

ASK/PSK-correspondence and the r-map

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Abstract

We formulate a correspondence between affine and projective special Kähler manifolds of the same dimension. As an application, we show that, under this correspondence, the affine special Kähler manifolds in the image of the rigid r-map are mapped to one-parameter deformations of projective special Kähler manifolds in the image of the supergravity r-map. The above one-parameter deformations are interpreted as perturbative α' -corrections in heterotic and type-II string compactifications with $N = 2$ supersymmetry. Also affine special Kähler manifolds with quadratic prepotential are mapped to one-parameter families of projective special Kähler manifolds with quadratic prepotential. We show that the completeness of the deformed supergravity r-map metric depends solely on the (well-understood) completeness of the undeformed metric and the sign of the deformation parameter.

Keywords: special real manifolds, special Kähler manifolds, r-map

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1 Preliminaries

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Introduction

The supergravity c-map, described in [FS90, CFG89, Hit09], can be understood as a special case of a more general construction, the HK/QK-correspondence. In fact, the supergravity c-map can be reduced to the much simpler rigid c-map. The corresponding manifolds and maps are summarized in the following diagram:

$$\begin{array}{ccccc}
 M & \xrightarrow{c} & N & \xrightarrow{\text{con}} & \hat{N} \\
 \downarrow SC & & \searrow HK/QK & & \downarrow SC \\
 \bar{M} & \xrightarrow{\bar{c}} & & & \bar{N}
 \end{array}$$

In this diagram the scalar manifolds \bar{M} of four-dimensional vector multiplets coupled to $N = 2$ supergravity, which are projective special Kähler, are related to the scalar manifolds \bar{N} of three-dimensional hypermultiplets coupled to supergravity, which are quaternionic-Kähler, by the supergravity c-map, which is induced by spacelike dimensional reduction from four to three dimensions. In the superconformal formulation of supergravity, the scalar manifolds \bar{M} and \bar{N} are obtained as superconformal quotients, denoted by SC in the diagram, from the scalar manifolds M and \hat{N} of associated rigid superconformal

theories. From this viewpoint reducing the supergravity c-map to the rigid c-map requires to associate to hyper-Kähler manifolds N in the image of the rigid c-map a hyper-Kähler cone \hat{N} . This operation is denoted in the diagram by con and is known as conification [ACM13]. The resulting relation between hyper-Kähler and quaternionic Kähler manifolds of the same dimension in the image of the rigid and local c-maps, respectively, is obtained from the HK/QK-correspondence [Hay08, ACM13, ACDM15]. It turns out that to apply the HK/QK-correspondence it is not essential that the hyper-Kähler manifold is in the image of the rigid c-map but what is required is essentially a function generating a certain isometric Hamiltonian flow. As a result one obtains not only the supergravity c-map metric but a one-parameter deformation thereof.

When attempting to apply this approach to the supergravity r-map introduced in [dWVP92], which is induced by the dimensional reduction of five-dimensional vector multiplets to four dimensions, one runs into the following problem. Although there exists a conification procedure for Kähler manifolds carrying an isometric Hamiltonian flow, which could potentially be applied to our problem, it turns out that the manifolds in the image of the rigid r-map do not carry a distinguished isometric Hamiltonian flow. Even worse, applying the Kähler conification to any of the generically existing Hamiltonian flows does not yield the desired metric.

In this paper, we will solve this puzzle by establishing an ASK/PSK-correspondence, see Theorem 4.11 and Definition 4.12, relating affine special Kähler to projective special Kähler manifolds of the same dimension. This is achieved by a new conification procedure which maps affine special Kähler manifolds to conical affine special Kähler manifolds and does not require a Hamiltonian flow. The relations between the rigid and local r -maps, superconformal quotients, conification, and the ASK/PSK correspondence are summarized in the following diagram.

$$\begin{array}{ccccc}
 U & \xrightarrow{r} & M & \xrightarrow{con} & \hat{M} \\
 SC \downarrow & & & \searrow ASK/PSK & \downarrow SC \\
 \mathcal{H} & \xrightarrow{\bar{r}} & & & \bar{M}
 \end{array}$$

Superconformal quotients map conical affine special real manifolds U to projective special real manifolds \mathcal{H} , and conical affine special Kähler manifolds \hat{M} to projective special Kähler manifolds \bar{M} . While U and \hat{M} are the scalar target manifolds of five- and four-dimensional superconformal vector multiplets, \mathcal{H} and \bar{M} are the target manifolds of the gauge equivalent theories of five- and four-dimensional vector multiplets coupled to (Poincaré) $N = 2$ supergravity. The lift of the supergravity r -map \bar{r} to the scalar manifolds of the associated superconformal vector multiplets is the composition $con \circ r$ of the rigid r -map r with the new conification map con , which will be defined and an-

alyzed in detail in this paper. In short, by applying the rigid r -map to a conical affine special real manifold U one obtains a Kähler manifold M which is affine special, but not conical. To relate M to the projective special Kähler manifold \bar{M} obtained by the supergravity r -map, we will construct a conical affine special Kähler manifold \hat{M} of dimension $\dim_{\mathbb{R}} \hat{M} = \dim_{\mathbb{R}} M + 2 = \dim_{\mathbb{R}} \bar{M} + 2$ using the conification map con . This provides us with a ‘superconformal lift’ $U \mapsto \hat{M}$ of the supergravity r -map and with a correspondence $M \mapsto \bar{M}$ between affine and projective special Kähler manifolds of the same dimension, which are in the image of the respective r -map. This is a special case of the ASK/PSK correspondence.

Now we explain the geometric idea underlying the ASK/PSK-correspondence. The initial affine special Kähler manifold of complex dimension n can be locally realized as a Lagrangian submanifold of \mathbb{C}^{2n} with induced geometric data, whereas a projective special Kähler manifold of complex dimension n is locally realized as projectivization of a Lagrangian cone in \mathbb{C}^{2n+2} , see [ACD02] for these statements. So basically we have to map a Lagrangian submanifold $\mathcal{L} \subset \mathbb{C}^{2n}$ to a Lagrangian cone in $\hat{\mathcal{L}} \subset \mathbb{C}^{2n+2}$. This is done in two steps. First, we embed \mathcal{L} into the affine hyperplane $\{z^0 = 1\} \subset \mathbb{C}^{2n+1} = \mathbb{C} \times \mathbb{C}^{2n}$, where z^0 denotes the coordinate on the first factor. Then we take $\hat{\mathcal{L}} \subset \mathbb{C}^{2n+2}$ to be the cone over the graph of certain function

$$\mathbb{C}^{2n+1} \supset \{1\} \times \mathcal{L} \cong \mathcal{L} \xrightarrow{f} \mathbb{C}.$$

The function f is what we call a Lagrangian potential, see Definition 2.3, and is unique up to an additive constant C . This constant plays a role analogous to the freedom in the choice of the Hamiltonian function in the HK/QK-correspondence [ACM13, ACDM15]. Whereas the real part of C has no effect on the resulting geometry, changing the imaginary part gives rise to a family of projective special Kähler manifolds (\bar{M}_c, \bar{g}_c) depending on the real parameter $c = \text{Im}(C)$. We discuss some global aspects of this construction in terms of a flat principal bundle with structure group $G_{\text{SK}} = \text{Sp}(\mathbb{R}^{2n}) \ltimes \text{Heis}_{2n+1}(\mathbb{C})$. This group acts on the set of pairs (\mathcal{L}, f) , where $\mathcal{L} \subset \mathbb{C}^{2n}$ is a Lagrangian submanifold and f is a Lagrangian potential, and acts simply transitively on the set of special Kähler pairs (ϕ, F) consisting of a (pseudo-)Kählerian Lagrangian immersion $\phi : M \rightarrow \mathbb{C}^{2n}$ and a corresponding holomorphic prepotential F , see Definition 1.5. For the close relation between Lagrangian potentials and holomorphic prepotentials, see Lemma 2.9. Note that the group G_{SK} is a central extension of the affine group $\text{Aff}_{\text{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n}) = \text{Sp}(\mathbb{R}^{2n}) \ltimes \mathbb{C}^{2n}$. The latter group acts simply transitively on Kählerian Lagrangian immersions, and the central extension is necessary to extend this action to the holomorphic prepotentials. It turns out that the group G_{SK} , contrary to the group $G = \text{Sp}(\mathbb{R}^{2n}) \ltimes \text{Heis}_{2n+1}(\mathbb{R})$, is not compatible with the induced Kähler metrics on the Lagrangian cones. It includes transformations

which change the holomorphic prepotential F by terms of the form $\sqrt{-1}(a_k Z^k + c)$, where a_k and c are real, which are not compatible with the induced metrics.

Our main application of the ASK/PSK-correspondence is a one-parameter deformation of the supergravity r -map obtained by applying the ASK/PSK-correspondence to an affine special Kähler manifold which is obtained from a conical affine special real manifold $U \subset \mathbb{R}^n$ via the rigid r -map, see Theorem 6.2. We give a global description of the resulting projective special Kähler manifolds $(\overline{M}_c, \overline{g}_c)$, where $(\overline{M}_0, \overline{g}_0) = (\overline{M}, \overline{g})$ is the manifold in the image of the supergravity r -map. The manifold \overline{M}_c is a domain in \mathbb{C}^n of the form $\overline{M}_c = \mathbb{R}^n + iU_c$, where $U_c \subset U$. We analyze when $(\overline{M}_c, \overline{g}_c)$ is a complete Riemannian manifold. First of all, the undeformed Riemannian manifold $(\overline{M}, \overline{g})$ is complete if and only if the underlying projective special real manifold $\mathcal{H} \subset \mathbb{R}^n$ is a connected component of a global level set $\{x \in \mathbb{R}^n \mid h(x) = 1\}$ of a homogeneous cubic polynomial h [CHM12, CNS16]. Recall that the level set is required to be locally strictly convex for \mathcal{H} to be a projective special real manifold (with positive definite metric). Assuming the undeformed metric to be complete we prove that the deformed manifold $(\overline{M}_c, \overline{g}_c)$, $c \neq 0$, is Riemannian and complete if and only if c is negative, see Theorem 6.2. These results should be contrasted with the more involved completeness theorems for one-loop deformed c -map spaces [CDS16]. In the case of projective special Kähler manifolds with cubic prepotential the completeness of the supergravity c -map metric was shown to be preserved precisely under one-loop deformations with positive deformation parameter. In case of general c -map spaces, however, this result has been established only under the additional assumption of regular boundary behavior for the initial projective special Kähler manifold, which is satisfied, for instance, for quadratic prepotentials. As in the case of the one-loop deformed c -map the isometry type of the deformed r -map space $(\overline{M}_c, \overline{g}_c)$ depends only on the sign of c (positive, negative or zero). Note that the completeness of \overline{M}_0 implies that \overline{M}_1 is neither isometric to \overline{M}_0 nor to \overline{M}_{-1} , since the latter 2 manifolds are then complete whereas \overline{M}_1 is incomplete. Computing the scalar curvature in examples, see Examples 6.4 and 6.5, we complete this analysis by showing that \overline{M}_0 and \overline{M}_{-1} are in general not isometric. Incidentally, most, but not all, of the above results extend from cubic polynomials to general homogeneous functions, say of degree $k > 1$, see Remark 6.3. For instance, it is not known whether the above necessary and sufficient completeness criterion for projective special real manifolds [CNS16, Theorem 2.5] holds for polynomials of quartic and higher degree.

Let us now explain how our deformation of the supergravity r -map can be interpreted physically as a ‘stringy deformation.’ Five-dimensional supergravity coupled to $n_V = n - 1$ vector multiplets (and as well hypermultiplets, which are not relevant for our discussion) can be obtained by compactification of the heterotic string on $K3 \times S^1$, together with

a choice of an $E_8 \times E_8$ or $SO(32)$ vector bundle V [AFT96, Asp96, LSTY96], referred to as the gauge bundle, or by compactification of eleven-dimensional supergravity on a Calabi-Yau threefold [CCDF95]. The vector multiplet couplings are encoded in a cubic, homogeneous polynomial (sometimes called cubic prepotential),

$$h = -\frac{1}{6}C_{ijk}x^i x^j x^k, \quad i, j, k = 1, \dots, n,$$

which can be identified up to a sign with the Hesse potential $-h = \frac{1}{6}C_{ijk}x^i x^j x^k$ of a projective special real manifold (with positive definite metric). The coefficients C_{ijk} depend on the details of the compactification. For Calabi-Yau compactifications they are the triple-intersection numbers of four-cycles, while for heterotic compactifications they depend on the number of vector multiplets and the gauge bundle.

