Fractional Differential Equations in Risk Theory

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A thesis submitted for the degree of

Doctor of Philosophy

November 2017

To my parents

Abstract

This thesis considers one of the central topics in the actuarial mathematics literature, deriving the probability of ruin in the collective risk model. The classical risk model and renewal risk models are focused in this project, where the claim number processes are assumed to be Poisson counting processes and any general renewal counting processes, respectively. The first part of this project is about the classical risk model. We look at the case when claim sizes follow a gamma distribution. Explicit expressions for ruin probabilities are derived via Laplace transform and inverse Laplace transform approach. The second half is about the renewal risk model. Very general assumptions on inter-arrival times are possible for the renewal risk model, which includes the classical risk model, Erlang risk model and fractional Poisson risk model. A new family of differential operators are defined in order to construct the fractional integro-differential equations for ruin probabilities in such renewal risk models. Through the characteristic equation approach, specific fractional differential equations for the ruin probabilities can be solved explicitly, allowing for the analysis of the ruin probabilities.

Acknowledgement

I would like to express my first and foremost appreciation and thanks to my PhD supervisor, **Dr. Corina Constantinescu**. Corina has been a tremendous mentor for me over the past five years and will also be for the rest of my life. She always makes research so appealing to me, which has motivated my decision to pursue an academic career. I would like to thank her for encouraging my research and for training me as an actuarial mathematician. She is not only an academic adviser but also a life mentor to me. Her advice has been very helpful and valuable. Her enthusiasm and love for students is infectious. Without her careful attention, encouragement and guidance I would have not been where I am now. I am grateful that over the years she has been devoting patience and confidence in my abilities.

In addition, I would like to thank my second PhD supervisor, **Prof. Jorge Ramirez**. Jorge has taught me what is the real passion for mathematics and he always makes the learning process very intriguing. Special thanks to him for fully taking care of me during my stay in Colombia. His way of teaching has given me a lot of inspiration. I have been strongly encouraged by his liveliness, enthusiasm and energy as an academic.

I would also like to mention the coauthor **Prof. Gennady Samorodnitsky** on my first research paper. Many thanks for his scientific advice and knowledge and many insightful discussions and suggestions. Without his high standards for research, our paper would not have been proceeded so smoothly.

Furthermore, I would also like to thank my PhD thesis committee members, the external examiner, **Prof. José Garrido**, and the internal examiner, **Dr. Apostolos Papaioannou** for agreeing to be my committee members. I also want to thank them for making my viva an unforgettable moment, and for their brilliant comments and suggestions.

I would like to thank all the host professors and friends in the RARE-318984 project, a Marie Curie IRSES Fellowship within the 7th European Community Framework Program, for their hospitality and friendly support. Special thanks to **Prof. Jiro Akahori** at Ritsumeikan University. Jiro is one of the nicest persons I have met in my life. He has been very supportive and has provided insightful advises on my research during my three visits to Japan. I would like to thank **Dr. Yuri Imamura**, for giving us informative lectures in Liverpool and taking care of me as well as my friends while we were in Tokyo and Minami Kusatsu. I would like to thank **Prof. Junyi Guo**, for involving me in all weekly research seminars and events in Nankai University, Tianjin. Special thanks to **Dr. Bo Li**, who has been my primary resource of abundant technical and analytical knowledge in theory. I would like to thank **Prof. Kais Hamza**, and **Dr. Andrea Collevecchio**, for their friendly and thoughtful host at Monash University in the beautiful city of Melbourne. Thanks to **Prof. Anna Panorska** and **Prof. Tomasz Kozubowski** for inviting me to visit the University of Nevada, Reno.

I would also like to thank the staff at the Institute for Financial and Actuarial Mathematics, University of Liverpool. Each one of them has been there to support me whenever I need any help. Thanks to **Dr. Corina Constantinescu**, **Dr. Olivier Menoukeu Pamen**, **Dr. Apostolos Papaioannou**, **Dr. Leonardo Rojas Nandayapa**, **Dr. Carmen Boado Penas** and **Dr. Bujar Gashi**. I would also like to express my special gratitude to **Prof. Kurt Langfeld**, for his substantial support on our summer research internship programs.

I wish to thank my very dear math friends. Special thanks to **Dr. Weihong Ni**, for her selfless help on both of my research and administrative issues. Thanks to (in alphabetical order) **Marcel Bräutigam**, **Yunzhou Chen**, **Suhang Dai**, **Daiyun Huang**, **Haoyu Qian**, **Lewis Ramsden**, **Zhongyang Sun** who made the PhD study an unforgettable part of my life.

Most of all I would like to thank my family who supported me through this long process and never ceased to believe in me. Words cannot express how grateful I am to my **mother** and **father** for all of their sacrifices. At the end I would like to express appreciation to my beloved girlfriend (now wife) **Maoqi Hu** who has always been my support from all aspects in life. Her love and understanding was very crucial for my success.

Contents

List of Symbols

$R(t)$	surplus process in risk model
X_i	individual claim size
T_i	inter-arrival time
$N(t)$	number of claims
$D(t)$	surplus process in dual risk model
Y_i	individual gain size
$\mathbb{R}$	real line
$\mathbb{C}$	complex plane
$\mathbb{N}$	natural numbers (including zero)
$\mathbb{Z}$	integer numbers
$\mathbb{Q}$	rational numbers
f_X	probability density function of random variable X
F_X	distribution function of random variable X
$\bar{F}_X$	tail distribution function of random variable X
$\mathbb{E}(X)$	expected value of random variable X
$\mathbb{V}\text{ar}(X)$	variance of random variable X
$\mathcal{L}\{f(x);\, s\}$	Laplace transform of function $f(x)$ with argument s
$\hat{f}$	simple version of the Laplace transform of function f
M_X	moment-generating function of random variable X
G_X	probability-generating function of random variable X
$f * g$	convolution of functions f and g
$\Gamma(z)$	gamma function

E_α	one-parameter Mittag-Leffler function
$E_{\alpha,\beta}$	two-parameter Mittag-Leffler function
$\lceil x \rceil$	ceiling function
$\lfloor x \rfloor$	floor function
$L^p([a,b])$	L^p spaces on $[a,b]$
${}_aI_x^r$	left Riemann-Liouville fractional integral
${}_xI_b^r$	right Riemann-Liouville fractional integral
${}_{-\infty}I_x^r$	left Weyl fractional integral
${}_xI_\infty^r$	right Weyl fractional integral
${}_aD_x^r$	left Riemann-Liouville fractional derivative
${}_xD_b^r$	right Riemann-Liouville fractional derivative
${}_{-\infty}D_x^r$	left Weyl-Liouville fractional derivative
${}_xD_\infty^r$	right Weyl-Liouville fractional derivative
${}_a^CD_x^r$	left Caputo fractional derivative
${}_x^CD_b^r$	right Caputo fractional derivative
$\mathcal{L}\left(\frac{d}{dx}\right)$	linear differential operator
$\mathcal{L}_T^*\left(\frac{d}{dy}\right)$	formal adjoint of $\mathcal{L}\left(\frac{d}{dx}\right)$
$\bigodot$	composition of operators
${}_a^\alpha R_x^r$	left Rock operator
${}_x^\alpha R_b^r$	Right Rock operator
$f^{m,n}$	probability density function of sum of m gamma and n ML random variables
$\mathcal{A}_{m,n}\left(\frac{d}{dx}\right)$	the fractional differential operator on $f^{m,n}$
$\mathcal{A}_{m,n}^*\left(\frac{d}{dy}\right)$	formal adjoint of $\mathcal{A}_{m,n}\left(\frac{d}{dx}\right)$
$R_{m,n}(t)$	surplus process in renewal risk model with inter-arrival time density $f_T^{m,n}$
$D_{m,n}(t)$	surplus process in renewal dual risk model with inter-arrival time density $f_T^{m,n}$
$N_{m,n}(t)$	counting process with inter-arrival time density $f_T^{m,n}$

List of Figures

List of Tables

Chapter 1

Introduction and Literature Review

1.1 Risk theory

The story of collective risk theory starts from the days of Lundberg and has been developed by Cramér, Segerdahl, Esscher, Ammeter, de Finetti and many other actuarial and mathematical researchers. The traditional approach is to consider the model of the surplus of an insurance company and find the probability that the risk reserve drops below zero, which is known as the ruin probability. The most fundamental model is the Cramér-Lundberg model or classical risk process,

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad \left(\sum_{i=1}^{0} X_i := 0 \right), \quad t > 0. \tag{1.1}$$

This model was proposed by Lundberg (1903, 1926) and Cramér (1930). The classical risk model describes the surplus of an insurance company over time, $R(t)$, where the non-negative constant u is the initial capital, the positive constant c is the income cashflow rate, the Poisson process $N(t)$ describes the total claim number before or at time t and the random variable X_i represents the i-th claim size. Moreover, all random variables in the classical model are assumed to be independent.

Deriving the ruin probability (see definition in expression (2.23)) is a central topic in the risk theory literature. Lundberg (1926) derived a bound and the asymptotic behavior for the ruin probability in the classical model, making use of an equation that in the risk theory literature is commonly referred to as the Lundberg's equation (see in (2.24)). The simplest case of a classical risk model is when claim sizes are exponentially distributed. Cramér (1930) used a differential argument to derive an equation for the non-ruin probability and solved it under the assumption of exponential claim sizes. As a sequential survey, Cramér (1955) gave a version of the differential argument. Since then, many actuarial mathematicians have analysed the ruin problem. One direction of research analyses ruin results for particular claims' distributions. Numerous approximations (Beekman, 1969; Kingman, 1962; Bloomfield and Cox, 1972; De Vylder, 1978; Willmot and Lin, 2001) and asymptotic results (Klüppelberg et al., 2004; Palmowski and Pistorius, 2009; Albrecher et al., 2012) have been derived, especially for heavy-tailed claims (Ramsay, 2003). However, ever since the explicit form of ruin probabilities in the case of exponential claim sizes was established, searching for explicit formulas for other (light-tailed) distributions has become a frequent direction of research.

The literature of deriving explicit expressions for the ruin probability of the classical compound Poisson risk model for various claims distributions is abundant in methods and results. Feller (2008) and Cramér (1955) separately derived the non-ruin probability $\phi(u)$ as a solution of an integro-differential equation, which, under some conditions, can be solved analytically when the claims are exponentially distributed by either differentiating both sides or taking the Laplace transform. Gerber (1973) used martingales to analyse the risk process with independent and stationary increments. Pakes (1975) derived the relationship between ruin probability and tail distribution of the claim severity. Thorin and Wikstad (1977) analysed the ruin problem when claims are log-normal distributed. Gerber et al. (1987) obtained the ruin probability for mixed Erlang-distributed claims by studying the severity of ruin, as well as its probability. Ramsay (2003) inverted the Laplace transforms over the complex domain to derive a closed-form solution of the ruin probability when the claim sizes follow a special Pareto distribution. Hubalek and Kyprianou (2011) considered a class of spectrally negative Lévy processes, called Gaussian tempered stable convolution, whose Lévy measure has a gamma component with shape parameter less than or equal to 1. They showed that their scale functions, which are essentially proportional to the non-ruin probability, admit expressions in terms of Mittag-Leffler functions.

The first half of this thesis falls into this category, exploring the classical ruin model with gamma-distributed claims, extending and generalising earlier results of Thorin (1973). Among the first distributions considered in the risk theory literature are the integer shaped gammas, or the so-called Erlang distributions. These have rational Laplace transforms and at the same time are phase-type distributed, dense in the class of continuous distributions. The ruin probability has closed-form expressions for classical risk models with phase-type claims, see e.g. Asmussen and Albrecher (2010) or rational Laplace transform distributions, see e.g. Albrecher et al. (2010). One interest of this thesis is to focus on gamma-distributed claim sizes with the shape parameter $r > 0$ and the rate parameter $\alpha > 0$ in classical risk model. When the shape parameter r is integer, namely when the claims are Erlang-distributed, then explicit expressions for non-ruin probabilities appear due to the fact that its Laplace transform can be written as the ratio of two polynomial functions. One can then use a partial fraction decomposition and further invert back to obtain a linear combination of exponential functions (Grandell, 1991a).

In Chapter 3 we go beyond Erlang distributions and derive results for gamma distributions that allow real shape parameters, using Laplace transform properties and deriving Pollaczeck-Khinchine type formulas. We first use shifting Laplace transform to get the expression of the non-ruin probability in closed-form when rational shape gamma claim sizes are assumed (see equation (3.4)). When the shape parameter is a rational number $r = m/n \in \mathbb{Q}$, the shifted Laplace transform of non-ruin probability is a ratio of polynomials of orders m and $m + n$ in $t = s^{1/n}$. This again permits a partial fraction decomposition which can further give an explicit expression of non-ruin probability consisting of Mittag-Leffler functions. However, extending these results to a real shape parameter r proves to be non-trivial. Prior to this work, the only known (to us) result for non-integer shape gamma-distributed claims is that of Thorin (1973) and it deals with a special case of the $\Gamma(1/b, 1/b)$, $b > 1$, distribution, although the results in general can only be calculated numerically. Then we comment on the merit of the series expressions obtained. More precisely, for real-shaped gamma claims, we first introduce two different methods, leading to two different series expressions in terms of Mittag-Leffler functions. The first approach is to extend the method of shifting Laplace transform to the real shape parameter case. Using geometric expansions, we present an explicit form in terms of an infinite sum of convolutions of exponential and Mittag-Leffler functions (see equation (3.8)). The second approach is also applying geometric expansion, but from a different aspect. We derive an explicit form in terms of an infinite sum of derivatives of Mittag-Leffler functions (see equation (3.12)), by carefully reconstructing geometric sum on the Laplace side. Moreover, for the third approach, we present a general Pollaczeck-Khinchine type form for the ruin probability in the classical Cramér-Lundberg model with light-tailed claims, in terms of moments, which in case of gamma claims reduces to a third, tractable expression (3.18). This last approach actually applies to any claim size distributions with finite moments, gamma distribution being a special case. All three results are shown to retrieve the classical exponential ruin probability result when reduced to exponential claims. Some numerical computations are presented afterwards to show the advantage of each expression. Independently, we also consider discrete claim sizes assumption in the classical risk model. Explicit expression (3.22) of ruin probability is obtained when geometric claims by using martingale method and optional stopping theorem.

Another direction that captured a lot of attention over the last hundred years is to consider altering the assumptions of either independence or memory loss of claim arrivals, thus analysing ruin probabilities in renewal risk models (Andersen, 1957) or models with various dependence structures (Albrecher and Boxma, 2004, 2005; Cheung et al., 2010; Constantinescu et al., 2013). Considering a gamma aggregate claims process, Dufresne et al. (1991) derived bounds for the ruin probabilities. Adding financial considerations to the model, such as returns in investments, see e.g. Garrido (1989); Paulsen (1998); Frolova et al. (2002); Kalashnikov and Norberg (2002); Paulsen (2008); Albrecher et al. (2012); Ramsden and Papaioannou (2017), interest rate models, see e.g. Cai and Dickson (2004) or perturbations in premium cash-flow, see e.g. Temnov (2014), asymptotics of ruin probabilities have been derived. Lévy risk models were considered and first passage and exit times were derived via fluctuation theory and scale functions, see e.g. Furrer et al. (1997); Furrer (1998); Yang and Zhang (2001); Avram et al. (2002); Kyprianou (2006); Garrido and Morales (2006); Palmowski and Pistorius (2009); Hubalek and Kyprianou (2011).

The renewal risk model in risk theory is also known as the Sparre Andersen model. The number of claims $N(t)$ is allowed to follow not only a Poisson counting process, but also a more general renewal process. The ruin probability $\psi(u)$ of a renewal risk model solves an integral equation, obtained from the renewal property (see e.g. Feller (2008)). Among renewal risk processes, Dickson (1998) and Dickson and Hipp (1998, 2001) considered the risk process with Erlang$(2, \beta)$ distributed or mixed 2-exponentially distributed inter-arrival times. They obtained an explicit expression for the Laplace transform of the ruin probability by solving a second-order differential equation. Lin and Willmot (1999) studied in detail the solution of a defective renewal equation which involves the time of ruin, the surplus immediately before ruin, and the deficit at the time of ruin. As a subsequent paper, Lin and Willmot (2000) derived explicitly the joint and marginal moments of the time of ruin, the surplus before ruin, and the deficit at ruin in the case where the inter-arrival times have a K_n distribution, for which the associated Laplace-Stieltjes transform is the ratio of a polynomial of degree $m < n$ to a polynomial of degree n. Examples are given for the cases when the claim size distribution is exponential, combinations of exponentials or mixtures of Erlangs. Dufresne (2002) obtained the Laplace transform of the non-ruin probability for a wide class of inter-arrival times with a rational Laplace transform representation. Li and Garrido (2004b) used a similar approach in Gerber and Shiu (1998) to derive a defective renewal equation for the expected discounted penalty due at ruin in a risk model with Erlang(n, β) inter-arrival times. Li and Garrido (2004a) considered a compound renewal risk process in the presence of a constant dividend barrier in which the claim inter-arrival times are generalised Erlang(n, β) distributed and derived an integro-differential equation with certain boundary conditions for the Gerber-Shiu function. Li and Garrido (2005) considered a compound renewal risk process whose inter-claim times that have a K_n distribution and derived the Laplace transform of the expected discounted penalty function at ruin. Gerber and Shiu (2005) analysed the renewal equation for Gerber-Shiu functions in Sparre Andersen models and consider the case of inter-arrival times following a sum of exponentials as a special case. Constantinescu (2006) used a unified approach to derive an integro-differential equation for the probability of ruin, under conditions regarding the claim sizes, claim arrivals and the returns from investment. Chen et al. (2007) derived a linear ordinary differential equation for the Gerber-Shiu function in a Poisson jump-diffusion process with phase-type jumps and solved it explicitly for penalty functions that depend only on the deficit at ruin. Chadjiconstantinidis and Papaioannou (2013) considered an extension to the compound Poisson risk process perturbed by diffusion in which two types of dependent claims, main claims and by-claims, are incorporated. An integro-differential equation system for the Gerber-Shiu expected discounted penalty functions is derived and solved by proving that the Gerber-Shiu function satisfies a defective renewal equation.

There are two algebraic structures for treating integral operators in conjunction with derivations, integro-differential operators and integro-differential polynomials. The name of integro-differential operator was first introduced in Rosenkranz and Regensburger (2008b) to describe a differential equation, its boundary conditions and its solution operator (Greens operator) of a BVP (linear boundary value problem for ordinary differential equations) in a uniform language. In Rosenkranz and Regensburger (2008a), the authors presented another terminology called "integro-differential polynomial", which allows them to deal with nonlinear integro-differential equations. This algebraic symbolic structure has nice applications in ruin theory. For instance, as an extension of the Erlang risk model, Albrecher et al. (2010) transformed the integral equation for the Gerber-Shiu function into an integro-differential equation whenever the inter-arrival time distributions exhibit rational Laplace transforms. If the claim size distributions also have rational Laplace transforms, the integro-differential equations can be further reduced to linear boundary value problems with the appropriate boundary conditions. This applies to random variables that are mixed or sums of Erlang random variables. Another application of these algebraic structures is to the renewal insurance model with premiums depending on the present surplus of the insurance portfolio. Albrecher et al. (2013) developed a symbolic technique to obtain asymptotic expressions for ruin probabilities and discounted penalty functions, based on boundary problems for linear ordinary differential equations with variable coefficients.

In Chapter 4 of this thesis, we present an approach to generalise the results from Albrecher et al. (2010). Inspired from Babenko (1986) and Podlubny (1998), we define a new class of operators, fractional differential operators, that we will refer to as the Rock operators in Section 4.1. Using these new operators, we are able to construct fractional differential equations for densities of random variables which are mixture or sums of heterogeneous gamma random variables and Mittag-Leffler random variables. These fractional differential equations can be regarded as extensions of homogeneous ordinary differential equations with constant coefficients. The Rock operators themselves possess many good properties since they are all defined by fractional derivatives, which are well developed in the fractional calculus literature. As an application, we consider a risk model (see in Section 4.2.1) with inter-arrival times from this family of distributions. The corresponding ruin probability will satisfy a fractional integro-differential equation due to the property of time density function. Furthermore, in this model, if the claim sizes are assumed to be a sum of heterogeneous gamma random variables, the ruin probability satisfies a fractional differential equation with constant coefficients.

In Section 4.3, we look into two specific risk models, in which the time density functions both solve fractional differential equations. The first model, gamma-time risk model, is a natural generalisation of Erlang risk model considered in Li and Garrido (2004b). Since the parameter estimation in practice always gives non-integer valued shape parameters for gamma random variables, it is quite necessary to study the ruin theory related to the non-integer-valued gamma random variables. Prior to this work, there are no obtained results for non-integer shape gamma-distributed time in risk theory literature. Another risk model considered is the (time-)fractional Poisson risk model. This model has been analysed by Beghin and Macci (2013) and Biard and Saussereau (2014), the fractional differential equation approach being applied for the first time on the fractional Poisson risk model. The ruin probabilities in these two models can be solved explicitly when the claim sizes are from the class of distributions with rational Laplace transforms and retrieved by our approach. Several concrete examples are discussed in details in Section 4.3.

The Rock operators can also be applied in the analysis of the dual risk model. The dual risk model, describes the surplus $D(t)$ of a company as

$$D(t) = u - ct + \sum_{i=1}^{N(t)} Y_i, \quad \left(\sum_{i=1}^{0} Y_i := 0\right), \quad t > 0, \tag{1.2}$$

where $u \geqslant 0$ is the initial capital and $c > 0$ represents the constant expense rate. The aggregate income up to time t is assumed to be a renewal counting process $N(t)$ and Y_i independent, identically distributed, representing the amount of individual gain. This model has been of increasing interest in ruin theory in recent years. The risk model (1.1) is appropriate for an insurance company, and the dual risk model (1.2) seems to be natural for companies that have occasional gains whose amount and frequency can be modeled by the process $\sum_{i=1}^{N(t)} Y_i$. Avanzi et al. (2007) explained thoroughly where applications of the dual model make sense, e.g. pharmaceutical or petroleum companies, and commission-based businesses. The ruin probability is also one of the main issues people consider. Among pioneering work on the subject we would like to cite Cramér (1955); Takács (1966); Seal (1969); Gerber (1979); Bühlmann (2007), and the references cited therein.

Recent works on the dual risk model mainly focus on the optimal dividend problem, along with the ruin problem analysis. By using Laplace transforms, Avanzi et al. (2007) studied the optimal dividend problem for a Lévy process which is skip-free downwards, whereas Avanzi and Gerber (2008) studied the similar dividend problem when the aggregate gains process is the sum of a shifted compound Poisson process and an independent Wiener process. Two practical optimization problems in relation to venture capital investments and/or Research and Development (R&D) investments have been studied by Bayraktar and Egami (2008). Cheung and Drekic (2008) derived integro-differential equations for the moments of the total discounted dividends as well as the Laplace transform of the time of ruin, which can be solved explicitly assuming the jump size distribution has a rational Laplace transform. Gerber and Smith (2008) investigated and examined the De Vylder approximations, diffusion approximations, gamma approximations and the gamma process approximations in the dual model. Song et al. (2008) allowed the surplus process in the dual risk model to continue if the surplus falls below zero. By introducing the renewal measure of the defective renewal sequence constituted by the zero points of the surplus process, Song et al. (2008) obtained the probability of hitting the zero point and derived formulas for the Laplace transform, expectation and variance of the total duration of negative surplus. Zhu and Yang (2008) considered both the finite and infinite horizon ruin probabilities under a dual Markov-modulated risk model. Upper and lower bounds of Lundberg type are derived for these ruin probabilities and a time-dependent version of Lundberg type inequalities is obtained. Ng (2009) considered the compound Poisson dual risk model under a threshold dividend strategy and derived a set of two integro-differential equations satisfied by the expected total discounted dividends until ruin. Ng (2010) studied the compound Poisson dual risk model when gains follow a phase-type distribution and obtained explicit formulas for the expected total discounted dividends until ruin and the Laplace transform of the time of ruin under a variety of dividend strategies.

