyond Erlang distributions and derive results for gamma distributions that allow real shape parameters, using Laplace transform properties and deriving Pollaczeck-Khinchine type formulas. In our discussion section, we comment on the merit of the series expressions obtained. More precisely, for gamma claims, we first introduce two different methods, leading to two different series expressions in terms of Mittag-Leffler functions. Moreover, we present a general Pollaczeck-Khinchine type form for the ruin probability in the classical Cramér-Lundberg model with light-tail claims, in terms of moments, which in case of gamma claims reduces to a third, tractable expression.

Since some of our results are expressed in terms of Mittag-Leffler functions, we remind the reader that a Mittag-Leffler function is an extension of an exponential function e^z and plays a very important role in the theory of fractional differential equations. As a one-parameter generalization of an exponential, the function introduced by Mittag-Leffler (1903) is an infinite series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C},\, \Re(\alpha) > 0,\, z \in \mathbb{C},$$

where $\Gamma(z)$ denotes the gamma function $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$. The two-parameters generalization of an exponential, introduced by Agarwal (1953),

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C},\, \Re(\alpha) > 0, \Re(\beta) > 0,\, z \in \mathbb{C} \tag{1}$$

is also referred to as Mittag-Leffler function, see e.g., (Erdélyi et al., 1955). Also, recall that (Podlubny (1998))

$$\int_0^\infty e^{-sz} z^{\alpha k + \beta - 1} E^{(k)}_{\alpha,\beta}(\pm a z^\alpha)\, dz = \frac{k! s^{\alpha - \beta}}{(s^\alpha \mp a)^{k+1}}, \quad \Re(s) > |a|^{1/\alpha}, \tag{2}$$

which for any $k \geqslant 0$, gives us the Laplace transform of Mittag-Leffler type functions and their derivatives. Here $E_{\alpha,\beta}^{(n)}$ is the nth derivative of the Mittag-Leffler function, which can be computed as

$$E_{\alpha,\beta}^{(n)}(z) = \sum_{j=0}^{\infty} \frac{(j+n)!z^j}{j!\Gamma(\alpha j + \alpha n + \beta)}.$$

The classical collective risk Cramér-Lundberg model, describes the reserve process $U(t)$ of an insurance company as

$$U(t) = u + ct - \sum_{k=1}^{N(t)} X_k, \quad t > 0, \tag{3}$$

where $u > 0$ is the initial capital of the company and $c > 0$ represents the constant rate at which the premiums are accumulated. The aggregated paid claims by time t are modelled by a compound Poisson process, with $N(t)$ a Poisson process with intensity λt and X_k independent, identically distributed random variables with finite mean, representing the amount of individual claims paid. One assumes the *positive loading assumption* $c > \lambda E X_1$, and define the ruin probability as

$$\psi(u) = \mathbb{P}\left(\inf_{t>0} U(t) < 0\right) = \mathbb{P}\left(\tau_u < \infty\right), \quad u > 0, \tag{4}$$

where τ_u is the first hitting time

$$\tau_u = \inf\left\{t \geq 0 : \sum_{k=1}^{N(t)} X_k - ct > u\right\}.$$

The non-ruin, or survival probability, is denoted

$$\phi(u) = 1 - \psi(u), \quad u > 0. \tag{5}$$

Lundberg (1926) derived a bound and the asymptotic behavior for the ruin probability in the classical model, making use of an equation that in risk theory literature is commonly referred to as the Lundberg's equation

$$M_X(s) M_T(-cs) = 1, \tag{6}$$

where $M_X(s)$ and $M_T(s)$ are the moment generating functions of the claim size distribution and the waiting time distribution, respectively. Note that, in classical compound Poisson case, Lundberg's equation (6) reads

$$cs - \lambda + \lambda M_X(-s) = 0. \tag{7}$$

The literature of deriving explicit expressions for the ruin probability of the classical compound Poisson risk model for various claims distributions is abundant in methods and results. Cramér (1955) derives the non-ruin probability $\phi(u)$ as a solution of an integro-differential equation, which, under some conditions, can be solved analytically by either differentiating both sides or taking the Laplace transform, when the claims are exponentially distributed. Gerber (1973) uses martingales to analyze the risk process with independent and stationary increments. Pakes (1975) derives the relationship between ruin probability and claims' tail distribution. Thorin and Wikstad (1977) analyzes the ruin problem when claims are log-normal distributed. Gerber et al. (1987) obtains the ruin probability for mixture Erlang claims by studying the severity of ruin, as well as its probability. Ramsay (2003) inverts the Laplace transforms over the complex domain to derive a closed-form solution of the ruin probability when the claim sizes follow a special Pareto distribution. Hubalek and Kyprianou (2011) have considered a class of spectrally negative Lévy processes, called Gaussian temepered stable convolution, whose Lévy measure has a Gamma component with shape parameter ≤ 1. They showed that their scale functions, which are essentially proportional to the survival probability, admit expressions in terms of Mittag-Leffler functions (Theorem 2 and 3), which have similarities to our results (12) and (14).

The simplest case of classical risk model is when claims are exponential distributed with parameter α. Under the assumption of positive loading, the ruin probability is given by (Cramér, 1930)

$$\psi(u) = \frac{\lambda}{\alpha c} e^{-\left(\alpha - \frac{\lambda}{c}\right)u}, \quad u > 0. \tag{8}$$

The focus of this paper is on gamma distributed claim sizes, i.e., with the density

$$f_X(x) = \frac{\alpha^r}{\Gamma(r)} x^{r-1} e^{-\alpha x}, \quad x > 0, \tag{9}$$

where $r > 0$ is the shape paramater, and $\alpha > 0$ is the scale parameter. The starting point is the classical integro-differential equation for the survival probability, valid whenever the claim size distribution has a density that we denote by f:

$$\frac{d}{du}\phi(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u \phi(u-z) f(z)\, dz, \quad u > 0\,. \tag{10}$$

An immediate conclusion is that the Laplace transform of non-ruin probability,

$$\hat{\phi}(s) = \int_0^\infty e^{-su}\phi(u)\, du, \quad \Re(s) > 0\,,$$

is given by

$$\hat{\phi}(s) = \frac{c\phi(0)}{cs - \lambda + \lambda M_X(-s)} = \frac{c\phi(0)}{cs - \lambda + \lambda(\frac{\alpha}{s+\alpha})^r}, \quad \Re(s) > 0\,, \tag{11}$$

when the claim sizes follow the gamma distribution with scale α and shape r. We would like to mention that the non-ruin probability of the classical risk model (3) with zero initial capital equals to

$$\phi(0) = 1 - \frac{\lambda\mu}{c},$$

where μ denotes the expected claim size, , see e.g. Rolski et al. (2009). Throughout the paper, by $\phi(0)$ we refer to this expression. We introduce three different methods to invert back the Laplace transform of the survival probability.

Note that the denominator in (11) is the right hand-side of Lundberg equation (7). When the shape parameter r is integer, namely when the claims are Erlang distributed, the expression in the right hand side of (11) can be written as the ratio of two polynomial functions. One can then use the partial fraction decomposition and invert $\hat{\phi}$ to obtain a linear combination of exponential functions (Grandell, 1991).

Notice that for a rational shape parameter $r = m/n$, with $\Re(s) > \alpha$, one could shift the argument s to obtain

$$\hat{\phi}(s-\alpha) = \frac{c\phi(0)}{c(s-\alpha) - \lambda + \lambda(\frac{\alpha}{s})^{m/n}} = \frac{c\phi(0)s^{m/n}}{c(s-\alpha)s^{m/n} - \lambda + \lambda\alpha^{m/n}},$$

which is a ratio of polynomials of orders m and $(m+1)$ in $t = s^{1/n}$. This again permits a partial fraction decomposition. In this case, an explicit expression can be obtained as in Wei Zhu's MSc project at University of Liverpool, using the two parameter Mittag-Leffler function in (1):

$$\phi(u) = e^{-\alpha u} u^{\frac{1}{n}-1} \sum_{k=0}^{m+n-1} m_k E_{\frac{1}{n},\frac{1}{n}}\left(s_k u^{\frac{1}{n}}\right), \tag{12}$$

with s_k and m_k real constants, determined on a case-by-case basis.

