TDMA is Optimal for All-unicast DoF Region of TIM if and only if Topology is Chordal Bipartite

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***Abstract*—The main result of this work is that an orthogonal access scheme such as TDMA achieves the all-unicast degrees of freedom (DoF) region of the topological interference management (TIM) problem if and only if the network topology graph is chordal bipartite, i.e., every cycle that can contain a chord, does contain a chord. The all-unicast DoF region includes the DoF region for any arbitrary choice of a unicast message set, so e.g., the results of Maleki and Jafar on the optimality of orthogonal access for the sum-DoF of one-dimensional convex networks are recovered as a special case. The result is also established for the corresponding topological representation of the index coding problem.**

I. Introduction

The topological interference management problem (TIM), introduced in [1], studies the degrees of freedom (DoF) of partially connected one-hop wireless networks with no channel state information at the transmitters except the network topology. As a generalization of the classical optimal frequency reuse question, and for its use of robust interference alignment schemes, TIM is of much practical interest. It is also of great theoretical interest because of an essential equivalence between TIM and the index coding problem, established in [1]. The index coding problem is one of the most intriguing open problems in network information theory due to its rich connections to various other prominent problems ranging from distributed storage [2], [3], caching [4] and general instances of network coding [5], [6] to hat-guessing problems [7] in recreational mathematics.

On the surface these problems seem very simple. However, in spite of various graph-theoretic [8], random coding [9], rate-distortion [10], as well as interference alignment [11] approaches that have been brought to bear upon it, the index coding problem, and by association the TIM problem, remain open. The difficulty is evident in recent results for index coding that prove the necessity, in general, of non-linear coding schemes for achievability arguments [5], [11], [12], and of non-Shannon information inequalities for converses [13]–[15], neither of which are well understood. As such, much of the recent progress on these problems has come about from a divide and conquer approach aimed at identifying broad classes of solvable instances of the problem. For example, [8], [16] identify all instances where it is optimal to serve only one user at a time. All half-rate feasible instances are identified in [11], [13]. All instances for which the alignment graph has no cycles or no forks are solved in [1] and this class is further generalized to include instances for which the alignment graph has no overlapping cycles in [14]. All instances with 5 or fewer messages are solved in [9], all single uniprior instances are solved in [17], and all instances where each source has non-trivial communication capacity to at most two of the destinations are solved in [10].

Continuing with the divide and conquer strategy, a problem of great interest from both theoretical and practical perspectives, is to identify those instances where the simplest achievable schemes are also optimal. Such an approach has been quite successful recently in wireless interference networks where much progress has been made in identifying settings where the simple scheme of treating interference as noise is (in a generalized DoF sense) optimal [18]. For TIM, perhaps the simplest scheme is an orthogonal access scheme, such as TDMA. The corresponding scheme for index coding is graph coloring, i.e., simultaneous scheduling of only those messages that do not interfere with each other. Remarkably, in recent work by [19], TDMA and graph coloring are shown to achieve the sum-DoF and sum-capacity of a fairly broad class of TIM and index coding problems, respectively. Based on the constraints on the topology and message sets, this class is known as one-dimensional convex networks. This is our starting point.

A. One-Dimensional Convex Networks [19]

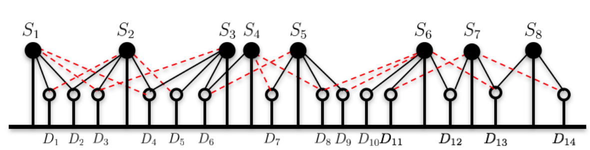

Fig. 1: A one-dimensional convex network instance of TIM (Fig. 2 of [19]). Each solid black edge indicates an independent desired message, each dashed red edge indicates no desired message, and the absence of an edge indicates 'out-of-range'. All edges together comprise the bipartite network topology graph.

For the TIM problem, [19] shows that orthogonal access (such as time division multiple access — TDMA) achieves the sum-DoF of a one-dimensional cellular network (all nodes placed on a straight line) that satisfies (i) source convexity, (ii)

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destination convexity, and (iii) message convexity. An example of such a network (with 19 messages) is shown in Fig. 1. The convexity assumptions are motivated by the observation that signals are stronger and communication is more likely to occur between nodes that are physically closer to each other than between nodes that are farther apart. For example, since Source S_1 is heard by Destination D_4, it must also be heard by destinations that are closer to it than D_4, e.g., destinations D_1, D_2, D_3. This is referred to as source convexity. Destination convexity is similarly defined. Further, if Source S_1 has a desired message for Destination D_2, it must have a desired message for Destination D_1 because D_1 is closer to S_1 than D_2. This is denoted as the message convexity assumption. Under these three assumptions, [19] shows that TDMA is optimal in terms of sum-DoF. For example, since the network in Fig. 1 satisfies these assumptions, an orthogonal scheduling scheme that schedules messages only between non-interfering source-destination pairs $S_1 \to D_1, S_3 \to D_5, S_5 \to D_7, S_6 \to D_{10}$ and $S_8 \to D_{14}$, achieves the optimal sum-DoF value (5 in this case).

The TIM result corresponds directly to an index coding result due to the remarkable association between the two problems identified in [1], such that there is a TIM instance associated with each index coding instance and vice versa (see Fig. 2). Corresponding instances of both problems are jointly described by a bipartite topology graph with sources on one side, destinations on the other, and edges representing the presence of a non-trivial communication channel whose communication capacity is not zero (TIM) or infinity (index coding). Reference [1] shows that (expressed in normalized units) the capacity region of any instance of the index coding problem acts as an outer bound on the DoF region of the corresponding instance of the TIM problem, and furthermore, the two are equivalent when restricted to linear schemes over the same field.

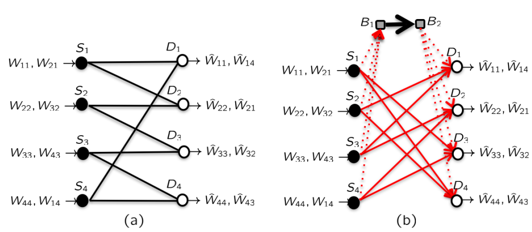

Fig. 2: (a) An instance of the TIM problem (edges constitute the bipartite network topology graph) and (b) the corresponding instance of the index coding problem. Red links (solid and dashed) have infinite capacity. Solid red links form the antidote graph, which is the complement of the network topology graph.

B. Beyond One-Dimensional Convex Networks

On the one hand, the optimality of TDMA for the physically motivated and fairly broad class of topologies represented by one-dimensional convex networks is surprising because it is known that simple schemes such as TDMA or CDMA (cf. (fractional) coloring and (partition) multicast) can be severely sub-optimal in general. For example, there exist instances of TIM (index coding) with K messages where optimal schemes involving interference alignment achieve a sum-DoF (sum-capacity) value that is a factor of $(1/3 + o(1))K^{1/4}$ higher than the best achievable through TDMA or CDMA [1], [13].

On the other hand, however, the result is limited by the assumption of a one-dimensional placement of nodes and the convexity constraints. For instance, even for physically motivated TIM topologies that satisfy all the convexity constraints, it is shown in [19] that going from one-dimensional settings to the much more realistic two-dimensional placements of sources and destinations, one immediately runs into examples where TDMA is no longer optimal[1] and interference alignment solutions significantly outperform conventional baselines. The convexity assumptions are also not applicable to heterogeneous networks, where a user may hear a distant high power base station, but still not be able to hear a closer but lower power base station. Moreover, beyond the TIM context, for the index coding problem in general, the one-dimensional node placements or convexity constraints are of little physical significance. Last but not the least, the focus on sum-DoF (capacity) is restrictive as well.