By establishing a proper connection between the compound Poisson dual risk model and the classical risk model, Afonso et al. (2013) showed and explained the dividends process dynamics for the dual risk model. Properties for the different random quantities involved and their relations lead to an analysis of different ruin and dividend probabilities, such as the calculation of the probability of a dividend, the distribution of the number of dividends, the expected and amount of dividends as well as the time of getting a dividend. Moreover, integro-differential equations for some of the above results and their Laplace transforms are obtained. Yang and Sendova (2014) studied the renewal dual risk model in which the times between positive gains are independent and identically distributed and have a generalised Erlang(n) distribution. They derived an explicit expression for the Laplace transform of the ruin time and obtained the expected discounted dividends with a threshold-dividend strategy. Rodríguez-Martínez et al. (2015) generalised the compound Poisson dual risk models to renewal dual risk models where waiting times are Erlang(n) distributed, and obtained expressions for the ruin probability and the Laplace transform of the time of ruin for an arbitrary single gain distribution by using the roots of the fundamental and the generalised Lundberg's equations. Furthermore, Rodríguez-Martínez et al. (2015) computed expected discounted dividends, as well as higher moments, when the individual common gains follow a phase-type distribution. Bergel et al. (2017) studied the renewal dual risk model when the waiting times are phase-type distributed. Using the roots of the fundamental and the generalised Lundbergs equations, Bergel et al. (2017) obtained expressions for the ruin probability and the Laplace transform of the time of ruin for an arbitrary single gain distribution and addressed the calculation of the expected discounted future dividends particularly when the individual common gains follow a phase-type distribution. In Chapter 4 of this thesis, we consider a dual risk model in which the inter-arrival times are assumed to be the sum of gamma-distributed random variables (see in Section 4.2.2). Fractional integro-differential equations for non-ruin probabilities are obtained using the Rock operators.

1.2 Fractional calculus

Fractional calculus was born in 1695, when Leibniz (1695) wrote to l'Hôpital in a letter about half derivatives. In his answer, Leibniz foresaw the beginning of the fractional calculus area. In fact, fractional calculus is as old as the traditional calculus proposed independently by Newton and Leibniz (Dzherbashyan and Nersesian, 1958; Oldham and Spanier, 1974; Miller and Ross, 1993). In the following centuries, lots of mathematicians tried to find the "proper" definition for fractional integrals and fractional derivatives (Samko et al., 1993; Rubin, 1996; Butzer and Westphal, 2000; Hilfer, 2008).

In classical calculus, the derivative has an important geometric interpretation, namely, it is associated with the concept of tangent. However, this interpretation failed in the fractional case until recent decades (Podlubny, 2004; Tavassoli et al., 2013; Karci, 2015). This difference can be seen as a problem for the slow progress of fractional calculus up to 1900. After Lebniz, it is Euler (1738) who constructed one kind of non-integer derivatives when he was introducing the gamma function. After almost another century, Fourier (1822) suggested an integral representation in order to define the fractional derivative. This version is the first definition for the derivative of arbitrary (positive) order in the literature. Meanwhile, fractional calculus has started to attract attention from the 1820s. The first application of fractional calculus is in solving an integral equation associated with the tautochrone problem (Abel, 1826). Liouville (1832) defined fractional derivatives for functions representable as a sum of exponentials as for differentiating the exponential function. This expression is known as the first version of Liouville's definition. The second definition formulated by Liouville is in terms of an integral and is now called the version by Liouville for fractional integral (see Definition 2.1.2). Liouville

(1832) also introduced the modified Riemann-Liouville fractional derivatives by applying fractional integrals on integer-order derivatives, which are lately named after Caputo (Caputo and Mainardi, 1971a). Riemann (1876) published an important paper in the history of fractional calculus after a series of works by Liouville, in which the Riemann-Liouville definition is formulated (see Definition 2.1.4). Both Liouville and Riemann formulations carry the so-called complementary function, which remains a problem to be solved. Independently of each other, Grünwald (1867) and Letnikov (1868) developed an approach to non-integer order derivatives in terms of a convenient convergent series, not by an integral as in the Riemann-Liouville approach. Specifically, Grünwald (1867) claimed that the fractional derivatives are integro-derivatives and he was the first to establish a general fractional derivative operator by taking limit of fractional difference quotients. Letnikov (1868) showed that his definition coincides with the versions formulated by Liouville, for particular values of the order, and with Riemann, under a convenient interpretation of the so-called non-integer order difference. Hadamard (1892) discussed the case when the non-integer order derivative of an analytic function must be taken in terms of its Taylor series.

The developments of fractional calculus afterwards have become much more systematic. Weyl (1917) introduced a derivative in order to circumvent a problem involving the periodic functions. In the same paper, the Weyl-Liouville fractional derivative is introduced, which is a special case of the Riemann-Liouville fractional derivative, with upper or lower infinite limit of integral (see Definition 2.1.5). Marchaud (1927) introduced a new definition for non-integer order of derivatives, which is equivalent to the Liouville version for some appropriate functions. Hardy and Littlewood (1928) presented several systematic treatments of certain theorems and properties of the Riemann-Liouville integrals and derivatives. A few standard classes were considered in Hardy and Littlewood (1928), including "Lebesgue classes", "Lipschitz classes" and more general classes of functions which satisfy "integrated Lipschitz conditions". Erdélyi (1940); Kober (1940); Erdélyi and Kober (1940) presented a distinct definition for non-integer order of integration that is applicable in problems involving integral and differential equations. Riesz (1939, 1949) proved the mean value theorem for fractional integrals and introduced another formulation that is associated with the Fourier transform. Caputo (1967) formulated a definition (see Definition 2.1.6), more restrictive than the Riemann-Liouville one but more appropriate to discuss problems involving a fractional differential equation with initial conditions. A detailed review of fractional calculus can be found in Hilfer (2008); De Oliveira and Tenreiro Machado (2014) and other references mentioned above.

Meanwhile, it has been shown that fractional-order models are more adequate than previously used integer-order models in various fields (Caputo and Mainardi, 1971b; Hilfer, 2000). Some of the areas of present-day applications of fractional models include fluid flows (He, 1998; Odibat and Momani, 2006), solute transport or dynamical processes in self-similar and porous structures (Liu et al., 2004; Baleanu et al., 2011), diffusive transport akin to diffusion (Metzler et al., 1994), material viscoelastic theory (Bagley and Torvik, 1983; Bagley and Calico, 1991; Koeller, 1984), electromagnetic theory (Zhang et al., 1989), dynamics of earthquakes (El-Misiery and Ahmed, 2006), control theory of dynamical systems (Vinagre et al., 2000; Baleanu et al., 2011), optics and signal processing (Sheng et al., 2011), bio-sciences (Magin, 2004, 2010), economics and finance (Mainardi et al., 2000; Scalas et al., 2000; Mendes, 2009), geology (DePaolo, 1981), astrophysics (Lee et al., 1996), probability and statistics (Carpinteri and Mainardi, 2014), chemical physics (Seki et al., 2003), and so on. The mathematical modelling and simulation of systems and processes, based on the description of their properties in terms of fractional derivatives, naturally leads to fractional differential equations. The question of the existence and uniqueness of solutions of initial value fractional differential equations has been considered in detail (Podlubny, 1998). Methods for solving fractional differential equations of rational order (Oldham and Spanier, 1974; Bagley and Calico, 1991; Miller and Ross, 1993) do not work in the cases of real order. Other authors used in their investigations the one-parameter Mittag-Leffler function (Bagley and Calico, 1991; Caputo and Mainardi, 1971b) or Fox H-function (Fox, 1961; Schneider and Wyss, 1989; Gloeckle and Nonnenmacher, 1991). In order to eliminate several disadvantages of previous known mehods, Podlubny (1998) has introduced a method which is suitable for a wide class of initial value problems for fractional differential equations. The method uses the Laplace transform technique and is based on the formula of the Laplace transforms (2.3) of the two-parameter Mittag-Leffler functions.

On the other hand, Meshkov (1974) used the notion of Green's function of a fractional differential equation for the first time. The definition of fractional Green's function is suggested and formally used by Miller and Ross (1993) when applying to fractional differential equations containing only derivatives of order $k\alpha$, where k is integer. Podlubny (1998) has given a more general definition of the fractional Green's function and discussed some of its properties, necessary for constructing solutions of initial value problems for fractional linear differential equations with constant coefficients. Some further analytical methods for solving fractional order integral and differential equations are also discussed in Podlubny (1998), namely the Mellin transform method, the power series method, and symbolic method which is firstly introduced by Babenko (1986). The Babenko's symbolic method itself is close to the Laplace transform method, but it can be used in more cases. The main idea of this method is to use some specific expansions, e.g. a binomial expansion or geometric expansion, on the differential operators, which will lead to an infinite sum of fractional derivatives. However, it is always necessary to check the validity of the formal solutions since the interchange of infinite summation and integration requires justification. In general, the justification of Babenko's approach is unknown yet, and therefore it is needed to look for such justification on a case to case basis. However, it is a powerful tool for determining the possible form of the solution. Numerous examples of the application of this method appearing in heat and mass transfer problems are discussed by Babenko (1986). The way of how to construct the Rock operators is actually a specific application of Babenko's approach. The idea behind the Rock operator is based on the binomial expansion, which will be discussed in detail in Chapter 4.

All the methods mentioned in the above paragraph are for solving fractional differential equations equipped with left fractional derivatives. In this thesis we are going to deal with right fractional derivatives. The Laplace transforms of right fractional derivatives do not have any analytic expressions. Thus, we need to find other approaches to solve these kind of initial (or terminal) value problems for fractional differential equations. Since the exponential function is the eigenfunction of the right Weyl-Liouville fractional derivatives, we apply the characteristic equation method to solve these fractional Weyl-Liouville differential equations.

1.3 Fractional Poisson process

The Poisson process is one of the fundamental processes in stochastic analysis. The fractional Poisson process $N_\mu(t)$ is a fractional non-Markovian generalisation of the Poisson process. The idea of this process is first raised by Repin and Saichev (2000), who discussed the "fractional Poisson law" by generalising the standard Poisson process for which the Laplace transform of the interval distribution between jumps has the form $\hat{f}(s) = (1 + s^\mu)^{-1}$. This further inverts back to a Mittag-Leffler density function with intensity parameter 1 and fraction parameter $0 < \mu \leqslant 1$. Jumarie (2001) constructed the fractional master equation for a long-range dependence process via Riemann-Liouville fractional derivative. The name "fractional Poisson process" is formally introduced by Laskin (2003), who introduced this fractional non-Markovian process by solving a fractional generalisation of the Kolmogorov-Feller equation (2.17). The fractional Kolmogorov-Feller equation is one of the fractional master equations constructed by Jumarie (2001).

Besides Laskin's contribution, there have been quite a few other researchers interested in the fractional Poisson process in the literature. Wang and Wen (2003) proposed a class of non-Gaussian stationary increment processes, named Poisson fractional processes, which were defined as a 'moving average' representation of the fractional Brownian motion. Mainardi et al. (2004) analysed some properties of the fractional Poisson process and compared it with other renewal processes. Mainardi et al. (2005) compared the classical Poisson process with the renewal process of Mittag-Leffler type and the renewal process of Wright type, furthermore considering corresponding renewal processes with rewards and calculated numerically their long-time behavior. Beghin and Orsingher (2009) presented three different fractional versions of the Poisson process and some related results concerning the distribution of the order statistics and the compound Poisson process. Beghin et al. (2010) showed that the so-called "generalised Mittag-Leffler functions" $E^k_{\alpha,\beta}(x)$ arise as solutions of some fractional extensions of the recursive differential equation governing the Poisson process. The corresponding processes are proved to be of renewal type, with the density of the intearrival times possessing power instead of exponential decay for $t \to \infty$. Cahoy et al. (2010) proposed a formal estimation procedure for parameters of the fractional Poisson process and established the asymptotic normality of the estimators for the two parameters appearing in the fractional Poisson model. Scalas (2011) showed that the functional limit of the compound fractional Poisson process is an α-stable Lévy process subordinated to the fractional Poisson process. Meerschaert et al. (2011) showed that a traditional Poisson process, with the time variable replaced by an independent inverse stable subordinator, is a fractional Poisson process. Another characterisation of the process was proposed by Politi et al. (2011): they introduced formulae for its finite-dimensional distribution functions, fully characterising the process. Beghin and Macci (2012) studied different fractional versions of the compound Poisson process, whose fractionalities are introduced in the counting process representing the number of jumps as well as in the density of the jumps themselves. Orsingher and Polito (2012) introduced the space-fractional Poisson process by applying the fractional backward operator in the master equation. Biard and Saussereau (2014) established the long-range dependence property of this non-stationary process. Rao (2015) introduced a class of processes termed as filtered fractional Poisson processes and filtered fractional Lévy processes. Di Crescenzo et al. (2016) considered a fractional counting process with jumps of amplitude 1, 2, . . . , k, whose probabilities satisfy a suitable system of fractional difference-differential equations and obtained the moment-generating function and the probability law of the resulting process in terms of generalised Mittag-Leffler functions.

The fractional Poisson process has a wide application in different fields. For instance, the residual life-time in statistics is generalised using this process (Politi et al., 2011). As a physical application, a new family of quantum coherent states has been introduced and studied (Laskin, 2009) and anomalous diffusions started to draw more attention (Politi et al., 2011). In meteorology (Blender et al., 2015) and earthquake analysis, results using the fractional Poisson process have potential implications for the predictability of extreme phenomenons. In ruin theory, the fractional Poisson process is first considered by Beghin and Macci (2013), who presented large deviation estimates for the ruin probabilities of a fractional Poisson risk model with light-tailed claim sizes. The ruin probabilities in this case have asymptotic exponential decay with rate equals to the solution R of Lundberg's equation. Biard and Saussereau (2014) established the long-range dependence property of fractional Poisson risk model and obtained several results for ruin-related quantities, e.g. ruin time, ruin probability and finite ruin probability under exponential claim size and heavy-tailed claim size assumptions.

The thesis is organsied as follows. Chapter 2 gives several related preliminary results covering the fractional calculus, probability essentials and risk theory basics. Chapter 3 presents the results about ruin probabilities in the classical risk model when gamma claims or geometric claims. Chapter 4 defines a new class of fractional differential operators and discusses ruin problems in renewal risk models when inter-arrival time density solves fractional differential equations. Chapter 5 summarises this project and talks about several potential future work directions. A few long proofs will be put in Appendix.

Chapter 2

Preliminary Results

This chapter provides the foundation of the concepts to be presented in this thesis. As our goal is to apply fractional calculus in the context of actuarial theory, concepts from these two backgrounds will be introduced. Technically, probability distributions and some stochastic processes will be presented since they are strongly related to the contents of Chapter 3 and 4. Three sections in this chapter will cover fractional calculus, probability essentials and risk theory basics respectively. The definitions of fractional calculus theory are taken mainly from Samko et al. (1993) and Podlubny (1998). The definitions of probability theory follow Feller (2008), Haight (1967) and Rolski et al. (1999). The classical risk model is presented as in Asmussen and Albrecher (2010), a reference that could be considered for further risk theory results.

2.1 Fractional calculus

Fractional calculus is the theory of integrals and derivatives of arbitrary order, which unifies and generalises the notions of integer-order differentiation and n-fold integration (Podlubny, 1998). The definitions of several special functions, fractional integrals and fractional derivatives, used in this paper are listed below in this section.

2.1.1 Mittag-Leffler function

The Mittag-Leffler function was firstly introduced by Mittag-Leffler (1903) as a generalisation of an exponential function. Some other mathematicians managed to generalise further this function in the sense that more parameters may appear in the expressions (Anders, 1905; Bateman et al., 1955; Prabhakar, 1971; Shukla and Prajapati, 2007). In this paper, one-parameter and two-parameter Mittag-Leffler functions are considered, which are defined as below.

Definition 2.1.1. The one-parameter Mittag-Leffler function is defined on the complex plane, as the series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C},\ \Re(\alpha) > 0,\ z \in \mathbb{C}. \tag{2.1}$$

The two-parameter Mittag-Leffler function is a generalised form of the one-parameter version, which is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0,\ z \in \mathbb{C}. \tag{2.2}$$

Proposition 2.1.1. The Laplace transform of the function $z^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm a z^\alpha)$ is (Podlubny, 1998)

$$\int_0^\infty e^{-sz} z^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm a z^\alpha)\, dz = \frac{k!\, s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}}, \quad \Re(s) > |a|^{1/\alpha}. \tag{2.3}$$

2.1.2 Fractional integrals

A fractional integral is the generalisation of a classical integral, which allows one to express r-fold integrals, where r is a real number. Riemann-Liouville approach is one of two classical approaches to achieve this. Based on the limits of integrations, the Riemann-Liouville fractional integrals have a left definition and a right definition (Hilfer, 2008).

Definition 2.1.2. The left Riemann-Liouville fractional integral of order $r > 0$ with lower limit a is defined as

$$_aI_x^r f(x) = \frac{1}{\Gamma(r)} \int_a^x (x-y)^{r-1} f(y)\, dy, \quad x > a,$$

and the right Riemann-Liouville fractional integral of order $r > 0$ with upper limit b is defined as

$$_xI_b^r f(x) = \frac{1}{\Gamma(r)} \int_x^b (y-x)^{r-1} f(y)\, dy, \quad x < b.$$

These two operators are well defined on $L^{\lceil r \rceil}([a,b])$, a space of functions for which the $\lceil r \rceil$-th power of the absolute value is Lebesgue integrable on $[a,b]$ (Rubin, 1996). $\lceil r \rceil$ denotes the ceiling function, taking as input a real number r and giving as output the least integer that is grater than or equal to r.

Definition 2.1.3. The Weyl fractional integrals are special cases of the Riemann-Liouville integrals, when a is replaced by $-\infty$ or b is replaced by ∞, denoted respectively as

$$_{-\infty}I_x^r f(x) = \frac{1}{\Gamma(r)} \int_{-\infty}^x (x-y)^{r-1} f(y)\, dy, \quad x \in \mathbb{R},$$

and

$$_xI_\infty^r f(x) = \frac{1}{\Gamma(r)} \int_x^\infty (y-x)^{r-1} f(y)\, dy, \quad x \in \mathbb{R}.$$

2.1.3 Fractional derivatives

The fractional derivatives have been generalised by various approaches in the literature. One of the approaches starts from the Riemann-Liouville fractional integral. Once equipped with the fractional integral, one can define a fractional derivative either by taking integer order derivatives of a fractional order integral, named Riemann-Liouville fractional derivatives, or by applying fractional integration on integer order derivatives, called Caputo derivatives.

Definition 2.1.4. The left Riemann-Liouville fractional derivative of order $r > 0$ with lower limit a is defined as the integer order derivative of a fractional integral

$$_aD_x^r f(x) = \frac{1}{\Gamma(n-r)} \frac{d^n}{dx^n} \int_a^x (x-y)^{n-r-1} f(y)\, dy, \quad x > a, \tag{2.4}$$

and similarly, the right Riemann-Liouville fractional derivative of order $r > 0$ with upper limit b is defined as

$$ {}_xD_b^r f(x) = (-1)^n \frac{1}{\Gamma(n-r)} \frac{d^n}{dx^n} \int_x^b (y-x)^{n-r-1} f(y)\, dy, \quad x < b, \tag{2.5} $$

where $n = \lfloor r \rfloor + 1$ is the smallest integer larger than r. $\lfloor r \rfloor$ denotes the the floor function, taking as input a real number r and giving as output the greatest integer that is less than or equal to r.

The expression (2.4) is the generalisation of differential operator $\frac{d}{dx}$, while the expression (2.5) can be regarded as the generalisation of operator $-\frac{d}{dx}$.

Proposition 2.1.2. For suitable functions f, the Riemann-Liouville fractional derivatives have semigroup properties (Valério et al., 2013)

$$ \frac{d^k}{dx^k} \, {}_aD_x^r f(x) = {}_aD_x^{r+k} f(x), $$

and

$$ \frac{d^k}{dx^k} \, {}_xD_b^r f(x) = (-1)^k \, {}_xD_b^{r+k} f(x). $$

Proposition 2.1.3. The Riemann-Liouville fractional derivatives are the left inverse operators of corresponding fractional integrals (Valério et al., 2013)

$$ {}_aD_x^r \, {}_aI_x^r f(x) = f(x), $$

and

$$ {}_xD_b^r \, {}_xI_b^r f(x) = f(x). $$

This holds for any $r \in \mathbb{C}$.

Using the change of variables and the definition of the beta function,

$$ \mathrm{B}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt, \quad \Re(x), \Re(y) > 0, $$

one has the following proposition.

Proposition 2.1.4. The left Riemann-Liouville fractional integral ${}_aI_x^r$ and left fractional derivative ${}_aD_x^r$ of the power function $(x-a)^p$ are (Podlubny, 1998)

$$ {}_aI_x^r(x-a)^p = \frac{\Gamma(1+p)}{\Gamma(1+p+r)}(x-a)^{p+r}, $$

and

$$ {}_aD_x^r(x-a)^p = \frac{\Gamma(1+p)}{\Gamma(1+p-r)}(x-a)^{p-r}. $$

Since the Mittag-Leffler function is defined as an infinite series of power functions, its fractional integrals and derivatives can be computed by applying Proposition 2.1.4.

Proposition 2.1.5. The left Riemann-Liouville fractional integral ${}_0I_x^r$ and left fractional derivative ${}_0D_x^r$ of the Mittag-Leffler functions are (Podlubny, 1998)

$$ {}_0I_x^r \left(x^{\beta-1}E_{\alpha,\beta}(\lambda x^\alpha)\right) = x^{\beta+r-1}E_{\alpha,\beta+r}(\lambda x^\alpha) $$

and

$$ {}_0D_x^r \left(x^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\lambda x^\alpha)\right) = x^{\alpha k+\beta-r-1}E_{\alpha,\beta-r}^{(k)}(\lambda x^\alpha). $$

The next proposition is about an extension of the Leibniz integral rule to fractional calculus.

Proposition 2.1.6. The left Riemann-Liouville fractional derivative ${}_0D_x^r$ of an integral depending on a parameter is given by

$$ {}_0D_x^r \int_0^x K(x,t)\,dt = \int_0^\infty {}_tD_x^r K(x,t)\,dt + \lim_{t\to x-0} {}_tD_x^{r-1}K(x,t). $$

The following formula will be used when taking the left Riemann-Liouville fractional derivative ${}_0D_x^r$ on a convolution integral of two functions with positive supports

$$ [f*g](x) = [g*f](x) = \int_0^x f(t)g(x-t)\,dt, \quad x>0. $$

Proposition 2.1.7. The left Riemann-Liouville fractional derivative ${}_0D_x^r$ of the (positive density) convolution integral equals to

$$ {}_0D_x^r\,[g*f]\,(x) = \left[{}_0D_x^r g * f\right](x) + \lim_{t\to+0} f(x-t)\ {}_0D_t^{r-1}g(t), \quad x>0. $$

Definition 2.1.5. The Weyl-Liouville fractional derivatives (Samko et al., 1993; Butzer and Westphal, 2000) are special cases of the Riemann-Liouville definition when a is replaced by $-\infty$ or b is replaced by ∞, denoted as

$$_{-\infty}D_x^r f(x) = \frac{1}{\Gamma(n-r)} \frac{d^n}{dx^n} \int_{-\infty}^{x} (x-y)^{n-r-1} f(y)\, dy, \quad x \in \mathbb{R},$$

and

$$_xD_{\infty}^r f(x) = (-1)^n \frac{1}{\Gamma(n-r)} \frac{d^n}{dx^n} \int_{x}^{\infty} (y-x)^{n-r-1} f(y)\, dy, \quad x \in \mathbb{R}, \tag{2.6}$$

where $n = \lfloor r \rfloor + 1$.

Another widely used definition of fractional derivatives are the so-called Caputo fractional derivatives.

Definition 2.1.6. The Caputo fractional derivatives are modified Riemann-Liouville fractional derivatives in the sense of interchanging the order of fractional integration and integer-order differentiation

$$^{C}_{a}D_x^r f(x) = \frac{1}{\Gamma(n-r)} \int_{a}^{x} (x-y)^{n-r-1} f^{(n)}(y)\, dy, \quad x > a, \tag{2.7}$$

and

$$^{C}_{x}D_b^r f(x) = \frac{1}{\Gamma(n-r)} \int_{x}^{b} (y-x)^{n-r-1} f^{(n)}(y)\, dy, \quad x < b, \tag{2.8}$$

where $n = \lfloor r \rfloor + 1$.

Proposition 2.1.8. For suitable functions f, the Caputo fractional derivatives have semigroup properties with integer order derivatives, which are

$$^{C}_{a}D_x^r \frac{d^k}{dx^k} f(x) = {^{C}_{a}D_x^{r+k}} f(x),$$

and

$$^{C}_{x}D_b^r \frac{d^k}{dx^k} f(x) = {^{C}_{x}D_b^{r+k}} f(x).$$

Proposition 2.1.9. The Caputo fractional derivatives are the left inverse operators of corresponding fractional integrals (Valério et al., 2013)

$$^{C}_{a}D_x^r\, {_aI_x^r} f(x) = f(x),$$

and

$$ {}_x^C D_b^r \, {}_x I_b^r f(x) = f(x). $$

This holds for $r \in \mathbb{N}$ or $\Re(r) \notin \mathbb{N}$.