Upon reduction on a further circle the Hesse potential determines a holomorphic prepotential, with the real variables x^i being replaced by complex variables Z^i :

$$\hat{F} = \frac{1}{6}C_{ijk} \frac{Z^i Z^j Z^k}{Z^0}. \quad (0.1)$$

But while five-dimensional $N = 2$ (minimal) supersymmetry requires that the Hesse potential must be a polynomial, four-dimensional $N = 2$ supersymmetry only requires the prepotential to be holomorphic. This allows further terms in Eq. 0.1, and it turns out that such terms are created by α' -corrections. The dimensional reductions of the constructions discussed above give rise to heterotic string theory on $K3 \times T^2$ and type-IIA string theory on a Calabi-Yau threefold. The prepotential, including corrections takes the form [CdLOGP91, HKTY95, CDFVP95, dWKLL95, AFG⁺95, HM96]

$$\hat{F} = \frac{1}{6}C_{ijk} \frac{Z^i Z^j Z^k}{Z^0} - 2\sqrt{-1}c(Z^0)^2 + \dots,$$

where the omitted terms are exponentially small for large $\text{Re}(Z^i/Z^0)$ and the factor -2 corresponds to the factor of -2 in formula (6.1). In type-II Calabi-Yau compactifications the omitted terms are world-sheet instantons and, therefore, non-perturbative corrections in α' . The leading correction term $-2\sqrt{-1}c(Z^0)^2$ arises at four-loop level in α' perturbation theory [GVdVZ86, NS86, CdLOGP91], and the real coefficient c is proportional to $\zeta(3)\chi$, where ζ is the Riemann ζ -function and χ is the Euler number of the Calabi-Yau three-fold. The heterotic prepotential has an analogous structure, and the coefficient c is proportional to $\zeta(3)c_1(0)$, where $c_1(0)$, as well as the coefficients of the further correction terms, is obtained by expanding a (model-dependent) modular form [HM96].

We have mentioned that when performing the conification we can shift the Lagrangian potential (or, equivalently, the holomorphic prepotential $F = \frac{1}{6}C_{ijk}z^i z^j z^k$ of the initial affine special Kähler manifold) by an imaginary constant, which then deforms the resulting

prepotential by precisely the same type of term as is created by the leading α' -correction. Thus the resulting deformed supergravity r-map might be called a ‘stringy’ r-map. We remark that the further freedom to also include imaginary translations does not have an interpretation in the above string theory realizations. Imaginary translations correspond to adding terms

$$\Delta\hat{F} = ia_{0I}Z^0Z^I$$

to the prepotential, where a_{0I} are real constants. Such terms do not occur as quantum or stringy corrections in the above four-dimensional string models. Curiously, adding a term

$$\delta\hat{F} = \frac{1}{24}c_{2I}Z^0Z^I$$

to the type-IIA prepotential, where c_{2I} are the components of the second Chern class of X , has been discussed before in the literature. However, this term has a real coefficient and can be transformed away by a symplectic transformation. Conversely, it can be generated by a symplectic transformation, which was used in [BCdW⁺97] as a solution-generating technique for black hole solutions.

The reason why deformations of the form $\Delta\hat{F}$ do not occur in string theory is that they are forbidden by the so-called Peccei-Quinn symmetries, which are symmetries that act on scalar fields by continuous shifts. The physical scalar fields of the four-dimensional theory correspond to the coordinates $z^i = Z^i/Z^0 = y^i + \sqrt{-1}x^i$, where x^i encode the five-dimensional scalars and one scalar degree of freedom split off from the space-time metric, and where y^i are the components of the five-dimensional vector fields along the direction the theory is reduced over. The gauge symmetries of the five-dimensional theory imply that the reduced four-dimensional theory is invariant under continuous shifts of the scalars y^i , that is under continuous real shifts of z^i . Moreover, Peccei-Quinn symmetries are preserved by perturbative quantum corrections, see for example [dWKLL95]. In fact the only non-trivial deformation of the prepotential which has the form of a cubic polynomial in Z^0, Z^i divided by Z^0 , and which is consistent with the Peccei-Quinn symmetries is precisely the term proportional to $(Z^0)^2$ [BMR11]. This excludes terms of the form $\Delta\hat{F}$ which are generated by imaginary shifts.

1 Preliminaries

Definition 1.1. *An affine special Kähler manifold (M, J, g, ∇) is a pseudo-Kähler manifold (M, J, g) with symplectic form $\omega := g(\cdot, J\cdot)$ endowed with a flat torsion-free connection ∇ such that $\nabla\omega = 0$ and $d^\nabla J = 0$.*

Definition 1.2. Let M be a complex manifold of complex dimension n and consider the complex vector space $T^*\mathbb{C}^n = \mathbb{C}^{2n}$ endowed with the canonical coordinates $(z^1, \dots, z^n, w_1, \dots, w_n)$, standard complex symplectic form $\Omega = \sum_{i=1}^n dz^i \wedge dw_i$, standard real structure $\tau : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$ and Hermitian form $\gamma = \frac{\sqrt{-1}}{2}\Omega(\cdot, \tau\cdot)$. A holomorphic immersion $\phi : M \rightarrow \mathbb{C}^{2n}$ is called Lagrangian (respectively, Kählerian) if $\phi^*\Omega = 0$ (respectively, if $\phi^*\gamma$ is non-degenerate). ϕ is called totally complex if $d\phi(T_p M) \cap \tau d\phi(T_p M) = 0$ for all $p \in M$.

Proposition 1.3 ([ACD02]). Let $\phi : M \rightarrow \mathbb{C}^{2n}$ be a holomorphic immersion.

- (1) ϕ is totally complex if and only if its real part $\text{Re } \phi : M \rightarrow \mathbb{R}^{2n}$ is an immersion.
- (2) If ϕ is Lagrangian, then ϕ is Kählerian if and only if it is totally complex.

A Kählerian Lagrangian immersion $\phi : M \rightarrow \mathbb{C}^{2n}$ induces on M the structure of an affine special Kähler manifold. Locally, an affine special Kähler manifold can always be realized as a Kählerian Lagrangian immersion. This is reflected in the following proposition.

Proposition 1.4 ([ACD02]). Let (M, J, g, ∇) be a simply connected affine special Kähler manifold of complex dimension n . Then there exists a Kählerian Lagrangian immersion $\phi : M \rightarrow \mathbb{C}^{2n}$ inducing the affine special Kähler structure (J, g, ∇) on M . Moreover, ϕ is unique up to a transformation of \mathbb{C}^{2n} by an element in $\text{Aff}_{\text{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$.

More precisely, the action of the group $\text{Aff}_{\text{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$ on the set of Kählerian Lagrangian immersions $\phi : M \rightarrow \mathbb{C}^{2n}$ is simply transitive, as can be proven along the lines of the proof of simple transitivity in Proposition 2.10.

Definition 1.5. Let $\phi : M \rightarrow \mathbb{C}^{2n}$ be a Kählerian Lagrangian immersion of an affine special Kähler manifold M . Denote by $\lambda = w^t dz = \sum_{i=1}^n w_i dz^i$ the Liouville form of \mathbb{C}^{2n} . A function $F : M \rightarrow \mathbb{C}$ is called a prepotential of ϕ if $dF = \phi^*\lambda$.

Remark 1.6. (1) The function $K := \gamma(\phi, \phi)$ is a Kähler potential of the Kähler form ω , i.e., $\omega = -\frac{i}{2}\partial\bar{\partial}K$.

- (2) Let M be a local affine special Kähler manifold given as a Kählerian Lagrangian immersion $\phi : M \rightarrow \mathbb{C}^{2n}$. Then the pullback of the canonical coordinates of $T^*\mathbb{C}^n = \mathbb{C}^{2n}$ gives functions $z^1, \dots, z^n, w_1, \dots, w_n : M \rightarrow \mathbb{C}$ such that $\phi = (z, w) := (z^1, \dots, z^n, w_1, \dots, w_n)$. It can always be achieved that $z, w : M \rightarrow \mathbb{C}^n$ are holomorphic coordinate systems by replacing ϕ with $x \circ \phi$ for some $x \in \text{Sp}(\mathbb{R}^{2n})$ and restricting M if necessary [ACD02]. In this case, we call (z, w) a *conjugate pair of special holomorphic coordinates*.

- (3) Let $\phi = (z, w) : M \rightarrow \mathbb{C}^{2n}$ be a Kählerian Lagrangian immersion of an affine special Kähler manifold given by a conjugate pair of special holomorphic coordinates (z, w) and let $F : M \rightarrow \mathbb{C}$ be a prepotential of ϕ . Then we can identify $M \cong z(M) \subset \mathbb{C}^n$ and ϕ with $dF : M \rightarrow T^*M = \mathbb{C}^{2n}$. In particular, $\phi(M) = \{(z, w) \in \mathbb{C}^{2n} \mid w_i = \frac{\partial F}{\partial z^i}\}$ is the graph of dF over M . In this case, $M \subset \mathbb{C}^n$ is called an *affine special Kähler domain* and $K(p) = \sum_{i=1}^n \text{Im}(\bar{z}^i F_i)$ where $F_i := \frac{\partial F}{\partial z^i}$.

Definition 1.7. A conical affine special Kähler manifold $(\hat{M}, \hat{J}, \hat{g}, \hat{\nabla}, \xi)$ is an affine special Kähler manifold $(\hat{M}, \hat{J}, \hat{g}, \hat{\nabla})$ and a vector field ξ such that $\hat{g}(\xi, \xi) \neq 0$ and $\hat{\nabla}\xi = \hat{D}\xi = \text{id}$, where \hat{D} is the Levi-Civita connection of \hat{g} .

Note that contrary to [CHM12, Definition 3] here we are not making any assumptions on the signature of the metric \hat{g} .

A conical affine special Kähler manifold \hat{M} of complex dimension $n+1$ locally admits Kählerian Lagrangian immersions $\Phi : U \rightarrow \mathbb{C}^{2n+2}$ that are equivariant with respect to the local \mathbb{C}^* -action defined by $Z = \xi - iJ\xi$ and scalar multiplication on \mathbb{C}^{2n} [ACD02]. As a consequence, the function $\hat{K} := \frac{1}{2}\hat{g}(Z, \bar{Z}) = \hat{g}(\xi, \xi)$ is a globally defined Kähler potential of \hat{M} . Indeed, if $p \in U$ then $\hat{K}(p) = \hat{g}_p(\xi, \xi) = \hat{\gamma}(\Phi(p), \Phi(p))$, where $\hat{\gamma} = \frac{\sqrt{-1}}{2}\hat{\Omega}(\cdot, \cdot)$ for the standard symplectic form $\hat{\Omega}$ of \mathbb{C}^{2n+2} .

If the vector field Z generates a principal \mathbb{C}^* -action then the symmetric tensor field

$$g' := -\frac{\hat{g}}{\hat{K}} + \frac{(\partial\hat{K})(\bar{\partial}\hat{K})}{\hat{K}^2} \quad (1.1)$$

induces a Kähler metric \bar{g} on the quotient manifold $\bar{M} := \hat{M}/\mathbb{C}^*$, compare [CDS16, Proposition 2]. It follows that $\pi^*\bar{g} = g'$ and $\pi^*\bar{\omega} = \frac{i}{2}\partial\bar{\partial}\log|\hat{K}|$, where $\bar{\omega} = \bar{g}(\cdot, J\cdot)$ is the Kähler form of \bar{M} . Set $\mathcal{D} := \text{span}\{\xi, J\xi\}$. Note that if $\hat{K} > 0$, then the signature of \bar{g} is minus the signature of $\hat{g}|_{\mathcal{D}^\perp}$, whereas if $\hat{K} < 0$ then the signature of \bar{g} agrees with the signature of $\hat{g}|_{\mathcal{D}^\perp}$.

