Optimality of Treating Interference as Noise: A Combinatorial Perspective

Xinping Yi, *Member, IEEE* and Giuseppe Caire, *Fellow, IEEE*

***Abstract*—For single-antenna Gaussian interference channels, we reformulate the problem of determining the Generalized Degrees of Freedom (GDoF) region achievable by treating interference as Gaussian noise (TIN) derived in [3] from a combinatorial optimization perspective. We show that the TIN power control problem can be cast into an assignment problem, such that the globally optimal power allocation variables can be obtained by well-known polynomial time algorithms (e.g., centralized Hungarian method or distributed Auction algorithm). Furthermore, the expression of the TIN-Achievable GDoF region (TINA region) can be substantially simplified with the aid of maximum weighted matchings. We also provide conditions under which the TINA region is a convex polytope that relax those in [3]. For these new conditions, together with a channel connectivity (i.e., interference topology) condition, we show TIN optimality for a new class of interference networks that is not included, nor includes, the class found in [3].**

Building on the above insights, we consider the problem of joint link scheduling and power control in wireless networks, which has been widely studied as a basic physical layer mechanism for device-to-device (D2D) communications. Inspired by the relaxed TIN channel strength condition as well as the assignment-based power allocation, we propose a low-complexity GDoF-based distributed link scheduling and power control mechanism (ITLinQ+) that improves upon the ITLinQ scheme proposed in [4] and further improves over the heuristic approach known as FlashLinQ. It is demonstrated by simulation that ITLinQ+ without power control provides significant average network throughput gains over both ITLinQ and FlashLinQ, and yet still maintains the same level of implementation complexity. Furthermore, when ITLinQ+ is augmented by power control, it provides an energy efficiency substantially larger than that of ITLinQ and FlashLinQ, at the cost of additional complexity and some signaling overhead.

***Index Terms*—Gaussian Interference Channels, Treating Interference as Noise, Generalized Degrees of Freedom, Power Control, Device-to-Device Communications.**

I. Introduction

Power control and treating interference as Gaussian noise (TIN) is one of the most well-known, vastly employed, and yet most attractive interference management techniques, due to its low complexity, robustness to channel uncertainty, and to the fact that codes for the single-user Gaussian channel are well understood and efficiently implemented. Interestingly, it has also been shown that in some cases TIN is optimal or approximately optimal. For example, we know that TIN achieves the sum-capacity in the noisy regime of the two-user Gaussian interference channel [5]–[7]. In the general K-user single-antenna Gaussian interference channel, Geng *et. al* [3] have shown that, subject to certain conditions on the channel strengths, TIN achieves the optimal *Generalized Degrees of Freedom* (GDoF) region, and achieves the capacity region to within a constant gap, independent of the channel coefficients and the signal-to-noise ratio (SNR). The TIN optimality condition found in [3] is simply expressed in words as the fact that, for each user (i.e., intended transmitter-receiver pair) the desired signal strength level is no less than the sum of maximum strengths of all interfering signals from the transmitter to the other (unintended) receivers, and to the receiver from the other (unintended) transmitters, when all signal strengths are expressed in log-scale (e.g., in dB). For future reference, we indicate this condition as the "GNAJ" condition, from the initials of the authors of [3]. Under the GNAJ condition, the TIN-Achievable GDoF region (briefly referred to as "TINA region") is a convex polytope defined by the individual GDoF constraints and by the sum-GDoF inequalities corresponding to all possible ordered subsets of users. With the aid of a combinatorial tool named *potential graphs*, the K-user TINA region was characterized in [3] by $\sum_{m=2}^{K}\binom{K}{m}(m-1)! \approx (K-1)!$ constraints. More recently, it has been also shown by Sun and Jafar in [8] that, by a series of transformations of linear programs, the sum-GDoF characterization can be translated into a minimum weighted matching problem in combinatorial optimization. As such, the sum-GDoF under the GNAJ condition can be characterized as disjoint cycles partition of the interference network.

Such remarkable findings have inspired various related works, such as the TIN optimality of general X-channels [9], parallel interference networks [8], and compound interference networks [10]. In general, the TIN problem consists of two subproblems. Beyond the TINA region characterization, it is also important to find efficient methods to solve the TIN power control problem, that is, finding the (minimum) transmit powers that achieve a certain desired GDoF-tuple in the TINA region. The TIN power control problem has been open for a long time until a recent progress reported by Geng and Jafar in [10], where a simple yet elegant polynomial-time centralized iterative algorithm to find the globally optimal power allocation variables is provided. This centralized algorithm relies still on the representation by

Manuscript received September 16, 2015; revised April 3, 2016; accepted June 4, 2016. This work was presented in part at IEEE Int. Symp. Information Theory 2015, Hong Kong, China [1], and Asilomar Conference on Signals, Systems and Computers 2015, Pacific Grove, CA [2].

X. Yi and G. Caire are with Communication and Information Theory Chair in Department of Electrical Engineering and Computer Science at Technische Universität Berlin, 10587 Berlin, Germany (email: {xinping.yi, caire}@tu-berlin.de). The work of Giuseppe Caire is partially supported by an Alexander von Humboldt Professorship grant.

Communicated by S.-Y. Chung, Associate Editor for Shannon Theory.

potential graphs.

One may wonder if the potential graph representation is the only path to both TINA region characterization and TIN power control problems. Further, due to the distributed nature of interference channels, decentralized power allocation algorithms are more interesting, desirable and yet challenging. In addition, it is worth noting that the GNAJ condition was only proven to be sufficient. An interesting counter-example in [3] showed that there exist partially-connected (in the sense of channel strength levels) interference channels, such that TIN achieves the optimal GDoF region and yet the GNAJ condition is not satisfied. A natural question then arises as to whether there exists a larger class of networks, including partially-connected ones, such that TIN is GDoF-optimal (i.e., TIN with power control still achieves the optimal GDoF region of the channel). These questions motivate this work.

In this paper, the optimality of TIN is revisited. The TIN optimality problem was formulated in [3] by first eliminating power allocation variables using the potential theorem [11], to establish the TINA region in terms of GDoF variables only, and then by finding the optimal power allocation variables for a given GDoF-tuple in the TINA region [10]. In contrast, we reformulate this problem in a reversed way, from a combinatorial optimization perspective [11]. Interestingly, by first casting power allocation into an assignment problem, the globally optimal power allocation variables corresponding to any feasible GDoF tuple in the TINA region can be found by solving the equivalent assignment problem in polynomial time, either in a centralized manner (e.g., Hungarian method [12], [13]) or in a distributed one (e.g., Auction algorithm [14]). Inspired by the duality between the assignment and the maximum weighted matching problems in combinatorial optimization [15], we can express the TINA region characterization in terms of a maximum weighted matching problem. In doing so, the TINA region is significantly simplified, requiring only $2^K - 1$ constraints instead of $\approx (K-1)!$. Interestingly, such a representation also offers an interpretation of the disjoint cycle partition in [8]. By this new formulation, we show that the TINA region is a convex polytope under a novel channel strength condition that relaxes the GNAJ condition in [3]. This new condition requires that the desired signal strength of each user is no less than the *maximum difference* between the sum strength of any pair of incoming/outgoing interference signals and the strength of the link between such a pair (all in dB scale). Furthermore, together with a connectivity condition, we are able to establish the optimality of TINA region for a new class of networks. Such conditions are not included nor include the GNAJ condition [3].

Beyond the theoretical interest of characterizing the TINA region, we are also interested in translating these results into practical system optimization algorithms. Device-to-Device (D2D) communication is expected to play an important role in future wireless communication systems (e.g., 5G), including applications such as car-to-car, machine-to-machine, proximity-based services, and multi-hop infrastructureless mesh networks. The physical layer of D2D systems is usually modeled as a Gaussian interference channel. Under the practical constraint of treating interference as Gaussian noise for the sake of complexity and robustness,[1] a long-standing problem consists of controlling the power of the D2D links (transmit-receive pairs) in order to maximize the overall network throughput.[2] The usual approach of guaranteeing a target signal-to-interference-plus-noise ratio (SINR) to each link turns out to yield an operating point that can be arbitrarily far from optimal. This is because some bottleneck links may impose too stringent constraints to the overall network. In contrast, much better network throughput can be achieved by selecting a subset of active links in each slot and allocating positive power only to these selected links [4], [18], [19]. By scheduling the subsets of active links over time, it is possible to achieve individual throughputs such that some *network utility function* is maximized. In turn, the shape of the network utility function determines the desired fairness criterion (e.g., see [20], [21]). Link selection and scheduling has become the subject of intensive research. This problem is closely related to power control, since link selection corresponds to allocating either zero or positive power to the transmitters. For a general D2D network, this problem is non-convex and, as a matter of fact, has a combinatorial nature. For example, a well-known power control method consists of replacing $\log(1+\mathrm{SINR})$ with $\log(\mathrm{SINR})$ in the user rate expression, and using Geometric Programming (GP) [22]. However, by neglecting the "1+" inside the "$\log$" one has implicitly forced all links to use positive power, since assigning zero power to some links would drive the GP objective function to $-\infty$. Instead, it is known that generally much better solutions can be found by first selecting a "good" subset of active links, and then allocating (positive) power only to the selected links.

Various schemes for link selection have been proposed in the literature, e.g., [4], [18], [23]–[25] to name a few. For example, a large number of works is based on constructing an interference conflict graph [26], and then selecting maximal independent sets. These "maximal independent set scheduling" schemes are flawed by a fundamentally arbitrary choice of the threshold according to which two links are considered to be in conflict. Recently, a distributed link scheduling mechanism called FlashLinQ was proposed in [18]. Compared to "maximal independent set scheduling" schemes, FlashLinQ takes both signal and interference strength into account. In FlashLinQ, links are ranked in priority order and considered one by one. A candidate link is scheduled if it does not cause/receive too much interference to/from links of higher priority that have already been selected (i.e., declared active). It is also possible to enforce fairness among the links by changing the priority order at each scheduling slot, such that each link with a certain probability will be given the highest priority. More recently, inspired by the GNAJ condition in [3], the authors in [4] proposed a new distributed link scheduling mechanism (referred to as "ITLinQ") that provides sum throughput gains over FlashLinQ and yet maintains the same level of low-complexity. Instead of

[1] From [16] we know that this condition is essentially equivalent to imposing the use of minimum distance decoding at each receiver.

[2] Consistently with [17], we use the term "throughput" to indicate the time-averaged rate over a long sequence of scheduling time slots. In contrast, the instantaneous rate is the rate achieved in a single slot, for a given set of active users, i.e., links with positive transmit power.

comparing the ratio of signal to interference strength of each new candidate link with a fixed threshold as in FlashLinQ, ITLinQ compares the interference level caused to/received from existing links with *an appropriately chosen exponent* of the signal strength of the new candidate link. It was verified by simulation in [4] that ITLinQ outperforms FlashLinQ with 28%-110% gains for a scenario where up to 4096 links can be scheduled.

As a matter of fact, for general channel strength coefficients, the maximal subset of links satisfying the GNAJ condition may not lead to the maximal (weighted) sum throughput or sum-GDoF. As will be demonstrated later, our relaxed channel strength conditions provide a larger convex polytope TINA region. This provides a generally larger subset of links on which power control can be applied, resulting in generally higher weighted sum-GDoF. As a consequence, we are able to design a new distributed link scheduling and power control mechanism (named "ITLinQ+"), further fine-tuning the decision criterion of link selection. It is demonstrated by simulation that, when only link selection with constant power transmission for the selected links is used, ITLinQ+ gains 5%-20% average sum throughput improvement over ITLinQ with 1024 links, at the expense of slightly increased signaling overhead. A problem with both FlashLinQ and ITLinQ is that they do not consider power control and simply enforce constant transmit power for all the selected links. In contrast, thanks to the new understanding of the TINA region developed in this paper, we can augment ITLinQ+ with also a power control mechanism able to find the minimum power vector supporting the GDoF tuple corresponding to the selected links. When ITLinQ+ is applied together with this power control mechanism, the achieved average sum throughput is further enhanced. Most notably, the energy efficiency of ITLinQ+ can be substantially improved (e.g., 30 dB power saving to achieve 60 bit/s/Hz sum throughput for a 16-user D2D network), at the cost of additional computational complexity and some signaling overhead. In short, ITLinQ+ improves the sum throughput performance and yet requires much less energy consumption, which is desirable for battery-powered D2D communications. Notice that achieving better or equal throughput with less energy consumption is not a contradiction here, since the network is operated in an interference limited regime, such that rate is not immediately and obviously correlated to transmit power.

This paper is organized as follows. In the next section, we present the system model of the general K-user Gaussian interference channels, followed by a summary of the main existing results of the approximate optimality of treating interference as Gaussian noise. In Section III, we reformulate the TIN problem from a combinatorial optimization perspective and we obtain a simplified description of the TINA region. By the simplified TINA region, we are able to identify a relaxed channel strength condition under which the general TINA region is a convex polytope. In Section IV, we consider the GDoF-based link scheduling and power control problem, offering a framework in this regard. Driven by this framework, the new decentralized link scheduling and power control mechanism named ITLinQ+ is proposed in Section V with detailed implementations. Section VI provides numerical results and comparisons with ITLinQ and FlashLinQ for some scenarios of D2D networks. We conclude the paper in Section VII.

Notation: Throughout this paper, we define $\mathcal{K} \triangleq \{1, 2, \ldots, K\}$. Let A, $\mathcal{A}$, and $\mathbf{A}$ represent a variable, a set, and a matrix, respectively. In addition, $\mathcal{A}^c$ is the complementary set of $\mathcal{A}$, and $|\mathcal{A}|$ is the cardinality of the set $\mathcal{A}$. $\mathbf{A}_{ij}$ presents the ij-th entry of the matrix $\mathbf{A}$, and $\mathbf{A}_i$ is the i-th row of $\mathbf{A}$. $A_{\mathcal{S}} \triangleq \{A_i, i \in \mathcal{S}\}$, and $\mathcal{A}_{\mathcal{S}} \triangleq \cup_{i\in\mathcal{S}} \mathcal{A}_i$. Define $\mathcal{A}\backslash\{a\} \triangleq \{x | x \in \mathcal{A}, x \neq a\}$ and $\mathcal{A}_1\backslash\mathcal{A}_2 \triangleq \{x | x \in \mathcal{A}_1, x \notin \mathcal{A}_2\}$. Logarithms are in base 2. With a bit abuse of notation, $k \neq i \neq j$ means $k \neq i$, $i \neq j$ and $k \neq j$.

II. System Model

A. Channel Model

We consider a K-user interference channel where both transmitters (Tx) and receivers (Rx) are equipped with a single antenna each. We shall refer to the j-th Tx-Rx pair as the j-th *user or link*. At Rx-j ($\forall j \in \mathcal{K} \triangleq \{1, \ldots, K\}$), the received signal at the discrete-time instant t is given by

$$Y_j(t) = \sum_{i=1}^{K} h_{ij} \tilde{X}_i(t) + Z_j(t) \tag{1}$$

where $\tilde{X}_i(t)$ is the transmitted signal from Tx-i with power constraint $\mathbb{E}\left(|\tilde{X}_i(t)|^2\right) \leq P_i$, h_{ij} is the channel coefficient between Tx-i and Rx-j, $Z_j(t) \sim \mathcal{CN}(0,1)$ is the (normalized) additive white Gaussian noise at Rx-j. Following [3], we translate the signal model in (1) into an equivalent GDoF-friendly form, given by

$$Y_j(t) = \sum_{i=1}^{K} \sqrt{P^{\alpha_{ij}}} e^{j\theta_{ij}} X_i(t) + Z_j(t) \tag{2}$$

where $X_i(t) = \frac{\tilde{X}_i(t)}{\sqrt{P_i}}$ is the normalized transmitted signal with power constraint $\mathbb{E}\left(|X_i(t)|^2\right) \leq 1$, $\sqrt{P^{\alpha_{ij}}}$ and θ_{ij} are magnitude and phase of the channel coefficient between Tx-i and Rx-j, respectively, and the exponent α_{ij} is defined as the corresponding channel strength level

$$\alpha_{ij} = \frac{\log(\max\{1, |h_{ij}|^2 P_i\})}{\log P} \tag{3}$$

where $P > 1$ is the average power. Given a transmit power adjustment P^{r_i} with $r_i \leq 0$, by which the actual transmit power is $P^{r_i} P_i$ at Tx-i, the signal-to-interference-plus-noise ratio (SINR) achieved by TIN at Rx-j is given by $\frac{P^{\alpha_{jj}+r_j}}{1+\sum_{i:i\neq j} P^{\alpha_{ij}+r_i}}$. We assume that the transmitters know the channel strength levels perfectly for power control, and the receivers have access to both the magnitude and phase of channel coefficients.

B. Treating Interference as Noise

We follow standard definitions for encoding/decoding functions and achievable rates. The individual achievable GDoF of message W_k is defined as $d_k \triangleq \lim_{P\to\infty} \frac{R_k}{\log P}$ where R_k is the achievable rate for message W_k, associated to the k-th link. The GDoF region is the collection of all achievable GDoF-tuples $(d_1, d_2, \ldots, d_K)$. The TIN-Achievable GDoF (TINA) region defined in [3] is the set of all K-tuples $\mathbf{d} = (d_1, d_2, \ldots, d_K)$ with components satisfying

$$d_j \le \max\left\{0, \alpha_{jj} + r_j - \max\{0, \max_{i:i\neq j}(\alpha_{ij} + r_i)\}\right\}, \quad (4)$$

for some assignment of the power allocation variables $\mathbf{r} = (r_1, r_2, \ldots, r_K) \in \mathbb{R}_-^K$. In the following, we denote the TINA by $\mathcal{R}^{\mathrm{TINA}}$, where the dependence on the specific network defined by $\{\alpha_{ij} : i, j \in \mathcal{K}\}$ is clear from the context. From [3] we also know that the *polyhedral* TINA region is obtained by removing the positive part operator[3] from the right-hand side of (4). Using the potential theorem [11], the authors of [3] are able to find a convex polytope form for the polyhedral TINA region for any subnetwork formed by a subset $\mathcal{S} \subseteq \mathcal{K}$ and its associated desired and interfering links. We shall denote such a polytope by $\mathcal{P}_{\mathcal{S}}^{\mathrm{TINA}}$. Since removing the positive part in the right-hand side of (4) restricts the GDoF region, then $\mathcal{P}_{\mathcal{S}}^{\mathrm{TINA}}$ is achievable by switching off all users in $\mathcal{S}^c = \mathcal{K}\backslash\mathcal{S}$ and by using TIN for the users in $\mathcal{S}$. We also denote by $\mathcal{R}^*$ the *optimal* GDoF region of the interference network, i.e., the region of GDoF-tuples achievable over any possible coding scheme (not restricted to TIN).

