EXPONENTIAL GROWTH OF SOME ITERATED MONODROMY GROUPS

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Abstract. Iterated monodromy groups of postcritically-finite rational maps form a rich class of self-similar groups with interesting properties. There are examples of such groups that have intermediate growth, as well as examples that have exponential growth. These groups arise from polynomials. We show exponential growth of the IMG of several non-polynomial maps. These include rational maps whose Julia set is the whole sphere, rational maps with Sierpiński carpet Julia set, and obstructed Thurston maps. Furthermore, we construct the first example of a non-renormalizable polynomial with a dendrite Julia set whose IMG has exponential growth.

Contents

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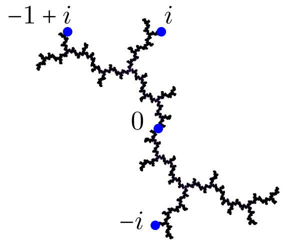
1. Introduction

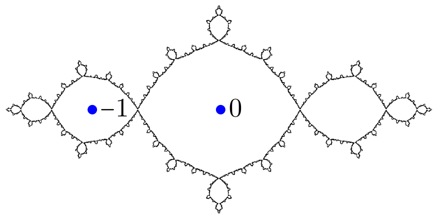
The *iterated monodromy group* (IMG) is a group that is defined in a natural way for certain dynamical systems, like iteration of a rational function on the Riemann sphere $\widehat{\mathbb{C}}$. It was defined by Nekrashevych, see for instance [BGN03], and independently by Kameyama in [Kam03], see also [Kam01]. Recently, the theory of IMG's, and its applications to the study of dynamical systems, has been developed rapidly, see in particular [Nek05, Nek11]. Many important problems in complex dynamics have been solved with the help of IMG's, such as the *Hubbard twisted rabbit problem* [BN06]. Iterated monodromy groups are in fact *self-similar groups*, that is, they act on a certain regular rooted tree in a "self-similar" fashion, and frequently have some interesting algebraic properties. Furthermore, their connection to dynamics provides new methods for the study of self-similar groups.

Iterated monodromy groups have been best understood for postcritically-finite polynomials (in dynamics a map is said to be *postcritically-finite* if each of its critical points has finite orbit). In particular, Nekrashevych gave a complete description of the IMG's of postcritically-finite polynomials in terms of *automata* generating them [Nek05, Nek09].

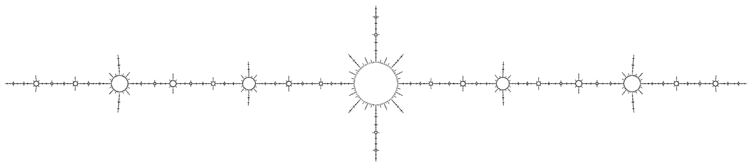
The study of algebraic properties of IMG's plays an important role in self-similar group theory as well as dynamics. For example, the *growth properties* of iterated monodromy groups have been investigated in the last decade. Recall that a finitely generated group has either *polynomial*, *intermediate*, or *exponential growth* depending on the volume growth of balls in the Cayley graph of the group, see Section 2.3. It has been known for a while that the iterated monodromy group of the polynomial $P_1(z) = z^2 + i$ is of intermediate growth, see [BP06]. Note that the *Julia set* of P_1 is a *dendrite*, see Figure 1a. On the other hand, the iterated monodromy group of the *Basilica map* $P_2(z) = z^2 - 1$ is of exponential growth; this was proved (for an isomorphic group) in [GŻ02]. Note that the closures of the Fatou components of P_2 containing 0 and -1 intersect, see Figure 1b. Having these two examples in mind, various people have been trying to establish connections between dynamical properties of a map and algebraic properties of its iterated monodromy group. For instance, it can be shown that any postcritically-finite polynomial of "Basilica type" has iterated monodromy group of exponential growth, see Theorem 2.5. At the same time, for the *airplane polynomial*, that is, the unique polynomial of the form $P_3(z) = z^2 + c_{air}$, where $c_{air} \neq 0$ is real and satisfies $P_3^3(c_{air}) = c_{air}$ ($c_{air} = -1.75488\ldots$), the growth of $\mathrm{IMG}(P_3)$ is unknown. Note that



(a) $P_1(z) = z^2 + i$.



(b) $P_2(z) = z^2 - 1$.



(c) $P_3(z) = z^2 + c_{air}$.

Figure 1. Julia sets of some quadratic polynomials.

two distinct bounded Fatou components of P_3 have disjoint closures, see Figure 1c.

For a postcritically-finite rational map f that is not a polynomial, the iterated monodromy group is much less understood. With the exception of *Lattès maps* (which have iterated monodromy group that is virtually $\mathbb{Z}^2$) and rational maps whose Julia set "contains a copy" of the Julia set of a polynomial (that is, they are *renormalizable* and their IMG's contain an IMG of a polynomial in a natural way), the growth of the IMG of a rational map f has previously been unknown.

The first result we show in this paper is the following.

Theorem 1.1. *There exists a postcritically-finite rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with Julia set equal to the whole sphere $\widehat{\mathbb{C}}$ and iterated monodromy group of exponential growth.*

We point out that this is the first non-Lattès example of a rational map whose Julia set is the whole Riemann sphere where the growth is known.

The key ingredient of this paper is to use geometric *tilings* of the sphere $\widehat{\mathbb{C}}$ associated with a rational (or more generally, branched covering) map f. We use them to describe the iterated monodromy action in a simple combinatorial way (via "rotation of tiles in flowers"). So these tilings may be viewed as representations of the *Schreier graphs* of the action of $\mathrm{IMG}(f)$ (on the levels of the dynamical preimage tree). To prove exponential growth of $\mathrm{IMG}(f)$, we find a free semigroup inside it using the properties of the constructed tilings. We also provide another proof, similar to the one in [GŻ02] for the Basilica group, which uses computations with the *wreath recursions* associated with the IMG, see Appendix A.2.

Next, we consider some rational maps whose Julia set is a Sierpiński carpet (that is, homeomorphic to the standard Sierpiński carpet). In particular, this means that distinct Fatou components have disjoint closures.

Theorem 1.2. *There exists a postcritically-finite rational map with Julia set equal to a Sierpiński carpet and iterated monodromy group of exponential growth.*

Again, this is the first example of such a rational map where the growth of the iterated monodromy group is known.

The methods that we develop here use rather the "combinatorial information" about the maps than their holomorphic nature. In particular, our methods apply to some *obstructed Thurston maps*, that is, postcritically-finite branched covering maps $f: S^2 \to S^2$ that are not "Thurston equivalent" to a rational map, see Section 2.1 for the definitions.

Theorem 1.3. *There exists an obstructed Thurston map $f: S^2 \to S^2$ with iterated monodromy group of exponential growth.*

Since it was shown that $P_1(z) = z^2 + i$ has iterated monodromy group of intermediate growth, it was conjectured that polynomials that are "similar to" P_1 have IMG of intermediate growth. However, the precise meaning of "similar to" is not clear, and indeed has changed over time. As already noted, the Julia set $\mathcal{J}$ of P_1 is a dendrite. Furthermore, every finite (that is, distinct from ∞) point in the postcritical set $\operatorname{post}(P_1) = \{i, -i, -1+i, \infty\}$ is a leaf of $\mathcal{J}$, that is, does not separate $\mathcal{J}$, see Figure 1a. Finally, P_1 is *not renormalizable*. Roughly speaking, this means that we cannot extract a simpler polynomial from any iterate P_1^n, see Section 10 for details. The following conjecture is quite natural and has been studied by the community for quite some time.

Conjecture. *Let P be a postcritically-finite non-renormalizable (quadratic) polynomial, such that its Julia set $\mathcal{J}_P$ is a dendrite. Then* $\operatorname{IMG}(P)$ *is of intermediate growth.*

This conjecture is supported by quite a few examples of polynomials: $P(z) = z^2 + i$ [BP06], $P(z) = z^3(-\frac{3}{2} + i\frac{\sqrt{3}}{2}) + 1$ [FG91], the quadratic polynomials with the kneading sequences $11(0)^\omega$ and $0(011)^\omega$ [DKR+12]. The hypothesis of non-renormalizability rules out the counterexamples that arise from *tuning*, a reverse operation to renormalization [McM94]. The tuning operation allows to construct examples of polynomials with dendrite Julia set whose IMG's are of exponential growth, see [Nek08, Section 5.5]. However, those maps will be renormalizable.

In this paper we show the following result.

Theorem 1.4. *There exists a postcritically-finite polynomial P with the following properties.*

(1) *The Julia set $\mathcal{J}$ of P is a dendrite.*
(2) *Every finite postcritical point of P is a leaf of $\mathcal{J}$.*
(3) *P is not renormalizable.*
(4) *The iterated monodromy group of P is of exponential growth.*

Theorem 1.4 and its proof show that the conjecture stated above is not valid for all polynomials, namely for the ones of sufficiently high degree. However it may be still true for quadratic polynomials. This shows that the question when the IMG of a polynomial is of intermediate growth is even more subtle than previously thought.

The maps in Theorems 1.1-1.4 are explicitly constructed in a combinatorial fashion. The proof of exponential growth relies strongly on this combinatorial description and is essentially the same for all considered maps. In each case it is easy to generalize the construction to obtain infinite families of maps with IMG's of exponential growth.

Based on our examples, we provide a quite general criterion for exponential growth of the IMG's of Thurston maps, see Theorem 6.3. This sufficient condition is not the most general that we can obtain, but rather was formulated to be easily applicable to various examples.

The paper is organized as follows. In Section 2 we review some standard material about Thurston maps, complex dynamics, growth of groups, and iterated monodromy groups. This section also contains an overview of the main properties of IMG's that are relevant for our paper, in particular, in the context of growth of groups. In Section 3 the map f_1 that serves as the example in Theorem 1.1 is constructed. In Section 4 we review the cell decompositions associated with f_1 and describe the iterated monodromy action in terms of them. In Section 5 we show exponential growth of $\mathrm{IMG}(f_1)$, that is, prove Theorem 1.1. In Section 6 we extract from the proof of Theorem 1.1 a general criterion for exponential growth of IMG's in our setting. In Section 7 we consider a family of postcritically-finite rational maps with Sierpiński carpet Julia set, in particular, we prove Theorem 1.2. In Section 8 we consider an obstructed Thurston map that proves Theorem 1.3. In Section 9 we construct the polynomial P from Theorem 1.4 and prove exponential growth of its iterated monodromy group. In Section 10 we review renormalization theory and prove that P is not renormalizable. In Appendix A.1 we review some standard material about actions on rooted trees and self-similar groups. In Appendix A.2 we give an alternative proof of exponential growth of $\mathrm{IMG}(f_1)$, show that it is a regular branch group, and conclude with some further properties.

1.1. **Notation.** We denote by $\mathbb{N}$ the set of positive integers, while $\mathbb{N}_0$ denotes the set of non-negative integers, that is, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let X be a topological space and $U \subset X$. We denote by $\overline{U}$, $\mathrm{int}(U)$, and ∂U the topological closure, the interior, and the boundary of the set U, respectively.

The n-th iterate of a map $f\colon S \to S$ on a set S is denoted by f^n for $n \in \mathbb{N}$. We set $f^0 \coloneqq \mathrm{id}_S$. Suppose a subset $U \subset S$ is given. We denote by $f^{-n}(U)$ the preimage of U under f^n, that is, $f^{-n}(U) = \{x \in S : f^n(x) \in U\}$. For simplicity, we denote $f^{-n}(y) \coloneqq f^{-n}(\{y\})$ for $y \in S$. Also we denote the restriction of f to U by $f|U$.

The identity element of a group is denoted by 1. The notation "$H < G$" means that H is a subgroup of G, as usual.

We consider right group actions. So if a group G acts on a set X, then the image of $x \in X$ under the action of an element $g \in G$ is denoted by x^g, and in a product $g_1 g_2$ the element g_1 acts first, that is, $x^{g_1 g_2} = (x^{g_1})^{g_2}$. We therefore write $[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2$ and $g_1^{g_2} = g_2^{-1} g_1 g_2$.

In a group G, $\mathrm{ord}(g)$ denotes the order (or period) of a group element $g \in G$, that is, the smallest positive integer n such that $g^n = 1$. If no such n exists, g is said to have infinite order and we set $\mathrm{ord}(g) = \infty$.

2. Background

2.1. **Thurston maps.** We provide a brief overview here, but refer the reader to [BM, Chapter 2] for details. Let $f\colon S^2 \to S^2$ be a *branched covering map* of the topological 2-sphere S^2. That is, f is a continuous and surjective map, that we can write locally around each point $p \in S^2$ as $z \mapsto z^d$ for some $d \in \mathbb{N}$ (depending on p) in orientation-preserving homeomorphic coordinates in domain and target. The integer $d \geq 1$ is uniquely determined by f and p, and called the *local degree* of the map f at p, denoted by $\deg(f, p)$.

A point $c \in S^2$ with $\deg(f, c) \geq 2$ is called a *critical point* of f. The image of a critical point is called a *critical value*. The set of all critical points of f is finite and denoted by $\mathrm{crit}(f)$. The union $\mathrm{post}(f) = \bigcup_{n=1}^{\infty} f^n(\mathrm{crit}(f))$ of the orbits of critical points is called the *postcritical set* of f. The map f is said to be *postcritically-finite* if its postcritical set $\mathrm{post}(f)$ is finite, in other words, the orbit of every critical point of f is finite.

Definition 2.1. A *Thurston map* is an orientation-preserving, postcritically-finite, branched covering $f\colon S^2 \to S^2$ of topological degree $d \geq 2$.

Natural examples are given by postcritically-finite rational maps on the Riemann sphere $\widehat{\mathbb{C}}$ and postcritically-finite polynomial maps on the complex plane $\mathbb{C}$. In this paper we consider Thurston maps defined by a subdivision rule on the sphere S^2. See for example the construction of the map f_1 in Section 3; in greater generality this may be found in [CFKP03] and [BM, Chapter 12].

The *ramification function* of a Thurston map $f : S^2 \to S^2$ is a function $\alpha_f\colon S^2 \to \mathbb{N} \cup \{\infty\}$ such that $\alpha_f(p)$ is the lowest common multiple of all local degrees $\deg(f^n, q)$, where $q \in f^{-n}(p)$ and $n \in \mathbb{N}$ are arbitrary. Thus $\alpha_f(p) = 1$ for all $p \in S^2 \setminus \mathrm{post}(f)$.

