Post Global Financial Crisis Modelling: Credit Risk For Firms That Are Too Big To Fail

Ephraim Clark[2], Sovan Mitra[1] and Octave Jokung[3]

Abstract

Since the start of the Global Financial Crisis the validity of all financial models have come under serious questioning, with firms that are 'too big to fail' being frequently discussed in the media. Such firms are systemically too important to the economy to allow them to fail, as well as posing significant contagion risk. In such firms, the standard credit risk models are not sufficient because systemically important firms do not default under standard circumstances. In fact it was frequently observed during the Global Financial Crisis that such firms were able to continue borrowing when other firms would normally default.

In this paper we propose a new model for credit risk for firms that are too big to fail. We propose a structural model of credit risk but model credit risk as a real option. We derive a closed form solution for the option to default and take into account the borrowing practices of systemically important firms. We develop our model to take into account economic factors using regime switching, and derive an option pricing solution under such a process. Finally we obtain solutions for hedging the option to default, which takes into account the market incompleteness of such options. We conduct numerical experiments to calculate the option to default at different debt values and volatility.

Keywords: credit risk; real options; too big to fail; financial crisis; hedging.

[1] University of Liverpool,

Liverpool,

United Kingdom.

[2] Middlesex University,

London,

United Kingdom.

[3] Université de Valenciennes,

Valenciennes,

France.

1. Introduction

The Merton model of credit risk [22] has been one of the most successful and widely used model for credit risk analysis. It has been the starting point of the development of many credit risk models [5], and has been successfully applied in industry (see for instance the KMV model [21]). The Merton model has useful features, such as the ability to forecast default probabilities, equations relating credit risk to share prices, and accounting for dividends in credit risk analysis.

Since the start of the GFC (global financial crisis) the validity of all financial models have come under serious questioning. In particular, credit risk models have been subject to particular scrutiny, as prior to the commencement of the GFC few models forecasted such frequent and high magnitude credit default events. One particular example was the use of tranches in collateralised debt obligations [24], which encouraged lending that led to high credit risk.

One issue that received particular attention since the start of the GFC is the issue of firms that are 'too big to fail' [7], or systemically important firms. Such firms can have a fundamental impact on the entire economy of a country, hence such firms cannot be allowed to fail (that is to default on its liabilities). The failure of such firms could result in other firms and members of the public having to sustain such high losses that they themselves would cause further problems in the economy. Additionally, with the growth of globalisation there could be potential contagion effects where such risks and losses spread across international markets and sectors. It was comprehensively demonstrated during the GFC that many Governments would not allow such firms to default or become bankrupt. A case in point is Northern Rock [25], where the UK Government seized ownership to prevent bankruptcy.

In cases where firms are too big to fail, Merton's model is no longer sufficient. Merton's model predicts that default occurs if debt increases too much or firm value declines too much. However, this was not consistent with firms that are too big to fail during the GFC. Firstly, the debt can continue to increase without causing defaulting, with firms continuing to borrow regardless of previous borrowing amounts amounts. Secondly, it was observed during the GFC that many firms suffered significant declines in firm value, due to various assets being over-valued and suffering the onset of the GFC. However firms were still able to continue to borrow with declining firm value.

Thirdly, firms that are too big to fail are commonly assisted by Governments through various initiatives, such as ' bail outs', debt rescheduling, loans (or in the case of Northern Rock it involved buying the entire company). At the start of the GFC it was frequently reported in the media that many companies were bailed out by the Government. In fact, firms have now come to expect to be bailed out by Governments, with the term 'Greenspan put' [23] being frequently mentioned.

Essentially, the Greenspan put implies that firms that are too big to fail firms will be bailed out by the Government. This 'put option' provided by the Government is not incorporated in any way in Merton's model, and in fact neutralises Merton's default conditions.

Fourthly, the Merton model does not recognise strategic default. The Merton model assumes the debtholder always repays his debt, even if the debtholder believes it is rationally optimal to default. In strategic defaulting, the debtholder is able to repay the debt but considers it is financially more optimal to default instead [2]. For instance, if the debtholder considers that the value of the debt will always be larger than the value of his assets in the future, the rationally optimal choice would be to default. Although defaulting would mean loss of all the debholder's assets, the debtholder can retain the cash from the debt and overall would be wealthier.

Strategic defaulting was precisely observed during the start of the GFC, where individuals had mortgages that heavily exceeded the value of their properties (and the property values were significantly declining in value). For systemically important firms strategic defaulting is a realistic possibility, since such firms borrow large sums and so can find themselves in the situation that the debt is worth more than the assets. Moreover, Krugman [1] has mentioned that strategic default is a necessary possibility that must be considered since the start of the GFC. Additionally, including strategic default is consistent with optimal capital structure theory [7].

Finally, the Merton model is unrealistic in that it assumes a purely binary outcome at the time of debt repayment: debt is either fully repaid or the firm defaults with the lenders taking full control of the entire company. Whilst this may have been a more appropriate description when Merton's model [22] was first proposed (in 1974) the legislation and functioning of markets have significantly changed since that time. Nowadays, a defaulting firm would not typically lose ownership of the entire company. The lenders would try to assist the firm before the firm defaulted, or help the firm improve its financial performance. In both cases the actions of the lender would improve the likelihood of repaying the debt, hence this would be in the lender's interests. Moreover, the lender is less interested in taking ownership of a defaulting firm than a non-defaulting, since a defaulting firm is worth far less. In fact the legislation on bankruptcy is now more directed to encouraging debt restructuring, that is to reschedule debt to be paid at another point in time.

In this paper we propose a new credit risk model, for systemically important firms or firms that are too big to fail. We continue along the same concept of defaulting in Merton's model by applying option theory, however we model credit risk as a real option. Additionally, we allow firms to continue to borrow regardless of debt value and asset value, unlike in Merton's model. We allow debt to be rolled over, reflecting the possibility of debt restructuring (unlike in Merton's model) and

we also take into account that firms that are too big to fail are likely to be bailed out subsidised in some way (which was demonstrated many times during the GFC).

The paper is organised as follows: in the next section we provide an introduction to Merton's model and a literature review of credit risk modelling. In the next section we introduce our model, derive a closed form solution for the option to default and develop our model under regime-switching. This allows us to model economic factors affecting default and we derive a closed form solution for the option to default under regime switching. In the next section we provide a method of hedging the option to default, using a hedging portfolio to minimise risk and takes into account the market incompleteness of the option to default. We then conduct some numerical experiments to evaluate our option to default, and finally we end with a conclusion.

2. Introduction and Related Literature

The Merton model of credit risk [22] is based on the Black-Scholes option pricing theory [6], which model stock prices $X(t)$ as a stochastic process at time t. Let there exist a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ where Ω denotes the sample space, $\mathcal{F}$ denotes a collection of events in Ω with probability measure $\mathbb{P}$, and we have a filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$. The set $\{\mathcal{F}_t\}$ denote the set of information that is available to the observer up to time t and we have

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}_T, \forall s, t \text{ with } s < t < T.$$

The set $\{\mathcal{F}_t\}, t \in [0, T]$ is also known as a filtration. Furthermore, for a given stochastic process $X(t)$, as more information is revealed to an observer as time t progresses, we introduce the filtration $\mathcal{F}_t^X$ which denotes the information generated by process $X(t)$ on the interval $[0, t]$. Finally, assume we have the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ then we define a change of measure from $\mathbb{P} \sim \mathbb{Q}$ to be the probability space $\{\Omega, \mathcal{F}, \mathbb{Q}\}$.