Definition 1.8. The quotient (\bar{M}, \bar{g}) is called a projective special Kähler manifold.

Remark 1.9. Let $\Phi = (Z, W) : M \rightarrow \mathbb{C}^{2n+2}$ be an equivariant Kählerian Lagrangian immersion such that (Z, W) is a conjugate pair of special holomorphic coordinates. Identify $M \cong Z(M) \subset \mathbb{C}^{n+1}$. Then the prepotential $F : M \rightarrow \mathbb{C}$ can be chosen to be homogeneous of degree 2 such that $\Phi = dF$.

2 Symplectic group actions

2.1 Linear representation of the central extension of the affine symplectic group

Let $G = \mathrm{Sp}(\mathbb{R}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{R})$ be the extension of the real Heisenberg group by the group of automorphisms $\mathrm{Sp}(\mathbb{R}^{2n})$. The complexification of G is the group $G_{\mathbb{C}} = \mathrm{Sp}(\mathbb{C}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{C})$ which contains G as a real subgroup. Given two elements $x = (X, s, v)$ and $x' = (X', s', v') \in G_{\mathbb{C}}$, where $X, X' \in \mathrm{Sp}(\mathbb{C}^{2n})$, $s, s' \in \mathbb{C} = Z(G)$, $v, v' \in \mathbb{C}^{2n}$, their product in $G_{\mathbb{C}}$ is given by

$$x \cdot x' = \left(XX', s + s' + \frac{1}{2}\Omega(v, Xv'), Xv' + v \right),$$

where Ω is the symplectic form on \mathbb{C}^{2n} .

The group $G_{\mathbb{C}}$ is a central extension of the group $\mathrm{Aff}_{\mathrm{Sp}(\mathbb{C}^{2n})}(\mathbb{C}^{2n})$ of affine transformations of \mathbb{C}^{2n} with linear part in $\mathrm{Sp}(\mathbb{C}^{2n})$. The two groups are related by the quotient homomorphism

$$G_{\mathbb{C}} \rightarrow \mathrm{Aff}_{\mathrm{Sp}(\mathbb{C}^{2n})}(\mathbb{C}^{2n}) = G_{\mathbb{C}}/Z(G_{\mathbb{C}}), \quad (X, s, v) \mapsto (X, v).$$

This induces an affine representation $\bar{\rho}$ of $G_{\mathbb{C}}$ on \mathbb{C}^{2n} with image $\mathrm{Aff}_{\mathrm{Sp}(\mathbb{C}^{2n})}(\mathbb{C}^{2n})$ whose restriction to the real group G has the image $\bar{\rho}(G) = \mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{R}^{2n})$. In the complex symplectic vector space \mathbb{C}^{2n} we use standard coordinates $(z^1, \dots, z^n, w_1, \dots, w_n)$ in which the complex symplectic form is $\Omega = \sum dz^i \wedge dw_i$.

We will now show that $\bar{\rho}$ can be extended to a linear symplectic representation

$$\rho : G_{\mathbb{C}} \rightarrow \mathrm{Sp}(\mathbb{C}^{2n+2})$$

in the sense that the group $\rho(G_{\mathbb{C}})$ preserves the affine hyperplane $\{z^0 = 1\} \subset \mathbb{C}^{2n+2}$ with respect to standard coordinates $(z^0, w_0, z^1, \dots, z^n, w_1, \dots, w_n)$ on $\mathbb{C}^{2n+2} = \mathbb{C}^2 \oplus \mathbb{C}^{2n}$ and the distribution spanned by ∂_{w_0} inducing on the symplectic affine space $\{z^0 = 1\}/\langle \partial_{w_0} \rangle \cong \mathbb{C}^{2n}$ the symplectic affine representation $\bar{\rho}$.

Remark 2.1. Notice that $\{z^0 = 1\}/\langle \partial_{w_0} \rangle$ is precisely the symplectic reduction of \mathbb{C}^{2n+2} with respect to the holomorphic Hamiltonian group action generated by the vector field ∂_{w_0} . The group $\rho(G_{\mathbb{C}}) \subset \mathrm{Sp}(\mathbb{C}^{2n+2})$ preserves the Hamiltonian z^0 of that action and, hence, ρ induces a symplectic affine representation on the reduced space. Similarly, we will consider the initial real symplectic affine space \mathbb{R}^{2n} as the symplectic reduction of the real symplectic vector space \mathbb{R}^{2n+2} in the context of the real group G .

Proposition 2.2. (i) *The map*

$$x = (X, s, v) \mapsto \rho(x) = \begin{pmatrix} 1 & 0 & 0 \\ -2s & 1 & \hat{v}^t \\ v & 0 & X \end{pmatrix}, \quad \hat{v} := X^t \Omega_0 v = \Omega_0 X^{-1} v,$$

where $\Omega_0 = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$ is the matrix representing the symplectic form on \mathbb{C}^{2n} , defines a faithful linear symplectic representation $\rho : G_{\mathbb{C}} \rightarrow \mathrm{Sp}(\mathbb{C}^{2n+2})$, which induces the affine symplectic representation $\bar{\rho} : G_{\mathbb{C}} \rightarrow \mathrm{Aff}_{\mathrm{Sp}(\mathbb{C}^{2n})}(\mathbb{C}^{2n})$ in the sense explained above.

(ii) *The image $\rho(G_{\mathbb{C}}) \subset \mathrm{Sp}(\mathbb{C}^{2n+2})$ consists of the transformations in $\mathrm{Sp}(\mathbb{C}^{2n+2})$ which preserve the hyperplane $\{z^0 = 1\} \subset \mathbb{C}^{2n+2}$ and the complex rank one distribution $\langle \partial_{w_0} \rangle$. The image $\rho(G) \subset \mathrm{Sp}(\mathbb{R}^{2n+2}) \subset \mathrm{Sp}(\mathbb{C}^{2n+2})$ is the group that additionally preserves the real structure of \mathbb{C}^{2n+2} .*

Proof: We first observe that, for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, an element of $\mathrm{GL}(2n+2, \mathbb{K})$ preserves $\{z^0 = 1\}$ and $\langle \partial_{w_0} \rangle$ if and only if it is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ -2s & c & w^t \\ v & 0 & X \end{pmatrix},$$

where $s \in \mathbb{K}$, $0 \neq c \in \mathbb{K}$, $v, w \in \mathbb{K}^{2n}$, and $X \in \mathrm{GL}(2n, \mathbb{K})$. One then checks that such a transformation is symplectic if and only if $X \in \mathrm{Sp}(\mathbb{K}^{2n})$, $c = 1$, and $w = \hat{v}$. Clearly an element in $\mathrm{GL}(2n, \mathbb{K})$ preserves the real structure of \mathbb{C}^{2n} if and only if $\mathbb{K} = \mathbb{R}$. This proves (ii) and shows that the linear transformation $\rho(x)$ induces the affine transformation $\bar{\rho}(x) \in \mathrm{Aff}_{\mathrm{Sp}(\mathbb{C}^{2n})}(\mathbb{C}^{2n})$ for all $x \in G_{\mathbb{C}}$.

To check that ρ is a representation we put $\mu(x) := -2s$, $\gamma(x) := \hat{v} = X^t \Omega_0 v$. Then we compute

$$\mu(xx') = \mu(x) + \mu(x') - \omega(v, Xv') = \mu(x) + \mu(x') + \hat{v}^t v',$$

which coincides with the matrix element of $\rho(x)\rho(x')$ in the second row and first column. Next we compute the column vector

$$\gamma(xx') = (XX')^t \Omega_0 (v + Xv') = (X')^t (\gamma(x) + \Omega_0 v') = (X')^t \gamma(x) + \gamma(x'),$$

the entries of which coincide with the last $2n$ entries of the second row of $\rho(x)\rho(x')$. From these properties one sees immediately that ρ is a representation. It is obviously faithful, since X , s , and v appear in the matrix $\rho(x)$. \square

We define the subgroup $G_{\text{SK}} = \text{Sp}(\mathbb{R}^{2n}) \ltimes \text{Heis}_{2n+1}(\mathbb{C}) \subset G_{\mathbb{C}}$ to be the extension of the complex Heisenberg group by $\text{Sp}(\mathbb{R}^{2n})$. It contains the real group G as a subgroup and is a central extension of the affine group $\bar{\rho}(G_{\text{SK}}) = \text{Aff}_{\text{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$. We will show that G_{SK} acts on pairs (ϕ, F) of Kählerian Lagrangian immersions and prepotentials. This gives a transformation formula, see Eq. (2.3), of prepotentials of affine special Kähler manifolds which generalizes de Wit's formula (9) in [dW96] from the case of linear to affine symplectic transformations.

2.2 Representation of $G_{\mathbb{C}}$ on Lagrangian pairs

Let $\mathcal{L} \subset \mathbb{C}^{2n}$ be a Lagrangian submanifold and denote by η be the canonical $\text{Sp}(\mathbb{R}^{2n})$ -invariant 1-form given by $\eta_q := \Omega(q, \cdot)$, for $q \in \mathbb{C}^{2n}$. In Darboux coordinates $(z^1, \dots, z^n, w_1, \dots, w_n)$ we can write η as $\eta = \sum z^i dw_i - w_i dz^i$. Since $d\eta = 2\Omega$, this form is closed when restricted to \mathcal{L} .

Definition 2.3. *We call a function $f : \mathcal{L} \rightarrow \mathbb{C}$ a Lagrangian potential of \mathcal{L} if $df = -\eta|_{\mathcal{L}}$ and a pair (\mathcal{L}, f) a Lagrangian pair if $\mathcal{L} \subset \mathbb{C}^{2n}$ is a Lagrangian submanifold and f is a Lagrangian potential of \mathcal{L} .*

Proposition 2.4. *The group $G_{\mathbb{C}}$ acts on the set of pairs (\mathcal{L}, f) , where $\mathcal{L} \subset \mathbb{C}^{2n}$ is a Lagrangian submanifold and f is a holomorphic function on \mathcal{L} . The action is defined as follows. Given $x = (X, s, v) \in G_{\mathbb{C}}$ and a pair (\mathcal{L}, f) as above, we define*

$$x \cdot (\mathcal{L}, f) := (x\mathcal{L}, x \cdot f), \quad (2.1)$$

where $x\mathcal{L} := \bar{\rho}(x)\mathcal{L}$ and $x \cdot f$ is function on $x\mathcal{L}$ defined as

$$x \cdot f := f \circ x^{-1} + \Omega(\cdot, v) - 2s.$$

Moreover, if f is a Lagrangian potential of \mathcal{L} , then $x \cdot f$ is a Lagrangian potential of $x\mathcal{L}$.

Proof. For the neutral element $e \in G_{\mathbb{C}}$, clearly $e \cdot (\mathcal{L}, f) = (\mathcal{L}, f)$. Let $q \in \mathcal{L}$ and $x, x' \in G_{\mathbb{C}}$ with $x = (X, s, v)$ and $x' = (X', s', v')$. Then

$$\begin{aligned} x \cdot (x' \cdot f)(xx'q) &= (x' \cdot f)(x'q) + \Omega(xx'q, v) - 2s \\ &= f(q) + \Omega(x'q, v') + \Omega(xx'q, v) - 2s - 2s' \\ &= f(q) + \Omega(xx'q, v + Xv') - 2 \left(s + s' + \frac{1}{2} \Omega(v, Xv') \right) \\ &= (xx') \cdot f(xx'q), \end{aligned}$$

where we have used the second-to-last equation that

$$\begin{aligned}\Omega(x'q, v') &= \Omega(Xx'q, Xv') \\ &= \Omega(xx'q - v, Xv') \\ &= \Omega(xx'q, Xv') - \Omega(v, Xv').\end{aligned}$$

This shows that Eq. (2.1) defines an action of $G_{\mathbb{C}}$. Now let f be a Lagrangian potential of \mathcal{L} and set $\tilde{q} = xq$. Then

$$\begin{aligned}d_{\tilde{q}}(x \cdot f) &= d_q f \circ d(x^{-1}) + d_{\tilde{q}}(\Omega(\cdot, v)) \\ &= -\eta_q \circ X^{-1} + \Omega(\cdot, v) \\ &= -\Omega(q, X^{-1}\cdot) + \Omega(\cdot, v) \\ &= -\Omega(Xq + v, \cdot) = -\eta_{\tilde{q}},\end{aligned}$$

hence, $x \cdot f$ is a Lagrangian potential of $x \cdot \mathcal{L}$. \square

Definition 2.5. We call a Lagrangian submanifold $\mathcal{L} \subset \mathbb{C}^{2n}$ Kählerian if the Hermitian form $\gamma = \sqrt{-1}\Omega(\cdot, \tau\cdot)$ is non-degenerate when restricted to \mathcal{L} . Similarly, a Lagrangian pair (\mathcal{L}, f) is called Kählerian if \mathcal{L} is Kählerian.