The *orbifold* associated with f is $\mathcal{O}_f \coloneqq (S^2, \alpha_f)$. The *Euler characteristic* of $\mathcal{O}_f$ is

$$\chi(\mathcal{O}_f) \coloneqq 2 - \sum_{p \in \operatorname{post}(f)} \left(1 - \frac{1}{\alpha_f(p)}\right).$$

It satisfies $\chi(\mathcal{O}_f) \leq 0$. We call $\mathcal{O}_f$ *hyperbolic* if $\chi(\mathcal{O}_f) < 0$, and *parabolic* if $\chi(\mathcal{O}_f) = 0$. A *Lattès map* is a rational Thurston map with parabolic orbifold that does not have periodic critical points (a point $p \in S^2$ is called *periodic* if $f^n(p) = p$ for an $n \in \mathbb{N}$).

Two Thurston maps $f \colon S^2 \to S^2$ and $g \colon \widetilde{S}^2 \to \widetilde{S}^2$, where $\widetilde{S}^2$ is another topological 2-sphere, are called *Thurston equivalent* if there are homeomorphisms $h_0, h_1 \colon S^2 \to \widetilde{S}^2$ that are isotopic rel. $\operatorname{post}(f)$ such that $h_0 \circ f = g \circ h_1$.

2.2. **Julia and Fatou sets.** The reader is referred to [Mil06] for background in complex dynamics.

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map. Then the *Julia set* $\mathcal{J}_f$ of f is the closure of the set of *repelling periodic points.* If f is postcritically-finite then $\mathcal{J}_f$ coincides with the set of limit points of the *full backwards orbit* $\bigcup_{n \geq 0} f^{-n}(t)$ of any point $t \in \widehat{\mathbb{C}} \setminus \operatorname{post}(f)$. The *Fatou set* of f is the set $\mathcal{F}_f = \widehat{\mathbb{C}} \setminus \mathcal{J}_f$. A *Fatou component* is a connected component of the Fatou set.

If f is a polynomial, then $\mathcal{J}_f$ coincides with the boundary of the set

$$\mathcal{K}_f \coloneqq \{z \in \mathbb{C} : \{f^n(z)\}_{n \geq 0} \text{ is bounded}\},$$

called the *filled Julia set.*

2.3. **Growth of groups.** Given a finitely generated group G with symmetric set of generators S one defines the *word length* of an element $g \in G$ with respect to S by

$$\ell_{G,S}(g) \coloneqq \min\{n \in \mathbb{N}_0 : g = s_1 \dots s_n, \text{ where } s_j \in S \text{ for } j = 1, \dots, n\};$$

and the *growth function* of G with respect to S by

$$\operatorname{growth}_{G,S}(n) \coloneqq \#\{g \in G : \ell_{G,S}(g) \leq n\}.$$

The group G is said to be of

(1) *polynomial growth*, if $\operatorname{growth}_{G,S}(n)$ is bounded above by a polynomial, that is, $\operatorname{growth}_{G,S}(n) \leq Cn^k$ for some constants $C > 0$, $k \in \mathbb{N}$;
(2) *exponential growth*, if $\operatorname{growth}_{G,S}$ is bounded below by an exponential function, that is, $\operatorname{growth}_{G,S}(n) \geq c\exp(\alpha n)$ for some constants $c, \alpha > 0$;
(3) *intermediate growth*, otherwise.

It can be shown that whether G has polynomial, intermediate, or exponential growth does not depend on the choice of the generating set S.

Milnor was the first to ask whether groups of intermediate growth exist [Mil68]. This was answered in the positive by Grigorchuk in [Gri83]. The example of a group of intermediate growth constructed by Grigorchuk in this paper, now called the *Grigorchuk group*, is a *self-similar group*. This means it acts on a binary rooted tree in a certain "self-similar" fashion, see Definition 11.2 and [Nek05, Chapter 1]. Self-similar groups often exhibit very interesting behavior and have been studied intensely in the last decades. For more information on the theory of growth of groups we refer the reader to the recent survey [Gri14] by Grigorchuk, see also [BGN03], a survey on self-similar groups and their properties.

2.4. **Iterated monodromy action and group.** Let $f\colon S^2 \to S^2$ be a Thurston map and $\operatorname{post}(f)$ be its postcritical set. Since $\operatorname{post}(f) \subset f^{-1}(\operatorname{post}(f))$, f induces a covering

$$f : \mathcal{M}_1 = S^2 \smallsetminus f^{-1}(\operatorname{post}(f)) \to \mathcal{M} = S^2 \smallsetminus \operatorname{post}(f).$$

Let d be the topological degree of f. Fix a basepoint $t \in \mathcal{M}$. We consider the backward orbit of t, meaning the formal disjoint union $\mathbf{T}_f = \bigsqcup_{n=0}^{\infty} f^{-n}(t)$. Then $\mathbf{T}_f$ has a natural structure of a d-ary rooted tree: the root, that is, the unique point on the level 0, is the basepoint $t \in f^{-0}(t) = \{t\}$, and a vertex $p \in f^{-n}(t)$ (of the n-th level) is connected by an edge to the vertex $f(p) \in f^{-(n-1)}(t)$ (of the $(n-1)$-th level), for $n \in \mathbb{N}$. The set $\mathbf{T}_f$ viewed as a rooted tree is called the *dynamical preimage tree* of f at the basepoint t.

The fundamental group $\pi_1(\mathcal{M}, t)$ acts naturally on the set of preimages $f^{-n}(t)$, $n \in \mathbb{N}_0$: the image $p^{[\gamma]}$ of a point $p \in f^{-n}(t)$ under the action of a loop $[\gamma] \in \pi_1(\mathcal{M}, t)$ is equal to the endpoint of the unique f^n-lift of γ that starts at p. It is easy to see that the action of the fundamental group on the vertices of the dynamical preimage tree $\mathbf{T}_f$ preserves the tree structure, that is, the fundamental group acts on $\mathbf{T}_f$ by automorphisms of the rooted tree. Thus, we have defined a group homomorphism

$$\phi_f : \pi_1(\mathcal{M}, t) \to \operatorname{Aut}(\mathbf{T}_f)$$

from the fundamental group of $\mathcal{M} = S^2 \smallsetminus \operatorname{post}(f)$ to the automorphism group $\operatorname{Aut}(\mathbf{T}_f)$ of the d-ary rooted tree $\mathbf{T}_f$.

Definition 2.2. The *iterated monodromy action* is the action of $\pi_1(\mathcal{M}, t)$ on the dynamical preimage tree $\mathbf{T}_f$. The quotient of $\pi_1(\mathcal{M}, t)$ by the kernel of its action on $\mathbf{T}_f$ is called the *iterated monodromy group* of f and is denoted by $\operatorname{IMG}(f)$. That is,

$$\operatorname{IMG}(f) = \pi_1(\mathcal{M}, t) / \operatorname{Ker}(\phi_f) \cong \phi_f(\pi_1(\mathcal{M}, t)).$$

2.5. **Selected properties of IMG's.** The iterated monodromy group of a Thurston map f is self-similar [Nek05, Proposition 5.2.2]. Furthermore, if f is a postcritically-finite rational map, then the iterated monodromy group (together with the associated wreath recursion, see Appendix A.1) contains all the "essential" information about the dynamics of the map f: one can reconstruct from $\mathrm{IMG}(f)$ the action of f on its Julia set $\mathcal{J}_f$. In this case the *limit space* of the iterated monodromy group is homeomorphic to the Julia set of the map [Nek05, Theorem 6.4.4]. Moreover, one can approximate the Julia set $\mathcal{J}_f$ by a certain sequence of finite graphs.

Definition 2.3. Let G be a group generated by a finite set S and acting on a set X. The *labeled Schreier graph* $\Gamma(G, S, X)$ is a labeled directed (multi)graph with the set of vertices X and the set of directed edges $X \times S$, where the edge (x, s) starts at x, ends at x^s, and is labeled by s, for each $x \in X$ and $s \in S$.

Nekrashevych showed that given a postcritically-finite rational map f, the sequence of the Schreier graphs of the action of $\mathrm{IMG}(f)$ on the n-th level of the dynamical preimage tree $\mathbf{T}_f$ converges to the Julia set of f, see [Nek05, Chapters 3 and 6].

It was observed that even very simple maps generate iterated monodromy groups with complicated structure and exotic properties which are hard to find among groups defined by more "classical" methods, see [BGN03]. For instance, $\mathrm{IMG}(z^2 + i)$ is a group of intermediate growth [BP06] and $\mathrm{IMG}(z^2 - 1)$ is an amenable group of exponential growth [GŻ02, BV05]. Below we list the most important results relevant to the growth theory of IMG's that are known at the moment.

Theorem 2.4 ([Nek11])**.** *Let f be a postcritically-finite rational map. Then* $\mathrm{IMG}(f)$ *does not contain a free group of rank* 2*.*

Theorem 2.5 ([Nek11])**.** *If a postcritically-finite polynomial f has two finite Fatou components with intersecting closures, then* $\mathrm{IMG}(f)$ *contains a free semigroup of rank 2 and is of exponential growth.*

Theorem 2.6 ([BKN10])**.** *If f is a postcritically-finite polynomial, then* $\mathrm{IMG}(f)$ *is amenable.*

For more information on the theory of iterated monodromy groups we refer the reader to [Nek05, Chapters 5–6] and [Nek11].

3. Construction of the map f_1

Here we describe the map $f_1 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that will serve as our main example. It is a postcritically-finite rational map, such that no critical point is periodic. This means that the Julia set of f_1 is the whole Riemann sphere $\widehat{\mathbb{C}}$ (see [Mil06, Corollary 16.5]). Since we are mainly interested in the combinatorial behavior of f_1, we will construct f_1 in a combinatorial fashion.

We consider two polyhedral surfaces Δ and Δ' constructed as follows. Let T be a Euclidean triangle with angles $\pi/2, \pi/3, \pi/6$. The surface

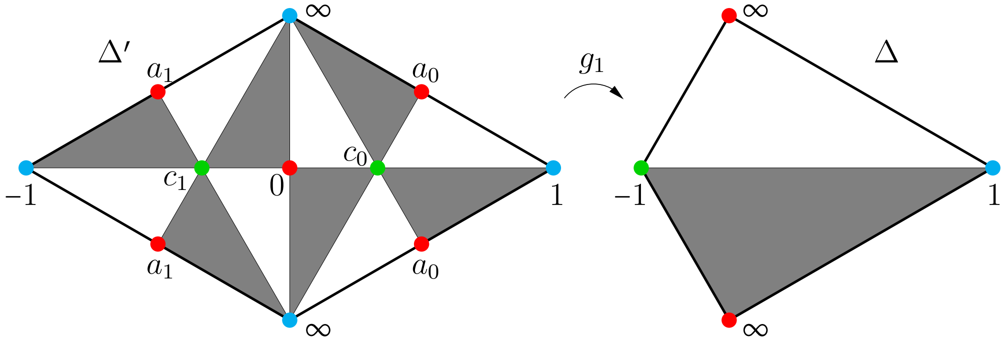


FIGURE 2. The polyhedral surfaces Δ' (left) and Δ (right) and the map $g_1: \Delta' \to \Delta$.

Δ is obtained by gluing two identical copies of T together along their boundaries. The surface Δ' is obtained by gluing two Euclidean triangles with angles $2\pi/3, \pi/6, \pi/6$ together along their boundaries. The two triangles of Δ, as well as Δ', are called the top and bottom faces. They correspond to the top and bottom triangles in Figure 2. The vertices of each such triangle are labeled $-1, 1, \infty$; they correspond to the vertices of Δ and Δ'. We color the top face of Δ white, and the bottom one black. Each face of Δ' can be divided into 6 triangles T' that are similar to T; we color them black and white as shown in Figure 2.

The map $g_1: \Delta' \to \Delta$ is now constructed as follows. Each of the 6 white triangles $T' \subset \Delta'$ is mapped by a similarity to the white face of Δ, that is, each vertex of T' is mapped to the one of the same angle. Similarly, each of the 6 black triangles $T' \subset \Delta'$ is mapped to the black face of Δ in the same fashion. To illustrate the mapping behavior, the vertices of Δ are colored red, blue, and green in Figure 2, and each vertex v of a triangle $T' \subset \Delta'$ is colored the same as $g_1(v)$.

It is a standard fact that every polyhedral surface can be equipped with a conformal structure in a natural way, see for example [Bea84, Section 3.3]. By the uniformization theorem this means that there are conformal maps $\varphi: \Delta \to \widehat{\mathbb{C}}$ and $\varphi': \Delta' \to \widehat{\mathbb{C}}$. To normalize these maps, we demand that the vertices of Δ and Δ' labeled $-1, 1, \infty$ are mapped to $-1, 1, \infty \in \widehat{\mathbb{C}}$ respectively. The symmetry then implies that the top face of Δ, as well as the top face of Δ', are mapped to the upper half-plane $\mathbb{H}^+$; the other face of Δ, and respectively of Δ', is mapped to the lower half-plane $\mathbb{H}^-$ (otherwise, the map $p \mapsto \overline{\varphi(\overline{p})}$ is distinct from φ violating its uniqueness[1]).

In the case at hand, we can actually construct the maps φ and φ' explicitly. Indeed, φ maps the white triangle in Δ to the upper half-plane by the Riemann map normalized so that the vertices labeled $-1, 1, \infty$ are mapped to the points $-1, 1, \infty \in \widehat{\mathbb{C}}$. Also φ maps the

[1]Here $\overline{p}$ denotes the point in Δ obtained from $p \in \Delta$ by reflection along the edge $[-1, 1]$, see Figure 2.

black triangle to the lower half-plane by a Riemann map with same normalization. Similarly, φ' is constructed by mapping the top and bottom face of Δ' to the upper and lower half-plane with the same normalization.

The map $f_1 \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by $f_1 \coloneqq \varphi \circ g_1 \circ (\varphi')^{-1}$ is now our desired map[2]. It is elementary to check that it is indeed a rational map.

By a slight abuse of notation, let us denote the images of the vertices in Δ' labeled by a_0, a_1, c_0, c_1 under φ' again by a_0, a_1, c_0, c_1. Note also that by symmetry, the vertex in Δ' labeled 0 is mapped to $0 \in \widehat{\mathbb{C}}$ by φ'.