This brings us to the motivation of this work, which is to go beyond these limitations, to answer the question — *what is the fundamental topological structure that determines the optimality of TDMA (fractional coloring) for the TIM (index coding) problem, making other sophisticated schemes redundant?*

C. Summary of Contribution

The main contribution of this work,[2] as highlighted in the title, is to show that TDMA (fractional coloring) achieves the all-unicast DoF (capacity) region of the TIM (index coding) problem if and only if the network topology graph is chordal bipartite, i.e., any cycle that can contain a chord, contains a chord.[3] Note that network topology graphs are always bipartite, and for such graphs, cycles of length 4 cannot have a chord. So a chordal bipartite graph is one in which any cycle of length 6 or more (such cycles are called long cycles) must contain a chord. For example, consider the network topology graph in Fig. 3, it is chordal bipartite because it only contains one long cycle (formed by nodes $S_1, D_2, S_3, D_3, S_5, D_4$), and this cycle has a chord (the edge connecting S_1 and D_3). Note that for a chordal bipartite graph, it is acceptable for a long cycle to have *some* chords missing (e.g., in Fig. 3, the chord connecting nodes D_2 and S_5 is not present in the only long cycle), provided it does not have *all* its chords missing. Furthermore, whether or not a network is chordal bipartite is easy to check (polynomial time) [20].

The all-unicast setting is defined as a setting where we have an independent unicast message between each source and each

[1]Fig. 2 is such an example. It corresponds to a 2 dimensional convex network (see [1]) with optimal sum-DoF value of 8/3, achieved by interference alignment, whereas orthogonal schemes cannot achieve more than 2 DoF.

[2]Frequently in this paper, we will use a compact notation where we merge corresponding statements for TIM and index coding by using parantheses for the respective alternatives.

[3]Note that chords are only defined for cycles, so e.g., tree topologies are also chordal bipartite graphs.

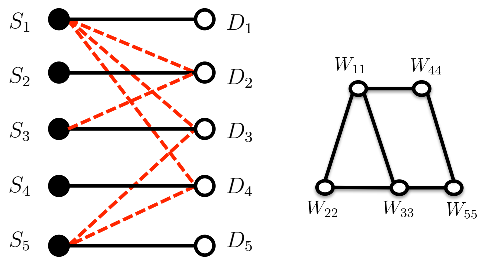

Fig. 3: A 5-message TIM instance (left), and its conflict graph (right). See Section II for details.

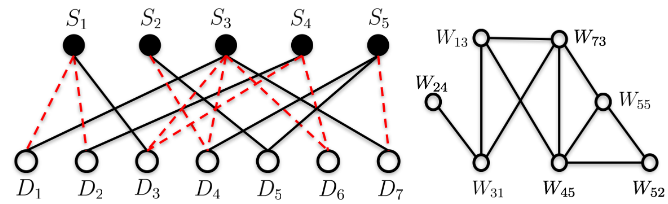

Fig. 4: A 7-message TIM instance (left) and its conflict graph (right). See Section II for details.

destination, i.e., all possible unicast messages are considered.[4] A unicast message is one that originates at only one source and is desired by only one destination. Characterizing the all-unicast DoF (capacity) region automatically characterizes the DoF (capacity) region for any arbitrary subset of unicast messages, as well as other traditional metrics such as symmetric or sum DoF (capacity). So, when TDMA achieves the all-unicast DoF region, it achieves the DoF region of any arbitrary subset of messages as well. For example, consider again Fig. 3. Since the network topology graph is chordal bipartite, our result implies that TDMA achieves the all-unicast DoF region and therefore also the DoF region of arbitrary subset of messages (e.g., the interference channel message setting, where each Source $S_i, i \in \{1, 2, \cdots, 5\}$ has an independent message only for its corresponding Destination $D_i, i \in \{1, 2, \cdots, 5\}$.)

While not restricted to the one-dimensional convex topologies studied in [19], chordal bipartite networks do include them as a special case. Since convex message sets are included in all-unicasts, and the DoF region includes sum-DoF, the results of [19] are recovered as a special case of our result. Within the context of one-dimensional networks, our result generalizes the results of [19] — it shows that for one-dimensional placement of sources and destinations, if the network satisfies either source convexity *or* destination convexity (without requiring *both*, turns out either assumption is sufficient (not necessary) to imply a chordal bipartite topology), then for any arbitrary unicast message set (without requiring message convexity), the entire DoF region (without restriction to only sum-DoF) is achieved by TDMA. For the setting illustrated in Fig. 1 (note that the topology is chordal bipartite), the result shows that the entire DoF (capacity) region for the 19 messages (as well as any other set of unicast messages possible in this setting) is achieved by TDMA (fractional coloring). Other examples that further highlight the generality of this result are presented in Fig. 3 (does not satisfy destination convexity) and Fig. 4 (does not satisfy source or destination convexity) where also the network topology graph is chordal bipartite and TDMA (fractional coloring) is optimal for the DoF (capacity) region under all possible unicast message sets. On the other hand, Fig. 2 is an example where the topology is not chordal bipartite, and so TDMA (fractional coloring) schemes cannot achieve the DoF (capacity) region (as indeed is shown in [1]).

[4] A message between a source and a destination that are topologically not connected can only have rate zero, so such messages may be ignored without loss of generality.

So to answer the question that motivates this work — *chordal bipartite network topology is the fundamental topological structure that determines the optimality of TDMA (fractional coloring), making all other sophisticated schemes unnecessary.*

II. System Model

Corresponding instances of TIM and index coding problems, each with M sources and N destinations, are simultaneously specified by a topology matrix $\boldsymbol{T}$ and a message set $\mathcal{M}$. $\boldsymbol{T}$ is an $N \times M$ matrix with elements $t_{ji} \in \{0, 1\}$. If there exists a non-trivial channel from Source i to Destination j, i.e., a channel whose capacity is not zero (TIM) or infinity (index coding) then $t_{ji} = 1$, otherwise $t_{ji} = 0$. Since communication is non-trivial only if the channel capacity is not zero or infinity, the set of messages $\mathcal{M}$ is a subset of $\overline{\mathcal{M}} \triangleq \{W_{ji} : t_{ji} = 1, i \in \{1, \ldots, M\}, j \in \{1, \ldots, N\}\}$, with W_{ji} representing an independent unicast message that originates at Source i and is intended for Destination j. If $\mathcal{M} = \overline{\mathcal{M}}$, the setting is called the all-unicast setting. Corresponding instances of TIM and index coding share the same topology matrix $\boldsymbol{T}$ and the same message set $\mathcal{M}$. The remaining description for each problem is provided next.

A. Topological Interference Management Problem (TIM)

An arbitrary instance of the TIM problem [1] is represented as a partially connected network with M sources, labeled S_1, S_2, ..., S_M, and N destinations, labeled D_1, D_2, ..., D_N. All sources and destinations are equipped with a single antenna each. The received signal for Destination D_j at time instant t is:

$$Y_j(t) = \sum_{i=1}^{M} t_{ji} h_{ji}(t) X_i(t) + Z_j(t) \tag{1}$$

where $X_i(t)$ is the transmitted signal from Source S_i. All transmitted signals are subject to a power constraint P. $Z_j(t)$ is the Gaussian noise with zero-mean and unit-variance at Destination D_j, $h_{ji}(t)$ is the channel coefficient between Source S_i and Destination D_j. The topology matrix $\boldsymbol{T} = [t_{ji}]_{N \times M}$ is known by all sources and destinations. Only the channel coefficients of those links over which the desired messages are received are assumed to be known by destinations. The channel coefficients and network topology are assumed to be fixed throughout the duration of communication. The relaxation to time-varying channels will be discussed in Section VI-B.

