ON THE NOTIONS OF MATING

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Abstract: The different notions of matings of pairs of equal degree polynomials are introduced and are related to each other as well as known results on matings. The possible obstructions to matings are identified and related. Moreover the relations between the polynomials and their matings are discussed and proved. Finally holomorphic motion properties of slow-mating are proved.

Resumé: Les différentes notions de accouplements de paires de polynômes de mêmes degré sont introduits et sont reliées les uns aux autres ainsi que les résultats connus concernant les accouplements. Les obstructions possibles à les accouplements sont identifiés et liés. En plus, les relations entre les polynômes et leurs accouplements sont discutées et prouvés. Enfin des propriétés de mouvement holomorphes de l'accouplement lente sont prouvés.

1. Introduction

The notion of mating of two polynomials of the same degree $d > 1$ was introduced by Douady and Hubbard in order to understand and classify dynamically certain rational maps. Since its introduction and the first proven results by Tan Lei, Mary Rees and Mitsuhiro Shishikura, several more or less equivalent notions of matings have been introduced and in use. This paper aims at introducing the different notions of matings and their applications. We will discuss along the way the different issues which naturally arise in connection with the different definitions of matings.

Fundamentally there are two different views on mating.

- The constructive approach: Mating is a procedure to construct new rational maps by combining two polynomials.
- The descriptive approach: Mating is a way to understand the dynamics of certain rational maps in terms of pairs of polynomials.

We shall pursue both views. In order to set the scene for the discussion properly let us start with reviewing the relevant background definitions and theorems. The reader novel to holomorphic dynamics will find full details and enlarged discussions in any of the monographs [Mi1], [C-G].

2. Equivalence relations

In the construction of the mating we take the quotient of a highly non-trivial equivalence relation. We shall thus take our starting point in a discussion of equivalence relations and the properties of quotients by equivalence relations.

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2.1. **The lattice of equivalence relations.** Let $R \subset S \times S$ be a relation on the set S. We shall follow the usual custom of writing xRy, $x \sim y$ or $x \sim_R y$ for $(x, y) \in R$. The three forms will be used interchangeably. Relations on S are naturally *partially ordered* by inclusion, i.e., the relation R is *bigger* than the relation Q if $Q \subset R$ or

$$\forall\, x, y \in S : \qquad xQy \Rightarrow xRy,$$

The set of all relations on S forms a *lattice* when equipped with this partial ordering. This means that *join* and *meet* are well defined. Recall that the join $Q \vee R = Q \cup R$ is the smallest relation bigger than Q and R. Similarly the *meet* $Q \wedge R = Q \cap R$ is the biggest relation smaller than Q and R. Similarly the join and meet of an arbitrary family of relations is defined. Note that the meet of a family of equivalence relations is again an equivalence relation.

Let R be a relation on S. The *equivalence relation generated* by R is the smallest equivalence relation bigger than R, i.e., the meet of all equivalence relations bigger than R.

2.2. **Closed equivalence relations.** The set of equivalence classes of an equivalence relation R on a set S forms a *decomposition*, i.e., a family of pairwise disjoint subsets of S whose union is S. Clearly each decomposition of S induces an equivalence relation on S (two points of S are equivalent if and only if they are contained in the same set of the decomposition). We denote by $[x]$ or more detailed $[x]_R$ the R-equivalence class of x. We denote by $S/{\sim} = S/R = \{[x] \mid x \in S\}$ the space of equivalence classes or quotient space and by $\Pi = \Pi_R \colon S \longrightarrow S/{\sim}$ the natural projection $\Pi(x) = [x]$. When S is a topological space we shall always assume that the quotient space $S/{\sim}$ is equipped with the quotient topology. That is a subset $U \subset S/{\sim}$ is open if and only if $\Pi^{-1}(U)$ is open in S.

Instead of quotient spaces one talks in geometric topology about *decomposition spaces*. The standard reference is Daverman's book [Dav]. We will however stick to talking about equivalence relations, instead of decompositions.

Definition 2.1 (saturation)**.** Let $\sim$ be an equivalence relation on a set S. A set $U \subset S$ is called *saturated* if $x \in U, y \sim x \Rightarrow y \in U$, equivalently $U = \Pi^{-1}(\Pi(U))$, equivalently U is a union of equivalence classes.

The *saturated interior* of a set $V \subset S$ is the set

$$V^* := \bigcup \{[x] \mid [x] \subset V\},$$

i.e., the biggest saturated set contained in V. Note that V^* may be empty, even if V is non empty.

The *saturation* of a set $A \subset S$ is the set

$$A^\dagger := \bigcup \{[x] \mid x \in A\} = \Pi^{-1}(\Pi(A)),$$

i.e., the smallest saturated set containing A.

Recall that a topological space S is Hausdorff if and only if every two distinct points have disjoint neighborhoods or equivalently if and only if the diagonal $\Delta = \Delta_S = \{(x, x) \mid x \in S\} \subset S \times S$ is closed. Here as elsewhere we suppose the later equipped with the product topology.

In general there is very little that can be said about the quotient space. However the standard assumption is that the equivalence relation is *closed*, the importance of which is shown by the following.

Definition and Lemma 2.2. Let S be a compact metric space. An equivalence relation $\sim$ on S is *closed* if each $[x]$ is compact and one (hence all) of the following equivalent conditions is satisfied.

(1) The set $\{(s,t) \mid s \sim t\} \subset S \times S$ is closed.
(2) Let $(s_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}}$ be convergent sequences in S. Then
$$s_n \sim t_n \text{ for all } n \in \mathbb{N}, \text{ implies } \lim s_n \sim \lim t_n.$$
(3) For any compact subset $C \subseteq S$: If C is the Hausdorff limit of a sequence of equivalence classes $([x_n])_{n\in\mathbb{N}}$, then $C \subseteq [x]$ for some $x \in S$.
(4) For any equivalence class $[x]$ and any neighborhood U of $[x]$ there is a neighborhood $V \subset U$ of $[x]$, s.t.
$$[y] \cap V \neq \emptyset \Rightarrow [y] \subset U.$$
(5) Each neighborhood U of any equivalence class $[x]$ contains a *saturated* neighborhood V of $[x]$.
(6) For each open set U the set saturated interior U^* is open.
(7) The quotient topology on $S/\sim$ is Hausdorff.
(8) For any closed set $K \subset S$ the saturation
$$K^\dagger := \bigcup \{[x] \mid x \in K\}$$
is closed.
(9) The quotient map $\Pi\colon S \to S/\sim$ is closed.
(10) The quotient topology on $S/\sim$ is metrizable.

Proof. A round robin style proof of the equivalence of (1) to (7) is:
(1) $\Rightarrow$ (2) is a general property of closed sets.
(2) $\Rightarrow$ (3), because any two points $z, w \in C$ are limits of sequences $(z_n), (w_n)$ with $z_n, w_n \in [x_n]$ for all n.
(3) $\Rightarrow$ (4), by contra position, if not then there exists an equivalence class $[x]$, a neighborhood U of $[x]$, which we can suppose to be open, a sequence of equivalence classes $[x_n]$ and points $z_n, w_n \in [x_n]$ such that z_n converges to $z \in [x]$ and $S\backslash U \ni w_n$. Passing to a subsequence if necessary we can suppose $[x_n]$ converges to a compact set C, which by construction intersects both $[x]$ and the compact complement $S\backslash U$ and thus is not a subset of any equivalence class.
(4) $\Rightarrow$ (5), let $[x]$, U and V be as in (4), then the saturation $V^\dagger$ fulfills the requirements.
(5) $\Rightarrow$ (6) follows immediately.
(6) $\Rightarrow$ (7). Let $[x]$ and $[y]$ be distinct and hence disjoint equivalence classes. Let $U([x])$ and $U([y])$ be corresponding disjoint open neighborhoods in S. Then $\Pi(U^*([x]))$ and $\Pi(U^*([y]))$ are disjoint neighborhoods of $\Pi(x)$ and $\Pi(y)$.
(7) $\Rightarrow$ (1) Let $\Phi = \Pi \times \Pi\colon S \times S \longrightarrow S/\sim \times S/\sim$ i.e., $\Phi(x,y) = (\Pi(x), \Pi(y))$, then $R = \Phi^{-1}(\Delta_{S/\sim})$ is closed, because Φ is continuous and $S/\sim$ is Hausdorff.
(6) $\Leftrightarrow$ (8) By definition, if $U = S\backslash K$ then $U^* = S\backslash K^\dagger$.
(8) $\Leftrightarrow$ (9) by definition of the quotient topology, as $\Pi^{-1}(\Pi(K)) = K^\dagger$.
Finally trivially (10) $\Rightarrow$ (7) whereas
(7) and its equivalent (6) $\Rightarrow$ (10), because a compact Hausdorff space is metrizable if and only if it is second countable, so that we just need to exhibit a countable basis for the quotient topology on $S/\sim$. For $\epsilon > 0$ and $x \in S$ write $U([x], \epsilon) = \{x' \mid d(x', [x]) < \epsilon\}$. Then for each fixed $m \in \mathbb{N}$ the open saturated sets $U^*([x], 1/m) \subseteq$

$U([x], 1/m)$ form a covering of the compact space S. Hence we may extract a finite sub-covering $V_{m,1} = U^*([x_{m,1}], m), \ldots, V_{m,n_m} = U^*([x_{m,n_m}], m)$. The projected sets $\Pi(V_{m,i})$ form a countable basis for the quotient topology on $S/\sim$: An open neighborhood of a point $\Pi(x)$ is the projection of a saturated neighborhood V of $[x]$. We shall thus find m and i such that $V_{m,i} \subset V$. Let $0 < \delta = \inf\{(x', [x]) \mid x' \in S \backslash V\}$. And let $0 < \epsilon = \inf\{(x', [x]) \mid x' \in S \backslash U^*([x], \delta/2)\}$. For m with $1 < m\epsilon$ choose i such that $[x] \subset V_{m,i}$. Then there exists $y \in [x_{m,i}]$ with $(y, [x]) < 1/m < \epsilon$ and hence $[x_{m,i}] \subset U^*([x], \delta/2) \subseteq U([x], \delta/2)$. Thus $V_{m,i} \subset \dot{U}([x], \delta) \subseteq V$, as $1/m < \epsilon \leq \delta/2$. □

Property (3) shows that in the case when $\sim$ is closed "small equivalence classes can converge to bigger equivalence classes" (but not vice versa!). This is the reason that closed equivalence relations are also called *upper semi-continuous*. The standard definition for upper semi-continuity (in general topological spaces) is (4).

Let $\varphi\colon S \to S'$ be a map. Then φ *induces* an equivalence relation $\sim = \sim_\varphi$ on S:

$$s \sim t \quad \text{if and only if} \quad \varphi(s) = \varphi(t),$$

for all $s, t \in S$. That is $\sim = (\varphi \times \varphi)^{-1}(\Delta_{S'})$. Evidently the induced map $\psi\colon S/\sim \longrightarrow \varphi(S)$ given by $\psi([x]) = \varphi(x)$ is a bijection. Moreover if S, S' are topological spaces and φ is continuous, then ψ is continuous.

The following is a standard topological fact, for which the reader shall easily provide a proof.

Lemma 2.3. *Let $\varphi\colon S \to S'$ be a continuous surjective map between topological spaces and $\sim$ be the equivalence relation on S induced by φ. Then the induced map $\psi\colon S/\sim \longrightarrow S'$ is a homeomorphism, if and only if $\varphi(U)$ is open for every saturated open set. In particular ψ is a homeomorphism if S is compact and S' is Hausdorff. In this case $\sim$ is furthermore closed.*

Lemma 2.4 (Closure of equivalence relation)**.** *Let $\sim$ be an equivalence relation on a compact metric space S. Then there is a unique smallest closed equivalence relation $\widehat{\sim}$ bigger than $\sim$. We call $\widehat{\sim}$ the* closure *of $\sim$.*

Proof. Let $(R_j)_{j \in J}$ be the family of all closed equivalence relations bigger than $\sim$. Then we define $\widehat{\sim}$ as the meet (i.e., the intersection) of all R_j. Clearly $\widehat{\sim}$ is a closed equivalence relation, bigger than $\sim$, and the smallest such relation. Uniqueness is evident as well. □

Note that $\{(s,t) \mid s \widehat{\sim} t\}$ is generally *not* the closure of $\{(s,t) \mid s \sim t\}$, which may fail to be transitive. An explicit description of $\widehat{\sim}$ is given in the proof of Proposition 2.8.

2.3. **Equivalence relations and (semi-)conjugacies.** Let S_1 and S_2 be spaces equipped with equivalence relations $\sim_1$ and $\sim_2$ respectively. A mapping $f\colon S_1 \longrightarrow S_2$ is called a *semi-conjugacy for the equivalence relations* $\sim_1$ and $\sim_2$ if $x \sim_1 y \Rightarrow f(x) \sim_2 f(y)$ for all $x, y \in S_1$. Equivalently $f^{-1}(U)$ is $\sim_1$-saturated for any $\sim_2$-saturated set $U \subset S_2$. The map f descends to (or induces) a map $F\colon S_1/\sim_1 \longrightarrow S_2/\sim_2$ between the quotients $S_1/\sim_1$ and $S_2/\sim_2$ given by

$$F([x]_1) := [f(x)]_2,$$

for all $x \in S_1$, if and only if f semi-conjugates $\sim_1$ to $\sim_2$.

If $S_1 = S_2 = S$ and $\sim_1 = \sim_2 = \sim$ we also say that $\sim$ is *f-invariant* and write $f/\!\sim$ for the quotient map F above.

Lemma 2.5. *Suppose $f\colon S_1 \longrightarrow S_2$ is continuous and that f semi-conjugates $\sim_1$ to $\sim_2$. Then $F := f/\!\sim\, : S_1/\!\sim_1 \longrightarrow S_2/\!\sim_2$ is continuous.*

Proof. Let $[U'] \subset S_2/\!\sim_2$ be open. Then $U := \Pi_2^{-1}([U']) \subset S_2$ is open and $\sim_2$ saturated. Thus $f^{-1}(U) \subset S_1$ is open and $\sim_1$ saturated so that $\Pi_1(f^{-1}(U)) = F^{-1}([U']) \subset S_1/\!\sim_1$ is open. □

Lemma 2.6. *Let R be a relation on a set S that is invariant with respect to $f\colon S \to S$, i.e., $xRy \Rightarrow f(x)Rf(y)$ for all $x, y \in S$. Then the equivalence relation $\sim$ generated by R is f-invariant.*

The proof is left as an easy exercise.

Let $f\colon S \to S$, $g\colon S' \to S'$ be self-maps of the sets S, S' respectively, which we consider as dynamical systems. A *semi-conjugacy* from f to g is a surjection $\varphi\colon S \to S'$ such that $\varphi \circ f = g \circ \varphi$ on S, i.e., the following diagram commutes

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\varphi"'] & S \arrow[d, "\varphi"] \\ S' \arrow[r, "g"] & S'. \end{tikzcd}$$

We say f is semi-conjugate to g by φ. The map $g\colon S' \to S'$ is called a *factor* of the dynamical system $f\colon S \to S$. If S, S' are topological spaces and φ is continuous, then φ is called a *topological semi-conjugacy*. If φ is a homeomorphism φ is called a *topological conjugacy*. Similarly if φ is (quasi-) conformal, we call φ a (quasi-) conformal conjugacy.