Then the set of critical points of f_1 is $\operatorname{crit}(f_1) = \{a_0, a_1, c_0, c_1, 0, \infty\} \subset \widehat{\mathbb{C}}$ and the set of postcritical points is $\operatorname{post}(f_1) = \{-1, 1, \infty\} \subset \widehat{\mathbb{C}}$.

Recall that the *ramification portrait* of a Thurston map $f : S^2 \to S^2$ is a directed graph with the vertex set $V = \bigcup_{n \geq 0} f^n(\operatorname{crit}(f))$ consisting of the union of all orbits of all critical points of f. For $v, w \in V$ there is a directed edge from v to w in the graph if and only if $f(v) = w$. Moreover, if $\deg(f, v) = d_v \geq 2$, that is, v is a critical point of f of degree d_v, we label the directed edge from v to $w = f(v)$ by "$d_v : 1$". Put differently, the ramification portrait illustrates how critical points are mapped by the map f. For instance, the ramification portrait of f_1 is shown below.

$$\begin{tikzcd}
c_0 \arrow[dr, "3:1"] & & & a_0 \arrow[d, "2:1"] & \\
& -1 \arrow[r] & 1 \arrow[loop above] & \infty \arrow[l, "4:1"'] & 0 \arrow[l, "2:1"'] \\
c_1 \arrow[ur, "3:1"] & & & a_1 \arrow[u, "2:1"'] &
\end{tikzcd} \tag{3.1}$$

We conclude that the ramification function α_{f_1} of f_1 (see Section 2.1) is given by

$$\alpha_{f_1}(1) = 24, \quad \alpha_{f_1}(-1) = 3, \quad \alpha_{f_1}(\infty) = 2.$$

This means that the orbifold associated with f_1 is hyperbolic.

Remark 3.1. The map f_1 may be given explicitly in the following two forms:

$$f_1(z) = 2\left(\frac{3}{4}\right)^3 \frac{z^2 - 1}{z^2(z^2 - \frac{9}{8})^2} + 1 = \frac{2(z^2 - \frac{3}{4})^3}{z^2(z^2 - \frac{9}{8})^2} - 1. \tag{3.2}$$

It is elementary to check that the two expressions agree. Thus the critical points are in fact $c_0 = \frac{\sqrt{3}}{2}$, $c_1 = -\frac{\sqrt{3}}{2}$, $a_0 = \frac{3}{2\sqrt{2}}$, $a_1 = -\frac{3}{2\sqrt{2}}$ (as

[2]The map f_1 was originally constructed in [Mey02] (where is was called R_6). Its purpose then was to construct a quasisymmetric map from a certain fractal surface to $\widehat{\mathbb{C}}$. This, however, will not be relevant here.

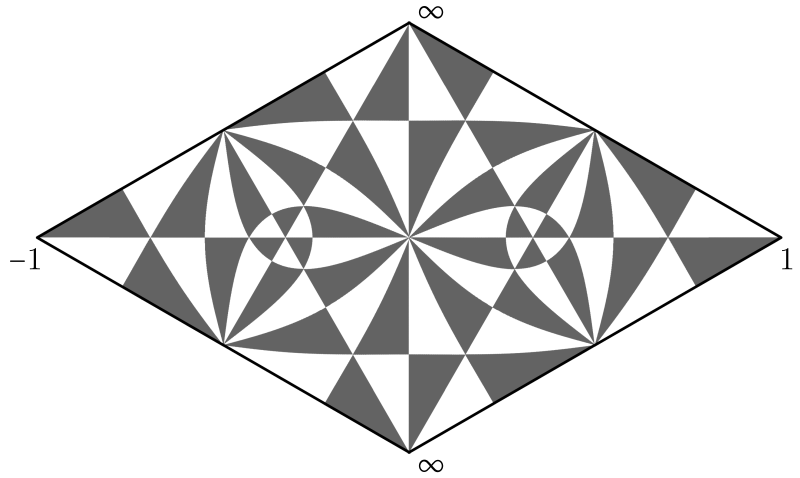


FIGURE 3. The 2-tiles of f_1.

well as 0 and ∞). Nevertheless, these precise values and the explicit formula (3.2) will be of no importance to us.

4. Tiles, flowers, and the iterated monodromy group

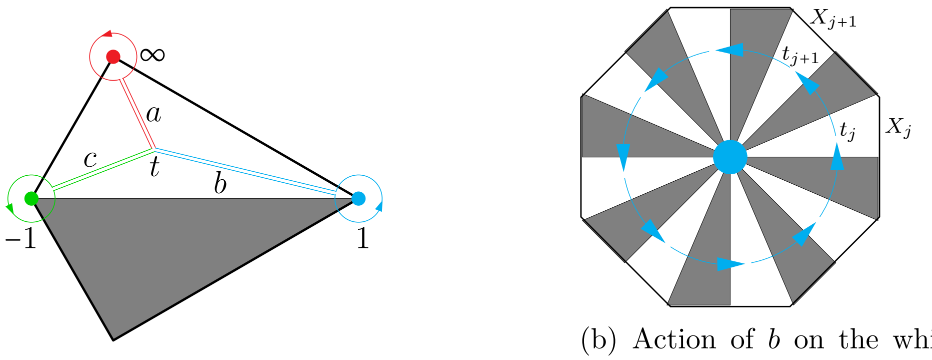
In this section we describe tilings and the iterated monodromy action associated with a Thurston map $f : S^2 \to S^2$ which has an f-invariant Jordan curve $\mathcal{C}$, such that $\operatorname{post}(f) \subset \mathcal{C}$. For simplicity, here we only discuss the case $f = f_1$. However, all the definitions and statements can be naturally adapted to the general case.

Note that the extended real line $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is f_1-invariant, meaning that $f_1(\widehat{\mathbb{R}}) \subset \widehat{\mathbb{R}}$, since f_1 is a real function. Furthermore $\operatorname{post}(f_1) \subset \widehat{\mathbb{R}}$. The closures of the components of $\widehat{\mathbb{C}} \smallsetminus \widehat{\mathbb{R}}$, that is, of the upper and lower half-planes, are called *0-tiles*. The one containing the upper half-plane is colored white and denoted by $X^0_{\mathtt{w}}$; the one containing the lower half-plane is colored black and denoted by $X^0_{\mathtt{b}}$.

Let n be a non-negative integer. The closure of a component of $\widehat{\mathbb{C}} \smallsetminus f_1^{-n}(\widehat{\mathbb{R}})$ is called an *n-tile*. Note that for any such n-tile X the set $f_1^n(X)$ is one of the two 0-tiles. We color X by the color of $f_1^n(X)$, that is, black or white. The *subdivision* of the sphere $\widehat{\mathbb{C}}$ in 2-tiles is shown in Figure 3 (where $\widehat{\mathbb{C}}$ is identified with Δ' as in the previous section).

Note that $f_1(\widehat{\mathbb{R}}) \subset \widehat{\mathbb{R}}$ is equivalent to $f_1^{-1}(\widehat{\mathbb{R}}) \supset \widehat{\mathbb{R}}$, which in turn implies $f^{-(n+1)}(\widehat{\mathbb{R}}) \supset f^{-n}(\widehat{\mathbb{R}})$ for all $n \in \mathbb{N}_0$. It follows that each $(n+1)$-tile is contained in an n-tile. For example, when comparing Figure 2 and Figure 3 one sees that every 1-tile contains (is subdivided into) six 2-tiles.

Each point in $f_1^{-n}(\operatorname{post}(f_1))$ is called an *n-vertex*. Note that the set of n-vertices contains all critical points of f_1^n. Furthermore, each postcritical point is an n-vertex for any $n \in \mathbb{N}_0$, since $f_1(\operatorname{post}(f_1)) \subset \operatorname{post}(f_1)$. We say that the n-vertex v is of *type* a, b, or c if $f_1^n(v) = \infty$, 1,



(a) Generators of $\mathrm{IMG}(f_1)$.

(b) Action of b on the white tiles of a b-flower.

FIGURE 4. The iterated monodromy action for f_1.

or -1, respectively. Note that each n-vertex v is also an $(n+1)$-vertex, but the type of v as an n-vertex may be different from the type of v as an $(n+1)$-vertex.

The closure of any component of $f_1^{-n}(\widehat{\mathbb{R}}) \setminus f_1^{-n}(\mathrm{post}(f_1))$ is called an *n-edge*. Thus, the postcritical points $-1, 1, \infty$ of f_1, which we called 0-vertices, divide $\widehat{\mathbb{R}}$ into the three 0-edges $[-1,1]$, $[1,\infty]$, and $[-\infty,-1]$.

Let $v \in \widehat{\mathbb{C}}$ be an n-vertex, and let $d_v := \deg(f^n, v)$. Then v is contained in, or incident to, d_v white as well as d_v black n-tiles. Furthermore, the colors of n-tiles alternate when going around v. Consider the union of all such n-tiles,

$$W^n(v) := \bigcup_{\substack{n\text{-tile } X, \\ \text{s.t. } v \in X}} X.$$

This set is called the *flower of level n* centered at v. We call d_v the *degree* of $W^n(v)$. We say that the flower $W^n(v)$ is an a-, b-, or c-*flower*, if the n-vertex v is of type a, b, or c, respectively. Note that the terminology is slightly different from the one in [BM], where flowers are always open.

Let us now choose generators of the iterated monodromy group of f_1. We first choose an arbitrary basepoint t in the (interior of the) upper half-plane. In particular $t \notin \mathrm{post}(f_1)$. Let $a = \gamma_\infty$, $b = \gamma_1$, and $c = \gamma_{-1}$ be loops based at t around ∞, 1, and -1, respectively. More precisely, we fix a small simple positively oriented circle around ∞ (Here, "small circle" means that it is inside a neighborhood U of ∞ such that $\mathrm{post}(f_1) \setminus \{\infty\} \subset \widehat{\mathbb{C}} \setminus U$). The loop a connects t to this circle in the upper half-plane, traverses it, and returns to t in the upper half-plane. The loops b and c are defined in an analogous fashion, see Figure 4a. By abuse of notation we identify the equivalence classes of a, b, and c in $\mathrm{IMG}(f_1) = \pi_1(\widehat{\mathbb{C}} \setminus \mathrm{post}(f_1), t)/\mathrm{Ker}(\phi_{f_1})$ (see Section 2.4) with a, b, and c, respectively. Since a, b, c generate $\pi_1(\widehat{\mathbb{C}} \setminus \mathrm{post}(f_1), t)$, they generate $\mathrm{IMG}(f_1)$. In fact, any two of the elements a, b, c generate $\mathrm{IMG}(f_1)$, since $acb = 1$.

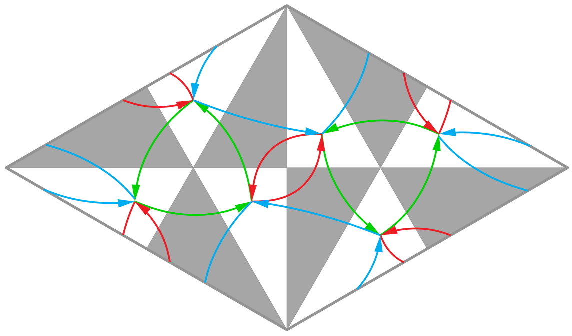


FIGURE 5. The first level Schreier graph of $\mathrm{IMG}(f_1)$.

Note that for any white n-tile X the map $f_1^n\colon X \to X^0_{\mathtt{w}}$ is a homeomorphism, see [BM, Proposition 5.17(i)]. Thus each white n-tile contains exactly one point from $f_1^{-n}(t)$. So we may identify the set of white n-tiles with the n-th level of the dynamical preimage tree $\mathbf{T}_{f_1}$ on which $\mathrm{IMG}(f_1)$ acts. Furthermore, each white n-tile, and thus any $p \in f_1^{-n}(t)$, is contained in exactly one a-flower, exactly one b-flower, as well as exactly one c-flower of level n. Next we are going to describe how to read off the action of $\mathrm{IMG}(f_1)$ on tiles from the tiling picture.

Consider a b-flower $W^n(v)$ of level n. From (3.1) we see that its degree is $d_v \in \{1,3,4,8\}$. Note that $\alpha_{f_1}(1) = 24$ is the lowest common multiple of these degrees by definition of the ramification function. Let $X_0, \dots, X_{d_v-1}$ be the white n-tiles contained in $W^n(v)$ labeled mathematically positively around v. Fix one such white n-tile $X_j \subset W^n(v)$. Let us consider the lift $\widetilde{b}$ of the loop b starting at the point $t_j \in f_1^{-n}(t) \cap X_j$. Then its endpoint is the unique point $t_{j+1} \in X_{j+1} \cap f_1^{-n}(t)$ (here the index is taken $\mathrm{mod}(d_v)$). We conclude that *b acts on white n-tiles by rotations around the centers of b-flowers.* This is illustrated in Figure 4b, where blue arrows represent lifts of b (up to homotopy). The analog description holds for the generators a and c.

The Schreier graph for $\mathrm{IMG}(f_1)$ acting on $f^{-1}(t)$ (the first level of the dynamical preimage tree) is shown in Figure 5. Here, we colored the edges red, blue, and greed instead of labeling them by generators a, b, and c, respectively. In this way, we may think of the sphere tiling generated by the n-tiles as a graphical representation of the Schreier graph of the action of $\mathrm{IMG}(f_1)$ on the n-th level of the dynamical preimage tree $\mathbf{T}_{f_1}$.

Lemma 4.1. *The order of each chosen generator of* $\mathrm{IMG}(f_1)$ *is given by the value of the ramification function at the corresponding postcritical point:*

$$\mathrm{ord}(a) = \alpha_{f_1}(\infty) = 2, \quad \mathrm{ord}(b) = \alpha_{f_1}(1) = 24, \quad \mathrm{ord}(c) = \alpha_{f_1}(-1) = 3.$$

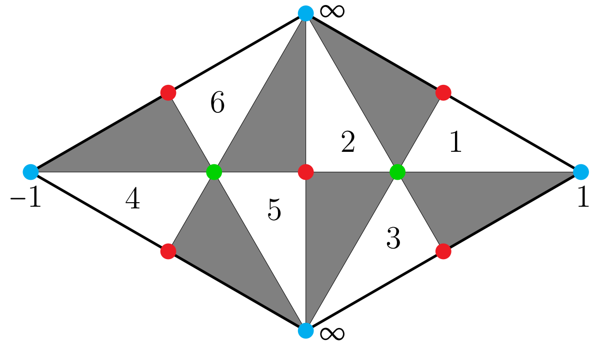


FIGURE 6. Labeling of the 1-tiles of f_1.