Extending these results to real shape parameter r proves to be non-trivial and different approaches are presented here. Prior to this work, the only known (to us) result for non-integer shape gamma distributed claims is that of Thorin (1973) and it deals with a special case of the $\Gamma(1/b, 1/b)$, $b > 1$, distribution. Namely, for the classical collective risk model with Poisson arrival intensity $\lambda = 1$, $\Gamma(1/b, 1/b)$, $b > 1$, distributed claims and positive loading $c > 1$, the ruin probability for $u \geqslant 0$ is

$$\begin{aligned}\psi(u) =& \frac{(c-1)(1-bR)e^{-Ru}}{1-cR-c(1-bR)} \\ &+ \frac{c-1}{b\pi}\sin\frac{\pi}{b}\int_0^\infty \frac{x^{1/b}e^{-(x+1)u/b}}{\left[x^{1/b}\left(1+c\frac{x+1}{b}\right)-\cos\frac{\pi}{b}\right]^2+\sin^2\frac{\pi}{b}}\,dx,\end{aligned}$$

where R is the positive solution of Lundberg equation (6). This approach explores the properties of completely monotone functions. When $b = 2$, the expression of ruin probability becomes a linear combination of exponentials and error functions, which expression (12) can recover when $r = 1/2$ (see Appendix A for details). However, note that the general form of the integral term appearing in the result can only be calculated numerically.

The paper is organised as follows. Section 2 extends the method of shifting Laplace transform to the real shape parameter case. Using geometric expansions, one can present an explicit form in terms of an infinite sum of convolutions of exponential and Mittag-Leffler functions. Section 3 derives an explicit form in terms of an infinite sum of derivatives of Mittag-Leffler functions, by carefully reconstructing geometric sum on the Laplace side. Section 4 uses induction and recursive formulas to derive the ruin probability in terms of integrals of sum of moments. This last result applies to any claim size distributions with finite moments, gamma distribution being a special case. All three results are shown to retrieve the classical exponential ruin probability result when reduced to exponential claims. Section 5 discusses the advantages or disadvantages of each one of the expression derived in the paper. For ease of reading, some of the calculations are deferred to the appendix.

2 Method One - Infinite Sum of Convolutions of Mittag-Leffler Functions

In this section we use our ability to recognise certain geometric expansions present in the Laplace transform of the survival probability when the claim sizes are gamma distributed. These expansions can be inverted to obtain an explicit form of the survival probability. The result is in terms of an infinite sum of convolutions. Recall that for two locally integrable functions, f, g on $(0, \infty)$ the convolution is defined by

$$f * g(x) = \int_0^x f(y)g(x-y)\,dy,\ x > 0\,, \tag{13}$$

and it is a locally integrable function. The convolution power of a locally integrable function f is defined recursively by $f^{*1} = f$, $f^{*n} = f^{*(n-1)} * f$, $n \geq 2$.

Theorem 2.1. For a classical compound Poisson risk model (3) with claim sizes X_k having the gamma distribution (9) with shape parameter $r > 0$ and scale parameter $\alpha > 0$, the non-ruin probability is

$$\phi(u) = \phi(0) + e^{-\alpha u}\phi(0)\left\{ e^{\alpha u} * \left(\sum_{n=1}^{\infty} \left(\frac{\lambda}{c}\right)^n \left[e^{\alpha u} - (\alpha u)^r E_{1,1+r}(\alpha u)\right]^{*n} \right) \right\}, \tag{14}$$

for any $u > 0$.

Remark 2.1. Note that the Mittag-Leffler functions in the expression (14) can be expressed in terms of incomplete gamma functions (Simon, 2015)

$$\begin{aligned} E_{1,\beta}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+\beta)} = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta-1)\Gamma(k+1)} B(\beta-1, k+1) \\ &= \frac{1}{\Gamma(\beta-1)} \int_0^1 (1-t)^{\beta-2} \sum_{k=0}^{\infty} \frac{(xt)^k}{\Gamma(k+1)}\,dt \\ &= \frac{1}{\Gamma(\beta-1)} \int_0^1 (1-t)^{\beta-2} e^{xt}\,dt = x^{1-\beta} e^x \frac{\gamma(\beta-1, x)}{\Gamma(\beta-1)} \end{aligned} \tag{15}$$

with the lower incomplete gamma function $\gamma(r, z) = \int_0^z t^{r-1}e^{-t}\,dt$.

Proof. Rearranging the expression (11), one can identify a geometric series with general term easily set to be between 0 and 1 for any $s > 0$,

$$\frac{\lambda}{c}\left(\frac{1}{s} - \frac{M_X(-s)}{s}\right) < 1,$$

so that we can write

$$\hat{\phi}(s) = \frac{\phi(0)}{s}\frac{1}{1 - \frac{\lambda}{c}\left(\frac{1}{s} - \frac{M_X(s)}{s}\right)} = \frac{\phi(0)}{s}\sum_{n=0}^{\infty}\left(\frac{\lambda}{c}\right)^n\left(\frac{1}{s} - \frac{(\frac{\alpha}{s+\alpha})^r}{s}\right)^n.$$

For $s > \alpha$ we can shift the argument as explained above, to obtain

$$\hat{\phi}(s-\alpha) = \frac{\phi(0)}{s-\alpha}\sum_{n=0}^{\infty}\left(\frac{\lambda}{c}\right)^n\left(\frac{1}{s-\alpha} - \frac{\alpha^r}{(s-\alpha)s^r}\right)^n.$$

Note that

$$\begin{aligned}\frac{1}{s-\alpha} - \frac{\alpha^r}{(s-\alpha)s^r} &= \frac{1}{s-\alpha} - \frac{\alpha^r}{s^{r+1}}\sum_{i=0}^{\infty}\left(\frac{\alpha}{s}\right)^i \\ &= \int_0^{\infty} e^{-su}\left(e^{\alpha u} - \sum_{i=0}^{\infty}\frac{\alpha^{r+i}}{\Gamma(r+i+1)}u^{r+i}\right) du\,,\end{aligned}$$

the Laplace transform of a positive function. Therefore,

$$\begin{aligned}e^{-\alpha u}\phi(u) &= \phi(0)\left\{e^{\alpha u} + e^{\alpha u} * \left(\sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n\left[e^{\alpha u} - \sum_{i=0}^{\infty}\frac{\alpha^{r+i}}{\Gamma(r+i+1)}u^{r+i}\right]^{*n}\right)\right\} \\ &= \phi(0)\left\{e^{\alpha u} + e^{\alpha u} * \left(\sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n\left[e^{\alpha u} - (\alpha u)^r E_{1,1+r}(\alpha u)\right]^{*n}\right)\right\},\end{aligned}$$

as required. $\square$

Remark 2.2. Note that Theorem 2.1 is an exponentially tilted variant of the Pollaczeck-Khinchine (Beekman) formula for gamma claims, see Rolski et al. (2009) and Asmussen and Albrecher (2010). To clarify this connection, consider the upper tail of claims $\bar{F}_X(u) = \mathbb{P}[X > u]$, which, as in Remark 2.1, identity (15), can be regarded as

$$e^{\alpha u}\bar{F}_X(u) = e^{\alpha u} - \alpha^r u^r E_{1,1+r}(\alpha u),$$

so that the equation (14) becomes

$$\begin{aligned}\phi(u) &= \phi(0) + e^{-\alpha u}\phi(0)\left(\sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n e^{\alpha u} * \left[e^{\alpha u}\bar{F}_X^{*n}(u)\right]\right) \\ &= \phi(0) + \phi(0)\sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n\int_0^u \bar{F}_X^{*n}(y)\,dy.\end{aligned}$$

This is equivalent to

$$\psi(u) = 1 - \left(1 + \phi(0)\sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n\int_0^u \bar{F}_X^{*n}(y)\,dy\right)$$

$$= \phi(0) \sum_{n=1}^{\infty} \left(\frac{\lambda}{c}\right)^n \int_u^{\infty} \bar{F}_X^{*n}(y)\, dy,$$

the Pollaczeck-Khinchine formula for the ruin probability, as in, e.g. Rolski et al. (2009).