Lemma 2.7. *Let f be semi-conjugate to g by φ as above. Then the following holds.*

- *The equivalence relation $\sim$ induced by φ is f-invariant.*
- *Assume φ is a topological semi-conjugacy, S compact, and S' is Hausdorff. Then $f/\!\sim\, : S/\!\sim\, \to S/\!\sim$ is continuous and topologically conjugate to $g\colon S' \to S'$.*

Proof. The first statement follows immediately from the definitions. Indeed, let $x \sim y$ for some $x, y \in S$. Then $\varphi(x) = \varphi(y)$. Thus $\varphi \circ f(x) = g \circ \varphi(x) = g \circ \varphi(y) = \varphi \circ f(y)$, which means that $f(x) \sim f(y)$.

To see the second statement, note that $S/\!\sim$ is homeomorphic to S' by Lemma 2.3, $f/\!\sim$ is continuous by Lemma 2.5. It is easily verified that the homeomorphism $\psi\colon S/\!\sim \to S'$ satisfies $g \circ \psi = \psi \circ f/\!\sim$. □

Proposition 2.8. *Let S be a compact metric space, $f\colon S \to S$ be a continuous map, and $\sim$ be an f-invariant equivalence relation. Then the closure $\widehat{\sim}$ of $\sim$ is f-invariant.*

Proof. Let $f\colon S \to S$ be continuous.

We first give an explicit description of the closure $\widehat{\sim}$. We will use ordinal numbers, i.e., transfinite induction.

Let $\sim$ be any equivalence relation on S. Here we think of $\sim$ as a subset of $S \times S$. Let $\eqsim$ be the closure of $\sim$ in $S \times S$. Note that $\eqsim$ is not necessarily an equivalence relation.

Now we define $\sim'$ as the equivalence relation generated by $\eqsim$. Note that this equivalence relation is not necessarily closed.

Claim 1. If $\sim$ is f-invariant then $\sim'$ is f-invariant as well.

Proof of Claim 1. We first note that $\eqsim$ is given by the following. For all $x, y \in S$ it holds $x \eqsim y$ if and only if there are sequences $x_n \to x$, $y_n \to y$ in S, such that $x_n \sim y_n$ for all $n \in N$. Since $\eqsim$ is closed in $S \times S$, as well as bigger than $\sim$ (which is f-invariant), it follows that

$$f(x) = \lim f(x_n) \eqsim \lim f(y_n) = f(y).$$

Thus $\eqsim$ is f-invariant.

Note that $\sim'$ is given by the following. For all $x, y \in S$ it holds $x \sim' y$ if and only if there is a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ such that

$$x_0 \eqsim x_1 \eqsim \ldots x_{n-1} \eqsim x_n.$$

Since $\eqsim$ is f-invariant it follows that $\sim'$ is f-invariant. This finishes the proof of Claim 1. □

We now define $\sim_\alpha$ for any ordinal number α as follows. Let $\sim_0 := \sim$ be the equivalence relation from the statement of the proposition. Assume $\sim_\alpha$ has been defined for all ordinals $\beta < \alpha$. If α is not a limit ordinal, i.e., $\alpha = \beta + 1$ (for an ordinal β), then

$$\sim_\alpha := \sim_\beta{}'.$$

If α is a limit ordinal we define

$$\sim_\alpha := \bigvee_{\beta<\alpha} \sim_\beta,$$

i.e., the join of all $\sim_\beta$ with $\beta < \alpha$. Recall that this is the smallest equivalence relation bigger than all $\sim_\beta$ (for $\beta < \alpha$). Note that $(\sim_\alpha)$ is an increasing sequence of equivalence relations, i.e., $\beta \leq \alpha \Rightarrow \sim_\beta \leq \sim_\alpha$. Thus in the case that α is a limit ordinal it holds for all $x, y \in S$ that

$$\begin{aligned} &x \sim_\alpha y \quad \text{if and only if} \\ &\text{there is a } \beta < \alpha \text{ such that } x \sim_\beta y. \end{aligned} \tag{2.1}$$

Thus $\sim_\alpha$ has been defined for all ordinals α (by transfinite induction).

We now show that all equivalence relations $\sim_\alpha$ are f-invariant. This holds for $\sim_0$ by assumption. Assume $\sim_\beta$ is f-invariant for all $\beta < \alpha$. If $\alpha = \beta + 1$, i.e., if α is not a limit ordinal, it follows from Claim 1 above that $\sim_\alpha$ is f-invariant. Finally if α is a limit ordinal, it follows from the description (2.1) that $\sim_\alpha$ is f-invariant. By transfinite induction it follows that $\sim_\alpha$ is f-invariant for all ordinals α.

Clearly $\sim_\alpha$ is closed if and only if $\sim_{\alpha+1} = \sim_\alpha$. In this case the closure of $\sim$ is $\widehat{\sim} = \sim_\alpha$. Furthermore it then follows that $\sim_\alpha = \sim_\gamma$ for all $\gamma \geq \alpha$. Thus the proof is finished with the following.

Claim 2. There exists an ordinal α such that $\sim_{\alpha+1} = \sim_\alpha$.

If this would not be true then all equivalence relations $\sim_\alpha$ would be distinct. This is impossible, since the cardinality of the set of all equivalence relations on S is bounded by the cardinality of the power set of $S \times S$.

□

Remark 2.9. The previous proof does not use the axiom of choice, since the family of equivalence relations $\sim_\alpha$ constructed in the proof is well-ordered by construction.

2.4. **Pseudo-isotopies and Moore's theorem.** Moore's theorem is of central importance in the theory of matings. It gives a condition when an equivalence relation on the 2-sphere $\mathbb{S}^2$ yields a quotient space that is again (homeomorphic to) a 2-sphere.

Definition 2.10. An equivalence relation $\sim$ on $\mathbb{S}^2$ is called of *Moore-type* if

(1) $\sim$ is *not trivial*, i.e., there are at least two distinct equivalence classes;
(2) $\sim$ is *closed*;
(3) each equivalence class $[x]$ is *connected*;
(4) no equivalence class *separates* $\mathbb{S}^2$, i.e., $\mathbb{S}^2 \setminus [x]$ is connected for each equivalence class $[x]$.

Definition 2.11. A homotopy $H\colon X \times [0,1] \to X$ is called a *pseudo-isotopy* if $H\colon X \times [0,1) \to X$ is an isotopy (i.e., $H(\cdot,t)$ is a homeomorphism for all $t \in [0,1)$). We will always assume that $H(x,0) = x$ for all $x \in X$.

Given a set $A \subset \mathbb{S}^2$, we call H a pseudo-isotopy *rel.* A if H is a homotopy rel. A, i.e., if $H(a,t) = a$ for all $a \in A$, $t \in [0,1]$. We call the map $h := H(\cdot,1)$ the *end* of the pseudo-isotopy H. We interchangeably write $H(\cdot,t) = H_t(\cdot)$ to unclutter notation.

The following is *Moore's Theorem.* A weaker version was proved by R. L. Moore in [Mo], see also [Ti].

Theorem 2.12 (Moore, 1925)**.** *Let $\sim$ be an equivalence relation on $\mathbb{S}^2$. Then $\sim$ is of Moore-type if and only if $\sim$ can be realized as the end of a pseudo-isotopy $H\colon \mathbb{S}^2 \times [0,1] \to \mathbb{S}^2$ (i.e., $x \sim y \Leftrightarrow H_1(x) = H_1(y)$).*

The "if-direction", i.e., the easy direction, can be found in [Mey3, Lemma 2.4]. A proof of the other direction can be found in [Dav, Theorem 25.1 and Theorem 13.4]. From Lemma 2.3 we immediately recover the original form of Moore's theorem, i.e., that $\mathbb{S}^2/\!\sim$ is homeomorphic to $\mathbb{S}^2$.

3. Polynomials

3.1. **Background from complex dynamics.** Let $R(z) = p(z)/q(z) : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map, where the polynomials p and q are without common factors. The degree of R, i.e., the maximum of the degrees of p and q, will be assumed to be at least 2. We consider the dynamical system given by iteration of R, i.e., with orbits:

$$z_0, z_1, \ldots, z_n = R(z_{n-1}), \ldots.$$

A point $z \in \widehat{\mathbb{C}}$ is called a *fixed point* if $R(z) = z$. We then call $\lambda := DR(z) = R'(z)$ the multiplier. The point z is called super-attracting, attracting, neutral, and repelling respectively if $\lambda = 0, 0 < |\lambda| < 1, |\lambda| = 1, |\lambda| > 1$.

A point z is periodic if it is a fixed point for some iterate R^k (for some $k \geq 1$), and (super-) attracting, neutral, and repelling if z is a (super-) attracting, neutral, and repelling fixed point of R^k respectively.

The Fatou set F_R is the open set of points in $\widehat{\mathbb{C}}$, for which the family of iterates $\{R^n\}_n$ form a normal family in the sense of Montel on some neighborhood of the point. The Julia set J_R is the compact complement. Equivalently the Julia set is the closure of the set of repelling periodic points.

For a polynomial

$$P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

the point ∞ is a super-attracting fixed point. Consequently the Julia set is a compact subset of $\mathbb{C}$. The set

$$K_f = \{z \in \mathbb{C} \mid f^n(z) \not\to \infty, \text{as } n \to \infty\}$$

is called the filled-in Julia set for f. Its topological boundary is the Julia set, $J_f = \partial K_f$. The set J_f and hence K_f is connected precisely if no (finite) critical point escapes or iterates to ∞.

3.2. **Böttcher's theorem.** The dynamics of polynomials is much better understood than the dynamics of general rational maps. The main reason is the following theorem.

Theorem 3.1 (Böttcher)**.** *Let $P(z) = z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$ be a monic polynomial. Then*

- *P is conformally conjugate to z^d in a neighborhood of ∞.*
- *If the Julia set J (equivalently the filled-in Julia set K) of P is connected the conjugacy extends conformally to the Riemann map $\varphi\colon \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus K$. This means the following diagram commutes*

$$\begin{tikzcd} \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \arrow[r, "z^d"] \arrow[d, "\varphi"'] & \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \arrow[d, "\varphi"] \\ \widehat{\mathbb{C}} \setminus K \arrow[r, "P"] & \widehat{\mathbb{C}} \setminus K. \end{tikzcd}$$

Note that in the above $\varphi(\infty) = \infty$. Furthermore we can and shall assume φ chosen such that $\lim_{z\to\infty} z/\varphi(z) = 1$. Note that $G(z) := \log|\varphi^{-1}(z)|$ is the *Green's function* of the domain $\widehat{\mathbb{C}} \setminus K$ (with pole at ∞).

Definition 3.2. The *external ray* $R(\zeta)$ of angle $\zeta \in \mathbb{S}^1 = \{|z| = 1\}$ is the arc

$$R(\zeta) := R_P(\zeta) = \varphi(\{r\zeta \mid r > 1\})$$

3.3. **The Carathéodory loop.** The following theorem of Carathéodory is fundamental to the study of the boundary of simply connected proper subsets of the plane.

Theorem 3.3 (Carathéodory)**.** *A univalent map $\phi\colon \mathbb{D} \longrightarrow \widehat{\mathbb{C}}$ has a continuous extension to $\mathbb{S}^1 = \partial\mathbb{D}$ if and only if $\partial\phi(\mathbb{D}) \subset \widehat{\mathbb{C}}$ is locally connected.*

Corollary 3.4. *Let the Julia set of the polynomial P be connected and locally connected. Then the Böttcher conjugacy φ from Theorem 3.1 extends to a continuous map*

$$\overline{\varphi}\colon \widehat{\mathbb{C}} \setminus \mathbb{D} \to \widehat{\mathbb{C}} \setminus \mathring{K}.$$

This extension is a semi-conjugacy from z^d to P where defined.

In this case the map $\sigma = \sigma_f := \varphi|_{\mathbb{S}^1} : \mathbb{S}^1 \to \partial K = J$ is called the *Carathéodory loop* or *Carathéodory semi-conjugacy* as it semi-conjugates z^d to P. Recall that this means it satisfies the following commutative diagram

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, "z^d"] \arrow[d, "\sigma"'] & \mathbb{S}^1 \arrow[d, "\sigma"] \\ J \arrow[r, "P"] & J, \end{tikzcd} \tag{3.1}$$

i.e., $\sigma(z^d) = P \circ \sigma(z)$ for all $z \in \mathbb{S}^1$. It follows from Lemma 2.7 that for the equivalence relation $\sim = \sim_\sigma$ on $\mathbb{S}^1$ induced by σ the quotient map $z^d/\sim : \mathbb{S}^1/\sim \to \mathbb{S}^1/\sim$ is topologically conjugate to $P\colon J \to J$, because $\mathbb{S}^1$ is compact and $J \subset \mathbb{C}$ is Hausdorff.

3.4. **The lamination of a polynomial.** Let P be a monic polynomial, J its Julia set, and K its filled Julia set. We assume that J (equivalently K) is connected and locally connected.

Denote as above by $\sigma\colon \mathbb{S}^1 \to J$ its Carathéodory semi-conjugacy. We have seen in (3.1) that the equivalence relation induced by σ allows to understand $P\colon J \to J$ as a factor of $z^d\colon \mathbb{S}^1 \to \mathbb{S}^1$. There is a closely related construction that allows to understand $P\colon K \to K$ as a factor of a self-map of the closed unit disk $\overline{\mathbb{D}}$, see e.g. [Do, Theorem 1, page 433] for a construction of K. The success of this approach of studying the topology of, and dynamics on the filled Julia sets of polynomials by making pinched disk models of K, may serve as a motivation for defining and studying matings. In fact one way to define matings is to glue two pinched disk models along the circle $\mathbb{S}^1$ by the map $z \mapsto \overline{z}$.

It will be useful to keep in mind below that the cardinality of any equivalence class $[z] \cap \mathbb{S}^1$ is finite. In fact the rational points $\exp(i2\pi(\mathbb{Q}/\mathbb{Z}))$ are the (pre)-periodic points in $\mathbb{S}^1$ under the map $z \mapsto z^d$. The rationals are thus the arguments of the (pre)-periodic external rays for f. By classical results of Sullivan, Douady and Hubbard, the (pre)-periodic rays land on (pre)-periodic points which are either repelling or parabolic. And conversely the Douady be-landing Theorem asserts that when K_f is connected, every repelling or parabolic (pre)-periodic point is the landing point of at least one and at most finitely many (pre)-periodic rays. Moreover a theorem of Kiwi, states that the maximum number of non-(pre)-periodic rays co-landing on a single non-pre-critical and non pre-periodic point is d, [Ki]. For a more recent and enlarged discussion see also the paper by Blokh et al. [Bl].

4. Mating definitions

There are many definitions of mating in use, e.g. topological mating, formal mating, intermediate forms such as slow matings as introduced by Milnor and explored by Buff and Cheritat (see also the contribution by Cheritat in this volume) and Shishikuras degenerate matings, geometric or conformal mating and Douady and Hubbards original definition of mating used by Zakeri-Yampolsky. We start with the topological mating which is the simplest to formulate and which readily exhibits the difficulties related to matings.