Proof. If $k = \alpha_{f_1}(1) = 24$, then k is a multiple of the degree of each b-flower. Hence b^k acts trivially on each white n-tile, $n \in \mathbb{N}_0$. This means that $b^k = 1$ in $\mathrm{IMG}(f_1)$. Conversely, if $k \geq 1$ is not a multiple of 24, there is a b-flower $W^n(v)$ whose degree d_v does not divide k. Then b^k does not act trivially on the white n-tiles in $W^n(v)$, so $b^k \neq 1$. Thus $\mathrm{ord}(b) = \alpha_{f_1}(1) = 24$. The argument is completely analogous for the generators a and c. □

Let us now give the *wreath recursions* of the generators a, b, c, see Appendix A.1. To do this we label the white 1-tiles as indicated in Figure 6. Let $t_j \in f_1^{-1}(t)$ be the preimage of t contained in the 1-tile labeled by $j \in \{1, \dots, 6\}$. Note that $f_1([-\infty, -1] \cup [1, \infty]) \subset [1, \infty]$, that is, $[-\infty, -1] \cup [1, \infty]$ forms a forward-invariant tree in $\widehat{\mathbb{C}}$ joining the postcritical points of f_1. This allows us to naturally choose *connecting paths* from the basepoint t to the points in $f_1^{-1}(t) = \{t_1, \dots, t_6\}$ and define a *labeling* on the dynamical preimage tree $\mathbf{T}_{f_1}$. Namely, we connect t to t_j, $j \in \{1, \dots, 6\}$, by a path ℓ_j that does not intersect $[-\infty, -1] \cup [1, \infty] \subset \widehat{\mathbb{R}} \subset \widehat{\mathbb{C}}$, meaning that ℓ_j stays in the interior of the domain in Figure 6. These choices define the label of every vertex of $\mathbf{T}_{f_1}$ uniquely by iterative lifting of the paths $\ell_1, \dots, \ell_6$, see [Nek05, Chapter 5.2] for more details. With respect to this labeling the action of $\mathrm{IMG}(f_1)$ on the regular rooted tree $\mathbf{T}_{f_1}$ becomes self-similar, see Definition 11.2. The wreath recursions of a, b, and c are then given by

$$\begin{aligned} a &= \langle\!\langle b^{-1}, 1, b, c^{-1}, 1, c \rangle\!\rangle\, (13)(25)(46) \\ b &= \langle\!\langle b, b^{-1}, 1, c, c^{-1}, 1 \rangle\!\rangle\, (2356) \\ c &= (123)(456). \end{aligned} \tag{4.1}$$

We will however not need the wreath recursions to prove exponential growth of $\mathrm{IMG}(f_1)$.

5. EXPONENTIAL GROWTH OF $\mathrm{IMG}(f_1)$

In this section we prove that the iterated monodromy group of the map f_1 has exponential growth. More precisely, we construct a free

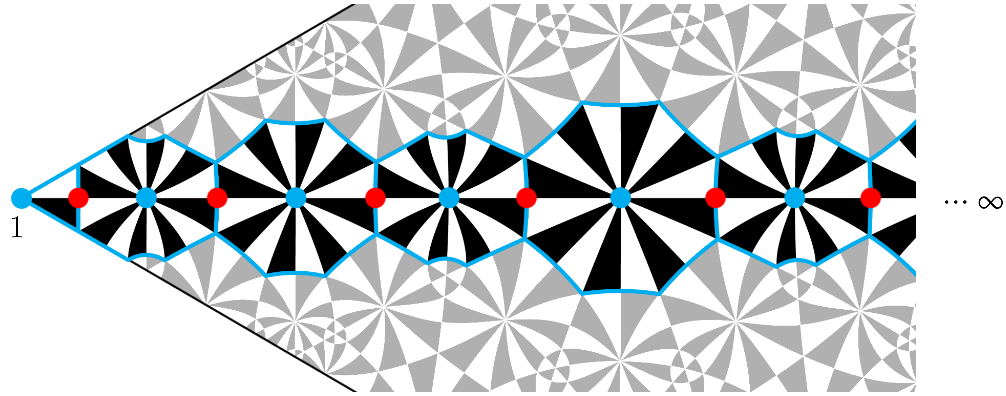


FIGURE 7. b-flowers on $[1, \infty]$.

semigroup inside $\mathrm{IMG}(f_1)$. We remind the reader that a more classical, a la *Basilica*, proof which only uses computations with the wreath recursions (4.1) can be found in Appendix A.2.

Recall that the map f_1 has three postcritical points -1, 1, and ∞, which we call 0-vertices, that divide $\widehat{\mathbb{R}}$ into the three 0-edges $[-1, 1]$, $[1, \infty]$, and $[-\infty, -1]$. The reader is advised to consult Figure 2. The 0-edge $[1, \infty]$ will be of special importance to us.

Lemma 5.1. *Any a-flower of level $n \geq 1$ has degree 2.*

Proof. From (3.1) we see that every preimage of ∞ is a critical point of local degree 2, which is not a postcritical point. The statement follows. □

Note that Lemma 5.1 is stronger than the statement that $\mathrm{ord}(a) = 2$, since the latter does not rule out a-flowers of degree 1.

Lemma 5.2 (Flowers on $[1, \infty]$)**.** *Let $n \in \mathbb{N}_0$. The 0-edge $[1, \infty]$ has the following properties.*

1. $f_1([1, \infty]) = [1, \infty]$. *Consequently, the 0-edge $[1, \infty]$ is (forward) f_1-invariant.*
2. *For every n-vertex $v \in (1, \infty)$ of type b the degree of $W^n(v)$ is 8. The degree of the b-flower $W^n(1)$ is 1.*
3. *There are exactly $2^n + 1$ n-vertices on $[1, \infty]$. Moreover, their type alternates between b and a.*
4. *For any n-vertex $v \in (1, \infty)$, the number of white n-tiles in the flower $W^n(v)$ that are contained in $X^0_{\mathtt{w}}$ equals the number of white n-tiles in $W^n(v)$ that are contained in $X^0_{\mathtt{b}}$.*
5. *For any n-vertex $v \in [1, \infty)$ there is a unique white n-tile $X(v) \in W^n(v) \cap X^0_{\mathtt{w}}$ that intersects $\widehat{\mathbb{R}}$ in an n-edge.*

The situation is illustrated in Figure 7. The n-vertices on $[1,\infty]$ of type b are marked as blue dots, the ones of type a as red dots. Also the b-flowers on $[1,\infty]$ are outlined in blue.[3]

Proof.

(1) Note that $[1,\infty]=[1,a_0]\cup[a_0,\infty]$, and f_1 maps $[1,a_0]$ as well as $[a_0,\infty]$ homeomorphically to $[1,\infty]$, see Figure 2.

(2) Note that $(1,\infty)$ contains no 1-vertices of type b (that is, there is no point $v\in(1,\infty)$ with $f(v)=1$). The only 2-vertex of type b on $(1,\infty)$ is a_0 (which satisfies $f_1(a_0)=\infty$ and $f_1^2(a_0)=1$). It follows from (1) that for any n-vertex $v\in(1,\infty)$ of type b the orbit $v, f_1(v), f_1^2(v),\dots,f_1^n(v)=1$ contains exactly 2 critical points of f_1, namely a_0 and ∞. Thus $\deg(f_1^n,v)=\deg(f_1,a_0)\deg(f_1,\infty)=8$ is the degree of $W^n(v)$ as desired. Clearly, $\deg(f_1^n,1)=1$ for all $n\in\mathbb{N}_0$.

(3) The statement follows from the description above and an elementary induction.

(4) Recall that $X^0_{\mathtt{w}}$ and $X^0_{\mathtt{b}}$ are the closures of the upper and lower half-planes, respectively. So we need to show that each flower $W^n(v)$, $v\in(1,\infty)$, contains as many white n-tiles above the real line as below.

First note that f_1 is a real function, meaning that $\overline{f_1(z)}=f_1(\overline{z})$. Thus the n-tiles are symmetric with respect to the real axis. Since $\deg(f_1^n,v)$ is even, the number of n-tiles in $W^n(v)$ below and above the real line is the same even number. As colors of tiles around v alternate the statement follows.

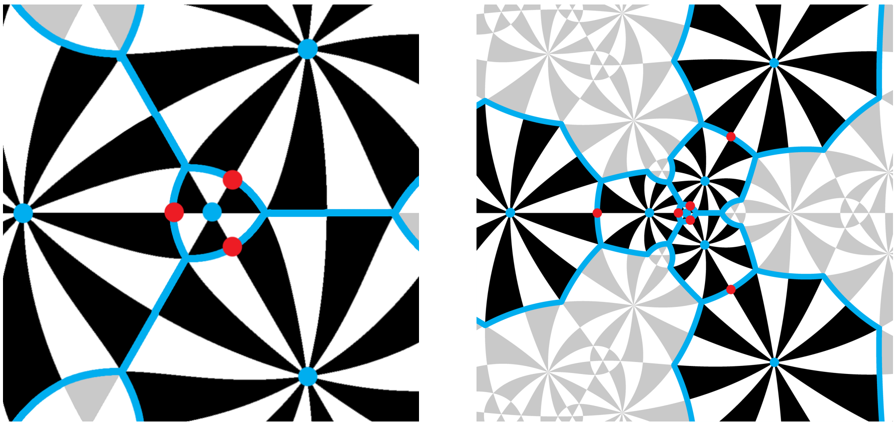
(5) The statement follows from the above considerations. $\square$

Corollary 5.3. *ab^4, ab^{12}, and ab^{20} are distinct elements of infinite order in* $\mathrm{IMG}(f_1)$.

Proof. Since $\mathrm{ord}(b)=24$ it follows that ab^4, ab^{12}, and ab^{20} are distinct in $\mathrm{IMG}(f_1)$.

Let $n\geq 2$ be an integer. Consider an n-vertex $v\in[1,\infty)$ of type b and the white n-tile $X(v)$ as in Lemma 5.2(5). Let v' be the n-vertex of type b to the right of v on $[1,\infty]$ and $X(v')\subset W^n(v')\cap X^0_{\mathtt{w}}$ be the white n-tile according to Lemma 5.2(5). In the latter we assume that v' is different from ∞. Then by the description of the iterated monodromy action on tiles from Section 4 as well as Lemmas 5.1 and 5.2 it follows that ab^4 maps $X(v)$ to $X(v')$. Put differently, ab^4 "shifts white n-tiles in $X^0_{\mathtt{w}}$ on the 0-edge $[1,\infty)$ to the right". Since the degree of $W^n(v')$ is 8, it follows that ab^{12} and ab^{20} act on $X(v)$ in exactly the same way as ab^4, that is, by shifting $X(v)$ to $X(v')$ on the right.

[3]For purely aesthetic reasons we have cut the Riemann sphere $\widehat{\mathbb{C}}$ along $[-\infty,1]$ and applied a 6-th root, that is, the picture shows the tiles after applying the map $z\mapsto(z-1)^{1/6}$. This ensures that the b-flowers on $[1,\infty]$ have roughly the same size.



(a) The flower centered at c_0. (b) The b-flowers around c_0.

FIGURE 8. Flowers around c_0.

Note that by Lemma 5.2(3) there are 2^{n-1} n-vertices of type b on $[1,\infty)$. From the above considerations, the elements ab^4, ab^{12}, and ab^{20} are of infinite order in $\mathrm{IMG}(f_1)$. □

Lemma 5.4. *The elements ab^4, ab^{12}, and ab^{20} generate a free semigroup in* $\mathrm{IMG}(f_1)$.

Put differently, we consider the words of the form

$$ab^{k_1}ab^{k_2}\dots ab^{k_N}, \tag{5.1}$$

where $N \in \mathbb{N}_0$ and $k_j \in \{4,12,20\}$ for $j = 1,\dots,N$. We will show that if two such words are distinct, then they are distinct as elements of $\mathrm{IMG}(f_1)$.

Proof. Suppose that $n \geq 2$ is an integer. Let us consider the critical point c_0, see Figure 2 and (3.1). Note that $f_1^n(c_0) = 1$ and $\deg(f_1^n, c_0) = 3$. Thus, c_0 is an n-vertex of type b and the degree of $W^n(c_0)$ is 3. Figure 8a shows such a flower $W^n(c_0)$.

Note that the 0-edge $[1,\infty]$ has three f_1^2-preimages at c_0. More precisely, there are three analytic closed arcs A_1, A_2, A_3, such that they intersect (pairwise) precisely in c_0 and $f_1^2\colon A_j \to [1,\infty]$ is a homeomorphism for $j = 1,2,3$. Denote by A_j' the arc A_j with the endpoint different from c_0 removed.

Let $j \in \{1,2,3\}$. It follows that on each arc A_j' the combinatorial picture of level n is the same as on $[1,\infty)$ for level $n-2$ (the latter one is described by Lemma 5.2). The only difference is that the degree of $W^n(c_0)$ is 3, while the degree of $W^n(1)$ is 1. In particular, we can associate to each n-vertex $v \in A_j'$ of type b a white n-tile $X(v) \subset W^n(v)$, so that $X(v)$ is mapped by ab^k to $X(v')$, for $k = 4,12,20$. Here, v' is the next n-vertex of type b on A_j that follows v, when traversing A_j starting from c_0. Figure 8b shows the b-flowers on the arcs A_1, A_2, A_3

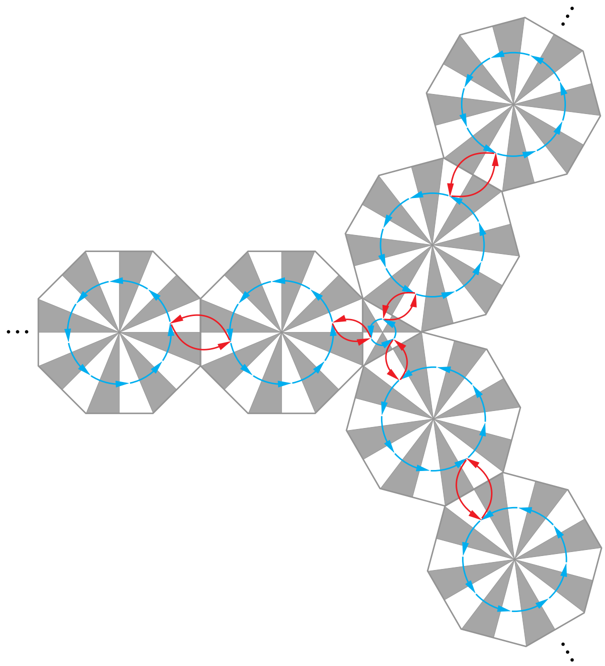


Figure 9. A subgraph of the Schreier graph at c_0.

for sufficiently large n (Figure 8a is a close-up of Figure 8b). For the convenience of the reader we also show a subgraph of the n-level Schreier graph in Figure 9, which may be viewed as a schematic version of Figure 8b.