Let us define two graphs for the TIM problem.

Definition 1. *Given the TIM problem with M sources and N destinations, topology matrix $\boldsymbol{T}$, and message set $\mathcal{M} \subseteq \overline{\mathcal{M}}$, define the following two graphs:*

- **Network Topology Graph***: An undirected bipartite graph with sources on one side, destinations on the other, and an edge between S_i and D_j whenever $t_{ji} = 1$.*
- **Message Conflict Graph***: An undirected graph where each message $W_{ji} \in \mathcal{M}$ is a vertex, and edges exist between two vertices if and only if the two messages conflict with each other. Two messages $W_{ji}, W_{j'i'}$ conflict if $(t_{j'i}, t_{ji'}) \neq (0,0)$, i.e., they originate from the same source ($i = i'$), or are intended for the same destination ($j = j'$), or if the source of one message interferes with the destination of the other message.*

Note that the network topology graph depends only on network connectivity, regardless of the message demand. The message conflict graph depends on both the network connectivity and the message demand. A connection between a source S_i and a destination D_j is a desired link if a desired message exists from S_i to D_j, and an interfering link otherwise. If we take all all messages $\overline{\mathcal{M}}$ into account and construct the message conflict graph $\mathcal{H}$, then for a specific message set $\mathcal{M} \subseteq \overline{\mathcal{M}}$, the corresponding message conflict graph is the induced subgraph $\mathcal{H}[\mathcal{M}]$. (See definitions in Section III-A). The message conflict graph indicates the conflict between two messages, but it loses some information of conflicting source and destination.

For TIM we use DoF region as our figure of merit. Common terms such as the coding scheme, achievable rate tuple $(\bar{R}_{ji} : W_{ji} \in \mathcal{M})$, capacity region $\mathcal{C}$ are used in the standard Shannon theoretic sense. Unfamiliar readers can refer to [1] for details.

Definition 2 (DoF Region [1])**.**

$$\mathcal{D}_{\mathcal{M}} = \Big\{ (d_{ji} : W_{ji} \in \mathcal{M}) \in \mathbb{R}_+^{|\mathcal{M}|} : \\ d_{ji} = \lim_{P \to \infty} \frac{\bar{R}_{ji}}{\log P}, \text{ s.t. } (\bar{R}_{ji} : W_{ji} \in \mathcal{M}) \in \mathcal{C} \Big\} \quad (2)$$

B. Index Coding Problem (IC)

The index coding problem consists of M sources, labeled $S_1, S_2, \ldots, S_M$, N destinations, labeled $D_1, D_2, \ldots, D_N$ and two additional nodes B_1, B_2, that are connected by a unit capacity edge from B_1 to B_2. There is an infinite capacity link from every source to B_1, and from B_2 to every destination. We also have infinite capacity links between sources and destinations identified by the antidote matrix $\boldsymbol{A} = [a_{ji}]_{N \times M}$, where $a_{ji} = 1$ means that Source S_i is connected to Destination D_j through an infinite capacity link, and $a_{ji} = 0$ otherwise. The antidote matrix $\boldsymbol{A}$ is simply the complement of the topology matrix $\boldsymbol{T}$ and is obtained as $\boldsymbol{A} = [1]_{N \times M} - \boldsymbol{T}$, where $[1]_{N \times M}$ denotes the $N \times M$ matrix with all entries equal to 1.

Define these graphs for the index coding problem.

Definition 3. *Given the index coding problem with M sources and N destinations, antidote matrix $\boldsymbol{A}$, and message set $\mathcal{M} \subseteq \overline{\mathcal{M}}$, define the following graphs:*

- **Antidote Graph***: A directed bipartite graph with sources on one side, and destinations on the other, and an arc (i.e., directed edge) from S_i to D_j whenever $a_{ji} = 1$.*
- **Side Information Digraph**[5]*: A directed graph where each message $W_{ji} \in \mathcal{M}$ is a vertex, and there exists an arc (i.e., directed edge) from W_{ji} to $W_{j'i'}$ if $a_{j'i} = 1$.*
- **Network Topology Graph***: An undirected bipartite graph with sources on one side, destinations on the other, and an edge between S_i and D_j whenever $a_{ji} = 0$.*
- **Message Conflict Graph***: An undirected graph where each message $W_{ji} \in \mathcal{M}$ is a vertex, and edges exist between two vertices if and only if the two messages conflict with each other. Two messages $W_{ji}, W_{j'i'}$ conflict if $(a_{j'i}, a_{ji'}) \neq (1,1)$, i.e., they originate from the same source ($i = i'$), or are intended for the same destination ($j = j'$), or if one source is not connected to the other destination through an infinite capacity link.*

Note that given the topology matrix $\boldsymbol{T}$ and the message set $\mathcal{M}$, corresponding instances of TIM and index coding have the same network topology graph and the same message conflict graph.

Coding schemes, achievable rate tuple $(R_{ji} : W_{ji} \in \mathcal{M})$ and capacity region are used in the standard Shannon theoretic sense.

C. Relationship between TIM and index coding

As shown in [1], for corresponding instances of the TIM problem and the index coding problem, described by the same topology matrix $\boldsymbol{T}$ and message set $\mathcal{M}$, the index coding capacity region is an outer bound on the TIM DoF region, and the two are equivalent under linear solutions [1]. Fig. 2 shows an example.

III. Background and Key Definitions

A. Background

In what follows, some of the most basic definitions pertaining to graph theory and polyhedral combinatorics are now recalled, for the readers who are not familiar with these two fields [21], [22].

Throughout this paper, we consider simple graphs, which are unweighted, containing no loops for a single vertex or multiple edges between two vertices.

A subgraph of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ containing a subset of vertices $\mathcal{S}$ ($\mathcal{S} \subseteq \mathcal{V}$) is said to be an *induced subgraph*, denoted by $\mathcal{G}[\mathcal{S}]$, if for any pair of vertices u and v in $\mathcal{S}$, uv is an edge of $\mathcal{G}[\mathcal{S}]$ if and only if uv is an edge of $\mathcal{G}$. Note that a subgraph is not sufficiently an induced subgraph.

The *chromatic number* of $\mathcal{G}$, denoted by $\chi(\mathcal{G})$, is the smallest number of colors that can be assigned to the vertices of $\mathcal{G}$, such that no two adjacent vertices have the same color. Such an assignment method is referred to ordinary *graph coloring*. A graph $\mathcal{G}$ is said to be $n : m$*-colorable* if each vertex in $\mathcal{G}$ can be assigned a set of m distinct colors in which the colors are drawn

[5]Note here that we adopt the more intuitive representation driven by the interference management perspective, such that the side information digraphs have arcs with directions opposite to what was originally used in [8].

from a palette of n colors, such that any adjacent vertices have no colors in common. This coloring method is also referred to as *fractional coloring*, which is a generalization of ordinary (non-fractional) graph coloring. Specifically, fractional coloring is a relaxation of non-fractional coloring with respect to linear programming formulations. Alternatively, fractional coloring schemes can be seen as extensions of non-fractional coloring schemes by allowing time sharing among them.