Remark 2.3. For $r = 1$ the expression (14) reduces to the classical result (8) of Cramér (1930).

Proof. When $r = 1$ the expression in the square bracket in (14) equal to 1 for all $u > 0$, and its n-fold convolution power is the function $u^{n-1}/(n-1)!$, $u > 0$. Therefore, one has

$$\begin{aligned}
\phi(u) &= \phi(0) + e^{-\alpha u}\phi(0)\left\{ e^{\alpha u} * \left(\sum_{n=1}^{\infty} \left(\frac{\lambda}{c}\right)^n \frac{u^{n-1}}{(n-1)!} \right) \right\} \\
&= \phi(0) + e^{-\alpha u}\phi(0)\left\{ e^{\alpha u} * \left(\frac{\lambda}{c} e^{\frac{\lambda}{c}u} \right) \right\} \\
&= \phi(0) + e^{-\alpha u}\phi(0)\left\{ \frac{\lambda}{c}\left(\alpha - \frac{\lambda}{c}\right)^{-1} e^{\frac{\lambda}{c}u} \left[e^{\left(\alpha - \frac{\lambda}{c}\right)u} - 1 \right] \right\} \\
&= \phi(0) \frac{\alpha}{\alpha - \frac{\lambda}{c}} \left[1 - \frac{\lambda}{\alpha c} e^{-\left(\alpha - \frac{\lambda}{c}\right)u} \right].
\end{aligned}$$

Since $\phi(0) = 1 - \lambda/\alpha c$, one concludes that

$$\phi(u) = 1 - \frac{\lambda}{\alpha c} e^{-\left(\alpha - \frac{\lambda}{c}\right)u},$$

which coincides with equation (8). □

Remark 2.4. For an integer number r, recall from Podlubny (1998) that

$$E_{1,1+r}(\alpha u) = \frac{1}{(\alpha u)^r} \left(e^{\alpha u} - \sum_{k=0}^{r-1} \frac{(\alpha u)^k}{k!} \right), \tag{16}$$

and so by (14) the survival probability equals to

$$\phi(u) = \phi(0) + e^{-\alpha u}\phi(0)\left\{ e^{\alpha u} * \left(\sum_{n=1}^{\infty} \left(\frac{\lambda}{c}\right)^n \left[\sum_{k=0}^{r-1} \frac{(\alpha u)^k}{k!} \right]^{*n} \right) \right\}. \tag{17}$$

Consider the case $r = 2$. The n-fold convolution in expression (17) becomes

$$(1 + \alpha u)^{*n} = \sum_{i=0}^{n} \binom{n}{i} \frac{\alpha^i u^{n+i-1}}{(n+i-1)!}, \tag{18}$$

which needs to be further convolved with $e^{\alpha u}$. Recall that the convolution of an exponential function and a power function is given by

$$e^{\alpha u} * u^k = \int_0^u e^{\alpha(u-s)} s^k\, ds = \frac{k!}{\alpha^{k+1}} e^{\alpha u} - \sum_{j=0}^{k} \frac{k!\, u^j}{\alpha^{k+1-j}\, j!}. \tag{19}$$

Using the linearity of the convolution, one may conclude from identities (18) and (19) that

$$
\begin{aligned}
e^{\alpha u} * (1+\alpha u)^{*n} &= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{e^{\alpha u}}{\alpha^n} - \sum_{j=0}^{n+i-1} \frac{u^j}{\alpha^{n-j}\, j!} \right) \\
&= \frac{2^n e^{\alpha u}}{\alpha^n} - \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{j=0}^{n+i-1} \frac{u^j}{\alpha^{n-j}\, j!} \right),
\end{aligned}
$$

which leads to the survival probability

$$
\begin{aligned}
\phi(u) &= e^{-\alpha u}\phi(0) \sum_{n=0}^{\infty} \left(\frac{\lambda}{c}\right)^n \left[\frac{2^n e^{\alpha u}}{\alpha^n} - \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{j=0}^{n+i-1} \frac{u^j}{\alpha^{n-j}\, j!} \right) \right] \\
&= \phi(0) \frac{1}{1 - \frac{2\lambda}{c\alpha}} - e^{-\alpha u}\phi(0) \sum_{n=0}^{\infty} \left(\frac{\lambda}{c}\right)^n \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{j=0}^{n+i-1} \frac{u^j}{\alpha^{n-j}\, j!} \right) \\
&= 1 - \left(1 - \frac{2\lambda}{\alpha c}\right) e^{-\alpha u} \sum_{n=0}^{\infty} \left(\frac{\lambda}{c}\right)^n \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{j=0}^{n+i-1} \frac{u^j}{\alpha^{n-j}\, j!} \right).
\end{aligned}
$$

To deal with the infinite series term $\sum_{n=0}^{\infty} \left(\frac{\lambda}{c}\right)^n \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{j=0}^{n+i-1} \frac{u^j}{\alpha^{n-j}\, j!} \right)$ in the above expression, first take its Laplace transform to obtain the following expression for $s > \alpha$,

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \left(\frac{\lambda}{c}\right)^n \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{j=0}^{n+i-1} \frac{\alpha^j}{\alpha^n\, s^{j+1}} \right) \\
&= \frac{1}{s} \sum_{n=0}^{\infty} \left(\frac{\lambda}{\alpha c}\right)^n \sum_{i=0}^{n} \binom{n}{i} \frac{1 - (\alpha/s)^{n+i}}{1 - \alpha/s} \\
&= \frac{1}{s} \sum_{n=0}^{\infty} \left(\frac{\lambda}{\alpha c}\right)^n \left(\frac{2^n}{1 - \alpha/s} - \left(\frac{\alpha}{s}\right)^n \frac{(1+\alpha/s)^n}{1 - \alpha/s} \right),
\end{aligned}
$$

where one detects a sum of two geometric series with general terms $\frac{2\lambda}{\alpha c}$ and $\frac{\lambda}{cs}\left(1 + \frac{\alpha}{s}\right)$ respectively. Therefore, the term of infinite series can be further expressed as

$$
\begin{aligned}
&\frac{1}{(s-\alpha)\left(1 - \frac{2\lambda}{\alpha c}\right)} - \frac{1}{(s-\alpha)\left(1 - \frac{\lambda}{cs}\left(1 + \frac{\alpha}{s}\right)\right)} \\
=& \frac{\alpha c}{\alpha c - 2\lambda} \left(\frac{\left(1 - \frac{\lambda}{cs}\left(1 + \frac{\alpha}{s}\right)\right) - \left(1 - \frac{2\lambda}{\alpha c}\right)}{(s-\alpha)\left(1 - \frac{\lambda}{cs}\left(1 + \frac{\alpha}{s}\right)\right)} \right) \\
=& \frac{2\lambda}{\alpha c - 2\lambda} \frac{s + \frac{\alpha}{2}}{s^2 - \frac{\lambda}{c}(s+\alpha)} = \frac{2\lambda}{\alpha c - 2\lambda} \left(\frac{m_1}{s - s_1} + \frac{m_1}{s - s_2} \right),
\end{aligned}
$$

where the last step involves a partial fraction decomposition, with $s_{1,2} = \frac{\lambda \pm \sqrt{\lambda^2 + 4\lambda\alpha c}}{2c}$.