The following proposition illustrates the relationships between Riemann-Liouville fractional derivatives and Caputo fractional derivatives (Almeida and Torres, 2011).

Proposition 2.1.10. For suitable functions f, one has

$$ {}_a D_x^r f(x) = {}_a^C D_x^r f(x) + \sum_{k=0}^{\lfloor r \rfloor} \frac{(x-a)^{k-r}}{\Gamma(k-r+1)} f^{(k)}(a), \quad x > a, $$

and

$$ {}_x D_b^r f(x) = {}_x^C D_b^r f(x) + \sum_{k=0}^{\lfloor r \rfloor} \frac{(b-x)^{k-r}}{\Gamma(k-r+1)} f^{(k)}(b), \quad x < b. \tag{2.9} $$

These two identities show that Riemann-Liouville fractional derivatives and Caputo fractional derivatives are equivalent if and only if, for all $0 \leqslant k \leqslant \lfloor r \rfloor$, the derivatives $f^{(k)}(a)$ or $f^{(k)}(b)$ equal to zeros respectively.

In the classical integration, there are lots of useful rules in calculus, like the chain rule, the Leibniz rule, the integration by parts and so on. In order to apply these rules in fractional calculus, one has to adapt each formula. The main difference consists in the appearance of more residual terms.

Proposition 2.1.11. The fractional integration by parts rules are (Almeida and Torres, 2011)

$$ \begin{aligned} \int_a^b g(x) \, {}_x^C D_b^r f(x) \, dx = & \int_a^b f(x) \, {}_a D_x^r g(x) \, dx \\ & + \sum_{j=0}^{\lfloor r \rfloor} \left[(-1)^{\lfloor r \rfloor + 1 + j} \left({}_a D_x^{r+j-\lfloor r \rfloor - 1} g(x) \right) \left({}_a D_x^{\lfloor r \rfloor - j} f(x) \right) \right]_a^b \end{aligned} \tag{2.10} $$

and

$$ \begin{aligned} \int_a^b g(x) \, {}_a^C D_x^r f(x) \, dx = & \int_a^b f(x) \, {}_x D_b^r g(x) \, dx \\ & + \sum_{j=0}^{\lfloor r \rfloor} \left[\left({}_x D_b^{r+j-\lfloor r \rfloor - 1} g(x) \right) \left({}_x D_b^{\lfloor r \rfloor - j} f(x) \right) \right]_a^b . \end{aligned} \tag{2.11} $$

In the case that $0 < r < 1$ and $f(a) = f(b) = 0$, the results become

$$\int_a^b g(x)\, {}_x^C D_b^r f(x)\, dx = \int_a^b f(x)\, {}_a D_x^r g(x)\, dx$$

and

$$\int_a^b g(x)\, {}_a^C D_x^r\, dx = \int_a^b f(x)\, {}_x D_b^r g(x)\, dx. \tag{2.12}$$

The fractional integration by parts rules (2.10) to (2.12) offer one of the connections between left derivatives and right derivatives.

The next proposition gives the possibility that we can adopt a characteristic equation approach to solve a specific kind of fractional differential equations.

Proposition 2.1.12. The eigenfunction of the fractional derivative ${}_x D_\infty^r$ is $e^{-\lambda x}$, where $\lambda \in \mathbb{R}^+$.

Proof. In fact, one has

$$\begin{aligned}
{}_x D_\infty^r e^{-\lambda x} =& (-1)^k \frac{d^k}{dx^k} \int_x^\infty \frac{1}{\Gamma(k-r)} (t-x)^{k-r-1} e^{-\lambda t}\, dt \\
=& (-1)^k \frac{d^k}{dx^k} \int_0^\infty \frac{1}{\Gamma(k-r)} s^{k-r-1} e^{-\lambda(s+x)}\, ds \\
=& (-1)^k \frac{d^k}{dx^k} e^{-\lambda x} \int_0^\infty \frac{1}{\Gamma(k-r)} s^{k-r-1} e^{-\lambda s}\, ds \\
=& (-1)^k (-\lambda)^k e^{-\lambda x} \lambda^{r-k} \\
=& \lambda^r e^{-\lambda x}.
\end{aligned}$$

□

There are many useful transformations for fractional integrals and derivatives, e.g. Laplace transforms, Fourier transforms and Mellin transforms (Podlubny, 1998). The next proposition is one of them which will be used in the thesis.

Proposition 2.1.13. The Laplace transform of the left Riemann-Liouville fractional derivative of order $r > 0$ is

$$\mathcal{L}\{{}_0 D_x^r f(x);\, s\} = s^r \hat{f}(s) - \sum_{k=0}^{\lfloor r \rfloor} s^k \left[{}_0 D_x^{r-k-1} f(x)\right]\Big|_{x=0}. \tag{2.13}$$

2.2 Probability theory and statistics

In this section, several discrete and continuous random variables used in the thesis are listed along with their probability-generating functions (for discrete random variable X)

$$G_X(z) := \mathbb{E}\left(z^X\right), \quad z \in \{z \,|\, G_X(z) < \infty\},$$

moment-generating functions (for continuous random variable Z)

$$M_Z(s) := \mathbb{E}\left(e^{sZ}\right), \quad s \in \{s \,|\, M_Z(s) < \infty\},$$

or the (unilateral) Laplace transforms of density functions $f_Z(z)$ supported on $t > 0$

$$\mathcal{L}\left\{f_Z(z);\, s\right\} = \hat{f}_Z(s) := \int_0^\infty e^{-sz} f(z)\, dz, \quad s \in \left\{ s \in \mathbb{C} \,\middle|\, \int_0^\infty \left|e^{-\Re(s)z} f(z)\right|\, dz < \infty \right\}.$$

2.2.1 Discrete and continuous distributions

The **geometric random variable** X describes either the number of Bernoulli trials needed to get one success or the failures before the first success. In the following context, the second definition is chosen, i.e., the probability mass function is

$$\mathbb{P}(X = k) = p \cdot (1-p)^k, \quad k \in \mathbb{N} \tag{2.14}$$

where $p \in (0, 1]$ denotes the probability of success on each single trial. The mean and variance of a geometric random variable are equal to $\frac{1-p}{p}$ and $\frac{1-p}{p^2}$. The probability-generating function of a geometric random variable is

$$G_X(z) = \frac{p}{1-(1-p)z}, \quad |z| < (1-p)^{-1}. \tag{2.15}$$

The **Poisson random variable** N expresses the number of events occurring in a fixed time interval if these events occur with a known average rate $\lambda > 0$ and independently of the time since the last event. Its probability mass function is given as

$$\mathbb{P}(N = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k \in \mathbb{N}.$$

The mean and variance of a Poisson random variable are both equal to λ. The probability-generating function of a Poisson random variable is

$$G_N(z) = e^{\lambda(z-1)}, \quad z \in \mathbb{R}.$$

The **exponential random variable** is a continuous random variable which is defined as the time between events in Poisson process. For an exponential random variable $Z \sim \exp(\lambda)$ with parameter $\lambda > 0$, the probability density function is

$$f_Z(z) = \lambda e^{-\lambda z}, \quad z > 0,$$

and cumulative distribution function

$$F_Z(z) = \int_0^z f(s)\,ds = 1 - e^{-\lambda z}, \quad z > 0.$$

The mean and variance of an exponential random variable are are equal to $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$, respectively. The moment-generating function of an exponential random variable is

$$M_Z(s) = \frac{\lambda}{-s+\lambda}, \quad s < \lambda.$$

The Laplace transform of the exponential density function equals to

$$\mathcal{L}\{f_Z(z);\, s\} = \hat{f}_Z(s) = \frac{\lambda}{s+\lambda}, \quad \Re(s) > -\alpha.$$

The **gamma random variable** is a continuous random variable with shape parameter $\alpha > 0$ and rate parameter $\beta > 0$. When α takes integer values n, it is also known as an Erlang random variable, which could be regarded as the sum of n independent and identically distributed exponential random variables with parameter β. The density function of a gamma random variable $Y \sim \Gamma(\alpha, \beta)$ is

$$f_Y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}, \quad y > 0,$$

with mean and variance equal to $\frac{\alpha}{\beta}$ and $\frac{\alpha}{\beta^2}$. The moment-generating function of a gamma random variable is

$$M_Y(s) = \left(\frac{\beta}{-s+\beta}\right)^\alpha, \quad s < \beta.$$

The Laplace transform of the gamma density function equals to

$$\mathcal{L}\left\{f_Y(y);\, s\right\} = \hat{f}_Y(s) = \left(\frac{\beta}{s+\beta}\right)^{\alpha}, \quad \Re(s) > -\beta.$$

The **Mittag-Leffler random variable** refers to two different family of distributions. The first family of Mittag-Leffler distributions is defined by a relation between the Mittag-Leffler function and their cumulative distribution functions. For a first kind of Mittag-Leffler random variable $X \sim \mathrm{ML}(\mu, \lambda)$ with parameters $\mu \in (0, 1]$ and $\lambda > 0$, the probability density function is

$$f_X(x) = \lambda x^{\mu-1} E_{\mu,\mu}(-\lambda x^{\mu}), \quad x > 0,$$

and cumulative distribution function

$$F_X(x) = 1 - E_{\mu}(-\lambda x^{\mu}), \quad x > 0,$$

where E_μ and $E_{\mu,\mu}$ are given in (2.1) and (2.2). When $\mu \in (0, 1)$, its mean and variance do not exist. The Laplace transform of the first kind Mittag-Leffler density function equals to

$$\mathcal{L}\left\{f_X(x)\,;s\right\} = \hat{f}_X(s) = \frac{\lambda}{s^{\mu} + \lambda}, \tag{2.16}$$

The second family of Mittag-Leffler distributions is defined by a relation between the Mittag-Leffler function and their moment-generating function. For a second kind of Mittag-Leffler random variable X' with parameter $\mu \in [0, 1]$, its moment-generating function equals to

$$\mathbb{E}\left(e^{zX'}\right) = E_{\mu}(Cz),$$

for some constant $C > 0$, where the convergence stands for all $z \in \mathbb{C}$ if $\alpha \in (0, 1]$, and all $|z| < 1/C$ if $\alpha = 0$. In this thesis we are going to use the first family.

2.2.2 Stochastic processes

The **homogeneous Poisson point process**, Poisson process in short, is usually denoted as $\{N(t), t \geqslant 0\}$. A counting process is a Poisson counting process with rate $\lambda > 0$ if it has the following three properties:

- $N(0) = 0$;
- has independent increments; and
- the number of events in any interval of length t is a Poisson random variable with parameter λt.

This definition has important features:

- the number of points in each finite interval has a Poisson distribution;
- the number of points in disjoint intervals are independent random variables;
- the distribution of each interval $(a+t, b+t]$ only depends on the interval's length $b-a$;
- the distance between two consecutive points of a Poisson process is an exponential random variable with parameter λ.

The **renewal counting process**, renewal process in short, is a generalisation of the Poisson process. It allows that the inter-arrival times take on a more general independent and identical distribution.

The **fractional Poisson process** denoted by $N_\mu(t)$, $t > 0$, $\mu \in (0,1]$, is a fractional non-Markovian generalisation of Poisson process $N(t)$, $t > 0$. The distribution of a fractional Poisson process $P_\mu(n,t) = \mathbb{P}\left(N_\mu(t) = n\right)$ is defined as the solution of a fractional generalisation of the Kolmogorov-Feller equation (Laskin, 2003)

$$ {}_0D_t^\mu P_\mu(n,t) = \lambda(P_\mu(n-1,t) - P_\mu(n,t)) + \frac{t^{-\mu}}{\Gamma(1-\mu)}\delta_{n,0}, \quad t > 0 \tag{2.17} $$

where λ is the intensity parameter and $\delta_{n,0}$ is the Kronecker symbol. The solution of equation (2.17) is given by

$$ P_\mu(n,t) = \frac{\lambda^n t^{\mu n}}{n!} \sum_{j=0}^{\infty} \frac{(n+j)!(-\lambda t^\mu)^j}{j!\Gamma(\mu j + \mu n + 1)} = \left.\left(\frac{(-z)^n}{n!}\frac{d^n}{dz^n}E_\mu(z)\right)\right|_{z=-\lambda t^\mu}, \quad t > 0, $$

where $E_\mu(z)$ is the one-parameter Mittag-Leffler function. Moreover, Laskin (2003) showed that the inter-arrival time t_μ of the fractional Poisson process has as cumulative distribution function

$$F_\mu(t) = 1 - P_\mu(0,t) = 1 - E_\mu(-\lambda t^\mu), \quad t > 0$$

and as probability density function

$$f_\mu(t) = \lambda t^{\mu-1} E_{\mu,\mu}(-\lambda t^\mu), \quad t > 0. \tag{2.18}$$

The Laplace transform of the inter-arrival time $f_\mu(t)$ of the fractional Poisson process is given in equation (2.16). It follows that the density function of the k-th event, τ_k, possesses the Laplace transform

$$\mathcal{L}\left\{f_\mu^{*k}(t)\,;s\right\} = \left(\frac{\lambda}{s^\mu + \lambda}\right)^k,$$

yielding that (see equation (2.3))

$$f_\mu^{*k}(t) = \lambda^k t^{k\mu-1} E_{\mu,k\mu}^{(k)}(-\lambda t^\mu), \quad t > 0. \tag{2.19}$$

The moment-generating function $H_\mu(s,t)$ of a fractional Poisson process is given by

$$H_\mu(s,t) = \sum_{n=0}^{\infty} e^{-sn} P_\mu(n,t) = E_\mu(\lambda t^\mu(e^{-s} - 1)), \quad t > 0.$$

The mean and variance of $N_\mu(t)$ equal to (Laskin, 2003)

$$\mathbb{E}\left(N_\mu(t)\right) = \frac{\lambda t^\mu}{\Gamma(\mu+1)} \quad , \quad \mathbb{V}\mathrm{ar}\left(N_\mu(t)\right) = \frac{2(\lambda t^\mu)^2}{\Gamma(2\mu+1)} - \frac{(\lambda t^\mu)^2}{(\Gamma(\mu+1))^2} + \frac{\lambda t^\mu}{\Gamma(\mu+1)}. \tag{2.20}$$

In general, the m-th order moment of a fractional Poisson process $N_\mu(t)$ is given by (Laskin, 2009)

$$\mathbb{E}\left((N_\mu(t))^m\right) = \sum_{i=0}^{\infty} S_\mu(m,i) \cdot (\lambda t^\mu)^i,$$

where

$$S_\mu(m,i) = \frac{1}{\Gamma(\mu i + 1)} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^m, \quad m \in \mathbb{Z}^+,\ i \in \mathbb{N}$$

is the fractional Stirling number (Laskin, 2009). In Repin and Saichev (2000), it is proved that, for $t \to \infty$, there exists a constant C, such that

$$\mathbb{P}\left(t_\mu > t\right) \sim C t^{-\mu}.$$

Consequently the inter-arrival times t_μ have heavy tails and infinite mean for $0 < \mu < 1$. The fractional Poisson process has long-range dependence property when $0 < \mu < 1$ (Biard and Saussereau, 2014), namely,

$$\limsup_{t\to\infty} \frac{\mathbb{V}\mathrm{ar}\left(N_\mu(t)\right)}{t} = \infty.$$

2.3 Actuarial concepts

Here are some basic insurance mathematical concepts which will be seen through out the thesis.

- An *insurance premium* is simply referred to as a "premium". The premium rate, denoted by c in the thesis, is the amount of money that policyholders pay to an insurance company per unit time.

- *Claims* are the amount of losses an insurer needs to pay for an insured product. The value of a claim is referred to as the *claim size* and it is considered as a non-negative random variable, denoted by X_i, with common distribution function F_X.

- The *number of claims* that occur in a certain period is a non-negative integer-valued random variable. The *claim counting process* is often denoted by $\{N(t), t > 0\}$ where $N(t)$ is the number of claims up to time t.

- An *epoch of a claim*, or sometimes called a *claim arrival time* is the time at which a claim happens. We denote the epochs by τ_1, τ_2, ... and the *inter-arrival times* or *waiting times* by $T_i = \tau_i - \tau_{i-1}$, with common distribution function F_T.

- A *risk surplus* denoted here by $R(t)$ is the amount of capital of an insurance company by time t. It increases by collecting premiums and drops by the payment of claims.

- The *net profit condition* for a risk model is

$$c \cdot \mathbb{E}(T_i) > \mathbb{E}(X_i), \tag{2.21}$$

  describing the situation where the insurance company can avoid certain ruin. If the net profit condition is not satisfied, i.e., $c \cdot \mathbb{E}(T_i) \leqslant \mathbb{E}(X_i)$, then the ruin occurs almost surely irrespective of the large value of the initial surplus.

2.4 Ruin probabilities

We start with formulating the risk model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space carrying the the following objects:

1. a counting point process $N = \{N(t); t > 0\}$ with $N(0) = 0$;

2. a sequence $\{X_i\}$ of independent and identically distributed positive random variables.

The stochastic process

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t > 0, \tag{2.22}$$

is called a risk process with initial capital u. This model describes the amount of surplus $R(t)$ of an insurance portfolio at time t. The ruin probability of a company having initial capital u is defined by

$$\psi(u) = \mathbb{P}\left(\inf_{t>0} R(t) < 0 \middle| R(0) = u\right) = \mathbb{P}\left(\tau_u < \infty\right), \quad u \geqslant 0, \tag{2.23}$$

where τ_u is the first hitting time

$$\tau_u = \inf\left\{ t \geq 0 : \sum_{k=1}^{N(t)} X_k - ct > u \right\}.$$

The non-ruin, or survival probability, is denoted by

$$\phi(u) = 1 - \psi(u), \quad u \geqslant 0.$$

Lundberg (1926) identified a very important equation whose solution is used to find the asymptotic behavior of the ruin probability. The equation is referred to as the Lundberg's equation:

$$M_X(s)M_T(-cs) = 1, \tag{2.24}$$

where $M_X(s)$ and $M_T(s)$ are the moment-generating functions of the claim size distribution and the inter-arrival time distribution respectively given these functions exist. The smallest positive solution of equation (2.24) is referred to the adjustment coefficient R, which is used quite often when doing asymptotic analysis (Asmussen and Albrecher, 2010).

The risk process $R(t)$ in (2.22) is referred to as the "classical risk process" if $N(t)$ is a homogeneous Poisson process. Cramér (1930) used the properties of Poisson processes and a differential argument to derive an integro-differential equation for the non-ruin probability. Consider $R(t)$ in a sufficiently small time interval $(0, \Delta]$ and separate the four possible cases as follow (Grandell, 1991a):

1. no claim occurs in $(0, \Delta]$;
2. one claim occurs in $(0, \Delta]$, but ruin does not happen;
3. one claim occurs in $(0, \Delta]$, and ruin happens;
4. more than one claims occurs in $(0, \Delta]$.

By applying the law of total probability assuming the differentiability of the non-ruin probability $\phi(u)$, one obtains the integro-differential equation

$$\frac{d}{du}\phi(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u \phi(u-y)f_X(y)\,dy, \quad u \geqslant 0, \tag{2.25}$$

where λ is the parameter of Poisson process. This method is very intuitive but not mathematically rigorous. Instead of using a differential argument, Feller (2008) derived an integral equation for the non-ruin probability by a renewal argument, which can be used not only for the classical risk model, but also for the renewal risk process (formally proposed and analysed seven years later by Andersen (1957)). As a renewal model, the classical risk process is a discrete-time Markov process since it "renews" at every jump time. The renewal argument states, that the non-ruin probability solves the following equation

$$\phi(u) = \mathbb{E}\left[\phi(u + cT_1 - X_1)\right] = \int_0^\infty \lambda t^{-\lambda t} \int_0^{u+ct} \phi(u + ct - y)\, dF_X(y)\, dt. \tag{2.26}$$

After changing variables and further simplification, one obtains

$$\phi(u) = \phi(0) + \int_0^u \phi(u - y)(1 - F_X(y))\, dy,$$

with initial value $\phi(0) = 1 - \frac{\lambda}{c} \cdot \mathbb{E}(X)$.

The simplest classical risk model scenario consists of claim sizes following an exponential distribution with parameter α. Under the net profit condition, the ruin probability is obtained by solving an ordinary differential equation with boundary conditions (Cramér, 1930)

$$\psi(u) = \frac{\lambda}{\alpha c} e^{-\left(\alpha - \frac{\lambda}{c}\right)u}, \quad u \geqslant 0. \tag{2.27}$$

Laplace transformation is another way to derive the expression of non-ruin probability. Taking a Laplace transform on the integro-differential equation (2.25) leads to

$$\hat{\phi}(s) = \frac{c\phi(0)}{cs - \lambda + \lambda \hat{f}_X(s)}. \tag{2.28}$$

When the claim sizes are Erlang-distributed, i.e., the Laplace transform of the claim size density equals

$$\hat{f}_X(s) = \left(\frac{\alpha}{s + \alpha}\right)^n, \quad \Re(s) > -\alpha,$$

then the expression on the right hand side of (2.28) can be written as the ratio of two polynomial functions in s. In the Erlang-distributed claim case, one can then use the partial fraction decomposition and invert $\hat{\phi}$ to obtain a linear combination of exponential functions (Grandell, 1991a; He et al., 2003).

Notice that for a rational shape parameter $r = m/n \in \mathbb{Q}$ where m and n are both positive integers, with $\Re(s) > \alpha$, one could shift the argument s to obtain

$$\hat{\phi}(s-\alpha) = \frac{c\phi(0)}{c(s-\alpha) - \lambda + \lambda(\frac{\alpha}{s})^{m/n}} = \frac{c\phi(0)s^{m/n}}{c(s-\alpha)s^{m/n} - \lambda + \lambda\alpha^{m/n}},$$

which is a ratio of polynomials of orders m and $(m+1)$ in $t = s^{1/n}$. This again permits a partial fraction decomposition which can further give an explicit expression for the non-ruin probability. In this case, an explicit expression can be obtained as in Zhu (2013), using the two parameter Mittag-Leffler function:

$$\phi(u) = e^{-\alpha u} u^{\frac{1}{n}-1} \sum_{k=0}^{m+n-1} m_k E_{\frac{1}{n},\frac{1}{n}}\left(s_k u^{\frac{1}{n}}\right), \quad u \geqslant 0 \tag{2.29}$$

with s_k and m_k real constants, determined on a case-by-case basis.

Extending these results to positive real shape parameters r proves to be non-trivial and different approaches are presented in this thesis. Prior to this work, the only known (to us) result for non-integer shape gamma-distributed claims is that of Thorin (1973), which deals with the classical collective risk model with Poisson arrival intensity $\lambda = 1$, $\Gamma(1/b, 1/b)$-distributed claims, $b > 1$, and positive loading $c > 1$, the ruin probability for $u \geqslant 0$ is

$$\begin{aligned}\psi(u) =& \frac{(c-1)(1-bR)e^{-Ru}}{1-cR-c(1-bR)} \\ &+ \frac{c-1}{b\pi}\sin\frac{\pi}{b}\int_0^\infty \frac{x^{1/b}e^{-(x+1)u/b}}{\left[x^{1/b}\left(1+c\frac{x+1}{b}\right)-\cos\frac{\pi}{b}\right]^2+\sin^2\frac{\pi}{b}}\,dx,\end{aligned}$$

where R is the positive solution of Lundberg's equation (2.24). This approach explores the properties of completely monotone functions. This class of gamma distribution, $\Gamma(1/b, 1/b)$, fits in the completely monotone class of distributions of the form

$$P(y) = \begin{cases} \int_0^\infty (1-e^{-\alpha y})\,dV(\alpha), & y \geq 0, \\ \\ 0, & y < 0, \end{cases}$$

where $V(\alpha)$ is a distribution function with $V(0) = 0$, which means its tail distribution is completely monotone. Thorin (1973) then derived the ruin probability as a function of V'. For $\Gamma(1/b, 1/b)$ distributed, with $b > 1$,

$$P(y) = \frac{1}{\Gamma(1/b)} \int_0^{y/b} x^{1/b-1} e^{-x}\, dx, \quad y \geq 0, \quad b > 1,$$

is completely monotone and

$$\frac{d}{dx} V(x) = \frac{1}{\pi} \sin \frac{\pi}{b} x^{-1} (bx-1)^{-1/b}, \quad x > 1/b.$$

When $b = 2$, the expression of ruin probability becomes a linear combination of exponentials and error functions, which expression (2.29) can recover when $r = 1/2$. However, notice that the general form of the integral term appearing in the result can only be calculated numerically.