4.1. **Definition of the Topological Mating.** Let $P_{\mathtt{w}}$, $P_{\mathtt{b}}$, $\mathtt{w}$ for 'white' and $\mathtt{b}$ for 'black', be two monic polynomials of the same degree $d \geq 2$ with connected and

locally connected filled-in Julia sets $K_{\mathtt{w}}, K_{\mathtt{b}}$. We consider the disjoint union $K_{\mathtt{w}} \sqcup K_{\mathtt{b}}$ and the map $P_{\mathtt{w}} \sqcup P_{\mathtt{b}} \colon K_{\mathtt{w}} \sqcup K_{\mathtt{b}} \to K_{\mathtt{w}} \sqcup K_{\mathtt{b}}$ given by

$$P_{\mathtt{w}} \sqcup P_{\mathtt{b}}|_{K_{\mathtt{w}}} = P_{\mathtt{w}}, \quad P_{\mathtt{w}} \sqcup P_{\mathtt{b}}|_{K_{\mathtt{b}}} = P_{\mathtt{b}}.$$

Let $\sigma_j \colon \mathbb{S}^1 \to \partial K_j$ be their Carathéodory loops (here $j = \mathtt{w}, \mathtt{b}$). Let $\sim$ be the equivalence relation on $K_{\mathtt{w}} \sqcup K_{\mathtt{b}}$ generated by

$$\sigma_{\mathtt{w}}(z) \sim \sigma_{\mathtt{b}}(\bar{z}).$$

for all $z \in \mathbb{S}^1$. Note that $\sigma_{\mathtt{w}}(z) \in J_{\mathtt{w}} \subset K_{\mathtt{w}}$, $\sigma_{\mathtt{b}}(\bar{z}) \in J_{\mathtt{b}} \subset K_{\mathtt{b}}$. Furthermore from (3.1) it follows that

$$P_{\mathtt{w}}(\sigma(z)) = \sigma_{\mathtt{w}}(z^d) \sim \sigma_{\mathtt{b}}(\bar{z}^d) = P_{\mathtt{b}}(\sigma(\bar{z})).$$

Thus $\sim$ is $P_{\mathtt{w}} \sqcup P_{\mathtt{b}}$ invariant (see Lemma 2.6) and hence this map descends to the quotient, see Section 2.3.

Definition 4.1 (Topological mating)**.** Let $\sim$ be as above. Then

$$K_{\mathtt{w}} \amalg K_{\mathtt{b}} := K_{\mathtt{w}} \sqcup K_{\mathtt{b}} / \sim .$$

The *topological mating* of the polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ is the map

$$P_{\mathtt{w}} \amalg P_{\mathtt{b}} := P_{\mathtt{w}} \sqcup P_{\mathtt{b}} / \sim \; : K_{\mathtt{w}} \amalg K_{\mathtt{b}} \to K_{\mathtt{w}} \amalg K_{\mathtt{b}}.$$

4.2. **Obstructions and equivalences.** From Definition 4.1 it looks very surprising that $P_{\mathtt{w}} \amalg P_{\mathtt{b}}$ is "often" (topologically conjugate to) a rational map. Rather it seems that there is no reason to assume that $P_{\mathtt{w}} \amalg P_{\mathtt{b}}$ has any nice properties at all. We list the different *obstructions*.

Definition 4.2 (Mating obstructions)**.**

- The equivalence relation $\sim$ may fail to be closed. In this case $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ is not a Hausdorff space (see Lemma 2.2). We then say that the mating of $P_{\mathtt{w}}, P_{\mathtt{b}}$ is *Hausdorff-obstructed*.
- The space $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ may fail to be a topological sphere. In this case we call the mating *Moore-obstructed*.
- If $P_{\mathtt{w}} \amalg P_{\mathtt{b}}$ is a post-critically finite branched covering, it may fail to be Thurston-equivalent to a rational map, the mating then is *Thurston-obstructed*.

If in the last case the Thurston obstruction happens to be a Levy cycle, we also call the mating *Levy-obstructed*. Note that one possible obstruction is omitted in the above, since the following holds.

Proposition 4.3. *Assume the topological mating of $P_{\mathtt{w}}, P_{\mathtt{b}}$ is not Moore obstructed, i.e., $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ is topologically $\mathbb{S}^2$. Then the mating $P_{\mathtt{w}} \amalg P_{\mathtt{b}} \colon \mathbb{S}^2 \to \mathbb{S}^2$ is an orientation-preserving branched covering.*

The proof is postponed to Section 4.6.

The two views on matings is reflected in the following definitions:

Definition 4.4 (Matings as rational maps)**.** Let $P_{\mathtt{w}}, P_{\mathtt{b}}$ be two degree $d > 1$ polynomials for which there exists a homeomorphism $h \colon K_{\mathtt{w}} \amalg K_{\mathtt{b}} \longrightarrow \mathbb{S}^2$.

- The polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ are called *combinatorially mateable* if they are post-critically finite and if the branched covering $h \circ P_{\mathtt{w}} \amalg P_{\mathtt{b}} \circ h^{-1}$ is Thurston equivalent to a rational map R.
- The polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ are called *topologically mateable* if the homeomorphism h can be so chosen that $R = h \circ P_{\mathtt{w}} \amalg P_{\mathtt{b}} \circ h^{-1}$ is a rational map.

- The polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ are called *geometrically mateable* if they are topologically mateable and h can additionally be chosen to be conformal on the interior of $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$.

Conversely we say that a rational map R is *combinatorially a mating, topologically a mating or geometrically a mating* if there exist polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ satisfying the corresponding property above with $R = h \circ P_{\mathtt{w}} \amalg P_{\mathtt{b}} \circ h^{-1}$.

Note that the first two definitions make sense for Thurston maps as well.

Definition 4.2 (**Cont.**)**.** In the notation of Definition 4.4:

- If h can not be chosen so that R is rational we say that the mating is *topologically obstructed*.
- If h can not be chosen so that R is rational and h is conformal on the interior of $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ we say the mating is *geometrically obstructed*.

Conjecturally any pair of topologically mateable polynomials are geometrically mateable, i.e., there are no purely geometrically obstructed pairs.

It is not known whether there are polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$, whose matings have a Hausdorff obstruction, i.e., which results in an equivalence relation $\sim$ that is not closed. We can however take the closure $\widehat{\sim}$ of $\sim$, and then $P_{\mathtt{w}} \amalg P_{\mathtt{b}}$ descends to the closure by Proposition 2.8.

Definition 4.5 (Closed topological mating)**.** Let

$$K_{\mathtt{w}} \widehat{\amalg} K_{\mathtt{b}} := (K_{\mathtt{w}} \sqcup K_{\mathtt{b}})/\widehat{\sim}.$$

The *closed topological mating* is defined as

$$P_{\mathtt{w}} \widehat{\amalg} P_{\mathtt{b}} := (P_{\mathtt{w}} \sqcup P_{\mathtt{b}})/\widehat{\sim} : K_{\mathtt{w}} \widehat{\amalg} K_{\mathtt{b}} \to K_{\mathtt{w}} \widehat{\amalg} K_{\mathtt{b}}.$$

However a priori this is not enough to guarantee that the quotient is homeomorphic to $\mathbb{S}^2$.

4.3. **Semi-conjugacies associated to the topological mating.** There are several semi-conjugacies naturally associated with matings. Assume the rational map $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is topologically the mating of the polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$.

We first note that both Julia sets $J_{\mathtt{w}}, J_{\mathtt{b}} \subset K_{\mathtt{w}} \sqcup K_{\mathtt{b}}$ are mapped by the quotient map $K_{\mathtt{w}} \sqcup K_{\mathtt{b}} \to (K_{\mathtt{w}} \sqcup K_{\mathtt{b}})/\sim = K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ to the same set. In fact the following Lemma is an easy exercise left to the reader:

Lemma 4.6. *Let the rational map $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be topologically the mating of the polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$.*

- *Then the maps $P_j\colon K_j \longrightarrow K_j$, $j = \mathtt{w}, \mathtt{b}$ are semi-conjugate to R on the image of the quotients $K_j/\sim$. More precisely there are maps $\varphi_j\colon K_j \hookrightarrow \widehat{\mathbb{C}}$ for $j = \mathtt{w}, \mathtt{b}$ (in general neither injective nor surjective) such that $\sim_{\varphi_j} = \sim|_{K_j}$ and the following diagram commutes.*

$$\begin{tikzcd} K_j \arrow[r, "P_j"] \arrow[d, hook, "\varphi_j"'] & K_j \arrow[d, hook, "\varphi_j"] \\ \widehat{\mathbb{C}} \arrow[r, "R"] & \widehat{\mathbb{C}} \end{tikzcd}$$

- *The dynamics of R on its Julia set J_R, i.e., $R\colon J_R \to J_R$, is a factor of both $P_j\colon J_j \longrightarrow J_j$, $j = \mathtt{w}, \mathtt{b}$.*

Topological mating is quite flexible, when the polynomials admits dynamics preserving topological deformations, e.g. when one of the polynomials has a critical point with an infinite orbit contained in hyperbolic components. In contrast the conformal mating gives very strong ties between the pair of polynomials $P_{\mathtt{w}} \amalg P_{\mathtt{b}}$ and the rational map R realizing the geometrical mating. The geometrical mating thus gives rise to enumerative type questions:

- If two polynomials are geometrically mateable. Is the resulting rational map R then unique up to Möbius conjugacy? If not are there finitely many such R? And then how many? If there are infinitely many, are there natural parametrizations?
- How many ways can a rational map be obtained as a geometrical mating of two polynomials?
- When does the mating depend continuously/measureably on input data?

4.4. **The Formal Mating.** The fine print of the above discussion is that while the topological mating is quite easily defined, it is often difficult to visualize. For example when the filled Julia sets $K_{\mathtt{w}}, K_{\mathtt{b}}$ are dendrites it is quite counterintuitive that "often" $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ is a topological sphere.

The formal mating introduced in the following, circumvents some of the difficulties. We always obtain a branched covering right from the start. And taking a further quotient yields again the topological mating.

Denote by $\overline{\mathbb{C}} := \mathbb{C} \cup \{(\infty, w) | w \in \mathbb{S}^1\}$ the compactification of $\mathbb{C}$ obtained by adjoining a circle at infinity. Note that each monic polynomial $P = z^d + \dots$ extends continuously to $P\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by $P((\infty, w)) = (\infty, w^d)$. We denote the extended map again by P for convenience.

Definition 4.7 (Formal Mating)**.** Let $P_{\mathtt{w}}\colon \overline{\mathbb{C}}_{\mathtt{w}} \to \overline{\mathbb{C}}_{\mathtt{w}}$, $P_{\mathtt{b}}\colon \overline{\mathbb{C}}_{\mathtt{b}} \to \overline{\mathbb{C}}_{\mathtt{b}}$ be two monic polynomials of the same degree. Here $\overline{\mathbb{C}}_i$ are the compactifications as above of the dynamical planes $\mathbb{C}_i$ for each polynomial P_i, $i = \mathtt{w}, \mathtt{b}$. Define

$$\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}} = (\overline{\mathbb{C}}_{\mathtt{w}} \sqcup \overline{\mathbb{C}}_{\mathtt{b}})/((\infty_{\mathtt{w}}, w) \sim (\infty_{\mathtt{b}}, \overline{w})).$$

We write (∞, w) for the equivalence class containing $(\infty_{\mathtt{w}}, w) \sim (\infty_{\mathtt{b}}, \overline{w})$ and equip $\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$ with the quotient topology. We call this set the *formal mating sphere*. The set $\mathbb{S}^1_\infty := \{(\infty, w) \mid w \in \mathbb{S}^1\} \subset \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$ is called the *equator* of the formal mating sphere.

The *formal mating* $P_{\mathtt{w}} \uplus P_{\mathtt{b}}\colon \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}} \longrightarrow \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$ is the map

$$P_{\mathtt{w}} \uplus P_{\mathtt{b}}(z) = \begin{cases} P_{\mathtt{w}}(z), & \text{if } z \in \mathbb{C}_{\mathtt{w}}, \\ P_{\mathtt{b}}(z), & \text{if } z \in \mathbb{C}_{\mathtt{b}}, \\ (\infty, z^d), & \text{for } (\infty, z). \end{cases}$$

Evidently $\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$ is homeomorphic to $\mathbb{S}^2$ and $P_{\mathtt{w}} \uplus P_{\mathtt{b}}$ is a branched covering.

Definition 4.8 (Ray-equivalence)**.** The external rays $R_{\mathtt{w}}(\zeta) \subset \overline{\mathbb{C}}_{\mathtt{w}}, R_{\mathtt{b}}(\bar{\zeta}) \subset \overline{\mathbb{C}}_{\mathtt{b}}$ of the polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ are naturally contained in the formal mating sphere $\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$. The *extended external ray* of angle $\zeta \in \mathbb{S}^1$ is the closure of $R_{\mathtt{w}}(\zeta) \cup R_{\mathtt{b}}(\bar{\zeta})$ in the formal mating sphere $\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$, denoted by $R(\zeta)$. Note that $R(\zeta)$ contains the point (∞, ζ) of the equator. If the filled Julia sets $K_{\mathtt{w}}, K_{\mathtt{b}}$ are connected and locally connected, then $R(\zeta)$ contains exactly one point of the Julia set $J_{\mathtt{w}}$, as well as exactly one point of the Julia set $J_{\mathtt{b}}$. The *ray-equivalence* $\sim_{\text{ray}}$ on the formal mating sphere is defined

to be the smallest equivalence relation, such that all points of the formal mating sphere that are contained in the same extended external ray are equivalent.

Lemma 4.9. *The formal mating is never topologically conjugate to a rational map.*

Proof. Consider the point $(\infty, 1)$ on the equator of the formal mating sphere. It is also contained in the extended external ray $R(1)$. Note that this is a fixed point of the formal mating $P_{\mathtt{w}} \uplus P_{\mathtt{b}}$. As a consequence of Böttcher's theorem (Theorem 3.1), all points in the interior of $R(1)$ (i.e., all points except the endpoints) are converging under iteration of $P_{\mathtt{w}} \uplus P_{\mathtt{b}}$ to $(\infty, 1)$. However points in a small neighborhood of $(\infty, 1)$ on the equator are repelled from $(\infty, 1)$ under iteration, since $P_{\mathtt{w}} \uplus P_{\mathtt{b}}$ on $\mathbb{S}^1$ is topologically conjugate to $z^d \colon \mathbb{S}^1 \to \mathbb{S}^1$. For rational maps such a behavior can only occur at a parabolic fixed point. However the same behavior occurs for the n-th iterate of the formal mating at each point $(\infty, \mathrm{e}^{2\pi i k/(d^n-1)})$ for all $k = 0, \dots, d^n - 1$. A rational map cannot have infinitely many parabolic periodic points. □

If we ask whether the formal mating is in some sense a rational map, we thus need a notion that is weaker than topological conjugacy. The most successful in this context is Thurston equivalence. Clearly, the formal mating is post-critically finite if and only if both polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ are post-critically finite. We may then ask if $P_{\mathtt{w}} \uplus P_{\mathtt{b}}$ is Thurston equivalent to a rational map.

The most important aspect of the formal mating however is that we can reconstruct the topological mating from it.

Proposition 4.10 (Topological mating from formal mating)**.** *Let $P_{\mathtt{w}}, P_{\mathtt{b}}$ be two monic polynomials of the same degree $d \geq 2$ with connected and locally connected Julia sets.*

(1) *The formal mating $P_{\mathtt{w}} \uplus P_{\mathtt{b}} \colon \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$ descends to the quotient $\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}} / \sim_{\mathrm{ray}}$.*
(2) *The quotient map is (topologically conjugate to) the topological mating $P_{\mathtt{w}} \amalg P_{\mathtt{b}}$. In particular $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ is homeomorphic to $\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}} / \sim_{\mathrm{ray}}$.*
(3) *The space $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ is a topological sphere if and only if the ray-equivalence $\sim_{\mathrm{ray}}$ is of Moore-type (see Definition 2.10).*

Proof. (1) The formal mating $P_{\mathtt{w}} \uplus P_{\mathtt{b}}$ clearly maps each extended external ray to another extended external ray. It follows that $\sim_{\mathrm{ray}}$ is invariant with respect to $P_{\mathtt{w}} \uplus P_{\mathtt{b}}$, hence this map descends to the quotient $\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}} / \sim_{\mathrm{ray}}$ (see Lemma 2.6).