A clique is a subgraph of a graph $\mathcal{G}$ such that any two vertices in this subgraph are adjacent. The size of a clique is the number of vertices. A maximum clique is the clique with the maximum possible size in $\mathcal{G}$. The *clique number* of $\mathcal{G}$, denoted by $\omega(\mathcal{G})$, is the number of vertices in the maximum clique. $\chi(\mathcal{G}) \geq \omega(\mathcal{G})$ for any undirected graph. The independent set of a graph $\mathcal{G}$ is a set of vertices such that any two vertices are not adjacent. The *independent set number*, denoted by $\alpha(\mathcal{G})$, is the cardinality of the largest independent set.

A *perfect graph* is a graph $\mathcal{G}$ in which the chromatic number of every induced subgraph $\mathcal{H}$ of $\mathcal{G}$ equals the clique number of this subgraph, i.e., $\chi(\mathcal{H}) = \omega(\mathcal{H})$. A chordless cycle is a cycle with no edges between any non-consecutive vertices. The length of a cycle is the number of vertices in this cycle. A *hole* is a chordless cycle with five or more vertices, and an *antihole* is the complement of a hole. The *complement* of a graph, is another graph containing the same vertices set, but an edge in one graph if and only if it is not in the complement. The underlying undirected graph of a directed graph is obtained by ignoring the direction in the directed graph. An undirected graph is *perfect* if and only if it contains neither odd holes nor odd antiholes as induced subgraphs. The odd hole is a hole with odd length, and the odd antihole is its complement.

A graph is *chordal* (or triangulated) if there is no induced subgraph with chordless cycles of length greater than three, i.e., every cycle with length greater than three has a chord. A graph is *weakly chordal* (or weakly triangulated) if it is hole-free and antihole-free in its induced subgraphs. Both chordal and weakly chordal graphs are subclasses of perfect graphs.

A graph is *chordal bipartite* if it is an undirected bipartite graph and there is no induced subgraph with chordless cycles of length greater than four, i.e., every cycle of length at least six has a chord. The chordal bipartite graph is both bipartite and chordal, and hence weakly chordal. For a graph $\mathcal{G}$, if $\mathcal{G}$ is weakly chordal, then $\mathcal{G}_e^2$ is also weakly chordal [27], and hence perfect.

Given a graph $\mathcal{G}$, the *line graph* of $\mathcal{G}$ is another graph, denoted by $\mathcal{G}_e$, such that each vertex of $\mathcal{G}_e$ represents an edge of $\mathcal{G}$, and any two vertices in $\mathcal{G}_e$ are adjacent if and only if their corresponding edges in $\mathcal{G}$ have an endpoint in common. The *square* of a graph $\mathcal{G}$, denoted by $\mathcal{G}^2$, is another graph that has the same set of vertices in $\mathcal{G}$, but in which two vertices are adjacent when their distance in $\mathcal{G}$ is at most 2. The *distance* between two vertices is the number of edges in a shortest path connecting them. Graphical illustrations are presented in Fig. 5 to show the relation among $\mathcal{G}$, its line graph $\mathcal{G}_e$, and the square of its line graph $\mathcal{G}_e^2$. Specifically, the vertex labeled ij in the line graph $\mathcal{G}_e$ corresponds to the edge between vertices i and j in $\mathcal{G}$.

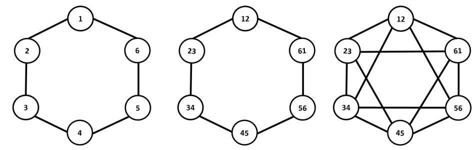

Fig. 5: The graph $\mathcal{G}$ (left), its line graph $\mathcal{G}_e$ (center), and the squared of its line graph $\mathcal{G}_e^2$ (right).

In a directed graph (digraph), a set of vertices forms a *di-clique* if, for any two vertices u and v, there is an arc from u to v and an arc from v to u. Clique covering is to partition the digraph into di-cliques such that all vertices are covered at least once. A chordless directed cycle (di-cycle) refers to the induced sub-digraph with n nodes $\{v_0, v_1, \cdots, v_{n-1}\}$ and arcs $\{(v_0, v_1), (v_1, v_2), \cdots, (v_{n-2}, v_{n-1}), (v_{n-1}, v_0)\}$, beyond which there do not exist any other arcs. A directed graph (or subgraph) is acyclic if it does not contain any di-cycles.

Given a clique in a graph, the *clique inequality* refers to the inequality

$$\sum_{j \in Q} x_j \leq 1,$$

where Q is the set of vertices in the clique, and the variable x_j corresponds to the vertex $j \in Q$. Given an n-vertex graph $\mathcal{G}$ and the collection of all cliques $\mathcal{Q}(\mathcal{G})$, the *polytope* defined by the clique inequalities is

$$\mathcal{P} = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}_n^+ : \sum_{j \in Q} x_j \leq 1, \forall Q \in \mathcal{Q}(\mathcal{G})\}$$

where Q refers to any cliques, including each single vertex and any two adjacent vertices. It is clear that $0 \leq x_j \leq 1$ for all $j \in \{1, \ldots, n\}$ in the polytope $\mathcal{P}$, and that some of the clique inequalities are redundant. The corner points of the polytope are also known as the vertices or the extreme points.

B. Key Definitions

Definition 4 (All-Unicast)**.** *The all-unicast setting means that from each Source S_i there is an independent message W_{ji} to each Destination D_j if $t_{ji} = 1$ ($a_{ji} = 0$), i.e., if there exists a non-trivial channel between them. Note that this includes arbitrary message sets as special cases, by setting the rates of some messages to zero.*

Definition 5 (Cycle)**.** *A cycle is a set of vertices and edges that form a closed loop. The length of a cycle is the number of vertices in this cycle. A cycle of length 6 or more is called a long cycle.*

Definition 6 (Chord)**.** *A chord is an edge that connects two non-adjacent vertices of a cycle.*

Definition 7 (Chordal Bipartite Network)**.** *A TIM (index coding) instance is referred to as a chordal bipartite network if its network topology graph does not contain a chordless long cycle. Chordal bipartite networks are exactly the graph class of chordal bipartite graphs in graph theory.*

Note that the network topology graph is always bipartite, and cannot contain odd length cycles. These graphs are simple, so there are no multiple edges and thus no length-2 cycles. In addition, because a source (destination) is not connected to any other sources (destinations), it is not possible for a length-4 cycle to contain a chord in the network topology graphs. Thus the chordal bipartite property in the network topology graph can be interpreted as – *any cycle that can contain a chord, must contain a chord.*

Remark 1. *Chordal bipartite networks include some special network topology graphs, such as forests and trees, convex/biconvex bipartite graphs, bipartite permutation graphs, bipartite distance hereditary graphs, and difference graphs [21].*

Definition 8 (TDMA [19])**.** *TDMA refers to a time-slotted transmission scheme with time sharing across time slots, such that in each time slot it schedules messages $\mathcal{W}_o$ for transmission simultaneously only if they are non-interfering (orthogonal), i.e., $t_{j'i} = t_{ji'} = 0$ for TIM ($a_{j'i} = a_{ji'} = 1$ for index coding), $\forall\ W_{ji}, W_{j'i'} \in \mathcal{W}_o,\ i \neq i', j \neq j'$.*

TDMA is also known as orthogonal access and link scheduling. The duration of each time slot can be varying to facilitate time sharing across time slots. In the message conflict graph, two adjacent vertices (messages) are interfering, and should be scheduled orthogonally in different slots. If we associate each slot with a different color, TDMA can be done using vertex coloring on the message conflict graphs, where two adjacent vertices (messages) should be assigned different colors. Associating the slots with a number of different colors, we generalize it to fractional coloring (see Section III-A). As a result, TDMA is also referred to as fractional coloring on the conflict graphs under the index coding context.