One can invert the Laplace transform back to obtain

$$\sum_{n=0}^{\infty}\left(\frac{\lambda}{c}\right)^n\sum_{i=0}^{n}\binom{n}{i}\left(\sum_{j=0}^{n+i-1}\frac{u^j}{\alpha^{n-j}\,j!}\right)=\frac{2\lambda}{\alpha c-2\lambda}\left(\delta(u)+m_1e^{s_1u}+m_2e^{s_2u}\right),$$

and so the non-ruin probability for $r=2$ is

$$\begin{aligned}\phi(u)&=1-\left(1-\frac{2\lambda}{\alpha c}\right)e^{-\alpha u}\frac{2\lambda}{\alpha c-2\lambda}\left(m_1e^{s_1u}+m_2e^{s_2u}\right)\\&=1-\frac{2\lambda}{\alpha c}\left(m_1e^{(s_1-\alpha)u}+m_2e^{(s_2-\alpha)u}\right),\end{aligned}$$

where $s_{1,2}$ are given above, and $m_{1,2}$ can be calculated from the fraction decomposition step. This result agrees with the elementary partial fraction inversion mentioned in Grandell (1991).

3 Method Two - Infinite Sum of Derivatives of Mittag-Leffler Functions

In this section, we present a different method to derive the survival probability which leads to an explicit form in terms of an infinite sum of derivatives of Mittag-Leffler functions.

Theorem 3.1. For a classical compound Poisson risk model (3) with claim sizes X_k following gamma distribution (9) with shape parameter $r>0$ and scale parameter $\alpha>0$, the non-ruin probability can be written as

$$\phi(u)=e^{-\alpha u}\phi(0)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\lambda\alpha^r}{c}\right)^k u^{(r+1)k}E_{1,rk+1}^{(k)}\left(\left(\alpha+\frac{\lambda}{c}\right)u\right),\qquad(20)$$

where $E_{\alpha,\beta}^{(n)}$ is the nth derivative of the Mittag-Leffler function.

Proof. Let $\beta>\alpha$. The first step is to find a function G whose Laplace transform is, for sufficiently large $s>0$,

$$g(s)=\frac{1}{as^\beta+bs^\alpha+c},$$

where a,b,c are non-zero constants. One can rewrite

$$g(s)=\frac{1}{c}\frac{c}{as^\beta+bs^\alpha}\frac{as^\beta+bs^\alpha}{as^\beta+bs^\alpha+c}=\frac{1}{c}\frac{\frac{c}{a}s^{-\alpha}}{s^{\beta-\alpha}+\frac{b}{a}}\frac{1}{1+\frac{\frac{c}{a}s^{-\alpha}}{s^{\beta-\alpha}+\frac{b}{a}}}.$$

Denoting $P=\frac{\frac{c}{a}s^{-\alpha}}{s^{\beta-\alpha}+\frac{b}{a}}$, which is a number in $(0,1)$, for large s, the expression becomes

$$g(s)=\frac{1}{c}\frac{P}{1-(-P)}=\frac{1}{c}\sum_{k=0}^{\infty}(-1)^kP^{k+1}=\frac{1}{c}\sum_{k=0}^{\infty}(-1)^k\left(\frac{c}{a}\right)^{k+1}\frac{s^{-\alpha k-\alpha}}{\left(s^{\beta-\alpha}+\frac{b}{a}\right)^{k+1}}.$$

Recognizing the Laplace transform formula (2), one can invert this expression term by term, to see that g is the Laplace transform of the function (Podlubny, 1998)

$$G(t) = \frac{1}{a}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{c}{a}\right)^k t^{\beta(k+1)-1}E^{(k)}_{\beta-\alpha,\beta+\alpha k}\left(-\frac{b}{a}t^{\beta-\alpha}\right).$$

Recall that for a classical risk model with gamma distributed claim sizes, the Laplace transform of survival probability after shifting the argument becomes, when s is large enough,

$$\begin{aligned}\hat{\phi}(s-\alpha) =& \frac{c\phi(0)s^r}{cs^{r+1}-(c\alpha+\lambda)s^r+\lambda\alpha^r}\\ =& \frac{c\phi(0)s^r}{cs^{r+1}-(c\alpha+\lambda)s^r}\sum_{k=0}^{\infty}(-1)^k\left(\frac{\lambda\alpha^r}{cs^{r+1}-(c\alpha+\lambda)s^r}\right)^k\\ =& \sum_{k=0}^{\infty}(-1)^k\frac{\phi(0)\left(\frac{\lambda}{c}\alpha^r\right)^k s^{-rk}}{\left(s-\left(\alpha+\frac{\lambda}{c}\right)\right)^{k+1}},\end{aligned}$$

which permits to invert term-by-term back to

$$\phi(u) = e^{-\alpha u}\phi(0)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\lambda\alpha^r}{c}\right)^k u^{(r+1)k}E^{(k)}_{1,rk+1}\left(\left(\alpha+\frac{\lambda}{c}\right)u\right),$$

as required. The last expression can be rewritten in the form

$$\phi(u) = e^{-\alpha u}\phi(0)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\lambda\alpha^r}{c}\right)^k u^{(r+1)k}\sum_{j=0}^{\infty}\frac{(j+k)!\left(\left(\alpha+\frac{\lambda}{c}\right)u\right)^j}{j!\Gamma(k(r+1)+1+j)}. \quad (21)$$

□

Remark 3.1. For $r = 1$, note that expression (20) also reduces, as it should, to the classical result (8) of Cramér (1930). We leave out a somewhat longish calculation.

4 Method Three - Tail Convolutions

In this section, we start with the classical risk model with any light-tail distributed claims. The non-ruin probability can be obtained as integral of an infinite sum of moments of claim size distributions. When the claims are gamma distributed, the resulting formulas can be relatively efficiently evaluated.

Recall the form (11) of the Laplace transform of the ruin probability in a compound Poisson process with a generic claim size X and the moment generating function M_X:

$$\hat{\phi}(s) = \phi(0)\frac{1}{s}\frac{1}{1-\frac{\lambda}{c}\frac{1-M_X(-s)}{s}}. \quad (22)$$

Notice that the term in the denominator,

$$\hat{g}(s) = \frac{1-M_X(-s)}{s}, \quad s > 0,$$

is the Laplace transform of the distributional tail

$$g(x) = P(X > x), \quad x \geqslant 0. \tag{23}$$

By the positive loading assumption we have

$$\hat{\phi}(s) = \phi(0)\frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{\lambda}{c}\right)^n (\hat{g}(s))^n, \tag{24}$$

since the ratio in the series is smaller than 1. Inverting the Laplace transforms in (24) gives us immediately the first statement of the next theorem. The key part of the theorem is the expression (26) for the ingredients in (25).

Theorem 4.1. The non-ruin probability in classical risk model can be written in the form

$$\phi(u) = \phi(0)\left(1 + \int_0^u \sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n g^{*n}(y)\,dy\right), \quad u > 0. \tag{25}$$

Here g^{*n} is the nth convolution of the tail distribution of claim X_j. It can be computed for $n \geq 2$ as

$$\begin{aligned} g^{*n}(x) = &\frac{1}{(n-1)!}\mathbb{E}\left[\left(\sum_{j=1}^{n} X_j - x\right)^{n-1} \mathbb{1}\left(\sum_{j=1}^{n} X_j > x\right)\right] \\ &-\frac{1}{(n-1)!}\sum_{i=1}^{n-1}\binom{n-1}{n-i-1} b_{n-i}(F)\mathbb{E}\left[\left(\sum_{j=1}^{i} X_j - x\right)^{i} \mathbb{1}\left(\sum_{j=1}^{i} X_j > x\right)\right], \end{aligned} \tag{26}$$

$n = 1, 2, \ldots$. Here $X_1, X_2, \ldots$ are the i.i.d. claim sizes. The sequence $\big(b_i(F),\, i = 1, 2\ldots\big)$ depends on the distribution F of claim sizes. It is defined recursively by

$$\begin{aligned} b_1(F) &= 1, \\ b_{m+1}(F) &= \mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^m - \sum_{i=1}^{m}\binom{m}{i-1} b_i(F)\mathbb{E}\left(\sum_{j=1}^{n-i} X_j\right)^{m+1-i}, \end{aligned} \tag{27}$$

for $m = 1, \ldots, n-1$. Thus defined, b_m is independent of $n > m$.