(2) We identify $K_{\mathtt{w}} \sqcup K_{\mathtt{b}}$ with $K_{\mathtt{w}} \sqcup K_{\mathtt{b}} \subset \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$. Then the map $P_{\mathtt{w}} \uplus P_{\mathtt{b}}$ clearly agrees with $P_{\mathtt{w}} \sqcup P_{\mathtt{b}}$ on $K_{\mathtt{w}} \sqcup K_{\mathtt{b}}$.

Note that the extended external ray $R(\zeta)$ contains the points $\sigma_{\mathtt{w}}(\zeta) \in J_{\mathtt{w}}, \sigma_{\mathtt{b}}(\bar{\zeta}) \in J_{\mathtt{b}}$. Thus the equivalence relation $\sim_{\mathrm{ray}}$ restricted to $K_{\mathtt{w}} \sqcup K_{\mathtt{b}}$ (viewed as subsets of the formal mating sphere) is equal to the equivalence relation $\sim$ from the topological mating.

Each point $x \in \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$ not contained in the filled Julia sets $K_{\mathtt{w}}, K_{\mathtt{b}} \subset \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$ is contained in one extended external ray. Put differently, each equivalence class $[x]_{\mathrm{ray}}$ of $\sim_{\mathrm{ray}}$ contains a representative $z \in K_{\mathtt{w}} \sqcup K_{\mathtt{b}} \subset \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$. The map

$$h \colon (\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}) / \sim_{\mathrm{ray}} \to K_{\mathtt{w}} \amalg K_{\mathtt{b}}$$

is now defined as follows. Each ray-equivalence class $[x]_{\mathrm{ray}} \in (\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}) / \sim_{\mathrm{ray}}$ is mapped by h to $[z] \in K_{\mathtt{w}} \amalg K_{\mathtt{b}} = (K_{\mathtt{w}} \sqcup K_{\mathtt{b}}) / \sim$, where $z \in K_{\mathtt{w}} \sqcup K_{\mathtt{b}}$ is a representative of $[x]_{\mathrm{ray}}$. This is a well-defined bijection. Furthermore h conjugates $(P_{\mathtt{w}} \uplus P_{\mathtt{b}}) / \sim_{\mathrm{ray}}$ to the topological mating $P_{\mathtt{w}} \amalg P_{\mathtt{b}}$.

One subtle point still needs to be verified however. Namely that the quotient topologies induced by $\sim$ and $\sim_{\mathrm{ray}}$ agree. More precisely we need to verify that h maps the quotient topology on $(\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\mathrm{ray}}$ to the quotient topology on $K_{\mathtt{w}} \amalg K_{\mathtt{b}} = (K_{\mathtt{w}} \sqcup K_{\mathtt{b}})/\sim$.

We denote by $\Pi_{\mathrm{ray}}\colon \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}} \to (\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\mathrm{ray}}$, $\Pi\colon K_{\mathtt{w}} \sqcup K_{\mathtt{b}} \to K_{\mathtt{w}} \amalg K_{\mathtt{b}} = (K_{\mathtt{w}} \sqcup K_{\mathtt{b}})/\sim$ the quotients maps.

Let $U \subset (\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\mathrm{ray}}$ be open. This is the case if and only if $[U]_{\mathrm{ray}} := \Pi_{\mathrm{ray}}^{-1}(U) \subset \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$ is open. Then $[U]_{\mathrm{ray}} \cap (K_{\mathtt{w}} \sqcup K_{\mathtt{b}})$ is open. Note that $[U]_{\mathrm{ray}} \cap (K_{\mathtt{w}} \sqcup K_{\mathtt{b}}) = \Pi^{-1}(h(U))$. Thus $h(U) \subset K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ is open.

Now let $h(U) \subset K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ be open. Then $\Pi^{-1}(h(U)) \subset K_{\mathtt{w}} \sqcup K_{\mathtt{b}}$ is open. From (3.1) it follows that $\Pi_{\mathrm{ray}}^{-1}(U) \cap \mathbb{S}^1_\infty \subset \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$ is an open subset of the equator $\mathbb{S}^1_\infty$ of the formal mating sphere. From this it follows that $\Pi_{\mathrm{ray}}^{-1}(U)$ is open, hence $U \subset (\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\mathrm{ray}}$ is open. Thus the map h is a homeomorphism.

(3) Assume first that $\sim_{\mathrm{ray}}$ is of Moore type. From (2) as well as Moore's theorem (Theorem 2.12, see also Lemma 2.3) it immediately follows that $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ is a topological sphere.

Assume now that $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ is a topological sphere. Since $(\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\mathrm{ray}}$ is homeomorphic to the topological sphere $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ by (2), it follows that $\sim_{\mathrm{ray}}$ is not trivial, i.e., has at least two distinct equivalence classes.

Each equivalence class $[x]_{\mathrm{ray}}$ of $\sim_{\mathrm{ray}}$ is compact. Otherwise $(\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\mathrm{ray}}$ is not Hausdorff, contradicting the fact that $\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$ is homeomorphic to the sphere $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ by (2).

Each equivalence class $[x]_{\mathrm{ray}}$ is connected by construction.

Assume now that there is an equivalence class $[x]_{\mathrm{ray}}$ that separates the sphere $\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$. Then $[x]_{\mathrm{ray}} \in (\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\mathrm{ray}}$ separates, i.e., $(\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\mathrm{ray}} \setminus \{[x]_{\mathrm{ray}}\}$ is not connected. This shows that $(\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\mathrm{ray}}$ is not homeomorphic to the sphere $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$.

□

Assume now that the two monic polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ of the same degree $d \geq 2$ have connected and locally connected Julia set. Let $\sim_\infty$ be the restriction of $\sim_{\mathrm{ray}}$ to the equator $\mathbb{S}^1_\infty$. This equivalence relation may be described as follows. Let $\sim_{\mathtt{w}}, \sim_{\mathtt{b}}$ be the equivalence relations on $\mathbb{S}^1$ induced by the Carathéodory loop $\sigma_{\mathtt{w},\mathtt{b}}\colon \mathbb{S}^1 \to J_{\mathtt{w},\mathtt{b}}$, i.e., for all $\zeta, \xi \in \mathbb{S}^1$ it holds

$$\begin{aligned} \zeta \sim_{\mathtt{w}} \xi \quad &:\Leftrightarrow \quad \sigma_{\mathtt{w}}(\zeta) = \sigma_{\mathtt{w}}(\xi), \text{ and} \\ \zeta \sim_{\mathtt{b}} \xi \quad &:\Leftrightarrow \quad \sigma_{\mathtt{b}}(\bar{\zeta}) = \sigma_{\mathtt{b}}(\bar{\xi}). \end{aligned}$$

Note that from Lemma 2.3 it follows that $\sim_{\mathtt{w}}, \sim_{\mathtt{b}}$ are closed. Identifying $\mathbb{S}^1$ with $\mathbb{S}^1_\infty$ it holds that $\sim_\infty$ is the equivalence relation generated by $\sim_{\mathtt{w}}$ and $\sim_{\mathtt{b}}$ (i.e., the join of $\sim_{\mathtt{w}}, \sim_{\mathtt{b}}$ in the lattice of equivalence relations). This means that $\zeta \sim_\infty \xi$ if and only if there is a finite sequence $w_0, \ldots, w_N \in \mathbb{S}^1{}_\infty$ such that

$$\zeta = w_0 \sim_{\mathtt{w}} w_1 \sim_{\mathtt{b}} \cdots \sim_{\mathtt{w}} w_{N-1} \sim_{\mathtt{b}} w_N = \xi.$$

Note that in the above we may choose $w_0 = w_1$ and/or $w_{N-1} = w_N$. It is sometimes conceptually easier to deal with $\sim_\infty$ instead of $\sim_{\mathrm{ray}}$. We have the following.

Lemma 4.11. *The ray-equivalence $\sim_{\mathrm{ray}}$ (on the formal mating sphere $\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$) is closed if and only if $\sim_\infty$ is closed (on $\mathbb{S}^1_\infty$).*

Proof. Clearly $\sim_{\text{ray}}$ being closed implies that $\sim_\infty$ is closed.

Assume now that $\sim_\infty$ is closed. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be convergent sequences in the formal mating sphere $\mathbb{S}^2$ such that $x_n \sim_{\text{ray}} y_n$ for all $n \in \mathbb{N}$. We need to show that $x := \lim x_n \sim_{\text{ray}} y := \lim y_n$. We can assume without loss of generality that $x \in \overline{\mathbb{C}}_{\mathtt{w}} \subset \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$.

If $x_n = y_n$ for infinitely many $n \in \mathbb{N}$ there is nothing to prove. Thus we can assume $x_n \neq y_n$ for all $n \in \mathbb{N}$. This implies that $[x_n]_{\text{ray}} = [y_n]_{\text{ray}}$ consist of more than a point, i.e., contains an extended external ray. Thus

$$[x_n]_\infty := [x_n]_{\text{ray}} \cap \mathbb{S}^1_\infty \neq \emptyset.$$

If x is in the interior of $K_{\mathtt{w}}$ then x_n is in the interior of $K_{\mathtt{w}}$ for sufficiently large n. By the definition of $\sim_{\text{ray}}$ it follows that $x_n = y_n$ for large n. Thus x is not in the interior of $K_{\mathtt{w}}$ by our assumptions. Hence $[x]_\infty = [x]_{\text{ray}} \cap \mathbb{S}^1_\infty \neq \emptyset$.

We now prove that there is a sequence $(w_n)_{n\in\mathbb{N}} \subset \mathbb{S}^1_\infty$ with $w_n \in [x_n]_\infty$ (for all $n \in \mathbb{N}$) such that for a subsequence $(w_k)_{k\in I}$ ($I \subset \mathbb{N}$ infinite) it holds

$$w_k \to w \in [x]_\infty \text{ as } k \in I \to \infty. \tag{4.1}$$

This is clear if $x \notin K_{\mathtt{w}}$. Assume now that $x \in J_{\mathtt{w}}$. Consider the extension of the Böttcher map $\overline{\varphi}_{\mathtt{w}} \colon \widehat{\mathbb{C}} \setminus \mathbb{D} \to \widehat{\mathbb{C}} \setminus \overset{\circ}{K}_{\mathtt{w}}$. It is continuous, see Corollary 3.4. The equivalence relation on $\widehat{\mathbb{C}} \setminus \mathbb{D}$ induced by $\overline{\varphi}_{\mathtt{w}}$ is still denoted by $\sim_{\mathtt{w}}$. By Lemma 2.3 $\sim_{\mathtt{w}}$ is closed.

Note that $(\overline{\varphi})^{-1}(x) = [x]_{\mathtt{w}} \in \mathbb{S}^1 = \partial(\widehat{\mathbb{C}} \setminus \mathbb{D})$. Let $U \subset (\widehat{\mathbb{C}} \setminus \mathbb{D})$ be an open neighborhood of $[x]_{\mathtt{w}} \in (\widehat{\mathbb{C}} \setminus \mathbb{D})$. Let U^* be the saturated interior, i.e., the set of all equivalence classes contained (of $\sim_{\mathtt{w}}$) in U. This is open by Lemma 2.2. By definition of the quotient topology it follows that $V := \overline{\varphi}(U^*) \subset \overline{\mathbb{C}}_{\mathtt{w}}$ is an open set containing x. Since $x_n \to x$ it follows that $x_n \in V$ for sufficiently large $n \in \mathbb{N}$. It follows that $[x_n]_{\mathtt{w}}$ is contained in U (for sufficiently large $n \in \mathbb{N}$). Pick an arbitrary $w_n \in [x_n]_{\mathtt{w}}$. Since $[x]_{\mathtt{w}}$ is compact, we can extract a convergent subsequence of (w_n) converging to a point $w \in [x]_{\mathtt{w}}$. Finally we note that $[x]_{\mathtt{w}} \subset [x]_\infty$, finishing the proof of (4.1).

Using the exact same argument we construct a sequence $u_k \in [x_k]_\infty = [y_k]_\infty$ for $k \in I$, and a subsequence $(u_m)_{m\in I'}$ converging to $u \in [y]_\infty$ ($I' \subset I \subset \mathbb{N}$ is an infinite set).

By construction it follows that $w_m \sim_\infty u_m$ for all $m \in I'$. Since $\sim_\infty$ is closed by assumption it follows that $w \sim_\infty u$. Thus $[x]_\infty = [w]_\infty = [u]_\infty = [y]_\infty$. Thus $[x]_{\text{ray}} = [y]_{\text{ray}}$ or $x \sim_{\text{ray}} y$ as desired. □

4.5. **Shishikuras degenerate Matings.** It may happen that two post-critically finite polynomials are topologically mateable, but that the formal mating is not Thurston equivalent to a rational map. More specifically some post-critical points of $P_{\mathtt{w}}$ may be ray-equivalent to some post-critical points of $P_{\mathtt{b}}$. In this case the formal mating will have so-called Levi-cycles. Shishikura constructs in the paper [S] a mating intermediate between the topological and the formal mating. For this mating the appropriate post-critical points of the formal mating have been merged, so that the resulting mapping no longer has any Levi-cycles.

4.6. **More on the ray-equivalence.** Each ray-equivalence class $[x]_{\text{ray}}$ may be viewed as a graph. Namely the set of edges is given by the set of all extended external rays $R(\zeta)$ contained in $[x]_{\text{ray}}$. The points in the Julia sets $J_{\mathtt{w}}, J_{\mathtt{b}}$ contained

in $[x]_{\mathrm{ray}}$ are the vertices, i.e., the set of vertices is given by $[x]_{\mathrm{ray}} \cap (J_{\mathtt{w}} \sqcup J_{\mathtt{b}})$. Such a vertex $v \in [x]_{\mathrm{ray}} \cap (J_{\mathtt{w}} \sqcup J_{\mathtt{b}})$ is incident to the extended external ray $R(\zeta) \subset [x]_{\mathrm{ray}}$ (viewed as an edge) if and only if $v \in R(\zeta)$. Clearly each such graph is connected. We call $[x]_{\mathrm{ray}}$ a *tree* if the just defined corresponding graph is a tree, otherwise $[x]_{\mathrm{ray}}$ is called *cyclic*. The diameter of $[x]_{\mathrm{ray}}$ is defined to be the diameter of the set viewed as a graph. If $[x]_{\mathrm{ray}}$ has infinite diameter, we say $[x]_{\mathrm{ray}}$ is an *infinite ray-connection*. Recall from the discussion in Section 3.4 that the number of edges incident to a vertex is finite. It follows that each $[x]_{\mathrm{ray}}$ viewed as a graph is at most countable, i.e., has at most countably many edges. Put differently $[x]_\infty = [x]_{\mathrm{ray}} \cap \mathbb{S}^1_\infty$ is at worst countable.

Evidently cyclic ray-equivalence classes do exists: Take as $P_{\mathtt{w}}$ any polynomial P for which the Carathéodory loop $\sigma\colon \mathbb{S}^1 \to J$ is not injective and let $P_{\mathtt{b}} := \overline{P(\bar{z})}$. On the other hand it is not known whether infinite ray-connections exist.

Proposition 4.12 (A. Epstein, unpublished)**.** *Let $P_{\mathtt{w}}, P_{\mathtt{b}}$ be two monic polynomials of the same degree $d \geq 2$ with connected and locally connected Julia sets. Then the following holds.*

- *If all equivalence classes of $\sim_{\mathrm{ray}}$ are ray-trees and of uniformly bounded diameter, then the topological space $K_{\mathtt{w}} \perp\!\!\!\perp K_{\mathtt{b}}$ is homeomorphic to $\mathbb{S}^2$.*
- *If $\sim_{\mathrm{ray}}$ has a cyclic ray-equivalence class or an infinite ray-connection, then $K_{\mathtt{w}} \perp\!\!\!\perp_{\mathrm{ray}} K_{\mathtt{b}}$ is not homeomorphic to $\mathbb{S}^2$.*

Proof. To show the first statement we first note that $K_{\mathtt{w}} \perp\!\!\!\perp K_{\mathtt{b}}$ is homeomorphic to $(\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\mathrm{ray}}$ by Lemma 4.10 (2). Thus we will prove the statement by showing that the equivalence relation $\sim_{\mathrm{ray}}$ (on the topological sphere $\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$) is of Moore-type.