The main results in [3] are summarized as below.

Theorem 1. [*GNAJ [3]*] *Consider a K-user single-antenna Gaussian interference channel with channel strengths $\{\alpha_{ij} : i, j \in \mathcal{K}\}$.*

1) *For any subnetwork formed by users in $\mathcal{S} \subseteq \mathcal{K}$, $\mathcal{P}_{\mathcal{S}}^{\mathrm{TINA}}$ can be described by*[4]

$$\begin{aligned}
&0 \le d_k \le \alpha_{kk}, \forall k \in \mathcal{S}, \quad d_i = 0, \forall i \in \mathcal{S}^c \\
&\sum_{k=0}^{m-1} d_{i_k} \le \sum_{k=0}^{m-1} (\alpha_{i_k i_k} - \alpha_{i_{[k-1]_{\mathrm{mod}\, m}} i_k}), \\
&\forall \text{ ordered subsets } (i_0, \ldots, i_{m-1}) \in \mathcal{S}, \forall\ m \in \{2, \ldots, |\mathcal{S}|\}.
\end{aligned} \quad (5)$$

2) *The TINA region of the whole network is given by*

$$\mathcal{R}^{\mathrm{TINA}} = \bigcup_{\mathcal{S}\subseteq\mathcal{K}} \mathcal{P}_{\mathcal{S}}^{\mathrm{TINA}}, \quad (6)$$

which is generally non-convex, since it is the union of convex polytopes, each one corresponding to a specific set of active users.

3) *If $\forall k \in \mathcal{K}$,*

$$\alpha_{kk} \ge \max_{i:i\neq k}\{\alpha_{ik}\} + \max_{j:j\neq k}\{\alpha_{kj}\}, \quad (7)$$

then TIN is GDoF-optimal, i.e., $\mathcal{R}^* = \mathcal{R}^{\mathrm{TINA}} = \mathcal{P}_{\mathcal{K}}^{\mathrm{TINA}}$ (the whole region is a single convex polytope).

Remark 1. *It is easy to see that, for $\mathcal{S} = \mathcal{K}$, there are in total $\sum_{m=2}^{K} \binom{K}{m}(m-1)! \approx (K-1)!$ constraints in (5). Since the sum-GDoF $\sum_{k=0}^{m-1} d_{i_k}$ does not depend on the order of the indices, for each unordered set of indices $\{i_0, \ldots, i_{m-1}\}$ there are $(m-1)!$ inequalities, of which only one is relevant. However, finding which one is relevant involves, in general, extensive search, such that finding a general more compact form that eliminates redundant inequalities is non-trivial.* $\Diamond$

[3] The positive part of x is $\max\{0, x\}$.

[4] We use the term *ordered subset* to indicate that order matters, but elements are not repeated. For example, $(1, 2, 3)$ and $(1, 3, 2)$ are two rising such subsets for $m = 3$, but $(1, 2, 2)$ is not valid, because it contains repeated elements.

III. TIN Problem Reformulation from a Combinatorial Perspective

The expression of the TINA region in (5) involves a huge number of constraints, some of which are redundant. However, it is unclear which one is unnecessary and which one is required. To make progress in this regard, we reformulate the TIN problem of [3], [10] from a combinatorial optimization perspective. By casting the power allocation into an assignment problem, we find an alternative form for the TINA region via its dual – the maximum weighted matching problem [15]. Some basic definitions of weighted matching are recalled in Appendix A.

A. Casting Power Allocation into Assignment Problems

In what follows, we consider a feasible GDoF tuple in $\mathcal{P}_{\mathcal{S}}^{\mathrm{TINA}}$ for any user set $\mathcal{S} \subseteq \mathcal{K}$, where [5]

$$d_j = \alpha_{jj} + r_j - \max\{0, \max_{i:i\neq j}(\alpha_{ij} + r_i)\},\ j \in \mathcal{S} \quad (8)$$

given power allocation parameters $\{r_j, j \in \mathcal{S}\}$. In general, a given GDoF-tuple in $\mathcal{P}_{\mathcal{S}}^{\mathrm{TINA}}$ may be achieved by different assignments of the power control variables $\{r_j : j \in \mathcal{S}\}$. The componentwise minimum configuration corresponding to a given target GDoF tuple is referred to as the *globally optimal power control assignment*. In this case, no user can reduce its transmit power while still achieving the same GDoF-tuple.

Thus, for $\mathcal{S} \subseteq \mathcal{K}$, the globally optimal power control assignment problem can be formulated as a multi-objective minimization problem:

$$\min_{\{r_j\}} \quad \{r_j,\ \forall j \in \mathcal{S}\} \quad (9a)$$

$$\text{s.t.} \quad d_j = \alpha_{jj} + r_j - \max\{0, \max_{i:i\neq j}(\alpha_{ij} + r_i)\}, \quad (9b)$$

$$d_j \ge 0,\ r_j \le 0,\ \forall j \in \mathcal{S}. \quad (9c)$$

By introducing two sets of non-negative auxiliary variables, namely, left labels $\{y_{u_j}\}$ and right labels $\{y_{v_j}\}$

$$y_{u_j} = -r_j \quad (10)$$

$$y_{v_j} = \max\{0, \max_{i:i\neq j}(\alpha_{ij} + r_i)\}, \quad (11)$$

the individual achievable GDoF can be rewritten as

$$d_j = \alpha_{jj} - (y_{u_j} + y_{v_j}). \quad (12)$$

Using also the fact that, for all $i \neq j$,

$$\begin{aligned}
y_{u_i} + y_{v_j} &= -r_i + \max\{0, \max_{i':i'\neq j}(\alpha_{i'j} + r_{i'})\} && (13a)\\
&\ge -r_i + \max_{i':i'\neq j}(\alpha_{i'j} + r_{i'}) && (13b)\\
&\ge \alpha_{ij}, && (13c)
\end{aligned}$$

[5] Note that we consider $d_i > 0,\ \forall\ i \in \mathcal{S}$. If $d_i = 0$, user pair i will be not activated and we simply remove it from $\mathcal{S}$ without affecting others.

we have the following relaxed optimization problem:

$$\max_{\{y_{u_j}, y_{v_j}\}} \{y_{u_j}, \ \forall j \in \mathcal{S}\} \tag{14a}$$
$$\text{s.t. } y_{u_j} + y_{v_j} = \alpha_{jj} - d_j, \forall j \in \mathcal{S} \tag{14b}$$
$$y_{u_i} + y_{v_j} \geq \alpha_{ij}, \forall i, j \in \mathcal{S}, i \neq j \tag{14c}$$
$$y_{u_j} \geq 0, \ y_{v_j} \geq 0, \ \ \forall j \in \mathcal{S} \tag{14d}$$

which can be tightened as long as $\max_{i:i \neq j}(\alpha_{ij} + r_i)$ is always made non-negative, such that for all $i \neq j$, the constraints in (14c) implied by the definition of $\{y_{u_i}\}$ and $\{y_{v_j}\}$ inherit the relation between r_i and r_j in (9b).

Taking a closer look at the optimization problem (14), we found that the multi-objective optimization problem can be transformed through linear scalarization [27] to a single-objective one, given by

$$\max_{\{y_{u_j}, y_{v_j}\}} \sum_{j \in \mathcal{S}} y_{u_j} \tag{15a}$$
$$\text{s.t. } (14b), (14c), (14d) \tag{15b}$$

such that the optimal solution to (15) is the Pareto optimal solution to (14). As shown in [10], the unique Pareto optimal power control solution is also globally optimal.

In addition, the equality in (14b) can be further relaxed to $y_{u_j} + y_{v_j} \geq \alpha_{jj} - d_j$ whereas is tightened back through minimizing $\sum_{j \in \mathcal{S}}(y_{u_j} + y_{v_j})$. As such, the global minimum power allocation problem in (9) can be reformulated as the following two-objective optimization problem:

$$\min_{\{y_{u_j}, y_{v_j}\}} \sum_{j \in \mathcal{S}} (y_{u_j} + y_{v_j}), \qquad \max_{\{y_{u_j}, y_{v_j}\}} \sum_{j \in \mathcal{S}} y_{u_j} \tag{16a}$$
$$\text{s.t. } y_{u_j} + y_{v_j} \geq \alpha_{jj} - d_j, \forall j \in \mathcal{S} \tag{16b}$$
$$(14c), (14d). \tag{16c}$$

Note that we are not really interested in the minimization of the first objective function (as the solution is known) but the parameters that achieve the minimum. We can further relax the optimization problem in (16) by dropping the second objective function, so that the resulting optimization problem can be recognized as a dual formulation of an assignment problem (see later development and definitions in Appendix A).

The globally optimal power allocation parameters in (9) can be found through solving such an assignment problem, and choosing among all the solutions that minimize the sum of the overall labels (i.e., $\min \sum_j (y_{u_j} + y_{v_j})$), that particular solution for which the sum of the left labels is maximum (i..e, $\max \sum_j y_{u_j}$). Such a solution is referred to as the maximum left label equilibrium. Notice that because of constraints (16b) and (16c) there is tension between left and right labels. This means that the same minimum overall sum can be achieved in many ways, and here we are interested in the maximum left label equilibrium (which coincides with the minimum right label equilibrium). Thus, the above optimization problem formulation can be summarized as follows.

Theorem 2. *For any $(d_j : j \in \mathcal{K}) \in \mathcal{P}_{\mathcal{S}}^{\text{TINA}}$, the globally optimal power control assignment $(r_j, j \in \mathcal{S})$ can be found as the maximum left label equilibrium of the following linear program:*

$$(AP): \quad \min_{\{y_{u_j}, y_{v_j}\}} \sum_{j \in \mathcal{S}} (y_{u_j} + y_{v_j}) \tag{17a}$$
$$\text{s.t. } y_{u_i} + y_{v_j} \geq \alpha_{ij}, \forall i \neq j \tag{17b}$$
$$y_{u_j} + y_{v_j} \geq \alpha_{jj} - d_j, \forall j \in \mathcal{S} \tag{17c}$$
$$y_{u_j} \geq 0, \ y_{v_j} \geq 0, \ \ \forall j \in \mathcal{S} \tag{17d}$$

where $r_j = -y_{u_j}, \ \forall j \in \mathcal{S}$.

Remark 2. *This linear program (17) can be recognized as a dual formulation of an assignment problem [15], so that the unique globally-optimal power allocation can be found in polynomial time (i.e., $O(K^3)$) using e.g., the (centralized) Hungarian method [12], [13] or the (distributed) Auction algorithm [14].* $\Diamond$

As most of the implementations of the Hungarian method are dedicated merely the first objective function in (16) (i.e., (17a)), they may not lead to the maximum left label equilibrium and thus not find the optimum power allocation in (9). Fortunately, the Kuhn-Munkres algorithm also targets the second objective function in (16) and tightens the relaxations by means of the following three ingredients: (1) The left label $y_{u_i} = -r_i$ is initialized as $\max\{\max_{j:j \neq i}(\alpha_{ij}), \alpha_{ii} - d_i\}$ and keeps decreasing, whereas the right label y_{v_j} is initialized as 0 and keeps increasing, through which $\max_{i:i \neq j}(\alpha_{ij} + r_i)$ is always made non-negative, and in turn the relaxation in (14) is tightened; (2) In each iteration, $\{y_{u_j}\}$ are made to decrease and $\{y_{v_j}\}$ are made to increase carefully step by step, in which the maximum left label equilibrium of $\{y_{u_j}\}$ is gradually explored; (3) The algorithm is terminated when the equalities in (16b) hold for all $j \in \mathcal{S}$, by which the relaxation of constraint from (14) to (16) is tightened. In other words, the algorithm terminates when the solution to the assignment problem is $\{(j, j), \ \forall j \in \mathcal{S}\}$, and outputs the maximum equilibrium of left labels $\{y^*_{u_j}\}$, which gives the global minimum power allocation $\{r_j = -y^*_{u_j}\}$. A detailed implementation of the Kuhn-Munkres algorithm with some parameters specified to fit our problem is relegated to Appendix B (see Algorithm 2). It is also worth noting that, according to the equality in (12), the optimal solution to (17) is achieved when the equality of (17c) holds. This observation is also added to Algorithm 2 as the new termination criterion. In particular, for an assignment problem with size K, the Kuhn-Munkres algorithm requires at most K rounds of iteration to converge to the optimal assignment solution. This fact can be also used to check the feasibility of GDoF tuples. If Algorithm 2 does not converge to the optimal solution for a given GDoF tuple within K iterations, then this GDoF tuple is infeasible.

It is also worthwhile to mention that a distributed Auction algorithm, originally due to Demange, Gale, and Sotomayor [14], achieves the minimum right label equilibrium, whose values are componentwise smaller than any other feasible ones, leading to the global optimality of power allocation in a decentralized manner. A detailed implementation is presented in Section V-C (see Algorithm 1).

Hereafter, we refer to the centralized Kuhn-Munkres algorithm (see Algorithm 2) and the distributed Demange-Gale-Sotomayor Auction algorithm (see Algorithm 1) as the *assignment-based* power control algorithms.

B. TINA Region Representation

In the following, starting from the power allocation solution of Theorem 2 and exploiting the duality between assignment and maximum weighted matching problems (see Appendix A), we shall reformulate the TINA region in a more useful and compact form. First, given a GDoF tuple $(d_1,\dots,d_K)$ and channel strength level values $\{\alpha_{ij}, i,j\in\mathcal{K}\}$, we define the following matrix associated with the assignment problem (17):

$$\mathbf{A}_{ij}=\begin{cases}\alpha_{ij}, & i\neq j\\ \alpha_{jj}-d_j, & i=j\end{cases}. \tag{18}$$

We refer to the subnetwork involving $\mathcal{S}\subseteq\mathcal{K}$ with channel strength $\{\mathbf{A}_{ij}, i,j\in\mathcal{S}\}$ as the *original* subnetwork $\mathcal{S}$.

By the duality theory in linear programming, we observe that the dual problem of (17) is given by

$$\max \sum_{(i,j)\in\mathcal{E}} \mathbf{A}_{ij}x(i,j), \tag{19a}$$

$$\text{s.t.} \sum_{i\in\mathcal{U}:(i,j)\in\mathcal{E}} x(i,j)\le 1, \tag{19b}$$

$$\sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} x(i,j)\le 1, \tag{19c}$$

$$x(i,j)\in[0,1]. \tag{19d}$$

It is known that this linear program has integer-valued optimal solutions for bipartite graphs [28]. Since in our case the graph associated to the transmitters and receivers in $\mathcal{S}$ and corresponding intended and interfering links is bipartite by construction, then (19) coincides with a maximum weighted matching problem, obtained by replacing (19d) with $x(i,j)\in\{0,1\}$. By the strong duality theorem, the minimum of the sum of all left and right labels $\{y_{u_j}, y_{v_j}\}$ in the primal problem (17) is equal to the maximum weight sum over all matchings of the original network in the dual problem (19).

Next, due to the complementary slackness condition [15], an edge (i,j) belongs to the maximum-weight matching, i.e., $x(i,j)=1$, if and only if $y_{u_i}+y_{v_j}=\mathbf{A}_{ij}$. For all given $\mathcal{S}\subseteq\mathcal{K}$, due to (12) we have that a feasible GDoF-tuple implies equality in (17c). Hence, it follows that the set of feasible GDoF-tuples (i.e, the region $\mathcal{P}_{\mathcal{S}}^{\text{TINA}}$) coincides with the set of all $(d_j : j\in\mathcal{S})$ for which the maximum matching solution to (19) is $\{(j,j), j\in\mathcal{S}\}$ in the original subnetwork $\mathcal{S}$. This key observation enables us to provide a more compact form for the TINA region.

For notational convenience, we construct a weighted fully-connected bipartite graph $\mathcal{G}=(\mathcal{K},\mathcal{K},\mathcal{K}\times\mathcal{K})$, where the weight α'_{ij} is modified upon the original network and specified as

$$\alpha'_{ij}=\begin{cases}\alpha_{ij}, & i\neq j\\ 0, & i=j\end{cases}. \tag{20}$$

For any $\mathcal{S}\subseteq\mathcal{K}$, we define the subgraph $\mathcal{G}[\mathcal{S}]=(\mathcal{S},\mathcal{S},\mathcal{S}\times\mathcal{S})$ with weights $\{\alpha'_{ij} : i,j\in\mathcal{S}\}$, which is referred to as the modified subnetwork $\mathcal{S}$. According to Appendix A, we denote by $\mathcal{M}_{\mathcal{S}}$ a matching in $\mathcal{G}[\mathcal{S}]$, and by $\mathcal{M}^*_{\mathcal{S}}$ the matching with the maximum weight in the modified subnetwork $\mathcal{S}$. The observation made above implies that, for all $\mathcal{S}'\subseteq\mathcal{S}\subseteq\mathcal{K}$, the matching $\{(j,j),\ j\in\mathcal{S}'\}$ has the maximal sum weight among all matchings in the original subnetwork $\mathcal{S}'$. Hence, it follows that

$$\sum_{j\in\mathcal{S}'}\mathbf{A}_{jj}=\sum_{j\in\mathcal{S}'}(\alpha_{jj}-d_j)\ge \max w(\mathcal{M}_{\mathcal{S}'})=w(\mathcal{M}^*_{\mathcal{S}'}),\quad \forall\ \mathcal{S}'\subseteq\mathcal{S}\subseteq\mathcal{K} \tag{21}$$

where $w(\cdot)$ is the weight sum of a matching. The inequality (21) is the necessary and sufficient condition that $\{(j,j),\ j\in\mathcal{S}'\}$ is the maximum weighted matching in the original subnetwork $\mathcal{S}'$. For the necessity, as the maximum weighted matching $\mathcal{M}^*_{\mathcal{S}'}$ in each modified subnetwork $\mathcal{S}'$ is also a valid matching in the corresponding original subnetwork $\mathcal{S}'$, the matching of sum weight $w(\{(j,j),\ j\in\mathcal{S}'\})$ is no less than any other matchings in the original subnetwork $\mathcal{S}'$ including $\mathcal{M}^*_{\mathcal{S}'}$. For the sufficiency, as (21) holds for any $\mathcal{S}'\subseteq\mathcal{S}\subseteq\mathcal{K}$, the matching of sum weight $w(\{(j,j),\ j\in\mathcal{S}'\})$ is always larger than any other matchings of sum weight $w(\{(i,j),\ i\neq j\in\mathcal{S}'\})$ in every original subnetwork $\mathcal{S}'$. As such, as long as (21) holds for *every* subnetwork $\mathcal{S}'\subseteq\mathcal{S}\subseteq\mathcal{K}$, $\{(j,j),\ j\in\mathcal{S}'\}$ is the maximum weighted matching in *every* original subnetwork $\mathcal{S}'$. This will also be further clarified in Appendix C through the proof of the following theorem.