Consider now two distinct words w_1 and w_2 of the form (5.1). By multiplying from the left with the inverse of the common initial word, it is enough to assume that

$$w_1 = b^{k_1}ab^{k_2}\dots ab^{k_N}, \quad w_2 = b^{m_1}ab^{m_2}\dots ab^{m_N},$$

where $k_1,\dots,k_N,m_1,\dots,m_N \in \{4,12,20\}$, and $k_1 \neq m_1$. Fix a sufficiently large n (in fact it will be enough to demand that $2^{n-3} > N$). Fix one white n-tile X in $W^n(c_0)$. Let us apply the two words w_1 and w_2 to X. Note that b^{k_1} and b^{m_1} map this tile to distinct n-tiles X_1 and X_2 in $W^n(c_0)$, since 8 is not a multiple of 3. We conclude that the remaining subwords $ab^{k_2}\dots ab^{k_N}$ and $ab^{m_2}\dots ab^{m_N}$ of w_1 and w_2 shift these n-tiles along two distinct arcs among A_1, A_2, A_3. Thus w_1 and w_2 map X to distinct n-tiles. Consequently, w_1 and w_2 are distinct elements in $\mathrm{IMG}(f_1)$.

The previous argument can be applied to show that distinct words of the form (5.1) are distinct in $\mathrm{IMG}(f_1)$ even if they have different length. However, we note that there is a special case when after cancellation of the common initial part of the two given words we are left with $w_1 = 1$ and $w_2 \neq 1$. In this case w_1 fixes the n-tile X selected above, while w_2 shifts it along one of the arcs A_1, A_2, A_3. Thus w_1 and w_2 are distinct elements in $\mathrm{IMG}(f_1)$. Hence we have proved that the elements ab^4, ab^{12}, ab^{20} generate a free semigroup in $\mathrm{IMG}(f_1)$. □

From the previous lemma it follows immediately that $\mathrm{IMG}(f_1)$ is of exponential growth. This proves Theorem 1.1.

6. A criterion for exponential growth

Here we analyze the essential ingredients used in Section 5 to prove exponential growth of the iterated monodromy group of the map f_1. This will result in a somewhat general sufficient condition for the IMG of a Thurston map to be of exponential growth. However, we point out that this criterion is far from being necessary. Moreover, the imposed conditions on the Thurston map can be further relaxed.

Let $g\colon S^2 \to S^2$ be a Thurston map (see Definition 2.1). We fix a Jordan curve $\mathcal{C} \subset S^2$ with $\mathrm{post}(g) \subset \mathcal{C}$. As in Section 4, the 0-edges are the closed arcs into which $\mathrm{post}(g)$ divides $\mathcal{C}$. For the map f_1 we considered the 0-edge $[1, \infty]$ which was f_1-invariant. So in general we demand that

(a) there is a g-invariant 0-edge E with endpoints $p, q \in \mathrm{post}(g)$ such that $g|E\colon E \to E$ is not a homeomorphism.

Put differently, $g(E) = E$ and $d_E := \deg(g|E) \geq 2$. It follows that the 0-edge E is subdivided into d_E^n n-edges by n-vertices for each $n \in \mathbb{N}_0$.

Further, we assume that

(b) $$g(p) = p.$$

Note that from $g(E) = E$ it follows that $g(\{p, q\}) \subset \{p, q\}$, which means that at least one of the points p, q is a fixed point of g^2. So the main purpose of condition (b) is to fix the notation, since p and q will play somewhat different roles.

Let v be an n-vertex on E for some $n \in \mathbb{N}_0$, meaning that $v \in g^{-n}(\mathrm{post}(g)) \cap E$. Since E is g-invariant, it follows that either $g^n(v) = p$ or $g^n(v) = q$. In the first case we say that the n-vertex v is of type p, otherwise we say that it is of type q. We again note that v is an $(n+1)$-vertex as well, and as such may be of different type. Recall that each n-edge e in E is mapped by g^n homeomorphically to E. If follows that one endpoint of e is an n-vertex of type p and the other one is of type q. Put differently, the types of the n-vertices on E alternate.

The second essential ingredient for the example f_1 was that for all n-vertices $v \in (1, \infty)$ of a fixed type the degrees of the flowers $W^n(v)$ were the same. In general we demand that there are constants $k_p, k_q \in \mathbb{N}$, such that for each $n \in \mathbb{N}$ every n-vertex $v \in E \setminus \{p, q\}$ of type p, and every n-vertex $w \in E \setminus \{p, q\}$ of type q, satisfies

(c) $$\deg(g^n, v) = k_p \text{ and } \deg(g^n, w) = k_q.$$

There is an elementary way to check that condition (c) is true. Let $V_p \coloneqq g^{-1}(p) \cap (E \setminus \{p,q\})$ and $V_q \coloneqq g^{-1}(q) \cap (E \setminus \{p,q\})$. These are the 1-vertices on $E \setminus \{p,q\}$ of type p and q respectively. We first note that (c) implies that there are constants $k_p, k_q \in \mathbb{N}$ with

$$\text{(c1)} \qquad k_p = \deg(g,v) \text{ and } k_q = \deg(g,w)$$

for all $v \in V_p$ and $w \in V_q$.

Let $v \in E \setminus \{p,q\}$ be an n-vertex of type p, then by (b) it is an $(n+1)$-vertex of type p as well. Assuming (c) it follows that

$$k_p = \deg(g^{n+1}, v) = \deg(g^n, v)\deg(g,p) = k_p \deg(g,p).$$

Thus (c) implies

$$\text{(c2)} \qquad \deg(g,p) = 1.$$

The other 0-vertex q satisfies either $g(q) = q$ or $g(q) = p$. In the first case we obtain from an exactly analogous argument that (c) implies

$$\text{(c3)} \qquad \deg(g,q) = 1 \quad \text{in the case} \quad g(q) = q.$$

Furthermore, in this case the three conditions (c1), (c2), (c3) imply (c). Indeed, for any n-vertex $v \in E \setminus \{p,q\}$ of type p the sequence $v, g(v), \dots, g^n(v) = p$ contains exactly one critical point of g, which is contained in V_p, and thus has local degree k_p. Hence $\deg(g^n, v) = k_p$. Similarly, $\deg(g^n, w) = k_q$ for each n-vertex $w \in E \setminus \{p,q\}$ of type q.

Let us now consider the second case $g(q) = p$. Suppose $w \in V_q$. Then $g^2(w) = p$. From (c) it follows that $k_p = \deg(g^2, w) = \deg(g,w)\deg(g,q) = k_q \deg(g,q)$. Thus we obtain the condition

$$\text{(c4)} \qquad k_p = k_q \deg(g,q) \quad \text{in the case} \quad g(q) = p.$$

Conversely, in this case the conditions (c1), (c2), (c4) imply that for any n-vertex $v \in E \setminus \{p,q\}$ of type p, the sequence $v, g(v), \dots, g^n(v) = p$ contains either exactly one point in V_p or the point q and exactly one point in V_q. In both situations $\deg(g^n, v) = k_p$. The argument that $\deg(g^n, w) = k_q$ for each n-vertex $w \in E \setminus \{p,q\}$ of type q is the same as in the first case.

In conclusion, we have seen that

$$(c1),\ (c2),\ (c3),\ (c4) \Leftrightarrow (c).$$

In practice we often consider the *ramification portrait of g restricted to E*. Let $v \in V_p \cup V_q \subset E \setminus \{p,q\}$. Since v is incident to two 1-edges contained in E, both of which are mapped to E, it follows that v is a critical point of g. Thus the ramification portrait of g does indeed contain all 1-vertices in E, and we may restrict it to the set of these vertices. We note that for a (critical) point $v \in V_p \cup V_q \cup \{p,q\}$ of degree $d_v = \deg(g,v)$ we label the directed edge from v to $g(v)$ by "$d_v : 1$". Note that d_v differs in general from $\deg(g|E, v)$ (which is always 2 for $v \in V_p \cup V_q$).

For example, the ramification portrait of f_1 restricted to $[1,\infty]$ is

$$a_0 \xrightarrow{2:1} \infty \xrightarrow{4:1} 1.$$

Given the restricted ramification portrait of a g-invariant 0-edge, it is immediate to verify that conditions (c1), (c2), (c3) (c4) (as well as (b)) are satisfied.

Condition (c) is not yet sufficient to ensure that a suitable word (which was ab^4 for the example f_1) acts by "shifting white n-tiles along E". Let $v \in E \setminus \{p,q\}$ be an n-vertex. Then there are n-edges $e, e' \subset E$ that intersect in v. Then any n-tile in the flower $W^n(v)$ of level n centered at v is either contained in the sector between e and e' or in the sector between e' and e. We refer to these two sectors as the *sectors into which E divides $W^n(v)$*. In this setting we demand that

(d) for each n-vertex $v \in E \setminus \{p,q\}$ the two sectors into which E divides $W^n(v)$ contain the *same number of* n-tiles.

Let $d_v = \deg(g^n, v)$. Then there are $2d_v$ n-edges that contain v. Let $e_0, e_1, \dots, e_{2d_v-1}$ be these n-edges labeled cyclically around v so that $e = e_0 \subset E$. Then (d) is equivalent to the requirement that $e' = e_{d_v} \subset E$.

Since E is invariant, $g^n(e_0) = g^n(e_{d_v}) = E$. At the same time g^n maps e_j to E if and only if j is even. It follows that d_v is even. Consequently, conditions (a) and (d) imply that the two sectors into which E divides $W^n(v)$ contain an (equal) even number of n-tiles. Thus the numbers k_p and k_q from condition (c) are even.

Let us color the 0-tiles black and white. As in Section 4, the dynamics of g defines a coloring on the n-tiles respecting the above choice for all $n \in \mathbb{N}$. The colors of the n-tiles containing v (that is, the n-tiles in the flower $W^n(v)$) alternate cyclically around v. From the discussion above, it follows that conditions (a) and (d) mean that the numbers of white n-tiles in the two sectors into which E divides $W^n(v)$ are equal.

Note that (d) is automatically satisfied if $E \subset \widehat{\mathbb{R}}$ and g is a real rational function, that is, $g(\widehat{\mathbb{R}}) \subset \widehat{\mathbb{R}}$.

As for condition (c), there is an equivalent condition that only involves 1-vertices, 1-tiles, and 1-flowers. More precisely, we demand that

(d') for each 1-vertex $v \in E \setminus \{p,q\}$ the two sectors into which E divides $W^1(v)$ contain the *same number of* 1-tiles.

Lemma 6.1. *Let E be a g-invariant* 0*-edge with the endpoints p and q as in* (a)*. Then condition* (d) *is equivalent to condition* (d').

Proof. The implication (d) $\Rightarrow$ (d') is trivial.

Let us show the other implication. First consider a 1-vertex $v \in E \setminus \{p,q\}$; recall that v then is an n-vertex as well for all $n \in \mathbb{N}$.

Since the 0-edge E is g-invariant, $g(v) \in \{p, q\}$. Assume that $g(v) = p$. The other case is completely analogous and will not be treated separately. Let E and $\widetilde{E}$ be the 0-edges incident to p. Then the 1-edges containing v are alternatingly mapped to E and $\widetilde{E}$ by g. Suppose that the 1-edges around v are $e_0, \widetilde{e}_0, e_1, \widetilde{e}_1, \dots, e_{d_v} = e_0$ in cyclic order. Here $d_v = \deg(g, v)$, $g(e_j) = E$, and $g(\widetilde{e}_j) = \widetilde{E}$ for $j = 0, \dots, d_v - 1$. Let $e_0 \subset E$, then by conditions (a) and (d') it follows that the other 1-edge in E containing v is $e_{d_v/2}$.

Fix an $n \in \mathbb{N}$ and view the 1-vertex $v \in E \smallsetminus \{p, q\}$ as an n-vertex. Consider all the $(n-1)$-edges containing $g(v) = p$. Denote these $(n-1)$-edges by $K_0, \dots, K_{m-1}$ so that $K_0 \subset E$, where $m = 2\deg(g^{n-1}, g(v))$. Then every 1-edge e_j, defined above, contains an n-edge $k_{j,0}$ incident to v, such that $g(k_{j,0}) = K_0$. Moreover, between e_j and e_{j+1} there are exactly $(m-1)$ n-edges containing v. It follows that the two sectors in $W^n(v)$ between the two n-edges in E containing v, namely $k_{0,0}$ and $k_{d_v/2,0}$, contain the same number of n-edges, proving the lemma in this case.

Consider now an n-vertex $v \in E \smallsetminus \{p, q\}$ that is not a 1-vertex. Then there is a $j \leq n-1$ such that $w := g^j(v) \in E \smallsetminus \{p, q\}$ is a 1-vertex. Since w is not a postcritical point of g, it follows that v is not a critical point of g^j. Thus there is a neighborhood of v on which g^j is a homeomorphism. Consequently, g^j maps n-edges containing v to $(n-j)$-edges containing w bijectively, and at the same time E to E. The desired statement for v follows now from the corresponding statement for the 1-vertex w already proved above. □

Finally, we assume the following condition.

(e) There is a point $c \in S^2$ with $g^k(c) = p$ for a $k \in \mathbb{N}$ such that $d_c := \deg(g^k, c)$ is not a divisor of k_p.

As for the map f_1 we define the elements a and b of $\mathrm{IMG}(g)$ as being represented by loops around q and p, respectively. More precisely, we fix a basepoint t in the interior of the white 0-tile denoted by $X^0_{\mathtt{w}}$. Then the loop a is represented by connecting t in $X^0_{\mathtt{w}}$ to a small circle around q. Similarly, the loop b is represented by connecting t in $X^0_{\mathtt{w}}$ to a small circle around p.