Assume now that the equivalence classes $[x]_{\mathrm{ray}}$ are trees of uniformly bounded diameter. Any non-trivial equivalence class $[x]_{\mathrm{ray}}$ of bounded diameter is connected as well as compact, since it is a finite union of compact sets (i.e., of the extended external rays). Furthermore each $[x]_{\mathrm{ray}}$ is by assumption a tree, thus $(\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}) \setminus [x]_{\mathrm{ray}}$ is connected.

We now show that $[x]_{\mathrm{ray}}$ is closed. By Lemma 4.11 it is enough to show that the restriction of $\sim_{\mathrm{ray}}$ to the equator $\mathbb{S}^1_\infty$, i.e., $\sim_\infty$, is closed. Recall that we assumed that the diameter of the equivalence classes $[x]_{\mathrm{ray}}$ is uniformly bounded. This is equivalent to the property that the size of $[x]_\infty$ is uniformly bounded.

Let $(x_n), (y_n) \in \mathbb{S}^1_\infty$ be convergent sequences with $x_n \sim_\infty y_n$ for all $n \in \mathbb{N}$. This means for each $n \in \mathbb{N}$ there are points $w_n^0, \dots, w_n^N \in \mathbb{S}^1_\infty$ with

$$x_n = w_n^0 \sim_{\mathtt{w}} w_n^1 \sim_{\mathtt{b}} \cdots \sim_{\mathtt{b}} w_n^N = y_n.$$

Taking a subsequence we can assume that all sequences $(w_n^j)_{n\in\mathbb{N}}$ converge. Since $\sim_{\mathtt{w}}, \sim_{\mathtt{b}}$ are closed it follows that $x = \lim x_n \sim_\infty y = \lim y_n$ as desired.

Since each equivalence class $[x]_\infty$ is finite, it follows that $\sim_{\mathrm{ray}}$ is not trivial. Thus $\sim_{\mathrm{ray}}$ is of Moore-type. By Moore's Theorem, i.e., Theorem 2.12, it follows that $(\widehat{\mathbb{C}} \uplus \widehat{\mathbb{C}})/\sim_{\mathrm{ray}}$ is homeomorphic to $\mathbb{S}^2$.

We now prove the second statement. First assume that there exists a cyclic equivalence class $[x]_{\mathrm{ray}}$. Removing the point $[x]_{\mathrm{ray}}$ from the space $(\widehat{\mathbb{C}}_{\mathtt{w}} \uplus \widehat{\mathbb{C}}_{\mathtt{b}})/\sim_{\mathrm{ray}}$ yields a disconnected space. Since the sphere $\mathbb{S}^2$ with a single point removed is connected, it follows that $(\widehat{\mathbb{C}}_{\mathtt{w}} \uplus \widehat{\mathbb{C}}_{\mathtt{b}})/\sim_{\mathrm{ray}}$ is not homeomorphic to $\mathbb{S}^2$.

Finally assume that there exists an equivalence class $[x]_{\text{ray}}$ of infinite diameter. Then we can find in $[x]_{\text{ray}}$ an infinite ray-path, i.e., a continuous injective map $\gamma\colon [0,\infty) \to [x]_{\text{ray}}$ that covers infinitely many (distinct) extended external rays. Furthermore we choose the parametrization of γ as follows for convenience. For each $n \in \mathbb{N}_0$ the set $\gamma([n, n+1])$ is an extended external ray, $\gamma(n) \in K_{\mathtt{w}}$ for all even $n \in \mathbb{N}_0$, $\gamma(n) \in K_{\mathtt{b}}$ for all odd $n \in \mathbb{N}_0$. Finally we require that $\gamma(n+1/2) \in \mathbb{S}^1_\infty$ for all $n \in \mathbb{N}_0$.

Assume that the restriction of $[x]_{\text{ray}}$ to the equator $\mathbb{S}^1_\infty$, i.e., $[x]_\infty = [x]_{\text{ray}} \cap \mathbb{S}^1_\infty$, is not closed. Then $\sim_{\text{ray}}$ is not closed (see Lemma 4.11), which implies that $(\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\text{ray}}$ is not Hausdorff by Lemma 2.2, hence not homeomorphic to $\mathbb{S}^2$.

Thus we assume now that $[x]_\infty$ is closed. This means that all the accumulation points of the sequence $\{\gamma(n+\frac{1}{2})\}_{n\in\mathbb{N}_0} \subset \mathbb{S}^1_\infty$ are contained in $[x]_\infty$. In this case note first that by possibly replacing an initial segment of γ by some other ray-path and reparametrizing, we can suppose $\gamma(\frac{1}{2})$ is an accumulation point of the sequence $\{\gamma(n+\frac{1}{2})\}_{n\in\mathbb{N}}$. However not all the points of this sequence can be accumulation points of the sequence. Because then the set of accumulation points of the sequence would contain a Cantor set, which by closedness of $[x]_\infty$ would also be a subset of this set, contradicting that this set is countable. Thus there is some value $m > 0$ for which $\gamma(m+\frac{1}{2})$ is an isolated point of $[x]_\infty$.

Let I_0 and I_1 denote the open intervals of $\mathbb{S}^1_\infty \setminus [x]_\infty$ neighboring $\gamma(m+\frac{1}{2})$. We now show that these two intervals belong to two disjoint connected components of the open set $\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}} \setminus [x]_{\text{ray}}$. If not there exists a curve $\kappa\colon [0,1] \longrightarrow \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}} \setminus [\gamma(\frac{1}{2})]_{\text{ray}}$ with $\kappa(0) \in I_0$ and $\kappa(1) \in I_1$. And such a curve together with the arc of $\mathbb{S}^1_\infty$ between $\kappa(0)$ and $\kappa(1)$ would separate $\gamma([0, m+\frac{1}{2}[)$ from $\gamma(]m+\frac{1}{2}, \infty[)$, which contradicts that the later accumulates $\gamma(\frac{1}{2})$. Hence the class $[\gamma(\frac{1}{2})]_{\text{ray}}$ is separating and thus is an obstruction to $K_{\mathtt{w}} \perp\!\!\!\perp_{\text{ray}} K_{\mathtt{b}}$ being homeomorphic to $\mathbb{S}^2$. □

Remark 4.13. Note that the previous theorem does not cover all cases. Namely the case when each ray-equivalence class has bounded diameter, but the bound is not uniform, is not included. This means there is a sequence $\{[x_n]_{\text{ray}}\}_{n\in\mathbb{N}}$ of ray-equivalence classes whose diameters tend to ∞. It seems likely that in this case one may obtain from a Cantor diagonal type argument a ray-equivalence class with infinite diameter. We do not have a proof for this however.

We are now ready to prove that in the absence of a Moore obstruction the topological mating results in a branched covering of the sphere.

Proof of Proposition 4.3. Let $f = P_{\mathtt{w}} \perp\!\!\!\perp P_{\mathtt{b}}$ be the topological mating of the monic polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ of the same degree d, and $K_{\mathtt{w}}, K_{\mathtt{b}}$ their filled Julia sets. We assume that $K_{\mathtt{w}} \perp\!\!\!\perp K_{\mathtt{b}}$ is topologically a sphere, which is furthermore identified with $(\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\text{ray}}$ by Proposition 4.10. The formal mating is denoted by $F = (P_{\mathtt{w}} \uplus P_{\mathtt{b}})/\sim_{\text{ray}}$.

Consider first a point $[y] \in K_{\mathtt{w}} \perp\!\!\!\perp K_{\mathtt{b}}$ such that the corresponding ray-equivalence class $[y]_{\text{ray}} \subset \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$ does not contain any critical value of $P_{\mathtt{w}}, P_{\mathtt{b}}$.

Let $[x]_{\text{ray}}$ be a preimage (i.e., a component of the preimage) of $[y]_{\text{ray}}$ by F. Note that $[x]_{\text{ray}}$ does not contain any critical point of $P_{\mathtt{w}}$ or $P_{\mathtt{b}}$. This means that distinct extended external rays $R(\zeta), R(\zeta') \subset [x]_{\text{ray}}$ incident to the same vertex $v \in J_{\mathtt{w}} \sqcup J_{\mathtt{b}}$ are mapped by the formal mating $P_{\mathtt{w}} \uplus P_{\mathtt{b}}$ to distinct external rays. Equivalently the angles ζ, ζ' are mapped to distinct angles by z^d.

Let $R(\zeta), R(\zeta')$ be two distinct extended external rays in $[x]_{\text{ray}}$ not intersecting in a vertex $v \in J_{\mathtt{w}} \sqcup J_{\mathtt{b}}$. If they are mapped to the same extended external ray by F it follows that $[y]_{\text{ray}} = F([x]_{\text{ray}})$ is cyclic. This cannot happen by Proposition 4.12. Equivalently the angles ζ, ζ' are mapped to distinct points by z^d. This means that $[x]_{\text{ray}}$ is mapped homeomorphically to $[y]_{\text{ray}}$ by F.

Consider the restriction to the equator, i.e., $[x]_\infty = [x]_{\text{ray}} \cap \mathbb{S}^1_\infty$ and $[y]_\infty = [y]_{\text{ray}} \cap \mathbb{S}^1_\infty$. It follows from Proposition 4.12 that $[y]_\infty$ contains only finitely many points, i.e., $\#[y]_\infty = n < \infty$. Furthermore from the above it follows that $F|\mathbb{S}^1_\infty$ maps $[x]_\infty$ to $[y]_\infty$ homeomorphically. Recall that $F|\mathbb{S}^1_\infty$ is (conjugate to) the map $z^d \colon \mathbb{S}^1 \to \mathbb{S}^1$.

For each angle $\xi \in \mathbb{S}^1_\infty$ let $V(\xi) \subset \mathbb{S}^1_\infty$ be an open arc containing ξ (i.e., an open connected neighborhood of ξ). Let $V = V[y] = \bigcup_{\xi \in [y]_\infty} V(\xi)$. Here the sets $V(\xi)$ in this union are chosen to be pairwise disjoint. Consider now the set $V' = (F|\mathbb{S}^1_\infty)^{-1}(V)$ (i.e., the preimage of V under the map $z^d \colon \mathbb{S}^1 \to \mathbb{S}^1$). This set consists of nd open arcs (in $\mathbb{S}^1_\infty$). Given any point $\zeta \in (F|\mathbb{S}^1_\infty)^{-1}([y]_\infty)$, there is a component $V'(\zeta)$ of V' which is a neighborhood of ζ. Conversely each component of V' is a neighborhood of such a point ζ. Let $V'[x] := \bigcup_{\zeta \in [x]_\infty} V'(\zeta)$. From the above it follows that $F|\mathbb{S}^1_\infty \colon V'[x] \to V[y]$ is a homeomorphism.

We can find an open neighborhood $U = U[y] \subset \overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}}$ of $[y]_{\text{ray}}$ such that $U \cap \mathbb{S}^1_\infty = V$.

Consider now the saturated interior $U^* = U^*[y]$ of U, i.e., the set of all ray-equivalence classes contained in $U[y]$. By Lemma 2.2 (6) it follows that U^* is an open, saturated neighborhood of $[y]_{\text{ray}}$. It follows that each preimage $U^*[x]$ of U^* by F is an open, saturated neighborhood of some preimage $[x]_{\text{ray}}$ of $[y]_{\text{ray}}$ by F. Note that $F|\mathbb{S}^1_\infty$ maps $U^*[x] \cap \mathbb{S}^1_\infty$, i.e. $V'[x]$, homeomorphically to $U^* \cap \mathbb{S}^1_\infty$, i.e. to V. It follows that $U^*[x]$ is mapped homeomorphically to U^* by F.

Let $[U^*] \subset (\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\text{ray}} = K_{\mathtt{w}} \perp\!\!\!\perp K_{\mathtt{b}}$ be the image of U^* under the quotient map. This is an open neighborhood of $[y]_{\text{ray}} \in (\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\text{ray}}$. Each component $[U^*[x]]$ of $f^{-1}[U^*]$ is the image of a saturated set $U^*[x]$ under the quotient map. Note that f maps $[U[x]]$ homeomorphically to $[U^*]$.

Let $\mathcal{V}$ be the set of ray-equivalence classes that contain critical values, $[\mathcal{V}] \subset (\overline{\mathbb{C}}_{\mathtt{w}} \uplus \overline{\mathbb{C}}_{\mathtt{b}})/\sim_{\text{ray}} = K_{\mathtt{w}} \perp\!\!\!\perp K_{\mathtt{b}}$ be image of this set under the quotient map. We have shown that $f \colon (K_{\mathtt{w}} \perp\!\!\!\perp K_{\mathtt{b}}) \setminus f^{-1}[\mathcal{V}] \to (K_{\mathtt{w}} \perp\!\!\!\perp K_{\mathtt{b}}) \setminus [\mathcal{V}]$ is a covering map. Thus f is a branched covering.

That f is orientation preserving follows from the fact the formal mating F is orientation-preserving and f may be viewed as a pseudo-isotopic deformation of F. □

4.7. **Conformal mating revisited.** There are also stronger notions of Conformal/Geometric Mating in use. We start with the original definition of Douady and Hubbard, which is easily seen to be equivalent to geometric mateability as defined in Definition 4.4. This definition was used by Zakeri and Yampolsky (see Theorem 4.18 below):

Definition 4.14 (Conformal Mating Ia)**.** A rational map $R \colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ of degree $d > 1$ is the conformal mating of two degree d polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ with connected and locally connected filled-in Julia sets $K_{\mathtt{w}}, K_{\mathtt{b}}$, if and only if there exists two

semi-conjugacies

$$\phi_i : K_i \to \widehat{\mathbb{C}}, \quad \text{with} \quad \phi_i \circ P_i = R \circ \phi_i,$$

conformal in the interior of the filled Julia sets, with $\phi_{\mathtt{w}}(K_{\mathtt{w}}) \cup \phi_{\mathtt{b}}(K_{\mathtt{b}}) = \widehat{\mathbb{C}}$ and with $\phi_i(z) = \phi_j(w)$ for $i, j \in \{\mathtt{w}, \mathtt{b}\}$ if and only if $z \sim w$. Here $\sim$ is the equivalence relation on $K_{\mathtt{w}} \sqcup K_{\mathtt{b}}$ which defines the topological mating, i.e., the one defined in Section 4.1.

The reader should compare the previous definition with Lemma 4.6.

Definition 4.15 (Conformal Mating II)**.** A rational map $R\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ of degree $d > 1$ is the conformal mating of two degree d polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ with connected and locally connected filled-in Julia sets $K_{\mathtt{w}}, K_{\mathtt{b}}$, if and only if there exists disjoint conformal embeddings $K^0_{\mathtt{w}}, K^0_{\mathtt{b}}$ in $\widehat{\mathbb{C}}$ of $K_{\mathtt{w}}, K_{\mathtt{b}}$ and a pseudo-isotopy H, such that each H_t, $t < 1$ is conformal on the interior of $K^0_{\mathtt{w}} \cup K^0_{\mathtt{b}}$ and such that $\phi_i = H_1|_{K^0_i}$, P_i, $i = \mathtt{w}, \mathtt{b}$ and R realizes the conformal mating Ia of Definition 4.14.

At least in case there exists polynomials with locally connected, positive area Julia sets carrying an invariant line field, one should consider also the following strengthened version of conformal mating.