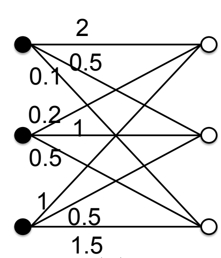
Theorem 3. *Consider a K-user single-antenna Gaussian interference channel with channel strengths $\{\alpha_{ij} : i,j\in\mathcal{K}\}$. For any user subset $\mathcal{S}\subseteq\mathcal{K}$, $\mathcal{P}_{\mathcal{S}}^{\text{TINA}}$ is given by:*

$$\left\{(d_k : k\in\mathcal{K}) : \begin{array}{l} d_k\ge 0,\ \forall k\in\mathcal{S},\quad d_i=0,\forall i\in\mathcal{S}^c\\ \sum_{k\in\mathcal{S}'} d_k\le\sum_{k\in\mathcal{S}'}\alpha_{kk}-w(\mathcal{M}^*_{\mathcal{S}'}),\ \ \forall\mathcal{S}'\subseteq\mathcal{S}\end{array}\right\} \tag{22}$$

where $w(\mathcal{M}^*_{\mathcal{S}'})=0$ if $|\mathcal{S}'|=1$. This simplified representation is equivalent to the expression in (5).

Proof. See Appendix C. □

Remark 3. *For individual users, i.e., $|\mathcal{S}'|=1$, we have individual GDoF constraints, i.e., $d_k\le\alpha_{kk}$. Using Theorem 3 into (6), we find that we need only 2^K-1 non-trivial inequalities, one for each non-trivial subset of $\mathcal{K}$, to describe the K-user TINA region $\mathcal{P}_{\mathcal{K}}^{\text{TINA}}$, which is significantly less than $\approx(K-1)!$ in [3].*



$$\begin{bmatrix}1.5 & 0.5 & 0.1\\ 0.2 & 0.4 & 0.5\\ 1 & 0.5 & 0.8\end{bmatrix}$$

(a) (b)

Fig. 1: (a) A 3-user interference channel, and (b) the input weight matrix of Hungarian method for the GDoF tuple (0.5, 0.6, 0.7).

Example 1. We consider the example in [10, Fig. 8] to show the efficiency of our formulation, as shown in Fig. 1(a). According to Theorem 3, the TINA GDoF region is immediately given as

$$\begin{aligned}\mathcal{P}^{\mathrm{TINA}}_{\{1,2,3\}} = \{(d_1,d_2,d_3):\ & 0 \le d_1 \le 2, 0 \le d_2 \le 1, 0 \le d_3 \le 1.5,\\ & d_1+d_2 \le 2.3, d_2+d_3 \le 1.5\\ & d_1+d_3 \le 2.4, d_1+d_2+d_3 \le 2.5\},\end{aligned}$$

which is identical to the expression found in [10]. In order to solve the power allocation for a given GDoF-tuple (say $(0.5, 0.6, 0.7)$ in this case), we take the weight matrix in Fig. 1(b) as the input of the Kuhn-Munkres algorithm (see Algorithm 2 in Appendix B) and we obtain:

$$y_{u_1} = 1.2,\ y_{u_2} = 0.4,\ y_{u_3} = 0.7,\ y_{v_1} = 0.3,\ y_{v_2} = 0,\ y_{v_3} = 0.1.$$

Thus, the globally optimal power allocation assignment is $r_1 = -1.2, r_2 = -0.4, r_3 = -0.7$, which coincide with what found in [10]. The details are relegated to Appendix B.

Clearly, to start Algorithm 2, y_{u_j} and y_{v_j} are initialized respectively with the maximum value of the j-th row of $\mathbf{A}$ and 0. Following the procedure in Algorithm 2, we gradually decrease y_{u_j} and increase y_{v_j} to make sure the constraints in (17) satisfied. Note that $r_j = -y_{u_j}$ is increasing during this procedure. Once we find one solution, it will be the global optimum assignment, because it is impossible to decrease r_j (correspondingly increase y_{u_j}) and find another solution in the region that we have already explored. ◊

C. A New TIN Optimality Condition

Besides the significant reduction of the number of inequalities of the TINA region as stated in Theorem 3, this new formulation enables us to identify a relaxed channel strength condition, by which the TINA region is a convex polytope.

Theorem 4. *Consider a K-user single-antenna Gaussian interference channel with channel strengths $\{\alpha_{ij} : i, j \in \mathcal{K}\}$. If*

$$\alpha_{kk} \ge \max_{i,j:\ i,j\neq k} \{\alpha_{ik} + \alpha_{kj} - \alpha'_{ij}\},\ \ \forall\ k \in \mathcal{K}, \tag{23}$$

where α'_{ij} is defined in (20), then $\mathcal{P}^{\mathrm{TINA}}_{\mathcal{S}}$ is monotonically non-decreasing with respect to $\mathcal{S}$, i.e., if $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{K}$ then $\mathcal{P}^{\mathrm{TINA}}_{\mathcal{S}_1} \subseteq \mathcal{P}^{\mathrm{TINA}}_{\mathcal{S}_2}$. Also, $\mathcal{R}^{\mathrm{TINA}} = \mathcal{P}^{\mathrm{TINA}}_{\mathcal{K}}$ is a convex polytope.

Proof. See Appendix D. □

Remark 4. *The newly found channel strength condition is a relaxed version of the GNAJ condition (7), because α'_{ij} is non-negative such that if (7) is satisfied, then (23) is satisfied automatically. When $i = j \neq k$, (23) reduces to $\alpha_{kk} \ge \alpha_{ik} + \alpha_{ki}$, $\forall\ k, i$, s.t. $k \neq i$, which is the same as that induced by (7). When $i \neq j \neq k$, it reduces to a quadrangular inequality $\alpha_{kk} \ge \alpha_{ik} + \alpha_{kj} - \alpha_{ij}$, for all $k \neq i$, $k \neq j$ and $i \neq j$, saying that the desired signal strength of each user is no less than the* maximum difference *between the sum strength of any pair of incoming/outgoing interference signals and the strength of the link between such a pair (all in dB scale).* ◊

With solely the condition (23), a relaxation of (7), we are not able to prove the optimality of the corresponding TINA region (although a convex polytope) through the techniques at hand (e.g., the cyclic bounds [3], [29]). Nevertheless, we can exhibit a class of networks different from the class identified in [3], for which the TINA region is GDoF-optimal. This is a special class of partially connected interference channels satisfying a topological condition given below. Interestingly, this class of networks is not included nor includes the class defined by the GNAJ condition. The converse proof follows the approach in [3] and is presented in Appendix E.

Theorem 5. *Consider a K-user single-antenna Gaussian interference channel with channel strengths $\{\alpha_{ij} : i, j \in \mathcal{K}\}$. Assume that (23) holds and, in addition, that for every $\mathcal{S} \subseteq \mathcal{K}$ with $|\mathcal{S}| > 2$, and the corresponding fully connected weighted subgraph $\mathcal{G}[\mathcal{S}] = (\mathcal{S}, \mathcal{S}, \mathcal{S}\times\mathcal{S})$ with weights $\{\alpha'_{ij} : i, j \in \mathcal{S}\}$,*

$$\exists\ (i,j) \in \mathcal{M}^*_{\mathcal{S}},\ \text{s.t. } \alpha_{ij} = 0. \tag{24}$$

Then, $\mathcal{R}^* = \mathcal{R}^{\mathrm{TINA}} = \mathcal{P}^{\mathrm{TINA}}_{\mathcal{K}}$.

Proof. See Appendix E. □

Remark 5. *As the maximum weighted matching may not be unique, Theorem 5 holds as long as (24) holds for any one of the maximum matchings. Condition (24) allows us to establish the optimality of TINA since, under this condition, we can prove that the converse is tight. This, however, is only a sufficient condition and there might be a larger class of networks, including both the subclass defined by Theorem 1 and the one defined by Theorem 5, for which TIN is GDoF-optimal. It is also worth noting that our new TIN optimality condition does not violate the conjecture in [3] that the GNAJ condition is also necessary "except for a set of channel gain values with measure zero."* ◊

Example 2. We illustrate the relaxed channel strength condition by the example in Fig. 2. It is easy to verify that the condition (23) holds for the entire network, while the original GNAJ condition (7) does not hold for users 1 and 2. Note that $\mathcal{M}^* = \{(1,3),(2,1),(3,2)\}$ is a (non-unique) maximum weighted matching and contains $\alpha'_{13} = \alpha_{13} = 0$, such that also condition (24) holds. Thus, from Theorems 4 and 3, the optimal TINA region $\mathcal{R}^*$ of this network is the polytope defined by:

$$\begin{aligned}\mathcal{P}^{\mathrm{TINA}}_{\{1,2,3\}} = \{(d_1,d_2,d_3):\ & 0 \le d_i \le 1, \forall i \in \{1,2,3\}\\ & d_1+d_2 \le 1.1, d_2+d_3 \le 1.3\\ & d_1+d_3 \le 1.2, d_1+d_2+d_3 \le 1.8\}.\end{aligned}$$

◊

Remark 6. *A subclass of network topologies for which (24) holds is those with no perfect matchings in any unweighted subgraph of $\mathcal{G}$ where zero-weight edges removed. A bipartite graph has no perfect matchings if Hall's condition does not hold [28]. The so-called triangular networks in [30] belong to this category.* ◊

IV. A GDoF-based Link Scheduling and Power Control Framework

In this section, we capitalize on the insight about the TINA region obtained before, in order to develop a framework for link

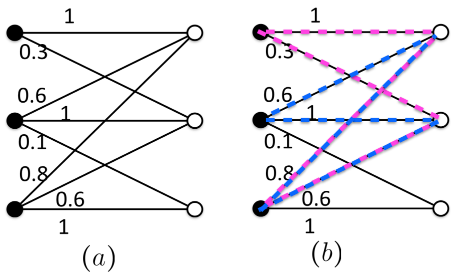


Fig. 2: (a) A 3-user Gaussian interference channel whose TINA region is a convex polytope. The value associated with each link represents the channel strength level α_{ij} and the missing links correspond to $\alpha_{ij} = 0$. (b) The links where the relaxed channel strength condition (23) is satisfied while the GNAJ condition does not hold (marked in blue and purple).

scheduling and power control in Gaussian K-user interference channels with the constraint that receivers treat interference as (Gaussian) noise. In general, the goal is to activate simultaneously a subset of links (i.e., transmitter-receiver pairs) with nonzero transmit power, aiming at maximizing some desired system utility functions. The classical *power control* problem (e.g., as formulated in [31]–[33]) finds the componentwise minimum transmit power vector that achieves given target SINRs at the receivers, when such target SINRs are feasible. However, this approach does not take into account that in modern TDMA systems the links may not be active in all scheduling slots. In contrast, by selecting a subset of links on each slot (scheduling), higher user throughput (i.e., time-averaged rate) can be achieved. As anticipated in Section I, a direct application of GP [22] also does not solve the scheduling problem, since implicitly all links must be allocated positive power. Intuitively, these approaches work well when SINRs significantly larger than 1 (0 dB) can be achieved for all the K links.

A general scheduling framework is provided by considering the user throughputs $T_k = \lim_{t\to\infty} \frac{1}{t}\sum_{\tau=1}^{t} R_k(\tau)$, where $R_k(\tau)$ indicates the rate achieved by link k during scheduling slot τ. Let $U(T_1, \dots, T_K)$ denote a concave componentwise non-decreasing *Network Utility Function* (see [20], [21], [34], [35] and references therein) of the user throughputs (e.g., weighted sum throughput), and let $\mathcal{T}$ denote the achievable throughput region of the system. Then, a general *Network Utility Maximization* (NUM) problem is given as

$$(NUM): \quad \max \ U(T_1, \dots, T_K) \tag{25a}$$
$$\text{s.t. } (T_1, \dots, T_K) \in \mathcal{T}. \tag{25b}$$

It is possible to systematically maximize any concave componentwise non-increasing network utility function $U(T_1, \dots, T_K)$ of the user throughputs by using the Lyapunov Drift-plus-Penalty method [21], [36], [37]. This decomposes the convex NUM problem over the region of achievable throughputs,[6] into a sequence of instantaneous weighted sum rate maximization problems of the form $\max_k \sum_k w_k(\tau) R_k(\tau)$ for $\tau = 1, 2, 3 \dots$, where the weights $\{w_k(\tau)\}$ are iteratively computed. It is clear that, depending on the weights $\{w_k(\tau)\}$, the sum rate may be maximized by turning off some particularly bad links, and by allocating positive power on subsets of favorable links. This process is referred to as "link selection", or "link scheduling" and power control.

[6]Notice that this region is always convex, since it is given by the convex hull of all instantaneously achievable rate K-tuples. However, the convex throughput region is very difficult to describe, therefore a direct solution of the underlying NUM problem is not possible in general, and we have to resort to the Lyapunov method.

In our case, we shall consider a GDoF criterion and replace T_k with $\overline{d}_k = \lim_{P\to\infty} \frac{T_k}{\log P}$. Through an immediate time-sharing argument, we have that the achievable region of throughput-GDoF is the convex hull of $\mathcal{R}^{\text{TINA}}$, denoted by $\text{conv}(\mathcal{R}^{\text{TINA}})$. In general, $\mathcal{R}^{\text{TINA}}$ is the union of convex polytopes (see Theorem 1), such that it is not generally convex. However, when (23) in Theorem 4 holds, then $\mathcal{R}^{\text{TINA}} = \text{conv}(\mathcal{R}^{\text{TINA}}) = \mathcal{P}_{\mathcal{K}}^{\text{TINA}}$. Using the GDoF criterion, the corresponding NUM problem becomes

$$(NUM - GDoF): \quad \max \ U(\overline{d}_1, \dots, \overline{d}_K) \tag{26a}$$
$$\text{s.t. } (\overline{d}_1, \dots, \overline{d}_K) \in \text{conv}(\mathcal{R}^{\text{TINA}}). \tag{26b}$$

It turns out that the above problem can be solved by iterating over time (i.e., over the scheduling slot) a sequence of "instantaneous" subproblems. The following result is quite standard and follows as corollary of the general theory developed for example in [21], [36]–[38] (and references therein), and shall be stated without proof here for the sake of space limitation.

Theorem 6. *Consider a K-user single-antenna Gaussian interference channel with channel strengths $\{\alpha_{ij} : i, j \in \mathcal{K}\}$, and corresponding TINA region $\mathcal{R}^{\text{TINA}}$. For a sequence of scheduling slots indexed by $t = 1, 2, 3, \dots$, consider the following iterative procedure:*

1) *Initialize weights $w_k(1) = 1$ for all $k \in \mathcal{K}$.*
2) *For $t = 1, 2, \dots$, repeat the following two steps:*
   - *Compute the GDoF-tuple $(d_1^*(t), \dots, d_K^*(t))$ solution of the max weighted sum-GDoF problem*

$$(SUM - GDoF): \quad \max \sum_{k\in\mathcal{K}} w_k(t) d_k \tag{27a}$$
$$\text{s.t. } (d_1, \dots, d_K) \in \mathcal{R}^{\text{TINA}}. \tag{27b}$$

   - *Update the weights according to*

$$w_k(t+1) = \max\{0, w_k(t) - d_k^*(t) + a_k^*(t)\}, \tag{28}$$

   where $(a_1^*(t), \dots, a_k^*(t))$ is the solution of the convex optimization problem

$$\max \ VU(a_1, \dots, a_K) - \sum_{k\in\mathcal{K}} w_k(t) a_k \tag{29a}$$
$$\text{s.t. } (a_1, \dots, a_K) \in [0, A_{\max}]^K, \tag{29b}$$

   where $V > 0$ and $A_{\max} > 0$ are control parameters of the algorithm.

Then, for sufficiently large $A_{\max}$ we have that

$$\lim_{t\to\infty} U\left(\frac{1}{t}\sum_{\tau=1}^{t} d_1^*(\tau), \dots, \frac{1}{t}\sum_{\tau=1}^{t} d_K^*(\tau)\right) \geq U(\overline{d}_1^*, \dots, \overline{d}_K^*) - \frac{\kappa}{V}, \tag{30}$$

where $(\bar{d}_1^*, \dots, \bar{d}_K^*)$ is the solution of the NUM-GDoF problem (26), and κ is a constant that depends on the system parameters but is independent of V. Hence, the above iterative scheduling algorithm can approach the optimal value of (26) by any desired accuracy.

Remark 7. *It is also possible to show that the time over which the limit in (30) is closely approached grows as $O(V)$. Therefore, in practice there is a tradeoff between how close we can approach the optimal network utility function value, and how quickly the scheduling algorithm converges. Nevertheless, here we are not concerned with this problem, and we use Theorem 6 as a general tool to translate a NUM problem in terms of the long-term time averaged rates (or GDoF, in our case) into a sequence of "instantaneous" (i.e., to be solved at each scheduling slot) max weighted sum-GDoF problem.* $\Diamond$

It follows that, from now on, we shall be concerned with solving the maximum weighted sum-GDoF problem (27) for an arbitrary set of weights $(w_1, \dots, w_K)$. The "power control" aspect of the problem resides in the fact that, when a solution $(d_1^*(t), \dots, d_K^*(t))$ of (27) is found, we must also provide the powers at which the links have to transmit in order to realize such a GDoF point in the TINA region. As anticipated in Section I, such transmit powers are generally not unique, and in this case we aim at finding the globally optimal power control assignment for the desired GDoF-tuple.