Proof. The proof is postponed to Appendix B. □

Remark 4.1. Method 3 computes the convolution powers for the tail-distribution in a Pollaczeck-Khinchine formula via a sequence of integrals of increasing dimension which simplify to one-dimensional integrals in the gamma case. See the discussion part for details.

Remark 4.2. Once again, for $r = 1$, the expression (25) can be checked to reduce to the classical result (8) of Cramér (1930). We leave out the calculation.

Remark 4.3. Even though the statement of Theorem 4.1 is valid for a general classical risk model, the calculation of the numbers $\big(b_i(F)\big)$ in (27) and functions $\big(g^{*n}\big)$ in (26) requires, in general, integration of increasing dimension. In the case where the claims follow a gamma distribution, the sums of the type $X_1+\ldots+X_i$ themselves also follow a gamma distribution, and the integrals always stay one-dimensional. The same is true in other cases where the distributions of such sums are known.

5 Discussion of Three Results

The results shown in this paper refer to extensions of the classical ruin models and all of them coincide with the classical result when claim sizes are exponentially distributed. In this section, we will present the advantages of each method and its result, including some numerical examples. Since all three expressions present infinite sums, we truncate those to their first 20 terms to be able to obtsin a numerical value. The corresponding truncated errors are less than 10^{-5} for all three expressions, and we have noticed that considering more than the first 20 terms will not decrease substantially the numerical errors.

The first expression in Theorem 2.1 is an infinite sum of convolution terms. When r takes integer value, the expression reduces to a sum of finite terms due to property (16) of the Mittag-Leffler function and thus explicit results can be implemented. As long as r is not integer, numerical methods are needed to calculate the probability. One choice is to use the relationship between the Mittag-Leffler function and incomplete gamma function, mentioned in Remark 2.1, identity (15), since the incomplete gamma functions are available in most numerical libraries and systems. The other choices would be to use "Mittag-Leffler function" MATLAB codes by Igor Podlubny (which calculates the Mittag-Leffler function with desired accuracy) or "MittagLeffleR" R package by Gurtek Gill and Peter Straka (whihc provides probability density, distribution function, quantile function and random variate generation for the Mittag-Leffler distributions, and the Mittag-Leffler function). For instance, we will take the sum of the first 20 convolutions in the expression for a numerical result for the survival probability

$$\phi(u) = \phi(0) + e^{-\alpha u}\phi(0)\left\{ e^{\alpha u} * \left(\sum_{n=1}^{20} \left(\frac{\lambda}{c}\right)^n \left[e^{\alpha u} - (\alpha u)^r E_{1,1+r}(\alpha u)\right]^{*n} \right) \right\}. \tag{28}$$

The second expression (20) is a quite time efficient method, which is very easy to implement with accurate results. Due to the fact that the derivative of a Mittag-Leffler function is an infinite series, this expression contains two-fold infinite sums. Moreover, inside each series, only gamma functions and power functions are needed to be calculated. Therefore, any software having 'addition' and loop functions can handle this expression. Compared with the first result, which contains convolution terms, this one is more time efficient in a numerical sense. The disadvantage is that we have no instance where we can get exact result, for $r \neq 1$. In this case, we could evaluate the first 20 derivatives in the expression to obtain a numerical approximation of the survival probability

$$\phi(u) = e^{-\alpha u}\phi(0)\sum_{k=0}^{20}\frac{(-1)^k}{k!}\left(\frac{\lambda\alpha^r}{c}\right)^k u^{(r+1)k}E^{(k)}_{1,rk+1}\left(\left(\alpha+\frac{\lambda}{u}\right)u\right).$$

The third result in Theorem 4.1 is presented in terms of moments of the claim size distribution. In principle, this method is valid for any claim distribution, but in the case of gamma claims, the distribution of the sum of X's are known analytically, so the computations are tractable. Note that since

$$\int_0^u \sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n g^{*n}(y)\,dy = \sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n \int_0^u g^{*n}(y)\,dy,$$

one needs to be able to compute efficiently

$$\int_0^u g^{*n}(y)\,dy = \frac{1}{(n-1)!}\mathbb{E}\int_0^u \left(\sum_{j=1}^n X_j - x\right)^{n-1} \mathbb{1}\left(\sum_{j=1}^n X_j > x\right)dx$$
$$-\frac{1}{(n-1)!}\sum_{i=1}^{n-1}\binom{n-1}{n-i-1}b_{n-1}(F)\mathbb{E}\int_0^u\left(\sum_{j=1}^i X_j - x\right)^i \mathbb{1}\left(\sum_{j=1}^i X_j > x\right)dx.$$

As it is easy to compute the sequence $(b_n(F))$ for gamma claims, one only needs to evaluate efficiently the functions

$$a_{n,k}(u) = \mathbb{E}\int_0^u\left(\sum_{j=1}^n X_j - x\right)^k \mathbb{1}\left(\sum_{j=1}^n X_j > x\right)dx,$$

for $k = n-1$ and n. However, these functions can be further expressed in terms of incomplete gamma functions as

$$a_{n,k}(u) = \frac{\alpha^{nr}}{\Gamma(nr)(k+1)}\left[\frac{\Gamma(nr+k+1)}{\alpha^{nr+k+1}} - \sum_{j=0}^{k+1}\binom{k+1}{j}(-u)^{k+1-j}\frac{\Gamma(nr+j,\alpha u)}{\alpha^{nr+j}}\right]$$

for $k = n-1$ and n. Therefore, the whole calculation consists on evaluating some incomplete gamma functions, and those have already been efficiently implemented. The first 20 convolutions in expression (25) would be sufficient when implementing the survival probability numerically.

Other claim size distributions to which the third method can be applied reasonably efficiently include the mixed exponential distribution with density function

$$f(x) = \alpha\lambda_1 e^{-\lambda_1 x} + (1-\alpha)\lambda_2 e^{-\lambda_2 x}.$$

The calculations become more involved but still practical.

Figure 1 shows the difference on accuracy of these three results.

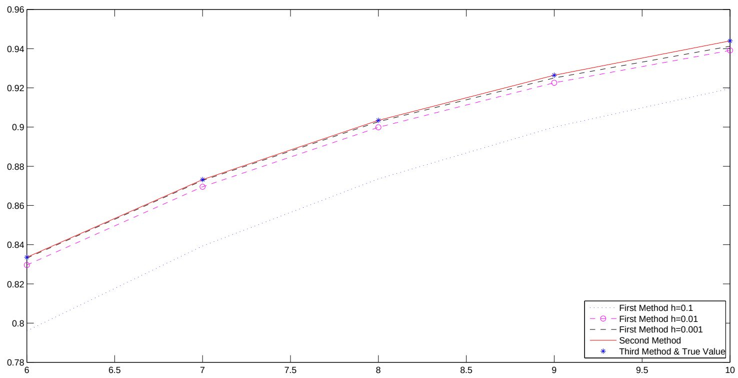

Figure 1: Difference on accuracy of three results

In order to test the accuracy of the three results, we choose the parameter values $\lambda = 1$, $c = 1$, $r = 2$ and $\alpha = 2.4$. All the dash lines stand for the first method and solid one for the method two and three. In order to numerically evaluate the convolution integral in equation (28), we break up the interval $[0, 10]$ into subintervals of length h, then apply the trapezoid rule and use Newton-Cotes formula to realize the numerical integration. We compare results obtained for different lengths h. One can see that using the second method, the results converge fast to the true value. Moreover, for the third method, all the moments of gamma random variables have explicit expressions. Note that for $r = 2$, one retrieves the case of Erlang(2) claims. Several equivalent results under this model assumption have been obtained in the past and the formula chosen in this test comes from (He et al., 2003)

$$\phi(u) = 1 + \frac{v_2(v_1+\alpha)^2}{(v_1-v_2)\alpha^2}e^{v_1 u} + \frac{v_1(v_2+\alpha)^2}{(v_2-v_1)\alpha^2}e^{v_2 u},$$

where

$$v_1 = \frac{\lambda - 2c\alpha + \sqrt{\lambda^2 + 4c\alpha\lambda}}{2c},$$

$$v_2 = \frac{\lambda - 2c\alpha - \sqrt{\lambda^2 + 4c\alpha\lambda}}{2c}.$$

The corresponding results are put in Table 1.