Definition 9 (Set of Cliques of Conflict Graph ($\mathcal{Q}$))**.** *A clique Q of the message conflict graph is a set of vertices (messages) such that any two are adjacent. The set of all cliques of the message conflict graph is denoted as $\mathcal{Q}$.*

In the rest of this paper, we identify the optimality of TDMA (fractional coloring) through the structural property of the bipartite network topology graphs, and characterize the DoF (capacity) region via vertex coloring on the undirected message conflict graphs.

IV. Main Result

Our main result for the TIM setting is stated in the following theorem.

Theorem 1. *TDMA achieves the all-unicast DoF region of the TIM problem if and only if the network topology is chordal bipartite. For chordal bipartite networks, the DoF region of the TIM problem is characterized through the cliques $Q \in \mathcal{Q}$ of the message conflict graph, as follows:*

$$\forall\ Q \in \mathcal{Q},\quad \textstyle\sum_{W_{ji} \in Q} d_{ji} \leq 1$$

where d_{ji} is the DoF of the message W_{ji}.

Proof. See Section V-B. □

Remark 2. *There are various characterizations of the structural properties for chordal bipartite graphs in terms of perfect elimination orderings, hypergraphs and adjacency matrices [23, Chapter 9]. Particularly, it has been shown that, for a graph with n vertices and m edges, the determination whether it is chordal bipartite or not can be made in time $O(\min(n^2, (n+m)\log n))$ [23, Chapter 9], [24].*

Remark 3. *The DoF region as specified above is comprised of clique inequalities corresponding to all possible cliques in the message conflict graph. There may be exponentially many cliques, many of them redundant because they are contained within larger cliques. Therefore, the representation may be simplified by restricting $\mathcal{Q}$ to be the set of maximal cliques. A maximal clique is the clique with the maximum possible number of vertices (messages). Any induced subgraph of a maximal clique is still a clique. Such a simplification is valid because the inequality associated with the maximal clique implies all other inequalities of the cliques with smaller size. The Bron-Kerbosch algorithm is an efficient method to enumerate all maximal cliques in an undirected graph [25]. It guarantees linear-time complexity with respect to the number of cliques. It has been shown that its worst-case running time is $O(3^{n/3})$, which matches the maximum possible number of maximal cliques in all n-vertex graphs, as there are at most $3^{n/3}$ maximal cliques [26].*

Next we illustrate this result through some examples.

Example 1. *Consider the one-dimensional TIM network instance with message W_{ii} from S_i to D_i as shown in Fig. 3, where convexity applies to the sources only. Note here that we only consider a subset of messages $\mathcal{M} = \{W_{11}, W_{22}, W_{33}, W_{44}, W_{55}\}$ rather than the all-unicast setting, so we construct the message conflict graph only with respect to $\mathcal{M}$. Because the network topology graph is chordal bipartite, Theorem 1 applies and TDMA achieves the DoF region. Note that the 6-vertex cycle $S_1 - D_2 - S_3 - D_3 - S_5 - D_4 - S_1$ is not chordless, because it indeed contains a chord $S_1 - D_3$. Considering all sets of conflicting messages, i.e., all cliques of the conflict graph shown in Fig. 3, and removing the redundant ones, we have the DoF region*

$$\mathcal{D} = \left\{ (d_{11}, \ldots, d_{55}) \in \mathbb{R}_+^5 \;\middle|\; \begin{array}{l} d_{11} + d_{22} + d_{33} \leq 1 \\ d_{11} + d_{44} \leq 1 \\ d_{33} + d_{55} \leq 1 \\ d_{44} + d_{55} \leq 1 \end{array} \right\}$$

As said, this DoF region can be simply obtained by listing all the maximal cliques, i.e., the message sets $\{W_{11}, W_{22}, W_{33}\}$, $\{W_{11}, W_{44}\}$, $\{W_{33}, W_{55}\}$, and $\{W_{44}, W_{55}\}$.

Example 2. *Consider the TIM instance in Fig. 4, with message set $\mathcal{M} = \{W_{13}, W_{24}, W_{31}, W_{45}, W_{52}, W_{55}, W_{73}\}$ (which is a subset of the all-unicast message set). Since the network is chordal bipartite, Theorem 1 applies and we have the DoF region by listing all the inequalities associated with the maximal cliques, where the constraints of smaller cliques are* *automatically implied.*

$$\mathcal{D} = \left\{ d_{\mathcal{M}} \in \mathbb{R}^7_+ \;\middle|\; \begin{array}{c} d_{24} + d_{31} \leq 1 \\ d_{31} + d_{13} + d_{73} \leq 1 \\ d_{13} + d_{45} + d_{73} \leq 1 \\ d_{45} + d_{55} + d_{73} \leq 1 \\ d_{52} + d_{45} + d_{55} \leq 1 \end{array} \right\}$$

Additionally, we are able to show (proof relegated to Section VI-A) that one-dimensional TIM instances with either source convexity or destination convexity (not necessarily both) are chordal bipartite. Thus, the optimality of TDMA shown in [19] for one-dimensional convex networks, applies with either source *or* destination convexity, to any arbitrary unicast message set, and to the entire DoF region.

Clique covering on the side information digraph is one of the most basic coding schemes in the index coding problem. Since the messages in the di-clique of the side information digraph form an independent set in the (undirected) message conflict graph, the (fractional) clique covering on the side information digraph is equivalent to vertex (fractional) coloring on the undirected message conflict graph [8]. As such, we conclude this section by translating the result for TIM to the index coding setting, as stated in the following theorem.

Theorem 2. *Fractional coloring achieves the all-unicast capacity region of the index coding problem associated with a network topology graph if and only if the network topology is chordal bipartite. For chordal bipartite networks, the capacity region of the index coding problem is the set of rate tuples comprised of R_{ji} in $\mathbb{R}^+$ that satisfy the following constraints:*

$$\forall\ Q \in \mathcal{Q}, \quad \textstyle\sum_{W_{ji} \in Q} R_{ji} \leq 1$$

where R_{ji} is the rate of the message W_{ji}.

Proof. See Section V-B. □

We note that this theorem characterizes the capacity region for a class of index coding problems with all possible message settings, for which the commonly studied broadcast rate in the multiple-unicast setting (see definitions in e.g., [8]) is a special case.

V. Proofs

A. Preliminaries

Consider the undirected network topology graph $\mathcal{G}$, which is bipartite, its line graph $\mathcal{G}_e$, and the square of its line graph $\mathcal{G}_e^2$. Interestingly, $\mathcal{G}_e^2$ is the message conflict graph corresponding to the network topology graph $\mathcal{G}$. We state this as the following lemma, which establishes the connection between the network topology and the message conflict graph. This connection is one of the most crucial elements of the paper, by which the structure of network topology can be transferred to the conflict graphs. By doing so, the achievability and the converse could be bridged through the message conflict graphs. The proof is presented in Section V-C.