In what follows, we first introduce the exact GDoF-based solution, and then subsequently simplify it until we could obtain an approximation with polynomial-time complexity. In turn, the exact or approximate solver of (27) can be plugged into the iterative scheduling algorithm of Theorem 6 in order to obtain a scheme that works for any suitable network utility function. For example, if throughput max-min fairness is desired, we can choose $U(\bar{d}_1, \dots, \bar{d}_K) = \min_k \bar{d}_k$. Instead, if proportional fairness is desired, we can choose $U(\bar{d}_1, \dots, \bar{d}_K) = \sum_{k\in\mathcal{K}} \log(\bar{d}_k)$.

A. Exact Joint Solution is Hard

We rewrite (27) more conveniently in the form:

$$(DP): \quad \max_{\{d_k\}} \quad \sum_{k\in\mathcal{K}} w_k d_k \tag{31a}$$

$$\text{s.t.} \quad (d_1, \dots, d_K) \in \bigcup_{\mathcal{S}\subseteq\mathcal{K}} \mathcal{P}_{\mathcal{S}}^{\text{TINA}}, \tag{31b}$$

which can be categorized as an instance of Disjunctive Programming (DP) [39]. The union involves $2^K - 1$ nontrivial polyhedra, and $\mathcal{P}_{\mathcal{S}}^{\text{TINA}}$ is described by $2^{|\mathcal{S}|}-1$ linear inequalities. As mentioned earlier, the union is not necessarily leading to a convex polytope, and thus the problem is not a convex optimization problem in general. Nevertheless, it can be transformed to an equivalent convex optimization problem by replacing $\bigcup_{\mathcal{S}\subseteq\mathcal{K}} \mathcal{P}_{\mathcal{S}}^{\text{TINA}}$ with its convex hull $\mathcal{Q} = \text{conv}\left(\bigcup_{\mathcal{S}\subseteq\mathcal{K}} \mathcal{P}_{\mathcal{S}}^{\text{TINA}}\right)$.

The full description of $\mathcal{Q}$ may require an exponential number of inequalities, yet $\mathcal{Q}$ has a compact representation in a higher-dimensional space. The so-called lift-and-project cutting plane method [40], [41] can be employed to offer an exact solution to this problem. The principle consists of three steps: (1) lift the subspace spanned by GDoF tuples into a higher-dimensional space by introducing some auxiliary variables, (2) obtain the compact representation in the form of a set of lift-and-project cutting planes, and (3) project the compact representation onto the original GDoF spanned subspace. These cutting planes are valid for the closure of the convex hull $\mathcal{Q}$, and can be generated by solving cutting generating linear programs derived from the higher dimensional representation (see [40], [41] and references therein).

Once we obtain the GDoF tuple $\mathbf{d}$ from (DP) maximizing the weighted sum-GDoF, the second step is to use either the centralized algorithm found in [10] or the assignment-based algorithms found in this paper to find the globally optimal power allocation vector $\mathbf{r}$. This is illustrated in Fig. 3.

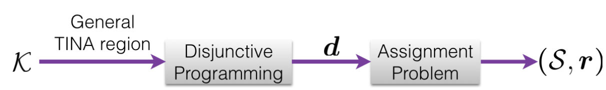


Fig. 3: The exact solution with the optimal link scheduling and assignment-based power control, where the globally optimal GDoF tuple is obtained via disjunctive programming over the union of all convex polytopes.

Unfortunately, the cutting generating linear programs still involve exponential number of constraints. This fact prohibits the application of this exact solution via disjunctive programming for network of practical size (e.g., a few tens to a few hundreds of D2D links). As such, reasonable approximation and heuristic approaches are desirable, although the global optimality is not guaranteed.

B. Separated Link Scheduling and Power Control

In view of the complexity of the exact solution, we resort to separated link scheduling and power control. We can first select heuristically a subset of links $\mathcal{S}$ whose TINA region is easy to describe and, at the same time, it leads to a reasonably large weighted sum-GDoF. In particular, we look for user subsets $\mathcal{S}$ whose TINA region coincides with the convex polytope $\mathcal{P}_{\mathcal{S}}^{\text{TINA}}$. Hence, finding the optimal GDoF tuple $\mathbf{d} \in \mathcal{P}_{\mathcal{S}}^{\text{TINA}}$ consists of solving an LP. Given the optimal $\mathbf{d}$ (restricted to $\mathcal{S}$), we can use assignment-based algorithms to obtain the corresponding optimal power allocation vector $\mathbf{r}$. This procedure is illustrated in Fig. 4.

Fig. 4: The separated solution with a heuristic link scheduling and assignment-based power control, where the desired GDoF tuple is obtained via linear programming over the selected convex polytope.

1) Link Scheduling: Recall that in the disjunctive programming (31), the objective function can be regarded as a moving hyperplane, and the constraint is the union of convex polytopes. A locally optimal weighted sum-GDoF solution is met when the hyperplane touches one of the vertices as a tangent plane to one convex polytope. The vertices of the largest convex polytope meet such hyperplanes with high probability, such that the weighted sum-GDoF can be maximized with high probability in such a polytope. As such, a heuristic link scheduling consists of selecting the largest subset of links whose corresponding TINA region is a convex polytope.

By analogy with information theoretic independent sets (ITIS) introduced in [4], we define an independent set as a set of links whose TINA region is a convex polytope. Using the conditions of Theorem 4, we have

Definition 1. *A user subset $\mathcal{S}$ is called an improved information theoretic independent set ("ITIS+"), if for any link $k \in \mathcal{S}$*

$$\alpha_{kk} \geq \max_{i,j \in \mathcal{S}\setminus\{k\}} \{\alpha_{ik} + \alpha_{kj} - \alpha'_{ij}\}. \tag{32}$$

Remark 8. *Comparing Definition 1 with the analogous definition of information-theoretic independent set (ITIS) in [4] we notice that any $\mathcal{S}$ satisfying the ITIS condition automatically satisfies also the ITIS+ condition, but the converse is not true. Hence, an ITIS+ contains no less links than an ITIS, and thus the TINA region (convex polytope) of the former includes that of the latter.* ◊

Of course, we would like to find the subset $\mathcal{S}$ over which the global maximum of $\sum_k w_k d_k$ is found. However, as already observed, this turns out to be very difficult and we shall content ourselves with good heuristics to select the ITIS+ set of active links $\mathcal{S}$.

2) Power Control: For a given ITIS+ $\mathcal{S}$, the GDoF tuple with the maximum weighted sum-GDoF can be identified by solving the following linear problem:

$$(LP): \quad \max_{d_i} \quad \sum_{i \in \mathcal{S}} w_i d_i \tag{33a}$$
$$\text{s.t.} \quad (d_1, \dots, d_K) \in \mathcal{P}_{\mathcal{S}}^{\text{TINA}}. \tag{33b}$$

Once the GDoF tuple is identified, various power allocation algorithms can be applied in order to find the corresponding power scaling vector $\mathbf{r}$. The LP (33) can be solved by using simplex, or interior-point method. However, since $\mathcal{P}_{\mathcal{S}}^{\text{TINA}}$ is defined by an exponential number of constraints (i.e., $2^{|\mathcal{S}|} - 1$), even the complexity solving (33) is impractical as soon as $|\mathcal{S}|$ is of the order of a few tens. The problem of reducing the size of constraints by exploiting the special structure of linear program is an interesting one, yet beyond the scope of this paper.

C. Replacing Linear Programming by Geometric Programming

In order to prevent the exponential complexity of solving (33), we propose to replace the LP by solving the power control problem on $\mathcal{S}$ using GP [22], which can be solved with polynomial-time complexity. In general, GP does not lead to the minimum power vector supporting the given (local optimum) GDoF vector. Therefore, we shall obtain the minimum power vector through the assignment-based algorithms. This idea is illustrated in the conceptual block diagram of Fig. 5.

Given a selected subset of links $\mathcal{S}$, GP is used to maximize the weighted sum rate at high SNR subject to the constraint that only links in $\mathcal{S}$ are active. We know that the GDoF region corresponding to the subnetwork $\mathcal{S}$ is a convex polytope $\mathcal{P}_{\mathcal{S}}^{\text{TINA}}$. Hence, there is no bottleneck link that forces the system to work in a bad operating point. It follows that we expect that the GP is close to the global optimum provided that $\mathcal{S}$ is well selected. Let

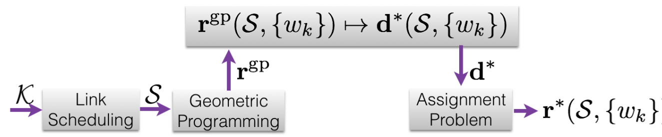


Fig. 5: The separated solution with a heuristic link scheduling and assignment-based power control, where the desired GDoF tuple is obtained instead via geometric programming.

the GP solution be denoted by $\mathbf{r}^{\text{gp}}(\mathcal{S},\{w_k\})$. In order to find the optimal power allocation vector, i.e., the componentwise minimum vector $\mathbf{r}^*(\mathcal{S},\{w_k\})$ yielding to the same weighted sum-GDoF, we first calculate the resulting (locally) optimal GDoF tuple by mapping $\mathbf{r}^{\text{gp}}(\mathcal{S},\{w_k\}) \mapsto \mathbf{d}^*(\mathcal{S},\{w_k\})$. Then, we use the assignment-based algorithms to find $\mathbf{r}^*(\mathcal{S},\{w_k\})$ from $\mathbf{d}^*(\mathcal{S},\{w_k\})$.

In what follows, we establish the equivalence of such a replacement. Given the weights $\{w_i, i \in \mathcal{S}\}$ of a selected use subset $\mathcal{S}$, we initially aim at solving the following optimization problem:

$$\max_{\{\bar{P}_i\}} \quad \sum_{i \in \mathcal{S}} w_i \log(1 + \text{SINR}_i) \tag{34a}$$
$$\text{s.t.} \quad \text{SINR}_i = \frac{G_{ii}\bar{P}_i}{1 + \sum_{j \neq i} G_{ji}\bar{P}_j} \tag{34b}$$
$$0 \leq \bar{P}_i \leq 1 \tag{34c}$$

where $G_{ij} = |h_{ij}|^2 P_i$ is the effective channel gain between Tx-i and Rx-j with power constraint P_i integrated, and $\bar{P}_i$ is the normalized transmit power allocation adjustment, by which $P_i\bar{P}_i$ is the actual transmit power. Let us introduce an auxiliary variable $t_i = \frac{1}{\text{SINR}_i}$ where t_i is a posynomial function of $\{\bar{P}_i, i \in \mathcal{S}\}$. Thus, the optimization problem at high SNR can be approximated to

$$(GP): \quad \min_{\{\bar{P}_i, t_i\}} \quad \prod_{i \in \mathcal{S}} t_i^{w_i} \tag{35a}$$
$$\text{s.t.} \quad \frac{1 + \sum_{j \neq i} G_{ji}\bar{P}_j}{G_{ii}\bar{P}_i} \leq t_i, \quad \forall i \in \mathcal{S} \tag{35b}$$
$$0 \leq \bar{P}_i \leq 1 \tag{35c}$$

which is a geometric program with respect to $\{\bar{P}_i, t_i, i \in \mathcal{S}\}$. A brief description of GP can be found in Appendix A. The equivalence between (LP) and (GP) is due to the following proposition.

Proposition 1. *Given an ITIS+ $\mathcal{S}$ and the weights $\{w_i, i \in \mathcal{S}\}$, the GP power allocation is equivalent to the LP power allocation in the sense that both approaches achieve the same optimal weighted GDoF.*

Proof. Given an ITIS+ $\mathcal{S}$, the condition (23) in Theorem 4 holds, and thus the TINA region is a convex polytope. As such, each transmitter-receiver pair is active, and thus each transmitter is allocated with positive power. This is the condition when GP works well. Let $G_{ij} = P^{\alpha_{ij}}$ and $\bar{P}_i = P^{r_i} \leq 1$. Substituting them into GP, we have

$$\min_{\{r_i,t_i\}} \quad \prod_{i\in\mathcal{S}} t_i^{w_i} \tag{36a}$$
$$\text{s.t.} \quad \frac{1+\sum_{j\neq i} P^{\alpha_{ji}+r_j}}{P^{\alpha_{ii}+r_i}} \le t_i \tag{36b}$$
$$-\infty \le r_i \le 0 \tag{36c}$$

By replacing t_i with P^{-d_i}, we have an equivalent formulation:

$$\max_{\{r_i,d_i\}} \quad \sum_{i\in\mathcal{S}} w_i d_i \tag{37a}$$
$$\text{s.t.} \quad 1+\sum_{j\neq i} P^{\alpha_{ji}+r_j} \le P^{-d_i+\alpha_{ii}+r_i} \tag{37b}$$
$$r_i \le 0, \quad \forall i \in \mathcal{S} \tag{37c}$$

Note that the log-sum-exp function can be rewritten as

$$\begin{aligned}\log(1+\sum_{j\neq i} P^{\alpha_{ji}+r_j}) &= (\max_j z_j^i)\log P + \log\sum_j P^{z_j^i - \max_{j'} z_{j'}^i} \\ &= (\max_j z_j^i + \epsilon_i)\log P \end{aligned}\tag{38}$$

where $\epsilon_i = \frac{\log\sum_j P^{z_j^i - \max_{j'} z_{j'}^i}}{\log P}$ and

$$z_j^i = \begin{cases} \alpha_{ji}+r_j, & j\neq i \\ 0, & j = i \end{cases}. \tag{39}$$

It is easily verified that $0 \le \epsilon \le \frac{\log|\mathcal{S}|}{\log P}$, and thus the second term in RHS is always bounded within $[0, \log K]$. Thus, we can rewrite the linear program as:

$$\max_{\{r_i,d_i\}} \quad \sum_{i\in\mathcal{S}} w_i d_i \tag{40a}$$
$$\text{s.t.} \quad d_i \le \alpha_{ii} + r_i - \max_{j\neq i}\{0, (\alpha_{ji}+r_j)\} - \epsilon_i \tag{40b}$$
$$d_i \ge 0,\ r_i \le 0, \quad \forall i \in \mathcal{S}. \tag{40c}$$

At high SNR ($P\to\infty$), $\epsilon_i \to 0$. It is not hard to verify that the feasible region of $(d_i : i \in \mathcal{S})$ in the above linear program is exactly the one in (33) formulated by taking GDoF metric into account. This completes the proof. □

Remark 9. *Admittedly, given the power allocation obtained by GP and the corresponding (suboptimal) rate tuple, one can apply the distributed power control algorithm developed in [31], [32] to obtain the globally optimal transmit power allocation to such a suboptimal rate tuple. Note however that, such an optimal solution is the* real *transmit power allocation, which is unique to a specific rate tuple, and unfortunately its power* exponent *may be not globally optimal with respect to the corresponding GDoF tuple [10, Appendix D]. In view of the fact that the weighted sum-GDoF maximization is of our interest and that the link scheduling also targets the largest achievable GDoF region rather than the rate region, the globally optimal power* exponents *fit our target better.*

As mentioned in [10], while there only exists a unique locally optimal power vector (and thus globally optimal) for a rate tuple, there are multiple locally optimal power exponent vectors for a GDoF tuple. It is because the real power linked to a specific rate tuple is unique, while the power exponent concerned for a specific GDoF tuple can be distinct. In other words, multiple locally optimal power exponent vectors lead to the same GDoF tuple, but only one locally optimal power exponent vector is globally optimal, which can be obtained by assignment-based algorithms or the algorithms in [10]. The concatenation of the geometric program and assignment-based algorithms offers maximal sum throughput at high SNR as well as minimal power consumption. Note that they are not contradicting, since the network is operated in an interference limited regime such that the sum rate is not immediately and obviously correlated to transmit power. ◊

V. ITLinQ+: A Decentralized Implementation in D2D Communications

In D2D communications, as smart devices are distributively located, a decentralized or semi-decentralized implementation of the link scheduling and power control framework illustrated in Section IV is desirable. The semi-decentralized refers to the case where some global information can be obtained through a controlling cellular base station, in base-station-assisted D2D [42].

In this section, we propose a decentralized mechanism (referred to as "ITLinQ+") for the link scheduling and power control framework aforementioned in Fig. 5. ITLinQ+ consists of three ingredients: (1) a decentralized implementation of link scheduling to find a large ITIS+ $\mathcal{S}$, (2) a decentralized GP implementation to find the power allocation vector $\mathbf{r}^{\text{gp}}(\mathcal{S},\{w_k\})$ and the corresponding GDoF tuple $\mathbf{d}^*(\mathcal{S},\{w_k\})$, and (3) a distributed Auction algorithm to solve the assignment problem and yield the globally minimal power allocation vector $\mathbf{r}^*(\mathcal{S},\{w_k\})$ corresponding to $\mathbf{d}^*(\mathcal{S},\{w_k\})$. These ingredients are detailed in the following subsections.

A. Decentralized Implementation of Link Scheduling

To find a large ITIS+, we design a decentralized link scheduling criterion similar to a greedy independent sets selection, requiring protocol signaling overhead and complexity comparable to that of FlashLinQ and ITLinQ. The decentralized link scheduling of ITLinQ+ is comprised of two phases: link scheduling and signaling.

Link Scheduling Phase: We first sort the links in non-increasing order of their weights w_k. If we wish to maximize the sum-throughput, we let $w_k = 1$ for all $k \in \mathcal{K}$ and sort the links in the order of their channel strengths. Notice that this global coordination information relative to the link priority order is needed also in FlashLinQ and ITLinQ. Links are added to the selected set by considering them sequentially in the above defined priority order. After some steps of the sequential selection, let $\mathcal{S} = \{i_1,\dots,i_{k-1}\}$ denote the currently selected set. Whether or not a new candidate link i_k is suitable to be scheduled depends on the following criteria. For the convenience of comparison with FlashLinQ and ITLinQ, we adopt the notations $\text{SNR}_k \triangleq P^{\alpha_{kk}}$ and $\text{INR}_{ij} \triangleq P^{\alpha_{ij}}$, where the noise power is normalized.