Lemma 2.6. A Lagrangian submanifold $\mathcal{L} \subset \mathbb{C}^{2n}$ is Kählerian if and only if \mathcal{L} is totally complex, i.e., $T_q\mathcal{L} \cap \tau T_q\mathcal{L} = \{0\}$ for all $q \in \mathcal{L}$.

Proof. Since the inclusion $\iota : \mathcal{L} \rightarrow \mathbb{C}^{2n}$ is a holomorphic Lagrangian immersion, the statement follows from Prop. 1.3. \square

Corollary 2.7. The group $G_{\text{SK}} \subset G_{\mathbb{C}}$ acts on the set of Kählerian Lagrangian pairs.

Proof. The group G_{SK} acts on \mathbb{C}^{2n} as the group $\bar{\rho}(G_{\text{SK}}) = \text{Aff}_{\text{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$ which is the affine linear group that leaves invariant the complex symplectic form Ω and the real structure τ and, hence, also the Hermitian form $\gamma = \sqrt{-1}\Omega(\cdot, \tau\cdot)$. This shows that if (\mathcal{L}, f) is a Kählerian Lagrangian pair, then $x \cdot (\mathcal{L}, F) = (\bar{\rho}(x)\mathcal{L}, x \cdot f)$ is again a Kählerian Lagrangian pair for all $x \in G_{\text{SK}}$. \square

2.3 Representation of G_{SK} on special Kähler pairs

Definition 2.8. Let (M, J, g, ∇) be a connected affine special Kähler manifold of complex dimension n and let $U \subset M$ be an open subset of M . We call a pair (ϕ, F) a special Kähler pair of U if $\phi : U \rightarrow \mathbb{C}^{2n}$ is a Kählerian Lagrangian immersion inducing on U the restriction of the special Kähler structure (J, g, ∇) and F is a prepotential of ϕ . We denote the set of special Kähler pairs of U by $\mathcal{F}(U)$.

The following Lemma shows how the notions of prepotentials and Lagrangian potentials are related.

Lemma 2.9. *Let M be a special Kähler manifold together with a Kählerian Lagrangian embedding $\phi : M \rightarrow \phi(M) \subset \mathbb{C}^{2n}$ inducing the special Kähler structure of M . Set $\mathcal{L} := \phi(M)$ and $(z, w) := \phi$. Then a Lagrangian potential f of \mathcal{L} defines a prepotential F of ϕ via*

$$F = \frac{1}{2}(\phi^* f + z^t w),$$

and vice versa.

Proof. Let f be a Lagrangian potential of \mathcal{L} . We compute

$$\begin{aligned} dF &= \frac{1}{2}(\phi^* df + d(z^t w)) \\ &= \frac{1}{2}(-\phi^* \eta + w^t dz + z^t dw) \\ &= w^t dz. \end{aligned}$$

Since ϕ is a biholomorphism onto its image, the converse follows easily. \square

Proposition 2.10. *Let M be a connected affine special Kähler manifold of complex dimension n and $U \subset M$ an open subset such that $\mathcal{F}(U) \neq \emptyset$. Then the group G_{SK} acts simply transitively on $\mathcal{F}(U)$. The action is defined as follows. Given $x = (X, s, v) \in G_{\text{SK}}$ and a special Kähler pair (ϕ, F) of U , we define*

$$x \cdot (\phi, F) := (x\phi, x \cdot F), \tag{2.2}$$

where $x\phi := \bar{\rho}(x) \circ \phi$ and

$$x \cdot F := F - \frac{1}{2}z^t w + \frac{1}{2}z'^t w' + \frac{1}{2}(x\phi)^* \Omega(\cdot, v) - s, \tag{2.3}$$

where $(z, w) := \phi$ and $(z', w') := x\phi$ are the components of ϕ and $x\phi$, respectively.

Proof. We begin by showing that eq. (2.2) defines a G_{SK} -action on $\mathcal{F}(U)$. Clearly, the neutral element of G_{SK} acts trivially. We can locally rewrite eq. (2.3) as

$$\begin{aligned} 2x \cdot F - z'^t w' &= 2F - z^t w + (x\phi)^* \Omega(\cdot, v) - 2s \\ &= (x\phi)^*(f \circ x^{-1} + \Omega(\cdot, v) - 2s) \\ &= (x\phi)^*(x \cdot f) \end{aligned}$$

where f is the Lagrangian potential locally corresponding to F according to Lemma 2.9, i.e., $\phi^* f = 2F - z^t w$. This shows that $x \cdot F$ is a prepotential, namely the prepotential

locally corresponding to the Lagrangian potential $x \cdot f$ via $x\phi$. The remaining group action axioms now follow easily from Proposition 2.4.

It remains to show that the action is simply transitive. Let $(\phi, F), (\phi', F')$ be two special Kähler pairs of U . Since ϕ and ϕ' are both Kählerian Lagrangian immersions inducing same special Kähler structure, we know from Prop. 1.4 that there is an element $(X, v) \in \text{Aff}_{\text{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$ such that $\phi' = (X, v) \circ \phi$. Since prepotentials are unique up to a constant, there is an $s \in \mathbb{C}$ such that $x \cdot F = F'$ for $x = (X, s, v) \in G_{\text{SK}}$. This shows $x \cdot (\phi, F) = (\phi', F')$ and, hence, the transitivity. To see that the action is free, assume that $x \cdot (\phi, F) = (\phi, F)$ for some $x = (X, s, v) \in G_{\text{SK}}$. Then $X \circ \phi + v = \phi$. Differentiating and taking the real part gives $(X - \mathbf{1}_{2n}) \circ \text{Re } d\phi = 0$. Since ϕ is Kählerian, $\text{Re } \phi$ is an immersion and this implies $X = \mathbf{1}_{2n}$. But then from $X \circ \phi + v = \phi$ it also follows that $v = 0$. Finally, $x \cdot (\phi, F) = (\phi, F - s)$ implies $s = 0$ and, hence, x is the identity of G_{SK} . \square

Corollary 2.11. *Under the assumptions of Prop. 2.10, the subgroup $\text{Sp}(\mathbb{R}^{2n}) \subset G_{\text{SK}}$ acts by*

$$x \cdot (\phi, F) = \left(\phi' = x\phi, F' = x \cdot F = F - \frac{1}{2}z^t w + \frac{1}{2}z^t w' \right)$$

on the set of special Kähler pairs (ϕ, F) . In particular, in the case of conical affine special Kähler manifolds, $\text{Sp}(\mathbb{R}^{2n})$ acts on the set of homogeneous prepotentials of degree 2.

Remark 2.12. By Corollary 2.11, the function $F - \frac{1}{2}z^t w$ is invariant under the above action of $\text{Sp}(\mathbb{R}^{2n})$ in the sense that

$$F' - \frac{1}{2}z^t w' = F - \frac{1}{2}z^t w. \quad (2.4)$$

This is precisely the statement of de Wit, see eq. (10) in [dW96], that $F - \frac{1}{2}z^t w$ transforms as a symplectic function under linear symplectic transformations.

In terms of the Lagrangian potentials f and f' corresponding to F and F' , eq. (2.4) is equivalent to

$$f \circ \phi = f' \circ \phi'.$$

3 Conification of Lagrangian submanifolds

The aim is to associate (under some assumptions) a Lagrangian cone $\hat{\mathcal{L}} \subset \mathbb{C}^{2n+2}$ with a Lagrangian submanifold $\mathcal{L} \subset \mathbb{C}^{2n}$, and vice versa.

Fix a linear symplectic splitting $\mathbb{C}^{2n+2} = \mathbb{C}^2 \times \mathbb{C}^{2n}$ of the symplectic vector space \mathbb{C}^{2n+2} with its standard symplectic form $\hat{\Omega}$ and linear Darboux coordinates z^0, w_0 in \mathbb{C}^2

such that the symplectic form on \mathbb{C}^2 is given by $dz^0 \wedge dw_0$. Then the symplectic vector space \mathbb{C}^{2n} with its standard symplectic form Ω is recovered as the symplectic reduction with respect to the Hamiltonian flow of the function z^0 as explained in Rem. 2.1. Let $\pi : \{z^0 = 1\} \rightarrow \{z^0 = 1\}/\langle \partial_{w_0} \rangle = \mathbb{C}^{2n}$ be the quotient map and $\iota : \{z^0 = 1\} \hookrightarrow \mathbb{C}^{2n+2}$ the inclusion.

In one direction, let \mathcal{L} be a Lagrangian submanifold of \mathbb{C}^{2n} . A submanifold $\hat{\mathcal{L}}_1 \subset \{z^0 = 1\} \subset \mathbb{C}^{2n+2}$ is called a *lift* of \mathcal{L} if the projection

$$\pi|_{\hat{\mathcal{L}}_1} : \hat{\mathcal{L}}_1 \rightarrow \mathcal{L}$$

is a diffeomorphism. Equivalently, a lift is a section over \mathcal{L} of the trivial \mathbb{C} -bundle $\pi : \{z^0 = 1\} \rightarrow \mathbb{C}^{2n}$. Hence, a lift $\hat{\mathcal{L}}_1$ is of the form $\hat{\mathcal{L}}_1 = \{(1, f(q), q) \mid q \in \mathcal{L}\}$ for a function $f : \mathcal{L} \rightarrow \mathbb{C}$.

Proposition 3.1. *Let $\hat{\mathcal{L}}_1$ be a lift of a Lagrangian submanifold $\mathcal{L} \subset \mathbb{C}^{2n}$ with respect to the function $f : \mathcal{L} \rightarrow \mathbb{C}$. Then the cone $\hat{\mathcal{L}} := \mathbb{C}^* \cdot \hat{\mathcal{L}}_1$ is Lagrangian if and only if f is a Lagrangian potential.*

Proof. By the above $\hat{\mathcal{L}}_1 = \{(1, f(q), q) \mid q \in \mathcal{L}\}$. To show that $\hat{\mathcal{L}} := \mathbb{C}^* \cdot \hat{\mathcal{L}}_1$ is Lagrangian it is sufficient to show that $\hat{\Omega}(p, \hat{X}_p) = 0$ for all $p \in \hat{\mathcal{L}}_1$ and $\hat{X}_p \in T_p \hat{\mathcal{L}}_1$. A tangent vector $\hat{X}_p \in T_p \hat{\mathcal{L}}_1$ is of the form $\hat{X}_p = df(X)\partial_{w_0} + X$ for $X \in T_q \mathcal{L}$ with $q = \pi(p)$. Then

$$\begin{aligned} \hat{\Omega}(p, \hat{X}_p) &= \hat{\Omega}(\partial_{z^0} + f(q)\partial_{w_0} + q, \hat{X}_p) \\ &= dz^0 \wedge dw_0 (\partial_{z^0} + f(q)\partial_{w_0}, df(X)\partial_{w_0}) + \Omega(q, X) \\ &= df(X) + \eta_q(X). \end{aligned}$$

Hence, $\hat{\mathcal{L}}$ is Lagrangian if and only if $df = -\eta|_{\mathcal{L}}$. \square

Definition 3.2. *Let $\hat{\mathcal{L}}_1$ be the lift of the Lagrangian pair (\mathcal{L}, f) . We call the Lagrangian cone $\text{con}(\mathcal{L}, f) := \mathbb{C}^* \cdot \hat{\mathcal{L}}_1$ the conification of (\mathcal{L}, f) .*

Conversely, let $\hat{\mathcal{L}} \subset \mathbb{C}^{2n+2}$ be a Lagrangian cone such that the submanifold $\hat{\mathcal{L}}_1 := \hat{\mathcal{L}} \cap \{z^0 = 1\}$ is transverse to the Hamiltonian vector field ∂_{w_0} and each integral curve intersects $\hat{\mathcal{L}}_1$ at most once. We will call Lagrangian cones with this property *regular*. Then we define $\mathcal{L} \subset \mathbb{C}^{2n}$ as the image of $\hat{\mathcal{L}}_1$ under the quotient map $\pi : \{z^0 = 1\} \rightarrow \{z^0 = 1\}/\langle \partial_{w_0} \rangle = \mathbb{C}^{2n}$. Since the pullback $\pi^*\Omega$ of the symplectic form Ω on \mathbb{C}^{2n} is given by $\pi^*\Omega = \iota^*\hat{\Omega}$, it follows that \mathcal{L} is Lagrangian. By the regularity, the function $f := w_0 \circ (\pi|_{\hat{\mathcal{L}}_1})^{-1}$ is a well-defined function on \mathcal{L} and $\hat{\mathcal{L}}_1$ is of the form $\hat{\mathcal{L}}_1 = \{(1, f(q), q) \mid q \in \mathcal{L}\}$. In particular, $\hat{\mathcal{L}}_1$ is the lift of \mathcal{L} with respect to the function f .