Black and Scholes [6] determined the closed form solution of European call options $C_{BS}(X(t))$ on the assumption of no arbitrage:

$$C_{BS}(X(0), T, r, \sigma, K) = X(0)\Psi(d_1) - Ke^{-rT}\Psi(d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{X(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

The $X(t)$ is the stock price that follows Geometric Brownian motion

$$X(t) = X(0) + \int_0^t \mu dt + \int_0^t \sigma dW(s),$$

or more conveniently,

$$dX(t)/X(t) = \mu dt + \sigma dW(t),$$

where μ denotes drift, σ denotes volatility and $W(t)$ is a Wiener process. In $C_{BS}(X(t), T, r, \sigma, K)$, T is the expiration date, $\Psi(\cdot)$ is the standard normal cumulative distribution function, r is the riskfree rate of interest and K is the strike price. The price of a European call option is also determined by risk neutral valuation:

$$C_{RN}(X(t), K, T) = e^{-rT} E^{\mathbb{Q}}[X(T) - K]^+,$$

where $\mathbb{Q}$ is the risk neutral probability measure.

The Merton model of credit risk (also known as the structural model) was published in [22] and is based on assuming a option based model for credit risk. The model is heavily inspired by the Black-Scholes option pricing model in [6]. In the Merton model we specify a future point in time $T > 0$, such that the underlying stochastic process follows $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}\}$. We also make the simplistic assumption that a firm is financed by equity and debt $V(t)$ only, where debt has face value L.

The firm's asset value $A(t)$ (which is separate from the firm equity or debt value) is also modelled by a stochastic differential equation, under a risk neutral measure $\mathbb{Q}$, by the following equation:

$$dA(t) = (r - \delta)A(t)dt + \sigma_A A(t)dW, \tag{1}$$

where σ_A is the volatility in asset value $A(t)$ and δ is the firm's dividend rate. As dividends result in payments to the shareholder the value of the firm is reduced by an equivalent amount. We note that $\delta \in \mathbb{R}$, that is δ can have positive and negative values, where negative values imply cash is paid into the company.

In the Merton model we assume the firm must repay the debt amount L at time T, default can only occur at a single point in time T, and that only two outcomes are possible:

$$\begin{aligned} V(T) &= L, \ A(T) \geq L, \\ V(T) &= A(T), \ A(T) < L. \end{aligned}$$

In the first situation the firm value exceeds L (that is $A(T) \geq L$), and we assume the firm has sufficient funds to repay the debt L. Hence the value of the debt $V(T)$

is also L. However, in the second situation the firm value drops below the face value of debt L at time T ($A(T) < L$) and so according to Merton's model we assume we have defaulted on the debt. The lenders then assume ownership of the entire firm, and so the value of the debt is now equal to the value of the firm rather than L, that is $V(T) = A(T)$. We note that threshold values other than L can be chosen to trigger default (for instance using a combination of values from short and long term liabilities [29]) but the main idea is retained in those models that default is triggered when $A(t)$ falls below a threshold value.

In the event that the lenders take ownership of the company at time T, all shares in the company would therefore belong to the lenders. This process is also consistent with standard financial theory, hence the Merton model has strong theoretical appeal. The time of default τ is defined by

$$\tau = T\mathbb{1}_{A(T)<L} + \infty\mathbb{1}_{A(T)\geq L},$$

where $\mathbb{1}_{\{.\}}$ denotes the indicator function. Merton's model of credit risk also implies that we can model the equity value $E(t)$ (that is the market capitalisation) as a European call option:

$$E(T) = max(A(T) - L, 0).$$

Under standard financial theory the residual value remaining in the firm, after the debt L has been paid off, will revert to the shareholders. However a defaulting firm (so that $A(T) < L$) implies the lender owns the entire firm and shareholders own nothing (therefore $E(T) = 0$. Consequently the equity value $E(t)$ equates to a call option on the firm value, and can be valued using the Black-Scholes option pricing equation:

$$E(0) = A(0)\Psi(d_1) - Le^{-rT}\Psi(d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{A(0)}{L}\right) + \left(r - \delta + \frac{\sigma_A^2}{2}\right)T}{\sigma_A\sqrt{T}},$$
$$d_2 = d_1 - \sigma_A\sqrt{T}.$$

Moreover, the debt $V(t)$ can be modelled as a put option

$$V(T) = L - max(L - A(T), 0).$$

Since the introduction of the Merton model [22] a number of credit risk models have been developed to improve credit risk analysis. For an extensive literature

review the reader is referred to [5], [4], [11] and [9]. In [17] Ramaswamy and Sundaresan develop a credit risk model to value debt in the form of corporate bonds $G(t)$, they also take into account stochastic interest rates (rather than constant interest rates as in Merton's model). The interest rates in [17] follow

$$dr = \kappa(\mu - r)dt + \sigma_2\sqrt{r}dW(t),$$

where μ, κ and σ_2 are constants. An additional advantage of the model in [17] is that it takes into account the term structure of interest rates and the uncertainty in interest rate values over time. This provides a more realistic model of credit risk since the cost of borrowing is heavily influenced by credit risk. The Ramaswamy and Sundaresan model derives a partial differential equation that originates from Brennan and Schwarz [8]

$$\begin{aligned}\frac{\partial A}{\partial \tau} &= \frac{\sigma_1^2 A^2}{2}\frac{\partial^2 G}{\partial A^2} + \Gamma\sigma_1\sigma_2\sqrt{r}A\frac{\partial^2 G}{\partial A \partial r} + \frac{r\sigma_2^2}{2}\frac{\partial^2 G}{\partial r^2} + \kappa(\mu - r)\frac{\partial A}{\partial r} \\ &+ (r-\gamma)A\frac{\partial G}{\partial A} - rA + \upsilon,\end{aligned}$$

where σ_1 is a volatility, Γ is the correlation of Wiener processes, γ relates to a parameter of the firm value process, and υ is the coupon rate. The model produces credit yield spreads that are more consistent with empirically observed data compared to the Merton model [22].

One credit risk model that takes into account strategic default is the work by Leland, specifically in [20],[19]. In [19] Leland considers debt under the proposition that strategic default is exercised by economic agents and that debt is modelled as an optimal capital structure problem (this is consistent with Miller-Modigliani Capital Structure Theory [7]). However, Leland assumes that the debt has infinite maturity, consequently default can never occur. To address this highly restrictive assumption Leland in [20] modifies his model to include finite maturity debt, but also debt that can be rolled over at each maturity period. The debt is financed by bonds and the value of all the bonds $D(T)$ is given by

$$\begin{aligned}D(T) &= \frac{\lambda}{r} + \left(\chi - \frac{\lambda}{r}\right)\left(\frac{1-e^{-rT}}{rT} - \frac{1}{T}\int_{t=0}^{T} e^{-rt}\psi(t)dt\right) \\ &+ \left((1-\iota)A^* - \frac{\lambda}{r}\right)\frac{1}{T}\int_0^T e^{-rt}\psi'(t)dt,\end{aligned}$$

where λ is the coupon payment per year, χ total principal value of all bonds, $\psi(t)$ is the cumulative distribution function of the passage time of bankruptcy, $\psi'(t)$ is the associated probability density function, A^* is the asset value that triggers default if $A(t) = A^*$. If default occurs then bondholders receive $(1-\iota)A^*$, hence ι specifies the fraction of the asset value distributed to the bondholders in default. However,

the model [20] still assumes (just as in Merton's model [22]) that a fall in value below a specific threshold will trigger default. As mentioned previously, firms close to default may typically restructure debt, enter negotiations or consider a range of possibilities rather than just default. Therefore this model does not enable one to take into account such different scenarios.

Other credit risk models include Zhou [30], who develops a credit risk model that incorporates jumps in the underlying asset:

$$A(t)dA = (\mu - \lambda\nu)dt + \sigma dW + (\pi - 1)dY,$$

where μ is the expected return on the firm's assets, ν, λ, σ are constants, dY is a Poisson process with intensity λ, and π represents the jump process. One group of influential credit risk models are the intensity based models, where the time of default τ is a random variable whereby

$$\begin{aligned} \mathbb{Q}(\tau < \infty) &= 1, \\ \mathbb{Q}(\tau = 0) &= 0, \end{aligned}$$

under some risk neutral measure $\mathbb{Q}$. As an example one could adopt

$$\mathbb{Q}(\tau > t) = exp\left(-\int_0^t \gamma(s)ds\right),$$

where $\gamma(s)$ can be a deterministic or stochastic process, and represents the intensity for a Poisson process. The time of the first jump in the Poisson process corresponds to τ.