- At Tx-i_k, check if the following condition is satisfied:

$$\text{SNR}_{i_k}^{\eta} \ge \frac{\text{INR}_{i_k i_j}}{(\min_{s<k, s\neq j}\{\text{INR}_{i_s i_j}\})^{\gamma}}, \quad \forall\ j<k, i_j \in \mathcal{S} \tag{41}$$

where $\eta, \gamma \in [0,1]$ are design parameters, and $\min_{s<k, s\neq j}\{\mathrm{INR}_{i_s i_j}\}$ is the least channel strength level of incoming interfering links of Rx-i_j.

- At Rx-i_k, check if the following condition is satisfied:

$$\mathrm{SNR}_{i_k}^{\eta} \geq \frac{\mathrm{INR}_{i_j i_k}}{(\min_{s<k, s\neq j}\{\mathrm{INR}_{i_j i_s}\})^{\gamma}}, \quad \forall\ j<k, i_j \in \mathcal{S} \tag{42}$$

where $\min_{s<k, s\neq j}\{\mathrm{INR}_{i_j i_s}\}$ is the least channel strength level of outgoing interfering links of Tx-i_j.

Note that the minimum value of INR's is initialized to be 1 such that the second link i_2 is selected if $\mathrm{SNR}_{i_2}^{\eta} \geq \max\{\mathrm{INR}_{i_1 i_2}, \mathrm{INR}_{i_2 i_1}\}$ is satisfied. In the k-th $(k\geq 3)$ link selection, if these two conditions are satisfied, then this new candidate link i_k can be scheduled, i.e., $\mathcal{S} \leftarrow \mathcal{S}\cup\{i_k\}$. In words, a candidate link is scheduled if the interference caused to/received from the already selected higher-priority links is smaller than the product of the signal strength with *an exponent* η of this link and the signal strength with *an exponent* γ of the weakest interfering link in the already selected subset.

We point out that the above two criteria for $\gamma=\eta=0.5$ imply that the user subset $\mathcal{S}$ forms an ITIS+. Because of the fact that $\min_{s<k, s\neq j}\{\mathrm{INR}_{i_s i_j}\} \leq \mathrm{INR}_{i_i i_j}$, $\forall i<k, i\neq j$ and $\min_{s<k, s\neq i}\{\mathrm{INR}_{i_i i_s}\} \leq \mathrm{INR}_{i_i i_j}$, $\forall j<k, j\neq i$, the selection rule with $\eta=\gamma=0.5$ implies that, $\forall i_i, i_j \in \mathcal{S}\backslash\{i_k\}$,

$$\mathrm{INR}_{i_k i_j} \leq \sqrt{\mathrm{SNR}_{i_k}\mathrm{INR}_{i_i i_j}}, \tag{43a}$$

$$\mathrm{INR}_{i_i i_k} \leq \sqrt{\mathrm{SNR}_{i_k}\mathrm{INR}_{i_i i_j}}. \tag{43b}$$

From Definition 1 we have that inequalities of (43) imply that adding i_k to the current set $\mathcal{S}$ yields an ITIS+. As $\mathrm{INR}_{i_i i_j}$ is usually unavailable at Tx-i_k and Rx-i_k, we replace it by the minimum of local interference strengths, which can be obtained by signaling. Nevertheless, choosing $\eta=\gamma=0.5$, (43) may not lead to the best performance (because of no guarantee of a large enough ITIS+). To compensate for this, we leave η and γ as design parameters, which can be tuned by simulation taking into account the channel statistics (path loss law) due to the specific physical network topology.

Link Signaling Phase: Before the link scheduling phase, we have two rounds of signaling to inform transmitters and receivers the channel strength information. This can be done in two rounds of pilot signals, as in FlashLinQ [18] and ITLinQ [4].

- In the first round, similarly to FlashLinQ, each Tx-i $(\forall i)$ sends pilot signal with full power P_i in a different frequency band, such that each receiver is able to measure the received signal levels and obtain SNR's and INR's.
- In the second round, similarly to ITLinQ, each Rx-j $(\forall j)$ sends pilot signal with full power P_j in a different frequency band, such that the transmitters measure the signal levels with a certain adjustment and obtain SNR's and INR's.

At the end of this procedure, for any link k, the local channel strength information $\{\mathrm{INR}_{ki}, \forall\ i\}$ and SNR_k are accessible at transmitter k, and $\{\mathrm{INR}_{jk}, \forall\ j\}$ and SNR_k at receiver k.

Another additional signaling cost happens at the end of each successful link selection. The transmitter and receiver i_j ($i_j \in \mathcal{S}$) have to inform the next being checked links the local minimum interfering channel strength (i.e., $\min_{i_s\in\mathcal{S}, s\neq j}\{\mathrm{INR}_{i_j i_s}\}$ and $\min_{i_s\in\mathcal{S}, s\neq j}\{\mathrm{INR}_{i_s i_j}\}$, respectively). This signaling can be done similarly as the above two-round signaling procedure.

To reduce this additional signaling overhead, we can replace both $\min_{i_s\in\mathcal{S}, s\neq j}\{\mathrm{INR}_{i_s i_j}\}$ and $\min_{i_s\in\mathcal{S}, s\neq j}\{\mathrm{INR}_{i_j i_s}\}$ by the global minimum value $\min_{i_s, i_j\in\mathcal{S}, s\neq j}\{\mathrm{INR}_{i_s i_j}\}$, which is the minimum value over all cross links in the selected subset. This minimum value can be broadcast after each successful link selection by an assisting base station. If it is not changed with the newly selected link, then the signaling overhead with this new link is avoided.

B. Decentralized GP Implementation to Find the Optimal GDoF-tuple

In what follows, we will consider a distributed implementation of GP [22] where Rx-i has access to the local knowledge $\{r'_{ji}=\alpha_{ji}+r_j\}_j$ only, i.e., the exponent of received signal power from Tx-j. From the maximum weighted sum rate problem in (34), replacing the variables with their exponents, we obtain an approximate linear program from GP taking into account the local knowledge

$$\max_{r_i} \quad \sum_{i\in\mathcal{S}} w_i(\alpha_{ii}+r_i-\max_{j\neq i}\{0, r'_{ji}\}) \tag{44a}$$

$$\text{s.t.} \quad r'_{ji}=\alpha_{ji}+r_j, \forall j\in\mathcal{S}\backslash\{i\},\ \forall i \tag{44b}$$

$$r_j \leq 0, \forall j\in\mathcal{S}. \tag{44c}$$

Introducing Lagrange multipliers $\{\kappa_{ji}\}$ only for the coupling constraints, we form a partial Lagrangian

$$\begin{aligned} L = &-\sum_{i\in\mathcal{S}} w_i(\alpha_{ii}+r_i-\max_{j\neq i}\{0, r'_{ji}\}) \\ &+\sum_{i\in\mathcal{S}}\sum_{j\in\mathcal{S}\backslash\{i\}} \kappa_{ji}(r'_{ji}-\alpha_{ji}-r_j) \end{aligned} \tag{45}$$

and thus, each user only has to take care of its local partial Lagrangian term, given by

$$\begin{aligned} L_i(r_i, \{r'_{ji}\}_{j\neq i}; \{\kappa_{ji}\}_j) = &-w_i(\alpha_{ii}+r_i-\max_{j\neq i}\{0, r'_{ji}\}) \\ &+\sum_{j\in\mathcal{S}\backslash\{i\}} \kappa_{ji}r'_{ji} - \left(\sum_{j\in\mathcal{S}\backslash\{i\}} \kappa_{ij}\right) r_i. \end{aligned} \tag{46}$$

The minimization of the partial Lagrangian can be solved by alternating optimization. As initialization, $\{\kappa_{ji}\}_j$ are set to be zero, and in each iteration we alternately do as follows. First, given the local knowledge of $\{\kappa_{ji}\}_j$ at user i, the minimization can be done locally by each user i in parallel with respect to the primal local variables r_i and $\{r'_{ji}, j\in\mathcal{S}\backslash\{i\}\}$. Then, given the updated local variables r_i and $\{r'_{ji}, j\in\mathcal{S}\backslash\{i\}\}$, we update $\{\kappa_{ji}\}_j$ again, and $\sum_{j\in\mathcal{S}\backslash\{i\}}\kappa_{ij}$ can be calculated base on the signaling of $\{\kappa_{jk}\}_j$ from user k. Keep doing this until it converges.

In particular, the dual variable $\{\kappa_{ji}\}_j$ can be obtained by solving the dual problem

$$\max_{\{\kappa_{ji}\}_j} g(\{\kappa_{ji}\}_j) \triangleq \sum_i \min_{r_i, \{r'_{ji}\}_{j\neq i}} L_i(r_i, \{r'_{ji}\}_{j\neq i}; \{\kappa_{ji}\}_j). \tag{47}$$

A simple solution of $\{\kappa_{ji}\}_j$ is to update iteratively with the updating rule in t-th iteration being:

$$\kappa_{ji}(t+1) = \kappa_{ji}(t) + \delta(t)(r'_{ji}(t) - r^e_{ji}(t)) \tag{48}$$

where $r^e_{ji}(t) = \alpha_{ji} + r_j(t)$ is the estimation of received signal power exponent, and $\delta(t)$ is a carefully chosen stepsize. As said, signaling of $\{\kappa_{jk}\}_j$ is needed from user k to other coupled users through, e.g., broadcasting, in each iteration, and the reduction of such signaling overhead can be similarly done as in [22].

C. Decentralized Auction Algorithm for Power Allocation

As shown earlier, the feasible power allocation vector $\mathbf{r}$ can be found by solving an assignment problem. The Auction algorithm is an efficient way to solve the assignment problem in a distributed manner. It mimics the sales auction in the business activities in which bids are compared in multiple rounds to make the best offer to the products, with each product going to the highest bidder. It has many variants, and interestingly the algorithm originally proposed by Demange, Gale, and Sotomayor [14] (referred to as "DGS Auction") adopts an ascending pricing strategy and converges to the minimum price equilibrium [14].

In our setting, the transmitters are bidders, and the receivers represent products. Let us look at the assignment problem from an auction perspective. The bidders have access to the local channel strength knowledge, i.e., bidder i only knows $\{\mathbf{A}_{ij}, j \in \mathcal{K}\}$. Here the right label y_{v_j} can be regarded as the price of the product j, meaning that a bidder must pay as much as y_{v_j} to obtain the product j. For a given price y_{v_j}, $\mathbf{A}_{ij} - y_{v_j}$ can be regarded as the benefit of the bidder i regarding the product j. We define profit margin by $y_{u_i} = \max_j\{\mathbf{A}_{ij} - y_{v_j}\}$. The objective is to determine the best assignment given this local information, such that each bidder i is happy to be assigned to a product j with the lowest price y_{v_j} and in turn highest profit margin y_{u_i}. Specifically, it is to minimize the price while maximizing the profit margin, satisfying $y_{u_i} + y_{v_j} \geq \mathbf{A}_{ij}, \forall i, j$.

An algorithm inspired by DGS Auction is detailed in Algorithm 1 where ϵ is a design parameter. It consists of multiple rounds of auction process. In each round, the bidder who is not assigned any product will bid his most profitable product, i.e., $j^* = \arg\max_j\{\mathbf{A}_{ij} - y_{v_j}\}$ with j^* being the most profitable product of the bidder i. If the associated profit $\mathbf{A}_{ij^*} - y_{v_{j^*}}$ is negative or j^* is already assigned to the bidder i, then we skip this bidder and consider the next unassigned bidder. Otherwise, bidding process starts. If the product j^* was already assigned to another bidder, then this bidder will be added to the demand set and reconsidered later. If this product j^* is free, then it will be assigned to the bidder i, and at the same time the price of the product j^* will be raised by ϵ. If ϵ is small, it requires more rounds of iteration to achieve a reasonably "almostly optimal" solution. While ϵ is large, as in real auctions, the bidder may take risks to pay an unnecessarily high price, leading to a suboptimal solution with a faster convergence. A demand set $\mathcal{D}$ is maintained among bidders to indicate which bidders are not assigned with any product, and O_j represents the owner of the product j who successively bids this product. Keep doing this auction process until every bidder has his product without competitors. The final assignment of bidders and products is the optimal solution to the assignment problem. Each assigned product j should inform each bidder the raised price y_{v_j} in each round, which can be done similarly as the signaling procedure in link scheduling.

Algorithm 1 A Decentralized Power Allocation Algorithm via Auction Algorithm

---

Require: The bidder i only has the local knowledge $\mathbf{A}_i = \begin{bmatrix}\alpha_{i1} & \dots & \alpha_{ii} - d_i & \dots & \alpha_{iK}\end{bmatrix}$.

Initialization: Set $y_{v_j} = 0$, $O_j = 0, \forall j$, and $\mathcal{D} = \{1, 2, \dots, K\}$.
while $\mathcal{D} \neq \emptyset$ **do**
    Choose a bidder i from the demand set, i.e., $\mathcal{D} \leftarrow \mathcal{D} \backslash \{i\}$
    For bidder i, find the best values in $\{\mathbf{A}_{ij} - y_{v_j}, \forall j\}$

$$z_i = \max_j\{\mathbf{A}_{ij} - y_{v_j}\}, \quad j^* = \arg\max_j\{\mathbf{A}_{ij} - y_{v_j}\}$$

    if $z_i \geq \epsilon$ && $O_{j^*} \neq i$ **then**
        if $O_{j^*} \neq 0$ **then**
            Add i into the demand set, i.e., $\mathcal{D} \leftarrow \mathcal{D} \cup \{O_{j^*}\}$
        end if
        Assign the bidder i with the product j^*, i.e., $O_{j^*} = i$
        Product j^* raises the price by ϵ, i.e., $y_{v_{j^*}} \leftarrow y_{v_{j^*}} + \epsilon$
    end if
end while

For any set of feasible power allocation parameters $\{r_j = -y_{u_j}\}_j$, there exists an equilibrium price vector $\mathbf{y}_v = \{y_{v_j}\}_j$. The minimum price equilibrium always exists as long as the corresponding GDoF tuple is feasible. It has been proved in [14] that this DGS Auction algorithm converges to the minimum price equilibrium $\mathbf{y}^*_v$, meaning that any other feasible price vector $\mathbf{y}_v$ is componentwise larger than this minimum equilibrium, i.e., $\mathbf{y}^*_v \leq \mathbf{y}_v$. As such, the globally minimal power allocation parameters can be obtained from $\{r_{i^*} = y_{v_{j^*}} - \mathbf{A}_{i^*j^*}\}$ where (i^*, j^*) belongs to the optimal bidder-product assignment. If we make ϵ sufficiently small such that the price is raised carefully in each round of auction, $\{y_{v_j}\}_j$ obtained by Algorithm 1 can achieve arbitrarily close to the minimum equilibrium price. The algorithm converges within a finite number of iterations. An adaptive price increasing strategy [43] can be applied to speed up the convergence.

Remark 10 (Additional Computational Complexity and Signaling Overhead)**.** *In general, ITLinQ+ has an improved sum throughput and energy efficiency over ITLinQ at the expense of additional signaling overhead as well as computational complexity. In what follows, we summarize the additional signaling and computation in both link scheduling and power control phases.*

For the link scheduling, the additional signaling overhead of ITLinQ+ consists in the broadcasting of the minimum interfering link strength in the selected user set after a new candidate link is successfully scheduled. This can be done in such a way that the newly selected transmitter/receiver sends pilot signal on a reserved frequency band while the *receivers/transmitters in the selection pool measure the received signal level and estimate such a value. There are at most $2|\mathcal{S}|$ rounds of additional signaling, where $\mathcal{S}$ is the finally selected user set. Except for this signaling overhead, the link scheduling phase of ITLinQ+ has the same low-complexity computation as ITLinQ.*

The power control in ITLinQ+ involves the calculation of the GDoF via GP and the corresponding minimum power exponent vector via either the Hungarian (Algorithm 2) or the Auction (Algorithm 1) algorithms. This can be achieved with polynomial-time complexity in $|\mathcal{S}|$. If such schemes are implemented in a distributed fashion, they involve signaling overhead due to the exchange of the Lagrange multipliers (for GP) and prices (for Auction) at each iteration. As an alternative, in a Base-Station assisted D2D system, the power and rate assignment can be centrally computed by the Base-Station. In this case the signaling overhead consists of sending link strength information to the Base-Station for the selected nodes in $\mathcal{S}$, and sending back the rate and power allocation control signals from the Base-Station to the selected nodes. Notice that if the channel strengths do not change rapidly over time (e.g., for a proximity D2D network with nomadic users or in industrial IoT applications), the Base-Station needs to know only the set of selected links $\mathcal{S}$, and update the channel strength knowledge on a slower pace. Furthermore, the power control command can be broadcast to the users over some downlink control channel on a different frequency band, by piggyback on existing cellular control channels which are implemented anyway. Given all these options available to the system designer, a more precise assessment of the signaling overhead related to power control must rely on the specific system architecture and are out of the scope of this paper. ◊

VI. Numerical Results

To demonstrate the gains of our ITLinQ+ mechanism over FlashLinQ and ITLinQ, we perform numerical analysis under a similar network setup as in [4], [18].