Another research direction in risk theory is to consider a general renewal risk model. The risk process $R(t)$ in (2.22) is referred to as the "renewal risk process" when $N(t)$ is a renewal counting process. Instead of the classical methods (Grandell, 1991a) described before equation (2.25), we would like to apply the renewal argument (2.26) in its full generality. The ruin probability in a renewal risk model satisfies (Feller, 2008)

$$\psi(u) = \int_0^\infty f_T(t) \left(\int_0^{u+ct} \psi(u+ct-y)\, dF_X(y) + \int_{u+ct}^\infty dF_X(y) \right) dt, \quad u \geqslant 0 \tag{2.30}$$

with the universal boundary condition $\lim_{u \to \infty} \psi(u) = 0$.

A renewal risk process is a continuous stochastic process, but has Markov property conditional on discrete jump times. Therefore one needs to find its generator, instead of an infinitesimal generator. However, once the inter-arrival time distribution is given, through similar probabilistic arguments, the equations for the ruin probability could be obtained as follows.

1. Exp(λ) distributed inter-arrival times (classical risk model). Let $f_T(t) = \lambda e^{-\lambda t}$, for $t > 0$, be the density function of inter-arrival times. The ruin probability satisfies

$$\left(-c\frac{d}{du} + \lambda \right) \psi(u) = \lambda \left(\int_0^u \psi(u-y)\, dF_X(y) + \int_u^\infty dF_X(y) \right), \quad u \geqslant 0.$$

2. Erlang$(2, \lambda)$ distributed inter-arrival times (Erlang(2) risk model considered by Dickson (1998)). Let $f_T(t) = \lambda^2 t e^{-\lambda t}$, for $t > 0$, be the density function of inter-arrival times. The ruin probability satisfies

$$\left(-c\frac{d}{du} + \lambda\right)^2 \psi(u) = \lambda^2 \left(\int_0^u \psi(u-y)\, dF_X(y) + \int_u^\infty dF_X(y)\right), \quad u \geqslant 0.$$

3. Erlang(n, λ) distributed inter-arrival times (Erlang(n) risk model considered by Li and Garrido (2004b)). Let $f_T(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}$, for $t > 0$, be the density function of inter-arrival times. The ruin probability satisfies

$$\left(-c\frac{d}{du} + \lambda\right)^n \psi(u) = \lambda^n \left(\int_0^u \psi(u-y)\, dF_X(y) + \int_u^\infty dF_X(y)\right), \quad u \geqslant 0.$$

4. Sum of n exponentially distributed inter-arrival times (Gerber and Shiu, 2005). Let inter-arrival times T_i be the sum of n independent heterogeneous exponential random variables, with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$. The ruin probability satisfies

$$\bigodot_{i=1}^n \left(-c\frac{d}{du} + \lambda_i\right) \psi(u) = \prod_{i=1}^n \lambda_i \left(\int_0^u \psi(u-y)\, dF_X(y) + \int_u^\infty dF_X(y)\right), \quad u \geqslant 0,$$

where $\bigodot$ denotes left-composition of operators, namely

$$\bigodot_{j=1}^m \mathcal{L}_j[f] := (\mathcal{L}_m \circ \cdots \circ \mathcal{L}_1)[f].$$

In the literature of risk theory, the product sign $\prod$ usually denotes the composition of operators.

5. More general inter-arrival times (Albrecher et al., 2010). Let the inter-arrival time density function f_T satisfy

$$\mathcal{L}_T\left(\frac{d}{dt}\right)[f_T](t) = f_T^{(n)}(t) + a_{n-1} f_T^{(n-1)}(t) + \cdots + a_0 f_T(t) = 0,$$

with almost homogeneous initial conditions

$$f_T^{(k)}(0) = 0, \quad \text{for } k = 0, \ldots, n-2,$$
$$f_T^{(n-1)}(0) = a_0.$$

The ruin probability satisfies

$$\mathcal{L}_T^*\left(c\frac{d}{du}\right)[\psi](u) = \alpha_0\left(\int_0^u \psi(u-y)\,dF_X(y) + \int_u^\infty dF_X(y)\right), \quad u \geqslant 0,$$

where $\mathcal{L}_T^*$ is the formal adjoint of $\mathcal{L}_T$ defined by

$$\int_0^\infty \mathcal{L}_T[f](x)\,g(x)\,dx = \int_0^\infty f(x)\,\mathcal{L}_T^*[g](x)\,dx.$$

If moreover the claim size density f_X satisfies a linear ordinary differential equation with constant coefficients

$$\mathcal{L}_X\left(\frac{d}{dy}\right)[f_X](y) = f_X^{(m)}(y) + b_{m-1}f_X^{(m-1)}(y) + \cdots + b_0 f_X(y) = 0,$$

with almost homogeneous initial conditions

$$f_X^{(k)}(0) = 0, \quad \text{for } k = 0, \ldots, m-2,$$
$$f_X^{(m-1)}(0) = a_0,$$

the ruin probability further satisfies

$$\mathcal{L}_X\left(\frac{d}{du}\right)\mathcal{L}_T^*\left(c\frac{d}{du}\right)[\psi](u) = a_0 b_0 \cdot \psi(u), \quad u \geqslant 0.$$

Constantinescu (2006) mentioned in her thesis about one of the main venues of future research when considering general gamma inter-arrival times in risk theory. This problem has been left to be solved until Constantinescu et al. (2017) managed to extend the result to more general renewal risk models.

Chapter 3

Compound Poisson Risk Models

In this chapter we will consider the ruin problem in the classical risk model. Explicit ruin probabilities in the classical risk model when gamma-distributed claim sizes are derived in Section 3.1. Another small result in classical risk model is in the case of geometric-distributed claim sizes, whose explicit ruin probability will be presented in Section 3.2.

3.1 Explicit expressions for non-ruin probabilities in the case of gamma claims

The focus of this section is on gamma-distributed claim sizes, i.e., with the density

$$f_X(y) = \frac{\alpha^r}{\Gamma(r)} y^{r-1} e^{-\alpha y}, \quad y > 0, \tag{3.1}$$

where $r > 0$ is the shape parameter, and $\alpha > 0$ is the rate parameter. The gamma distribution $\Gamma(r, \alpha)$ has two positive parameters. When the shape parameter r takes value in $r = n \in \mathbb{N}^+$, it is also known as Erlang distribution and can be interpreted as the sum of n independent and identically distributed exponential random variables. However, this interpretation fails if the shape parameter has non-integer values. In the risk literature, explicit expressions for ruin probabilities in the classical risk model have been previously obtained only under Erlang claim sizes assumption. In this section, we consider the classical ruin problem with gamma-distributed claim sizes.

The starting point is the classical integro-differential equation for the non-ruin probability (2.25)

$$\frac{d}{du}\phi(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u \phi(u-y) f_X(y)\, dy, \quad u \geqslant 0\,.$$

An immediate conclusion is that the Laplace transform of non-ruin probability

$$\hat{\phi}(s) = \frac{c\phi(0)}{cs - \lambda + \lambda M_X(-s)} = \frac{c\phi(0)}{cs - \lambda + \lambda(\frac{\alpha}{s+\alpha})^r}, \quad \Re(s) > 0. \tag{3.2}$$

We would like to mention that the non-ruin probability of the classical risk model (1.1) with zero initial capital equals to

$$\phi(0) = 1 - \frac{\lambda \mathbb{E}(X)}{c},$$

where $\mathbb{E}(X)$ denotes the expected claim size, see e.g. Rolski et al. (1999). Throughout this chapter, by $\phi(0)$ we refer to this expression.

When the shape parameter r is integer, namely when the claims are Erlang-distributed, the expression in the right hand side of (3.2) can be written as the ratio of two polynomial functions. One can then use the partial fraction decomposition and invert $\hat{\phi}$ to obtain a linear combination of exponential functions (Grandell, 1991a).

3.1.1 Gamma-distributed claim sizes with rational shape parameters

The following theorem gives the explicit expression for non-ruin probability when claim sizes are gamma-distributed with rational shape parameters. Notice that for a rational shape parameter $r = m/n \in \mathbb{Q}$, with $\Re(s) > \alpha$, one could shift the argument s to obtain

$$\hat{\phi}(s-\alpha) = \frac{c\phi(0)}{c(s-\alpha) - \lambda + \lambda(\frac{\alpha}{s})^{m/n}} = \frac{c\phi(0)s^{m/n}}{c(s-\alpha)s^{m/n} - \lambda + \lambda\alpha^{m/n}}\,, \tag{3.3}$$

which is a ratio of polynomials of orders m and $m+n$ in $t = s^{1/n}$. This again permits a partial fraction decomposition.

Theorem 3.1.1. For a classical compound Poisson risk model (1.1) with claim sizes distributed as $\Gamma\left(\frac{m}{n}, \alpha\right)$, the probability of non-ruin is given by

$$\phi(u) = e^{-\alpha u} u^{\frac{1}{n}-1} \sum_{k=0}^{m+n-1} m_k E_{\frac{1}{n},\frac{1}{n}}\left(s_k u^{\frac{1}{n}}\right), \quad u \geqslant 0, \tag{3.4}$$

where $E_{\frac{1}{n},\frac{1}{n}}$ is the two-parameter Mittag-Leffler function defined in (2.2) with s_k and m_k are constants (see Remark 3.1.1 and 3.1.2 below).

Proof. Since the Laplace transform of f_X is $\left(\frac{\alpha}{s+\alpha}\right)^{\frac{m}{n}}$, equation (3.2) leads to the expression of $\hat{\phi}(s)$ as

$$\hat{\phi}(s) = \frac{c\phi(0)}{cs - \lambda + \lambda\left(\frac{\alpha}{s+\alpha}\right)^{\frac{m}{n}}}, \quad \Re(s) > -\alpha.$$

Shifting the Laplace argument gives (3.3) which has as denominator a polynomial in $s^{\frac{1}{n}}$ and thus can be further decomposed in partial fractions:

$$\hat{\phi}(s-\alpha) = \frac{\phi(0) s^{\frac{m}{n}}}{\prod_{k=0}^{m+n-1}\left(s^{\frac{1}{n}} - s_k\right)} = \sum_{k=0}^{m+n-1} \frac{m_k}{s^{\frac{1}{n}} - s_k},$$

where for each $k = 0, 1, \ldots, m+n-1$, s_k, m_k are constants to be determined on a case-by-case basis. Once the values of above parameters are obtained, one could invert it back to the sum of Mittag-Leffler function due to Proposition 2.1.1. Thus, the non-ruin probability for claim size with m/n shape gamma distribution has the form (3.4). □

Remark 3.1.1. All of the constants s_k in (3.4) are the roots of Lundburg equation (2.24)

$$x^{m+n} = \left(\alpha + \frac{\lambda}{c}\right) x^m - \frac{\lambda}{c}\alpha^{\frac{m}{n}}. \tag{3.5}$$

Note that both sides of (3.5) are analytic and $s_0 = \alpha^{\frac{1}{n}}$ is always a root of this equation. For $x = a + b\,i$ on the closed contour $\Gamma = \{x : |x| = \alpha^{1/n}\}$, i.e.,

$$a^2 + b^2 = \alpha^{\frac{2}{n}},$$

the maximum norm of the right-hand side of equation (3.5) equals

$$\max_{x\in\Gamma}\left|\left(\alpha + \frac{\lambda}{c}\right) x^m - \frac{\lambda}{c}\alpha^{\frac{m}{n}}\right| = \alpha^{\frac{m+n}{n}}$$

when $a = \alpha^{1/n}$ and $b = 0$. This means that on the closed contour $\Gamma = \{x : |x| = \alpha^{1/n}\}$ except the real value $x = \alpha^{1/n}$ we have

$$\left|x^{m+n}\right| > \left|\left(\alpha + \frac{\lambda}{c}\right) x^m - \frac{\lambda}{c}\alpha^{\frac{m}{n}}\right|.$$

By Rouché's theorem, equation (3.5) and $x^{m+n} = 0$ have the same number of roots $m+n$ inside the circle $\{x : |x| = \alpha^{1/n}\}$. We conclude that the other roots $s_1, s_2 \ldots, s_{m+n-1}$ have real parts less than $\alpha^{1/n}$.

Remark 3.1.2. All of the constants s_k in (3.4) are the solutions of systems of linear equations by adopting the method of undetermined coefficients.

Remark 3.1.3. There are no repeated roots among $s_0, s_1, \ldots, s_{m+n-1}$ in (3.4). In order to validate this argument, we check the derivative

$$\begin{aligned}
&\frac{d}{dx}\left(cx^{m+n} - (c\alpha + \lambda)x^m + \lambda\alpha^{\frac{m}{n}}\right) \\
=&c(m+n)x^{m+n-1} - (c\alpha + \lambda)mx^{m-1} \\
=&cx^{m-1}\left((m+n)x^n - \left(\alpha + \frac{\lambda}{c}\right)m\right).
\end{aligned}$$

Assuming there exist repeated roots $s_i = s_j = \eta$, the value of η must simultaneously satisfy the following two equations (the original function and its derivative at this point both equal to zero)

$$\begin{cases}
c\,\eta^{m+n} - (c\alpha + \lambda)\eta^m + \lambda\alpha^{\frac{m}{n}} = 0, \\
\\
(m+n)\eta^n - \left(\alpha + \frac{\lambda}{c}\right)m = 0.
\end{cases}$$

Starting from the second equation and substituting it into the first one gives

$$\frac{(c\alpha + \lambda)n}{m+n}\eta^m = \lambda\alpha^{\frac{m}{n}},$$

which has different roots of the second equation, a contradiction.

Remark 3.1.4. The case of positive integer $r = \frac{m}{1}$ in Theorem 3.1.1 reverts to the classical case of Erlang(m, α) distributed claim sizes. The non-ruin probability becomes

$$\phi(u) = e^{-\alpha u}u^{\frac{1}{1}-1}\sum_{k=0}^{m} m_k E_{1,1}(s_k u) = \sum_{k=0}^{m} m_k e^{(s_k-\alpha)u}, \quad u \geqslant 0,$$

coincides with the existed result in risk theory literature Li and Garrido (2005).

Remark 3.1.5. Thorin (1973) provides a closed-form expression for the ruin probability when the claims are gamma-distributed with parameters $k = \alpha$. When $k = \alpha = \frac{1}{2}$ and the inter-arrival times are exponentially distributed with parameter $\lambda = 1$, this becomes

$$\psi(u) = \frac{(c-1)(1-2R)e^{-Ru}}{1+c(3R-1)} + \frac{c-1}{2\pi}\int_0^\infty \frac{\sqrt{x}e^{-(x+1)u/2}}{(x+1)\left[\frac{c^2}{4}x^2 + (\frac{c^2}{4}+c)x + 1\right]}\,dx, \quad u \geqslant 0, \tag{3.6}$$

where R is the unique positive solution of Lundberg's equation

$$(1+cR)\sqrt{1-2R} = 1,$$

which can be solved to be $R = \frac{c-4+\sqrt{c^2+8c}}{4c}$.

On the other hand, we can show that in our result (2.29), for $r = 1/2$, the non-ruin probability equals

$$\phi(u) = e^{-\alpha u}u^{-\frac{1}{2}}\sum_{k=0}^{2} m_k E_{\frac{1}{2},\frac{1}{2}}\left(s_k u^{\frac{1}{2}}\right), \quad u \geqslant 0. \tag{3.7}$$

Here the roots s_0, s_1, s_2 and coefficients m_0, m_1, m_2 can be calculated explicitly,

$$s_0 = \sqrt{\alpha}, \quad s_1 = -\frac{\sqrt{\alpha}}{2} + \sqrt{\frac{\alpha}{4}+\frac{\lambda}{c}}, \quad s_2 = -\frac{\sqrt{\alpha}}{2} - \sqrt{\frac{\alpha}{4}+\frac{\lambda}{c}},$$

and

$$m_0 = \frac{s_0\left(1-\frac{\lambda}{2c\alpha}\right)}{(s_1-s_0)(s_2-s_0)} = \frac{\sqrt{\alpha}\left(c-\frac{\lambda}{2\alpha}\right)}{2c\alpha+\lambda},$$

$$m_1 = \frac{s_1\left(1-\frac{\lambda}{2c\alpha}\right)}{(s_0-s_1)(s_2-s_1)} = \frac{\left(-\frac{\sqrt{\alpha}}{2}+\sqrt{\frac{\alpha}{4}-\frac{\lambda}{c}}\right)\left(1-\frac{\lambda}{2c\alpha}\right)}{\left(\frac{3}{2}\sqrt{\alpha}-\sqrt{\frac{\alpha}{4}-\frac{\lambda}{c}}\right)\left(-2\sqrt{\frac{\alpha}{4}-\frac{\lambda}{c}}\right)},$$

$$m_2 = \frac{s_2\left(1-\frac{\lambda}{2c\alpha}\right)}{(s_0-s_2)(s_1-s_2)} = \frac{\left(-\frac{\sqrt{\alpha}}{2}-\sqrt{\frac{\alpha}{4}-\frac{\lambda}{c}}\right)\left(1-\frac{\lambda}{2c\alpha}\right)}{\left(\frac{3}{2}\sqrt{\alpha}+\sqrt{\frac{\alpha}{4}-\frac{\lambda}{c}}\right)\left(2\sqrt{\frac{\alpha}{4}-\frac{\lambda}{c}}\right)}.$$

It can be proved that (3.7) is the same as (3.6). The proof will be put in the appendix chapter.

3.1.2 Gamma-distributed claim sizes with real shape parameters

In this section, we provide three equivalent expressions for ruin probabilities in a Cramér-Lundberg model with real-gamma distributed claims. The results are solutions of integro-differential equations, derived by means of (inverse) Laplace transforms. All three formulas have infinite series forms, two involving Mittag-Leffler functions and the third one involving moments of the claims distribution. This last result applies to any claim size distribution that exhibits finite moments. The content is this section is mainly from Constantinescu et al. (2017).

3.1.2.1 Method One - Infinite sum of convolutions of Mittag-Leffler functions

For the first approach, we recognise certain geometric expansions present in the Laplace transform of the non-ruin probability when the claim sizes are gamma-distributed. These expansions can be inverted to obtain an explicit form of the non-ruin probability. The result is in terms of an infinite sum of convolutions. The convolution power of a locally integrable function f is defined recursively by $f^{*1} = f$, $f^{*n} = f^{*(n-1)} * f$, $n \geq 2$.

Theorem 3.1.2. For a classical compound Poisson risk model with claim sizes $X_k \sim \Gamma(r, \alpha)$, the non-ruin probability is

$$\phi(u) = \phi(0) + e^{-\alpha u}\phi(0)\left\{ e^{\alpha u} * \left(\sum_{n=1}^{\infty} \left(\frac{\lambda}{c}\right)^n \left[e^{\alpha u} - (\alpha u)^r E_{1,1+r}(\alpha u)\right]^{*n} \right) \right\}, \tag{3.8}$$

for any $u \geqslant 0$.

Proof. Rearranging the expression (3.2) for $\hat{\phi}(s)$, one can identify a geometric series with general term easily set to be between 0 and 1 for any $s > 0$,

$$\frac{\lambda}{c}\left(\frac{1}{s} - \frac{M_X(-s)}{s}\right) < 1,$$

so we can write

$$\hat{\phi}(s) = \frac{\phi(0)}{s} \frac{1}{1 - \frac{\lambda}{c}\left(\frac{1}{s} - \frac{\hat{f}(s)}{s}\right)} = \frac{\phi(0)}{s} \sum_{n=0}^{\infty} \left(\frac{\lambda}{c}\right)^n \left(\frac{1}{s} - \frac{(\frac{\alpha}{s+\alpha})^r}{s}\right)^n.$$

For $\Re(s) > \alpha$ we can shift the argument as explained in (3.2), to obtain

$$\hat{\phi}(s-\alpha) = \frac{\phi(0)}{s-\alpha} \sum_{n=0}^{\infty} \left(\frac{\lambda}{c}\right)^n \left(\frac{1}{s-\alpha} - \frac{\alpha^r}{(s-\alpha)s^r}\right)^n .$$

Notice that

$$\begin{aligned} \frac{1}{s-\alpha} - \frac{\alpha^r}{(s-\alpha)s^r} &= \frac{1}{s-\alpha} - \frac{\alpha^r}{s^{r+1}} \sum_{i=0}^{\infty} \left(\frac{\alpha}{s}\right)^i \\ &= \int_0^{\infty} e^{-su} \left(e^{\alpha u} - \sum_{i=0}^{\infty} \frac{\alpha^{r+i}}{\Gamma(r+i+1)} u^{r+i} \right) du \,, \end{aligned}$$

which is the Laplace transform of a positive function. Therefore, we have

$$\begin{aligned} e^{-\alpha u}\phi(u) &= \phi(0) \left\{ e^{\alpha u} + e^{\alpha u} * \left(\sum_{n=1}^{\infty} \left(\frac{\lambda}{c}\right)^n \left[e^{\alpha u} - \sum_{i=0}^{\infty} \frac{\alpha^{r+i}}{\Gamma(r+i+1)} u^{r+i} \right]^{*n} \right) \right\} \\ &= \phi(0) \left\{ e^{\alpha u} + e^{\alpha u} * \left(\sum_{n=1}^{\infty} \left(\frac{\lambda}{c}\right)^n \left[e^{\alpha u} - (\alpha u)^r E_{1,1+r}(\alpha u) \right]^{*n} \right) \right\}, \end{aligned}$$

as required. □

Remark 3.1.6. Note that the Mittag-Leffler functions in the expression (3.8) can be expressed in terms of incomplete gamma functions (Simon, 2015)

$$\begin{aligned} E_{1,\beta}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+\beta)} = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta-1)\Gamma(k+1)} B(\beta-1, k+1) \\ &= \frac{1}{\Gamma(\beta-1)} \int_0^1 (1-t)^{\beta-2} \sum_{k=0}^{\infty} \frac{(xt)^k}{\Gamma(k+1)} \, dt \\ &= \frac{1}{\Gamma(\beta-1)} \int_0^1 (1-t)^{\beta-2} e^{xt} \, dt = x^{1-\beta} e^x \frac{\gamma(\beta-1, x)}{\Gamma(\beta-1)} \end{aligned} \tag{3.9}$$

with the lower incomplete gamma function $\gamma(r,z) = \int_0^z t^{r-1} e^{-t} \, dt$.

Remark 3.1.7. Note that Theorem 3.1.2 is an exponentially tilted variant of the Pollaczeck-Khinchine (Beekman) formula for gamma claims, see Rolski et al. (1999) and Asmussen and Albrecher (2010). To clarify this connection, consider the upper tail of claims $\bar{F}_X(u) = \mathbb{P}[X > u]$, which, as in Remark 3.1.6, identity (3.9), can be regarded as

$$e^{\alpha u} \bar{F}_X(u) = e^{\alpha u} - \alpha^r u^r E_{1,1+r}(\alpha u),$$

so that equation (3.8) becomes

$$\begin{aligned}\phi(u) &= \phi(0) + e^{-\alpha u}\phi(0)\left(\sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n e^{\alpha u} * \left[e^{\alpha u}\bar{F}_X^{*n}(u)\right]\right)\\ &= \phi(0) + \phi(0)\sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n \int_0^u \bar{F}_X^{*n}(y)\,dy.\end{aligned}$$

This is equivalent to

$$\begin{aligned}\psi(u) &= 1 - \left(1 + \phi(0)\sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n \int_0^u \bar{F}_X^{*n}(y)\,dy\right)\\ &= \phi(0)\sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n \int_u^{\infty} \bar{F}_X^{*n}(y)\,dy,\end{aligned}$$

the Pollaczeck-Khinchine formula for the ruin probability, as in, e.g. Rolski et al. (1999).

Remark 3.1.8. When $r = 1$ in (3.8), we recover (2.27).