3. Option Model For Default

In this section we introduce our (real) option model for default, derive its closed form solution and discuss its properties.

3.1. Debt Process and Default Model

Let $V(t)$ denote the debt of the firm at time t, where $V(t)$ follows a Geometric Brownian motion process, that is

$$dV(t) = \eta V(t)dt + \sigma V(t)dW, \tag{2}$$

where η is the drift and σ is the volatility of the debt value and both are constants, $W(t)$ is a Wiener process defined on a filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$. The modelling of debt growth as a stochastic differential equation is consistent with many models in credit risk (see for instance the literature review in the previous section for examples). We expect dV to exhibit debt growth arising from random

processes, since debt can fluctuate significantly based on market conditions that follow random processes. Our model incorporates this by the Wiener process $W(t)$, and σ determines the magnitude of the fluctuations.

In our model we have not limited debt $V(t)$ to any particular value other than $V(t) \geq 0, \forall t$. This differs from the Merton model which must always have $V(T) \leq A(T), \forall T$, otherwise default occurs and debt $V(t)$ would automatically stop growing. Our model therefore reflects the characteristic of too big to fail firms borrowing without having to incur default, purely on the value of assets alone. As was observed in the GFC, many firms with extremely low asset values were still able to borrow because they were too big to fail. Additionally, our model reflects the characteristic that firms frequently engage in debt negotiation, restructuring etc. nowadays, hence debt can continue to grow rather than entering default.

If we take expectations of the debt growth in our model then we have

$$\begin{aligned} \mathbb{E}[dV] &= \mathbb{E}[\eta V dt + \sigma V dW], \\ &= \eta V dt, \end{aligned}$$

and we also have expected debt value

$$\begin{aligned} \mathbb{E}[V(t)] &= \mathbb{E}\left[V(0)exp\left(\left(\eta - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right)\right], \\ &= V(0)exp(\nu t), \text{ where } \nu = \left(\eta - \frac{\sigma^2}{2}\right). \end{aligned}$$

Our model incorporates typical characteristics of debt value. Firstly, we expect change in debt value dV to increase according to ηV because larger debt values $(V(t))$ typically lead to larger changes in debt value. Moreover, debt is frequently taken to fund some long term activity (e.g. firm growth, expansion overseas, acquisition of another company etc.) hence we expect some long term factor to affect debt growth: this is incorporated by η. Secondly, the expected debt value $\mathbb{E}[V(t)]$ grows at a continuously compounded rate of ν. This property reflects the fact that we expect debt to generally grow over time and its effect is compounded, which typically happens in lending.

We now examine the decision to default in our model and we follow Merton's concept of credit risk by modelling default as an option. In our model let C denote a real call option to default and we assume $C = f(V(t))$ as default is fundamentally a function of debt $V(t)$. However, unlike Merton, we do not assume default occurs at some single point in time T (that is we do not model it as a European option). Instead we allow default to occur at any point in time $t \geq 0$ since default is realistically never restricted to a single point in time. This was particularly evident during the start of the GFC when many firms began defaulting at a range of times. The

ability to exercise at any time t is a key characteristic of real option theory, and using real option theory [10] we can model the the option to default to occur at any time. Additionally, we do not impose an expiration date T, since debt can always be rolled over for a firm that is too big to fail. Hence default time t can happen at any time.

We now introduce the option for default. The option to default is given by

$$C(V(t)) = \max_{V(t)\in\mathbb{R}^+} e^{-\eta t}\mathbb{E}[V(t) - K],$$

with boundary condition

$$V(t) = 0 \Rightarrow C(0) = 0, \ \forall t.$$

The time of default is given by time t and is unknown, that is when the option is exercised. The constant K is the option strike and represents the costs that are incurred when the defaulting action is undertaken (that is the option is exercised). If a firm defaults, it will incur penalties and costs (ranging from administrative costs to possible fines levied against it from financial institutions). This is typically a feature that is not incorporated within many structural credit risk models, but poses a significant and rational disincentive to default. In the event of default (when the option is exercised) the debtholder will keep the debt $V(t)$ minus the defaulting costs K, hence a larger K implies defaulting provides less incentive for defaulting.

We note that the strike K represents a 'sunk' cost in real options theory. This means that K is an irreversible cost, that is once this cost is expended then it can no longer be recovered. Secondly, the timing of the sunk cost can be delayed, that is we are not forced to spend K until $V(t) > K$. The flexibility or optionality to decide whether to exercise the option is extremely valuable, because we can avert using the option when $V(t) < K$ and so avoid negative payoffs. We also note that the boundary condition arises from standard option pricing theory. If the underlying asset is worthless ($V(t) = 0$) then the option will also be worthless for all t ($C(V(t))$).

In our model we assume $r > \eta$, where r is the riskless rate. This is because we have $r = \eta + \xi$, where ξ represents any subsidy or bailout funding that too big to fail firms receive. The rationale for $r > \eta$ is explained as follows. For firms that are categorised as too big to fail, such firms are typically able to borrow at very close to riskless rates, since firm size generally reduces the cost of borrowing. Hence smaller size businesses typically pay higher borrowing costs. Consequently, the debt $V(t)$ does not grow at a very high levels because it can borrow at extremely low rates.

Secondly, too big to fail firms do not simply borrow at very low rates but also they receive 'bailouts', funding and other subsidies to prevent default. Such assistance it typically provided by the Government and the assistance is essentially a reduction

in their debt. Consequently, for too big to fail firms we must model this assistance in the credit risk model, as it is an essential part of credit risk modelling, which was most apparent during the GFC. Hence we have

$$\eta dt = (r - \xi)dt.$$

Additionally, the term ξdt reflects the property that firms can gain more Government assistance over time. The firms do not receive all their Government assistance at once but over a period of time, hence it should increase over time. This also implies that firms should value delaying their option to benefit from this Government assistance. If the firm defaults then the firm can no longer receive subsidies to prevent default. Hence it is important to capture this time value of delaying in our credit risk model. In terms of Merton's model, ξ is equivalent δ (dividend payments paid out or received by the firm).

3.2. Partial Differential Equation For The Option To Default

Using our model of C(V(t)) and V(t) we now derive the partial differential equation that C(V(t)) must obey for option pricing. We now state this in the following theorem:

Theorem. *For a firm with debt V(t) where*

$$dV = \eta V dt + \sigma V dW(t),$$

the option to default $C = f(V(t))$ is given by the partial differential equation

$$0 = \eta V(t)\frac{\partial C}{\partial V} + \frac{1}{2}\sigma^2 V^2(t)\frac{\partial^2 C}{\partial V^2} - rC.$$

Proof. Firstly, we can derive dC by applying Ito's Lemma [16] since $C = f(V(t))$, a function of $V(t)$. We therefore have

$$dC = \sigma V(t)\frac{\partial C}{\partial V}dW + \left(\eta V(t)\frac{\partial C}{\partial V} + \frac{1}{2}\sigma^2 V^2(t)\frac{\partial^2 C}{\partial V^2}\right)dt. \tag{3}$$

Hence the change in option value has a deterministic component as well as a stochastic component. We now recall that the Government is the main entity (and in many cases the only entity) that assists firms that are too big to fail. Since the Government can invest at the riskless rate r, we can assume the Government can set aside a reserve of cash to hedge the option to default $C(V(t))$, and this will grow at a rate $rC(V(t))dt$. We would expect the Government to hedge the expected value of $dC(t)$. If we apply a dynamic hedging argument, therefore we have

$$rC(V(t))dt = \mathbb{E}[dC] = \eta V(t)\frac{\partial C}{\partial V}dt + \frac{1}{2}\sigma^2 V^2(t)\frac{\partial^2 C}{\partial V^2}dt.$$

If we now rearrange and divide throught by dt, we have

$$\begin{aligned} 0 &= \eta V(t)\frac{\partial C}{\partial V}dt + \frac{1}{2}\sigma^2 V^2(t)\frac{\partial^2 C}{\partial V^2}dt - rC(V(t))dt, \\ &= \eta V(t)\frac{\partial C}{\partial V} + \frac{1}{2}\sigma^2 V^2(t)\frac{\partial^2 C}{\partial V^2} - rC. \blacksquare \end{aligned}$$

Our option $C((V(t))$ must therefore satisfy this partial differential equation. This equation is a second order, linear, homogeneous partial differential equation. We note in passing that such partial differential equations frequently appear in many applications, for example economics, science and engineering to name a few examples.