Definition 4.16 (Conformal Mating III)**.** A rational map $R\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ of degree $d > 1$ is the conformal mating of two degree d polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ with connected and locally connected filled-in Julia sets $K_{\mathtt{w}}, K_{\mathtt{b}}$, if and only if there exists disjoint conformal embeddings $K^0_{\mathtt{w}}, K^0_{\mathtt{b}}$ in $\widehat{\mathbb{C}}$ of $K_{\mathtt{w}}, K_{\mathtt{b}}$ and a pseudo-isotopy H, such that each H_t, $t < 1$ is quasi-conformal with complex dilatation 0 a.e. on $K^0_{\mathtt{w}} \cup K^0_{\mathtt{b}}$ and such that $\phi_i = H_1|_{K^0_i}$, P_i, $i = \mathtt{w}, \mathtt{b}$ and R realizes the Conformal Mating Ia of Definition 4.14.

4.8. **Mating Questions.** We may summarize the basic mating questions as follows:

- When is the equivalence relation $\sim_{\text{ray}}$ closed, i.e., when is there no Hausdorff obstruction?
- When is there no Moore obstruction?
- If $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ is homeomorphic to $\mathbb{S}^2 \sim \overline{\mathbb{C}}$, when is there then a homeomorphism which conjugates $P_{\mathtt{w}} \amalg P_{\mathtt{b}}$ to a rational map?
- Are the diameters of the equivalence classes of $\sim_{\text{ray}}$ always finite? Or equivalently are the equivalence classes of $\sim$ always finite?
- If bounded can they be of arbitrary size? (This is the question of existence of long ray-connections)

4.9. **Existence of Matings.** To show that the theory of matings is not vacuous let us mention a few of the mating results obtained so far. The first result obtained by M. Rees, Tan Lei, and M. Shishikura (see [Ree92], [Tan], [S]) concerns the mating of quadratic post-critically finite polynomials.

Theorem 4.17 (Tan Lei, Rees, Shishikura)**.** *Let $P_{\mathtt{w}}(z) = z^2 + c_{\mathtt{w}}$ and $P_{\mathtt{b}}(z) = z^2 + c_{\mathtt{b}}$ be two post-critically finite quadratic polynomials. Then $P_{\mathtt{w}}$ and $P_{\mathtt{b}}$ are conformally mateable (in the strong sense II) if and only if $c_{\mathtt{w}}$ and $c_{\mathtt{b}}$ do not belong to conjugate limbs of the Mandelbrot set. Moreover if mateable the resulting rational map is unique up to Möbius-conjugacy.*

One implication in the above theorem is relatively easy to see. Namely if $c_{\mathtt{w}}, c_{\mathtt{b}}$ belong to conjugate limbs of the Mandelbrot set, the corresponding polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ are not matable. Let $c_{\mathtt{w}}$ be in the p/q and $c_{\mathtt{b}}$ be in the $-p/q$ limb. Then the unique p/q-cycle of rays for $P_{\mathtt{w}}$ is identified with the unique $-p/q$-cycle of rays for $P_{\mathtt{b}}$. Each ray in the cycles connects a fixed point $\alpha_{\mathtt{w}}$ for $P_{\mathtt{w}}$ to a fixed point $\alpha_{\mathtt{b}}$ for $P_{\mathtt{b}}$. The corresponding ray-equivalence class is cyclic, so that the polynomials are not even topologically mateable by Proposition 4.12.

Matings of polynomials that are not post-critically finite are much more difficult to understand. An important result in this setting was proved by M. Yampolsky and S. Zakeri (see [Y-Z]).

Theorem 4.18 (Yampolsky and Zakeri)**.** *Suppose $P_{\mathtt{w}}, P_{\mathtt{b}}$ are quadratic polynomials which are not anti-holomorphically conjugate and each with a bounded type Siegel fixed point. Then $P_{\mathtt{w}}$ and $P_{\mathtt{b}}$ are geometrically mateable in the sense of Definition 4.14.*

Here bounded type Siegel fixed point means that the arguments $\theta_i \in \mathbb{R}/\mathbb{Z}$ of the corresponding multipliers $\lambda_i = \exp(i2\pi\theta_i)$ have continuous fraction expansions with uniformly bounded partial fractions.

On the other hand, regarding the question whether a rational map arises as a mating of polynomials, we have the following result obtained in [Mey2], see also [Mey1] and [Mey3].

Theorem 4.19 (Meyer)**.** *Let $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a post-critically finite rational map such that its Julia set is the whole sphere $\widehat{\mathbb{C}}$. Then every sufficiently high iterate R^n of R arises as a mating (i.e., is topologically conjugate to the topological mating of two polynomials).*

In fact the previous statement remains true for *expanding Thurston maps*.

5. Slow Mating

Milnor defined in [Mi2] a notion of mating intermediate to the topological and the formal mating and depending on a complex parameter $\lambda \in \mathbb{C}\backslash\overline{\mathbb{D}}$. This notion has been explored recently by Buff and Cheritat (see also Cheritats contribution to this volume).

The advantage of Milnors construction is that it constructs directly rational maps $\widehat{R}_\lambda$ of the right degree. Moreover the domain and range comes with holomorphically embedded copies of the filled Julia sets. The disadvantage is that the domain and the range of the map are not the same and in particular the embedded copies of the filled Julia sets in the domain and range are different. The domain and range can however be identified via a d-quasi conformal homeomorphism χ_λ, which has complex dilatation zero a.e. on the embedded copies of the filled Julia sets. The composition $\widetilde{R}_\lambda = \chi_\lambda^{-1} \circ \widehat{R}_\lambda$ is a degree d, d-quasi-regular dynamical system, which preserves and is conformal on the embedded copies of the filled Julia sets. And which is a d-quasi-regular degree d covering of the separating annulus C^λ to itself. Moreover the restriction of $\widetilde{R}_\lambda$ to the filled Julia sets has polynomial-like (in particular holomorphic) extensions, which are conformally conjugate to appropriate polynomial-like restrictions of the polynomials.

5.1. **Definition of the slow mating.** Let $P_{\mathtt{w}}, P_{\mathtt{b}}$ be two monic polynomials of degree d with connected filled Julia sets $K_{\mathtt{w}}, K_{\mathtt{b}}$. We do however not assume that $K_{\mathtt{w}}, K_{\mathtt{b}}$ are locally connected. As in Theorem 3.1 we denote by $\varphi_i \colon \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus K_i$ the Böttcher conjugacy (which conjugates z^d to P_i) for $i = \mathtt{w}, \mathtt{b}$. Recall that the Green's function G_i of K_i (with pole at ∞) is the subharmonic function given by

$$G_i(z) = \begin{cases} 0, & \text{if} \quad z \in K_i \\ \log|\varphi_i^{-1}(z)|, & \text{if} \quad z \notin K_i. \end{cases}$$

For $t > 0$ write

$$U_i^t := \{z \in \mathbb{C} \mid G_i(z) < 2t\}, \qquad i = \mathtt{w}, \mathtt{b}$$

Then $P_i \colon U_i^{(t/d)} \longrightarrow U_i^t$ is polynomial-like of degree d, i.e., $U' = U_i^{(t/d)} \subset\subset U = U_i^t$ are isomorphic to $\mathbb{D}$ and P_i is holomorphic and proper of degree d.

Fix any $\lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}$, write $t = \log|\lambda| > 0$. Consider the annulus $A = A(\lambda^2) = \{w \in \mathbb{C} \mid 1 < w < |\lambda^2|\}$. Note that $\varphi_i \colon A(\lambda^2) \longrightarrow U_i^t \setminus K_i$ is a biholomorphic map. And denote by $\iota_\Lambda \colon A(\lambda^2) \longrightarrow A(\lambda^2)$ the biholomorphic involution $\iota_\lambda(w) = \lambda^2/w$, which fixes λ. We identify the two annuli $U_{\mathtt{w}}^t \setminus K_{\mathtt{w}}, U_{\mathtt{b}}^t \setminus K_{\mathtt{b}}$ via the Böttcher maps φ_i and ι_Λ, so that "the inner boundary of the first is identified with the outer of the second" (and vice versa). Formally we let $\overset{\lambda}{\sim}$ be the equivalence relation on $U_{\mathtt{w}}^t \sqcup U_{\mathtt{b}}^t$ generated by

$$\varphi_{\mathtt{w}}(w) \overset{\lambda}{\sim} \varphi_{\mathtt{b}}(\iota_\lambda(w)), \tag{5.1}$$

for all $w \in A(\lambda^2)$. Or equivalently for $z_{\mathtt{w}} \in U_{\mathtt{w}}^t \setminus K_{\mathtt{w}}$ and $z_{\mathtt{b}} \in U_{\mathtt{b}}^t \setminus K_{\mathtt{b}}$:

$$z_{\mathtt{w}} \overset{\lambda}{\sim} z_{\mathtt{b}} \quad \Longleftrightarrow \quad \varphi_{\mathtt{w}}^{-1}(z_{\mathtt{w}}) \cdot \varphi_{\mathtt{b}}^{-1}(z_{\mathtt{b}}) = \lambda^2.$$

Define $\widehat{\mathbb{C}}^\lambda = (U_{\mathtt{w}}^t \sqcup U_{\mathtt{b}}^t)/\overset{\lambda}{\sim}$. As usual let $\Pi^\lambda \colon U_{\mathtt{w}}^t \sqcup U_{\mathtt{b}}^t \longrightarrow \widehat{\mathbb{C}}^\lambda$ denote the natural projection. Furthermore equip $\widehat{\mathbb{C}}^\lambda$ with the complex structure given by the two local parameters $\Pi_i^\lambda = \Pi^\lambda \colon U_i^t \longrightarrow \widehat{\mathbb{C}}^\lambda$, $i = \mathtt{w}, \mathtt{b}$. Then the change of coordinates $U_{\mathtt{w}}^t \setminus K_{\mathtt{w}} \to U_{\mathtt{b}}^t \setminus K_{\mathtt{b}}$ is given by $\varphi_{\mathtt{b}}^{-1} \circ \iota_\lambda \circ \varphi_{\mathtt{w}}(z)$. The Riemann surface $\widehat{\mathbb{C}}^\lambda$ is simply connected and compact, hence isomorphic to $\widehat{\mathbb{C}}$.

For each $\lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}$ the polynomials $P_{\mathtt{w}}, P_{\mathtt{b}}$ induce a proper degree d holomorphic map $R_\lambda \colon \widehat{\mathbb{C}}^\lambda \longrightarrow \widehat{\mathbb{C}}^{\lambda^d}$ given by

$$R_\lambda(w) = \begin{cases} \Pi_{\mathtt{w}}^{\lambda^d} \circ P_{\mathtt{w}}(z), & \text{if } w = \Pi_{\mathtt{w}}^\lambda(z), \\ \Pi_{\mathtt{b}}^{\lambda^d} \circ P_{\mathtt{b}}(z), & \text{if } w = \Pi_{\mathtt{b}}^\lambda(z). \end{cases}$$

This map is well defined and hence proper holomorphic of degree d, since it follows from (Böttcher's) Theorem 3.1 that for all $z_{\mathtt{w}} \in U_{\mathtt{w}}^t \setminus K_{\mathtt{w}}, z_{\mathtt{b}} \setminus U_{\mathtt{b}}^t$ it holds

$$\Pi_{\mathtt{w}}^\lambda(z_{\mathtt{w}}) = w = \Pi_{\mathtt{b}}^\lambda(z_{\mathtt{b}}) \quad \Longleftrightarrow \quad \varphi_{\mathtt{w}}^{-1}(z_{\mathtt{w}}) \cdot \varphi_{\mathtt{b}}^{-1}(z_{\mathtt{b}}) = \lambda^2 \quad \Longrightarrow$$
$$\varphi_{\mathtt{w}}^{-1}(P_{\mathtt{w}}(z_{\mathtt{w}})) \cdot \varphi_{\mathtt{b}}^{-1}(P_{\mathtt{b}}(z_{\mathtt{b}})) = \lambda^{2d} \quad \Longleftrightarrow \quad \Pi_{\mathtt{w}}^{\lambda^d} \circ P_{\mathtt{w}}(z_{\mathtt{w}}) = \Pi_{\mathtt{b}}^{\lambda^d} \circ P_{\mathtt{b}}(z_{\mathtt{b}}).$$

If we fix conformal isomorphisms $\eta^\lambda \colon \widehat{\mathbb{C}}^\lambda \longrightarrow \widehat{\mathbb{C}}$, and express R_λ in these coordinates, then R_λ becomes a family of rational maps. These are however a priori only defined up to pre- and post-composition by Möbius transformations. We shall return to this discussion later.

Again it should be emphasized that this definition is well defined even for polynomials whose Julia sets are not locally connected. Thus this may serve as a starting point to define matings in this setting.

5.2. **Equivalence to the Topological Mating.** The topological mating may be recovered from $R_\lambda\colon \widehat{\mathbb{C}}^\lambda \longrightarrow \widehat{\mathbb{C}}^{\lambda^d}$, $\lambda > 1$ in a manner analogous to how the topological mating was obtained from the formal mating in Proposition 4.10.

Let $P_{\mathtt{w}}, P_{\mathtt{b}}$ be monic polynomials of degree d with connected and locally connected Julia sets. Recall that $R_i(\zeta) \subset \mathbb{C} \setminus K_i$ denotes the external ray with angle $\zeta \in \mathbb{S}^1$ for $i = \mathtt{w}, \mathtt{b}$. Let λ be real, i.e., $\lambda > 1$. Let $\overset{\lambda}{\sim}_M$ be the smallest equivalence relation on $\widehat{\mathbb{C}}^\lambda$ for which for each $\zeta \in \mathbb{S}^1$ the set

$$\overline{\Pi^\lambda(R_{\mathtt{w}}(\zeta) \cap U^t_{\mathtt{w}})} = \overline{\Pi^\lambda(R_{\mathtt{b}}(\overline{\zeta}) \cap U^t_{\mathtt{b}})}$$

is contained in one equivalence class. Then R_λ semi-conjugates the equivalence relation $\overset{\lambda}{\sim}_M$ to $\overset{\lambda^d}{\sim}_M$, i.e. $z \overset{\lambda}{\sim}_M z'$ implies $R_\lambda(z) \overset{\lambda^d}{\sim}_M R_\lambda(z')$.

Let $K_{\mathtt{w}} \amalg^\lambda_M K_{\mathtt{b}} = \widehat{\mathbb{C}}^\lambda / \overset{\lambda}{\sim}_M$ with natural projections $\Pi^\lambda_M\colon \widehat{\mathbb{C}}^\lambda \longrightarrow K_{\mathtt{w}} \amalg^\lambda_M K_{\mathtt{b}}$ and define for $\lambda > 1$

$$P_{\mathtt{w}} \amalg^\lambda_M P_{\mathtt{b}}\colon K_{\mathtt{w}} \amalg^\lambda_M K_{\mathtt{b}} \longrightarrow K_{\mathtt{w}} \amalg^{\lambda^d}_M K_{\mathtt{b}}$$

as the mapping induced by R_λ, i.e., the top square below commutes

$$\begin{array}{ccc}
\widehat{\mathbb{C}}^\lambda & \xrightarrow{R_\lambda} & \widehat{\mathbb{C}}^{\lambda^d} \\
{\scriptstyle \Pi^\lambda_M}\big\downarrow & & \big\downarrow{\scriptstyle \Pi^{\lambda^d}_M} \\
K_{\mathtt{w}} \amalg^\lambda_M K_{\mathtt{b}} & \xrightarrow{P_{\mathtt{w}} \amalg^\lambda_M P_{\mathtt{b}}} & K_{\mathtt{w}} \amalg^{\lambda^d}_M K_{\mathtt{b}} \\
{\scriptstyle h_\lambda}\big\downarrow & & \big\downarrow{\scriptstyle h_{\lambda^d}} \\
K_{\mathtt{w}} \amalg K_{\mathtt{b}} & \xrightarrow{P_{\mathtt{w}} \amalg P_{\mathtt{b}}} & K_{\mathtt{w}} \amalg K_{\mathtt{b}}.
\end{array} \tag{5.2}$$

Then for $\lambda > 1$ the spaces $K_{\mathtt{w}} \amalg^\lambda_M K_{\mathtt{b}}$ are canonically homeomorphic to $K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ by homeomorphisms h_λ such that $h_{\lambda^d} \circ (P_{\mathtt{w}} \amalg^\lambda_M P_{\mathtt{b}}) \circ h_\lambda^{-1} = P_{\mathtt{w}} \amalg P_{\mathtt{b}}$ is the topological mating (i.e., the bottom square commutes). The proof is similar to that of Lemma 4.10 and is left to the reader.