Definition 3.3. We call the pair $\text{red}(\hat{\mathcal{L}}) := (\mathcal{L}, f)$ the reduction of the regular Lagrangian cone $\hat{\mathcal{L}} \subset \mathbb{C}^{2n+2}$.

Proposition 3.4. With respect to a splitting $\mathbb{C}^{2n+2} = \mathbb{C}^2 \times \mathbb{C}^{2n}$ and linear Darboux coordinates z^0, w_0 of \mathbb{C}^2 , we obtain a one-to-one correspondence

$$\{\text{Lagrangian pairs } (\mathcal{L}, f) \text{ in } \mathbb{C}^{2n}\} \xleftrightarrow{1:1} \{\text{Regular Lagrangian cones in } \mathbb{C}^{2n+2}\}$$

given by conification and reduction.

Moreover, conification and reduction are equivariant with respect to the action of the group $G_{\mathbb{C}}$, i.e., $\text{con}(x \cdot (\mathcal{L}, f)) = \rho(x) \text{con}(\mathcal{L}, f)$ and $\text{red}(\rho(x)\hat{\mathcal{L}}) = x \cdot \text{red}(\hat{\mathcal{L}})$ for $x \in G_{\mathbb{C}}$.

Proof. Let $\hat{\mathcal{L}} \subset \mathbb{C}^{2n+2}$ be a regular Lagrangian cone. We have already seen that $\hat{\mathcal{L}}_1 = \hat{\mathcal{L}} \cap \{z^0 = 1\}$ is the same as the lift of the pair $(\mathcal{L}, f) := \text{red}(\hat{\mathcal{L}})$. Since the cone $\hat{\mathcal{L}} = \mathbb{C}^* \cdot \hat{\mathcal{L}}_1$ is Lagrangian, it follows from Prop. 3.1 that f is a Lagrangian potential and, hence, $\text{con}(\text{red}(\hat{\mathcal{L}})) = \hat{\mathcal{L}}$. Conversely, if (\mathcal{L}, f) is a Lagrangian pair and $\hat{\mathcal{L}}_1 \subset \{z^0 = 1\}$ is the lift of \mathcal{L} with respect to f , then $\text{con}(\mathcal{L}, f) = \mathbb{C}^* \cdot \hat{\mathcal{L}}_1$ is a regular Lagrangian cone by Prop. 3.1. Since $\text{con}(\mathcal{L}, f) \cap \{z^0 = 1\} = \hat{\mathcal{L}}_1$, it follows that $\text{red}(\text{con}(\mathcal{L}, f)) = (\mathcal{L}, f)$. This shows $\text{red} = \text{con}^{-1}$.

Now let $x = (X, s, v) \in G_{\mathbb{C}}$ and $\hat{\mathcal{L}}_1$ be the lift of a Lagrangian pair (\mathcal{L}, f) . Then

$$\begin{aligned} \rho(x)\hat{\mathcal{L}}_1 &= \rho(x)\{(1, f(q), q) \in \mathbb{C}^{2n+2} \mid q \in \mathcal{L}\} \\ &= \{(1, f(q) + \hat{v}^t q - 2s, xq) \in \mathbb{C}^{2n+2} \mid q \in \mathcal{L}\} \\ &= \{(1, f(q) + \Omega(xq, v) - 2s, xq) \in \mathbb{C}^{2n+2} \mid q \in \mathcal{L}\} \\ &= \{(1, f(x^{-1}q') + \Omega(q', v) - 2s, q') \in \mathbb{C}^{2n+2} \mid q' \in x\mathcal{L}\} \\ &= \{(1, x \cdot f(q'), q') \in \mathbb{C}^{2n+2} \mid q' \in x\mathcal{L}\}. \end{aligned}$$

This shows that $\rho(x)\hat{\mathcal{L}}_1$ is the lift of the Lagrangian pair $x \cdot (\mathcal{L}, f) = (x\mathcal{L}, x \cdot f)$. Since the action of $G_{\mathbb{C}}$ on \mathbb{C}^{2n+2} leaves level-sets of z^0 and the distribution spanned by ∂_{w_0} invariant, it follows that

$$\text{con}(x \cdot (\mathcal{L}, f)) = \mathbb{C}^* \cdot (\rho(x)\hat{\mathcal{L}}_1) = \rho(x)(\mathbb{C}^* \cdot \hat{\mathcal{L}}_1) = \rho(x) \text{con}(\mathcal{L}, \hat{w}_0).$$

The equivariance of red follows immediately from $\text{red} = \text{con}^{-1}$. \square

Proposition 3.5. Let (\mathcal{L}, f) be a Lagrangian pair such that \mathcal{L} is Kählerian. If there is a point $q \in \mathcal{L}$ such that q is real and $f(q) \notin \mathbb{R}$, then there is an open neighborhood $U \subset \mathcal{L}$ of q such that the Lagrangian cone $\hat{U} := \text{con}(U, f) \subset \hat{\mathcal{L}} := \text{con}(\mathcal{L}, f)$ is Kählerian.

Proof. Let $q \in \mathcal{L}$ be real such that $f(q) \notin \mathbb{R}$ and choose an arbitrary $\hat{v} \in T_p \hat{\mathcal{L}} \cap \tau T_p \hat{\mathcal{L}}$ for $p = (1, f(q), q) \in \hat{\mathcal{L}}$. Since $T_p \hat{\mathcal{L}} = \text{span}_{\mathbb{C}}(p) \oplus T_q \mathcal{L}$, we have $\hat{v} = \lambda(1, f(q), q) + (0, df(v), v)$ for $\lambda \in \mathbb{C}$ and $v \in T_q \mathcal{L}$. The condition $\hat{v} - \tau \hat{v} = 0$ gives three equations

$$\begin{aligned} 0 &= \lambda - \bar{\lambda}, \\ 0 &= \lambda f(q) - \overline{\lambda f(q)} + df(v) - \overline{df(v)}, \\ 0 &= \lambda q - \overline{\lambda q} + v - \bar{v}. \end{aligned}$$

From the first, we immediately see that $\lambda \in \mathbb{R}$. From the third we find $v - \bar{v} = \lambda(\bar{q} - q) = 0$ since q is a real point. But $v - \bar{v} = 0$ is only possible if $v = 0$ as \mathcal{L} is Kählerian. The second equation then implies $\lambda(f(q) - \overline{f(q)}) = 0$ which, as $f(q) \notin \mathbb{R}$, is only possible if $\lambda = 0$. Hence, $\hat{v} = 0$ and this shows $T_p \hat{\mathcal{L}} \cap \tau T_p \hat{\mathcal{L}} = 0$. Since $\hat{\mathcal{L}}$ is Lagrangian, this is equivalent to the Hermitian form $\hat{\gamma} = \hat{\Omega}(\cdot, \tau \cdot)$ being non-degenerate when restricted to $\hat{\mathcal{L}}$ at the point p . By continuity, it is then also non-degenerate on a neighborhood $\hat{U}_1 \subset \hat{\mathcal{L}}_1 = \hat{\mathcal{L}} \cap \{z^0 = 1\}$ of p . Non-degeneracy is invariant under multiplication by $z^0 \in \mathbb{C}^*$, which acts by homothety on the Hermitian form $\hat{\gamma}$. Therefore, $\hat{\gamma}|_{\hat{\mathcal{L}}}$ is non-degenerate on $\hat{U} := \mathbb{C}^* \cdot \hat{U}_1$ which is the conification of the Lagrangian pair (U, f) for $U = \pi(\hat{U}_1)$. \square

Proposition 3.6. *If (\mathcal{L}, f) is a Lagrangian pair and \mathcal{L} is Kählerian, then there is an open subset $U \subset \mathcal{L}$ and an element $x \in G_{\text{SK}}$ such that the cone $\text{con}(x \cdot (U, f))$ is Kählerian.*

Proof. Let (\mathcal{L}, f) be a Lagrangian pair such that \mathcal{L} is Kählerian. If \mathcal{L} does not have real points, set $\mathcal{L}' = \mathcal{L} - q$ for an arbitray $q \in \mathcal{L}$. Then $0 \in \mathcal{L}'$ is a real point and we can choose a Lagrangian potential f' such that $f'(0) \notin \mathbb{R}$. This determines an element $x \in G_{\text{SK}}$ such that $(\mathcal{L}', f') = x \cdot (\mathcal{L}, f)$. The statement now follows from Prop. 3.5. \square

4 Conification of affine special Kähler manifolds

4.1 Conification of special Kähler pairs

Since special Kähler pairs locally correspond to Lagrangian pairs we can use the results from the previous chapter to give a conification procedure for special Kähler pairs.

Proposition 4.1. *Let (ϕ, F) be a special Kähler pair of an affine special Kähler manifold M and denote by $(z, w) := \phi$ the components of ϕ as before. Set $\hat{M} := \mathbb{C}^* \times M = \{(z^0, p) \in \mathbb{C}^* \times M\}$ with \mathbb{C}^* -action defined by $\lambda \cdot (z^0, p) := (\lambda z^0, p)$. Then the map*

$$\begin{aligned} \Phi : \hat{M} &\rightarrow \mathbb{C}^{2n+2} \\ (z^0, p) &\mapsto z^0(1, (2F - z^t w)(p), \phi(p)) \end{aligned}$$

is a \mathbb{C}^ -equivariant Lagrangian immersion of \hat{M} .*

Proof. Consider open subsets \hat{U} of \hat{M} of the form $\hat{U} = \mathbb{C}^* \times U$ where $U \subset M$ is open such that $\phi|_U$ is an embedding. Let (\mathcal{L}, f) be the Lagrangian pair corresponding to $(\phi, F)|_U$ by Lemma 2.9. Then $\Phi(z^0, p) = z^0(1, f(\phi(p)), \phi(p))$ for all $(z^0, p) \in \hat{U}$, i.e., $\Phi(\hat{U}) = \text{con}(\mathcal{L}, f)$. This shows that Φ is a Lagrangian immersion. The equivariance is obvious. \square

Definition 4.2. Let (ϕ, F) be a special Kähler pair of an affine special Kähler manifold M . We call the complex manifold $\hat{M} = \mathbb{C}^* \times M$ together with the map $\Phi : \hat{M} \rightarrow \mathbb{C}^{2n+2}$ the conification of the special Kähler pair (ϕ, F) and we write $\Phi = \text{con}(\phi, F)$. We say that the special Kähler pair (ϕ, F) is non-degenerate if the immersion Φ is Kählerian and $\hat{\gamma}(\Phi, \Phi) \neq 0$.