3.3. Closed Form Solution For The Option To Default

In order to derive a closed form solution for $C(V(t))$, the option price for defaulting, we use the partial differential equation and additional boundary conditions to obtain a solution. We now state this in our Theorem.

Theorem. *The option to default $C(V(t))$ is subject to the following boundary conditions*

$$\begin{aligned} C(V(t)) &= 0, \forall t, \text{ if } V(t)=0, \\ C(V^\theta) &= V^\theta - K, \\ \frac{\partial C}{\partial V}\Big|_{V=V^\theta} &= 1, \end{aligned}$$

where K is the strike price, $V = V^\theta$ denotes the optimal value to exercise the option $C(V(t))$. The option $C(V(t))$ has closed form solution

$$C(V(t)) = \beta V(t)^\alpha,$$

where

$$\beta = \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha K^{\alpha-1}}.$$

and

$$\alpha = \frac{-\left(\eta - \frac{\sigma^2}{2}\right) + \sqrt{\left(\eta - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}}{\sigma^2},$$

where η and σ are defined in

$$dV(t) = \eta V(t)dt + \sigma V(t)dW(t).$$

Proof. To solve the partial differential equation we observe the partial differential equation is a second order, linear partial differential equation. The partial differential

equation is also homogeneous because the equation is equal to 0. Therefore the partial differential equation has a solution of the form

$$C(V(t)) = \beta V(t)^{\alpha} + \hat{\beta} V(t)^{\hat{\alpha}}, \tag{4}$$

where α, $\hat{\alpha}$, β, $\hat{\beta}$ are constants. We also have the equation

$$\zeta^2 \frac{\sigma^2}{2} + \zeta \left(\eta - \frac{\sigma^2}{2} \right) - r \quad = \quad 0,$$

where $\zeta = \{\alpha, \hat{\alpha}\}$, and $\zeta \in \mathbb{R}$. Therefore we have a quadratic in ζ, hence to determine $\alpha, \hat{\alpha}$, we solve

$$a\zeta^2 + b\zeta + c = 0,$$

where $a = \dfrac{\sigma^2}{2}$, $b = \left(\eta - \dfrac{\sigma^2}{2} \right)$, $c = -r$. Therefore we have

$$\zeta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

because $b^2 - 4ac$ is non-negative. The solutions are therefore

$$\begin{aligned} \alpha &= \frac{-\left(\eta - \frac{\sigma^2}{2} \right) + \sqrt{\left(\eta - \frac{\sigma^2}{2} \right)^2 + 2\sigma^2 r}}{\sigma^2}, \\ \hat{\alpha} &= \frac{-\left(\eta - \frac{\sigma^2}{2} \right) - \sqrt{\left(\eta - \frac{\sigma^2}{2} \right)^2 + 2\sigma^2 r}}{\sigma^2}. \end{aligned}$$

Let us take into consideration the boundary conditions to determine the constants β, $\hat{\beta}$. We have already stated that $C(0) = 0$, $\forall t$, and we observe $\hat{\alpha} < 0$ since $\eta, \sigma, r \in \mathbb{R}^+$, therefore $\hat{\beta} = 0$. Consequently, our solution is now reduced to

$$C(V(t)) = \beta V(t)^{\alpha}.$$

Hence we do not need to determine $\hat{\alpha}$. To determine β we require the second and third boundary conditions. To obtain them, firstly we deduce that there exists an optimal value in the underlying asset to exercise the option. This essentially corresponds to an optimal stopping problem in option theory [27]. For our model, let the optimal exercise value of $V(t)$ be denoted by V^{θ}, that is at V^{θ} the value of the option C will be at its optimal.

At $V(t) < V^{\theta}$ the option's value (in other words the value of the right to default) is worth more than the net value of default (and this value would be equal to $V^{\theta} - K$). As holding the option (or equivalently holding the right to default) is more valuable than the net value to default, it will be in the firm's interest *not* to default *as well*

as continue to borrow. This property also reflects the strategic defaulting behaviour of firms, which is not incorporated within Merton's model. The option value at V^θ would be given by

$$C(V^\theta) = V^\theta - K. \tag{5}$$

Since we have already deduced that $C(V(t)) = \beta V(t)^\alpha$, therefore our second boundary condition is

$$\beta(V^\theta)^\alpha = V^\theta - K. \tag{6}$$

In option theory this condition is also referred to as the value-matching condition [10].

The third boundary condition is obtained from option theory and the smooth-pasting condition [10] (and also known as the high-contact condition [18]). At $V = V^\theta$ we have the partial derivative:

$$\frac{\partial C}{\partial V}\Big|_{V=V^\theta} = 1. \tag{7}$$

Essentially, the smooth pasting condition means that at optimal exercise (V^θ), the value of the option C should be equal to the intrinsic value, that is $C = V(t) - K$, or equivalently $C = V^\theta - K$. So for $V(t) = V^\theta$ we therefore have

$$C = V(t) - K \Rightarrow \frac{\partial C}{\partial V}\Big|_{V=V^\theta} = 1.$$

We determine β in the second boundary condition as follows:

$$\beta(V^\theta)^\alpha = V^\theta - K \Rightarrow \beta = \frac{V^\theta - K}{(V^\theta)^\alpha}.$$

We already know that $C(V(t)) = \beta V(t)^\alpha$, the third boundary condition gives

$$\begin{aligned}
\frac{\partial(\beta V(t)^\alpha)}{\partial V}\Big|_{V=V^\theta} &= 1, \\
(\alpha\beta V(t)^{\alpha-1})\Big|_{V=V^\theta} &= 1, \\
V^\theta &= \alpha\beta(V^\theta)^\alpha, \\
&= \alpha(V^\theta - K).
\end{aligned}$$

We have

$$V^\theta(1-\alpha) = -K\alpha \Rightarrow V^\theta = \frac{K\alpha}{\alpha - 1}.$$

Now substituting our expression of $V^\theta = \dfrac{K\alpha}{\alpha - 1}$ into our expression for β gives

$$\begin{aligned}
\beta &= \frac{\dfrac{K\alpha}{\alpha-1} - K}{\left(\dfrac{K\alpha}{\alpha-1}\right)^{\alpha}}, \\
&= \left(\frac{K\alpha}{\alpha-1}\right)^{-\alpha+1} - \left(\frac{\alpha K}{\alpha-1}\right)^{-\alpha+1}\left(\frac{\alpha-1}{\alpha}\right), \\
&= \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}K^{\alpha-1}}. \blacksquare
\end{aligned}$$

We can deduce that since $V(t) \geq 0$, $\forall t$ (by properties of Geometric Brownian motion), $C(V(t)) \geq 0$, $\forall t$ (since we have the right but not the obligation to exercise the option), therefore $C(V(t)) = \beta V(t)^\alpha \Rightarrow \beta \geq 0$. Furthermore, given that $K \in \mathbb{R}^+$ we can deduce

$$\beta = \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}K^{\alpha-1}} \Rightarrow \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}K^{\alpha-1}} \geq 0 \Rightarrow \alpha \geq 1.$$

Our model now explains why firms do not default. The option to default increases with $V(t)$, so increasing debt actually increases the option value to default and deters firms from defaulting. Additionally, β is a decreasing function of K, therefore decreasing K will increase the option value to default, or the cost of defaulting directly impacts the option value to default. On the other hand if K is increased then the option value to default is worth less and so discourage firms to hold on to this option, that is to exercise default. In either case, if the option value is worth more than defaulting itself, it is therefore better for the firm to continue holding the option to default rather than exercising the option and so defaulting. Thus borrowing, debt restructuring, and evading default are all the result of rational decision making.