Proof. The expression in the square bracket in (3.8) equals to 1 for all $u \geqslant 0$, and its n-fold convolution power is the function $u^{n-1}/(n-1)!$, $u \geqslant 0$. Therefore, one has

$$\phi(u) = \phi(0) + e^{-\alpha u}\phi(0)\left\{e^{\alpha u} * \left(\sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n \left[e^{\alpha u} - \int_0^u \alpha e^{\alpha x}dx\right]^{*n}\right)\right\}, \tag{3.10}$$

where the term in the square bracket is

$$H(u) = e^{\alpha u} - \int_0^u \alpha e^{\alpha x}dx = \begin{cases} 1, & u \geq 0,\\ 0, & u < 0.\end{cases}$$

The convolution of $H(x)$ and itself is $\int_{-\infty}^{\infty} 1\,ds = u$. This implies that the n-th convolution in will become some power function times constants. In this case, the infinite sum in (3.10) just converges to the exponential function, which means

$$\begin{aligned}\phi(u) &= \phi(0) + e^{-\alpha u}\phi(0)\left\{e^{\alpha u} * \left(\sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n \frac{u^{n-1}}{\Gamma(n-1)}\right)\right\}\\ &= \phi(0) + e^{-\alpha u}\phi(0)\left\{e^{\alpha u} * \left(\frac{\lambda}{c}e^{\frac{\lambda}{c}u}\right)\right\}.\end{aligned}$$

The convolution term can be calculated by definition directly,

$$\begin{aligned}\phi(u) &= \phi(0) + e^{-\alpha u}\phi(0)\left\{\frac{\lambda}{c}\left(\alpha - \frac{\lambda}{c}\right)^{-1} e^{\frac{\lambda}{c}u}\left[e^{\left(\alpha-\frac{\lambda}{c}\right)u} - 1\right]\right\} \\ &= \phi(0)\frac{\alpha}{\alpha - \frac{\lambda}{c}}\left[1 - \frac{\lambda}{\alpha c}e^{-\left(\alpha-\frac{\lambda}{c}\right)u}\right].\end{aligned}$$

Since $\phi(0) = 1 - \lambda/\alpha c$, one concludes that

$$\phi(u) = 1 - \frac{\lambda}{\alpha c}e^{\left(\frac{\lambda}{c}-\alpha\right)u}, \quad u \geqslant 0,$$

which coincides with equation (2.27). □

Remark 3.1.9. For an integer number r, recall from Podlubny (1998) that

$$E_{1,1+r}(\alpha u) = \frac{1}{(\alpha u)^r}\left(e^{\alpha u} - \sum_{k=0}^{r-1}\frac{(\alpha u)^k}{k!}\right), \tag{3.11}$$

and so by (3.8) the non-ruin probability equals to

$$\phi(u) = \phi(0) + e^{-\alpha u}\phi(0)\left\{e^{\alpha u} * \left(\sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n\left[\sum_{k=0}^{r-1}\frac{(\alpha u)^k}{k!}\right]^{*n}\right)\right\}, \quad u \geqslant 0. \tag{3.12}$$

Consider the case $r = 2$. The expression (3.12) agrees with the elementary partial fraction inversion mentioned in Grandell (1991a). Proof will be put in the appendix chapter.

3.1.2.2 Method Two - Infinite sum of derivatives of Mittag-Leffler functions

Now we present a different method to derive the non-ruin probability when claim sizes are gamma-distributed, which leads to an explicit form in terms of an infinite sum of derivatives of Mittag-Leffler functions.

Theorem 3.1.3. For a classical compound Poisson risk model (1.1) with claim sizes $X_k \sim \Gamma(r, \alpha)$, the non-ruin probability can be written as

$$\phi(u) = e^{-\alpha u}\phi(0)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\lambda\alpha^r}{c}\right)^k u^{(r+1)k}E^{(k)}_{1,rk+1}\left(\left(\alpha + \frac{\lambda}{c}\right)u\right), \quad u \geqslant 0, \tag{3.13}$$

where $E^{(n)}_{\alpha,\beta}$ is the n-th derivative of the Mittag-Leffler function.

Proof. Let $\beta > \alpha$. The first step is to find a function G whose Laplace transform is, for sufficiently large $s > 0$,

$$g(s) = \frac{1}{as^\beta + bs^\alpha + c},$$

where a, b, c are non-zero constants. One can rewrite

$$g(s) = \frac{1}{c}\frac{c}{as^\beta + bs^\alpha}\frac{as^\beta + bs^\alpha}{as^\beta + bs^\alpha + c} = \frac{1}{c}\frac{\frac{c}{a}s^{-\alpha}}{s^{\beta-\alpha} + \frac{b}{a}}\frac{1}{1 + \frac{\frac{c}{a}s^{-\alpha}}{s^{\beta-\alpha}+\frac{b}{a}}}.$$

Denoting $P = \frac{\frac{c}{a}s^{-\alpha}}{s^{\beta-\alpha}+\frac{b}{a}}$, which is a number in $(0,1)$ for large s, the expression becomes

$$g(s) = \frac{1}{c}\frac{P}{1-(-P)} = \frac{1}{c}\sum_{k=0}^{\infty}(-1)^k P^{k+1} = \frac{1}{c}\sum_{k=0}^{\infty}(-1)^k \left(\frac{c}{a}\right)^{k+1} \frac{s^{-\alpha k-\alpha}}{\left(s^{\beta-\alpha} + \frac{b}{a}\right)^{k+1}}.$$

Recognizing the Laplace transform formula (2.1.1), one can invert this expression term by term, to see that g is the Laplace transform of the function (Podlubny, 1998)

$$G(t) = \frac{1}{a}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{c}{a}\right)^k t^{\beta(k+1)-1}E^{(k)}_{\beta-\alpha,\beta+\alpha k}\left(-\frac{b}{a}t^{\beta-\alpha}\right).$$

Recall from (3.3) that for a classical risk model with gamma-distributed claim sizes, the Laplace transform of non-ruin probability after shifting the argument becomes, when s is large enough,

$$\begin{aligned}
\hat{\phi}(s-\alpha) =& \frac{c\phi(0)s^r}{cs^{r+1} - (c\alpha+\lambda)s^r + \lambda\alpha^r} \\
=& \frac{c\phi(0)s^r}{cs^{r+1} - (c\alpha+\lambda)s^r}\sum_{k=0}^{\infty}(-1)^k\left(\frac{\lambda\alpha^r}{cs^{r+1} - (c\alpha+\lambda)s^r}\right)^k \\
=& \sum_{k=0}^{\infty}(-1)^k\frac{\phi(0)\left(\frac{\lambda}{c}\alpha^r\right)^k s^{-rk}}{\left(s - \left(\alpha + \frac{\lambda}{c}\right)\right)^{k+1}},
\end{aligned}$$

which permits term-by-term inversions

$$\phi(u) = e^{-\alpha u}\phi(0)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\lambda\alpha^r}{c}\right)^k u^{(r+1)k}E^{(k)}_{1,rk+1}\left(\left(\alpha+\frac{\lambda}{c}\right)u\right),$$

as required. The last expression can be rewritten in the form

$$\phi(u) = e^{-\alpha u}\phi(0)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\lambda\alpha^r}{c}\right)^k u^{(r+1)k}\sum_{j=0}^{\infty}\frac{(j+k)!\left(\left(\alpha+\frac{\lambda}{c}\right)u\right)^j}{j!\Gamma(k(r+1)+1+j)}. \tag{3.14}$$

□

Remark 3.1.10. For $r = 1$, note that expression (3.13) also reduces, as it should, to the classical result (2.27). The proof is put in the appendix chapter.

3.1.2.3 Method Three - Tail convolutions

Finally, we start with the classical risk model with any light-tailed distributed claims. The non-ruin probability ϕ will be obtained as integral of an infinite sum of moments of claim size distributions. When the claims are gamma-distributed, the resulting formulas can be relatively efficiently evaluated.

Recall the form (3.2) of the Laplace transform of the ruin probability in a compound Poisson process with a generic claim size X and the moment-generating function M_X:

$$\hat{\phi}(s) = \phi(0)\frac{1}{s}\frac{1}{1-\frac{\lambda}{c}\frac{1-M_X(-s)}{s}}. \tag{3.15}$$

Notice that the term in the denominator,

$$\hat{g}(s) = \frac{1-M_X(-s)}{s}, \quad s > 0,$$

is the Laplace transform of the distributional tail

$$g(x) = P(X > x), \quad x \geqslant 0. \tag{3.16}$$

By the positive loading assumption (2.21) we have

$$\hat{\phi}(s) = \phi(0)\frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{\lambda}{c}\right)^n (\hat{g}(s))^n, \tag{3.17}$$

since the ratio in the series is smaller than 1. Inverting the Laplace transforms in (3.17) gives us immediately the first statement of the next theorem. The key part of the theorem is the expression (3.19) for the ingredients in (3.18).

Theorem 3.1.4. The non-ruin probability in classical risk model can be written in the form

$$\phi(u) = \phi(0)\left(1 + \int_0^u \sum_{n=1}^{\infty}\left(\frac{\lambda}{c}\right)^n g^{*n}(y)\,dy\right), \quad u \geqslant 0. \tag{3.18}$$

Here g^{*n} is the n-th convolution of the tail distribution of claim X_j. It can be computed for $n \geq 2$ as

$$\begin{aligned} g^{*n}(x) =& \frac{1}{(n-1)!}\mathbb{E}\left[\left(\sum_{j=1}^{n} X_j - x\right)^{n-1} \mathbb{1}\left(\sum_{j=1}^{n} X_j > x\right)\right] \\ &- \frac{1}{(n-1)!}\sum_{i=1}^{n-1}\binom{n-1}{n-i-1} b_{n-i}(F_X)\mathbb{E}\left[\left(\sum_{j=1}^{i} X_j - x\right)^{i} \mathbb{1}\left(\sum_{j=1}^{i} X_j > x\right)\right], \end{aligned} \tag{3.19}$$

$n \in \mathbb{Z}^+$. The sequence $(b_i(F_X),\, i \in \mathbb{Z}^+)$ depends on the distribution F_X of claim sizes. It is defined recursively by

$$\begin{aligned} b_1(F_X) &= 1, \\ b_{m+1}(F_X) &= \mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^{m} - \sum_{i=1}^{m}\binom{m}{i-1} b_i(F_X)\mathbb{E}\left(\sum_{j=1}^{n-i} X_j\right)^{m+1-i}, \end{aligned} \tag{3.20}$$

for $m = 1, \ldots, n-1$. Thus defined, b_m is independent of $n > m$.

Proof. The proof is postponed to appendix chapter. □

Remark 3.1.11. The third method computes the convolution powers for the tail-distribution in a Pollaczeck-Khinchine formula via a sequence of integrals of increasing dimension which simplify to one-dimensional integrals in the gamma case. See the discussion part for details.

3.1.3 Numerical result of three methods

The results shown in this chapter refer to extensions of the classical ruin models and all of them coincide with the classical result when claim sizes are exponentially distributed. In this section, we will present the advantages of each method and its result, including some numerical examples. Since all three expressions present infinite sums, we truncate those to their first 20 terms to be able to obtain a numerical value. The corresponding truncation errors are less than 10^{-5} for all three expressions, and we have noticed that considering more than the first 20 terms will not decrease substantially the numerical errors.

The first expression in Theorem 3.1.2 is an infinite sum of convolution terms. When r takes integer value, the expression reduces to a sum of finite terms due to Proposition (3.11) of the Mittag-Leffler function and thus explicit results can be implemented. As long as r is not integer, numerical methods are needed to calculate the probability. One choice is to use the relationship between the Mittag-Leffler function and incomplete gamma function, mentioned in Remark 3.1.6, identity (3.9), since the incomplete gamma functions are available in most numerical libraries and systems. The other choices would be to use "Mittag-Leffler function" MATLAB codes by Igor Podlubny (which calculates the Mittag-Leffler function with desired accuracy) or "MittagLeffleR" R package by Gurtek Gill and Peter Straka (which provides probability density, distribution function, quantile function and random variate generation for the Mittag-Leffler distributions, and the Mittag-Leffler function). For instance, we will take the sum of the first 20 convolutions in the expression for a numerical result for the non-ruin probability

$$\phi(u) = \phi(0) + e^{-\alpha u}\phi(0)\left\{ e^{\alpha u} * \left(\sum_{n=1}^{20} \left(\frac{\lambda}{c}\right)^n \left[e^{\alpha u} - (\alpha u)^r E_{1,1+r}(\alpha u)\right]^{*n} \right) \right\}. \qquad (3.21)$$

The second expression (3.13) is a quite time efficient method, which is very easy to implement with accurate results. Due to the fact that the derivative of a Mittag-Leffler function is an infinite series, this expression contains two-fold infinite sums. Moreover, inside each series, only gamma functions and power functions are needed to be calculated. Therefore, any software having 'addition' and loop functions can handle this expression. Compared with the first result, which contains convolution terms, this one is more time efficient in a numerical sense. The disadvantage is that we have no instance where we can get exact result, for $r \neq 1$. In this case, we could evaluate the first 20 derivatives in the expression to obtain a numerical approximation of the non-ruin probability

$$\phi(u) = e^{-\alpha u}\phi(0) \sum_{k=0}^{20} \frac{(-1)^k}{k!} \left(\frac{\lambda \alpha^r}{c}\right)^k u^{(r+1)k} E_{1,rk+1}^{(k)} \left(\left(\alpha + \frac{\lambda}{u}\right) u \right).$$

The third result in Theorem 3.1.4 is presented in terms of moments of the claim size distribution. In principle, this method is valid for any claim distribution, but in the case of gamma claims, the distribution of the sum of X is- known analytically, so the computations are tractable. Note that since

$$\int_0^u \sum_{n=1}^{\infty} \left(\frac{\lambda}{c}\right)^n g^{*n}(y)\, dy = \sum_{n=1}^{\infty} \left(\frac{\lambda}{c}\right)^n \int_0^u g^{*n}(y)\, dy,$$

one needs to be able to compute efficiently

$$\int_0^u g^{*n}(y)\, dy = \frac{1}{(n-1)!} \mathbb{E} \int_0^u \left(\sum_{j=1}^n X_j - x\right)^{n-1} \mathbb{1}\left(\sum_{j=1}^n X_j > x\right) dx$$
$$-\frac{1}{(n-1)!} \sum_{i=1}^{n-1} \binom{n-1}{n-i-1} b_{n-1}(F_X) \mathbb{E} \int_0^u \left(\sum_{j=1}^i X_j - x\right)^i \mathbb{1}\left(\sum_{j=1}^i X_j > x\right) dx.$$

As it is easy to compute the sequence $(b_n(F_X))$ for gamma claims, one only needs to evaluate efficiently the functions

$$a_{n,k}(u) = \mathbb{E} \int_0^u \left(\sum_{j=1}^n X_j - x\right)^k \mathbb{1}\left(\sum_{j=1}^n X_j > x\right) dx,$$

for $k = n-1$ and n. However, these functions can be further expressed in terms of incomplete gamma functions as

$$a_{n,k}(u) = \frac{\alpha^{nr}}{\Gamma(nr)(k+1)} \left[\frac{\Gamma(nr+k+1)}{\alpha^{nr+k+1}} - \sum_{j=0}^{k+1} \binom{k+1}{j} (-u)^{k+1-j} \frac{\Gamma(nr+j, \alpha u)}{\alpha^{nr+j}}\right]$$

for $k = n-1$ and n. Therefore, the whole calculation consists on evaluating some incomplete gamma functions, and those have already been efficiently implemented. The first 20 convolutions in expression (3.18) would be sufficient when implementing the non-ruin probability numerically.

Note that in comparison with the other two results, this method can be used with any claim size distribution. The numerical complexity, however, can be higher than in the case of gamma-distributed claims. Mixed exponentially distributed is one of the other examples that would have explicit result by method three. For instance, we consider the claim sizes X_j follow a mixed exponential distribution with density function

$$f(x) = \alpha\lambda_1 e^{-\lambda_1 x} + (1-\alpha)\lambda_2 e^{-\lambda_2 x},$$

then the distribution of $\sum_{j=1}^{n} X_j$ can be obtained by its Laplace transform

$$\begin{aligned}
&\left(\frac{\alpha\lambda_1}{s+\lambda_1} + \frac{(1-\alpha)\lambda_2}{s+\lambda_2}\right)^n \\
=&\sum_{i=0}^{n} \binom{n}{i} \frac{\alpha^i \lambda_1^i}{(s+\lambda_1)^i} \frac{(1-\alpha)^{n-i}\lambda_2^{n-i}}{(s+\lambda_2)^{n-i}} \\
=&\frac{\alpha^n\lambda_1^n}{(s+\lambda_1)^n} + \frac{(1-\alpha)^n\lambda_2^n}{(s+\lambda_2)^n} + \sum_{i=1}^{n-1}\left(\sum_{m=1}^{i} \frac{\beta_m\lambda_1^m}{(s+\lambda_1)^m} + \sum_{m=1}^{n-i} \frac{\gamma_m\lambda_2^m}{(s+\lambda_2)^m}\right),
\end{aligned}$$

where all β_m, $m = 1, \ldots, i$ and γ_m, $m = 1, \ldots, n-i$ are computed from partial fraction decomposition. Thus, the sum of independent and identically distributed X_m in this case has density function

$$\begin{aligned}
f_\Sigma(x) =&\alpha^n \frac{\lambda_1^n}{\Gamma(n)} x^{n-1} e^{-\lambda_1 x} + (1-\alpha)^n \frac{\lambda_2^n}{\Gamma(n)} x^{n-1} e^{-\lambda_2 x} \\
&+ \sum_{i=1}^{n-1}\left(\sum_{m=1}^{i} \beta_m \frac{\lambda_1^m}{\Gamma(m)} x^{m-1} e^{-\lambda_1 x} + \sum_{m=1}^{n-i} \gamma_m \frac{\lambda_2^m}{\Gamma(m)} x^{m-1} e^{-\lambda_2 x}\right),
\end{aligned}$$

which can be regarded as a finite mixed gamma distribution. Thus, the expression for $a_{n,k}$ in this case is

$$\begin{aligned}
a_{n,k} =&\frac{\alpha^n\lambda_1^n}{\Gamma(n)(k+1)} \left[\frac{\Gamma(n+k+1)}{\lambda_1^{n+k+1}} - \sum_{j=0}^{k+1} \binom{k+1}{j} (-u)^{k+1-j} \frac{\Gamma(n+j, \lambda_1 u)}{\lambda_1^{n+j}}\right] \\
&+\frac{(1-\alpha)^n\lambda_2^n}{\Gamma(n)(k+1)} \left[\frac{\Gamma(n+k+1)}{\lambda_2^{n+k+1}} - \sum_{j=0}^{k+1} \binom{k+1}{j} (-u)^{k+1-j} \frac{\Gamma(n+j, \lambda_2 u)}{\lambda_2^{n+j}}\right] \\
&+\sum_{i=1}^{n-1}\left(\sum_{m=1}^{i} \frac{\beta_m\lambda_1^m}{\Gamma(m)(k+1)} \left[\frac{\Gamma(m+k+1)}{\lambda_1^{m+k+1}} - \sum_{j=0}^{k+1} \binom{k+1}{j} (-u)^{k+1-j} \frac{\Gamma(m+j, \lambda_1 u)}{\lambda_1^{m+j}}\right]\right. \\
&\left.+\sum_{m=1}^{n-i} \frac{\gamma_m\lambda_2^m}{\Gamma(m)(k+1)} \left[\frac{\Gamma(m+k+1)}{\lambda_1^{m+k+1}} - \sum_{j=0}^{k+1} \binom{k+1}{j} (-u)^{k+1-j} \frac{\Gamma(m+j, \lambda_2 u)}{\lambda_2^{m+j}}\right]\right)
\end{aligned}$$

for $k = n-1$ and n. These terms can again be efficiently evaluated.

The next figure shows the difference on accuracy of these three results.

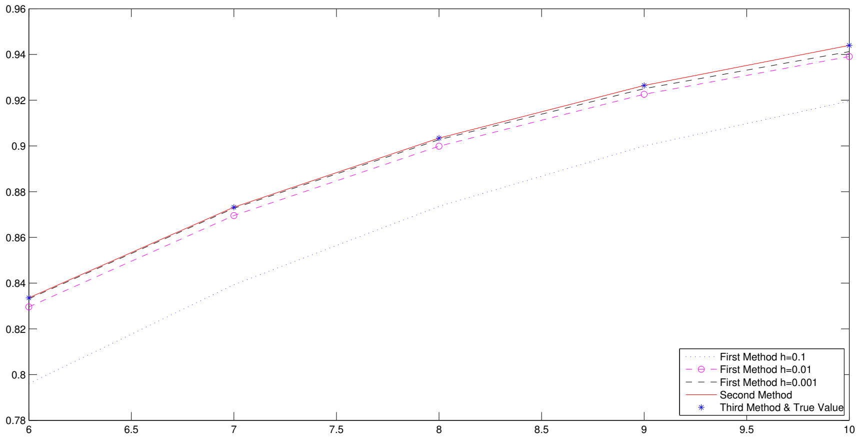

Figure 3.1: Difference on accuracy of three results

In order to test the accuracy of the three results, we choose the parameter values $\lambda = 1$, $c = 1$, $r = 2$ and $\alpha = 2.4$. All the dashed lines stand for the first method and solid one for the method two and three. In order to numerically evaluate the convolution integral in equation (3.21), we break up the interval $[0, 10]$ into subintervals of length h, then apply the trapezoid rule and use Newton-Cotes formula to realize the numerical integration. We compare results obtained for different lengths h. One can see that using the second method, the results converge relatively fast. Moreover, for the third method, the results are exactly the same as the true values because the moments of gamma random variables have explicit expressions. Note that for $r = 2$, one retrieves the case of Erlang(2) claims. Several equivalent results under this model assumption have been obtained in the past and the formula chosen in this test comes from (He et al., 2003)

$$\phi(u) = 1 + \frac{v_2(v_1+\alpha)^2}{(v_1-v_2)\alpha^2}e^{v_1 u} + \frac{v_1(v_2+\alpha)^2}{(v_2-v_1)\alpha^2}e^{v_2 u},$$

where

$$v_1 = \frac{\lambda - 2c\alpha + \sqrt{\lambda^2 + 4c\alpha\lambda}}{2c},$$

$$v_2 = \frac{\lambda - 2c\alpha - \sqrt{\lambda^2 + 4c\alpha\lambda}}{2c}.$$

Here is the corresponding table.

initial capital	Method 1 h=0.1	Method 1 h=0.01	Method 1 h=0.001	Method 2	Method 3 & True Value
u=0	0.167	0.167	0.167	0.167	0.167
u=1	0.340	0.350	0.352	0.352	0.352
u=2	0.480	0.503	0.505	0.506	0.506
u=3	0.588	0.620	0.623	0.623	0.623
u=4	0.674	0.709	0.713	0.713	0.713
u=5	0.742	0.777	0.781	0.782	0.782
u=6	0.796	0.830	0.833	0.834	0.834
u=7	0.839	0.870	0.873	0.873	0.873
u=8	0.874	0.900	0.903	0.903	0.903
u=9	0.900	0.923	0.925	0.926	0.926
u=10	0.920	0.939	0.941	0.944	0.944

Table 3.1: Difference on accuracy of three results

The errors between the results obtained from method two and true values are significantly smaller at 10^{-11} level.

As mentioned before, the second result is the most efficient one among all three, in numerical sense. In Table 3.2 are some results run by MATLAB using method two. These results can also be obtained using method one if one sets the step length to be as small as $h = 0.0001$, which takes more time. In Table 3.2, the parameter values are set to be $\lambda = 1$, $c = 1$ and safety loading $\theta = 0.2$. Because the safety loading is held constant, for each r, we choose an α such that the average claim size $\frac{r}{\alpha}$ stays the same.

Initial Capital	$r = 0.5$	$r = 1$	$r = 1.5$	$r = 2$	$r = 2.5$	$r = 3$
$u = 0$	0.167	0.167	0.167	0.167	0.167	0.167
$u = 1$	0.281	0.318	0.338	0.352	0.361	0.368
$u = 2$	0.371	0.441	0.481	0.506	0.523	0.536
$u = 3$	0.449	0.543	0.593	0.623	0.644	0.660
$u = 4$	0.517	0.626	0.680	0.713	0.735	0.750
$u = 5$	0.576	0.693	0.749	0.782	0.802	0.817
$u = 6$	0.628	0.749	0.803	0.834	0.852	0.865
$u = 7$	0.673	0.795	0.846	0.873	0.890	0.901
$u = 8$	0.713	0.832	0.879	0.903	0.918	0.927
$u = 9$	0.749	0.862	0.905	0.926	0.939	0.947
$u = 10$	0.779	0.887	0.926	0.944	0.954	0.961

Table 3.2: Non-ruin probabilities of classical risk model with safety loading is 0.2

The corresponding plotting figure is shown as follows.

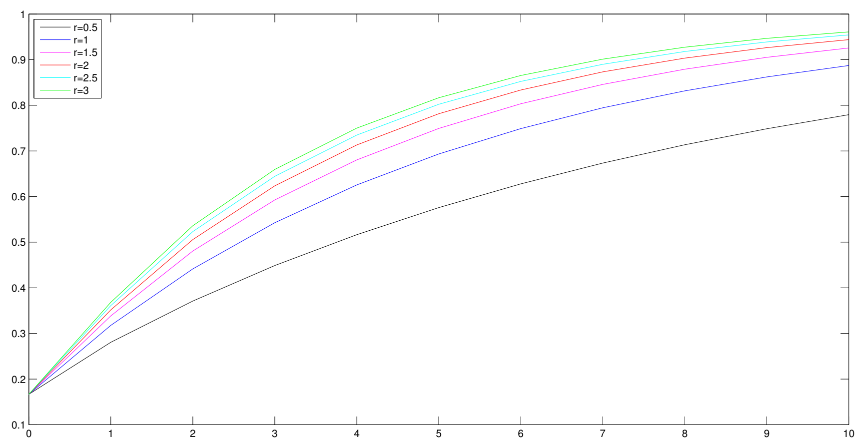

Figure 3.2: Non-ruin probabilities of classical risk model with safety loading is 0.2

One can observe that when the safety loading and other model parameters are fixed, the larger r is, the higher non-ruin probability the model has. The reason is that in this case, the expected claim size is fixed, further means that the ratio $\frac{r}{\alpha}$ is fixed, whereas the variance of claim size $\frac{r}{\alpha^2}$ decreases as r increases, i.e., the chance of having large claims will decrease. Since ruin is usually caused by some large claims, the model with a larger shape parameter r is more likely to survive.