Proposition 4.3. Let (ϕ, F) be a special Kähler pair of an affine special Kähler manifold M . Then conification is equivariant with respect to the action of G_{SK} in the sense that $\text{con}(x \cdot (\phi, F)) = \rho(x) \circ \text{con}(\phi, F)$ for $x \in G_{\text{SK}}$.

Proof. This follows since conification locally corresponds to the conification of Lagrangian pairs. \square

Theorem 4.4. Let (ϕ, F) be a non-degenerate special Kähler pair of an affine special Kähler manifold M . Then $\Phi = \text{con}(\phi, F)$ induces on \hat{M} the structure of a conical affine special Kähler manifold. This structure is independent of the representative of the equivalence class of (ϕ, F) in $\mathcal{F}(M)/G$.

Proof. Let Φ be the conification of a non-degenerate special Kähler pair (ϕ, F) . Then Φ is by definition a Kählerian Lagrangian immersion of \hat{M} inducing the special Kähler metric $\hat{g} = \text{Re } \Phi^*(\hat{\gamma})$. Since Φ is also equivariant with respect to the \mathbb{C}^* -action, it follows that the real part $\xi := \text{Re}(Z)$ of the vector field Z generating the \mathbb{C}^* action satisfies $\nabla \xi = D\xi = \text{id}$. Its length is given by

$$\hat{g}(\xi, \xi) = \hat{\gamma}(\Phi, \Phi) = |Z^0|^2 (\text{Im } f + K) \neq 0 \quad (4.1)$$

where $f = 2F - z^t w$ for $(z, w) := \phi$ and $K = \gamma(\phi, \phi)$. This shows that Φ induces on \hat{M} a conical affine special Kähler structure.

Let $(\phi', F') \in \mathcal{F}(M)$ with $\Phi' = \text{con}(\phi', F')$. Then $(\phi', F') = x \cdot (\phi, F)$ for a unique $x \in G_{\mathbb{C}}$ and by Proposition 4.3 $\Phi' = \rho(x) \circ \Phi$. Now Φ and Φ' induce the same conical affine Kähler structure on \hat{M} if and only if $\rho(x) \in \text{Sp}(\mathbb{R}^{2n+2})$ which is the case if and only if $x \in G$. \square

Proposition 4.5. Let (ϕ, F) be a special Kähler pair defined on $U \subset M$ and set $f = 2F - z^t w$ for $(z, w) := \phi$ and $K = \gamma(\phi, \phi)$. Then (ϕ, F) is non-degenerate if and only if $\text{Im } f + K \neq 0$ and $\bar{\omega} := \frac{i}{2} \partial \bar{\partial} \log |\text{Im } f + K|$ is non-degenerate.

Proof. This follows easily from eqs. (1.1) and (4.1) \square

Remark 4.6. A special Kähler domain $M \subset \mathbb{C}^n$ with coordinates z^1, \dots, z^n of \mathbb{C}^n and prepotential $F : M \rightarrow \mathbb{C}$ determines a special Kähler pair (ϕ, F) by setting $\phi = dF : M \rightarrow T^*\mathbb{C}^n = \mathbb{C}^{2n}$. Then the conification

$$\begin{aligned}\hat{M} &= \{(Z^0, Z^1, \dots, Z^n) \in \mathbb{C}^* \times \mathbb{C}^n \mid Z^i/Z^0 \in M, i = 1, \dots, n\}, \\ \Phi &= \text{con}(dF, F) : \hat{M} \rightarrow \mathbb{C}^{2n+2}\end{aligned}$$

is the graph of $d\hat{F}$, where \hat{F} is a holomorphic homogeneous function of degree 2 given by

$$\hat{F}(Z^0, \dots, Z^n) = (Z^0)^2 F\left(\frac{Z^1}{Z^0}, \dots, \frac{Z^n}{Z^0}\right).$$

The special Kähler pair (ϕ, F) is non-degenerate if and only if the matrix given by $\text{Im}\left(\frac{\partial^2 \hat{F}}{\partial Z^I \partial Z^J}\right)$ for $I, J = 0, \dots, n$ is invertible and

$$\begin{aligned}\hat{K}(Z^0, \dots, Z^n) &= \sum_{I=0}^n \text{Im}\left(\bar{Z}^I \frac{\partial F}{\partial Z^I}\right) \\ &= |Z^0|^2 (K(z^1, \dots, z^n) + \text{Im}(f(z^1, \dots, z^n)))\end{aligned}$$

is non-zero, where $z^i = Z^i/Z^0$, $f = 2F - \sum_{i=1}^n z^i \frac{\partial F}{\partial z^i}$, and $K = \sum_{i=1}^n \text{Im}(\bar{z}^i \frac{\partial F}{\partial z^i})$. Note that in this case, $\hat{K} = \hat{\gamma}(\Phi, \Phi)$ is the Kähler potential, $\text{Im}\left(\frac{\partial^2 \hat{F}}{\partial Z^I \partial Z^J}\right) = \frac{\partial^2 \hat{K}}{\partial Z^I \partial Z^J}$ are the components of the metric, and

$$K'(z^1, \dots, z^n) := -\log |K(z^1, \dots, z^n) + \text{Im}(f(z^1, \dots, z^n))| = -\log |\hat{K}(1, z^1, \dots, z^n)|$$

gives a Kähler potential of the projective special Kähler metric \bar{g} defined on $\hat{M}/\mathbb{C}^* \cong M$.

Example 4.7. Let $M \subset \mathbb{C}^n$ with standard coordinates (z^1, \dots, z^n) be an affine special Kähler domain with a holomorphic prepotential $F = \sum_{i,j=1}^n a_{ij} z^i z^j + \frac{1}{2}C$ for $a_{ij}, C \in \mathbb{C}$. Note how the parameter C does not affect the affine special Kähler geometry of M . We have $K = 2 \sum_{i,j=1}^n z^i \bar{z}^j \text{Im}(a_{ij})$ and $f = 2F - \sum_{i=1}^n z^i \frac{\partial F}{\partial z^i} = C$. Consider the conification of the special Kähler pair (dF, F) . We denote by (Z^0, \dots, Z^n) the homogeneous coordinates on $\mathbb{C}^* \times M$. The holomorphic prepotential \hat{F} of the conification is then given by $\hat{F}(Z^0, Z) = \sum_{i,j=1}^n a_{ij} Z^i Z^j + \frac{1}{2}C(Z^0)^2$. The matrix

$$\left(\text{Im} \frac{\partial^2 \hat{F}}{\partial Z^I \partial Z^J}\right)_{I,J=0,\dots,n} = \begin{pmatrix} \text{Im} C & 0 \\ 0 & 2(\text{Im} a_{ij})_{i,j=1,\dots,n} \end{pmatrix}$$

is non-degenerate if and only if $c := \text{Im} C \neq 0$. Thus (dF, F) is non-degenerate if and only if $c \neq 0$ and $K + \text{Im} f = K + c \neq 0$ on M .

Assuming (dF, F) is non-degenerate, then the projective special Kähler metric \bar{g} on M is given by

$$\begin{aligned}\bar{g} &= - \sum_{i,j=1}^n \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log |K + c| dz^i d\bar{z}^j \\ &= - \frac{1}{K + c} g + \frac{1}{(K + c)^2} (\partial K)(\bar{\partial} K),\end{aligned}$$

where g is the affine special Kähler metric of M .

4.2 The ASK/PSK-correspondence

In this section we will give a global description of the conification procedure of the previous section and establish the ASK/PSK-correspondence which will assign a projective special Kähler manifold to any affine special Kähler manifold given a non-degenerate special Kähler pair. For this, we will prove that every affine special Kähler manifold admits a flat principal G_{SK} -bundle. Using this bundle, we show that if the holonomy of the flat connection is contained in the group $G \subset G_{\text{SK}}$, then the local conification of a non-degenerate special Kähler pair (ϕ, F) can be extended to the largest domain on which analytic continuation of (ϕ, F) is non-degenerate.

Lemma 4.8. *Let G be a Lie group and \mathcal{F} be a presheaf on a manifold M with values in the category of principal homogeneous G -spaces. Then the disjoint union of stalks $P := \dot{\cup}_{p \in M} \mathcal{F}_p$ carries the structure of a principal G -bundle $\pi : P \rightarrow M$ with a flat connection 1-form θ such that the horizontal sections of P over U are given by $\mathcal{F}(U)$.*

Proof. Fix a point $p \in M$ and a neighborhood U of p such that $\mathcal{F}(U) \neq \emptyset$. We claim that evaluation of sections, i.e., the map taking a section $s \in \mathcal{F}(U)$ to its germ $[s]_p \in \mathcal{F}_p$, is a bijection. Let $[s_V]_p \in \mathcal{F}_p$, where $s_V \in \mathcal{F}(V)$ for some open neighborhood V of p . Without loss of generality, we can assume $V \subset U$. If $s \in \mathcal{F}(U)$ is a section, then there is a unique $x \in G$ such that $x \cdot s|_V = s_V$. Hence, $x \cdot s$ and s_V define the same germ at p . This shows the surjectivity. Now let $s, \tilde{s} = x \cdot s \in \mathcal{F}(U)$ such that $[s]_p = [\tilde{s}]_p$. Then there is a neighborhood $V \subset U$ of p such that $s|_V = \tilde{s}|_V$. Since $s = x \cdot \tilde{s}$ for a unique $x \in G$ this implies $x = e$, where $e \in G$ is the neutral element, showing the injectivity. It follows that the stalks of \mathcal{F} are also principal homogeneous G -spaces with G -action defined as $x \cdot [s]_p = [x \cdot s]_p$.

Set $P = \dot{\cup}_{p \in M} \mathcal{F}_p$ and $\pi : P \rightarrow M$, $[s]_p \mapsto p$. We can now consider a section $s \in \mathcal{F}(U)$ as a section of P over U by setting $s(p) := [s]_p$. Choose an open covering $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ such that $\mathcal{F}(U_\alpha) \neq \emptyset$ and for each U_α pick a section $s_\alpha \in \mathcal{F}(U_\alpha)$. Define G -equivariant

maps $\Psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$ such that $\Psi_\alpha(s_\alpha(p)) = (p, e)$. These maps are bijective by the first part of the proof. Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$ be a non-empty overlap. Then $\mathcal{F}(U_{\alpha\beta}) \neq \emptyset$ and by the simply transitive action of G on $\mathcal{F}(U_{\alpha\beta})$ there is a unique $x_{\alpha\beta} \in G$ such that $s_\alpha = x_{\alpha\beta}s_\beta$, showing that the transition maps

$$\Psi_{\alpha\beta}(p, g) := (\Psi_\beta \circ \Psi_\alpha^{-1})(p, g) = \Psi_\beta(g \cdot s_\alpha(p)) = \Psi_\beta(gx_{\alpha\beta} \cdot s_\beta(p)) = (p, gx_{\alpha\beta})$$

are smooth and the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \rightarrow G_{\text{SK}}$, $g_{\alpha\beta}(p) = x_{\alpha\beta}$ are constant. On a non-empty overlap $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ we have $s_\beta = x_{\beta\gamma} \cdot s_\gamma$ and $s_\alpha = x_{\alpha\beta} \cdot s_\beta = x_{\alpha\beta}x_{\beta\gamma} \cdot s_\gamma$. Hence, the transition functions satisfy $g_{\alpha\gamma} = g_{\alpha\beta}g_{\beta\gamma}$. This shows that $\pi : P \rightarrow M$ is a principal G_{SK} bundle, see, e.g., [KN63, Chapter 1, Proposition 5.2]).

The transformation rule for local connection 1-forms $\theta_\alpha \in \Omega^1(U_\alpha, \text{Lie}(G_{\text{SK}}))$ is

$$\theta_\beta = \text{Ad}(g_{\alpha\beta}^{-1})\theta_\alpha + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$$

for transition functions $g_{\alpha\beta} : U_{\alpha\beta} \rightarrow G$. In our case, the transition functions $g_{\alpha\beta}(p) = x_{\alpha\beta}$ are constant. Thus we see that setting $\theta_\alpha = 0$ defines a flat connection 1-form θ on P .