Our model offers an interesting implication for Government aid and subsidies, which are incorporated by ξ. If $\xi = 0$ then the Government never assists firms that are too big to fail and $r = \eta$, and so our option is equivalent to a perpetual call with zero dividends. For such an option, it is known that it is equivalent to holding an American call option on a non-dividend stock, with a large expiration date T [18]. Consequently, it is well known that such an option should never be exercised. Thus eliminating Government aid has the impact of discouraging default. Hence the 'Greenspan put' may have the impact of encouraging default in the long term, rather than discouraging them.

4. Option To Default Under Economic Cycles

The Geometric Brownian motion process has been widely used to model a range of economic and financial phenomena. However, it has been researched in many academic articles that such a process does not address more fundamental and longer term dynamics of economic cycles, for example a 'credit crunch'. This is particularly important to address for firms that are too big to fail, since different economic cycles can lead to significantly different credit risk. Moreover, it has been empirically demonstrated that volatility is related to fundamental economic factors and σ plays a critical role in our option to default $C(V(t) = \beta V(t)^{\alpha}$. Examples of such studies include [3] where it is claimed that volatility changes are caused by economic reforms, Schwert [26] empirically demonstrates that volatility increases during financial crises.

A common class of models that have been used to model such fundamental economic changes is the class of regime switching models. Such models are essentially Markov chains and do not violate Fama's 'Efficient Markets Hypothesis' [13]. Hamilton [14] pioneered the usage of regime switching models for economic applications; other examples of regime switching models include [28] and [12]. In fact Schwert [26] suggests that volatility changes during the Great Depression can be explained by regime switching models.

A regime switching model consists of Θ states where $\Theta \in \mathbb{N}^{+}$ is a countable state space; j denotes some state, $j \in \Theta$ and applies in some time interval $[t, t+s]$. In theory Θ can be any value, however $\Theta = 2$ is sufficient for many economic applications (see for example [15]), asset price $X(t)$ where $\{X(t) \in \Theta\}$ has probability

$$P(X(t+s) = j | X(u), u \in [0, s]) = P(X(t+s) = j | X(s)), \ \forall j \in \Theta, \ \forall s, t.$$

Additionally, a regime switching model has a transition probability p_{ij} where the state i at time s moves to state j at time $s+t$ is

$$p_{ij} = P(X(s+t) = j | X(s) = i), \ \forall i, j \in \Theta, \ \forall s, t.$$

To incorporate economic factors in our model, we introduce a regime switching volatility $\boldsymbol{\sigma}$, as has been done in [15]. We therefore have

$$\boldsymbol{\sigma} = [\sigma_1 \ \sigma_2], \ \text{where } \sigma_i \in \mathbb{R}^{+}, \ \forall i \in \Theta.$$

Consequently we have the following debt process

$$dV(t) = \eta V(t) dt + \sigma_i(t) V(t) dW, \ \forall i \in \Theta.$$

We note that the volatility is now a function of time because i changes with time, hence we do not need to include t in $\sigma_i(t)$ if we include i but we include it for clarity.

We can also derive the cumulative distribution function $F_i(.)$ for each state i

$$F_i(V(t) \leq x) = \Psi\left(\frac{ln(x) - \eta}{\sigma_i(t)}\right), \forall i \in \Theta,$$

and probability density function $f_i(.)$

$$f_i(V(t) \leq x) = \phi\left(\frac{ln(x) - \eta}{\sigma_i(t)}\right), \forall i \in \Theta,$$

where $\phi(.)$ is the standard normal probability density function.

To solve the option pricing equation for each state, it is convenient to introduce the following operator

$$\mathcal{L}_i = \eta V(t)\frac{\partial}{\partial V} + \frac{1}{2}\sigma_i^2(t)V^2(t)\frac{\partial^2}{\partial V^2} - r, \forall i \in \Theta.$$

Therefore for each state i we would have the same partial differential equation, that is

$$\mathcal{L}_i(C(V(t))) = 0, \forall i \in \Theta.$$

As we would have the same partial differential equation, we can therefore apply the same solution method to second order, linear, homogeneous partial differential equations. We would therefore obtain the option price for each state i

$$C(V(t)) = \beta_i V(t)^{\alpha_i},$$

where

$$\alpha_i = \frac{-\left(\eta - \frac{\sigma_i^2(t)}{2}\right) + \sqrt{\left(\eta - \frac{\sigma_i^2(t)}{2}\right)^2 + 2\sigma_i^2(t)r}}{\sigma_i^2(t)},$$

and

$$\beta_i = \frac{(\alpha_i - 1)^{\alpha_i - 1}}{\alpha_i^{\alpha_i} K^{\alpha_i - 1}}.$$

.

We have derived the option to default, taking into account economic cycles and conditions. To apply these option pricing equations, it would be conditional on knowing the state i. Although this may be possible to determine, since economic state exists for some time period before changing state, we would also want to know the option price to default unconditional on the state i. We therefore derive the unconditional option to default and state this in our following Theorem.

Theorem. *The option to default on the debt $V(t)$, where*

$$dV(t) = \eta V(t)dt + \sigma_i(t)V(t)dW, \ \forall i \in \Theta,$$

has the closed form solution

$$C(V(t)) = \overline{\beta} V(t)^{\overline{\alpha}},$$

where

$$\begin{aligned}
\overline{\alpha} &= \frac{-\left(\eta - \frac{\overline{\sigma}^2}{2}\right) + \sqrt{\left(\eta - \frac{\overline{\sigma}^2}{2}\right)^2 + 2\overline{\sigma}^2 r}}{\overline{\sigma}^2}, \\
\overline{\beta} &= \frac{(\overline{\alpha} - 1)^{\overline{\alpha}-1}}{\overline{\alpha}^{\overline{\alpha}} K^{\overline{\alpha}-1}}, \\
\overline{\sigma} &= \sqrt{\pi_1 \sigma_1^2(t) + \pi_2 \sigma_2^2(t)}, \\
\pi_j &= \lim_{t \to \infty} P(\sigma_i(t) = j), \ \forall i, j \in \Theta.
\end{aligned}$$

Proof. To find a closed form solution for the option price we first recognise that the option has no time limit for exercising, hence we can take the limit: $\lim t \to \infty$. In fact perpetual options are valued on the basis that we use American option equations with expiry $\lim t \to \infty$. For a Markov chain there exists a limiting distribution for each state, that is we have $\boldsymbol{\pi} = [\pi_1 \ \pi_2]$ where

$$\pi_j = \lim_{t \to \infty} P(X(t) = j), \ \forall j \in \Theta,$$

subject to

$$\sum_{j=1}^{j=\Theta} \pi_j = 1.$$

For a 2 state Markov chain it can be shown that

$$\boldsymbol{\pi} = [\pi_1 \ \pi_2] = \left[\left(\frac{p_{21}}{p_{21} + p_{12}} \right) \left(\frac{p_{12}}{p_{21} + p_{12}} \right) \right].$$

In the limit $t \to \infty$ we therefore have a mixture distribution with weights $\boldsymbol{\pi} = [\pi_1 \ \pi_2]$. The cumulative distribution function $\overline{F}(.)$ for the overall process is

$$\overline{F}(V(t) \leq x) = \sum_{i=1}^{i=2} \Psi \left(\frac{ln(x) - \eta}{\sigma_i} \right) \pi_i,$$

and the probability density function $\overline{f}(.)$ is

$$\overline{f}(V(t) \leq x) = \sum_{i=1}^{i=2} \phi \left(\frac{ln(x) - \eta}{\sigma_i} \right) \pi_i.$$