The next table and figure will show how the non-ruin probability changes with various premium rates and same safety loading when claim has gamma distribution with $r = 1.5$.

initial capital	$c = 1$	$c = 1.2$	$c = 1.4$	$c = 1.6$	$c = 1.8$	$c = 2$
u=0	0.167	0.167	0.167	0.167	0.167	0.167
$u = 1$	0.338	0.311	0.291	0.276	0.264	0.255
$u = 2$	0.481	0.437	0.403	0.377	0.356	0.338
$u = 3$	0.593	0.540	0.498	0.465	0.437	0.414
$u = 4$	0.680	0.624	0.578	0.540	0.508	0.481
$u = 5$	0.749	0.693	0.645	0.605	0.570	0.540
$u = 6$	0.803	0.749	0.702	0.660	0.624	0.593
$u = 7$	0.846	0.795	0.749	0.708	0.672	0.639
$u = 8$	0.879	0.833	0.789	0.749	0.713	0.680
$u = 9$	0.905	0.863	0.823	0.785	0.749	0.717
$u = 10$	0.926	0.888	0.851	0.815	0.781	0.749

Table 3.3: Non-ruin probabilities of classical risk model with safety loading is 0.2

The corresponding plotting figure is shown as follows.

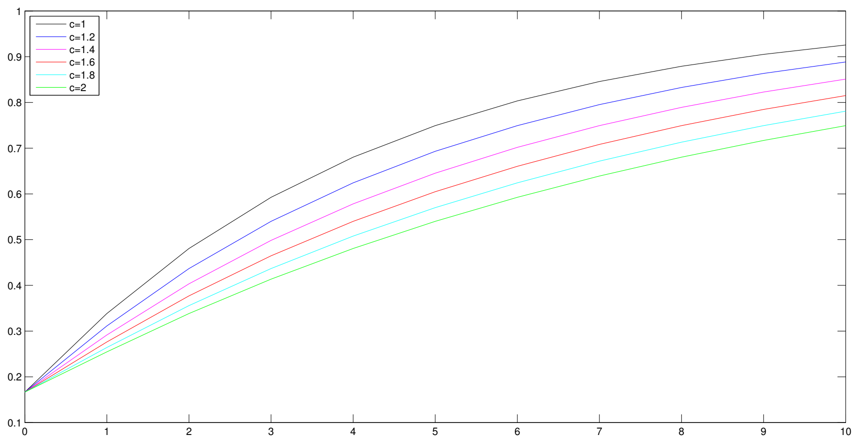

Figure 3.3: Non-ruin probabilities of classical risk model with safety loading is 0.2

Since the safety loading θ is fixed, the larger premium rate c is, the larger the expected claim size is. In this case, the shape parameter of claim distribution r is set to be constant 1.5, which means the larger expectation gives larger variance. Thus, similar to the previous test, this result is quite reasonable. When the safety loading and claim shape parameter are fixed, decreasing premium rate can make the company less likely to have ruin.

3.2 Explicit expressions for ruin probabilities in the case of geometric claims

In this section, the claim sizes are assumed to follow geometric distribution (see in (2.14)) in the classical risk model. Discrete claim sizes assumption usually rises in discrete-time risk models in the literature. While most theoretical risk models use the concept of time continuity, the practical reality is discrete. One widely used discrete risk model is the compound binomial model, first proposed by Gerber (1988), which is a discrete analog of the compound Poisson model in risk theory. It is a fully discrete-time model where premiums, claim amounts, and the initial surplus are assumed to be integer valued, but can be used as an approximation to the continuous-time compound Poisson model. Unlike continuous-time risk models, discrete-time risk models have not attracted much attention and the literature counts fewer contributions. A very detailed review can be found in Li et al. (2009).

However, only a few publications in risk theory have considered the case of discrete claim sizes appearing in continuous-time risk models. One interpretation of this assumption would be that some insurance policies might use step functions for claim payments. In this case, the claim sizes can only take finitely or at most countably many values. In this section, we use the martingale approach to derive the explicit ruin probability for classical risk model with geometric claims. The first use of martingales in risk theory is due to Gerber (1973), which is later studied by Gerber (1979); Delbaen and Haezendonck (1985); Dassios and Embrechts (1989); Grandell (1991a,b); Embrechts et al. (1993); Schmidli (1994, 2010) and discussed in detail in Asmussen and Albrecher (2010).

Theorem 3.2.1. In the classical risk model (1.1), if $(X_i)_{i\geqslant 1}$ are i.i.d. geometric random variables with probability mass function $p_k = p(1-p)^k$, the ruin probability has the form:

$$\psi(u) = \frac{1-(1-p)e^{-\gamma}}{p}e^{\gamma u} = \frac{1}{p}\left(e^{\gamma u} - (1-p)e^{\gamma u-\gamma}\right), \quad u \geqslant 0, \tag{3.22}$$

where γ is the negative root of

$$c\gamma + \lambda\left(\frac{p}{1-(1-p)e^{-\gamma}} - 1\right) = 0. \tag{3.23}$$

Proof. The moment-generating function of process $R(t)$ is

$$G_{R(t)}(s) = \mathbb{E}\left(e^{s\left(u+ct-\sum\limits_{i=1}^{N(t)} X_i\right)}\right) = e^{s(u+ct)+\lambda t\left(\frac{p}{1-(1-p)e^{-s}}-1\right)}.$$

Denote

$$\kappa(s) = cs + \lambda\left(\frac{p}{1-(1-p)e^{-s}} - 1\right),$$

then $M_t = e^{-sR_t + t\kappa(s)}$ is a martingale. Let γ be the negative root of $\kappa(\gamma)$. Applying the optional stopping theorem, one has

$$\begin{aligned} e^{\gamma u} =& \mathbb{E}(M_0) = \mathbb{E}(M_\tau) \\ =& \mathbb{E}\left(e^{\gamma R_\tau}\middle|\, \tau < \infty\right)\mathbb{P}(\tau < \infty) \\ =& \frac{p}{1-(1-p)e^{-\gamma}}\psi(u), \quad u \geqslant 0, \end{aligned}$$

which completes the proof. □

Remark 3.2.1. It is well known that a sequence of discrete geometric distributions converges to the exponential distribution. Denoting $np = \alpha$ and $x = k/n$, one will have

$$1 = \sum_{k=0}^{\infty}\mathbb{P}(X=k) = \sum_{k=0}^{\infty}\alpha\left(\left(1-\frac{\alpha}{n}\right)^{n\,k/n}\frac{1}{n}\right) \to \int_0^\infty \alpha e^{-\alpha x}\,dx.$$

Therefore, the moment-generating function of the geometric distribution also converges,

$$\frac{p}{1-(1-p)e^z} = \lim_{n\to\infty} \frac{\alpha}{n-(n-\alpha)e^{x/n}} = \frac{\alpha}{\alpha - x}.$$

Theorem 3.2.1 can be analysed by using this technique. In such case, γ will be the negative root of the limiting version of equation (3.23)

$$c\gamma + \lambda \left(\frac{\alpha}{\alpha - \gamma} - 1 \right) = 0,$$

which leads to $\gamma = \frac{\lambda}{c} - \alpha$. At the same time, the expression of ruin probability $\psi(u)$ becomes

$$\psi(u) = \frac{\alpha + \gamma}{\alpha} e^{\gamma u} = \frac{\lambda}{\alpha c} e^{\left(\frac{\lambda}{c} - \alpha\right)u},$$

which coincides with the classical result (2.27).

3.3 Summary

New results for the (non-)ruin probabilities in the classical risk model in the case of gamma or geometric-distributed claim sizes are obtained in this chapter. Classical results are retrieved when certain parameters take specific values or limits.

Chapter 4

Renewal Risk Models

The renewal risk model in risk theory, also known as Sparre Andersen model, is introduced and analysed by Andersen (1957)

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t > 0. \tag{4.1}$$

In this model it is allowed that the number of claims $N(t)$ could not only follow the Poisson counting process, but also a more general renewal counting process. The ruin probability $\psi(u)$ of a renewal risk model solves an integral equation (2.30) by using the renewal property (Feller, 2008).

In this chapter, we are going to consider a family of renewal risk models with the inter-arrival time density functions solving fractional differential equations. A new class of fractional differential operators are defined in order to tackle the ruin problems in these models. Explicit expressions for ruin probabilities are obtained when claim size distributions have rational Laplace transforms. The content in this chapter is mainly from the working paper (Constantinescu et al., 2017).

4.1 Fractional differential operators

Let $\mathcal{L}(y)$ denote an n-th degree polynomial $y^n + p_1 y^{n-1} + \cdots + p_{n-1} y + p_n$ and consider the following associated homogeneous ordinary differential equation with constant coefficients

$$\mathcal{L}\left(\frac{d}{dx}\right)[f](x) = f^{(n)}(x) + p_1 f^{(n-1)}(x) + \cdots + p_{n-1} f'(x) + p_n f(x) = 0. \tag{4.2}$$

Suppose further that equation (4.2) can be expressed in the form

$$\bigodot_{j=0}^{m} \left(\frac{d}{dx} + \lambda_j\right)^{k_j} [f](x) = 0 \tag{4.3}$$

for positive real number λ_j and integers k_j, $j = 1, \cdots, m$. The solution $f(x)$ to (4.3) is the probability density function of either a sum of Erlang random variables or a mixed Erlang random variable, depending on the boundary conditions (see examples in Albrecher et al. (2010)). For instance, one can express any density function which is a convolution of n exponential densities with parameters λ_j in equation (4.3), with almost homogeneous initial conditions

$$\begin{aligned} & f^{(k)}(0) = 0 \quad (\text{for } k = 0, 1, \ldots, m-2) \\ \text{and} \quad & f^{(m-1)}(0) = \prod_{j=1}^{m} \lambda_j . \end{aligned}$$

In the special case of exponentials with the same parameter λ, this is an Erlang(m) density function. We would like to generalise equation (4.3), and characterise its solutions, in the case where the exponents k_j are no longer integers.

4.1.1 Rock operators

In order to generalise expression (4.2), it is necessary to explore the world of fractional calculus. Solving fractional differential equations has become an essential issue as fractional-order models appear to be more adequate than previously used integer-order models in various fields. A large host of available analytical methods for solving fractional order integral and differential equations is discussed in Podlubny (1998), including the Mellin transform method, the power series method, and the symbolic method.

The symbolic method was first introduced in Babenko (1986) and generalises the Laplace transform method: it uses a specific expansion (e.g., binomial or geometric) on the differential operator and write it as an infinite sum of fractional derivatives. However, it is always necessary to check the validity of the formal expansion since the interchange of infinite summation and integration requires justification. It is nevertheless a powerful tool for determining the possible form of the solution. Numerous examples of the application of this method to heat and mass transfer problems are discussed by Babenko (1986).

In this section we define a new family of operators based on the binomial expansion. The important motivation underlying the following definition comes from realising that for positive integer n and $\alpha \in \mathbb{R}$,

$$\left(\frac{d}{dx} + \alpha\right)^n [f](x) = e^{-\alpha x} \frac{d^n}{dx^n} \left(e^{\alpha x} f(x)\right),$$

and similarly for $\left(-\frac{d}{dx} + \alpha\right)^n$. We thus introduce the Rock operators as the natural generalisation:

Definition 4.1.1. Let $r > 0$, $\alpha \in \mathbb{R}$, $a \in [-\infty, \infty)$ and $b \in (-\infty, \infty]$. The left Rock operator ${}^{\alpha}_{a}R^{r}_{x}$ is defined by

$${}^{\alpha}_{a}R^{r}_{x}\left[f\right](x) := e^{-\alpha x}\, {}_{a}D^{r}_{x}\left(e^{\alpha x}\, f(x)\right),$$

and the right Rock operator ${}^{\alpha}_{x}R^{r}_{b}$, $r > 0$, $\alpha \in \mathbb{R}$ by

$${}^{\alpha}_{x}R^{r}_{b}\left[g\right](x) := e^{\alpha x}\, {}^{C}_{x}D^{r}_{b}\left(e^{-\alpha x}\, g(x)\right).$$

The domain of definition of ${}^{\alpha}_{a}R^{r}_{x}$ and ${}^{\alpha}_{x}R^{r}_{b}$, $r > 0$, $\alpha \in \mathbb{R}$ are those of the left Riemann-Liouville fractional derivative ${}_{a}D^{r}_{x}$ and the right Caputo fractional derivative ${}^{C}_{x}D^{r}_{b}$ respectively, which are given in (2.4) and (2.8).

Remark 4.1.1. 1. The Rock operators have right inverses which we denote by ${}^{\alpha}_{a}R^{\blacklozenge r}_{x}$ and ${}^{\alpha}_{x}R^{\blacklozenge r}_{b}$. By Proposition 2.1.3 and 2.1.9 the right inverses are given by

$$\begin{aligned} & {}^{\alpha}_{a}R^{\blacklozenge r}_{x}[f](x) := e^{-\alpha x}\,{}_{a}I^{r}_{x}\left(e^{\alpha x} f(x)\right), \quad \forall\, r \in \mathbb{C}, \\ \text{and} \quad & {}^{\alpha}_{x}R^{\blacklozenge r}_{b}[g](x) := e^{\alpha x}\,{}_{x}I^{r}_{b}\left(e^{-\alpha x} g(x)\right), \quad \Re(r) \notin \mathbb{N} \vee r \in \mathbb{N}. \end{aligned}$$

2. The Rock operators generalise two families of differential operators

$$\left(\frac{d}{dx} + \alpha\right)^{n} \quad \text{and} \quad \left(-\frac{d}{dx} + \alpha\right)^{n}$$

when n could be any positive real numbers.

3. When $\alpha = 0$, the Rock operators can also be reduced to the fractional derivatives ${}_{a}D^{r}_{x}$ and ${}^{C}_{x}D^{r}_{b}$ respectively.

Furthermore, in the case $a = 0$, integration by parts yields the following characterisation of the formal adjoint of ${}^{\alpha}_{0}R^{r}_{x}$. Along with the integration by parts formula in Proposition 2.1.11, this is the key calculation needed for the proof of our main result.

Proposition 4.1.1. Let $\alpha \in \mathbb{R}$ and $r > 0$. The formal adjoint with respect to integration by parts of the left Rock operator ${}^{\alpha}_{0}R^{r}_{x}$ is the right Rock operator ${}^{\alpha}_{x}R^{r}_{\infty}$, namely,

$$\int_{0}^{\infty} {}^{\alpha}_{0}R^{r}_{x}[f](x)\, g(x)\, dx = \int_{0}^{\infty} f(x)\; {}^{\alpha}_{x}R^{r}_{\infty}[g](x)\, dx,$$

for appropriate functions f and g (see Definition 2.1.6).

4.1.2 Rock operators and some related distributions

The left Rock operator can be used to construct differential equations for density functions. Consider a gamma probability density function with shape parameter $r \in \mathbb{R}^{+}$ and rate parameter $\lambda \in \mathbb{R}^{+}$

$$f_r(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, \quad x > 0.$$

Recall that when r takes integer value, namely n, the gamma density function solves a homogeneous ordinary differential equation with constant coefficients

$$\left(\frac{d}{dx}+\lambda\right)^n f_r(x)=0, \quad x>0,$$

along with boundary conditions $f_r^{(n-1)}(0)=\lambda^n$ and $f_r^{(n-k)}(0)=0$ for $k=2,\dots,n$.

When r is no longer necessary integer, instead of an ordinary differential equation, the gamma density function solves a fractional differential equation

$$ {}_0^{\lambda}R_x^r[f_r](x)=e^{-\lambda x}\,{}_0D_x^r\left(e^{\lambda x}f_r(x)\right)=0, \quad x>0, \tag{4.4}$$

with boundary conditions ${}_0^{\lambda}R_x^{r-1}[f_r](0)=\lambda^r$ and ${}_0^{\lambda}R_x^{r-k}[f_r](0)=0$ for $k=2,\dots,\lceil r\rceil$. This can be proved by substituting the density function into the equation to have

$$e^{-\lambda x}\,{}_0D_x^r\left(e^{\lambda x}\frac{\lambda^r}{\Gamma(r)}x^{r-1}e^{-\lambda x}\right)=e^{-\lambda x}\frac{\lambda^r}{\Gamma(r)}\,{}_0D_x^r x^{r-1}.$$

It is known from Podlubny (1998) that the left Riemann-Liouville fractional derivative of a power function has an explicit form (see Proposition 2.1.4), which gives

$${}_0^{\lambda}R_x^r[f_r](x)=e^{-\lambda x}\frac{\lambda^r}{\Gamma(r-r)}x^{-1}.$$

Since the gamma function explodes at zero, one concludes that the above quantity equals to zero. Using similar calculation, for the boundary conditions we have

$${}_0^{\lambda}R_x^{r-k}[f_r](x)\big|_{x=0}=e^{-\lambda x}\frac{\lambda^r}{\Gamma(k)}x^{k-1}\bigg|_{x=0},$$

which equals to λ^r, when $k=1$ and 0, when $k>1$.

Another distribution related to the left Rock operator is the Mittag-Leffler distribution, which is the inter-arrival time distribution in the fractional Poisson process. The Mittag-Leffler probability density function with parameters $\mu\in(0,1]$ and $\lambda\in\mathbb{R}^+$ is

$$f_\mu(x)=\lambda x^{\mu-1}E_{\mu,\mu}(-\lambda x^\mu), \quad x>0,$$

and solves the following fractional differential equation

$$\left({}_0^0R_x^{\mu} + {}_0^{\lambda}R_x^0\right)[f_\mu](x) = ({}_0D_x^{\mu} + \lambda)[f_\mu](x) = 0, \quad x > 0, \tag{4.5}$$

with the boundary condition $\left({}_0^0R_x^{\mu-1} + {}_0^{\lambda}R_x^{-1}\right)[f_\mu](0) = {}_0D_x^{\mu-1}[f_\mu](0) = \lambda$. Equation (4.5) holds since that

$${}_0D_x^{\mu}\left(\lambda x^{\mu-1}E_{\mu,\mu}(-\lambda x^{\mu})\right) = -\lambda^2 x^{\mu-1}E_{\mu,\mu}\left(-\lambda x^{\mu}\right),$$

with the boundary condition proved by

$${}_0D_x^{\mu-1}f_\mu(0) = \frac{1}{\Gamma(1-\mu)}\int_0^x (x-y)^{\mu}\lambda y^{\mu-1}E_{\mu,\mu}(-\lambda y^{\mu})\,dy\bigg|_{x=0} = \lambda E_\mu(-\lambda x^{\mu})|_{x=0} = \lambda.$$

The next theorem considers a family of random variables to which the approach presented in this chapter applies to. In its full generality, we consider risk processes with waiting times that can be written as finite sums of independent heterogeneous gamma and Mittag-Leffler random variables. We now characterise the fractional boundary value problem satisfied by the density function of such waiting times.

Theorem 4.1.1. Consider a random variable T defined by

$$T = \sum_{i=1}^{m} Y_i + \sum_{j=1}^{n} Z_j, \tag{4.6}$$

in terms of gamma random variables $Y_i \sim \Gamma(r_i, \lambda_{1,i})$ and Mittag-Leffler random variables $Z_j \sim \mathrm{ML}(\mu_j, \lambda_{2,j})$, all independent of each other. Here r_i, $\lambda_{1,i}$, $\lambda_{2,j} \in \mathbb{R}^+$ and $\mu_j \in (0,1]$. Then the density function $f_T^{m,n}(t)$ of T solves the following fractional differential equation

$$\mathcal{A}_{m,n}\left(\frac{d}{dt}\right)[f_T^{m,n}](t) := \left(\bigodot_{j=1}^{n}\left({}_0D_t^{\mu_j} + \lambda_{2,j}\right)\bigodot_{i=1}^{m}{}_0^{\lambda_{1,i}}R_t^{r_i}\right)[f_T^{m,n}](t) = 0, \quad t > 0, \tag{4.7}$$

with the boundary conditions (when $n \neq 0$)

$$\left({}_0D_t^{\mu_1-1}\bigodot_{j=2}^{n}\left({}_0D_t^{\mu_j} + \lambda_{2,j}\right)\bigodot_{i=1}^{m}{}_0^{\lambda_{1,i}}R_t^{r_i}\right)[f_T^{m,n}](0) = \prod_{i=1}^{m}\lambda_{1,i}^{r_i}\prod_{j=1}^{n}\lambda_{2,j},$$

and

$$\left({}_0D_t^{\mu_1-k}\bigodot_{j=2}^{n}\left({}_0D_t^{\mu_j} + \lambda_{2,j}\right)\bigodot_{i=1}^{m}{}_0^{\lambda_{1,i}}R_t^{r_i}\right)[f_T^{m,n}](0) = 0,$$

for $k = 2, \ldots, \left\lceil \sum_{j=1}^{n}\mu_j + \sum_{i=1}^{m} r_i \right\rceil$.

Proof. In order to prove equation (4.7), we will use induction principle for two variables to validate equation (4.7) together with an extra statement: for any function g supported on $[0,\infty)$, $\mathcal{A}_{m,n}\left(\frac{d}{dt}\right)[f_T^{m,n} * g](t) = c_{m,n}\cdot g(t)$ where constant $c_{m,n} = \prod_{i=1}^{m}\lambda_{1,i}^{r_i}\prod_{j=1}^{n}\lambda_{2,j}$.

Base step: when $m=1, n=0$ or $m=0, n=1$, recall from equation (4.4) and (4.5) we have

$$\mathcal{A}_{1,0}\left(\frac{d}{dt}\right)\left[f_T^{1,0}\right](t) = 0 \quad \text{and} \quad \mathcal{A}_{0,1}\left(\frac{d}{dt}\right)\left[f_T^{0,1}\right](t) = 0.$$

Furthermore, by simple calculation we obtain

$$\begin{aligned}
&\mathcal{A}_{1,0}\left(\frac{d}{dt}\right)\left[f_T^{1,0} * g\right](x) = e^{-\lambda_{1,1}t}\cdot {}_0D_t^{r_1}\left(e^{\lambda_{1,1}t}\left[f_T^{1,0} * g\right]\right)(t)\\
=&\int_0^t e^{-\lambda_{1,1}y}\,{}_0D_y^{r_1}\left(e^{\lambda_{1,1}y}f_T^{1,0}(y)\right)\cdot g(t-y)\,dy + g(t)\cdot {}_0D_y^{r_1-1}\left(e^{\lambda_{1,1}y}f_T^{1,0}(y)\right)\Big|_{y=0}\\
=&\lambda_{1,1}^{r_1}\cdot g(t).
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{A}_{0,1}\left(\frac{d}{dt}\right)\left[f_T^{0,1} * g\right](t) = \left({}_0D_t^{\mu_1} + \lambda_{2,1}\right)\left[f_T^{0,1} * g\right](t)\\
=&\int_0^t {}_0D_y^{\mu_1}\left(f_T^{0,1}(y)\right)\cdot g(t-y)\,dy + g(t)\cdot {}_0D_y^{\mu_1-1}\left(f_T^{0,1}(y)\right)\Big|_{y=0}\\
&+\lambda_{2,1}\int_0^t f_T^{0,1}(y)\cdot g(t-y)\,dy\\
=&\lambda_{2,1}\cdot g(t)
\end{aligned}$$

The representations hold for $m=1$, $n=0$ and $m=0$, $n=1$.