Lemma 1. *For any network topology graph $\mathcal{G}$, the square of its line graph, $\mathcal{G}_e^2$, is its message conflict graph.*

Next, we carry over the chordal bipartite property of $\mathcal{G}$ to the perfect-graph property of $\mathcal{G}_e^2$. $\mathcal{G}$ is a chordal bipartite graph (see Chapter 12.4 in [21]), thus weakly chordal [27]. It is proved in [27] that if $\mathcal{G}$ is weakly chordal, then $\mathcal{G}_e^2$ is also weakly chordal. As weakly chordal graphs are subclass of perfect graphs (see Chapter 66.5d in [22]), $\mathcal{G}_e^2$ is perfect. We state this crucial result as the following lemma.

Lemma 2. *If $\mathcal{G}$ is chordal bipartite, then $\mathcal{G}_e^2$ is perfect.*

Remark 4. *When a subset of messages is considered, we have an induced subgraph of $\mathcal{G}_e^2$ as the message conflict graph. Let $\mathcal{M}$ be the message set of interest. The message conflict graph with respect to the message set $\mathcal{M}$ can be represented by the induced subgraph $\mathcal{G}_e^2[\mathcal{M}]$. The chordal bipartite, weakly chordal, and perfect properties are also inherited by the induced subgraphs [21].*

Given a perfect graph, we have the following property from polyhedral combinatorics.

Lemma 3 (From [28], [22, Chapter 65])**.** *If $\mathcal{H}$ is a perfect graph, the polytope defined by its clique inequalities has integral corner points, i.e., the coordinates of all corner points are integer-valued.*

Let us also recall the definition of a Demand Graph [1], [16] in both TIM and index coding problems.

Definition 10 (Demand Graph)**.** *The demand graph is a directed bipartite graph with messages on one side and destinations on the other, with a directed edge from a message to a destination if and only if this message is intended for this destination, and with a directed edge from a destination to a message if and only if the source from which this message originates is not connected to the destination in the network topology graph.*

Lemma 4 (From [1], [8], [16])**.** *For index coding, the sum-capacity of a set of messages that form an acyclic demand graph is upper bounded by 1.*

For a chordal bipartite network, an acyclic demand graph connects to a clique in $\mathcal{G}_e^2$, as stated in the following lemma. The proof is presented in Section V-D.

Lemma 5. *If $\mathcal{G}$ is chordal bipartite, then for each clique in $\mathcal{G}_e^2$, the associated messages form an acyclic demand graph.*

B. Proofs of Theorem 1 and Theorem 2

As Theorem 1 and Theorem 2 have similar forms, we present their proofs simultaneously.

Sufficiency: We prove that if a network is chordal bipartite, then TDMA (fractional coloring) achieves the all-unicast DoF (capacity) region.

First consider the outer bound. It suffices to prove the outer bound of the index coding problem (Theorem 2) because the capacity region of an index coding instance is an outer bound on the DoF region for the associated TIM instance [1]. Therefore the outer bound of the TIM problem (Theorem 1) is implied by that of the index coding problem.

We now prove the outer bound of the index coding problem. As Lemma 5 shows that each clique in $\mathcal{G}_e^2$ corresponds to an acyclic demand graph, and by Lemma 4 the sum capacity of associated messages in an acyclic demand graph is upper bounded by 1, we obtain the following outer bounds.

$$\forall\ Q \in \mathcal{Q}, \quad \textstyle\sum_{W_{ji} \in Q} R_{ji} \leq 1,$$

As mentioned in Section III-A, the above inequalities are called the clique inequalities [22].

Next we proceed to the achievability. It suffices to prove the achievability for the TIM problem (Theorem 1) because any achievable DoF tuple of a TIM instance translates to the same achievable rate tuple of the associated index coding instance [1]. Therefore the achievability of the index coding problem (Theorem 2) is implied by that of the TIM problem. We now prove the achievability of the TIM problem.

We show that the following outer bound DoF region is achievable by TDMA.

$$\forall\ Q \in \mathcal{Q}, \quad \textstyle\sum_{W_{ji} \in Q} d_{ji} \leq 1$$

To this end, we show that the outer bound region has integral corner points, meaning that each coordinate of the DoF tuples of the corner points is either 0 or 1. Note that the linear clique inequalities together with nonnegative constraints of the DoF tuple yield a convex polytope. The corner points of this polytope have integral (especially binary-valued) coordinates. If we treat the coordinate value 0 as switch-off of the corresponding message, and 1 as switch-on, each corner point corresponds to an on-off setting of messages. From the definition of the message conflict graph it follows that the switched-on messages belong to a set of orthogonal messages, such that they can be scheduled simultaneously over one time slot. Moreover, since each corner point of the outer bound region can be achieved by one shot scheduling, time sharing between these corner points can achieve the whole region, and the overall scheme is TDMA.

We are left to prove that the outer bound region has integral corner points. Since $\mathcal{G}_e^2$ is a perfect graph, by Lemma 3, it is known that the DoF region (polytope) defined by the clique inequalities has integral corner points.

The sufficiency proof is complete.

Necessity: The necessity proofs for Theorem 1 and Theorem 2 are identical, i.e., we only need to show that there exists at least one unicast message setting for which TDMA (fractional coloring) is suboptimal, when the network topology is not chordal bipartite. Specifically, we want to show that if a network is not chordal bipartite (contains chordless cycles with length $2n, n = 3, 4, \ldots$), then TDMA (fractional coloring) does not achieve the DoF (capacity) region for at least one message setting. Let us start with the TIM setting (Theorem 1).

Suppose now the network topology graph contains a chordless cycle with $2n$ vertices, which consist of n distinct source nodes, labeled $S'_1, \ldots, S'_n$ and n distinct destination nodes, labeled $D'_1, \ldots, D'_n$. We consider the sub-network induced by these nodes. As the cycle is chordless, the sub-network topology is cyclic where each Source $S'_i, i \in \{1, \ldots, n\}$ is connected to two destinations D'_{i-1}, D'_i and each Destination D'_i is connected to two sources S'_i, S'_{i+1} (source/destination indices are interpreted modulo-n, i.e., $n+1 = 1, 0 = n$). An example with $n = 4$ is shown in Fig. 2. To prove the desired claim, it suffices to find a DoF tuple that is achievable, thus inside the DoF region, but cannot be achieved by TDMA.

We consider the cases where n is odd or even separately.

When n is odd, we consider the interference channel message setting, i.e., there are n desired messages in the sub-network, one each from Source S'_i to Destination D'_i. As each destination only suffers interference from one non-desired source, CDMA (i.e., multicast) can achieve DoF tuple $(1/2, 1/2, \ldots, 1/2)$ with n elements [1]. This is because over two channel uses, each destination sees two independent linear equations in the two symbols (one desired and one interfering) that it is able to hear, from which it can resolve both. However, when n is odd, TDMA can achieve sum DoF at most $(n-1)/2$, which is strictly less than $n/2$, so that TDMA is unable to achieve this DoF tuple. This is due to the structure of the message conflict graph for odd cycles.

When n is even, we consider the all-unicast message setting and the DoF tuple $(1/3, 1/3, \ldots, 1/3)$ with $2n$ elements, which is achievable by interference alignment [1], but not by TDMA. TDMA cannot schedule more than $n/2$ (which is strictly less than $2n/3$) messages at the same time, e.g., if we schedule the message from S'_i to D'_i, then messages from S'_i to D'_{i-1} and from S'_{i+1} to D'_i cannot be scheduled. This is because they are either originated from the same source or intended for the same destination, and therefore form a clique in the message conflict graph. Because of the conflicts between them, if we schedule one of these three messages, we cannot simultaneously schedule the other two messages. Therefore, according to the message conflict graph, at most $n/2$ messages in total can be scheduled in one single time slot. Thus, TDMA is again sub-optimal.