The mixture distribution is therefore equivalent to

$$\overline{F}(V(t) \leq x) = \Psi\left(\frac{ln(x) - \eta}{\overline{\sigma}}\right), \text{ where } \overline{\sigma} = \sqrt{\pi_1\sigma_1^2 + \pi_2\sigma_2^2},$$

As $t \to \infty$ the Geometric Brownian motion associated with this cumulative distribution function is therefore

$$dV(t) \quad = \quad \eta V(t)dt + \overline{\sigma}V(t)dW.$$

Using our equation we can therefore obtain the same partial differential equation, that is

$$\mathcal{L}(C(V(t))) = 0,$$

where $V(t) = f(\overline{\sigma})$. Consequently we will have a second order, linear and homogeneous partial differential equation and so can apply the same solution method. We therefore obtain the option price $C(V(t)) = \overline{\beta}V(t)^{\overline{\alpha}}$, where

$$\overline{\alpha} = \frac{-\left(\eta - \frac{\overline{\sigma}^2}{2}\right) + \sqrt{\left(\eta - \frac{\overline{\sigma}^2}{2}\right)^2 + 2\overline{\sigma}^2 r}}{\overline{\sigma}^2},$$

and

$$\overline{\beta} = \frac{(\overline{\alpha} - 1)^{\overline{\alpha}-1}}{\overline{\alpha}^{\overline{\alpha}}K^{\overline{\alpha}-1}}. \blacksquare$$

Essentially the impact of economic cycles is that the volatility is proportionately impacted, which in turn affects the parameters α, β of the option pricing equation. Hence economic cycles can significantly impact the decision to default, or to continue borrowing more debt $V(t)$.

5. Hedging The Option To Default

It is typically necessary to hedge any derivative position, such as the option to default $C(V(t))$, so that in the event of a default the hedging entity has sufficient funds to cover the default. This is particularly important for Governments, who are typically responsible for assisting with defaults in firms that are too big to fail. Additionally, firms need to manage their own credit risk and so need to ensure they can hedge out their risk of defaults.

As the option to default for too big to fail firms would be a highly incomplete market, we cannot perfectly hedge $C(V(t))$ with a perfectly replicating portfolio. Consequently, we require an alternative method of hedging. It is highly likely that we can find $Z(t)$, a correlated asset with $C(V(t))$, which can be used for hedging purposes. A correlated asset is highly likely to exist because it is rare to find any

financial process that is completely independent from any other financial process. From a fundamental perspective, such correlations frequently exist because many financial processes and firms are all interrelated by economic factors, for example GDP growth rate, interest rates, exchanges rates etc..

To hedge in the incomplete markets means that there always exists some intrinsic risk in the hedging process that cannot be fully eliminated [5]. However, we can aim to minimise the risk by specifying some metric to minimise the risk. We apply a hedging that minimises the variance, as variance is one of the most common measures of risk. We now state our hedging method.

Theorem. *Let $C(V(t))$ denote the option to default. Let there exist a hedging portfolio $H \in (\Delta, \rho)$, where H consists of Δ number of shares in asset Z(t) and ρ number of riskless bonds $B(t)$, $Cov(Z(t), C(V(t))) \neq 0$ and $\mathbb{E}[Z(t)]$ is known. The hedging portfolio $H^* \in (\Delta_M, \rho)$ that minimises risk with respect to variance is given by:*
$\Delta_M \in \mathbb{R}$ is such that it solves

$$\frac{\partial Var(P(t))}{\partial \Delta} = 0,$$

and

$$\rho \quad = \quad \frac{\Delta \mathbb{E}[Z(t)]}{B(t)},$$

subject to

$$\mathbb{E}[P(t)] = \mathbb{E}[C(V(t))],$$

and

$$P(t) = \Delta Z(t) + C(V(t)) - \rho B(t).$$

Proof. Let us denote the correlated asset as $Z(t)$ whose expectation $\mathbb{E}[Z(t)]$ is known. This is not an unrealistic assumption given that Geometric Brownian motion processes are frequently used to model random processes and their expectations are known. To complete our hedging portfolio we require a riskless bond with value $B(t)$:

$$dB(t) = rB(t)dt.$$

We form part of our portfolio for hedging option $C(V(t))$:

$$\Delta Z(t) - \rho B(t),$$

where Δ, ρ are the quantity of $Z(t)$ and the number of bonds of $B(t)$, respectively.

We now have a hedged portfolio $P(t)$ where

$$P(t) = C(V(t)) + \Delta Z(t) - \rho B(t). \tag{8}$$

To minimise the risk of $P(t)$ we minimise $Var(P(t))$, however this could also affect the expected value of $C(V(t))$ and so disadvantage the benefit of hedging. Therefore to eliminate this possibility we impose the constraint:

$$\mathbb{E}[P(t)] = \mathbb{E}[C(V(t))].$$

Hence our hedging portfolio should provide the same expectation as $C(V(t))$. We can deduce that

$$\begin{aligned}
\mathbb{E}[P(t)] &= \mathbb{E}[C(V(t)) + \Delta Z(t) - \rho B(t)], \\
&= \mathbb{E}[C(V(t))] + \mathbb{E}[\Delta Z(t) - \rho B(t)], \\
0 &= \mathbb{E}[\Delta Z(t) - \rho B(t)], \\
\rho &= \frac{\Delta \mathbb{E}[Z(t)]}{B(t)}.
\end{aligned}$$

As we have already stated that $\mathbb{E}[Z(t)]$ is known, we can therefore calculate ρ if Δ is known (to be given later).

The variance of $P(t)$ is expressed as

$$\begin{aligned}
Var(P(t)) &= Var(C(V(t)) + \Delta Z(t) - \rho B(t)), \\
Var(P(t)) &= Var(C(V(t)) + \Delta(Z(t))),
\end{aligned}$$

since $Var(B(t)) = 0$, $Cov(B(t), Z(t)) = Cov(B(t), C(V(t))) = 0$ for a riskless bond (it has zero variance and correlation). Now for two random variables X, Y

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y), \forall a, b \in \mathbb{R}.$$

where a, b are constants, we therefore have in our hedging portfolio

$$Var(P(t)) = Var(C((V(t))) + \Delta^2 Var(Z(t)) + 2\Delta Cov(C(V(t)), Z(t)).$$

To minimise $Var(P(t))$ we differentiate $Var(P(t))$ with respect to $\Delta \in \mathbb{R}$ and maxmise the resultant equation. We note that we are not restricted in our choice of Δ. We therefore have the first order condition

$$\begin{aligned}
\frac{\partial Var(P(t))}{\partial \Delta} &= 2\Delta Var(Z(t)) + 2Cov(C(V(t)), Z(t)) = 0, \\
\implies \Delta_M &= -\frac{Cov(C(V(t)), Z(t))}{Var(Z(t))}.
\end{aligned}$$

We also have the second order condition

$$\frac{\partial^2 Var(P(t))}{\partial \Delta^2} = 2Var(Z(t)) \Rightarrow \frac{\partial^2 Var(P(t))}{\partial \Delta^2} > 0.$$

The second order condition is always attained because $Z(t)$ is not a riskless asset, therefore $Var(Z(t)) \geq 0, \forall t$. ■

To determine ρ we have

$$\begin{aligned} \rho &= \frac{\Delta_M \mathbb{E}[Z(t)]}{B(t)}, \\ &= \left(\frac{Cov(C(V(t)), Z(t))}{Var(Z(t))}\right)\left(\frac{\mathbb{E}[Z(t)]}{B(t)}\right). \end{aligned}$$

If we now substitute Δ_M into the equation for $Var(P(t))$ we can also determine the reduction in risk from hedging:

$$\begin{aligned} Var(P_M(t)) &= Var(C(V(t))) + \Delta_M^2 Var(Z(t)) + 2\Delta_M Cov(C(V(t)), Z(t)), \\ &= Var(C(V(t))) + \frac{Cov^2(C(V(t)), Z(t))}{Var(Z(t))} - 2\left(\frac{Cov^2(C(V(t)), Z(t))}{Var(Z(t))}\right), \\ &= Var(C(V(t))) - \left(\frac{Cov^2(C(V(t)), Z(t))}{Var(Z(t))}\right). \end{aligned}$$

Hence our hedging portfolio can reduce the variance by optimal choice of the correlated asset. The risk of our hedging portfolio can be improved further by increasing $|Cov(C(V(t)), Z(t))|$ and decreasing the variance of $Z(t)$.