In the above we have seen that a section $s \in \mathcal{F}(U)$ gives a local trivialization $\Psi : \pi^{-1}(U) \rightarrow U \times G$. A section \tilde{s} of $\pi^{-1}(U)$ is horizontal with respect to θ if and only if it is constant in this trivialization. Thus it is of the form $\tilde{s}(p) = [x \cdot s]_p$ for some $x \in G$. Under the identification $\mathcal{F}_p \cong \mathcal{F}(U)$, \tilde{s} thus corresponds to $x \cdot s \in \mathcal{F}(U)$, completing the proof. \square

Now let (M, J, g, ∇) be an affine special Kähler manifold of complex dimension n . Consider the map \mathcal{F} assigning to each open subset U of M the set $\mathcal{F}(U)$ of special Kähler pairs of U . The map \mathcal{F} is a sheaf with values in the category of G_{SK} -principal homogeneous spaces. The restriction map is given by $(\phi, F)|_V = (\phi|_V, F|_V)$. By Lemma 4.8 the sheaf \mathcal{F} thus defines a flat principal G_{SK} -bundle $\pi : P \rightarrow M$ with flat connection 1-form θ where $P = \dot{\cup}_{p \in M} \mathcal{F}_p$.

Definition 4.9. *We call the flat principal G_{SK} -bundle of germs of special Kähler pairs $\pi : P \rightarrow M$ the bundle of special Kähler pairs.*

Definition 4.10. (1) *We call a germ u in the fiber P_p non-degenerate if there is a non-degenerate special Kähler pair (ϕ, F) of an open neighborhood of p such that $[(\phi, F)]_p = u$. Note that every fiber contains at least one non-degenerate germ by Proposition 3.6.*

(2) *Let $u = [(\phi, F)]_p$ be a non-degenerate germ in the fiber P_p and (ϕ, F) be a non-degenerate special Kähler pair. Define $\text{dom}(u) \subset M$ to be the set of points in M*

that are connected to p via a path γ along which the analytic continuation of (ϕ, F) is non-degenerate. We call $\text{dom}(u)$ the domain of non-degeneracy of u .

Note that analytic continuation of a special Kähler pair (ϕ, F) defined on a neighborhood of a point p along a path γ corresponds to parallel transport of the germ $u = [(\phi, F)]_p \in P_p$ along γ . Therefore, if u is non-degenerate, then a point $p' \in M$ is in $\text{dom}(u)$ if and only if there is a horizontal path from u to the fiber over p' such that all points of γ are non-degenerate.

Theorem 4.11. *Let M be a connected affine special Kähler manifold of complex dimension n and $\pi : P \rightarrow M$ be the bundle of special Kähler germs of M with its flat connection 1-form θ . Assume that $\text{Hol}(\theta) \subset G$. Let $u \in P$ be a non-degenerate point. Then the manifold $\hat{M}_u := \mathbb{C}^* \times \text{dom}(u)$ carries a conical affine special Kähler structure.*

Proof. Due to the condition on the holonomy, we can reduce the bundle $\pi : P \rightarrow M$ and the connection 1-form θ to a $\text{Hol}(\theta)$ -bundle

$$P(u) := \{u' \in P \mid \text{there is a } \theta\text{-horizontal path connecting } u \text{ and } u'\} \subset P.$$

First note that if $u' \in P(u)_{p'}$ is a non-degenerate germ in the fiber over p' , then all germs in the fiber are non-degenerate. Indeed, if $u'' \in P(u)_{p'}$, then $u'' = x \cdot u'$ for some $x \in \text{Hol}(\theta) \subset G$. Thus if (ϕ', F') is the non-degenerate special Kähler pair corresponding to u' then $\text{con}(x \cdot (\phi', F')) = \rho(x) \text{con}(\phi', F')$ is Kählerian since $\rho(x) \in \text{Sp}(\mathbb{R}^{2n})$ for all $x \in G$.

By the definition of $\text{dom}(u)$ the fibers of $P(u)|_{\text{dom}(u)}$ are all non-degenerate. Hence, we can find an open covering $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of $\text{dom}(u)$ and non-degenerate special Kähler pairs $(\phi_\alpha, F_\alpha) \in \mathcal{F}(U_\alpha)$ such that $[(\phi_\alpha, F_\alpha)]_p \in P(u)_p$ for all $p \in \text{dom}(u)$. This gives a covering $\hat{\mathcal{U}} = (\hat{U}_\alpha) := (\mathbb{C}^* \times U_\alpha)_{\alpha \in I}$ and conic Kählerian Lagrangian immersions $\Phi_\alpha = \text{con}(\phi_\alpha, F_\alpha) : \hat{U}_\alpha \rightarrow \mathbb{C}^{2n+2}$. The induced conical affine special Kähler structure on \hat{U}_α is independent of the choice of special Kähler pairs (ϕ_α, F_α) for each $\alpha \in I$ by Theorem 4.4 and agrees on overlaps, since the transition functions take values in $\text{Sp}(\mathbb{R}^{2n+2})$. This shows that the Φ_α induce a well-defined conical affine special Kähler structure on $\hat{M}_u = \mathbb{C}^* \times \text{dom}(u)$. \square

The \mathbb{C}^* -action on \hat{M}_u is principal. Hence, the quotient $\overline{M}_u = \hat{M}_u / \mathbb{C}^*$ is projective special Kähler with metric \bar{g}_u given by eq. (1.1). In particular, a Kähler potential of \bar{g}_u is given by $K'_u(p) := -\log |\hat{K}_u(1, p)|$ for $p \in \text{dom}(u)$.

Definition 4.12. *We call the map taking the affine special Kähler manifold (M, g) and a special Kähler germ u of M to the projective special Kähler manifold $(\overline{M}_u, \bar{g}_u)$ the ASK/PSK-correspondence.*

5 Completeness of Hessian metrics associated with a hyperbolic centroaffine hypersurface

In this section we will prove a completeness result for a one-parameter deformation of a positive definite Hessian metric with Hesse potential of the form $-\log h$ where h is a homogeneous function on a domain in \mathbb{R}^n . The latter metric is isometric to a product of the form $dr^2 + g_{\mathcal{H}}$, where $g_{\mathcal{H}}$ is proportional to the canonical metric on a centroaffine hypersurface $\mathcal{H} \subset \mathbb{R}^n$. This will be specialized in section 6 to the case of a cubic polynomial h and related to the r-map.

Let $U \subset \mathbb{R}^n$ be a domain such that $\mathbb{R}^{>0} \cdot U \subset U$ and let $h : U \rightarrow \mathbb{R}$ be a smooth positive homogeneous function of degree $k > 1$. Then $\mathcal{H} := \{h = 1\} \subset U$ is a smooth hypersurface and $U = \mathbb{R}^{>0} \cdot \mathcal{H}$. We assume that for $g_U := -\partial^2 h$ the metric $g_{\mathcal{H}} := \iota^* g_U$ is positive definite, where $\iota : \mathcal{H} \hookrightarrow U$ is the inclusion. The manifold $(\mathcal{H}, \frac{1}{k} g_{\mathcal{H}})$ is a hyperbolic centroaffine hypersurface in the sense of [CNS16].

Definition 5.1. *If h is a cubic homogeneous polynomial, then the manifold $(\mathcal{H}, g_{\mathcal{H}})$, defined as above, is called a projective special real manifold.*

Let $g' := -\partial^2 \log h = \frac{1}{h} g_U + \frac{1}{h^2} (dh)^2$. Denote by $\xi := x^i \partial_{x^i}$ the position vector field on U and by $E \subset TU$ the distribution of tangent spaces tangent to the level sets of h . Then TU decomposes into

$$TU = E \oplus \langle \xi \rangle. \quad (5.1)$$

Proposition 5.2. *The bilinear form $\check{g} := g_U - \frac{g_U(\xi, \cdot)^2}{g_U(\xi, \xi)}$ is positive semidefinite with kernel $\mathbb{R}\xi$, and we can write*

$$g_U = \check{g} - \frac{k-1}{kh} (dh)^2, \quad (5.2)$$

$$g' = \frac{1}{h} \check{g} + \frac{1}{kh^2} (dh)^2. \quad (5.3)$$

In particular, g_U is a Lorentzian metric, g' is a Riemannian metric on U , and the decomposition (5.1) is orthogonal with respect to g_U and g' .

Proof. By homogeneity of h , we have $dh(\xi) = kh$, $g_U(\xi, \cdot) = -(k-1)dh$ and $g_U(\xi, \xi) = -k(k-1)h$. This implies $\check{g}|_{E \times E} = g_U|_{E \times E} > 0$ and, hence, $\ker \check{g} = \mathbb{R}\xi$. Observing that $\frac{g_U(\xi, \cdot)^2}{g_U(\xi, \xi)} = -\frac{(k-1)}{kh} (dh)^2$ we obtain the formulas for g_U and g' . The distributions E and $\mathbb{R}\xi$ are obviously orthogonal with respect to \check{g} and $(dh)^2$ and, therefore, also with respect to g_U and g' which are linear combinations (with functions as coefficients) of these two tensors. \square

Definition 5.3. For $c \in \mathbb{R}$ we define the bilinear symmetric form

$$g'_c := -\partial^2 \log(h+c) = \frac{1}{h+c} g_U + \frac{1}{(h+c)^2} (dh)^2 \quad (5.4)$$

on the set

$$U_c := \begin{cases} \{x \in U \mid h(x) + c > 0\} & \text{for } c \leq 0, \\ \{x \in U \mid h(x) - c(k-1) > 0\} & \text{for } c > 0. \end{cases} \quad (5.5)$$

Proposition 5.4. (1) As in Proposition 5.2 we can write

$$g'_c = \frac{1}{h+c} \check{g} + \frac{h-c(k-1)}{kh} \frac{1}{(h+c)^2} (dh)^2. \quad (5.6)$$

(2) The metric g'_c is Riemannian on U_c .

(3) If $cc' > 0$, then (U_c, g'_c) is isometric to $(U_{c'}, g'_{c'})$.

Proof. (1) Equation (5.6) follows by inserting (5.2) into (5.4).

(2) The positive definiteness of g'_c follows directly from eq. (5.6) since the coefficients of the two terms are positive.

(3) Scalar multiplication by $\lambda > 0$ is a diffeomorphism on U . Let $\phi_\lambda : U_c \rightarrow U$ be the restriction. Using the homogeneity of h it easily follows that $\phi_\lambda(U_c) = U_{\lambda^k c}$.

Computing

$$\begin{aligned} \phi_\lambda^* g'_c &= \phi_\lambda^* \left(\frac{1}{h+c} g_U + \frac{1}{(h+c)^2} (dh)^2 \right) \\ &= \frac{1}{\lambda^k h + c} \lambda^k g_U + \frac{1}{(\lambda^k h + c)^2} \lambda^{2k} (dh)^2 \\ &= \frac{1}{h + \lambda^{-k} c} g_U + \frac{1}{(h + \lambda^{-k} c)^2} (dh)^2 \\ &= g'_{\lambda^{-k} c} \end{aligned}$$

we see that for $\lambda = (c'/c)^{1/k}$ we have $\phi_\lambda^*(g'_{c'}) = g'_c$. Hence, ϕ_λ gives the required isometry. \square

Theorem 5.5. Assume that g' is a complete metric on U and $c < 0$. Then g'_c is a complete metric on U_c .

Remark 5.6. The metric g' on U is complete if and only if $g_{\mathcal{H}}$ is complete, since (U, g') is isometric to $(\mathbb{R} \times \mathcal{H}, dr^2 + g_{\mathcal{H}})$.

Proof. Denote by $L(\gamma)$ and $L'_c(\gamma)$ the Riemannian length of a curve γ in U_c with respect to g' and g'_c , respectively. Note first that

$$\begin{aligned} g'_c - g' &= \left(\frac{1}{h+c} - \frac{1}{h} \right) \check{g} + \frac{1}{k} \left(\underbrace{\frac{h-c(k-1)}{h}}_{>1} \frac{1}{(h+c)^2} - \frac{1}{h^2} \right) (dh)^2 \\ &\geq \frac{1}{k} \left(\frac{1}{(h+c)^2} - \frac{1}{h^2} \right) (dh)^2 \geq 0 \end{aligned} \tag{5.7}$$

on U' . Hence, $L'_c(\gamma) \geq L(\gamma)$ for any curve γ in U_c .

