

# A NOTE ON NON-UNIQUE ENHANCEMENTS

ALICE RIZZARDO AND MICHEL VAN DEN BERGH

ABSTRACT. We give an easy example of a triangulated category, linear over a field  $k$ , with two different enhancements, linear over  $k$ , answering a question of Canonaco and Stellari.

In their recent survey paper on enhancements for triangulated categories, Canonaco and Stellari pose the following question:

**Question.** [1, Question 3.13] *Are there examples of triangulated categories, linear over a field  $k$ , with non-unique  $k$ -linear enhancement?*

Below we give such an example. In fact, we simply observe that the topological graded field example (see [6, §2.1]) can be made to work in the algebraic case<sup>1</sup>.

**Example.** *Let  $K = k(x_1, \dots, x_{n+1})$  and  $F = K[t, t^{-1}]$ , where  $n > 0$  is even,  $t$  has cohomological degree  $n$  and  $K$  is concentrated in degree zero. Since all homogeneous elements in  $F$  are invertible, we call  $F$  a graded<sup>2</sup> field. Let  $0 \neq \eta \in \mathrm{HH}_k^{n+1}(K)$ . Then, by Lemma 1 below,  $\tilde{\eta} = \eta \otimes d/dt$  is a non-zero element of  $\mathrm{HH}_k^{n+2}(F)^{-n}$ . Since, also by Lemma 1,  $\mathrm{HH}_k^s(F) = 0$  for  $s > n + 2$ , we may construct a minimal  $A_\infty$  structure  $(0, m_2, 0, \dots, 0, m_{n+2}, m_{n+3}, \dots)$  on  $F$  such that the class of  $m_{n+2}$  is  $\tilde{\eta}$  (see [4, Lem. B.4.1]). Let  $F_\eta$  be the resulting  $A_\infty$ -algebra and let  $f_1 \in \mathrm{Aut}_k(F)$ . One checks using [4, Lem. B.4.2]) or directly that  $\tilde{\eta} \circ f_1$  is the first obstruction against extending  $f_1$  to an  $A_\infty$ -isomorphism  $f : F \rightarrow F_\eta$ . Since  $\tilde{\eta}$  is non-trivial, the same is true for  $\tilde{\eta} \circ f_1$  and so  $F$  and  $F_\eta$  are not  $A_\infty$ -isomorphic.*

*As in [5], we see that the triangulated category  $\mathrm{Perf}(F_\eta)$  of right perfect  $F_\eta$ -modules is equivalent, as a graded category, to the category of graded  $F$ -vector spaces of finite rank. Since the latter category is semi-simple, it has only one triangulated structure compatible with the graded structure. Hence  $\mathrm{Perf}(F)$  and  $\mathrm{Perf}(F_\eta)$  are equivalent as triangulated categories.*

---

1991 *Mathematics Subject Classification.* 13D09, 18E30, 14A22.

*Key words and phrases.* Uniqueness of enhancement.

The first author is a Postdoctoral Research Fellow at the University of Edinburgh. She is supported by EPSRC grant EP/N021649/1. The second author is a senior researcher at the Research Foundation - Flanders (FWO). He is supported by the FWO-grant G0D8616N ‘‘Hochschild cohomology and deformation theory of triangulated categories.’’

<sup>1</sup>Another, more complicated, but also potentially more interesting, example is presented in [3]. The techniques in loc. cit. are based on an ingenious direct manipulation of solutions to the Maurer-Cartan equation. However the authors feel that as it stands, the arguments are not fully complete. In particular while it is possible to ‘‘remove strict units’’ [3, §4.4]) from solutions to the Maurer-Cartan solution, the required  $A_\infty$ -isomorphism will in general be more complicated than the proof suggests. This makes the verification of ‘‘Condition (1)’’ in loc. cit. more delicate (if at all possible) and therefore the same is true for the claim that the constructed functor is exact. We are currently discussing these points with the author.

<sup>2</sup>Throughout all graded notions are interpreted in the ‘‘super’’ sense.

On the other hand,  $\text{Perf}(F)$  and  $\text{Perf}(F_\eta)$  have canonical  $A_\infty$ -enhancement given by the  $A_\infty$ -categories of twisted complexes  $\text{tw}(F)$  and  $\text{tw}(F_\eta)$  (see [4, Ch. 7]). We claim that  $\text{tw}(F)$  and  $\text{tw}(F_\eta)$  are not  $A_\infty$ -equivalent. Indeed, any  $A_\infty$ -equivalence between them would have to send the indecomposable (right)  $F$ -module  $F_F$  to an object in  $\text{tw}(F_\eta)$  which is  $A_\infty$ -isomorphic to  $\Sigma^u(F_\eta)_{F_\eta}$  for some  $u$ . Hence  $\text{tw}(F)(F_F, F_F) \cong F$  and  $\text{tw}(F_\eta)(\Sigma^u(F_\eta)_{F_\eta}, \Sigma^u(F_\eta)_{F_\eta}) \cong F_\eta$  would have to be  $A_\infty$ -isomorphic  $A_\infty$ -algebras (since they are both minimal). This is not the case as we have established above.