5.3. **Choosing conformal isomorphisms.** In this section we show that we may choose the isomorphisms $\eta^\lambda\colon \widehat{\mathbb{C}}^\lambda \longrightarrow \widehat{\mathbb{C}}$ such that the images of the filled Julia sets move holomorphically in $\widehat{\mathbb{C}}$ with $\lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}$.

To normalize the motion we will choose three distinguished points in $\widehat{\mathbb{C}}^\lambda$. Fix two points $z_{\mathtt{w}} \in K_{\mathtt{w}}$, $z_{\mathtt{b}} \in K_{\mathtt{b}}$ and let $w_{\mathtt{w}}(\lambda) := \Pi^\lambda(z_{\mathtt{w}})$, $w_{\mathtt{b}}(\lambda) := \Pi^\lambda(z_{\mathtt{b}})$ be their images in $\widehat{\mathbb{C}}^\lambda$. The third point is chosen as follows. Recall from (5.1) that $\varphi_{\mathtt{w}}(\lambda) \overset{\lambda}{\sim} \varphi_{\mathtt{b}}(\lambda^2/\lambda) = \varphi_{\mathtt{b}}(\lambda)$. Let $w_1(\lambda) := \Pi^\lambda(\varphi_{\mathtt{w}}(\lambda)) = \Pi^\lambda(\varphi_{\mathtt{b}}(\lambda))$.

We shall choose $\lambda_0 = \mathrm{e}^1$ as the base point of the motion. For $\Lambda > 0$ and K_i locally connected we can choose the points $w_{\mathtt{w}}(\lambda_0), w_{\mathtt{b}}(\lambda_0), w_1(\lambda_0) \in \widehat{\mathbb{C}}^{\lambda_0}$ so that they are not identified under the equivalence relation from Section 5.2, i.e., are in different equivalence classes of $\overset{\lambda_0}{\sim}_M$. That is $\varphi_{\mathtt{w}}(\mathrm{e}) = \varphi_{\mathtt{b}}(\mathrm{e}), z_{\mathtt{w}}, z_{\mathtt{b}}$ are not ray-equivalent in the sense of Definition 4.8. This is however not necessary for most of the following discussion.

For $\lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}$ let $\eta_\lambda\colon \widehat{\mathbb{C}}^\lambda \longrightarrow \widehat{\mathbb{C}}$ be the conformal isomorphism normalized by

$$\eta_\lambda(w_{\mathtt{w}}(\lambda)) = 0, \quad \eta_\lambda(w_{\mathtt{b}}(\lambda)) = \infty, \text{ and } \eta_\lambda(w_1(\lambda)) = 1,$$

and define $K^\lambda_{\mathtt{w}} = \eta_\lambda(\Pi^\lambda(K_{\mathtt{w}}))$, $K^\lambda_{\mathtt{b}} = \eta_\lambda(\Pi^\lambda(K_{\mathtt{b}}))$.

Theorem 5.1. *The map* $M\colon (\mathbb{C}\backslash\overline{\mathbb{D}}) \times (K_{\mathtt{w}}^{\lambda_0} \cup K_{\mathtt{b}}^{\lambda_0}) \longrightarrow \widehat{\mathbb{C}}$ *given by*

$$M(\lambda, z) = \eta_\lambda \circ \Pi^\lambda \circ (\Pi^{\lambda_0})^{-1} \circ \eta_{\lambda_0}^{-1}(z)$$

is a holomorphic motion with base point λ_0*, i.e.,*

(1) $M(\lambda_0, \cdot) = \mathrm{Id}$ *on* $K_{\mathtt{w}}^{\lambda_0} \cup K_{\mathtt{b}}^{\lambda_0}$*.*
(2) For each fixed $z \in K_{\mathtt{w}}^{\lambda_0} \cup K_{\mathtt{b}}^{\lambda_0}$ *the map* $\mathbb{C}\backslash\overline{\mathbb{D}} \ni \lambda \to M(\lambda, z)$ *is holomorphic.*
(3) For each fixed $\lambda \in \mathbb{C}\backslash\overline{\mathbb{D}}$ *the map* $K_{\mathtt{w}}^{\lambda_0} \cup K_{\mathtt{b}}^{\lambda_0} \ni z \mapsto M(\lambda, z)$ *is injective.*

Corollary 5.2. *The family of rational maps* $\widehat{R}_\lambda = \eta_{\lambda^d} \circ R_\lambda \circ \eta_\lambda^{-1}\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ *depends holomorphically on* $\lambda \in \mathbb{C}\backslash\overline{\mathbb{D}}$*.*

Proof. The zeros and poles of $\widehat{R}$ depend holomorphically on λ and $\widehat{R}$ fixes 1. □

Conjecture 1. *If the family of degree* d *rational maps* $\widehat{R}_\lambda$*,* $\lambda > 1$ *has a limit* R *of degree* d*, as* $\lambda \searrow 1$*, then* R *is a conformal mating of* $P_{\mathtt{w}}$ *and* $P_{\mathtt{b}}$ *in the strongest sense, Definition 4.16.*

Before we proceed to a proof of the theorem, let us introduce a few facts about polynomials with connected filled Julia set. It is well known see e.g. [B-H] or [P-T] that the almost complex structures on $\widehat{\mathbb{C}}$ which are given by the Beltrami forms

$$\sigma_\Lambda = \mu_\Lambda(z)\frac{d\overline{z}}{dz}, \qquad \mu_\Lambda(z) = \frac{\Lambda - 1}{\Lambda + 1}\frac{z}{\overline{z}}, \quad z \in \mathbb{C}^*$$

are invariant under $z \mapsto z^k$ for every $\Lambda \in \mathbb{H}_+$ and every $k \in \mathbb{Z}\backslash\{0\}$ and under $z \mapsto \alpha z$ for every $\alpha \in \mathbb{C}^*$. (Note that contrary to most conventions $\sigma_1 \equiv 0$.) Moreover the integrating q-c homeomorphism for σ_Λ, that fix 0, 1 and ∞ is the map $\zeta_\Lambda(z) = z|z|^{(\Lambda-1)}$. It restricts to the identity on the unit circle and conjugates $z \mapsto z^d$ to itself. In fact the lift to the logarithmic coordinate on $\mathbb{C}^*$ which fixes 0 is the real-linear map fixing i and sending 1 to Λ. The maps ζ_Λ form a group with neutral element $\zeta_1 = \mathrm{Id}$, with $\zeta_{\Lambda+it} = \zeta_\Lambda \circ \zeta_{1+it}$ for all $\Lambda \in \mathbb{H}$ and $t \in \mathbb{R}$ and with $\zeta_{ss'} = \zeta_s \circ \zeta_{s'}$ for all $s, s' > 0$. And thus defines a group action on $\widehat{\mathbb{C}}$. Consequently for every polynomial P with connected filled Julia set K, and every $\Lambda \in \mathbb{H}$ we obtain a P-invariant almost complex structure on $\widehat{\mathbb{C}}$ with Beltrami form $\sigma_\Lambda^P = \mu_\Lambda^P \frac{d\overline{z}}{dz}$, where σ_Λ^P and hence μ_Λ^P is equal to 0 on K_P and equal to $\psi_P^*(\sigma_\Lambda)$ on $\mathbb{C}\backslash K$, i.e.

$$\mu_\Lambda^P = \frac{\Lambda - 1}{\Lambda + 1}\frac{\psi_P(z)\overline{\psi_P'(z)}}{\overline{\psi_P(z)}\psi_P'(z)}, \quad c \in \mathbb{C}\backslash K,$$

where $\psi_P = \varphi_P^{-1}$. For this almost complex structure the map $\zeta_\Lambda^P\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ given by

$$\zeta_\Lambda^P(z) = \begin{cases} z, & z \in K_P \\ \varphi_P \circ \zeta_\Lambda \circ \psi_P(z), & z \notin K_P \end{cases}$$

is continuous and hence an integrating q-c homeomorphism. It acts on points in $\mathbb{C}\backslash K$ by multiplying potential t' by $\Re(\Lambda)$ and adding $\Im(\Lambda)t'$ to the argument. In particular for Λ real ζ_Λ^P preserves rays and maps points of potential t' to points of potential Λt. By construction the maps ζ_Λ^P form a group under composition. And this group is canonically isomorphic to the group formed by the maps ζ_Λ under composition, as follows from the formula above (see also [P-T] for further details).

Fix $\lambda_0 = \mathrm{e}^1$ and consider the almost complex structures on $\widehat{\mathbb{C}}^{\lambda_0}$ given by the Beltrami forms $\sigma_\Lambda^{\lambda_0} = (\Pi_i^{\lambda_0})_*(\sigma_\Lambda^{P_i})$, which is supported only on the separating annulus

$C^{\lambda_0} = \Pi_i^{\lambda_0}(U_i^1)$, $i = \mathtt{w}, \mathtt{b}$. Note that the invariance of σ_Λ under the involutions ι_λ, $\sigma_\Lambda = \iota_\lambda^*(\sigma_\Lambda)$ ensures that $\sigma_\Lambda^{\lambda_0}$ is well defined. Moreover $\sigma_\Lambda^{\lambda_0}$ depends complex analytically on $\Lambda \in \mathbb{H}_+$, since point wise the coefficient function $\mu_\Lambda^{\lambda_0}$ is a complex scalar multiple of norm 1 or 0 of the constant $(\Lambda - 1)/(\Lambda + 1)$.

For each $\Lambda \in \mathbb{H}_+$ write $\lambda = \mathrm{e}^\Lambda$ and let $\phi_\Lambda \colon \widehat{\mathbb{C}}^{\lambda_0} \longrightarrow \widehat{\mathbb{C}}$ be the integrating homeomorphism for $\sigma_\Lambda^{\lambda_0}$ which is normalized by

$$\phi_\Lambda(w_\mathtt{w}(\lambda_0)) = 0, \quad \phi_\Lambda(w_\mathtt{b}(\lambda_0)) = \infty, \text{ and } \phi_\Lambda(w_1(\lambda_0)) = 1.$$

Then $\phi_1 = \eta_{\lambda_0}$ and ϕ_Λ depends holomorphically on Λ, by the Ahlfors-Bers Theorem for almost complex structures depending analytically on a complex parameter.

Theorem 5.3. *The map $H \colon \mathbb{H}_+ \times \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ given by $H(\Lambda, z) = \phi_\Lambda \circ \phi_1^{-1}(z)$ is a holomorphic motion with base point 1 and its (z-variable) restriction to $(K_\mathtt{w}^{\lambda_0} \cup K_\mathtt{b}^{\lambda_0})$ is a $2\pi i$ periodic holomorphic motion (in the Λ-variable).*

Proof. The only statement not justified already is the $2\pi i$ periodicity. However for $\Lambda' = 1 + 2\pi i$ the integrating map $\zeta_{\Lambda'}$ for $\sigma_{\Lambda'}$ restricts to the identity on the circles of center 0 and radius e^k, $k \in \mathbb{Z}$. As a consequence $\zeta_{\Lambda'}^{P_i}$ restricts to the identity on K_i, on the equipotential $\varphi_i(\mathrm{e}\mathbb{S}^1)$ containing $\varphi_i(\lambda_0)$ and on the boundary of U_i^1 for $i = \mathtt{w}, \mathtt{b}$. It follows that the map $\phi \colon \widehat{\mathbb{C}}^{\lambda_0} \longrightarrow \widehat{\mathbb{C}}^{\lambda_0}$ given by

$$\phi(z) = \Pi_i^{\lambda_0} \circ \zeta_{\Lambda'}^{P_i} \circ (\Pi_i^{\lambda_0})^{-1}(z)$$

is an integrating qc-homeomorphism for $\sigma_{\Lambda'}^{\lambda_0}$ which is the identity on $\Pi^{\lambda_0}(K_\mathtt{w} \sqcup K_\mathtt{b})$ union the core geodesic $\Pi_i^{\lambda_0}(\varphi_i(\mathrm{e}\mathbb{S}^1))$ of the separating annulus C^{λ_0}. In particular ϕ fixes $w_1(\lambda_0), w_\mathtt{w}(\lambda_0)$ and $w_\mathtt{b}(\lambda_0)$. Thus $\phi_{\Lambda'} = \phi_1 \circ \phi$ and $\phi_{\Lambda'} = \phi_1$ on $\Pi^{\lambda_0}(K_\mathtt{w} \cup K_\mathtt{b})$. The $2\pi i$ periodicity then follows because the maps ζ_Λ form a group action on $\mathbb{C}^*$, so that $\phi_{\Lambda'} = \phi_1 \circ \phi$ implies $\phi_{\Lambda + i2\pi} = \phi_\Lambda \circ \phi$ for all $\Lambda \in \mathbb{H}$. □

Corollary 5.4. *The map $\widehat{H} \colon (\mathbb{C} \backslash \overline{\mathbb{D}}) \times (K_\mathtt{w}^{\lambda_0} \cup K_\mathtt{b}^{\lambda_0}) \longrightarrow \widehat{\mathbb{C}}$ given by $\widehat{H}(\lambda, z) := H(\log \lambda, z)$ is a holomorphic motion with base point λ_0.*

Theorem 5.5. *For every $\Lambda \in \mathbb{H}_+$ and $\lambda = \mathrm{e}^\Lambda$ $\eta_\lambda \circ \Pi^\lambda = \phi_\Lambda \circ \Pi^{\lambda_0}$ on $K_\mathtt{w} \sqcup K_\mathtt{b}$ and the map $\eta_\lambda^{-1} \circ \phi_\Lambda \colon \widehat{\mathbb{C}}^{\lambda_0} \longrightarrow \widehat{\mathbb{C}}^\lambda$ is a quasi conformal homeomorphism with complex dilatation 0 a.e. on $\Pi^{\lambda_0}(K_\mathtt{w} \sqcup K_\mathtt{b})$. Moreover for $\Lambda > 0$ and $K_\mathtt{w}, K_\mathtt{b}$ locally connected this map preserves the equivalence classes of $\overset{\lambda}{\sim}_M$.*