This completes the necessity proof for the TIM setting (Theorem 1). We now consider the necessity proof for the index coding setting (Theorem 2). The same proof applies by noting that the above cases only use linear schemes and index coding and TIM are equivalent under linear schemes [1].

Remark 5. *The prerequisite of the necessity is under the all-unicast setting, where all messages are taken into account. There exist some topologies that are not chordal bipartite, and TDMA still achieve the DoF region for a particular subset of messages. This does not contradict the necessity.*

Remark 6. *From the DoF region, it is not hard to verify that symmetric DoF value is given by $1/\chi(\mathcal{G}_e^2)$, where $\chi(\mathcal{G}_e^2)$ is the chromatic number of $\mathcal{G}_e^2$, and sum DoF value is given by the independence set number of $\mathcal{G}_e^2$, i.e., $\alpha(\mathcal{G}_e^2)$.*

C. Proof of Lemma 1

First, $\mathcal{G}_e^2$ and the message conflict graph of $\mathcal{G}$ have the same vertex set. In the message conflict graph, there is a vertex for each message in $\mathcal{G}$. In $\mathcal{G}_e^2$, there is a vertex for each edge in $\mathcal{G}$ and in the all-unicast message setting, each edge in $\mathcal{G}$ corresponds to a message. Thus the claim follows.

Second, we prove $\mathcal{G}_e^2$ and the message conflict graph of $\mathcal{G}$ have the same edge set. In the message conflict graph, two messages (vertices) $W_{ji}, W_{j'i'}$ are connected if and only if they originate from the same source ($i = i'$), or are intended for the same destination ($j = j'$), or one source interferes with the other destination ($a_{j'i} = 0 (t_{j'i} = 1)$ or $a_{ji'} = 0 (t_{ji'} = 1)$). When $i = i'$ or $j = j'$, the two edges representing $W_{ji}, W_{j'i'}$ in $\mathcal{G}$ share a common vertex such that these two messages (vertices) are connected in $\mathcal{G}_e$ (have distance 1, thus connected in $\mathcal{G}_e^2$). When $a_{j'i} = 0 (t_{j'i} = 1)$ or $a_{ji'} = 0 (t_{ji'} = 1)$, the two edges representing messages $W_{ji}, W_{j'i'}$ in $\mathcal{G}$ both connect to the edge representing message $W_{j'i}$ or $W_{ji'}$ such that these two messages (vertices) are both connected to message (vertex) $W_{j'i}$ or $W_{ji'}$ in $\mathcal{G}_e$ (have distance 2, thus connected in $\mathcal{G}_e^2$). Conversely, whenever an edge exists between two messages (vertices) in $\mathcal{G}_e^2$ (have distance 1 or 2 in $\mathcal{G}_e$), the messages conflict in $\mathcal{G}$ (thus connected in the message conflict graph). Then we have the desired claim.

Therefore, $\mathcal{G}_e^2$ is the message conflict graph of $\mathcal{G}$. Note that this result holds regardless of whether $\mathcal{G}$ is chordal bipartite or not.

D. Proof of Lemma 5

We show that if a set of messages forms a clique in $\mathcal{G}_e^2$ (they mutually conflict), the demand graph formed by these messages and their desired destinations must be acyclic, given $\mathcal{G}$ is chordal bipartite. To set up a proof by contradiction, suppose a set of messages forms a clique in $\mathcal{G}_e^2$ and there exists a directed cycle (di-cycle) in the induced demand graph comprised of these messages and their desired destinations. Let $\mathcal{G}_c$ be the shortest such di-cycle. $\mathcal{G}_c$ must be a chordless di-cycle because if $\mathcal{G}_c$ contains a chord, then the chord splits $\mathcal{G}_c$ into two cycles. One of the two cycles is a di-cycle and it is shorter than $\mathcal{G}_c$, contradicting the assumption that $\mathcal{G}_c$ is the shortest. The length of $\mathcal{G}_c$ is denoted as k.

We argue that $\mathcal{G}_c$ cannot contain vertices corresponding to two or more messages that originate at the same source node. To see this, let us assume the opposite, that $\mathcal{G}_c$ contains vertices corresponding to two messages (denoted as W'_i, W'_j) that originate from the same source (denoted as S^*). Since $\mathcal{G}_c$ is a di-cycle, there must be an arc from a destination in $\mathcal{G}_c$ (say D'_l) to W'_i. From the definition of the demand graph, we know that D'_l is not connected to S^* in the network topology graph. Since W'_j also originates at S^*, there must be an arc from D'_l to W'_j in the demand graph as well. Thus, we have two outgoing arcs in $\mathcal{G}_c$ from D'_l, to W'_i and W'_j. But this is not possible because a node in a chordless di-cycle cannot have two outgoing arcs. Therefore, we have proved that $\mathcal{G}_c$ cannot contain vertices corresponding to two or more messages that originate at the same source node. Similarly, $\mathcal{G}_c$ cannot contain vertices corresponding to two or more messages that are intended for the same destination node.

Next we proceed to show the contradiction, that the demand graph formed by a set of messages that forms a clique in $\mathcal{G}_e^2$ can not contain a di-cycle. First, because the demand graph is bipartite, k must be even such that $k = 2n, n \in \mathbb{N}$.

Second, $n \neq 1$ because such a length-2 di-cycle in the demand graph means that a destination wants a message and is not connected to the source that emits the message in the network topology graph, which is not possible.

Third, $n \neq 2$, because otherwise $\mathcal{G}_c$ contains two messages, W'_i, W'_j (which originate at sources S'_i and S'_j), and their desired destinations, D'_i, D'_j, respectively, such that in the network topology graph D'_i is not connected to S'_j, and D'_j is not connected to S'_i. Therefore, messages W'_i, W'_j do not conflict. This contradicts the assumption that these two messages form a clique in $\mathcal{G}_e^2$.

Finally, we consider $n = 3, 4, \ldots$. Since the sources from which the messages originate are distinct, let us replace each message node in $\mathcal{G}_c$ with the source node at which it originates. With this substitution, $\mathcal{G}_c$ is made up of n sources, denoted as $S'_1, \cdots, S'_n$ and n destinations, denoted as $D'_1, \cdots, D'_n$. Without loss of generality, let us assume that in this chordless di-cycle, there is an arc from S'_i to $D'_i, i \in \{1, \cdots, n\}$ and an arc from D'_i to S'_{i+1}, where the indices are interpreted modulo-n. As the di-cycle is chordless, it contains only these $2n$ arcs. Now compare $\mathcal{G}_c$ with the (undirected) network topology graph (denoted as $\mathcal{G}^*$) induced by sources $S'_1, \cdots, S'_n$ and destinations $D'_1, \cdots, D'_n$. In $\mathcal{G}^*$, direct links (edges between S'_i and D'_i) remain and cross links are the complements of those in $\mathcal{G}_c$, i.e., D'_i is connected to $S'_{i+2}, S'_{i+3}, \cdots, S'_{i+N-1}$. This is because there is an arc from Destination D'_i to message $W'_j, j \neq i$ in the demand graph if and only if D'_i is not connected to Source S'_j that emits W'_j in $\mathcal{G}^*$. $\mathcal{G}^*$ contains a cycle of length 6, $S'_1 - D'_1 - S'_3 - D'_n - S'_2 - D'_2 - S'_1$. This cycle is chordless because D'_1 is not connected to S'_2; D'_n is not connected to S'_1; and D'_2 is not connected to S'_3 in the network topology graph. This contradicts the assumption that the overall network topology graph $\mathcal{G}$ (which contains $\mathcal{G}^*$ as a subgraph) does not contain chordless cycles with length 6 or more.