As the first setup, we only consider the link scheduling without power control to show the benefit of our new decision criterion. We consider a 1 km × 1 km square area and randomly drop n transmitter-receiver pairs. As in [4], [18], the simulated channel follows the LoS model in ITU-1411, and the system parameters are listed in Table I. Two scenarios are considered: (1) Scenario 1: The distance of any two paired transmitter and receiver is uniformly distributed in $[5, 30]$m, all links are supposed to operate over a 5MHz bandwidth spectrum, and the maximum transmit power is 20 dBm; (2) Scenario 2: The range of distance is enlarged to $[10, 60]$m with a larger maximum transmit power 30 dBm and a wider bandwidth of 10MHz. We compare ITLinQ+ with no scheduling case where all links are activated, FlashLinQ, as well as ITLinQ with properly chosen parameters $\eta = 0.7$ and $M = 25$ dB as in [4]. In both scenarios, we use the same parameter $\eta = 0.9$ and $\gamma = 0.1$ for our ITLinQ+, where the impact of the desired links is more emphasized while the influence of the cross link is relatively lightened. Although this choice of two parameters is probably not leading to the optimum for a specific scenario, fortunately it leads to good performance which is insensitive to the variation of system parameters. To take the fairness into account, we also compare the fair versions for both ITLinQ and ITLinQ+. For both scenarios of fair ITLinQ, we follow the parameters carefully chosen in [4] with a threshold $\mathrm{SNR}_{th} = 110$ dB, $\bar{\eta} = 0.6$ and $\bar{M} = 20$ dB. For the fair ITLinQ+, we reduce η to 0.7 for both scenarios, while keeping other parameters unchanged.

TABLE I: System Parameters

Parameters	Values
Cell range	1km × 1km
Carrier Frequency	2.4GHz
Bandwidth	5MHz and 10 MHz
Distance (uniformly distributed)	$[5, 30]$ and $[10, 60]$ m
Transmit Power	20 and 30 dBm
Noise power spectral density	-174 dBm/Hz
Antenna Height	1.5 m
Antenna Gain per Device	-2.5 dB
Noise Figure	7 dB

Fig. 6 shows the average sum throughput (bits/s/Hz) versus the total number of links in the D2D networks ranging from 8 to 1024. The average performance is over 20 randomly chosen locations of n pairs and 20 randomly permuted orders of link priority. It demonstrates the significant improvement of ITLinQ+ over FlashLinQ and ITLinQ. For instance, with in total 1024 links, ITLinQ+ achieves 40% gain in Scenario 1, 60% gain in Scenario 2 over FlashLinQ, and 20% gain in Scenario 1 and 40% gain in Scenario 2 over ITLinQ. [7] Fig. 7 plots the comparison of Cumulative Distribution Function (CDF) versus the average per link rate for 1024 links based on the software implementation [44]. It is also shown that the fair version of our ITLinQ+ offers better average sum throughput than ITLinQ, and guarantees the comparable fairness as the fair ITLinQ. Better fairness could be achievable by further reducing η and increasing γ at the cost of sum throughput degradation.

As shown in Fig. 6, without power control, the benefit of the sole link scheduling is not significant in a network with a small number of links. So, we further consider smaller random networks with 16 and 64 link pairs where both link scheduling (with the same parameters η, γ and M respectively as before) and power control are enabled. The desired and cross link strength levels are uniformly distributed in [0,2] and [0,1], respectively. We compare in Fig. 8 the sum throughput versus the actually consumed power for different mechanisms. We also plot the curves for our ITLinQ+ with no power control, GP-based power control, and the assignment-based AP power control for reference.

Notably, to achieve the same sum throughput of for example 60 bits/s/Hz, our ITLinQ+ with AP power control saves transmit power 30 dB and 38 dB respectively compared with ITLinQ and FlashLinQ for the 16-user network, and it saves 30 dB and 32 dB respectively for the 64-user network. When it comes to the per link throughput, to achieve 2.5 bits/s/Hz per link for the 16-user network, it saves 23 dB and 25 dB compared to ITLinQ

[7] The gap between ITLinQ and ITLinQ+ might be reduced if one can further fine-tune the parameter M in ITLinQ. Unfortunately, there was no guideline on how to choose this parameter in [4].

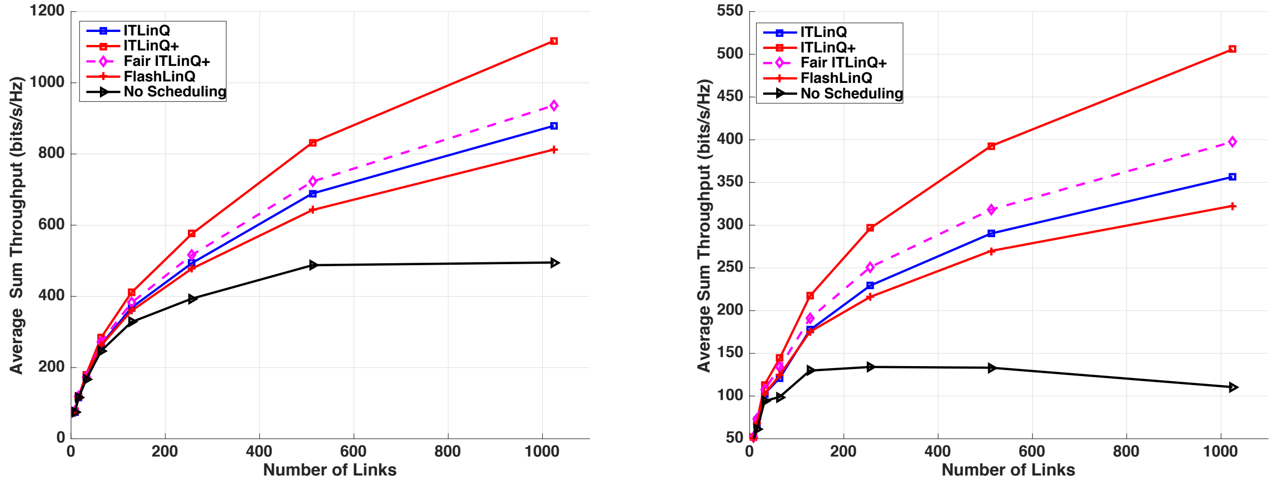


Fig. 6: Sum throughput comparison among no scheduling, FlashLinQ, ITLinQ, ITLinQ+ and Fair ITLinQ+ without power control. The left is for Scenario 1 and the right for Scenario 2.

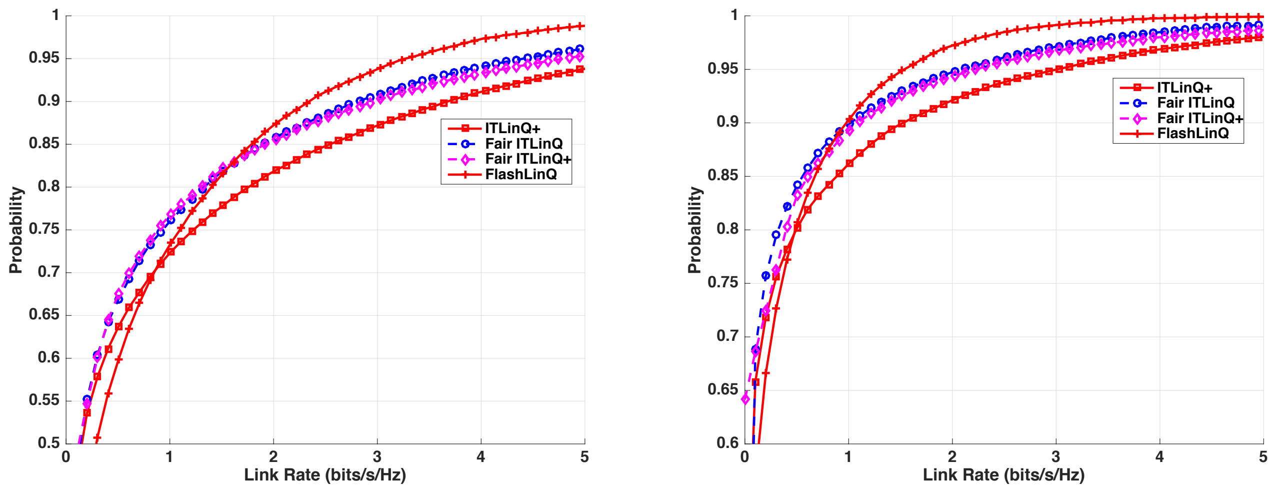


Fig. 7: Average per link rate comparison among FlashLinQ, Fair ITLinQ, ITLinQ+ and Fair ITLinQ+ without power control for 1024 links. The left is for Scenario 1 and the right for Scenario 2.

and FlashLinQ, respectively, and for the 64-user network, it saves 55 dB and more than 60 dB respectively. In the small network (e.g., 16-user case), ITLinQ+ and ITLinQ tend to select the same set of links. The improvement of ITLinQ+ over ITLinQ in small networks is mainly due to power control. For the larger network (e.g., 64-user case), pure link scheduling of ITLinQ+ offers 5 dB gains over ITLinQ to achieve sum throughput of 140 bits/s/Hz.

VII. Conclusion

The GDoF optimality problem of treating interference as noise for Gaussian interference channels has been reformulated from a combinatorial optimization perspective. Thanks to this new formulation, we cast power allocation into an assignment problem, which can be solved in polynomial time. A new expression for the TIN-Achievable GDoF region is provided, which is more compact and useful than what known before since it eliminates many redundant inequalities. A relaxed version of the condition in [3] on the channel coefficients is given, for which the TIN-Achievable GDoF region is a convex polytope. Finally, a new TIN optimality condition is also revealed, by which TIN still achieves the optimal GDoF region for a class of networks different from the one identified in [3].

We are also able to translate these insights into practical communication systems (e.g., D2D networks). With the newly found channel strength condition, we employed it as a new decision criterion in a distributed link scheduling mechanism. Together with the globally optimal distributed power allocation algorithms, we proposed a distributed spectrum sharing mechanism (ITLinQ+) for D2D networks. By simulation, we have shown that our ITLinQ+ mechanism achieves significant sum throughput improvement over FlashLinQ and ITLinQ with the same level implementation complexity. Moreover, ITLinQ+

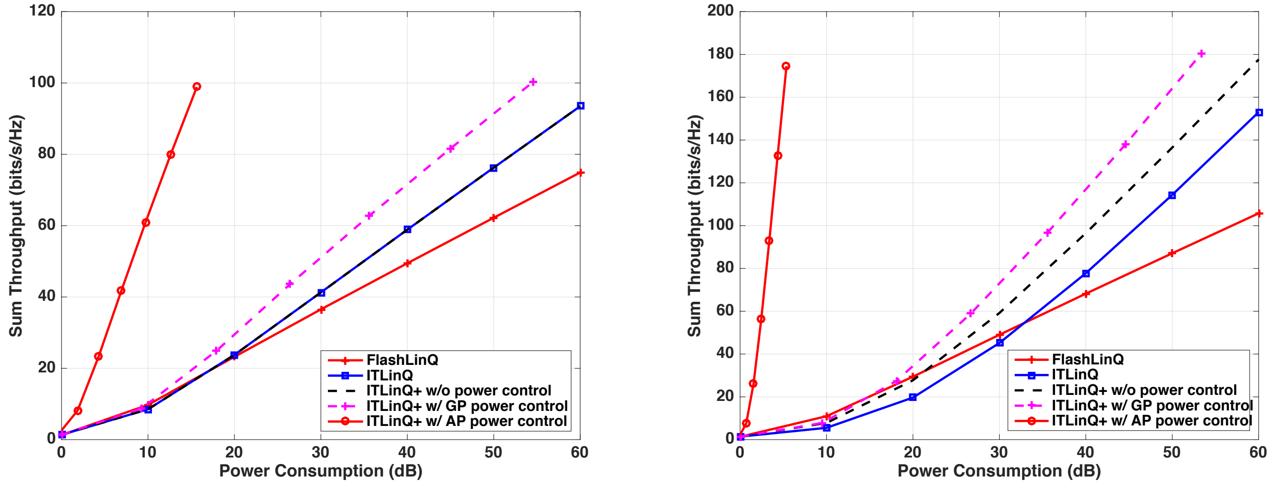


Fig. 8: Sum throughput versus actual power consumption for different spectrum sharing mechanisms. The left is for 16-user case and the right for 64-user case.

also promises a substantial improvement on power saving at the cost of additional complexity and some signaling overhead.

Appendix

A. Background

1) Weighted Matching: In this work we shall make extensive use of weighted matchings [28] of bipartite graphs. We recall here some basic definitions. Let $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$ denote a bipartite graph with left vertices $\mathcal{U}$, right vertices $\mathcal{V}$ and edges $\mathcal{E} \subseteq \mathcal{U} \times \mathcal{V}$. A matching $\mathcal{M} \subseteq \mathcal{E}$ is a set of edges, any two of which do not share the same vertex. When weights $w(u,v)$ are associated to the edges $(u,v) \in \mathcal{E}$, we denote by $w(\mathcal{M}) = \sum_{(u,v)\in\mathcal{M}} w(u,v)$ the weight of the matching $\mathcal{M}$, and we let $\mathcal{M}^* = \arg\max_{\mathcal{M}} w(\mathcal{M})$ denote the maximum weighted matching, i.e., the matching with maximum sum-weight. $\mathcal{M}^*$ can be characterized as the solution of the integer program:

$$\max \quad \sum_{(u,v)\in\mathcal{E}} w(u,v)x(u,v), \tag{49}$$

$$\text{s.t.} \quad \sum_{u\in\mathcal{U}:(u,v)\in\mathcal{E}} x(u,v) \le 1, \tag{50}$$

$$\sum_{v\in\mathcal{V}:(u,v)\in\mathcal{E}} x(u,v) \le 1, \tag{51}$$

$$x(u,v) \in \{0,1\}. \tag{52}$$

When equality holds in all constraints, the resulting solution is called a *perfect matching*, i.e., a matching that covers all vertices. The LP relaxation of (49)-(52), obtained by replacing (52) with $x(u,v) \in [0,1]$, is called fractional matching [45]. For bipartite graphs, the solution of this LP relaxation is integral, i.e., $x \in \{0,1\}$, meaning that, given a fractional matching, there exists a perfect matching such that the sum-weights of two matchings are equal. In other words, there always exists an integral solution to the LP relaxation problem.

A vertex/edge is called matched if it is involved in a matching; otherwise it is a free vertex/edge. A path is alternating if its edges alternate between matched and free edges. The augment operation $\text{aug}(\cdot)$ is to exchange matched and free edges in an alternating path that starts from and ends to free vertices. For instance, given an alternating path $\mathcal{P} = \{(i_0,i_1),(i_1,i_2),(i_2,i_3),\ldots,(i_{2n},i_{2n+1})\}$ consists of a matching $\mathcal{M} = \{(i_1,i_2),(i_3,i_4),\ldots,(i_{2n-1},i_{2n})\}$ and free edges $\mathcal{P}\backslash\mathcal{M}$, the augment of $\mathcal{P}$ results in a new matching $\mathcal{M}' = \text{aug}(\mathcal{P}) \triangleq \mathcal{P}\backslash\mathcal{M}$ and free edges $\mathcal{M}$.

2) Geometric Programming: Geometric programing is a powerful tool to solve a class of nonlinear optimization problems under a standard form

$$\min \quad f_0(\mathbf{x}) \tag{53a}$$

$$\text{s.t.} \quad f_i(\mathbf{x}) \le 1, \quad i = 1,\ldots,m \tag{53b}$$

$$g_i(\mathbf{x}) = 1, \quad i = 1,\ldots,p \tag{53c}$$

where $\{f_i(\mathbf{x}), i = 0,1,\ldots,m\}$ are posynomial functions $f_i(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ in a form of

$$f(x_1,\ldots,x_n) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}} \tag{54}$$

with $c_k \ge 0, \forall k$ and $\{g_i(\mathbf{x}), i = 1,\ldots,p\}$ are monomial functions $g_i(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ in the form of

$$g(x_1,\ldots,x_n) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \tag{55}$$

with $c \ge 0$.

B. The Kuhn-Munkres Algorithm

In what follows, we present the procedure of the Kuhn-Munkres algorithm together with an illustrative example. To ease the presentation, we construct a bipartite graph $\mathcal{G} = (\mathcal{U}, \mathcal{V})$ with weight of edge (i,j) being $\mathbf{A}_{ij}$ and $\mathcal{U}, \mathcal{V}$ being transmitter and receiver sets respectively. The Kuhn-Munkres algorithm is to find the maximum weighted matching in this bipartite graph. The input is the weight matrix $\mathbf{A}$ defined in (18), and the output is the matching with maximum sum weights and the corresponding left and right labels $\{y_{u_j}, y_{v_j}\}_j$, in which the left labels $\{y_{u_j}\}_j$ achieve the maximum left label equilibrium [12], [13].

As the initialization, we choose a feasible labeling with $y_{u_i} = \max_j \mathbf{A}_{ij}, \forall\ i$, and $y_{v_j} = 0, \forall\ j$. This labeling is feasible, because $y_{u_i} + y_{v_j} \geq \mathbf{A}_{ij}$ always holds for any pair of $i \in \mathcal{U}$ and $j \in \mathcal{V}$. We also construct an equality subgraph, $\mathcal{G}_E$, including all the vertices of $\mathcal{G}$ but only those edges (i,j) such that

$$y_{u_i} + y_{v_j} = \mathbf{A}_{ij}, \forall i \in \mathcal{U}, j \in \mathcal{V}. \tag{56}$$

It has been proved in [12], [13] that, once $\mathcal{G}_E$ has a perfect matching $\mathcal{M}$, then this matching $\mathcal{M}$ is a maximum weighted matching, and thus the corresponding labels are the final solution to the assignment problem.

The algorithm consists of multiple rounds of iterations. In each round, we first check if there exists a perfect matching in $\mathcal{G}_E$. Note that for a feasible GDoF tuple, the maximum matching always involves edges (j,j) according to (12). Thus, the perfect matching consists of edges $\{(j,j), \forall j \in \mathcal{K}\}$. If not, the left and right labels are carefully decreased and increased respectively, and the equality subgraph $\mathcal{G}_E$ is updated accordingly. Once $\mathcal{G}_E$ contains the perfect matching $\{(j,j), \forall j \in \mathcal{K}\}$, the resulting left labels $\{y_{u_j}\}_j$ achieve the maximum left label equilibrium, yielding the global minimal power allocation $r_j = -y_{u_j}$ for all j. The details of the Kuhn-Munkres algorithm are given in Algorithm 2, where $\mathcal{N}_L(\mathcal{S})$ is the neighborhood of a set of nodes of $\mathcal{S}$ in $\mathcal{G}_E$, i.e., $\mathcal{N}_L(\mathcal{S}) = \{j : (i,j) \in \mathcal{G}_E, \forall i \in \mathcal{S}, j \in \mathcal{V}\}$. An illustrative example is also given as follows.