Table 1: Difference on accuracy of three results

initial capital	Method 1 h=0.1	Method 1 h=0.01	Method 1 h=0.001	Method 2	Method 3
u=0	0.167	0.167	0.167	0.167	0.167
u=1	0.340	0.350	0.352	0.352	0.352
u=2	0.480	0.503	0.505	0.506	0.506
u=3	0.588	0.620	0.623	0.623	0.623
u=4	0.674	0.709	0.713	0.713	0.713
u=5	0.742	0.777	0.781	0.782	0.782
u=6	0.796	0.830	0.833	0.834	0.834
u=7	0.839	0.870	0.873	0.873	0.873
u=8	0.874	0.900	0.903	0.903	0.903
u=9	0.900	0.923	0.925	0.926	0.926
u=10	0.920	0.939	0.941	0.944	0.944

The errors between the results obtained from method two and true values are significantly smaller at 10^{-11} level.

Here are some results run by MATLAB using method two. These results can also be obtained using method one if one sets the step length to be as small as $h = 0.0001$, which takes more time. In table 2, the parameter values are set to be $\lambda = 1$, $c = 1$ and safety loading $\theta = 0.2$. Because the safety loading is held constant, for each r, we choose an α such that the average claim size $\frac{r}{\alpha}$ stays the same.

Table 2: Survival probabilities for different parameter values r, when the safety loading is 0.2

Initial Capital	$r = 0.5$	$r = 1$	$r = 1.5$	$r = 2$	$r = 2.5$	$r = 3$
$u = 0$	0.167	0.167	0.167	0.167	0.167	0.167
$u = 1$	0.281	0.318	0.338	0.352	0.361	0.368
$u = 2$	0.371	0.441	0.481	0.506	0.523	0.536
$u = 3$	0.449	0.543	0.593	0.623	0.644	0.660
$u = 4$	0.517	0.626	0.680	0.713	0.735	0.750
$u = 5$	0.576	0.693	0.749	0.782	0.802	0.817
$u = 6$	0.628	0.749	0.803	0.834	0.852	0.865
$u = 7$	0.673	0.795	0.846	0.873	0.890	0.901
$u = 8$	0.713	0.832	0.879	0.903	0.918	0.927
$u = 9$	0.749	0.862	0.905	0.926	0.939	0.947
$u = 10$	0.779	0.887	0.926	0.944	0.954	0.961

The corresponding plotting figure is shown in Figure 2.

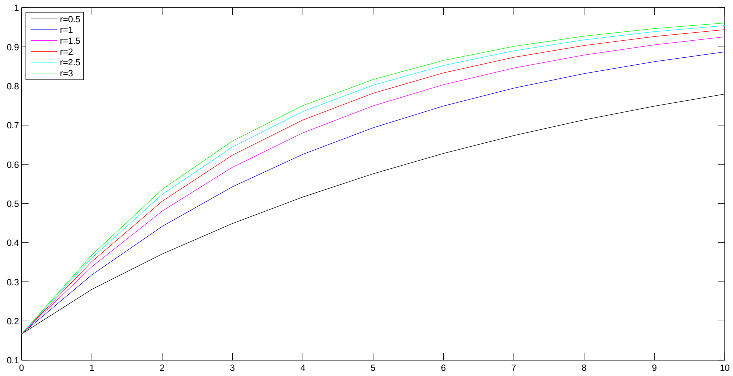

Figure 2: Survival probabilities for different parameter values r, when the safety loading is 0.2

One can observe that when the safety loading and other model parameters are fixed, the bigger r is, the higher survival probability the model has. The reason is that in this case, the expected claim size is fixed, further means that the ratio $\frac{r}{\alpha}$ is fixed, whereas the variance of claim size $\frac{r}{\alpha^2}$ decreases as r increases, i.e., the chance of having large claims will decrease. Since ruin is usually caused by some large claims, the model with a bigger shape parameter r is more likely to survive.

Table 3 and Figure 3 show how the survival probability changes with various premium rates and same safety loading when claim has gamma distribution with $r = 1.5$.

Table 3: Survival probabilities for different parameter values c, when the safety loading is 0.2

initial capital	$c = 1$	$c = 1.2$	$c = 1.4$	$c = 1.6$	$c = 1.8$	$c = 2$
u=0	0.167	0.167	0.167	0.167	0.167	0.167
$u = 1$	0.338	0.311	0.291	0.276	0.264	0.255
$u = 2$	0.481	0.437	0.403	0.377	0.356	0.338
$u = 3$	0.593	0.540	0.498	0.465	0.437	0.414
$u = 4$	0.680	0.624	0.578	0.540	0.508	0.481
$u = 5$	0.749	0.693	0.645	0.605	0.570	0.540
$u = 6$	0.803	0.749	0.702	0.660	0.624	0.593
$u = 7$	0.846	0.795	0.749	0.708	0.672	0.639
$u = 8$	0.879	0.833	0.789	0.749	0.713	0.680
$u = 9$	0.905	0.863	0.823	0.785	0.749	0.717
$u = 10$	0.926	0.888	0.851	0.815	0.781	0.749

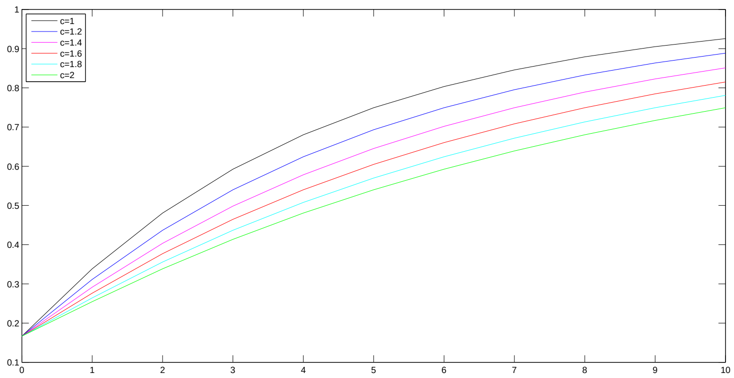

Figure 3: Survival probabilities for different parameter values c, when the safety loading is 0.2

Since the safety loading θ is fixed, the bigger premium rate c is, the bigger the expected claim size is. In this case, the shape parameter of claim distribution r is set to be constant 1.5, which means the larger expectation gives larger variance. Thus, similar to the previous test, this result is quite reasonable. When the safety loading and claim shape parameter are fixed, decreasing premium rate can make the company less likely to have ruin.

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6 Appendix A

Thorin (1973) provides an integral expression for the ruin probability when the claims are gamma distributed with parameters $k = \alpha$. When $k = \alpha = \frac{1}{2}$ and inter-arrival times are exponentially distributed with parameter $\lambda = 1$, this becomes

$$\psi(u) = \frac{(c-1)(1-2R)e^{-Ru}}{1+c(3R-1)} + \frac{\rho}{2\pi}\int_0^\infty \frac{\sqrt{x}e^{-(x+1)u/2}}{(x+1)\left[\frac{c^2}{4}x^2 + (\frac{c^2}{4}+c)x + 1\right]}\, dx, \quad (29)$$

where R is the unique positive solution of Lundberg equation

$$(1+cR)\sqrt{1-2R} = 1,$$

which can be solved to be $R = \frac{c-4+\sqrt{c^2+8c}}{4c}$.