Now, for some $T > 0$ let $\gamma : [0, T) \rightarrow U_c$ be a curve that is not contained in any compact set in U_c . If γ already has infinite length with respect to g' then it also has infinite length with respect to g'_c by eq. (5.7) and we are done.

Assume that $L(\gamma) < \infty$. Since g' is complete, there exists a compact set $K \subset U$ such that $\gamma \subset K$. Then $\{\gamma(t)\}$ has a limit point $p \in U$ that is not in U_c because otherwise $\overline{\{\gamma(t)\}} \subset U_c$ is a compact subset of U_c containing γ which is a contradiction. By continuity of h , this limit point lies in $\{h+c=0\}$. Hence, we can find a sequence $t_i \in [0, T)$, $t_i \rightarrow T$, such that $h(\gamma(t_i)) \rightarrow -c$.

Using the estimate

$$\begin{aligned} g'_c &= \frac{1}{h+c} \check{g} + \frac{h-c(k-1)}{kh} (d \log(h+c))^2 \\ &\geq \frac{1}{k} (d \log(h+c))^2 \end{aligned}$$

we find

$$\begin{aligned} L'_c(\gamma) &\geq \frac{1}{\sqrt{k}} \int_0^{t_i} \left| \frac{\partial}{\partial t} \log(h(\gamma(t)) + c) \right| dt \\ &\geq \frac{1}{\sqrt{k}} |\log(h(\gamma(t_i)) + c) - \log(h(\gamma(0)) + c)| \xrightarrow{t_i \rightarrow T} \infty \end{aligned}$$

Hence, any curve that is not contained in any compact set in U_c has infinite length with respect to g'_c . This is equivalent to the completeness of g'_c . \square

Remark 5.7. In the case of $c > 0$ the metric g'_c is not complete. One can find a curve with limit point in $\{h-c(k-1)=0\}$ that has finite length.

The following lemma will be used in the proof of Theorem 6.2 in the next section.

Lemma 5.8. *Let (M_1^n, g_1) be a complete Riemannian manifold. Then the metric*

$$g := \begin{pmatrix} g_1 & 0 \\ 0 & g_1 \end{pmatrix}$$

defined on the product $M = M_1 \times \mathbb{R}^n$ is complete.

Proof. This is a special case of [CHM12, Theorem 2]. \square

6 Application to the r-map

Let us first recall the definition of the supergravity r-map, following [CHM12].

Let $(\mathcal{H}, g_{\mathcal{H}})$ be a projective special real manifold defined by a homogeneous cubic polynomial h such that $\mathcal{H} \subset \{h = 1\}$. Set $U := \mathbb{R}^{>0} \cdot \mathcal{H}$ and define $g_U := -\partial^2 h$.

Define $\bar{M} = \mathbb{R}^n + iU \subset \mathbb{C}^n$ with coordinates $(z^i = y^i + \sqrt{-1}x^i)_{i=1,\dots,n} \in \mathbb{R}^n + iU$. We endow \bar{M} with a Kähler metric \bar{g} defined by the Kähler potential $K(z) = -\log h(x)$. As a matrix, this metric is given by $\bar{g} = \frac{1}{4} \begin{pmatrix} -\partial^2 \log h(x) & 0 \\ 0 & -\partial^2 \log h(x) \end{pmatrix}$. Take note that \bar{g} is positive definite and is the quotient metric of the conical affine special Kähler manifold $\mathbb{C}^* \times \bar{M}$ defined by the prepotential $\hat{F}(Z^0, \dots, Z^n) = -h(Z^1, \dots, Z^n)/Z^0$, where Z^0 is the coordinate in the \mathbb{C}^* -factor and $Z^i := Z^0 z^i$ for $i = 1, \dots, n$.

Definition 6.1. *The correspondence $(\mathcal{H}, g_{\mathcal{H}}) \mapsto (\bar{M}, \bar{g})$ is called the supergravity r-map.*

Related to the projective special real manifold $(\mathcal{H}, g_{\mathcal{H}})$ is the so-called conical affine special real manifold (U, g_U) . The rigid r-map assigns it to the affine special Kähler manifold $(M := \bar{M}, g)$ with metric g induced by the holomorphic prepotential $F(z) = -h(z)$. As a matrix with respect to the real coordinates (y^i, x^i) , this metric is given by $g = \begin{pmatrix} -\partial^2 h(x) & 0 \\ 0 & -\partial^2 h(x) \end{pmatrix}$.

Let U_c be defined as in eq. (5.5) and set $M_c = \mathbb{R}^n + iU_c \subset M$. Note that $M_0 = M$.

Theorem 6.2. *Applying the ASK/PSK-correspondence to the special Kähler pair*

$$(\phi_c, F_c) := (dF, F - 2\sqrt{-1}c) \quad (6.1)$$

defined on M_c with $F(z) = -h(z)$ and $c \in \mathbb{R}$ gives a projective special Kähler manifold (\bar{M}_c, \bar{g}_c) . If $c = 0$ we recover the supergravity r-map metric $\bar{g} = \bar{g}_0$. For any pair $c, c' \in \mathbb{R}$ such that $cc' > 0$ the obtained manifolds (\bar{M}_c, \bar{g}_c) and $(\bar{M}_{c'}, \bar{g}_{c'})$ are isometric. Moreover, if $c < 0$ and $(\mathcal{H}, g_{\mathcal{H}})$ is complete, then (\bar{M}_c, \bar{g}_c) is complete.

Proof. We will use Proposition 4.5 to show that $(dF, F - 2\sqrt{-1}c)$ is a non-degenerate special Kähler pair on M_c . Set $f(z) = 2(F - 2\sqrt{-1}c) - \sum_{i=1}^n z^i \frac{\partial F}{\partial z^i} = h(z) - 4\sqrt{-1}c$ and $K(z) = \sum_{i=1}^n \text{Im} \left(\bar{z}^i \frac{\partial F}{\partial z^i} \right)$. Using the identity

$$\text{Im} h(z) = \sum_{i=1}^n \text{Im} \left(\bar{z}^i \frac{\partial h}{\partial z^i} \right) - 4h(\text{Im } z),$$

we compute $\text{Im} f(z) + K(z) = -4(h(\text{Im } z) + c)$, which is nonzero on M_c . The function $K' := -\log |\text{Im} f + K| = -\log(4|h(\text{Im } z) + c|)$ defines a symmetric bilinear tensorfield

$\bar{g}_c = \sum_{i,j=1}^n \frac{\partial^2 K'}{\partial z^i \partial \bar{z}^j} dz^i d\bar{z}^j$ which, as a matrix, is of the form

$$\bar{g}_c = \frac{1}{4} \begin{pmatrix} -\partial^2 \log(h(x) + c) & 0 \\ 0 & -\partial^2 \log(h(x) + c) \end{pmatrix} = \frac{1}{4} \begin{pmatrix} g'_c(x) & 0 \\ 0 & g'_c(x) \end{pmatrix} \quad (6.2)$$

where ∂^2 is the real Hessian operator with respect to the real coordinates x and g'_c is the deformed metric of the previous section. Hence, we see that \bar{g}_c is positive definite by Proposition 5.4. This proves that $(dF, F - 2\sqrt{-1}c)$ is a non-degenerate special Kähler pair on M_c . In particular, \bar{g}_c is the projective special Kähler metric that is obtained via eq. (1.1) from the conical affine special Kähler metric \hat{g} on the cone $\mathbb{C}^* \times M_c$ with structure induced by $\text{con}(dF, F - 2\sqrt{-1}c)$. The supergravity r-map metric is recovered for $c = 0$. If $g_{\mathcal{H}}$ is complete and $c < 0$, then \bar{g}_c is complete by Theorem 5.5 and Lemma 5.8. It was proven in Proposition 5.4.(3) that scalar multiplication on U by $\lambda > 0$ induces a family of isometries $\phi_\lambda : (U_c, g'_c) \rightarrow (U_{\lambda^3 c}, g'_{\lambda^3 c})$. The differential defines a corresponding family of isometries $d\phi_\lambda : (\bar{M}_c = TU_c, \bar{g}_c) \rightarrow (\bar{M}_{\lambda^3 c} = TU_{\lambda^3 c}, \bar{g}_c)$. \square

Remark 6.3. The above proof shows that the family of Kähler manifolds (\bar{M}_c, \bar{g}_c) with \bar{g}_c given by eq. (6.2) is still defined when the projective special real manifold is replaced by a general hyperbolic centroaffine hypersurface associated with a homogeneous function \tilde{h} . The statements about completeness and isometries relating members of the family (\bar{M}_c, \bar{g}_c) remain true under the assumption that the centroaffine hypersurface is complete. However, the metrics \bar{g}_c are in general no longer projective special Kähler. In fact, the ASK/PSK-correspondence can not be applied, as the Kähler metric g obtained by the generalized r-map is in general no longer affine special Kähler. However, it turns out that the metrics g and \bar{g}_c are related by an elementary deformation, as defined in [MS14, Definition 1], with the symmetry replaced by the vector field $X = \text{grad } \tilde{h}(x)$ and $g_\alpha := g(X, \cdot)^2 + g(JX, \cdot)^2$. Indeed, the metric \bar{g}_c is of the form

$$\begin{aligned} \bar{g}_c &= f_1 g + f_2 g_\alpha \\ &= \frac{1}{4} \left(\frac{1}{\tilde{h} + c} g + \frac{1}{(\tilde{h} + c)^2} \left((d\tilde{h})^2 + (d\tilde{h} \circ J)^2 \right) \right), \end{aligned}$$

for $f_1 = \frac{1}{4(\tilde{h} + c)}$ and $f_2 = \frac{1}{4(\tilde{h} + c)^2}$. Its Kähler potential is $-\log(\tilde{h}(\text{Im } z) + c)$.

Example 6.4. Consider the complete projective special real manifold

$$\mathcal{H} = \{(x, y, z) \in \mathbb{R}^3 \mid x(xy - z^2) = 1, x > 0\}$$

and set $U = \mathbb{R}^{>0} \cdot \mathcal{H}$. Computing the scalar curvature of the metric $g'_c := -\partial^2 \log(h + c)$ for $h = x(xy - z^2)$ and $c \in \mathbb{R}$, for example with *Mathematica* [Wol] using the *RGTC* package [Bon03], gives

$$\text{scal}_{g'_c} = -\frac{3(h^2 - 11ch + 6c^2)}{4(h - 2c)^2}.$$

For $c = 0$ we find that $\text{scal}_{g'_c} = -\frac{3}{4}$ is constant. For $c \neq 0$ we can further substitute $u := h/c$ and find

$$\text{scal}_{g'_c} = -\frac{3(u^2 - 11u + 6)}{4(u - 2)^2}$$

which is constant only on the level sets of h . This shows that the deformed metrics are in general not isometric to the undeformed metric. Since the manifold (U_c, g'_c) is contained in $(\overline{M}_c, \overline{g}_c)$ as a totally geodesic submanifold, this shows that the deformed metrics are in general not isometric to the undeformed metric.

Example 6.5 (STU model). *Consider the complete projective special real manifold*

$$\mathcal{H} = \{(x, y, z) \in \mathbb{R}^3 \mid xyz = 1, x > 0, y > 0\}$$

and set $U = \mathbb{R}^{>0} \cdot \mathcal{H}$. Computing the scalar curvature of the metric $g'_c := -\partial^2 \log(h + c)$ for $h = xyz$ and $c \in \mathbb{R}$, gives

$$\text{scal}_{g'_c} = \frac{3c(4h^2 - 3ch + 2c^2)}{2h(h - 2c)^2}.$$

For $c = 0$ we find that $\text{scal}_{g'_c} = 0$ is constant. For $c \neq 0$ we can substitute $u := h/c$ and find

$$\text{scal}_{g'_c} = \frac{3(4u^2 - u + 2)}{2u(u - 2)^2}$$

which is constant only on the level sets of h .

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