Assuming conditions (a)–(d), one shows exactly analogous to Corollary 5.3 that $a^{k_q/2}b^{k_p/2}$ is of infinite order in $\mathrm{IMG}(g)$. If we also assume condition (e), it follows exactly as in Lemma 5.4 that $a^{k_q/2}b^{k_p/2}$ and $a^{k_q/2}b^{k_p/2}b^{k_p}$ generate a free semigroup in $\mathrm{IMG}(g)$. This means we have proved the following.

Theorem 6.2. *Let $g\colon S^2 \to S^2$ be a Thurston map, and $\mathcal{C} \subset S^2$ be a Jordan curve with* $\mathrm{post}(g) \subset \mathcal{C}$ *that satisfies conditions* (a)–(d). *Then* $\mathrm{IMG}(g)$ *contains an element of infinite order.*

Theorem 6.3. *Let $g: S^2 \to S^2$ be a Thurston map, and $\mathcal{C} \subset S^2$ be a Jordan curve with* $\operatorname{post}(g) \subset \mathcal{C}$ *that satisfies conditions* (a)–(e). *Then* $\operatorname{IMG}(g)$ *is of exponential growth.*

Let us provide some remarks on how our conditions may be relaxed.

Remarks 6.4.

(1) For any Thurston map $g: S^2 \to S^2$, $\operatorname{IMG}(g)$ is isomorphic to $\operatorname{IMG}(g^n)$ for any $n \in \mathbb{N}$. Thus to prove exponential growth of $\operatorname{IMG}(g)$ it is enough to check that our conditions are satisfied for some iterate g^n.

(2) Two Thurston maps $f: S^2 \to S^2$ and $g: S^2 \to S^2$ that are Thurston equivalent have isomorphic iterated monodromy groups [Nek05, Corollary 6.5.3]. Thus our conditions, in particular (a), need only be satisfied up to isotopy rel. $\operatorname{post}(g)$.

(3) For simplicity, let us denote $x := a^{k_p/2} b^{k_q/2}$. By (b) there exists a unique point $t_0 \in g^{-1}(t) \cap W^1(p)$. Conditions (a)–(d) imply that there is a sequence of distinct points $t_0, t_1, \ldots, t_{n-1} \in g^{-1}(t)$, such that $t_{j+1 \bmod (n)} = t_j^x$, that is, the lift x_j of x starting at t_j ends at $t_{j+1 \bmod (n)}$ for each $j = 0, \ldots, n-1$. In fact, $n = d_E = \deg(g|E)$. Let γ be the closed curve in $S^2 \smallsetminus \operatorname{post}(f)$ obtained by concatenation of the paths x_j, $j = 0, \ldots, n-1$. It follows that the curve γ is homotopic to x rel. $\operatorname{post}(g)$. Furthermore $\deg(g|\gamma) = n > 1$. Loosely speaking, we can say that x acts on the white n-tiles sharing an edge with E by "permuting them cyclically around E". In this way, one can relax conditions (a), (c), and (d) by requiring existence of an element $x \in \operatorname{IMG}(g)$ with the properties mentioned above. This element x will be automatically of infinite order in $\operatorname{IMG}(g)$. However, one will also need to adapt condition (e) in a certain way to conclude the exponential growth of $\operatorname{IMG}(g)$. The resulting criterion however is somewhat cumbersome and will not be formulated here.

7. Sierpiński carpet rational maps

In this section we present a family of postcritically-finite rational maps, such that each map in the family has a Sierpiński carpet as its Julia set, and iterated monodromy group of exponential growth. Here, we call a set $\mathcal{S} \subset \widehat{\mathbb{C}}$ a *Sierpiński carpet* if and only if it is homeomorphic to the standard Sierpiński carpet. By Whyburn's characterization, this is the case if and only if $\mathcal{S}$ is compact, connected, locally connected, has topological dimension 1, and no local cut-points. Equivalently, there exists a sequence $\{D_j\}_{j \in \mathbb{N}}$ of Jordan domains in $\widehat{\mathbb{C}}$ having pairwise disjoint closures, such that $\mathcal{S} = \widehat{\mathbb{C}} \smallsetminus \left(\bigcup_j D_j\right)$ has empty interior and $\operatorname{diam}(D_j) \to 0$ as $j \to \infty$, see [Why58]. These two characterizations show that a Sierpiński carpet is a universal object and justify the study of rational maps with Sierpiński carpet Julia set. The first example of such a rational map is due to Milnor and Tan Lei, see [Mil93, Appendix].

The rational maps we will consider now have originally been studied by Haïssinsky and Pilgrim in [HP12] (with one difference that we address at the end of the section). We briefly review the construction, which is similar to the way the map f_1 from Section 3 was defined.

To keep the discussion elementary, we first consider one concrete map from our family. Let Δ and Δ' be the two polyhedral surfaces shown in Figure 10. Here, Δ is a pillow obtained by gluing two copies of the unit square $[0,1]^2$, called faces of Δ, together along their boundaries. The polyhedral surfaces Δ' is constructed from 2×11 squares of side length $1/3$, called 1-*squares* of Δ' (they correspond to 1-tiles in the terminology of Section 4). More precisely we take two copies of

$$Y \coloneqq [0,1]^2 \;\cup\; [0,\tfrac{1}{3}] \times [-\tfrac{1}{3},0] \;\cup\; [-\tfrac{1}{3},0] \times [0,\tfrac{1}{3}] \;\subset \mathbb{R}^2$$

and glue them together along their boundaries. We call each copy of $Y \subset \Delta'$ a face of Δ'. Clearly Δ and Δ' are homeomorphic to S^2. As polyhedral surfaces, Δ and Δ' may both be naturally viewed as Riemann surfaces. With this point of view, there are conformal maps

$$\varphi\colon \Delta \to \widehat{\mathbb{C}} \text{ and } \varphi'\colon \Delta' \to \widehat{\mathbb{C}}.$$

By symmetry, we may assume that φ and φ' map one face of Δ and, respectively, Δ' to the upper half-plane and the other face to the lower half-plane, so that the points marked $-1, 0, 1, \infty$ in Figure 10 are mapped to $-1, 0, 1, \infty \in \widehat{\mathbb{R}} \subset \widehat{\mathbb{C}}$, respectively. In fact, φ and φ' may be constructed explicitly by mapping each face of Δ and, respectively, Δ' by a Riemann map to the upper or lower half-plane, where the four marked points are mapped as indicated.

The map $g\colon \Delta' \to \Delta$ is now given as follows. Each 1-square of Δ' is mapped (conformally) by a similarity that scales by the factor 3 to a face of Δ as indicated in Figure 10. We define the map

$$f \coloneqq \varphi \circ g \circ (\varphi')^{-1}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}.$$

Note that f is holomorphic, hence a rational map. Consider the points in Δ' where at least four 1-squares of Δ' intersect. Their images under φ' are exactly the critical points of f. Furthermore, the points $-1, 0, 1, \infty \in \widehat{\mathbb{C}}$, which are the images of the vertices of Δ under φ, are exactly the postcritical points of f. So, f is a rational Thurston map with $\operatorname{post}(f) = \{-1, 0, 1, \infty\}$. Moreover, the Julia set of f is a Sierpiński carpet, see [HP12].

We mention in passing that we may think of f as being constructed via a *two-tile subdivision rule* in the sense of Cannon-Floyd-Parry (see [CFKP03] and [BM, Chapter 12]). Alternatively, f is constructed from a Lattès map by "adding a flap" or "blowing up an arc" in the sense of [PT98].

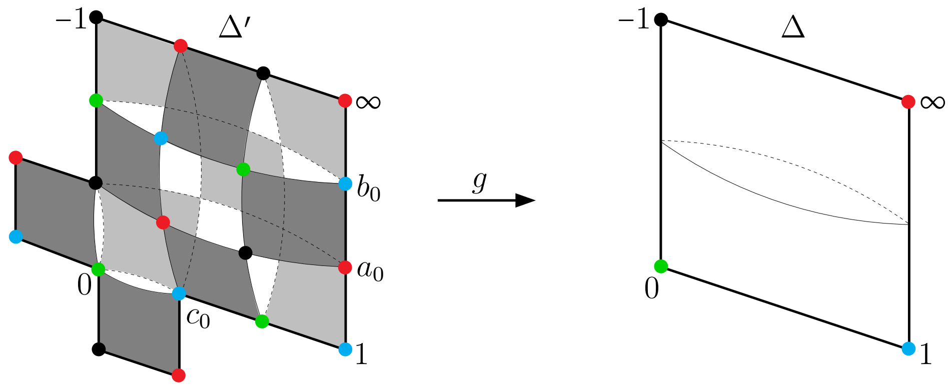


FIGURE 10. Construction of a Sierpiński carpet map.

Let $\mathcal{C} := \widehat{\mathbb{R}}$, $E := [1, \infty] \subset \mathcal{C}$, $p := 1$, and $q := \infty$. The ramification portrait of f restricted to E is

$$b_0 \xrightarrow{2:1} 1 \circlearrowleft \qquad a_0 \xrightarrow{2:1} \infty. \circlearrowleft$$

Again, by a slight abuse of notation, we denote the images of the points labeled by a_0, b_0, and c_0 in Figure 10 under φ' by a_0, b_0, and c_0, respectively.

Note that the critical point c_0 of f satisfies $\deg(f, c_0) = 3$, see Figure 10. It is now elementary to check that with the above choices conditions (a)–(e) from Section 6 are satisfied. It follows from Theorem 6.3 that $\mathrm{IMG}(f)$ is of exponential growth. Thus we have proved Theorem 1.2.

We may vary the construction of the map f. In particular, divide each of the two faces of the pillow Δ in $n \times n$ squares of side length $1/n$, where $n \geq 3$ is an odd number, and add two diagonally symmetric flaps as in Figure 10 to obtain a polyhedral surface Δ'. Then repeat the above construction. Again the resulting rational map f has a Sierpiński carpet as its Julia set, and $\mathrm{IMG}(f)$ is of exponential growth. The maps in [HP12] are constructed in the same fashion, but there $n \geq 2$ is even. This results in a *hyperbolic* rational map f, meaning that every critical point of f is contained in the Fatou set $\mathcal{F}_f$. For each such map f condition (c) is not satisfied. Consequently, Theorem 6.3 does not apply. We do not know if these hyperbolic maps have iterated monodromy groups of exponential growth. However, the argument in [HP12], showing that f has a Sierpiński carpet Julia set, is equally valid independently of whether n is even or odd.

8. A FAMILY OF OBSTRUCTED MAPS

Here, the example from the previous section is slightly modified to obtain an infinite family of obstructed Thurston maps with iterated monodromy groups of exponential growth.

The construction is very similar to the one in Section 7. The only difference is that instead of "adding two flaps" to a Lattès map we add only one; see Figure 11 and compare with Figure 10. As before, we obtain a map $g\colon \Delta' \to \Delta$ and define homeomorphisms $\varphi\colon \Delta \to \widehat{\mathbb{C}}$ and $\varphi'\colon \Delta' \to \widehat{\mathbb{C}}$ that are normalized as in the previous section. That is, the top and bottom faces of Δ (and Δ') are mapped to the upper and lower half-planes in $\widehat{\mathbb{C}}$, respectively, so that the points labeled by $-1, 0, 1, \infty$ in Δ (and Δ') are mapped to $-1, 0, 1, \infty \in \widehat{\mathbb{C}}$, respectively. We obtain a Thurston map $f \coloneqq \varphi \circ g \circ (\varphi')^{-1}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\mathrm{IMG}(f)$ is of exponential growth (using Theorem 6.3 in the exact same form as in the previous section).

However, we want to point out a crucial difference from the situation in Section 7. Namely, here each face of Δ' is not symmetric with respect to the (diagonal) geodesic joining the vertices labeled by 0 and ∞ (in the respective face). This means we cannot choose the map $\varphi'\colon \Delta' \to \widehat{\mathbb{C}}$ to be conformal with our normalization. Consequently, the constructed map f is not rational.

With a slight abuse of notation consider now an arbitrary Thurston map $f\colon S^2 \to S^2$. Thurston gave a criterion when the map f is Thurston equivalent to a rational map. We present it only in the case when f has 4 postcritical points and a hyperbolic orbifold. We refer to [DH93] for the general statement, as well as the terminology. Let $\gamma \subset S^2 \setminus \mathrm{post}(f)$ be a Jordan curve that is *non-peripheral*, meaning that each component of $S^2 \setminus \gamma$ contains at least 2 postcritical points (since $\#\,\mathrm{post}(f) = 4$ this means that each component contains exactly 2 postcritical points). Let $\gamma_1, \dots, \gamma_k$ be the components of $f^{-1}(\gamma)$ that are non-peripheral. The curve γ is called *invariant* if one (or, equivalently in our case, each) of the curves γ_j, $j = 1, \dots, k$, is homotopic to γ rel. $\mathrm{post}(f)$. Denote by d_j the degree of the restriction $f\colon \gamma_j \to \gamma$ for $j = 1, \dots, k$. Assume that γ is invariant and define

$$\lambda_f(\gamma) \coloneqq \sum_{j=1}^{k} \frac{1}{d_j}. \tag{8.1}$$

Then the curve γ is called a *Thurston obstruction* if $\lambda_f(\gamma) \geq 1$. Thurston's theorem now says that f is Thurston equivalent to a rational map if and only if f has no Thurston obstruction. Otherwise f is called an *obstructed* Thurston map.

It follows that the map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ constructed in this section has a Thurston obstruction γ as shown in Figure 11. Here, we choose to draw γ in Δ and its preimages $\gamma_1, \dots, \gamma_4$ in Δ' for simplicity. In reality, we consider the images of these curves under φ and φ' in $\widehat{\mathbb{C}}$, respectively. Then $\lambda_f(\gamma) = 1$ (note that the component γ_4 of $f^{-1}(\gamma)$ is peripheral, thus it does not contribute to the sum (8.1)). Consequently, γ is a Thurston obstruction. Thus f is not Thurston equivalent to a rational

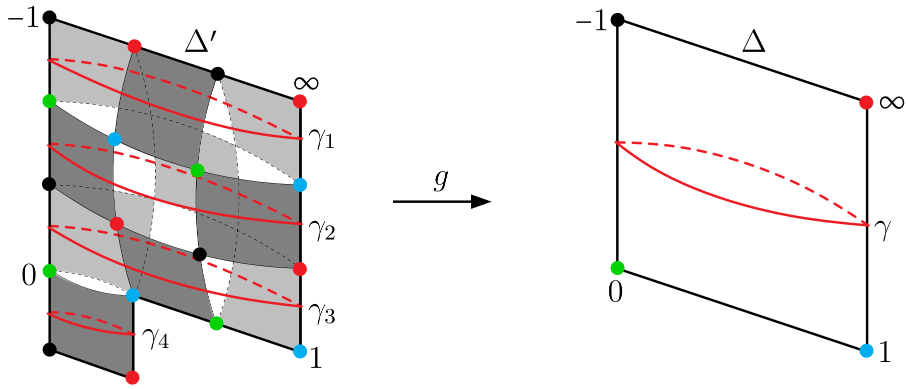


FIGURE 11. Construction of an obstructed map.

map by Thurston's theorem, meaning it is obstructed. We have proved Theorem 1.3.