Example 3. The detailed power allocation procedure according to Algorithm 2 is presented as follows.

As an initialization, we assign

$$y_{u_1} = 1.5,\ y_{u_2} = 0.5,\ y_{u_3} = 1,\ y_{v_1} = y_{v_2} = y_{v_3} = 0 \tag{57}$$

such that we construct the equality subgraph $\mathcal{G}_E$ with edges $\{(1,1),(2,3),(3,1)\}$ as in Fig. 9(b). Note that $\mathcal{G}_E$ does not contain a perfect matching. So we choose an arbitrary matching, e.g., $\mathcal{M} = \{(2,3),(3,1)\}$, as shown in Fig. 9(b).

In the first round, we choose a free vertex u_1 in $\mathcal{G}_E$ and set $\mathcal{S} = \{u_1\}$ and $\mathcal{T} = \emptyset$. Because $\mathcal{N}_L(\mathcal{S}) = \{v_1\} \neq \mathcal{T}$, we go to line-12 and pick $v_1 \in \mathcal{N}_L(\mathcal{S}) \backslash \mathcal{T}$. Note that v_1 is matched to u_3, and thus we update $\mathcal{S} = \{u_1, u_3\}$, $\mathcal{T} = \{v_1\}$. As $\mathcal{N}_L(\mathcal{S}) = \{v_1\} = \mathcal{T}$, we go to line-8 and obtain

$$\alpha_L = \min_{u_i \in \{u_1,u_3\}, v_j \in \{v_2,v_3\}} \{y_{u_i} + y_{v_j} - \mathbf{A}_{ij}\} = 0.2 \tag{58}$$

As such, we have

$$y_{u_1} = 1.3,\ y_{u_2} = 0.5,\ y_{u_3} = 0.8,\ y_{v_1} = 0.2,\ y_{v_2} = 0,\ y_{v_3} = 0. \tag{59}$$

and go to line-2.

In the second round, we update the equality subgraph $\mathcal{G}_E$ with edges $\{(1,1),(2,3),(3,1),(3,3)\}$ and there still does not contain a perfect matching, as shown in Fig. 9(c). Thus, we choose $\mathcal{M} = \{(2,3),(3,1)\}$ as a matching. Still, we pick the free vertex u_1, and set $\mathcal{S} = \{u_1\}$ and $\mathcal{T} = \emptyset$. Now, as $\mathcal{N}_L(\mathcal{S}) = \{v_1, v_3\} \neq \mathcal{T}$, we pick $v_1 \in \mathcal{N}_L(\mathcal{S}) \backslash \mathcal{T}$. Because v_1 is matched to u_3, we update $\mathcal{S} = \{u_1, u_3\}$, $\mathcal{T} = \{v_1\}$. At this point, $\mathcal{N}_L(\mathcal{S}) = \{v_1, v_3\} \neq \mathcal{T}$ again. Thus, we pick $v_3 \in \mathcal{N}_L(\mathcal{S}) \backslash \mathcal{T}$.

Algorithm 2 A Centralized Power Allocation Algorithm via the Hungarian Method

Require: Matrix $\mathbf{A}$ with ij-th element specified in (18).

1: Initialization: Set $y_{u_i} = \max_j \mathbf{A}_{ij}, \forall\ i$ and $y_{v_j} = 0, \forall\ j$. Construct $\mathcal{G}_E$ according to $\{y_{u_i}, y_{v_j}, \forall i, j\}$ and choose an arbitrary matching $\mathcal{M}$ contained in $\mathcal{G}_E$. Let $\mathcal{S} = \mathcal{T} = \emptyset$.
2: **if** $\exists\ \mathcal{M} = \{(j,j), \forall j \in \mathcal{K}\}$ in $\mathcal{G}_E$ **then**
3:     $r_i = -y_{u_i}, \forall\ i$, and **return**
4: **else**
5:     Pick a free vertex $u \in \mathcal{U}$
6:     $\mathcal{S} \leftarrow \{u\}$, $\mathcal{T} \leftarrow \emptyset$.
7: **end if**
8: **if** $\mathcal{N}_L(\mathcal{S}) = \mathcal{T}$ **then** $\alpha_L = \min_{i \in \mathcal{S}, j \notin \mathcal{T}} \{y_{u_i} + y_{v_j} - \mathbf{A}_{ij}\}$,
9:     Update $y_{u_k} \leftarrow y_{u_k} - \alpha_L$, if $k \in \mathcal{S}$,
10:     Update $y_{v_k} \leftarrow y_{v_k} + \alpha_L$, if $k \in \mathcal{T}$,
11:     Update $\mathcal{M} = \{(i,j) : y_{u_i} + y_{v_j} = \mathbf{A}_{ij}\}$
12: **else**
13:     Pick $v \in \mathcal{N}_L(\mathcal{S}) \backslash \mathcal{T}$
14:     **if** v is a free vertex **then**
15:         Augment the alternating path $u \to v$ that contains the matching $\mathcal{M}$
16:         Update $\mathcal{M} \leftarrow \mathrm{aug}(\{u \to v\})$ and goto 2
17:     **else**
18:         **if** v is matched to u' **then**
19:             $\mathcal{S} \leftarrow \mathcal{S} \cup \{u'\}, \mathcal{T} \leftarrow \mathcal{T} \cup \{v\}$
20:             goto 8
21:         **end if**
22:     **end if**
23: **end if**

Due to v_3 is matched to u_2, we update $\mathcal{S} = \{u_1, u_2, u_3\}$ and $\mathcal{T} = \{v_1, v_3\}$. Here $\mathcal{N}_L(\mathcal{S}) = \{v_1, v_3\} = \mathcal{T}$, we go to line-8 and have

$$\alpha_L = \min_{u_i \in \{u_1,u_2,u_3\}, v_j \in \{v_2\}} \{y_{u_i} + y_{v_j} - \mathbf{A}_{ij}\} = 0.1 \tag{60}$$

As such, the labels are updated as

$$y_{u_1} = 1.2,\ y_{u_2} = 0.4,\ y_{u_3} = 0.7,\ y_{v_1} = 0.3,\ y_{v_2} = 0,\ y_{v_3} = 0.1. \tag{61}$$

and then we go to line-2.

Till now, in the updated equality subgraph shown in Fig. 9(d), we have a perfect matching $\{(1,1),(2,2),(3,3)\}$. Thus, the algorithm returns with

$$r_1 = -1.2,\ r_2 = -0.4,\ r_3 = -0.7. \tag{62}$$

$\Diamond$

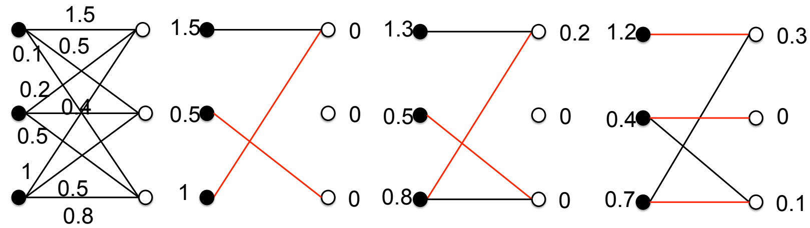
C. *Proof of Theorem 3*

For the sake of this proof, we denote by $\mathcal{P}_\mathcal{S}$ the region defined by (5), and by $\mathcal{P}_\mathcal{S}^{\mathrm{TINA}}$ the region defined by (22). Our goal is to show that $\mathcal{P}_\mathcal{S} = \mathcal{P}_\mathcal{S}^{\mathrm{TINA}}$ for any $\mathcal{S} \subseteq \mathcal{K}$.

$\underline{\mathcal{P}_\mathcal{S} \subseteq \mathcal{P}_\mathcal{S}^{\mathrm{TINA}}}$: To prove this, we show that for any inequality presented in $\mathcal{P}_\mathcal{S}^{\mathrm{TINA}}$, we can always find the same one in $\mathcal{P}_\mathcal{S}$.

Given a subnetwork $\mathcal{G}[\mathcal{S}]$, the matching with the maximum weight is a degree-1 subgraph. [8] Connecting the direct links

[8] If direct links are in the matching, we can eliminate them from $\mathcal{S}$, which does not affect our proof.



(a) (b) (c) (d)

Fig. 9: Illustration of the Hungarian method. (a) The weighted bipartite graph with weight of ij-th edge being $\mathbf{A}_{ij}$. (b) The equality subgraph with initiated labeling. (c) The updated equality subgraph with first round label update. (d) The update equality subgraph with second round label update. The edges in red give a matching in equality subgraphs.

will lead to single or multiple disjoint cycles, as all the nodes has degree-2.

For the single-cycle case, this cycle corresponds to a sum-GDoF constraint in $\mathcal{P}_{\mathcal{S}}$. For the multiple-cycle case, each cycle corresponds to a sum-GDoF constraint in $\mathcal{P}_{\mathcal{S}}$ of the users involved in this cycle. Thus, the sum-GDoF constraint with the maximum weighted matching in $\mathcal{P}_{\mathcal{S}}^{\text{TINA}}$ corresponds to the combination of these sum-GDoF constraints in $\mathcal{P}_{\mathcal{S}}$.

As such, $\mathcal{P}_{\mathcal{S}}$ contains or implies all the constraints in $\mathcal{P}_{\mathcal{S}}^{\text{TINA}}$. As $\mathcal{P}_{\mathcal{S}}$ has more constraints, it follows that $\mathcal{P}_{\mathcal{S}} \subseteq \mathcal{P}_{\mathcal{S}}^{\text{TINA}}$.

$\underline{\mathcal{P}_{\mathcal{S}}^{\text{TINA}} \subseteq \mathcal{P}_{\mathcal{S}}}$*:* To prove this, we show that, for any subset of users $\mathcal{S}$, the TINA GDoF region confined by $\mathcal{P}_{\mathcal{S}}$ is no smaller than that by $\mathcal{P}_{\mathcal{S}}^{\text{TINA}}$. It is clear that $\mathcal{P}_{\mathcal{S}}$ is determined by individual GDoF and sum-GDoF constraints of any subset of users in $\mathcal{S}$. The individual GDoF constraints of two regions are identical. Thus, our focus will be on the sum-GDoF constraints for users in $\mathcal{S}$ with $|\mathcal{S}| \geq 2$.

For the user set $\mathcal{S}$, the sum-GDoF constraints in $\mathcal{P}_{\mathcal{S}}$ only come from (1) the sum-GDoF constraints with all possible permutations of $\mathcal{S}$, and (2) the combination of a number of individual and/or sum-GDoF constraints of subsets of $\mathcal{S}$. For the first case, the sum-GDoF constraint in $\mathcal{P}_{\mathcal{S}}$ is dominated by the maximum weight of any possible matchings (associated with cyclic sequences). For the second case, suppose the combination involves a number of subnetworks $\mathcal{S}_1, \ldots, \mathcal{S}_p \subseteq \mathcal{S}$, where these subnetworks may have any intersections. This combination of constraints involves every user with equal times (say b times), i.e., $|\{i : j \in \mathcal{S}_i\}| = b$ for all $j \in \mathcal{S}$. Otherwise, the combination will not lead to a sum-GDoF constraint, because it is a weighted sum-GDoF constraint and can be implied by the combination of other sum-GDoF constraints. Each sum-GDoF constraint for a subnetwork involves a cyclic sequence and hence forms a matching, and thus the sum-GDoF for each subnetwork can be given by

$$\sum_{j \in \mathcal{S}_i} d_j \leq \sum_{j \in \mathcal{S}_i} \alpha_{jj} - w(\mathcal{M}_{\mathcal{S}_i}), \quad \forall i = \{1, \ldots, p\}. \tag{63}$$

The sum-GDoF constraint that comes from the linear combination [9] of these subnetworks can be written as

$$\sum_{i=1}^{p} \sum_{j \in \mathcal{S}_i} d_j \leq \sum_{i=1}^{p} \sum_{j \in \mathcal{S}_i} \alpha_{jj} - \sum_{i=1}^{p} w(\mathcal{M}_{\mathcal{S}_i}). \tag{64}$$

[9] As a subnetwork $\mathcal{S}_i$ is allowed to present in $\{\mathcal{S}_1, \ldots, \mathcal{S}_p\}$ many times, the weighted sum over subnetwork $\mathcal{S}_i$ in (63) can be done by repeating the subnetwork $\mathcal{S}_i$ multiple times. So, here we consider non-weighted sum of (63), which implies any linear combinations.

Due to the fact that $|\{i : j \in \mathcal{S}_i\}| = b$ for all $j \in \mathcal{S}$, we have

$$\sum_{j \in \mathcal{S}} d_j \leq \sum_{j \in \mathcal{S}} \alpha_{jj} - \frac{1}{b} \sum_{i=1}^{p} w(\mathcal{M}_{\mathcal{S}_i}). \tag{65}$$

According to the definition of the weighted matching in Appendix A, $\frac{1}{b} w(\mathcal{M}_{\mathcal{S}_i})$ corresponds to the sum weight of a fractional perfect matching in the subnetwork $\mathcal{S}_i$ by assigning $x(u,v)$ in (49)-(52) with $\frac{1}{b}$. The sum weights over all matchings for the subnetworks $\{\mathcal{S}_1, \ldots, \mathcal{S}_p\}$ can be regarded as a fractional perfect matching for the overall network $\mathcal{S}$ by assigning $x(u,v) = \frac{1}{b}$. Specifically, the constraint, e.g., (50), becomes $\sum_i \sum_{u \in \mathcal{U}:(u,v) \in \mathcal{E}_i} x(u,v) \leq 1$ where $\mathcal{E}_i$ is the edge set of subnetwork $\mathcal{S}_i$, and it looks as if each vertex can support b edges from subnetworks of $\{\mathcal{S}_1, \ldots, \mathcal{S}_p\}$. In bipartite graphs, the weight of any fractional perfect matching equals the weight of a perfect matching [28], [45], i.e., $\frac{1}{b} \sum_{i=1}^{p} w(\mathcal{M}_{\mathcal{S}_i}) = w(\mathcal{M}_{\mathcal{S}})$ for a matching $\mathcal{M}_{\mathcal{S}}$.

Thus, neither the weight of any matching nor of any fractional matching is greater than the maximum weighted matching, such that the sum-GDoF constraints in $\mathcal{P}_{\mathcal{S}}^{\text{TINA}}$ will be more restrictive than those or any combinations in $\mathcal{P}_{\mathcal{S}}$, i.e., $\mathcal{P}_{\mathcal{S}}^{\text{TINA}} \subseteq \mathcal{P}_{\mathcal{S}}$. This completes the proof.

D. *Proof of Theorem 4*

In what follows, we prove that under condition (23), $\mathcal{P}_{\mathcal{S}}^{\text{TINA}}$ is monotonically increasing. Hence, from (6) this immediately implies that $\mathcal{R}^{\text{TINA}} = \mathcal{P}_{\mathcal{K}}^{\text{TINA}}$ which, by inspection, is a convex polytope.

Let us start with $|\mathcal{S}| = 2$. Suppose without loss of generality $\mathcal{S} = \{k, j\}$. Due to the condition (23), $\min\{\alpha_{kk}, \alpha_{jj}\} \geq \alpha_{kj} + \alpha_{jk}$, then it is easy to verify that $\mathcal{P}_{k}^{\text{TINA}} \subseteq \mathcal{P}_{\{k,j\}}^{\text{TINA}}$ and $\mathcal{P}_{j}^{\text{TINA}} \subseteq \mathcal{P}_{\{k,j\}}^{\text{TINA}}$.

Then, we prove the general cases with the following lemma.

Lemma 1. *Given a subgraph $\mathcal{G}[\mathcal{S}]$ with weights $\{\alpha'_{ij}, i,j \in \mathcal{S}\}$, the difference of maximum weighted matching with and without the user k is bounded by*

$$w(\mathcal{M}^*_{\mathcal{S}}) - w(\mathcal{M}^*_{\mathcal{S}\setminus\{k\}}) \leq \max_{i,j \in \mathcal{S}, i,j \neq k} \{\alpha_{ik} + \alpha_{kj} - \alpha'_{ij}\}$$

Proof. Suppose without loss of generality that the maximum weighted matching of $\mathcal{G}[\mathcal{S}]$ $(k \in \mathcal{S})$ includes links (i,k) and (k,j) with weights α'_{ik} and α'_{kj} respectively and $i,j \neq k$. Note that whether $i = j$ or not does not affect our proof. After removing user k and edges (i,k), (k,j) from the matching, and adding the link (i,j) with weight α'_{ij}, we have a matching for $\mathcal{S}\backslash\{k\}$. Thus, for all $\{(i,k),(k,j)\}\in\mathcal{M}^*_{\mathcal{S}}$, we have

$$\begin{aligned} & w(\mathcal{M}^*_{\mathcal{S}}) - w(\mathcal{M}^*_{\mathcal{S}\backslash\{k\}}) \\ & \le \max_{\{(i,k),(k,j)\}\in\mathcal{M}^*_{\mathcal{S}},i,j\in\mathcal{S},i,j\neq k} \{\alpha'_{ik}+\alpha'_{kj}-\alpha'_{ij}\} \quad (66) \\ & \le \max_{i,j\in\mathcal{S},i,j\neq k} \{\alpha'_{ik}+\alpha'_{kj}-\alpha'_{ij}\} \quad (67) \\ & = \max_{i,j\in\mathcal{S},i,j\neq k} \{\alpha_{ik}+\alpha_{kj}-\alpha'_{ij}\}. \quad (68) \end{aligned}$$

□

Together with the condition (23), we have

$$\alpha_{kk} \ge w(\mathcal{M}^*_{\mathcal{S}}) - w(\mathcal{M}^*_{\mathcal{S}\backslash\{k\}}) \quad (69)$$

for any user $k\in\mathcal{S}$. According to Theorem 3, the sum-GDoF constraints of $\mathcal{P}^{\rm TINA}_{\mathcal{S}\backslash\{k\}}$ consist of

$$\sum_{j\in\mathcal{S}'} d_j \le \sum_{j\in\mathcal{S}'} \alpha_{jj} - w(\mathcal{M}^*_{\mathcal{S}'}), \quad \forall \mathcal{S}'\subseteq\mathcal{S}\backslash\{k\}, \quad (70)$$

while with an additional k, $\mathcal{P}^{\rm TINA}_{\mathcal{S}}$ consists of the above sum-GDoF constrains in (70) as well as the following ones

$$\begin{aligned} \sum_{j\in\mathcal{S}'\cup\{k\}} d_j \le & \sum_{j\in\mathcal{S}'\cup\{k\}} \alpha_{jj} - w(\mathcal{M}^*_{\mathcal{S}'\cup\{k\}}) \quad (71) \\ & = \sum_{j\in\mathcal{S}'} \alpha_{jj} + \alpha_{kk} - w(\mathcal{M}^*_{\mathcal{S}'\cup\{k\}}), \quad \forall\mathcal{S}'\subseteq\mathcal{S}\backslash\{k\} \quad (72) \end{aligned}$$

It is readily verified that as long as (23) is satisfied, for any $\mathcal{S}'\subseteq\mathcal{S}\backslash\{k\}$, the region confined by the sum-GDoF constraint in (72) with user k is larger than that by the sum-GDoF constraint in (70) without user k. Thus, it follows that the projection of $\mathcal{P}^{\rm TINA}_{\mathcal{S}}$ onto $\mathcal{P}^{\rm TINA}_{\mathcal{S}\backslash\{k\}}$ is no smaller than $\mathcal{P}^{\rm TINA}_{\mathcal{S}\backslash\{k\}}$ according to the above relation of sum-GDoF constraints. In other words, with the additional user k, the GDoF region is not decreasing. It follows immediately that $\mathcal{P}^{\rm TINA}_{\mathcal{S}\backslash\{k\}}\subseteq\mathcal{P}^{\rm TINA}_{\mathcal{S}}$ $(\forall k\in\mathcal{S})$. More generally, if $\mathcal{S}_1\subseteq\mathcal{S}_2$, then $\mathcal{P}^{\rm TINA}_{\mathcal{S}_1}\subseteq\mathcal{P}^{\rm TINA}_{\mathcal{S}_2}$. This completes the proof.