On the other hand, from our result (12), when $r = 1/2$, the survival probability equals

$$\phi(u) = e^{-\alpha u} u^{-\frac{1}{2}} \sum_{k=0}^{2} m_k E_{\frac{1}{2},\frac{1}{2}} \left(s_k u^{\frac{1}{2}} \right).$$

Here s_0, s_1, s_2, m_0, m_1 and m_2 can be calculated explicitly, since for these special parameter values, the Mittag-Leffler function equals to

$$E_{\frac{1}{2},\frac{1}{2}} \left(s_k u^{\frac{1}{2}} \right) = \frac{1}{\sqrt{\pi}} + s_k u^{\frac{1}{2}} e^{s_k^2 u} \frac{2}{\sqrt{\pi}} \int_{-s_k u^{\frac{1}{2}}}^{\infty} e^{-t^2}\, dt,$$

which leads to

$$\phi(u) = e^{-\alpha u} u^{-\frac{1}{2}} (m_0 + m_1 + m_2) \frac{1}{\sqrt{\pi}} + \sum_{k=0}^{2} s_k m_k e^{\left(s_k^2 - \alpha\right) u} \frac{2}{\sqrt{\pi}} \int_{-s_k u^{\frac{1}{2}}}^{\infty} e^{-t^2}\, dt.$$

The ruin probability can be expressed as

$$\begin{aligned} \psi(u) = & - \frac{2 s_1^2 \left(1 - \frac{1}{c}\right)}{(s_0 - s_1)(s_2 - s_1)} e^{-Ru} + \frac{1}{2} \mathrm{erfc} \left(s_0 u^{\frac{1}{2}} \right) \\ & + \frac{s_1^2 \left(1 - \frac{1}{c}\right)}{(s_0 - s_1)(s_2 - s_1)} \mathrm{erfc} \left(s_1 u^{\frac{1}{2}} \right) e^{\left(s_1^2 - \frac{1}{2}\right) u} \\ & - \frac{s_2^2 \left(1 - \frac{1}{c}\right)}{(s_0 - s_2)(s_1 - s_2)} \mathrm{erfc} \left(-s_2 u^{\frac{1}{2}} \right) e^{\left(s_2^2 - \frac{1}{2}\right) u}. \end{aligned} \qquad (30)$$

which by exploiting properties of the roots s_0, s_1 and s_2 leads to (29).

7 Appendix B: Proof of Theorem 4.1

Proof. We start by proving that the sequence $\big(b_i(F),\ i = 1, 2 \ldots\big)$ defined in (27) has the property that b_m is independent of $n \geq m$. Since the statement is clear for $m = 1$, we proceed by induction. Assume that $b_k(F)$ is independent of $n \geqslant k$ for all $k \leqslant m$, and let $n \geqslant m+1$. It is enough to show that the difference between the expressions in the right hand side of (27) computed for n and for $n+1$ is equal to zero. We have:

$$\begin{aligned} & \mathbb{E} \left(\sum_{j=1}^{n+1} X_j \right)^m - \sum_{i=1}^{m} \binom{m}{i-1} b_i(F) \mathbb{E} \left(\sum_{j=1}^{n+1-i} X_j \right)^{m+1-i} \\ & - \mathbb{E} \left(\sum_{j=1}^{n} X_j \right)^m + \sum_{i=1}^{m} \binom{m}{i-1} b_i(F) \mathbb{E} \left(\sum_{j=1}^{n-i} X_j \right)^{m+1-i} \\ = & \sum_{k=1}^{m} \binom{m}{k} \mathbb{E}\left(X^k\right) \mathbb{E} \left(\sum_{j=1}^{n} X_j \right)^{m-k} \\ & - \sum_{i=1}^{m} \binom{m}{i-1} b_i(F) \sum_{k=1}^{m+1-i} \binom{m+1-i}{k} \mathbb{E}(X^k) \mathbb{E} \left(\sum_{j=1}^{n-i} X_j \right)^{m+1-i-k} \end{aligned}$$

$$
\begin{aligned}
=&\sum_{k=1}^{m}\binom{m}{k}\mathbb{E}\left(X^k\right)\mathbb{E}\left(\sum_{j=1}^{n}X_j\right)^{m-k}\\
&-\sum_{k=1}^{m}\mathbb{E}(X^k)\sum_{i=1}^{m+1-k}\frac{m!}{(i-1)!k!(m+1-i-k)!}b_i(F)\mathbb{E}\left(\sum_{j=1}^{n-i}X_j\right)^{m+1-i-k}\\
=&\sum_{k=1}^{m}\binom{m}{k}\mathbb{E}\left(X^k\right)\cdot\\
&\left[\mathbb{E}\left(\sum_{j=1}^{n}X_j\right)^{m-k}-\sum_{i=1}^{m+1-k}\binom{m-k}{i-1}b_i(F)\mathbb{E}\left(\sum_{j=1}^{n-i}X_j\right)^{m-k+1-i}\right]\\
=&\sum_{k=1}^{m}\binom{m}{k}\mathbb{E}\left(X^k\right)\cdot\\
&\left[\mathbb{E}\left(\sum_{j=1}^{n}X_j\right)^{m-k}-\sum_{i=1}^{m-k}\binom{m-k}{i-1}b_i(F)\mathbb{E}\left(\sum_{j=1}^{n-i}X_j\right)^{m-k+1-i}-b_{m-k+1}\right]\\
=&\sum_{k=1}^{m}\binom{m}{k}\mathbb{E}\left(X^k\right)\left(b_{m-k+1}-b_{m-k+1}\right)\\
=&0
\end{aligned}
$$

by (27). This completes the induction step and, hence, proves that b_m is independent of $n \geq m$.

In order to prove the representation (26) we start from the cases $n=2$ and $n=3$, checking the structure of the formula in those cases and then proceed by induction. For $n=2$ we have

$$
\begin{aligned}
g^{*2}(x)=&\int_0^x g(y)g(x-y)dy=\int_0^x dy\int_y^\infty f(v)dv\int_{x-y}^\infty f(w)dw \qquad (31)\\
=&\int\int_{v+w>x}\left(\min(v,x)-(x-w)\right)f(v)f(w)\,dv\,dw\\
=&\int_{v>x}f(v)dv\int_0^\infty wf(w)dw+\int\int_{v\leqslant x,\,v+w>x}(v+w-x)f(v)f(w)\,dv\,dw\\
=&\mathbb{P}(X>x)\mathbb{E}(X)+\mathbb{E}\left[(X_1+X_2-x)\mathbb{1}(X_1+X_2>x)\right]\\
&-\int\int_{v>x}(v+w-x)f(v)f(w)\,dv\,dw\\
=&\mathbb{P}(X>x)\mathbb{E}(X)+\mathbb{E}\left[(X_1+X_2-x)\mathbb{1}(X_1+X_2>x)\right]\\
&-\mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right]-\mathbb{P}(X>x)\mathbb{E}(X)\\
=&\mathbb{E}\left[(X_1+X_2-x)\mathbb{1}(X_1+X_2>x)\right]-\mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right], \qquad (32)
\end{aligned}
$$

which coincides with (26) for $n=2$ with $b_1(F)=1$.

For a generic random variable Y with a finite mean consider the function

$$
h_1(x)=\mathbb{E}\left((Y-x)\mathbb{1}(Y>x)\right), \quad x>0.
$$

Note the appearance of such functions in the above expression for g^{*2}. We proceed with calculating the convolution of this function with g. The notation in the following calculation assumes that X and Y are defined on the same probability space and are independent.

$$
\begin{aligned}
g * h_1(x) = & \int_0^x g(y)\mathbb{E}\left((Y-(x-y))\mathbb{1}(Y > x-y)\right) \, dy \\
= & \int_0^\infty \int_0^\infty f_X(v)\, dv f_Y(w)\, dw \int_0^\infty (w-x+y)\mathbb{1}(x-w \leqslant y \leqslant \min(x,v))\, dy \\
= & \frac{1}{2}\int\int_{v+w>x} (\min(w, v+w-x))^2 \, f_X(v)f_Y(w)\, dv\, dw \\
= & \frac{1}{2}\mathbb{P}(X > x)\mathbb{E}(Y^2) + \frac{1}{2}\mathbb{E}\left((X+Y-x)^2\mathbb{1}(X+Y>x)\right) \\
& - \frac{1}{2}\int\int_{v>x} (v+w-x)^2 f_X(v)f_Y(w)\, dv\, dw \\
= & \frac{1}{2}\mathbb{E}\left((X+Y-x)^2\mathbb{1}(X+Y>x)\right) - \frac{1}{2}\mathbb{E}\left[(X-x)^2\mathbb{1}(X>x)\right] \\
& - \mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right]\mathbb{E}(Y),
\end{aligned}
$$

with the last step following by simple algebraic manipulations.