Inductive step: for non-negative m and n, we assume that the statement

$$\mathcal{A}_{m,n}\left(\frac{d}{dt}\right)\left[f_T^{m,n}\right](t) = 0 \quad \text{and} \quad \mathcal{A}_{m,n}\left(\frac{d}{dt}\right)[f_T^{m,n} * g](t) = c_{m,n}\cdot g(t)$$

hold. Left to prove that they hold for $m+1$ and $n+1$,

$$\mathcal{A}_{m+1,n}\left(\frac{d}{dt}\right)\left[f_T^{m+1,n}\right](t) = 0 \quad \text{and} \quad \mathcal{A}_{m+1,n}\left(\frac{d}{dt}\right)\left[f_T^{m+1,n} * g\right](t) = c_{m+1,n}\cdot g(t)$$

and

$$\mathcal{A}_{m,n+1}\left(\frac{d}{dt}\right)\left[f_T^{m,n+1}\right](t)=0 \quad \text{and} \quad \mathcal{A}_{m,n+1}\left(\frac{d}{dt}\right)\left[f_T^{m,n+1}*g\right](t)=c_{m,n+1}\cdot g(t).$$

Beginning with the "left statements" in both cases,

$$\mathcal{A}_{m+1,n}\left(\frac{d}{dt}\right)\left[f_T^{m+1,n}\right](t)=e^{-\lambda_{1,m+1}t}\cdot {}_0D_t^{r_{m+1}}\left(e^{\lambda_{1,m+1}t}\mathcal{A}_{m,n}\left(\frac{d}{dt}\right)\left[f_T^{m,n}*f_T^{1,0}\right]\right)(t)$$
$$=e^{-\lambda_{1,m+1}t}\cdot {}_0D_t^{r_{m+1}}\left(e^{\lambda_{1,m+1}t}\cdot c_{m,n}\cdot f_T^{1,0}(t)\right)=0.$$

and

$$\mathcal{A}_{m,n+1}\left(\frac{d}{dt}\right)\left[f_T^{m,n+1}\right](t)=\left({}_0D_t^{\mu_{n+1}}+\lambda_{2,n+1}\right)\mathcal{A}_{m,n}\left(\frac{d}{dt}\right)\left[f_T^{m,n}*f_T^{0,1}\right](t)$$
$$=\left({}_0D_t^{\mu_{n+1}}+\lambda_{2,n+1}\right)\left(c_{m,n}\cdot f_T^{0,1}(t)\right)=0$$

For the "right statements", we have

$$\begin{aligned}&\mathcal{A}_{m+1,n}\left(\frac{d}{dt}\right)\left[f_T^{m+1,n}*g\right](t)\\ =&e^{-\lambda_{1,m+1}t}\cdot {}_0D_t^{r_{m+1}}\left(e^{\lambda_{1,m+1}t}\mathcal{A}_{m,n}\left(\frac{d}{dt}\right)\left[f_T^{m,n}*\left(f_T^{1,0}*g\right)\right]\right)(t)\\ =&e^{-\lambda_{1,m+1}t}\cdot {}_0D_t^{r_{m+1}}\left(e^{\lambda_{1,m+1}t}c_{m,n}\cdot f_T^{1,0}*g\right)(t)=c_{m+1,n}\cdot g(t),\end{aligned}$$

and

$$\mathcal{A}_{m,n+1}\left(\frac{d}{dt}\right)\left[f_T^{m,n+1}*g\right](t)=\left({}_0D_t^{\mu_{n+1}}+\lambda_{2,n+1}\right)\mathcal{A}_{m,n}\left(\frac{d}{dt}\right)\left[f_T^{m,n}*\left(f_T^{0,1}*g\right)\right](t)$$
$$=\left({}_0D_t^{\mu_{n+1}}+\lambda_{2,n+1}\right)\left[c_{m,n}\cdot f_T^{0,1}*g\right](t)=c_{m,n+1}\cdot g(t),$$

thereby showing $m+1$ and $n+1$ cases are true. It holds for any m and n. This completes the mathematical induction proof.

To validate the boundary conditions, we compute

$$\left({}_0D_t^{\mu_1-k}\bigodot_{j=2}^{n}\left({}_0D_t^{\mu_j}+\lambda_{2,j}\right)\bigodot_{i=1}^{m}{}^{\lambda_{1,i}}_{\ 0}R_t^{r_i}\right)\left[f_T^{m,n-1}*f_T^{0,1}\right](0)$$

$$=\prod_{i=1}^{m}\lambda_{1,i}^{r_i}\prod_{j=2}^{n}\lambda_{2,j}\cdot{}_0D_t^{\mu_1-k}\left[f_T^{0,1}\right](0)$$

$$=\prod_{i=1}^{m}\lambda_{1,i}^{r_i}\prod_{j=2}^{n}\lambda_{2,j}\cdot\lambda_{2,1}\,t^{k-1}E_{\mu_1,k}(-\lambda_{2,1}t_1^{\mu})\big|_{t=0}=\begin{cases}\prod_{i=1}^{m}\lambda_{1,i}^{r_i}\prod_{j=1}^{n}\lambda_{2,j}, & k=1\\ \\ 0, & k>1.\end{cases}$$

This completes the proof. □

Remark 4.1.2. One can show that the boundary conditions in (4.1.1) have various equivalent expressions. For any positive integer number

$$k\leqslant\left\lceil\sum_{i=1}^{m}r_i+\sum_{j=1}^{n}\mu_j\right\rceil,$$

by choosing non-negative integers $k_{1,i}$ and $k_{2,j}$ such that

$$\sum_{i=1}^{m}k_{1,i}+\sum_{j=1}^{n}k_{2,j}=k,$$

we have the boundary conditions of equation (4.7) as

$$\left(\bigodot_{j=1}^{n}\left({}_0D_t^{\mu_j-k_{2,j}}+\lambda_{2,j}\cdot{}_0I_t^{k_{2,j}}\right)\bigodot_{i=1}^{m}{}^{\lambda_{1,i}}_{\ 0}R_t^{r_i-k_{1,i}}\right)[f_T^{m,n}](0)=\begin{cases}\prod_{i=1}^{m}\lambda_{1,i}^{r_i}\prod_{j=1}^{n}\lambda_{2,j}, & k=1\\ \\ 0, & k>1.\end{cases}$$

Remark 4.1.3. One can show that every operator ${}^{\lambda_{1,i}}_{\ 0}R_t^{r_i}$ or $\left({}_0D_t^{\mu_j}+\lambda_{2,j}\right)$ commute with each other for all $i=1,\ldots,m$ and $j=1,\ldots,n$ on density function $f_T^{m,n}$.

Remark 4.1.4. Equation (4.7) along with its boundary conditions can be regarded as the generalisation of a pair of boundary problems discussed in Rosenkranz and Regensburger (2008b). When the fractional differential algebra is properly defined these fractional-order boundary problems can be factorised and further solved by obtaining their corresponding Green's operators.

4.2 Fractional integro-differential equations for (non-)ruin probabilities in risk models and dual risk models

The Rock operators give us the ability to study a very general family of distributions that may find applications in various areas, e.g, queuing theory, risk theory and control theory. Although many of the available techniques for the analysis of the associated equations are numerical or asymptotic, the fractional differential approach still can offer an analytic insight of the related problems. In this section, we aim at accomplishing that with particular problems in the theory of risk that involve the Rock operators. A special family of renewal risk models will be considered, among which the Erlang(n) and fractional Poisson risk models are included. We will show that the ruin probabilities in these models solve fractional integro-differential equations involving Rock operators.

4.2.1 Applications of the Rock operators in risk models

Before moving to the main result, we introduce a lemma that allows us to change the argument of the Rock operators on a bivariate function under certain circumstances.

Lemma 4.2.1. For positive real numbers α, r and c, the following identity holds

$$ {}^{\alpha}_{x}R^{r}_{\infty}[f(x+cy)](x,y) = c^{-r} \cdot {}^{\alpha c}_{y}R^{r}_{\infty}[f(x+cy)](x,y). \tag{4.8} $$

Proof. We start from the left-hand side of equation (4.8). By definition we have

$$ {}^{\alpha}_{x}R^{r}_{\infty}[f(x+cy)](x,y) = e^{\alpha x}\frac{1}{\Gamma(n-r)}\int_{x}^{\infty}(t-x)^{n-r-1}\frac{d^n}{dt^n}\left(e^{-\alpha t}f(t+cy)\right)\,dt. $$

Letting $s = \frac{1}{c}(t-x)+y$ leads to

$$ {}^{\alpha}_{x}R^{r}_{\infty}[f(x+cy)](x,y) = \frac{1}{\Gamma(n-r)}\int_{y}^{\infty}e^{\alpha cy}(s-y)^{n-r-1}c^{-r}\frac{d^n}{dy^n}\left(e^{-\alpha cs}f(cs+x)\right)\,ds, $$

which is the right-hand side of equation (4.8). □

Lemma 4.2.1 allows us to change the argument of the Rock operators to a bivariate function, as long as certain conditions fulfilled.

Theorem 4.2.1. Consider the renewal risk model

$$R_{m,n}(t) = u + ct - \sum_{i=1}^{N_{m,n}(t)} X_i, \quad t > 0, \tag{4.9}$$

where the inter-arrival times T_k are assumed to be sum of independent gamma random variables $Y_i \sim \Gamma(r_i, \lambda_{1,i})$ and Mittag-Leffler random variables $Z_j \sim \mathrm{ML}(\mu_j, \lambda_{2,j})$ as in (4.6). Then the ruin probability $\psi(u)$ under model $R_{m,n}$, satisfies the following fractional integro-differential equation of the form

$$\begin{aligned}&\left(\bigodot_{j=1}^{n}\left(c^{\mu_j}\cdot {}_{u}^{C}D_{\infty}^{\mu_j}+\lambda_{2,j}\right)\bigodot_{i=1}^{m}\left(c^{r_i}\cdot {}^{\lambda_{1,i}/c}_{\quad u}R_{\infty}^{r_i}\right)\right)[\psi](u)\\&=\prod_{i=1}^{m}\lambda_{1,i}^{r_i}\prod_{j=1}^{n}\lambda_{2,j}\left(\int_0^u\psi(u-y)\,dF_X(y)+\int_u^{\infty}dF_X(y)\right),\quad u\geqslant 0,\end{aligned} \tag{4.10}$$

with the universal boundary condition $\lim_{u\to\infty}\psi(u)=0$.

Proof. For a general renewal risk model, the ruin probability solves the renewal equation (Feller, 2008)

$$\psi(u)=\int_0^{\infty}f_T^{m,n}(t)\left(\int_0^{u+ct}\psi(u+ct-y)\,dF_X(y)+\int_{u+ct}^{\infty}dF_X(y)\right)dt,\quad u\geqslant 0. \tag{4.11}$$

Denoting the terms in parentheses of (4.11) as

$$h(u+ct)=\int_0^{u+ct}\psi(u+ct-y)\,dF_X(y)+\int_{u+ct}^{\infty}dF_X(y),$$

we now apply

$$\bigodot_{j=1}^{n}\left(c^{\mu_j}\cdot {}_{u}^{C}D_{\infty}^{\mu_j}+\lambda_{2,j}\right)\bigodot_{i=1}^{m}\left(c^{r_i}\cdot {}^{\lambda_{1,i}/c}_{\quad u}R_{\infty}^{r_i}\right)$$

on both sides of (4.11) and use property 4.8 to obtain

$$\begin{aligned}&\left(\bigodot_{j=1}^{n}\left(c^{\mu_j}\cdot {}_{u}^{C}D_{\infty}^{\mu_j}+\lambda_{2,j}\right)\bigodot_{i=1}^{m}\left(c^{r_i}\cdot {}^{\lambda_{1,i}/c}_{\quad u}R_{\infty}^{r_i}\right)\right)[\psi](u)\\&=\int_0^{\infty}f_T^{m,n}(t)\left(\bigodot_{j=1}^{n}\left({}_{t}^{C}D_{\infty}^{\mu_j}+\lambda_{2,j}\right)\bigodot_{i=1}^{m}{}^{\lambda_{1,i}}_{\ \ t}R_{\infty}^{r_i}\right)[h(u+ct)](u,t)\,dt.\end{aligned}$$

The fractional integration by parts rule 2.10 is applicable here. We move the operator $\left({}^C_t D^{\mu_1}_\infty + \lambda_{2,1}\right)$ from function h to $f^{m,n}_T$, to reduce to

$$\begin{aligned}
&\int_0^\infty f^{m,n}_T(t)\left(\bigodot_{j=1}^{n}\left({}^C_t D^{\mu_j}_\infty + \lambda_{2,j}\right)\bigodot_{i=1}^{m} {}^{\lambda_{1,i}}{}_t R^{r_i}_\infty\right)[h(u+ct)](u,t)\,dt \\
&= \int_0^\infty \left({}_0 D^{\mu_1}_t + \lambda_{2,1}\right)[f^{m,n}_T](t)\cdot\left(\bigodot_{j=2}^{n}\left({}^C_t D^{\mu_j}_\infty + \lambda_{2,j}\right)\bigodot_{i=1}^{m} {}^{\lambda_{1,i}}{}_t R^{r_i}_\infty\right)[h(u+ct)](u,t)\,dt \\
&+ \sum_{k=0}^{\lfloor\mu_1\rfloor}\Bigg[(-1)^{\lfloor\mu_1\rfloor+1+k}\cdot {}_0 D^{\mu_1+k-\lfloor\mu_1\rfloor-1}_t[f^{m,n}_T](t) \\
&\quad\cdot\left(\bigodot_{j=2}^{n}\left({}^C_t D^{\mu_j}_\infty + \lambda_{2,j}\right)\bigodot_{i=1}^{m} {}^{\lambda_{1,i}}{}_t R^{r_i}_\infty\right)[h(u+ct)](u,t)\Bigg]_0^\infty
\end{aligned}$$

The boundary condition term evaluated at $t = 0$ could be computed using the initial value theorem of Laplace transforms,

$$\begin{aligned}
&{}_0 I^{1-\mu_1}_t[f^{m,n}_T](0) \\
&= \lim_{s\to\infty} s\int_0^\infty e^{-st}\,{}_0 I^{1-\mu_1}_t[f^{m,n}_T](t)\,dt \\
&= \lim_{s\to\infty}\left(s^{\mu_1}\cdot\prod_{j=1}^{n}\frac{\lambda_{2,j}}{s^{\mu_j}+\lambda_{2,j}}\cdot\prod_{i=1}^{m}\left(\frac{\lambda_{1,i}}{s+\lambda_{1,i}}\right)^{r_i}\right) = 0.
\end{aligned}$$

Another boundary condition term evaluated at $t = \infty$ also equals zero due to the fact that the definition of the right Caputo fractional derivative is an integral from t to ∞. Analogously, we are able to move the first n operators $\bigodot_{j=1}^{n}\left({}^C_t D^{\mu_j}_\infty + \lambda_{2,j}\right)$ from function h to $f^{m,n}_T$ with all boundary conditions vanishing, which leads to

$$\int_0^\infty \bigodot_{j=1}^{n}\left({}_0 D^{\mu_j}_t + \lambda_{2,j}\right)[f^{m,n}_T](t)\cdot\bigodot_{i=1}^{m} {}^{\lambda_{1,i}}{}_t R^{r_i}_\infty[h(u+ct)](u,t)\,dt.$$

Now we use the integration by parts formula in Proposition 4.1.1 to take the first Rock operator ${}^{\lambda_{1,1}}{}_t R^{r_1}_\infty$ off of h. Furthermore it can be shown that its adjoint ${}^{\lambda_{1,1}}{}_0 R^{r_1}_t$ commutes with $\left({}_0D_t^{\mu_j} + \lambda_{2,j}\right)$ for all $j = 1, \dots, n$ on density function $f_T^{m,n}$. We therefore get:

$$\int_0^\infty \left(\bigodot_{j=1}^{n} \left({}_0D_t^{\mu_j} + \lambda_{2,j}\right) {}^{\lambda_{1,1}}_{\ 0}R_t^{r_1} \right) [f_T^{m,,n}](t) \cdot \bigodot_{i=2}^{m} {}^{\lambda_{1,i}}_{\ t}R_\infty^{r_i}[h(u+ct)](u,t)\, dt$$
$$+ \sum_{k=0}^{\lfloor r_1 \rfloor} \Bigg[(-1)^{\lfloor r_1\rfloor+1+k} \cdot \bigodot_{i=2}^{m} {}^{\lambda_{1,i}}_{\ t}R_\infty^{r_i}[h(u+ct)](u,t)$$
$$\cdot \left(\bigodot_{j=1}^{n} \left({}_0D_t^{\mu_j} + \lambda_{2,j}\right) {}^{\lambda_{1,i}}_{\ 0}R_t^{r_1+k-\lfloor r_1\rfloor-1} \right) [f_T^{m,n}](t) \Bigg]_0^\infty .$$

The boundary condition at $t = 0$ can be computed by applying the initial value theorem

$$\left(\bigodot_{j=1}^{n} \left({}_0D_t^{\mu_j} + \lambda_{2,j}\right) {}^{\lambda_{1,1}}_{\ 0}R_t^{r_1+k-\lfloor r_1\rfloor-1} \right) [f_T^{m,n}](0)$$
$$= \prod_{j=1}^{n} \lambda_{2,j} \cdot \left(e^{-\lambda_{1,1}t}\, {}_0D_t^{r_1+k-\lfloor r_1\rfloor-1} \left(e^{\lambda_{1,1}t} f_T^{m,0}(t) \right) \right)\Big|_{t=0}$$
$$= \prod_{j=1}^{n} \lambda_{2,j} \cdot \lim_{s\to\infty} \left(s \int_0^\infty e^{-(s+\lambda_{1,1})t}\, {}_0D_t^{r_1+k-\lfloor r_1\rfloor-1} \left(e^{\lambda_{1,1}t} f_T^{m,0}(t) \right) dt \right)$$
$$= \prod_{j=1}^{n} \lambda_{2,j} \cdot \lim_{s\to\infty} \Bigg(\frac{\lambda_{1,1}^{r_1} \cdot s}{(s+\lambda_{1,1})^{\lfloor r_1\rfloor+1-k}} \prod_{i=2}^{m} \left(\frac{\lambda_{1,i}}{s+\lambda_{1,i}} \right)^{r_i}$$
$$- s \sum_{l=0}^{k-1} (s+\lambda_{1,1})^l \left[{}_0D_t^{r_1+k-\lfloor r_1\rfloor-l-2} \left(e^{\lambda_{1,1}} f_T^{m,0}(t) \right) \right]\Big|_{t=0} \Bigg) .$$

We continue to iteratively use the initial value theorem on the terms

$$s(s+\lambda_{1,1})^l \left[{}_0D_t^{r_1+k-\lfloor r_1\rfloor-l-2} \left(e^{\lambda_{1,1}t} f_T^{m,0}(t) \right) \right] \Big|_{t=0}$$

until it eventually gives us

$$s(s+\lambda_{1,1})^{\lfloor r_1\rfloor-1} \left[{}_0I_t^{\lfloor r_1\rfloor+1-r_1} \left(e^{\lambda_{1,1}t} f_T^{m,0}(t) \right) \right] \Big|_{t=0} = s(s+\lambda_{1,1})^{r_1-2} \prod_{i=1}^{m} \left(\frac{\lambda_{1,i}}{s} \right)^{r_i} ,$$

which tends to zero when $s \to \infty$. The boundary condition term evaluated at $t = \infty$ gives zero since the right Caputo derivatives vanish at infinity. Analogously, we are able to move the rest operators $\bigodot_{i=1}^{m} {}^{\lambda_{1,i}}_{\ t}R_\infty^{r_i}$ from function h to $f_T^{m,n}$ with all boundary conditions vanishing, which leads to

$$\int_0^\infty \left(\bigodot_{j=1}^{n} \left({}_0D_t^{\mu_j} + \lambda_{2,j} \right) \bigodot_{i=1}^{m} {}^{\lambda_{1,i}}_{0}R_t^{r_i} \right) [f_T^{m,,n}](t) \cdot [h(u+ct)](u,t)\, dt$$
$$+ \left[[h(u+ct)](u,t) \cdot \left(\bigodot_{j=1}^{n} \left({}_0D_t^{\mu_j} + \lambda_{2,j} \right) {}^{\lambda_{1,n}}_{0}R_t^{r_n-1} \bigodot_{i=1}^{m-1} {}^{\lambda_{1,i}}_{0}R_t^{r_i} \right) [f_T^{m,n}](t) \right]_0 .$$

Since the time density satisfies equation (4.7), the integral term of the above equation vanishes. The boundary conditions of $f_T^{m,n}$ ensure that the lower summand is, at $t = 0$,

$$\left(\bigodot_{j=1}^{n} \left({}_0D_t^{\mu_j} + \lambda_{2,j} \right) {}^{\lambda_{1,n}}_{0}R_t^{r_n-1} \bigodot_{i=1}^{m-1} {}^{\lambda_{1,i}}_{0}R_t^{r_i} \right) [f_T^{m,n}](0) = \prod_{i=1}^{m} \lambda_{1,i}^{r_i} \prod_{j=1}^{n} \lambda_{2,j}.$$

This completes the proof. □

Corollary 4.2.1. The non-ruin probability $\phi(u) = 1 - \psi(u)$ for the risk model in Theorem 4.2.1 satisfies the following fractional integro-differential equation

$$\left(\bigodot_{j=1}^{n} \left(c^{\mu_j} \cdot {}_u^CD_\infty^{\mu_j} + \lambda_{2,j} \right) \bigodot_{i=1}^{m} \left(c^{r_i} \cdot {}^{\lambda_{1,i}/c}_{u}R_\infty^{r_i} \right) \right) [\phi](u)$$
$$= \prod_{i=1}^{m} \lambda_{1,i}^{r_i} \prod_{j=1}^{n} \lambda_{2,j} \left(\int_0^u \phi(u-y)\, dF_X(y) \right), \quad u \geqslant 0, \tag{4.12}$$

with the universal boundary condition $\lim_{u\to\infty} \phi(u) = 1$.

Theorem 4.2.1 characterises a fractional integro-diferential equation satisfied by the ruin probability ψ for a large class of waiting times distributions. Whether or not one can solve for it depends on the particular form of the claim size distribution function F_X. We now restrict the rest of the analysis to claim sizes X_i distributed as a sum of an arbitrary number of independent Gamma random variables. The next theorem shows that, in this case, the whole equation (4.10) can be written as a boundary value problem with only fractional derivatives. It is important to note that if the claim sizes included any Mittag-Leffler components, as it is the case of T in Theorem 4.2.1, we would have $\mathbb{E}(X_i) = \infty$ and ruin would happen with probability one.

Theorem 4.2.2. Consider the same renewal risk model (4.9) in Theorem 4.2.1. Assume further that the claim sizes X_i are each distributed as a sum of l independent $\Gamma(s_k, \alpha_k)$ distributed random variables for some $s_k, \alpha_k > 0$, $k = 1, \dots, l$ i.e.,

$$\bigodot_{k=1}^{l} {}^{\alpha_k}_{0}R^{s_k}_{y} \left[f_X\right](y) = 0, \quad y > 0,$$

with the boundary conditions

$$\left({}^{\alpha_1}_{0}R^{s_1-1}_{y} \bigodot_{k=2}^{l} {}^{\alpha_k}_{0}R^{s_k}_{y} \right) \left[f_X\right](0) = \prod_{k=1}^{l} \alpha_k,$$

and

$$\left({}^{\alpha_1}_{0}R^{s_1-k'}_{y} \bigodot_{k=2}^{l} {}^{\alpha_k}_{0}R^{s_k}_{y} \right) \left[f_X\right](0) = 0,$$

for $k' = 2, \dots, \left\lceil \sum_{k=1}^{l} s_k \right\rceil$. The non-ruin probability $\phi(u)$ satisfies a fractional differential equation

$$\begin{aligned} &\left(\bigodot_{k=1}^{l} {}^{\alpha_k}_{0}R^{s_k}_{u} \bigodot_{j=1}^{n} \left(c^{\mu_j} \cdot {}^{C}_{u}D^{\mu_j}_{\infty} + \lambda_{2,j}\right) \bigodot_{i=1}^{m} \left(c^{r_i} \cdot {}^{\lambda_{1,i}/c}_{u}R^{r_i}_{\infty}\right) \right) [\phi](u) \\ &= \prod_{i=1}^{m} \lambda_{1,i}^{r_i} \prod_{j=1}^{n} \lambda_{2,j} \prod_{k=1}^{l} \alpha_k^{s_k} \cdot \phi(u), \quad u \geqslant 0, \end{aligned} \tag{4.13}$$

with the universal boundary condition $\lim_{u\to\infty} \phi(u) = 1$ and initial-value boundary conditions

$$\left({}^{\alpha_1}_{0}R^{s_1-k'}_{u} \bigodot_{k=2}^{l} {}^{\alpha_k}_{0}R^{s_k}_{u} \bigodot_{j=1}^{n} \left(c^{\mu_j} \cdot {}^{C}_{u}D^{\mu_j}_{\infty} + \lambda_{2,j}\right) \bigodot_{i=1}^{m} \left(c^{r_i} \cdot {}^{\lambda_{1,i}/c}_{u}R^{r_i}_{\infty}\right) \right) [\phi](0) = 0, \tag{4.14}$$

for $k' = 1, \dots, \left\lceil \sum_{k=1}^{l} s_k \right\rceil - 1$.