E. Proof of Theorem 5

Due to the fact that $\mathcal{R}^*\supseteq\mathcal{R}^{\rm TINA}$ and that, under condition (23), $\mathcal{R}^{\rm TINA}=\mathcal{P}^{\rm TINA}_{\mathcal{K}}$, achievability trivially follows.

For the converse, we follow the cyclic outer bounds first revealed in [29, Theorem 2] and later used to prove the optimality of TIN condition in [3, Theorem 3].

Thus, for the K-user Gaussian interference channel in the weak interference regime, the GDoF region under the condition (23) is included in the set of GDoF tuples $(d_1,d_2,\dots,d_K)$ such that

$$d_j \le \alpha_{jj}, \quad \forall j\in\mathcal{K} \quad (73)$$

$$\sum_{j=0}^{m-1} d_{i_j} \le \min\{f_\pi, g_{\pi,0},\dots,g_{\pi,m-1}\}, \quad (74)$$

for any ordered subset $\pi=(i_0,i_1,\dots,i_{m-1})\subset\mathcal{K}^m$, where we define

$$f_\pi \triangleq \sum_{j=0}^{m-1} \max\{0,\alpha_{i_ji_{j+1}},\alpha_{i_ji_j}-\alpha_{i_{j-1}i_j}\} \quad (75)$$

$$g_{\pi,k} \triangleq \sum_{j=0}^{m-1} (\alpha_{i_ji_j}-\alpha_{i_{j-1}i_j}) + \alpha_{i_{k-1}i_k}, \quad k=0,\dots,m-1, \quad (76)$$

and where the index subscript arithmetic is modulo m.

When $m=2$, then condition (23) is equivalent to the GNAJ condition and the bound is known to be tight. When $m>2$, let us first consider the bound formed by the "g" terms in (74). Notice that the left-hand side of (74) depends only on the indices in π but not on its order. Hence, letting $\mathcal{S}$ denote a given unordered subset of size m of $\mathcal{K}$ and using the short-cut notation $\pi\in\pi(\mathcal{S})$ to indicate the ordered sets formed with the elements of $\mathcal{S}$, i.e., the permutations of $\mathcal{S}$, we can write

$$\begin{aligned} & \min_{\pi\in\pi(\mathcal{S})} \min_{k=0,\dots,m-1} \{g_{\pi,k}\} \\ & = \min_{\pi\in\pi(\mathcal{S})} \left\{ \sum_{j=0}^{m-1} (\alpha_{i_ji_j}-\alpha_{i_{j-1}i_j}) + \min_{k=0,\dots,m-1} \{\alpha_{i_{k-1}i_k}\} \right\} \\ & = \sum_{j\in\mathcal{S}} \alpha_{jj} - \max_{\pi\in\pi(\mathcal{S})} \left\{ \sum_{j=0}^{m-1} \alpha_{i_{j-1}i_j} - \min_{k=0,\dots,m-1} \{\alpha_{i_{k-1}i_k}\} \right\} \\ & = \sum_{j\in\mathcal{S}} \alpha_{jj} - w(\mathcal{M}^*_{\mathcal{S}}) \quad (77) \end{aligned}$$

where (77) is due to the condition (24). If the maximum weighted matching involves multiple cycles, then the sum-GDoF outer bound can be the combination of multiple sum-GDoF constraints associated with the corresponding cyclic sequences. Thus, (77) still holds, because condition (24) holds for any subset of $\mathcal{S}\subseteq\mathcal{K}$. Due to the fact that

$$\sum_{j=0}^{m-1} (\alpha_{i_ji_j}-\alpha_{i_{j-1}i_j}) \le \sum_{j=0}^{m-1} \max\{0,\alpha_{i_ji_{j+1}},\alpha_{i_ji_j}-\alpha_{i_{j-1}i_j}\}$$

we have that

$$\begin{aligned} \sum_{j\in\mathcal{S}} d_j & \le \min_{\pi\in\pi(\mathcal{S})} \min\{f_\pi,g_{\pi,0},\dots,g_{\pi,m-1}\} \quad (78) \\ & = \sum_{j\in\mathcal{S}} \alpha_{jj} - w(\mathcal{M}^*_{\mathcal{S}}) \quad (79) \end{aligned}$$

which coincides with $\mathcal{P}^{\rm TINA}_{\mathcal{S}}$ for every $\mathcal{S}\subseteq\mathcal{K}$. Under the condition (23), the TINA is the largest polyhedral region, so the converse bound is tight.

REFERENCES

[1] X. Yi and G. Caire, "On the optimality of treating interference as noise: A combinatorial optimization perspective," in *Proc. IEEE International Symposium on Information Theory (ISIT)*, Hong Kong, Jun. 2015.

[2] ——, "ITLinQ+: An improved spectrum sharing mechanism for device-to-device communications," in *The 49th Asilomar Conference on Signals, Systems, and Computers (Invited)*, Nov. 2015, pp. 1310–1314.

[3] C. Geng, N. Naderializadeh, A. S. Avestimehr, and S. A. Jafar, "On the optimality of treating interference as noise," *IEEE Trans. Inf. Theory*, vol. 61, no. 4, pp. 1753–1767, Apr. 2015.

[4] N. Naderializadeh and A. Avestimehr, "ITLinQ: A new approach for spectrum sharing in device-to-device communication systems," *IEEE J. Sel. Areas Commun.*, vol. 32, no. 6, pp. 1139–1151, June 2014.

[5] X. Shang, G. Kramer, and B. Chen, "A new outer bound and the noisy-interference sum–rate capacity for Gaussian interference channels," *IEEE Trans. Inf. Theory*, vol. 55, no. 2, pp. 689–699, 2009.

[6] A. S. Motahari and A. K. Khandani, "Capacity bounds for the gaussian interference channel," *IEEE Trans. Inf. Theory*, vol. 55, no. 2, pp. 620–643, 2009.

[7] V. S. Annapureddy and V. V. Veeravalli, "Gaussian interference networks: Sum capacity in the low-interference regime and new outer bounds on the capacity region," *IEEE Trans. Inf. Theory*, vol. 55, no. 7, pp. 3032–3050, 2009.

[8] H. Sun and S. A. Jafar, "On the optimality of treating interference as noise for K user parallel Gaussian interference networks," *IEEE Trans. Inf. Theory*, vol. 62, no. 4, pp. 1911–1930, Apr. 2016.

[9] C. Geng, H. Sun, and S. A. Jafar, "On the optimality of treating interference as noise: General message sets," *IEEE Trans. Inf. Theory*, vol. 61, no. 7, pp. 3722–3736, Jul. 2015.
[10] C. Geng and S. A. Jafar, "On the optimality of treating interference as noise: Compound interference networks," *arXiv:1412.2711*, Dec. 2014.
[11] A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*. Springer, 2003, vol. 24.
[12] H. W. Kuhn, "The Hungarian method for the assignment problem," *Naval Research Logistics Quarterly*, vol. 2, no. 1-2, pp. 83–97, 1955.
[13] J. Munkres, "Algorithms for the assignment and transportation problems," *J. SIAM*, vol. 5, no. 1, pp. 32–38, 1957.
[14] G. Demange, D. Gale, and M. Sotomayor, "Multi-item auctions," *The Journal of Political Economy*, pp. 863–872, 1986.
[15] R. Burkard, M. Dell'Amico, and S. Martello, *Assignment Problems, Revised Reprint*. SIAM, 2012, vol. 106.
[16] A. Lapidoth, "Nearest neighbor decoding for additive non-gaussian noise channels," *IEEE Trans. Inf. Theory*, vol. 42, no. 5, pp. 1520–1529, 1996.
[17] D. Tse and S. Hanly, "Multiaccess fading channels. I. polymatroid structure, optimal resource allocation and throughput capacities," *IEEE Trans. Inf. Theory*, vol. 44, no. 7, pp. 2796–2815, Nov 1998.
[18] X. Wu, S. Tavildar, S. Shakkottai, T. Richardson, J. Li, R. Laroia, and A. Jovicic, "FlashLinQ: A synchronous distributed scheduler for peer-to-peer ad hoc networks," *IEEE/ACM Trans. Networking*, vol. 21, no. 4, pp. 1215–1228, 2013.
[19] A. Gjendemsj, D. Gesbert, G. E. Oien, and S. G. Kiani, "Binary power control for sum rate maximization over multiple interfering links," *IEEE Trans. Wireless Commun.*, vol. 7, no. 8, pp. 3164–3173, 2008.
[20] J. Mo and J. Walrand, "Fair end-to-end window-based congestion control," *IEEE/ACM Trans.Networking*, vol. 8, no. 5, pp. 556–567, 2000.
[21] L. Georgiadis, M. J. Neely, and L. Tassiulas, *Resource Allocation and Cross-layer Control in Wireless Networks*. Now Publishers Inc, 2006.
[22] M. Chiang, C. W. Tan, D. P. Palomar, D. O'Neill, and D. Julian, "Power control by geometric programming," *IEEE Trans. Wireless Commun.*, vol. 6, no. 7, pp. 2640–2651, 2007.
[23] B. Hajek and G. Sasaki, "Link scheduling in polynomial time," *IEEE Trans. Inf. Theory*, vol. 34, no. 5, pp. 910–917, 1988.
[24] S. A. Borbash and A. Ephremides, "Wireless link scheduling with power control and SINR constraints," *IEEE Trans. Inf. Theory*, vol. 52, no. 11, pp. 5106–5111, 2006.
[25] W. Wang, Y. Wang, X.-Y. Li, W.-Z. Song, and O. Frieder, "Efficient interference-aware TDMA link scheduling for static wireless networks," in *Proc. ACM MobiCom*. Los Angeles, CA, USA: ACM, Sept. 2006, pp. 262–273.
[26] K. Jain, J. Padhye, V. N. Padmanabhan, and L. Qiu, "Impact of interference on multi-hop wireless network performance," *Wireless networks*, vol. 11, no. 4, pp. 471–487, 2005.
[27] C.-L. Hwang and K. Yoon, *Multiple Attribute Decision Making – Methods and Applications: A State-of-the-Art Survey*. Springer Science; Business Media, 2012, vol. 186.
[28] L. Lovász and M. D. Plummer, *Matching Theory*, New York, 1986.
[29] L. Zhou and W. Yu, "On the capacity of the K-user cyclic Gaussian interference channel," *IEEE Trans. Inf. Theory*, vol. 59, no. 1, pp. 154–165, 2013.
[30] X. Yi and D. Gesbert, "Topological interference management with transmitter cooperation," *IEEE Trans. Inf. Theory*, vol. 61, no. 11, pp. 6107 – 6130, Nov. 2015.
[31] G. J. Foschini and Z. Miljanic, "A simple distributed autonomous power control algorithm and its convergence," *IEEE Trans. Vehic. Tech.*, vol. 42, no. 4, pp. 641–646, 1993.
[32] R. D. Yates, "A framework for uplink power control in cellular radio systems," *IEEE J. Sel. Areas Commun.*, vol. 13, no. 7, pp. 1341–1347, 1995.
[33] S. V. Hanly, "An algorithm for combined cell-site selection and power control to maximize cellular spread spectrum capacity," *IEEE J. Sel. Areas Commun.*, vol. 13, no. 7, pp. 1332–1340, 1995.
[34] D. P. Palomar and M. Chiang, "A tutorial on decomposition methods for network utility maximization," *IEEE J. Sel. Areas Commun.*, vol. 24, no. 8, pp. 1439–1451, 2006.
[35] X. Lin, N. B. Shroff, and R. Srikant, "A tutorial on cross-layer optimization in wireless networks," *IEEE J. Sel. Areas Commun.*, vol. 24, no. 8, pp. 1452–1463, 2006.
[36] H. Shirani-Mehr, G. Caire, and M. J. Neely, "MIMO downlink scheduling with non-perfect channel state knowledge," *IEEE Trans. Commun.*, vol. 58, no. 7, pp. 2055–2066, 2010.
[37] M. Neely, "Stochastic network optimization with application to communication and queueing systems," *Synthesis Lectures on Communication Networks*, vol. 3, no. 1, pp. 1–211, 2010.
[38] ——, "Universal scheduling for networks with arbitrary traffic, channels, and mobility," in *49th IEEE Conference on Decision and Control (CDC)*. IEEE, 2010, pp. 1822–1829.
[39] E. Balas, "Disjunctive programming," *Annals of Discrete Mathematics*, vol. 5, pp. 3–51, 1979.
[40] E. Balas, S. Ceria, and G. Cornuéjols, "A lift-and-project cutting plane algorithm for mixed 0–1 programs," *Mathematical programming*, vol. 58, no. 1-3, pp. 295–324, 1993.
[41] E. Balas, "Projection, lifting and extended formulation in integer and combinatorial optimization," *Annals of Operations Research*, vol. 140, no. 1, pp. 125–161, 2005.
[42] N. Golrezaei, P. Mansourifard, A. F. Molisch, and A. G. Dimakis, "Base-station assisted device-to-device communications for high-throughput wireless video networks," *IEEE Trans. Wireless Communications*, vol. 13, no. 7, pp. 3665–3676, 2014.
[43] M. Zandi, M. Dong, and A. Grami, "Dynamic spectrum access via channel-aware heterogeneous multi-channel auction with distributed learning," *IEEE Trans. Wireless Communications*, vol. 14, no. 11, pp. 5913 – 5926, Nov. 2015.
[44] N. Naderializadeh and A. S. Avestimehr, "ITLinQ," *[Online] Available: http://www-scf.usc.edu/ naderial/ITLinQ.html*.
[45] E. R. Scheinerman and D. H. Ullman, *Fractional Graph Theory*. DoverPublications.com, 2011.

Xinping Yi (S'12 – M'15) received the B.E. degree from Huazhong Univ. of Sci. & Tech., and the M.E. degree from Univ. of Electronic Sci. & Tech. of China, both in Electrical Engineering. He joined Institut Eurécom, Sophia Antipolis, France in 2011 and obtained his Ph.D. degree in Telecommunications from École Nationale Supérieure des Télécommunications (now Télécom ParisTech), Paris, France, in 2014. From 2009 to 2011, he was a research engineer at Huawei Technologies, Shenzhen, China. Since November 2014, he has been a research and teaching associate at Technische Universität Berlin, Berlin, Germany. His research interests include information theory, signal processing, combinatorial optimization, and their applications in wireless communications, index coding and data analytics.

X. Yi was a recipient of the 2014 Chinese Government Award for Outstanding Self-financed Students Abroad.

Giuseppe Caire (S '92 – M '94 – SM '03 – F '05) was born in Torino, Italy, in 1965. He received the B.Sc. in Electrical Engineering from Politecnico di Torino (Italy), in 1990, the M.Sc. in Electrical Engineering from Princeton University in 1992 and the Ph.D. from Politecnico di Torino in 1994. He has been a post-doctoral research fellow with the European Space Agency (ESTEC, Noordwijk, The Netherlands) in 1994-1995, Assistant Professor in Telecommunications at the Politecnico di Torino, Associate Professor at the University of Parma, Italy, Professor with the Department of Mobile Communications at the Eurecom Institute, Sophia-Antipolis, France, and he is currently a professor of Electrical Engineering with the Viterbi School of Engineering, University of Southern California, Los Angeles and an Alexander von Humboldt Professor with the Electrical Engineering and Computer Science Department of the Technical University of Berlin, Germany.

He served as Associate Editor for the IEEE Transactions on Communications in 1998-2001 and as Associate Editor for the IEEE Transactions on Information Theory in 2001-2003. He received the Jack Neubauer Best System Paper Award from the IEEE Vehicular Technology Society in 2003, the IEEE Communications Society & Information Theory Society Joint Paper Award in 2004 and in 2011, the Okawa Research Award in 2006, the Alexander von Humboldt Professorship in 2014, and the Vodafone Innovation Prize in 2015. Giuseppe Caire is a Fellow of IEEE since 2005. He has served in the Board of Governors of the IEEE Information Theory Society from 2004 to 2007, and as officer from 2008 to 2013. He was President of the IEEE Information Theory Society in 2011. His main research interests are in the field of communications theory, information theory, channel and source coding with particular focus on wireless communications.