6. Numerical Experiment

In this section we calculate the option to default $C(V(0))$ for different values of σ and $V(0)$. The debt $V(0)$ was varied from 70-140, with an option strike of $K = 90$, so that we calculated the value of in-the-money, out-of-the money and at-the-money options. The volatility σ was varied from 0.05 to 0.5, so that volatility covered a wide range of market conditions. The specification of the additional parameters do not significantly alter the conclusion of the results but we include them here for completeness:$\eta = 3\%, r = 5\%$. We also calculated the values of α and β for different σ values (all other parameters affecting α, β were kept constant). We now present the results

Figure 1: Graph Of Option Prices $C(V(0))$ For Different Debt Values $(V(0))$ And Volatility (σ)

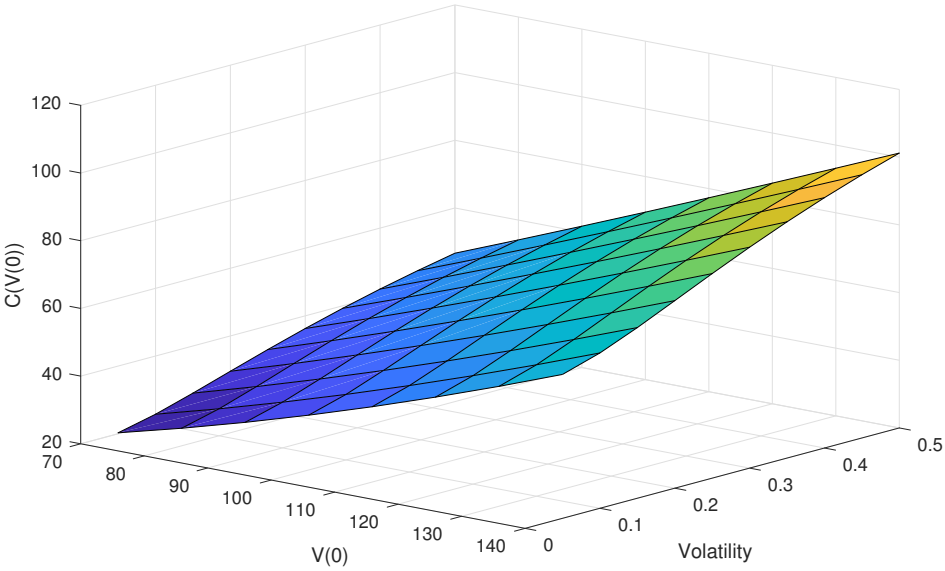

Figure 2: Graph Of Option Prices $C(V(0))$ For Different Debt Values $(V(0))$ And Volatility (σ) From Debt $V(0)$ Axis

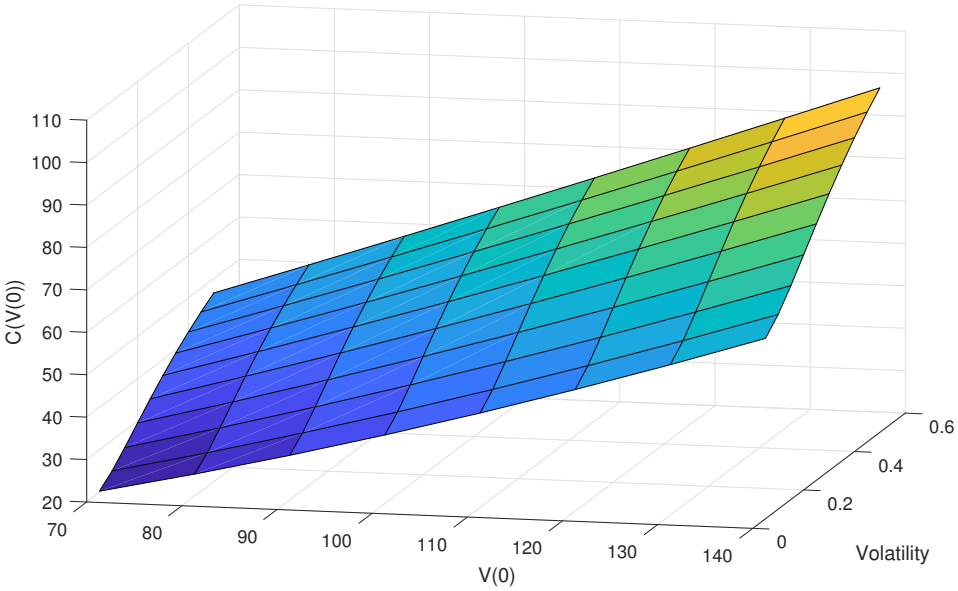

Figure 3: Graph Of Option Prices $C(V(0))$ For Different Debt Values $(V(0))$ And Volatility (σ) From Volatility Axis