Applying this result, first with $Y = X_1 + X_2$ and then with $Y = X$, to the right hand side of (32) we obtain the following expression for g^{*3}:

$$
\begin{aligned}
g^{*3}(x) = & g * g^{*2}(x) \\
= & \frac{1}{2}\mathbb{E}\left[(X_1+X_2+X_3-x)^2\mathbb{1}(X_1+X_2+X_3>x)\right] - \frac{1}{2}\mathbb{E}\left[(X-x)^2\mathbb{1}(X>x)\right] \\
& - 2\mathbb{E}(X)\mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right] - \frac{1}{2}\mathbb{E}\left[(X_1+X_2-x)^2\mathbb{1}(X_1+X_2>x)\right] \\
& + \frac{1}{2}\mathbb{E}\left[(X-x)^2\mathbb{1}(X>x)\right] + \mathbb{E}(X)\mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right] \\
= & \frac{1}{2}\mathbb{E}\left[(X_1+X_2+X_3-x)^2\mathbb{1}(X_1+X_2+X_3>x)\right] \\
& - \frac{1}{2}\mathbb{E}\left[(X_1+X_2-x)^2\mathbb{1}(X_1+X_2>x)\right] - \mathbb{E}(X)\mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right].
\end{aligned}
$$

This coincides with (26) for $n = 3$ with $b_1(F) = 1$, $b_2(F) = EX$. Accordingly, we are led to introduce, for a generic random variable Y, and $n \geq 1$, the function

$$
h_n(x) = \mathbb{E}\left[(Y-x)^n\mathbb{1}(Y>x)\right], \quad x > 0,
$$

and calculate its convolution with g. Once again, in the following calculation we assume that X and Y are defined on the same probability space and are independent.

$$
\begin{aligned}
g * h_n(x) = & \int_0^x g(y)\mathbb{E}\left[(Y-(x-y))^n\mathbb{1}(Y>(x-y))\right] \, dy \qquad (33) \\
= & \frac{1}{n+1}\int\int_{v+w>x} (\min(w, v+w-x))^{n+1} f_X(v)f_Y(w)\, dv\, dw \\
= & \frac{1}{n+1}\mathbb{P}(X>x)\mathbb{E}(Y^{n+1}) + \frac{1}{n+1}\mathbb{E}\left[(X+Y-x)^{n+1}\mathbb{1}(X+Y>x)\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{n+1}\int\int_{v>x}(v+w-x)^{n+1}f_X(v)f_Y(w)\,dv\,dw\\
=&\frac{1}{n+1}\mathbb{P}(X>x)\mathbb{E}(Y^{n+1})+\frac{1}{n+1}\mathbb{E}\left[(X+Y-x)^{n+1}\mathbb{1}(X+Y>x)\right]\\
&-\frac{1}{n+1}\sum_{j=0}^{n+1}\binom{n+1}{j}\mathbb{E}(Y^{n+1-j})\mathbb{E}\left[(X-x)^j\mathbb{1}(X>x)\right]\\
=&\frac{1}{n+1}\mathbb{E}\left[(X+Y-x)^{n+1}\mathbb{1}(X+Y>x)\right]\\
&-\frac{1}{n+1}\sum_{j=1}^{n+1}\binom{n+1}{j}\mathbb{E}(Y^{n+1-j})\mathbb{E}\left[(X-x)^j\mathbb{1}(X>x)\right].
\end{aligned}
$$

Assume now that the statement (26) holds for g^{*k} with all $k\le n$ for some $n\ge 3$. We will establish the validity of this formula for $k=n+1$. We have by (33):

$$
\begin{aligned}
g^{*(n+1)}(x)=&\frac{1}{(n-1)!}\frac{1}{n}\Bigg\{\mathbb{E}\left[\left(\sum_{j=1}^{n+1}X_j-x\right)^n\mathbb{1}\left(\sum_{j=1}^{n+1}X_j>x\right)\right]\\
&-\sum_{i=1}^{n}\binom{n}{i}\mathbb{E}\left[\left(\sum_{j=1}^{n}X_j\right)^{n-i}\right]\mathbb{E}\left[(X-x)^i\mathbb{1}(X>x)\right]\Bigg\}\\
&-\frac{1}{(n-1)!}\sum_{k=1}^{n-1}\binom{n-1}{n-k-1}b_{n-k}(F)\frac{1}{k+1}\Bigg\{\mathbb{E}\left[\left(\sum_{j=1}^{k+1}X_j-x\right)^{k+1}\mathbb{1}\left(\sum_{j=1}^{k+1}X_j>x\right)\right]\\
&-\sum_{i=1}^{k+1}\binom{k+1}{i}\mathbb{E}\left[\left(\sum_{j=1}^{k}X_j\right)^{k+1-i}\right]\mathbb{E}\left[(X-x)^i\mathbb{1}(X>x)\right]\Bigg\}\\
=&\frac{1}{n!}\mathbb{E}\left[\left(\sum_{j=1}^{n+1}X_j-x\right)^n\mathbb{1}\left(\sum_{j=1}^{n+1}X_j>x\right)\right]\\
&-\frac{1}{n!}\sum_{k=2}^{n}\frac{n}{k}\binom{n-1}{n-k}b_{n-k+1}(F)\mathbb{E}\left[\left(\sum_{j=1}^{k}X_j-x\right)^k\mathbb{1}\left(\sum_{j=1}^{k}X_j>x\right)\right]\\
&-\sum_{i=2}^{n}\mathbb{E}\left[(X-x)^i\mathbb{1}(X>x)\right]\left[\frac{1}{n!}\binom{n}{i}\mathbb{E}\left(\sum_{j=1}^{n}X_j\right)^{n-i}\right.\\
&\left.-\frac{1}{(n-1)!}\sum_{k=i-1}^{n-1}\binom{n-1}{n-k-1}b_{n-k}(F)\frac{1}{k+1}\binom{k+1}{i}\mathbb{E}\left(\sum_{j=1}^{k}X_j\right)^{k+1-i}\right]\\
&+\theta_n(F)\mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right]\\
=&\frac{1}{n!}\mathbb{E}\left[\left(\sum_{j=1}^{n+1}X_j-x\right)^n\mathbb{1}\left(\sum_{j=1}^{n+1}X_j>x\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{n!}\sum_{k=2}^{n}\frac{n}{k}\binom{n-1}{n-k}b_{n-k+1}(F)\mathbb{E}\left[\left(\sum_{j=1}^{k}X_j - x\right)^k \mathbb{1}\left(\sum_{j=1}^{k}X_j > x\right)\right] \\
&+ \theta_n(X)\mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right],
\end{aligned}
$$

with the cancellation due to the defining property (27). Here

$$
\begin{aligned}
\theta_n(F) =& -\frac{n}{n!}\mathbb{E}\left(\sum_{j=1}^{n}X_j\right)^{n-1} \\
&+ \frac{1}{(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{n-k-1}b_{n-k}(F)\frac{1}{k+1}\mathbb{E}\left(\sum_{j=1}^{k}X_j\right)^k(k+1) \\
=& -\frac{1}{(n-1)!}b_1(F),
\end{aligned}
$$

once again by the defining property (27). Therefore,

$$
\begin{aligned}
g^{*(n+1)}(x) =& \frac{1}{n!}\mathbb{E}\left[\left(\sum_{j=1}^{n+1}X_j - x\right)^n \mathbb{1}\left(\sum_{j=1}^{n+1}X_j > x\right)\right] \\
&- \frac{1}{n!}\sum_{i=1}^{n}\binom{n}{n-i}b_{n+1-i}(F)\mathbb{E}\left[\left(\sum_{j=1}^{i}X_j - x\right)^i \mathbb{1}\left(\sum_{j=1}^{i}X_j > x\right)\right].
\end{aligned}
$$

This completes the induction step.

□