Proof. For $i = \mathtt{w}, \mathtt{b}$ and $t = 2\Re(\Lambda)$ the maps

$$\xi_i = \phi_\Lambda \circ \Pi_i^{\lambda_0} \circ (\zeta_\Lambda^{P_i})^{-1} \colon U_i^t \longrightarrow \widehat{\mathbb{C}}$$

are quasi conformal with complex dilatation 0 and hence biholomorphic. Moreover they satisfy

$$\forall z_\mathtt{w} \in U_\mathtt{w}^t \backslash K_\mathtt{w}, \forall z_\mathtt{b} \in U_\mathtt{b}^t \backslash K_\mathtt{b} : \qquad \xi_\mathtt{w}(z_\mathtt{w}) = \xi_\mathtt{b}(z_\mathtt{b}) \Longleftrightarrow \varphi_\mathtt{w}(z_\mathtt{w})\varphi_\mathtt{b}(z_\mathtt{b}) = \lambda^2$$

and $\xi_i = \phi_\Lambda \circ \Pi_i^{\lambda_0}$ on K_i, because $\zeta_\Lambda^{P_i} = \mathrm{Id}$ on the filled Julia sets K_i. Thus it follows from the normalizations that $\xi_i = \eta_\lambda \circ \Pi_i^\lambda$ so that $\eta_\lambda \circ \Pi^\lambda = \phi_\Lambda \circ \Pi^{\lambda_0}$ on $K_\mathtt{w} \sqcup K_\mathtt{b}$ and hence the map $\eta_\lambda^{-1} \circ \phi_\Lambda \colon \widehat{\mathbb{C}}^{\lambda_0} \longrightarrow \widehat{\mathbb{C}}^\lambda$ is a quasi conformal homeomorphism, which has complex dilatation 0 a.e. on this set. Because $\zeta_\Lambda^{P_i} = \mathrm{Id}$ on the filled Julia sets

K_i.

$$\begin{tikzcd}
U_i^1, K_i \arrow[r, "\zeta_\Lambda^{P_i}, \mathrm{Id}"] \arrow[d, "\Pi_i^{\lambda_0}"'] & U_i^t, K_i \arrow[d, "\Pi_i^{\lambda}"] \\
\widehat{\mathbb{C}}^{\lambda_0}, \Pi_i^{\lambda_0}(K_i) \arrow[r, "\eta_\lambda^{-1}\circ\phi_\Lambda"] \arrow[d, "\phi_1=\eta_{\lambda_0}"'] \arrow[dr, "\phi_\Lambda"] & \widehat{\mathbb{C}}^{\lambda}, \Pi_i^{\lambda}(K_i) \arrow[d, "\eta_\lambda"] \\
\widehat{\mathbb{C}}, K_i^{\lambda_0} & \widehat{\mathbb{C}}, K_i^{\lambda}.
\end{tikzcd}$$

Finally preservation of $\overset{\lambda}{\sim}_M$ follows, because ζ_Λ preserves lines through the origin, when $\Lambda > 0$. □

Proof. (of Theorem 5.1): Theorem 5.1 follows immediately by combining Corollary 5.4 and Theorem 5.5 since for $z \in K_{\mathtt{w}}^{\lambda_0} \cup K_{\mathtt{b}}^{\lambda_0}$

$$M(\lambda, \cdot) = \eta_\lambda \circ \Pi^\lambda \circ (\Pi^{\lambda_0})^{-1} \circ \eta_{\lambda_0}^{-1} = \phi_\Lambda \circ \Pi^{\lambda_0} \circ (\Pi^{\lambda_0})^{-1} \circ \phi_1^{-1} = \phi_\Lambda \circ \phi_1^{-1}.$$

□

Theorem 5.6. *For every $\Lambda > 0$ and $\lambda = \mathrm{e}^\Lambda$ the map $\chi_\lambda \colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ given by $\chi_\lambda = \varphi_{d\Lambda} \circ \varphi_\Lambda^{-1}$ is a d-quasiconformal homeomorphism, with*

$$\chi_\lambda(K_{\mathtt{w}}^\lambda \cup K_{\mathtt{b}}^\lambda) = K_{\mathtt{w}}^{\lambda^d} \cup K_{\mathtt{b}}^{\lambda^d}$$

and with complex dilatation 0 almost everywhere on $K_{\mathtt{w}}^\lambda \cup K_{\mathtt{b}}^\lambda$.

Proof. For $\Lambda > 0$ the maps $z \mapsto z|z|^{(\Lambda'-1)}$ form a commutative group action on $\mathbb{C}^*$. Thus it follows that the real dilatation of the qc-homeomorphism $\chi_\lambda = \phi_{d\Lambda} \circ \phi_\Lambda^{-1}$ equals the dilation of ϕ_d at corresponding points. This is 0 on $\Pi^{\lambda_0}(K_{\mathtt{w}} \sqcup K_{\mathtt{b}})$ and an easy computation shows that it equals d on the separating annulus C^{λ_0}. □

Corollary 5.7. *For $\Lambda > 0$ and $\lambda = \mathrm{e}^\lambda$ the map $\widetilde{R}^\lambda := \chi_\lambda^{-1} \circ \widehat{R}_\lambda \colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is a quasi-regular degree d branched covering which is conformal a.e. on $K_{\mathtt{w}}^\lambda \cup K_{\mathtt{b}}^\lambda$ and which is a d-qc covering map from $\eta_\lambda(C^\lambda)$ to itself. Moreover for $i = \mathtt{w}, \mathtt{b}$ the map $\widetilde{R}^\lambda$ coincides on K_i^λ with the quadratic-like map*

$$\widetilde{P}_i^\lambda \colon \eta_\lambda(\Pi_i^\lambda(U_i^{\Lambda/d})) \longrightarrow \eta_\lambda(\Pi_i^\lambda(U_i^\Lambda)) = K_i^\lambda \cup \eta_\lambda(C^\lambda)$$

which is conformally conjugate from P_i by $\eta_\lambda \circ \Pi_i^\lambda$.

Let $\sim_1$ be the equivalence relation on $\widehat{\mathbb{C}}$ conjugate by $\eta_{\lambda_0}^{-1}$ to $\overset{\lambda_0}{\sim}$ on $\widehat{\mathbb{C}}^{\lambda_0}$, i.e. $z_1 \sim_1 z_2$ if and only if $\eta_{\lambda_0}^{-1}(z_1) \overset{\lambda_0}{\sim} \eta_{\lambda_0}^{-1}(z_2)$. In the following proposition we reverse the time orientation of the Moore-isotopy, in order to unclutter notation

Proposition 5.8. *If the restriction $H \colon]0,1] \times \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ of the holomorphic motion in Theorem 5.3, has a continuous extension $\widetilde{H} \colon [0,1] \times \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$, which is a time reversed Moore isotopy for $\sim_1$. Then both $\widehat{R}_\lambda$ and $\widetilde{R}_\lambda$, $\lambda = \mathrm{e}^\Lambda$ converge to a (the same) rational map R_0 of degree d as $\Lambda \to 0$. And R_0 is a conformal mating of $P_{\mathtt{w}}$ and $P_{\mathtt{b}}$ in the strong sense of Definition 4.16.*

Proof. For each $t \in]0,1]$ define $\widetilde{H}_t\colon [0,1]\times\widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ by $\widetilde{H}_t(s,z) = \widetilde{H}(st, \phi_1 \circ \phi_t^{-1}(z)) = \phi_{st}\circ\phi_t^{-1}$. Then each $\widetilde{H}_t$ is a Moore isotopy for the equivalence relation $\sim_t$ on $\widehat{\mathbb{C}}$ conjugate to $\overset{\lambda}{\sim}$ by η_λ^{-1}, where $\lambda = \mathrm{e}^t$. Let $\Theta_\lambda\colon K_{\mathtt{w}} \amalg^{\lambda} K_{\mathtt{b}} \longrightarrow \overline{\mathbb{C}}$ be the induced homeomorphism as given by Lemma 2.5. Thus for $0 < t < 1/d$ we have the commutative diagrams

$$
\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[d, "\widetilde{H}_t(0,\cdot)"'] & \widehat{\mathbb{C}}^{\lambda} \arrow[l, "\eta_\lambda"'] \arrow[r, "R_\lambda"] \arrow[d, "\Pi^{\lambda}"'] & \widehat{\mathbb{C}}^{\lambda^d} \arrow[r, "\eta_{\lambda^d}"] \arrow[d, "\Pi^{\lambda^d}"] & \widehat{\mathbb{C}} \arrow[d, "\widetilde{H}_{dt}(0,\cdot)"] \\
\widehat{\mathbb{C}} & K_{\mathtt{w}} \amalg^{\lambda} K_{\mathtt{b}} \arrow[l, "\Theta_\lambda"'] \arrow[r, "P_{\mathtt{w}} \amalg^{\lambda} P_{\mathtt{b}}"] \arrow[d, "h_\lambda"'] & K_{\mathtt{w}} \amalg^{\lambda^d} K_{\mathtt{b}} \arrow[r, "\Theta_{\lambda^d}"] \arrow[d, "h_{\lambda^d}"] & \widehat{\mathbb{C}} \\
 & K_{\mathtt{w}} \amalg K_{\mathtt{b}} \arrow[r, "P_{\mathtt{w}} \amalg P_{\mathtt{b}}"] & K_{\mathtt{w}} \amalg K_{\mathtt{b}} &
\end{tikzcd}
\tag{5.3}
$$

The homeomorphisms $h_\lambda \circ \Theta_\lambda^{-1}\colon \widehat{\mathbb{C}} \longrightarrow K_{\mathtt{w}} \amalg K_{\mathtt{b}}$ are independent of λ, because for any $z \in K_{\mathtt{w}} \sqcup K_{\mathtt{b}}$: $\widetilde{H}_t(0, \eta_\lambda \circ \Pi^{\lambda}(z)) = \widetilde{H}(0, \eta_{\lambda_0}\circ\Pi^{\lambda_0}(z))$ is independent of λ. Thus by Proposition 4.3 the map

$$R_0 = \Theta_{\lambda^d} \circ h_{\lambda^d}^{-1} \circ (P_{\mathtt{w}} \amalg P_{\mathtt{b}}) \circ h_\lambda \circ \Theta_\lambda^{-1}$$

is a fixed degree d branched covering. The map $\widetilde{H}$ is uniformly continuous, because it is continuous with compact domain. Thus the projections $\widetilde{H}_t(0,z)$ converge uniformly to the identity. And the holomorphic degree d branched coverings $\widehat{R}_\lambda$ converge uniformly to R_0, which shows convergence and that R_0 is holomorphic. Similarly $\chi_\lambda^{-1}(z) = \widetilde{H}_{d\Lambda}(1/d, z)$ converge uniformly to the identity, so that $\widetilde{R}_\lambda = \chi_\lambda^{-1}\circ\widehat{R}_\lambda$ also converges uniformly to R_0. By construction $\widetilde{H}\circ\eta_{\lambda_0}\circ\Pi_i^{\lambda_0}$, P_i and R_0 satisfies the requirements of the strong mating definition Definition 4.16. □

5.4. **Cheritat movies.** It is easy to see that $\widehat{R}_\lambda$ converges uniformly to the monomial z^d as $\lambda \to \infty$. Cheritat has used this to visualize the path of Milnor intermediate matings $\widehat{R}_\lambda$, $\lambda \in]1,\infty[$ of quadratic polynomials through films. Cheritat starts from λ very large so that $K_{\mathtt{w}}^{\lambda}$ and $K_{\mathtt{b}}^{\lambda}$ are essentially just two down scaled copies of $K_{\mathtt{w}}$ and $K_{\mathtt{b}}$, the first near 0, the second near ∞. From the chosen normalization and the position of the critical values in $K_{\mathtt{w}}^{\lambda} \cup K_{\mathtt{b}}^{\lambda}$ he computes $\widehat{R}_{\sqrt{\lambda}}$. From this $K_{\mathtt{w}}^{\sqrt{\lambda}} \cup K_{\mathtt{b}}^{\sqrt{\lambda}}$ can be computed by pull back of $K_{\mathtt{w}}^{\lambda} \cup K_{\mathtt{b}}^{\lambda}$ under $\widehat{R}_{\sqrt{\lambda}}$. Essentially applying this procedure iteratively one obtains a sequence of rational maps $\widehat{R}_{\lambda_n}$ and sets $K_{\mathtt{w}}^{\lambda_n} \cup K_{\mathtt{b}}^{\lambda_n}$, where $\lambda_n \searrow 1$ and $\lambda_n^2 = \lambda_{n-1}$. For more details see the paper by Cheritat in this volume.

6. Appendix: Branched Coverings

Definition 6.1. A branched covering $F\colon \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ is a map such that: For all $x \in \mathbb{S}^2$ there exists local coordinates $\eta\colon \omega(x) \longrightarrow \mathbb{C}$, $\zeta\colon \omega(F(x)) \longrightarrow \mathbb{C}$ and $d \geq 1$ such that

$$\zeta \circ F \circ \eta^{-1}(z) = z^d.$$

When $d > 1$ above the point x is called a critical point. The set of critical points for F is denoted Ω_F.

The branched covering F is called post-critically finite (PCF), if the post-critical set

$$P_F = \{F^n(x) | x \in \Omega_F, n > 0\}$$

is finite.

6.1. **Thurston Equivalence.**

Definition 6.2 (Thurston Equivalence)**.** Two post-critically finite branched coverings $F_1, F_2 \colon \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ are said to be *Thurston equivalent* if and only if there exists a pair of homeomorphisms $\Phi_1, \Phi_2 \colon \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ isotopic relative to the post critical set of F_1 such that

$$\begin{array}{ccc} \mathbb{S}^2 & \xrightarrow{F_1} & \mathbb{S}^2 \\ {\scriptstyle \Phi_1}\downarrow & & \downarrow{\scriptstyle \Phi_2} \\ \mathbb{S}^2 & \xrightarrow{F_2} & \mathbb{S}^2. \end{array}$$

6.2. **Multicurves.** Let $P \subset \mathbb{S}^2$ be a finite set.

- A simple closed curve $\gamma \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^2 \smallsetminus P$ is called *peripheral* if one of the complementary components $\mathbb{S}^2 \smallsetminus \gamma$ contains at most one point of P.
- A *multi curve* Γ in $\mathbb{S}^2 \smallsetminus P$ is a set or collection of mutually non homotopic, non-peripheral simple closed curves in $\mathbb{S}^2 \smallsetminus P$.
- Note that a multi curve has at most $\#P - 3$ elements.

6.3. **Thurston matrices.**

- Let $F \colon \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ be a PCF branched covering with post critical set P, $\#P > 3$
- A multicurve $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ in $\mathbb{S}^2 \smallsetminus P$ *is F-stable* if for every j and every connected component δ of $F^{-1}(\gamma_j)$, the simple closed curve δ is either homotopic to some γ_i or peripheral in $\mathbb{S}^2 \smallsetminus P$.
- The *Thurston Matrix* of F with respect to the F-stable multicurve Γ is the non negative $n \times n$ matrix $A = A_{i,j}$ given by

$$A_{i,j} = \sum_{\delta} 1/\deg(F : \delta \to \gamma_j)$$

  where the sum is over all connected components δ of $F^{-1}(\gamma_j)$ homotopic to γ_i relative to P, in $\mathbb{S}^2 \smallsetminus P$.

6.4. **Thurston obstructions.**

- Having only non negative entries, the Thurston matrix A has a positive *leading eigenvalue*, i.e. eigenvalue of maximal modulus.
- A *Thurston obstruction* to F is an F-stable multicurve Γ with leading eigenvalue of modulus at least 1.

6.5. **The fundamental theorem for post-critically finite rational maps.**

Theorem 6.3 (Thurston)**.** *Let $F \colon \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ be a post-critically finite branched covering with post-critical set P, and hyperbolic orbifold. Then F is Thurston equivalent to a rational map if and only if F has no Thurston obstruction.*

The Orbifold of F

- The orbifold $\mathcal{O}_F$ associated to F is the topological orbifold $(\mathbb{S}^2, \nu)$ with underlying space $\mathbb{S}^2$ and whose weight $\nu(x)$ at x is the least common multiple of the local degree of F^n over all iterated preimages $F^{-n}(x)$ of x.
- The orbifold $\mathcal{O}_F$ is said to be *hyperbolic* if its Euler characteristic $\chi(\mathcal{O}_F)$ is negative, that is if:

$$\chi(\mathcal{O}_F) := 2 - \sum_{x \in P} (1 - \frac{1}{\nu(x)}) < 0.$$

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