Therefore, for chordal bipartite networks $\mathcal{G}$, whenever we have a clique in $\mathcal{G}_e^2$, the associated messages form an acyclic demand graph. The proof is complete.

Remark 7. *Note that a clique in $\mathcal{G}_e^2$ may not correspond to an acyclic demand graph if $\mathcal{G}$ is not chordal bipartite. For a counterexample, consider a cyclic network with 3 sources, labeled S'_1, S'_2, S'_3 and 3 destinations, labeled D'_1, D'_2, D'_3. In the network topology graph, S'_1 is connected to D'_3, D'_1, S'_2 is connected to D'_1, D'_2, and S'_3 is connected to D'_2, D'_3. The network topology graph is not chordal bipartite as all vertices form a length-6 chordless cycle. Consider the messages $W'_{11}, W'_{22}, W'_{33}$. They mutually conflict and form a clique in $\mathcal{G}_e^2$. But the demand graph formed by them is not acyclic, and the sum DoF is not upper bounded by 1. In fact, CDMA can achieve DoF $1/2$ per message [1], as each receiver overhears one interference in addition to its desired signal.*

VI. Discussion

A. One-dimensional Convex Networks

In a one-dimensional network, the source and destination nodes are placed along a straight line. We define the relation $a < b$ between two nodes to indicate that node a is "to the left of" node b.

Definition 11 (Source Convexity)**.** *Source convexity refers to the property that if a source (say S_i) can be heard by two* *destination nodes (say D_j, D_k), then it must also be heard by all other destination nodes that are in between, i.e., $(D_j < D_l < D_k)$ AND $(t_{ki} = t_{ji} = 1) \Rightarrow t_{li} = 1$.*

Definition 12 (Destination Convexity)**.** *Destination convexity refers to the property that if a destination (say D_i) can hear two source nodes (say S_j, S_k), then it must also hear all other source nodes that are in between, i.e., $(S_j < S_l < S_k)$ AND $(t_{ij} = t_{ik} = 1) \Rightarrow t_{il} = 1$.*

Corollary 1. *One-dimensional network topology graphs with either source convexity or destination convexity (not necessarily both) are chordal bipartite, and thus TDMA achieves the all-unicast DoF region of these TIM instances.*

Proof. We only need to prove that one-dimensional network topology is chordal bipartite, by which the optimality of TDMA follows straightforwardly due to Theorem 1. Since the name of source or destination is entirely cosmetic, we consider a one-dimensional TIM instance with only source convexity, without loss of generality. Similar proof applies to cases with only destination convexity as well.

To set up a proof by contradiction, suppose the TIM instance is not chordal bipartite, i.e., its network topology graph contains a chordless cycle with length $2n, n = 3, 4, \ldots$, which corresponds to a cyclic sub-network with n distinct sources, labeled $S'_1, \ldots, S'_n$ and n distinct destinations, labeled $D'_1, \ldots, D'_n$. Source $S'_i, i \in \{1, \ldots, n\}$ is connected to two destinations D'_{i-1}, D'_i and Destination D'_i is connected to two sources S'_i, S'_{i+1} (source/destination indices are interpreted modulo-n, i.e., $n+1 = 1, 0 = n$). As the cycle is chordless, each source is connected to only two destinations. Further, because of source convexity, the destinations that are connected to the same source must be consecutive. For example, as S'_2 is connected to only D'_1 and D'_2, there cannot be any destination in the interval between D'_1 and D'_2. Similarly, there is no destination in between D'_i and D'_{i+1}, such that the order of the destinations in one straight line must appear as $D'_1 < D'_2 < \cdots < D'_n$ or $D'_1 > D'_2 > \cdots > D'_n$. In both cases, D'_1 and D'_n are not consecutive. We arrive at a contradiction. □

B. Coherence Time

We show that Theorem 1 holds regardless of channel coherence time. We assume that the network topology remains unchanged during the period of communication regardless of channel coherence time. Channel coefficient values may change even as the topology remains fixed. Coherence time refers to the change in channel coefficient values. The channel coefficient values can change with every channel use or remain fixed for arbitrary intervals as long as the network topology is unchanged.

Corollary 2. *TDMA achieves the all-unicast DoF region of the TIM problem if and only if the network is chordal bipartite, regardless of channel coherence time.*

Proof. We now do not require the channel coefficients $h_{ji}(t)$ to be constant. Instead, $h_{ji}(t)$ can vary in an arbitrary manner as long as the values are bounded away from zero and infinity, i.e., there is no requirement on the channel coherence time. Bounding the coefficients away from zero and infinity maintains a fixed network topology during the entire period of communication.

Sufficiency: We prove that if a TIM network is chordal bipartite, then TDMA achieves the all-unicast DoF region, regardless of channel coherence time.

The outer bounds provided in Section V-B hold regardless of channel coherence time. Also, TDMA scheme applies to both constant and time-varying channel setting, such that the achievability proof in Section V-B is not affected. Hence sufficiency is proved.

Necessity: We prove that if a TIM network is not chordal bipartite, then TDMA does not achieve the all-unicast DoF region, regardless of channel coherence time.

We follow the proof in Section V-B. Suppose the network topology graph is not chordal bipartite, such that it contains a chordless length-$2n$ cycle, $n = 3, 4, \ldots$. Such a chordless cycle corresponds to a cyclic sub-network. We show that TDMA can not achieve the DoF region for such a cyclic sub-network (and therefore the all-unicast DoF region).

We also consider the cases where n is odd or even separately.

When n is odd, we use the same interference channel message setting and consider DoF tuple $(1/2, 1/2, \ldots, 1/2)$. As CDMA can be applied to time-varying channels, the DoF tuple is still achievable, regardless of channel coherence time. However, TDMA can not schedule more than $(n-1)/2$ messages over one time slot, and is therefore unable to achieve this DoF tuple.

When n is even, we still consider the all-unicast message setting. In this case, the DoF tuple $(1/3, 1/3, \ldots, 1/3)$ does not work as the interference alignment scheme used to achieve this tuple requires the channels to be constant for 3 symbol periods [1]. Instead, we consider the sum-DoF value. As shown in Section V-B, TDMA can not achieve more than $n/2$ sum-DoF. However, the scheme in [29] can achieve $(n+1)/2$ sum-DoF, regardless of channel coherence time. See Fig. 9 in [29] for a pictorial illustration for $n = 4$ case. Thus, TDMA is again sub-optimal, regardless of channel coherence time. This completes the necessity proof. □

VII. Conclusion

We show that the necessary and sufficient condition for TDMA (fractional coloring) to achieve the all-unicast DoF (capacity) region of TIM (index coding) is that the network topology should be chordal bipartite. The absence/presence of chordless cycles prevents/creates opportunities for more sophisticated achievable schemes. Among other interesting observations, we note that for chordal bipartite networks while fractional coloring is needed to achieve the capacity region, the conventional (non-fractional) coloring suffices to achieve all the corner points, i.e., the extreme points of the capacity region polytope. Finally, in the TIM context, the main results hold regardless of channel coherence time. Potential directions for future work include exploring optimality of TDMA for other variants of TIM such as multilevel TIM [30], cellular TIM [31], MIMO TIM [32], TIM with transmitter cooperation

[33], with decoded message passing [34], with alternating connectivity [35]–[37], with reconfigurable antennas [38], and under constrained coherence patterns [39]–[41].

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