9. A NON-RENORMALIZABLE POLYNOMIAL P WITH IMG OF EXPONENTIAL GROWTH

Here, we present an example of a postcritically-finite, non-renormalizable polynomial $P: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with dendrite Julia set and iterated monodromy group of exponential growth. This polynomial serves as an example in Theorem 1.4. In fact, P is given by

$$P(z) = \frac{2}{27}(z^2+3)^3(z^2-1)+1 = \frac{2}{27}z^8 + \frac{16}{27}z^6 + \frac{4}{3}z^4 - 1.$$

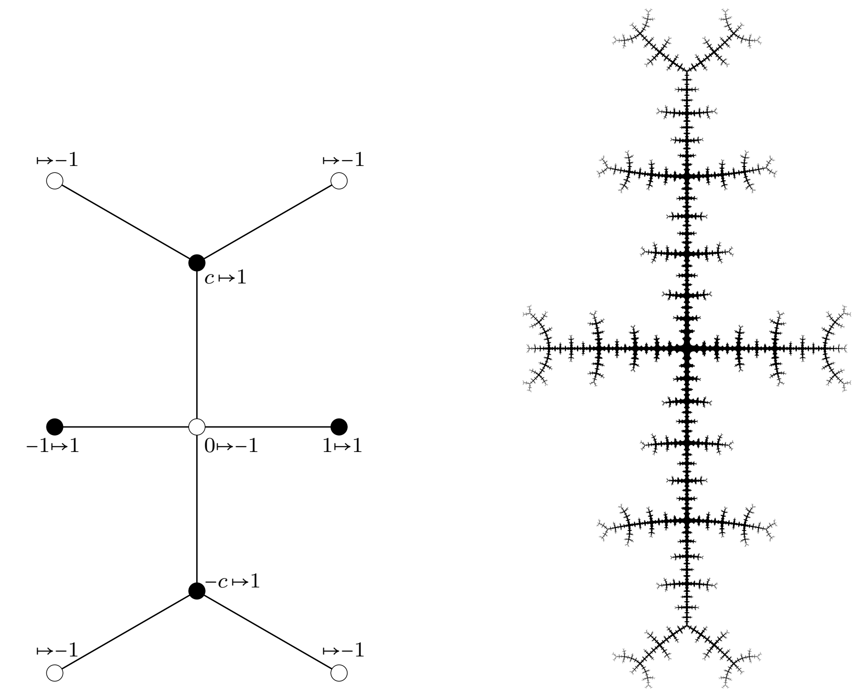
From these two expressions we immediately see that P has the (finite) critical points $\pm\sqrt{3}i$ of local degree 3, and 0 of local degree 4. Furthermore, they are mapped as follows

$$\pm\sqrt{3}i \xrightarrow{3:1} 1 \circlearrowleft \longleftarrow -1 \xleftarrow{4:1} 0.$$

Thus $\operatorname{post}(P) = \{-1, 1, \infty\}$. Let $p \coloneqq 1$, $q \coloneqq -1$, $\mathcal{C} \coloneqq \widehat{\mathbb{R}}$, $E \coloneqq [-1,1]$, and $c \coloneqq \sqrt{3}i$. With these choices conditions (a)–(e) are satisfied. From Theorem 6.3 it follows that $\operatorname{IMG}(P)$ has exponential growth. Hence P satisfies property (4) of Theorem 1.4.

Clearly P is postcritically-finite and has no (finite) periodic critical points. It follows that the Julia set $\mathcal{J}$ of P, shown in Figure 12b, is a *dendrite* (that is, a compact, connected, locally connected set with empty interior which does not separate the plane), see [Bea91, §11.2]. Consequently, P satisfies property (1) of Theorem 1.4.

Since $\mathcal{J}$ is a dendrite, the *Hubbard tree* of P may be defined as the smallest continuum in $\mathcal{J}$ containing all (finite) postcritical points (see [DH84]). From $P([-1,1]) = [-1,1]$ we conclude that $[-1,1]$ is contained in the Julia set $\mathcal{J}$ (because each orbit in $[-1,1]$ is bounded).



(a) The pre-Hubbard tree/ dessin d'enfant of P.

(b) The Julia set of P.

FIGURE 12. Visualizing P.

Hence, $H = [-1, 1]$ is the Hubbard tree of P and the Julia set $\mathcal{J}$ coincides with the closure

$$\overline{\bigcup_{n \geq 0} P^{-n}(H)}.$$

Let us color the point 1 black and -1 white. The preimage $H' \coloneqq P^{-1}(H)$ of the Hubbard tree H is schematically shown in Figure 12a. Here, we color the points in $P^{-1}(1)$ black and the points in $P^{-1}(-1)$ white, and indicate how they are mapped for convenience (even though this information is already contained in the coloring). We also label the critical and postcritical points.

Let us recall that a point x of a dendrite X is called a *leaf* of X if $X \setminus \{x\}$ is connected. Clearly, the points 1 and -1 are leaves of the Hubbard tree H and the pre-Hubbard tree H'. It follows that each set $P^{-n}(H)$, $n \in \mathbb{N}_0$, is a planar tree and its set of leaves contains points $1, -1$. Consequently, the postcritical points 1 and -1 are leaves of the dendrite Julia set $\mathcal{J}$. Thus P satisfies property (2) of Theorem 1.4.

The polynomial P is in fact a *Shabat* (or *Belyĭ*) *polynomial*. This means that P has exactly 2 finite critical values. The diagram in Figure 12a is the *dessin d'enfant* of P. For a general Shabat polynomial, its dessin d'enfant is obtained as the preimage of an arc connecting its two finite critical values. It is evident that the dessin d'enfant of any Shabat polynomial is a tree. Conversely, every planar tree $\mathcal{T}$ is the dessin d'enfant of a Shabat polynomial, see [LZ04, Theorem 2.2.9]. Roughly speaking, one considers a set of half-planes

$$\bigsqcup_{\text{edges } e \text{ of } \mathcal{T}} \{\mathbb{H}_e^+, \mathbb{H}_e^-\},$$

that is, two half-planes for each edge e of the tree $\mathcal{T}$. Then, one constructs an abstract Riemann surface Δ' by gluing these half-planes together as indicated by the structure of $\mathcal{T}$. There is a natural holomorphic map $g: \Delta' \to \mathbb{C}$ that maps the half-planes in Δ' alternatingly to the upper and lower half-planes in $\mathbb{C}$. Using the uniformization theorem one obtains a conformal map $\varphi: \Delta' \to \mathbb{C}$. The map $g \circ \varphi^{-1}$ is then the desired map, which can be shown to be a Shabat polynomial. Put differently, the construction of a Shabat polynomial is very similar to the construction of the rational maps in Section 3 and Section 7. The half-planes $\mathbb{H}_e^+$ and $\mathbb{H}_e^-$ correspond to the white and black 1-tiles from which Δ' was constructed there. Finally, we note that a general dessin d'enfant is not necessarily planar or a tree. The concept was introduced by Grothendieck in [Gro97] as a way to describe algebraic curves.

10. P is not renormalizable

Here, we show that the polynomial P from the previous section is not renormalizable. That is, we verify that P satisfies property (3) of Theorem 1.4. This means P is a counterexample to the the conjecture stated in Section 1. However the conjecture may be still valid for quadratic polynomials. We first recall some relevant definitions.

A continuous map $f: U \to V$ is said to be *proper* if $f^{-1}(K)$ is compact in U for every compact $K \subset V$. Assume now in addition that U and V are Jordan domains in $\mathbb{C}$ and that f is holomorphic. In this case f extends continuously to ∂U and this extension (still denoted by f) satisfies $f(\partial U) = \partial V$. The map $f: \partial U \to \partial V$ is topologically conjugate to $z^d: S^1 \to S^1$ for a $d \in \mathbb{N}$ (here we view the unit circle S^1 as the boundary of the unit disk $\mathbb{D}$). Furthermore, every point in V has exactly d preimages in U when counted with multiplicity (see [Mil06, Problem 15-c]). The number d is then called the *degree* of the proper map $f: U \to V$.

Definition 10.1. A *polynomial-like map of degree* $d \geq 1$ is a triple (f, U, V), where $U, V \subset \mathbb{C}$ are Jordan domains such that $\overline{U}$ is a compact subset of V and $f: U \to V$ is a proper holomorphic map of degree d.

Polynomial-like maps were introduced by Douady and Hubbard in [DH85]. The above definition differs slightly from the typical one found in the literature, as we allow the case $d = 1$. The *filled Julia set* of a polynomial-like map (f, U, V), denoted by $\mathcal{K}(f|U)$, is the set of points in U that never leave U under iteration of f, that is,

$$\mathcal{K}(f|U) = \bigcap_{n\geq 0}(f|U)^{-n}(U).$$

It is a compact subset of U.

Definition 10.2. Let $P\colon \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$ with connected Julia set. Let $n \in \mathbb{N}$, then $P^n\colon U \to V$ is called a *renormalization* of P if

(1) (P^n, U, V) is a polynomial-like map;
(2) $\mathcal{K}(P^n|U)$ is connected;
(3) the degree δ of (P^n, U, V) satisfies

$$2 \leq \delta < d^n.$$

If such a renormalization $P^n\colon U \to V$ exists for some $n \in \mathbb{N}$, then we call the polynomial P *renormalizable.*

Condition (3) in the above definition excludes trivial polynomial-like restrictions, that is, P itself or maps of degree 1. Renormalization has been mostly considered in the case of quadratic polynomials, see for example [McM94]. The higher degree case has been considered in [Ino02], see also [LMS15]. Note that in the non-quadratic case there are several distinct definitions of renormalization. We have chosen the most general one.

Theorem 10.3. *Let $P\colon \mathbb{C} \to \mathbb{C}$ be a polynomial with exactly two finite postcritical points that does not have finite periodic critical points. Then P is not renormalizable.*

For the proof we use the following elementary fact that is a consequence of the Riemann-Hurwitz formula [Mil06, Theorem 7.2]. Let $V \subset \widehat{\mathbb{C}}$ be a Jordan domain that contains a single postcritical point p of a polynomial P. Then every component of $P^{-1}(V)$ is a Jordan domain that contains exactly one point from $P^{-1}(p)$.

Proof. Let P be a polynomial as in the statement. Assume that P is renormalizable, that is, there exists a polynomial-like map (P^n, U, V) as in Definition 10.2 for some $n \in \mathbb{N}$. Denote by δ its degree. Let p and q be the two finite postcritical points of P and $\mathcal{K}(P^n|U)$ be the filled Julia set of (P^n, U, V).

Clearly, if V contains no postcritical point of P, then the polynomial-like map (P^n, U, V) has degree $\delta = 1$ by the Riemann-Hurwitz formula. Thus (P^n, U, V) is not a renormalization of P.

Without loss of generality, we may assume from now on that $p \in V$. Two cases remain, namely $q \in V$ and $q \notin V$.

First suppose that $q \in V$. This means that

$$\{p, q\} = \operatorname{post}(P) \setminus \{\infty\} = \operatorname{post}(P^n) \setminus \{\infty\} \subset V.$$

Thus the closed Jordan domain $A_\infty \coloneqq \widehat{\mathbb{C}} \setminus V$ contains a single postcritical point of P^n, namely ∞, in its interior. Note that every component of $P^{-n}(A_\infty)$ contains a preimage of ∞, hence there is only one such component. Using the remark before the proof, we conclude that $P^{-n}(A_\infty) = \widehat{\mathbb{C}} \setminus P^{-n}(V)$ is a Jordan domain. It follows that $P^{-n}(V)$ consists of a single component, namely U. This in turn implies that the degree of the polynomial-like map (P^n, U, V) is $\delta = \deg(P^n) = \deg(P)^n$. This contradicts condition (3) in Definition 10.2, meaning that (P^n, U, V) is not a renormalization of P.

Now suppose that $q \notin V$, that is, V contains exactly one postcritical point, namely p. By the remark after Theorem 10.3, U contains exactly one point in $P^{-n}(p)$, which we denote by c. Then c is a critical point of P^n; for otherwise the degree of (P^n, U, V) is $\delta = 1$, contradicting condition (3) in Definition 10.2. Since P has no periodic critical points, $c \neq p$. Thus the set $(P^n|U)^{-1}(c) = (P^n|U)^{-2}(p)$ contains exactly δ distinct points $c_1, \dots, c_\delta$. It follows that $(P^n|U)^{-1}(U) = (P^n|U)^{-2}(V)$ has exactly δ components $U_1, \dots, U_\delta$ (where $c_j \in U_j$ for $j = 1, \dots, \delta$). Furthermore, $\overline{U_j} \subset U$ for each j, since $\overline{U} \subset V$ and $P^n \colon U \to V$ is proper. Iterating this process, we see that each U_j contains δ components of $(P^n|U)^{-2}(U)$, and so on. It follows that $U_j \cap \mathcal{K}(P^n|U) \neq \varnothing$ for all $j = 1, \dots, \delta$. Hence $\mathcal{K}(P^n|U)$ is disconnected, contradicting condition (2) in Definition 10.2. Thus (P^n, U, V) is not a renormalization of P. $\square$

Theorem 10.3 implies that the polynomial P from Section 9 is not renormalizable. Thus P satisfies property (3) of Theorem 1.4. This finishes the proof of Theorem 1.4.

11. Appendix A

In the literature, one usually studies properties of a self-similar group G via its wreath recursion. In particular, see the proof of exponential growth of the Basilica group, that is, $\mathrm{IMG}(z^2 - 1)$, in [GŻ02, Proposition 4] or the study of iterated monodromy groups of postcritically-finite quadratic polynomials in [BN08]. The goal of this appendix is to show how these techniques apply in our setting, that is, when G is the iterated monodromy group of a Thurston map.