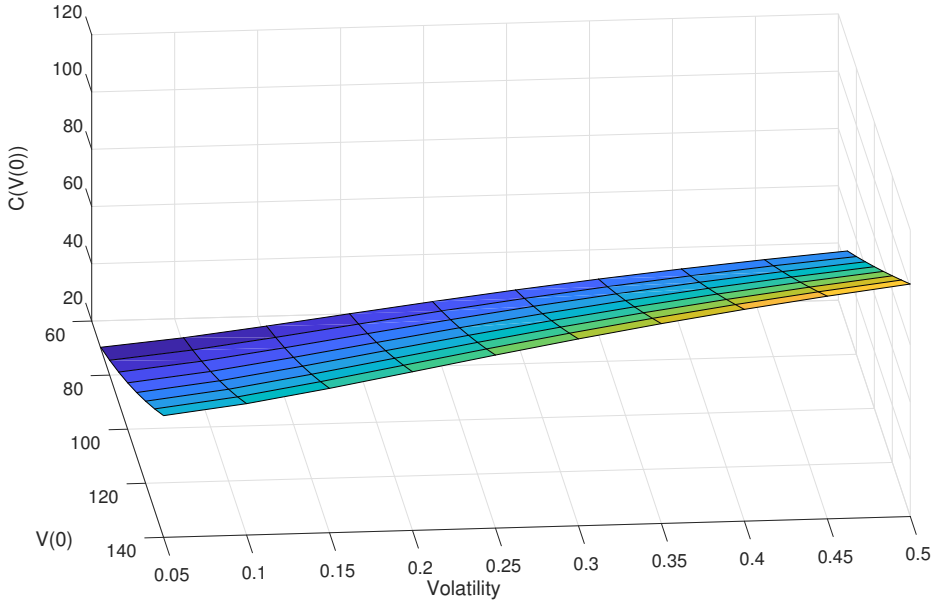

Figure 4: Graph Of α For Different Volatility (σ) Values

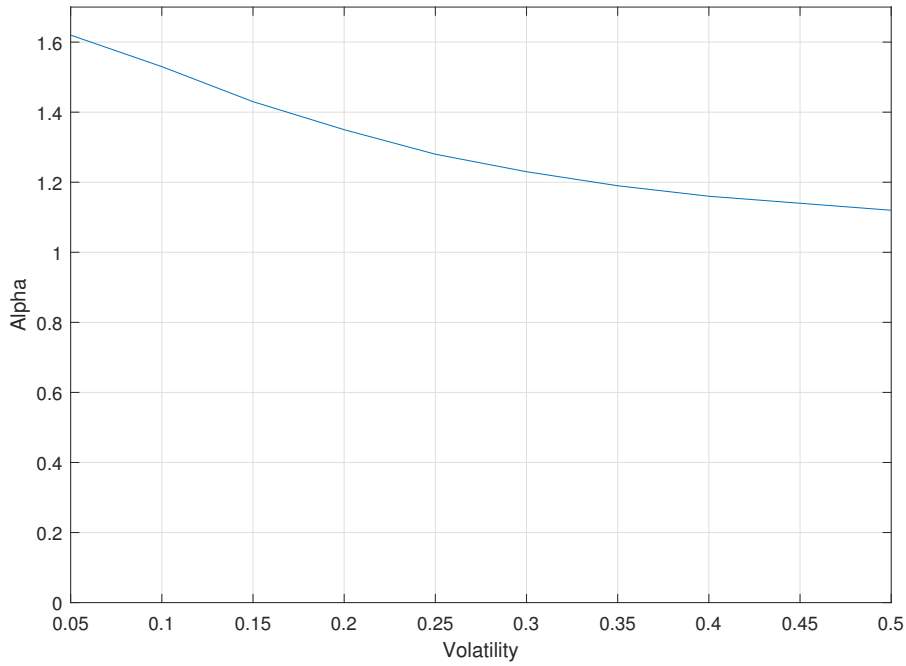

Figure 5: Graph Of β For Different Volatility (σ) Values

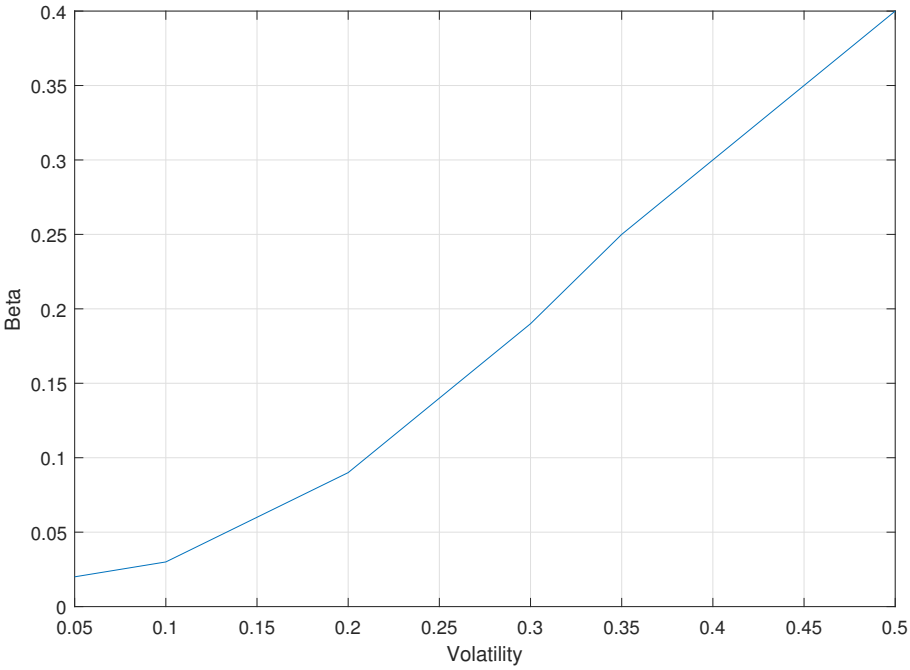

Table 1: Option Price Values $C(V(0))$ For Different Volatility σ And $V(0)$

	$C(V(0))$ **For Different** $V(0)$							
σ	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**
0.05	20.28	25.19	30.50	36.19	42.25	48.67	55.42	62.51
0.10	22.8	27.97	33.50	39.36	45.55	52.04	58.82	65.89
0.15	26.07	31.57	37.38	43.47	49.84	56.46	63.33	70.42
0.20	29.57	35.41	41.52	47.87	54.45	61.24	68.23	75.41
0.25	33.01	39.19	45.59	52.20	59.00	65.98	73.13	80.44
0.30	36.26	42.75	49.44	56.30	63.33	70.50	77.82	85.27
0.35	39.25	46.04	52.99	60.09	67.33	74.71	82.20	89.80
0.40	41.98	49.03	56.22	63.55	70.99	78.55	86.21	93.97
0.45	44.43	51.72	59.14	66.67	74.31	82.04	89.86	97.77
0.50	46.64	54.14	61.76	69.48	77.29	85.19	93.16	101.21

The Figures 1-3 demonstrate how the option price to default varies with debt $V(0)$ and volatility σ. The calculation values are also given in Table 1 for convenience. As one would expect from the option pricing equation, the option prices increase with $V(0)$ and σ. However, it is interesting to observe from the graphs that the impact of both $V(0)$ and σ have an almost linear impact on option price, which is not apparent from the option pricing equation. In particular, if we look at Figures 2 and 3 we note that the graphs have an approximately linear relationship with $V(0)$ and σ, respectively.

The graph in Figure 2 implies the option to default becomes more valuable as debt $V(0)$ increases. This is an interesting insight because it means that a company that continues to borrow will find that they have increasingly less incentive to default, since the option to default will be more valuable. This graph therefore may explain the unwillingness of firms to default and to continue borrowing instead. Furthermore in terms of strategic defaulting and Optimal Capital Theory, it suggests that defaulting is not the optimal decision for a firm to undertake.

The graph in Figure 3 shows that increasing volatility increases the option to default. In standard option theory it is known that volatility increases the value of the option, since there is more opportunity for the underlying asset to exceed the strike price. Similarly, the volatility of debt increases the value of the option to default, and this suggests that companies with volatile debt growth will also be encouraged to hold onto the option to default rather than default.

In Figure 4 we plot α against σ and this graph shows that α decreases with σ. Additionally, we know that $\alpha \geq 1$ hence we can observe from the graph that $\alpha \to 1$, as $\sigma \uparrow$. Thus the change in α values tends to decline as σ increases, in other words $\dfrac{\partial \alpha}{\partial \sigma} \to 0, \sigma \to 1$. In Figure 5 we plot β against σ and we observe that β increases with σ. We recall that $\beta \geq 0$, and we can see from the graph that σ results in an approximately linear increase in β. Hence the increase in option price $C(V(0))$ with σ is due to β increasing at a greater value than α decreasing.

An interesting observation from our results is that far in the money options have values greater than the intrinsic value. In other words $C(V(0)) > V(0) - K$, for $V(0) >> K$. We have set $K = 90$, and even for far in the money options (such as $V(0) = 140$) the intrinsic value is $V(0) - K = 50$ but the option to default is worth more, for all values of volatility. This implies that the option to default $C(V(0))$ is worth more than the value obtained from defaulting, and so defaulting is not the optimal choice for firms. It can be seen for far in the money options that the option value significantly increases with $V(0)$ and σ, hence increasing debt and debt volatility do not encourage firms to default.

7. Conclusion

Since the start of the GFC the credit risk of firms that are 'too big to fail' has been an issue of significant concern to the public and media. In this paper we developed a model to address the credit risk of firms that are too big to fail. We have achieved this by incorporating important credit risk factors such as modelling debt growth $V(t)$ without restriction on asset value, incorporating Government bail-outs and subsidies, and allowing firms to default at any point in time (rather than at a single point in time).

In this paper we have derived the partial differential equation for our option to default, and derived a closed form solution for the option to default. We have developed the model to include economic factors through the application of regime switching, and derived a closed form solution of option prices under such a process. We have also derived a method of hedging the option to default, which takes into account the market incompleteness and minimises hedging risk. We conducted numerical experiments to calculate the price of options at different debt values and volatility. In terms of future work, we aim to develop the model to include stochastic interest rates, to model the impact of any transaction costs upon hedging and incorporate the impact of liquidity risk within our models.

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