*Remark.* There is nothing special about the particular pair  $(K, \eta)$  we have used. The chosen  $(K, \eta)$  simply allows for the most trivial argument for the existence of an  $A_\infty$ -structure on  $F$  with the given  $m_{n+2}$ .

We have used the following basic lemma:

**Lemma 1.** *Let  $F = K[t, t^{-1}]$  be as above. The Hochschild cohomology of  $F$  is given by  $\text{HH}_k^*(F) \cong \text{HH}_k^*(K) \otimes_k \text{HH}_k^*(k[t, t^{-1}])$ . Moreover  $\text{HH}^i(k[t, t^{-1}]) = 0$  for  $i > 1$  and the derivation  $d/dt$  represents a non-trivial element of  $\text{HH}^1(k[t, t^{-1}])$ .*

*Proof.* By the next lemma we only have to understand  $\text{HH}^*(k[t, t^{-1}])$ . Since  $t$  has even degree,  $F$  is graded commutative and so the claim follows from the graded version of the HKR theorem.  $\square$

**Lemma 2<sup>3</sup>** *Let  $A, B$  be graded  $k$ -algebras such that the graded tensor product  $B^e := B \otimes_k B^o$  is noetherian. Then  $\text{HH}^*(A \otimes_k B) \cong \text{HH}^*(A) \otimes_k \text{HH}^*(B)$ .*

*Proof.* Let  $Q^\bullet$  be a resolution of  $B$  by finitely generated graded projective  $B$ -bimodules, and let  $P^\bullet$  be an arbitrary resolution of  $A$  by graded projective  $A$ -bimodules. Then  $P^\bullet \otimes_k Q^\bullet$  is a graded projective resolution of  $A \otimes_k B$  and we have

$$\begin{aligned} \text{HH}^*(A \otimes_k B) &= H^*(\text{Hom}_{A^e \otimes_k B^e}(P^\bullet \otimes_k Q^\bullet, A \otimes_k B)) \\ &\cong H^*(\text{Hom}_{A^e}(P^\bullet, A) \otimes_k \text{Hom}_{B^e}(Q^\bullet, B)) \\ &\cong \text{HH}^*(A) \otimes_k \text{HH}^*(B). \end{aligned} \quad \square$$

## REFERENCES

- [1] P. Stellari A. Canonaco, *A tour about existence and uniqueness of dg enhancements and lifts*, arXiv:1605.00490 [math.AG].
- [2] P. A. Bergh and S. Oppermann, *Cohomology of twisted tensor products*, J. Algebra **320** (2008), no. 8, 3327–3338.
- [3] H. Kajiura, *On  $A_\infty$ -enhancements for triangulated categories*, J. Pure Appl. Algebra **217** (2013), no. 8, 1476–1503.
- [4] K. Lefèvre-Hasegawa, *Sur les  $A_\infty$ -catégories*, Ph.D. thesis, Université Paris 7, 2003.
- [5] J. Lurie, *Chromatic homotopy theory (252x)*, Course Notes, Lecture 24.
- [6] S. Schwede, *The stable homotopy category has a unique model at the prime 2*, Adv. Math. **164** (2001), no. 1, 24–40.

---

<sup>3</sup>This lemma is stated in [2, Thm 4.7] without any hypotheses on  $A, B$ . However the proof in loc. cit. (essentially the proof we have given) requires some kind of finiteness hypothesis which seems to have been inadvertently omitted. Indeed the result is false if  $A, B$  are fields of infinite transcendence degree over  $k$ .

*E-mail address*, Alice Rizzardo: `alice.rizzardo@ed.ac.uk`

SCHOOL OF MATHEMATICS, THE UNIVERSITY OF EDINBURGH, JAMES CLERK MAXWELL BUILDING, THE KING'S BUILDINGS, PETER GUTHRIE TAIT ROAD, EDINBURGH, EH9 3FD, SCOTLAND, UK

*E-mail address*, Michel Van den Bergh: `michel.vandenbergh@uhasselt.be`

UNIVERSITEIT HASSELT, UNIVERSITAIRE CAMPUS, 3590 DIEPENBEEK, BELGIUM