11.1. Appendix A.1: Actions on rooted trees and self-similar groups.

For the convenience of the reader we review the definition of self-similar groups and some closely related notions, more details can be found in [Nek05, Chapters 1 and 2].

Choose an *alphabet* X of d letters. The set of all words in X of length $n \in \mathbb{N}$ is denoted by X^n. The empty word is the only word of length 0 and denoted by $\varnothing$. Consequently we set $X^0 = \{\varnothing\}$. Let $X^* = \bigcup_{n \in \mathbb{N}_0} X^n$ be the set of all finite words in the alphabet X. Then X^* has a natural structure of a d-ary rooted tree: we define the root to be $\varnothing$ and connect every word $v \in X^n$ to all words of the form $vx \in X^{n+1}$ for an arbitrary letter $x \in X$ and each $n \in \mathbb{N}_0$. The set X^* viewed as a rooted tree is called the *tree of words* in the alphabet X and is denoted by T. It is evident that the n-th level of the tree T is given by the words in X^n, $n \in \mathbb{N}_0$.

Let $\mathrm{Aut}(T)$ be the automorphism group of the tree T, that is, the group of all bijective maps $g : T \to T$ that preserve the adjacency of the vertices of T. We consider the *right action* of $\mathrm{Aut}(T)$ on the tree T. So, the image of a vertex v under the action of an element $g \in \mathrm{Aut}(T)$ is denoted by v^g, and in the product $g_1 g_2$ the element g_1 acts first.

Definition 11.1. Let G be a subgroup of $\mathrm{Aut}(T)$. The *n-th level stabilizer* is the subgroup $\mathrm{Stab}_G(n)$ of those elements of G that fix pointwise all vertices of the n-th level X^n of the tree T. That is,

$$\mathrm{Stab}_G(n) = \{g \in G : v^g = v \text{ for all } v \in X^n\}.$$

The n-th level stabilizer is a normal subgroup in G of finite index for all $n \in \mathbb{N}_0$.

Let $v \in X^*$ be an arbitrary vertex of the tree of words T. Denote by T_v the subtree of T rooted at v such that the vertex set of T_v is $\{vx : x \in X^*\}$. Clearly, T_v is isomorphic to the whole tree of words T via the shift $\iota_v : T_v \to T$ defined by $vu \mapsto u$ for $u \in X^*$.

For every $g \in \mathrm{Aut}(T)$ and $v \in X^*$, we define an automorphism $g|_v : T \to T$, called the *restriction* of g to the subtree T_v, by

$$g|_v = \iota_{v^g} \circ g \circ \iota_v^{-1}.$$

To simplify the notation assume that $X = \{1, ..., d\}$. Denote by $\Sigma(X)$ the symmetric group of permutations of the set X. Then every element $g \in \mathrm{Aut}(T)$ can be written in the following form, called the *wreath recursion* of g,

$$g = \langle\!\langle g|_1, \dots, g|_d \rangle\!\rangle \sigma_g,$$

where $\langle\!\langle g|_1, \dots, g|_d \rangle\!\rangle \in \mathrm{Stab}_{\mathrm{Aut}(T)}(1) \cong \mathrm{Aut}(T)^X$, that is, an element of the direct product $\mathrm{Aut}(T) \times \cdots \times \mathrm{Aut}(T)$ with $d = \#X$ factors, and $\sigma_g \in \Sigma(X)$ is the permutation equal to the action of g on X^1. Formally, there is a canonical isomorphism

$$\psi : \mathrm{Aut}(T) \to \mathrm{Aut}(T)^X \rtimes \Sigma(X), \tag{11.1}$$

where the semidirect product is taken with respect to the natural action of $\Sigma(X)$ on the factors of $\mathrm{Aut}(T)^X$ (an element $\sigma \in \Sigma(X)$ acts on $\mathrm{Aut}(T)^X$ by the permutation of the factors coming from its action on X, that is, $(g_1, \dots, g_d)^\sigma = (g_{1^\sigma}, \dots, g_{d^\sigma})$). In other words, the automorphism group $\mathrm{Aut}(T)$ is isomorphic to the *permutational wreath product*

$\mathrm{Aut}(T) \wr \Sigma(X)$. That is, the wreath recursions of arbitrary elements $g, h \in \mathrm{Aut}(T)$ are multiplied according to the rule

$$\langle\!\langle g_1, \dots, g_d \rangle\!\rangle \sigma \cdot \langle\!\langle h_1, \dots, h_d \rangle\!\rangle \tau = \langle\!\langle g_1 h_{1^\sigma}, \dots, g_d h_{d^\sigma} \rangle\!\rangle \sigma\tau.$$

Definition 11.2. A group $G < \mathrm{Aut}(T)$ acting faithfully on the tree $T = X^*$ is said to be *self-similar* if for every $g \in G$ and every $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$(xv)^g = y(v)^h$$

for all $v \in X^*$. Put differently, G is called self-similar if each restriction $g|_v$ belongs to G for all $g \in G$ and $v \in X^*$.

It is clear from the definition that a group $G < \mathrm{Aut}(T)$ is self-similar if and only if $g|_x \in G$ for all $x \in X^1$ and all generators g of G. Furthermore, every self-similar group G has an *associated wreath recursion*, that is, a homomorphism $\psi : G \to G \wr \Sigma(X)$ given by the restriction of the canonical isomorphism (11.1) to G.

Definition 11.3. Let $G < \mathrm{Aut}(T)$ be a self-similar group. G is said to be *recurrent* (or *self-replicating*) if its action is transitive on the first level X^1 of the tree $T = X^*$ and for some (and thus for all) $x \in X$ the homomorphism $\psi_x : G_x \to G$ given by $g \mapsto g|_x$ is onto, where $G_x = \{g \in G : x^g = x\}$ is the stabilizer of x in G.

Note that if a self-similar group is recurrent, then it is transitive on every level of the tree T (the group is then called *level-transitive*).

Definition 11.4. A self-similar group $G < \mathrm{Aut}(T)$ is called *regular branch* if there exists a finite index subgroup H of G such that $H^X < \psi(H)$, where $\psi : G \to G \wr \Sigma(X)$ is the wreath recursion associated with G. In such a case we say that G is regular branch over H.

11.2. **Appendix A.2: Further properties of** $\mathrm{IMG}(f_1)$**.** Here we revisit the iterated monodromy group of the map f_1 from Section 3. Let a, b, c be the generators of $\mathrm{IMG}(f_1)$ as in Section 4. That is, $\mathrm{IMG}(f_1)$ acts on the dynamical preimage tree $\mathbf{T}_{f_1}$ that is identified with $\{1, \dots, 6\}^*$ and the wreath recursions of the generators are given by (4.1). We use this to prove additional properties of $\mathrm{IMG}(f_1)$. The discussion is kept rather short, to avoid excessive details.

We start with giving alternative proofs of Corollary 5.3 and Lemma 5.4 (which implies exponential growth of $\mathrm{IMG}(f_1)$).

Proof. To save space we will only prove that $x \coloneqq ab^4$ and $y \coloneqq ab^{12}$ generate a free semigroup in $\mathrm{IMG}(f_1)$ (the proof is analogous if we introduce the third generator $z \coloneqq ab^{20}$).

Using (4.1) one can compute the following identities

$$\begin{aligned} x^2 &= \langle\!\langle x, \cdot, x^s, \cdot, \cdot, \cdot \rangle\!\rangle \\ y^2 &= \langle\!\langle y, \cdot, y^s, \cdot, \cdot, \cdot \rangle\!\rangle \\ xy &= \langle\!\langle y, \cdot, x^s, \cdot, \cdot, \cdot \rangle\!\rangle \\ yx &= \langle\!\langle x, \cdot, y^s, \cdot, \cdot, \cdot \rangle\!\rangle, \end{aligned} \tag{11.2}$$

where $s = ab^{-1}$ and $\cdot$ represents the omitted terms.

The first two identities imply that x and y are elements of infinite order in $\mathrm{IMG}(f_1)$. Indeed, the order of x cannot be an odd number, because x acts as the permutation $(13)(25)(46)$ on the first level of the dynamical preimage tree $\mathbf{T}_{f_1}$. If $\mathrm{ord}(x) = 2n$, $n \in \mathbb{N}$, then (11.2) imply that $x^n = 1$, contradiction. The same argument applies to y.

Now let w be an arbitrary word in x and y. Here and throughout the proof, $|w|$ denotes the length of the word w with respect to the alphabet $\{x, y\}$. If $|w|$ is even, w can be written uniquely as a product of x^2, y^2, xy, and yx. From (11.2) it follows that $\psi_1(w)$ is a word in $\{x, y\}$ and $|\psi_1(w)| = |w|/2$, where ψ_1 denotes the projection onto the first coordinate in the canonical isomorphism (11.1), that is, $\psi_1(w) = w|_1$. Moreover, $\psi_1(w)$ ends with the same letter (x or y) as w.

Consider now any two distinct words w_1 and w_2 in the alphabet $\{x, y\}$. We are going to show that they represent different group elements in $\mathrm{IMG}(f_1)$. The proof is by induction on $|w_1|+|w_2|$.

The base cases $|w_1| + |w_2| \in \{1, 2\}$ can easily be verified. Now let $|w_1| + |w_2| \geq 3$. By multiplying from the right with the inverse of the common ending word, it is enough to assume that w_1 and w_2 end with different letters. We can further assume that the parity of $|w_1|$ and $|w_2|$ is the same (for otherwise, $w_1 w_2^{-1}$ does not fix the first level). If $|w_i|$ is odd set $w_i' = x w_i$, otherwise set $w_i' = w_i$, for $i = 1, 2$. Then $\psi_1(w_1')$ and $\psi_1(w_2')$ are two distinct words in $\{x, y\}$. Indeed, they end with different letters since w_1' and w_2' do. Furthermore,

$$|\psi_1(w_1')| + |\psi_1(w_2')| \leq \frac{|w_1| + 1}{2} + \frac{|w_2| + 1}{2} < |w_1| + |w_2|.$$

Thus the induction hypothesis applies to $\psi_1(w_1')$ and $\psi_1(w_2')$, which finishes the proof. □

To simplify the notation we denote the iterated monodromy group $\mathrm{IMG}(f_1)$ by G from now on.

Lemma 11.5. *G is recurrent, consequently, it is level-transitive.*

Proof. It is evident that G acts transitively on the first level of $\mathbf{T}_{f_1}$, see Figure 5. Using (4.1), we check that $b, (b^4)^c \in G_1$ $(= \{g \in G : 1^g = 1\})$ and $\psi_1(b) = b$, $\psi_1((b^4)^c) = c^{-1}b^{-1}$. Since $\langle b, c^{-1}b^{-1} \rangle = G$, the statement follows. □

Proposition 11.6. *G is regular branch over*

$$H := \langle [b^2, c] \rangle^G = \langle [b^2, c]^g : g \in G \rangle.$$

Proof. First we verify that $H^6 < \psi(H)$. To this end, we use the wreath recursions (4.1) and the fact that $(bc)^2 = c^3 = 1$ to obtain the following identities. [4]

$$\begin{aligned}
b^{-8} &= \langle\langle b^{-8}, (cb)^2, (bc)^2, c^{-8}, (bc)^2, (cb)^2 \rangle\rangle \\
&= \langle\langle b^{-8}, 1, 1, c, 1, 1 \rangle\rangle \\
b^8 c^{-1} b^{-4} c &= \langle\langle b^9 c, (b^{-1}c^{-1})^2 b^{-4}, (c^{-1}b^{-1})^2 cb, c^9 b, (c^{-1}b^{-1})^2 c^{-4}, (b^{-1}c^{-1})^2 bc \rangle\rangle \\
&= \langle\langle b^9 c, b^{-4}, cb, b, c^{-1}, bc \rangle\rangle \\
(b^8 c^{-1} b^{-4} c)^2 &= \langle\langle (b^9 c)^2, b^{-8}, 1, b^2, c, 1 \rangle\rangle.
\end{aligned}$$

Then we consider the following commutator

$$\begin{aligned}
[(b^8 c^{-1} b^{-4} c)^2, b^{-8}] &= \langle\langle [(b^9 c)^2, b^{-8}], 1, 1, [b^2, c], 1, 1 \rangle\rangle \\
&= \langle\langle 1, 1, 1, [b^2, c], 1, 1 \rangle\rangle.
\end{aligned} \tag{11.3}$$

Here we use that $[(b^9c)^2, b^{-8}] = 1$ as one can verify (At the same time, $[b^9 c, b^{-8}] \neq 1$. For that reason H is not chosen to be the commutator subgroup $[G, G]$ of G).

Since G is recurrent, (11.3) implies that $1 \times 1 \times 1 \times H \times 1 \times 1 < \psi(H)$, consequently, $H^6 < \psi(H)$.

Thus, we are only left to check that H has finite index in G. By construction, H is a normal subgroup of G. Furthermore, a generic element of the quotient group G/H can be written in the form

$$(b) c^{\pm 1} b c^{\pm 1} \dots b c^{\pm 1} b^k, \quad k \in \{0, \dots, 23\}, \tag{11.4}$$

where the notation "(b)" means that b may or may not appear as the first letter in the word. Since $(bc)^2 = 1$, we can further normalize (11.4) to

$$(b) c^{\pm 1} b^k, \quad k \in \{0, \dots, 23\}.$$

Hence the quotient subgroup G/H is finite. □

We close the appendix with two corollaries of the previous proposition.

Corollary 11.7. *G contains a subgroup isomorphic to $\mathbb{Z}^n$ for each $n \in \mathbb{N}$.*

Proof. First, we observe that $[c, b^4] \in H$ is of infinite order. Indeed, $[c, b^4] \in \mathrm{Stab}_G(1)$ and $\psi_1([c, b^4]) = (ab^4)^{b^{-1}}$. Now the statement follows from $H^6 < \psi(H)$. □

[4] The "FR package", written by L. Bartholdi, for the computer algebra system GAP, is very helpful for such computations.

Corollary 11.8. $\mathrm{IMG}(f_1)$ *has a finite* endomorphic presentation*, that is, a finite recursive presentation as introduced in* [Bar03]*. However, it is not finitely presented.*

Proof. An immediate corollary of [Bar03, Theorem 1], since iterated monodromy groups of postcritically-finite rational maps are *contracting* [Nek05, Theorem 6.4.4]. ☐

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