Proof. Taking operators $\bigodot_{k=1}^{l} {}^{\alpha_k}_{0}R^{s_k}_{u}$ on two sides of (4.12) leads to

$$\begin{aligned} &\left(\bigodot_{k=1}^{l} {}^{\alpha_k}_{0}R^{s_k}_{u} \bigodot_{j=1}^{n} \left(c^{\mu_j} \cdot {}^{C}_{u}D^{\mu_j}_{\infty} + \lambda_{2,j}\right) \bigodot_{i=1}^{m} \left(c^{r_i} \cdot {}^{\lambda_{1,i}/c}_{u}R^{r_i}_{\infty}\right) \right) [\phi](u) \\ &= \prod_{i=1}^{m} \lambda_{1,i}^{r_i} \prod_{j=1}^{n} \lambda_{2,j} \cdot \bigodot_{k=1}^{l} {}^{\alpha_k}_{0}R^{s_k}_{u} \left(\int_0^u \phi(u-y) f_X(y)\, dy \right) \end{aligned}$$

Recall from Theorem 4.1.1, we know that the non-ruin probability function $\phi(u)$ is supported on $[0,\infty)$, so the identity

$$\bigodot_{k=1}^{l} {}^{\alpha_k}_{\ 0}R^{s_k}_u[\phi * f_X](u) = \prod_{k=1}^{l} \alpha_k^{s_k} \cdot \phi(u)$$

holds in this case, which gives equation (4.13).

For the boundary conditions, we compute

$$\begin{aligned}
&\left({}^{\alpha_1}_{\ 0}R^{s_1-k'}_u \bigodot_{k=2}^{l} {}^{\alpha_k}_{\ 0}R^{s_k}_u \bigodot_{j=1}^{n} \left(c^{\mu_j} \cdot {}^{C}_{u}D^{\mu_j}_{\infty} + \lambda_{2,j} \right) \bigodot_{i=1}^{m} \left(c^{r_i} \cdot {}^{\lambda_{1,i}/c}_{\quad u}R^{r_i}_{\infty} \right) \right) [\phi](0) \\
=&\prod_{i=1}^{m} \lambda_{1,i}^{r_i} \prod_{j=1}^{n} \lambda_{2,j} \cdot {}^{\alpha_1}_{\ 0}R^{s_1-k'}_u \bigodot_{k=2}^{l} {}^{\alpha_k}_{\ 0}R^{s_k}_u \left(\int_0^u \phi(u-y) f_X(y)\, dy \right) \Bigg|_{u=0} \\
=&\prod_{i=1}^{m} \lambda_{1,i}^{r_i} \prod_{j=1}^{n} \lambda_{2,j} \cdot {}^{\alpha_1}_{\ 0}R^{s_1-k'}_u \bigodot_{k=2}^{l} {}^{\alpha_k}_{\ 0}R^{s_k}_u \left(\phi(u) * f_{l-1}(u) * f_1(u) \ \right)|_{u=0} \\
=&\prod_{i=1}^{m} \lambda_{1,i}^{r_i} \prod_{j=1}^{n} \lambda_{2,j} \prod_{k=2}^{l} \alpha_k^{s_k} \cdot {}^{\alpha_1}_{\ 0}R^{s_1-k'}_u \left(\phi(u) * f_1(u) \ \right)|_{u=0}
\end{aligned}$$

where f_{l-1} stands for the density function of sum of $\Gamma(s_k, \alpha_k)$, $k = 2, \ldots, l$ and f_1 stands for the density function of $\Gamma(s_1, \alpha_1)$. Applying equation (2.1.7) gives

$$\begin{aligned}
&\prod_{i=1}^{m} \lambda_{1,i}^{r_i} \prod_{j=1}^{n} \lambda_{2,j} \prod_{k=2}^{l} \alpha_k^{s_k} \cdot e^{-\alpha_1 u}\, {}_0D^{s_1-k'}_u \left[\int_0^u e^{\alpha_1(u-y)} \phi(u-y) \cdot e^{\alpha_1 y} f_1(y)\, dy \right] \Bigg|_{u=0} \\
=&\prod_{i=1}^{m} \lambda_{1,i}^{r_i} \prod_{j=1}^{n} \lambda_{2,j} \prod_{k=2}^{l} \alpha_k^{s_k} \cdot e^{-\alpha_1 u} \left[e^{\alpha_1 u}\phi(u) * {}_0D^{s_1-k'}_u (e^{\alpha_1 u} f_1(u)) \right] \Bigg|_{u=0} \\
&+ \prod_{i=1}^{m} \lambda_{1,i}^{r_i} \prod_{j=1}^{n} \lambda_{2,j} \prod_{k=2}^{l} \alpha_k^{s_k} \cdot \phi(0)\, {}_0D^{s_1-k'-1}_y \left(e^{\alpha_1 y} f_1(y) \right) \Big|_{y=0} \\
=&\prod_{i=1}^{m} \lambda_{1,i}^{r_i} \prod_{j=1}^{n} \lambda_{2,j} \prod_{k=2}^{l} \alpha_k^{s_k} \left[e^{-\alpha_1 u} \left[e^{\alpha_1 u}\phi(u) * \frac{\alpha_1^{s_1}}{\Gamma(k')} u^{k'-1} \right] \Bigg|_{u=0} + \phi(0) \frac{\alpha_1^{s_1}}{\Gamma(k'+1)} y^{k'} \Bigg|_{y=0} \right] \\
=&0, \quad \text{for} \quad k' = 1, \ldots, \left\lceil \sum_{k=1}^{l} s_k \right\rceil - 1.
\end{aligned}$$

This completes the proof. □

Our next goal is solving the fractional differential boundary value problem in Theorem 4.2.2 via a characteristic equation from the ansatz $\phi(u) = e^{-zu}$. The main technical difficulty in the full generality of Theorem 4.2.2 arises from the fact that the operators in equation (4.13) combine two different types of differential derivatives, right Caputo fractional derivatives and left Rock operators which are ultimately defined in terms of left Riemann-Liouville fractional derivatives. The proposed ansatz is an eigenfunction only for the operators in

$$\bigodot_{j=1}^{n} \left(c^{\mu_j} \cdot {}_u^C D_\infty^{\mu_j} + \lambda_{2,j}\right) \bigodot_{i=1}^{m} \left(c^{r_i} \cdot {}^{\lambda_{1,i}/c}{}_u R_\infty^{r_i}\right)$$

so we will restrict to the case of $s_k \in \mathbb{N}$, $k = 1, \dots, l$ which simplifies things greatly since

$$\bigodot_{k=1}^{l} {}^{\alpha_k}{}_0 R_u^{s_k} = \bigodot_{k=1}^{l} \left(\frac{d}{du} + \alpha_k\right)^{s_k}$$

reduces to a combination of ordinary derivatives.

Note that assuming $s_k \in \mathbb{N}$, $k = 1, \dots, l$ is equivalent to assuming that the claim sizes X_i are each distributed as a sum of l independent Erlang random variables. Moreover, under this case, the operator

$$\bigodot_{k=1}^{l} {}^{\alpha_k}{}_0 R_u^{s_k} \bigodot_{j=1}^{n} \left(c^{\mu_j} \cdot {}_u^C D_\infty^{\mu_j} + \lambda_{2,j}\right) \bigodot_{i=1}^{m} \left(c^{r_i} \cdot {}^{\lambda_{1,i}/c}{}_u R_\infty^{r_i}\right)$$

on the left hand side of (4.13) is a composition of right Caputo fractional derivatives. Furthermore, with the ansatz $\phi(u) = e^{-zu}$, equation (4.13) yields the following characteristic equation for z:

$$\prod_{k=1}^{l} (-z + \alpha_k)^{s_k} \cdot \prod_{j=1}^{n} (c^{\mu_j} z^{\mu_j} + \lambda_{2,j}) \cdot \prod_{i=1}^{m} (cz + \lambda_{1,i})^{r_i} = \Lambda_{m,n} \cdot \prod_{k=1}^{l} \alpha_k^{s_k}. \tag{4.15}$$

Note that $z = 0$ is always a root of (4.15). If equation (4.15) has $N > 0$ additional distinct complex roots with positive real part, say $z_1, \dots, z_N$, then the non-ruin probability ϕ that solves (4.13) is

$$\phi(u) = 1 + \sum_{p=1}^{N} K_p e^{-z_p u} \tag{4.16}$$

The constants K_p, $p=1,\dots,N$ are to be determined from the boundary conditions (4.14) which we now characterise.

Proposition 4.2.1. Suppose $s_k \in \mathbb{N}$, $k=1,\dots,l$, in Theorem 4.2.2. The number of initial-value boundary conditions of $\phi(u)$ is $N=\sum_{k=1}^{l} s_k$ and they are given explicitly by:

$$\bigodot_{k=1}^{l} {}^{\alpha_k}_{\ 0}R_u^{s_{p,k}} \bigodot_{j=1}^{n} \left(c^{\mu_j}\cdot {}^{C}_{u}D_{\infty}^{\mu_j}+\lambda_{2,j}\right) \bigodot_{i=1}^{m} \left(c^{r_i}\cdot {}^{\lambda_{1,i}/c}_{\quad u}R_{\infty}^{r_i}\right)[\phi](0)=0,\ p=1,\dots,N \quad (4.17)$$

where the values of $s_{p,k}$ are to be computed as follows: let

$$L(p)=\inf\left\{\ell\in\mathbb{N}:\sum_{k=1}^{\ell} s_k \leqslant p\right\},\quad p=1,\dots,N \quad (4.18)$$

and define

$$s_{p,k}=\begin{cases} s_k, & \text{if } k<L(p),\\ p-\sum_{i=1}^{L(p)-1} s_i-1, & \text{if } k=L(p),\\ \vdots & \\ 0, & \text{if } k>L(p). \end{cases}$$

Proof. We consider the p-th boundary condition

$$\begin{aligned} &\bigodot_{k=1}^{l} {}^{\alpha_k}_{\ 0}R_u^{s_{p,k}} \mathcal{A}^*_{m,n}\left(c\frac{d}{du}\right)[\phi](0)\\ &=\prod_{i=1}^{m}\lambda_{1,i}^{r_i}\prod_{j=1}^{n}\lambda_{2,j}\prod_{k=1}^{L(p)-1}\alpha_k^{s_k}\ {}^{\alpha_{L(p)}}_{\quad 0}R_u^{s_{p,L(p)}}\left[\phi * f_{L(p)} * f_{L(p)+}\right](0), \end{aligned}$$

where $f_{L(p)}$ stands for the density function of a $\Gamma\left(s_{L(p)},\alpha_{L(p)}\right)$ random variable and $f_{L(p)+}$ for the density function of a sum of random varibales with distributions $\Gamma\left(s_k,\alpha_k\right)$, $k=L(p)+1,\dots,L$. Let $\Phi=\phi * f_{L(p)+}$ and apply (2.1.7) to compute

$$\begin{aligned} {}^{\alpha_{L(p)}}_{\quad 0}R_u^{s_{p,L(p)}}\left[\Phi * f_{L(p)}\right](u) &= \Phi(u)\ {}_0D_y^{s_{p,L(p)}-1}\left(e^{\alpha_{L(p)}y}f_{L(p)}(y)\right)\Big|_{y=0}\\ &\quad + e^{-\alpha_{L(p)}u}\left[e^{\alpha_{L(p)}(u)}\Phi(u) * {}_0D_u^{s_{p,L(p)}}e^{\alpha_{L(p)}u}f_{L(p)}(u)\right]. \end{aligned}$$

Note that $s_{p,L(p)-1} < s_{L(p)}$ and we have

$$ {}^{\alpha_{L(p)}}_{\quad 0}R_u^{s_{p,L(p)}} \left[\Phi * f_{L(p)}\right](0) = \int_0^u \Phi(u-y)\, {}^{\alpha_{L(p)}}_{\quad 0}R_y^{s_{p,L(p)}} f_{L(p)}(y)\, dy \Big|_{u=0} = 0. $$

Since this holds for all $1 \leqslant p \leqslant N$, we complete the proof. □

Substituting the expression (4.16) for $\phi(u)$ into the boundary conditions (4.17) yields explicit linear equations for the unknown constants K_p, $p = 1, \ldots, N$. First, denote

$$ \Delta_p := \prod_{j=1}^{n} (c^{\mu_j} z_p^{\mu_j} + \lambda_{2,j}) \prod_{i=1}^{m} (cz_p + \lambda_{1,i})^{r_i}, \quad p = 1, \ldots, N. \tag{4.19} $$

Then, the constants K_p, $p = 1, \ldots, N$ in (4.16) satisfy

$$ \begin{cases} \Lambda_{m,n} + \sum_{p=1}^{N} \Delta_p K_p = 0 \\ \alpha_1 \Lambda_{m,n} + \Delta \sum_{p=1}^{N} \left(-z_p + \alpha_1\right) K_p = 0 \\ \quad \cdots \\ \alpha_1^{s_1} \Lambda_{m,n} + \sum_{p=1}^{N} \Delta_p \left(-z_p + \alpha_1\right)^{s_1} K_p = 0 \\ \alpha_1^{s_1} \alpha_2 \Lambda_{m,n} + \sum_{p=1}^{N} \Delta_p \left(-z_p + \alpha_1\right)^{s_1} \left(-z_p + \alpha_2\right) K_p = 0 \\ \quad \cdots \\ \alpha_1^{s_1} \alpha_2^{s_2} \Lambda_{m,n} + \sum_{p=1}^{N} \Delta_p \left(-z_p + \alpha_1\right)^{s_1} \left(-z_p + \alpha_2\right)^{s_2} K_p = 0 \\ \quad \cdots \\ \prod_{k=1}^{l-1} \alpha_k^{s_k} \alpha_l^{s_l - 1} \Lambda_{m,n} + \sum_{p=1}^{N} \Delta_p \prod_{k=1}^{l-1} \left(-z_p + \alpha_k\right)^{s_k} \left(-z_p + \alpha_l\right)^{s_l - 1} K_p = 0, \end{cases} \tag{4.20} $$

where

$$ \Lambda_{m,n} = \prod_{i=1}^{m} \lambda_{1,i}^{r_i} \prod_{j=1}^{n} \lambda_{2,j}. $$

4.2.2 Applications of the Rock operators in dual risk models

The dual risk model describes the surplus $D(t)$ of a company as in (1.2). Due to the renewal property, the non-ruin probability ϕ_D of a dual risk model satisfies an integral equation (Afonso et al., 2013), for $u \geqslant 0$,

$$\phi_D(u) = \int_0^{u/c} f_T(t) \int_0^{\infty} \phi_D(u - ct + y)\, dF_Y(y)\, dt = \frac{1}{c} \int_0^u f_T\left(\frac{u-s}{c}\right) W(s)\, ds, \quad (4.21)$$

where

$$W(s) = \int_0^{\infty} \phi_D(s + y)\, dF_Y(y).$$

In this section, we consider a family of dual risk models, whose inter-arrival times are assumed to be the sum of gamma random variables.

Theorem 4.2.3. Consider the dual risk model

$$D_m(t) = u - ct + \sum_{i=1}^{N(t)} Y_i, \quad t > 0,$$

where the inter-arrival times T_k are assumed to be sum of m independent gamma random variables $Z_i \sim \Gamma(r_i, \lambda_i)$, $i = 1, \ldots, m$. The non-ruin probability, $\phi_D(u)$, satisfies, in this case, a fractional integro-differential equation of the form

$$\left(\bigodot_{i=1}^{m} \left(c^{r_i} \cdot {}^{\lambda_i/c}_{\;\;0}R_u^{r_i} \right) \right) [\phi_D](u) = \prod_{i=1}^{m} \lambda_i^{r_i} \int_0^{\infty} \phi_D(u + y)\, dF_Y(y), \quad u \geqslant 0, \quad (4.22)$$

with boundary conditions $\phi_D(0) = 0$ and other boundary conditions. When all $r_i \in \mathbb{N}$, we have explicitly $M = \sum_{i=1}^{m} r_k$ initial conditions, namely,

$$\left(\bigodot_{i=1}^{m} \left(c^{r_i} \cdot {}^{\lambda_i/c}_{\;\;0}R_u^{r_{q,i}} \right) \right) [\phi_D](0) = 0.$$

Like in renewal risk models, here $r_{q,i} \in \{0, 1, \ldots, r_i\}$ take different values, for different boundary conditions. For instance, for the q-th (out of M in total) boundary condition, we denote

$$M(q) = \inf \left\{ \ell \in \mathbb{N} : \sum_{i=1}^{\ell} r_i \leqslant q \right\},$$

then the values for $r_{q,i}$ in this case equal to

$$r_{q,i} = \begin{cases} r_i, & i < M(q), \\ q - \sum_{j=1}^{M(q)-1} r_j - 1, & i = M(q), \\ \vdots & \\ 0, & i > M(q). \end{cases}$$

Proof. Taking operators $\bigodot_{i=1}^{m} \left(c^{r_i} \cdot {}^{\lambda_i/c}_{\ \ 0}R^{r_i}_u \right)$ on both sides of (4.21) leads to

$$\left(\bigodot_{i=1}^{m} \left(c^{r_i} \cdot {}^{\lambda_i/c}_{\ \ 0}R^{r_i}_u \right) \right) [\phi_D](u) = \frac{1}{c} \bigodot_{i=1}^{m} \left(c^{r_i} \cdot {}^{\lambda_i/c}_{\ \ 0}R^{r_i}_u \right) \int_0^u f_T \left(\frac{u-s}{c} \right) W(s)\, ds$$

$$= \bigodot_{i=1}^{m} {}^{\lambda_i}_{0}R^{r_i}_v \int_0^v f_T\,(v-x)\, W(cx)\, dx = \bigodot_{i=1}^{m} {}^{\lambda_i}_{0}R^{r_i}_v \int_0^v f_T\,(v-x)\, V(x)\, dx,$$

where $v = \frac{u}{c}$ and $V(x) = W(cx)$. Recall from Theorem 4.1.1 that the function V is supported on $[0, \infty)$, hence we obtain

$$\bigodot_{i=1}^{m} {}^{\lambda_i}_{0}R^{r_i}_v [V * f_T](v) = \prod_{i=1}^{m} \lambda_i^{r_i} \cdot V(v) = \prod_{i=1}^{m} \lambda_i^{r_i} \int_0^\infty \phi_D(u+y)\, dF_Y(y).$$

For the q-th boundary condition, we have

$$\left(\bigodot_{i=1}^{m} \left(c^{r_i} \cdot {}^{\lambda_i/c}_{\ \ 0}R^{r_{q,i}}_u \right) \right) [\phi_D](u) = \prod_{i=1}^{M(q)-1} \lambda_i^{r_i} \cdot {}^{\lambda_{M(q)}}_{\ \ 0}R^{r_{q,M(q)}}_v \left[V * f_{M(q)} * f_{M(q)+} \right](v),$$

where $f_{M(q)}$ stands for the density function of $\Gamma\left(r_{M(q)}, \lambda_{M(q)}\right)$ random variable and $f_{M(q)+}$ for the the density function of sum of $\Gamma\left(r_i, \lambda_i\right)$ random variables, for $M(q) < i \leqslant L$. Let $\mathbf{V} = V * f_{M(q)+}$ and compute the following term

$$\begin{aligned} &{}^{\lambda_{M(q)}}_{\ \ 0}R^{r_{q,M(q)}}_v \left[\mathbf{V} * f_{M(q)} \right](v) \\ =& e^{-\lambda_{M(q)}v}\, {}_0D^{r_{q,M(q)}}_v \left(\int_0^v e^{\lambda_{M(q)}(v-y)} \mathbf{V}(v-y) \cdot e^{\lambda_{M(q)}y} f_{M(q)}(y)\, dy \right) \\ =& e^{-\lambda_{M(q)}v} \left[e^{\lambda_{M(q)}(v)} \mathbf{V}(v) * {}_0D^{r_{q,M(q)}}_v e^{\lambda_{M(q)}v} f_{M(q)}(v) \right] \\ &+ \mathbf{V}(v)\, {}_0D^{r_{q,M(q)}-1}_y \left(e^{\lambda_{M(q)}y} f_{M(q)}(y) \right)\Big|_{y=0} \end{aligned}$$

by applying the rule of fractional derivatives on a convolution integral. Note that since $r_{q,M(q)-1} < r_{M(q)}$, we conclude that

$$ {}^{\lambda_{M(q)}}_{\ \ 0}R_v^{r_{q,M(q)}} \left[\mathbf{V} * f_{M(q)}\right](0) = \left(\int_0^v \mathbf{V}(v-y) \cdot {}^{\lambda_{M(q)}}_{\ \ 0}R_y^{r_{q,M(q)}} f_{M(q)}(y)\, dy \right)\Bigg|_{v=0} = 0. $$

This holds for all $1 \leqslant q \leqslant M$, which completes the proof. □

Remark 4.2.1. When r is integer, say n, equation (4.22) reduces to

$$ \left(c\frac{d}{du} + \lambda \right)^n \phi_D(u) = \lambda^n \int_0^\infty \phi_D(u+y)\, dF_Y(y), \quad u \geqslant 0, $$

with the corresponding boundary conditions. This is equivalent to the existed result for ruin probability in Erlang-time dual risk model (see e.g. Rodríguez-Martínez et al. (2015)).

4.3 Explicit expressions for ruin probabilities in gamma-time and fractional Poisson risk models

The model we consider in Theorem 4.2.1 is a very general class in the renewal risk model. In this section, we focus on two specific models which might be more appealing than others. Explicit forms of ruin (non-ruin) probabilities are derived.

Remark 4.3.1. It has been shown Asmussen and Albrecher (2010) that for any renewal risk model, the ruin probability always has an exponential form when the claim distribution is exponential. However, the fractional differential equation approach bridges a solid connection between classical risk model and a class of renewal models, which might be applied in a more sophisticated model.

4.3.1 Gamma-time risk models

A gamma-time risk model, describes the reserve process $R_r(t)$ of an insurance company as

$$ R_r(t) = u + ct - \sum_{i=1}^{N_r(t)} X_i, \quad t > 0, $$

which replaces the Poisson process $N(t)$ in the classical model (1.1) by a renewal counting process $N_r(t)$ with $\Gamma(r,\lambda)$ inter-arrival times. This is a natural extension of an Erlang(n) risk model consiered for instance, by Li and Garrido (2004b). Being a special case of Theorem 4.2.1, the equation for ruin probability $\psi_r(u)$ in gamma-time risk model can be written as

$$c^r \cdot e^{\frac{\lambda}{c}u}\, {}_u^C D_\infty^r \left(e^{-\frac{\lambda}{c}u}\psi_r(u)\right) = \lambda^r \left(\int_0^u \psi_r(u-y)\, dF_X(y) + \int_u^\infty dF_X(y)\right), \quad u \geqslant 0. \tag{4.23}$$

When r takes integer values, say n, the Caputo fractional derivative reduces into the classical derivative

$${}_u^C D_\infty^n = (-1)^n \frac{d^n}{du^n}.$$

Thus, equation (4.23) reduces to the existed result of Li and Garrido (2004b),

$$\left(-c\frac{d}{du} + \lambda\right)^n \psi_n(u) = \lambda^n \left(\int_0^u \psi_n(u-y)\, dF_X(y) + \int_u^\infty dF_X(y)\right), \quad u \geqslant 0.$$

Equation (4.23) has an explicit solution whenever the claim size distribution has rational Laplace transform.

Example 4.3.1. In the risk model (4.1) with $\Gamma(r,\lambda)$ distributed inter-arrival times and Exp(α) distributed claim sizes, the ruin probability equals to

$$\psi_r(u) = \left(\frac{\lambda}{cx_2}\right)^r e^{\left(\frac{\lambda}{c}-x_2\right)u}, \quad u \geqslant 0, \tag{4.24}$$

where $x_2 > \frac{\lambda}{c}$ is the larger root of equation

$$c^r x^r \left(x - \left(\frac{\lambda}{c} + \alpha\right)\right) + \alpha\lambda^r = 0. \tag{4.25}$$

Proof. This example falls into the category of Theorem 4.2.2. In order to solve the equation

$$\left({}_0^\alpha R_u^1 \left(c^r \cdot {}_u^{\lambda/c} R_\infty^r\right)\right)[\phi_r](u) = \alpha\lambda^r \cdot \phi_r(u), \tag{4.26}$$

one needs to assume that the ruin probability satisfies

$$\left(\frac{d^j}{du^j}\left(e^{-\frac{\lambda}{c}u}\phi_r(u)\right)\right)\Bigg|_{u\to\infty} = 0, \quad j = 0, 1, \ldots, [r]+1.$$

Using the identity (2.9), we replace the Caputo derivative by a Riemann-Liouville derivative, in equation (4.26), thus having

$$c^r \left(\frac{d}{du} + \alpha \right) \left(e^{\frac{\lambda}{c}u} \ {}_uD^r_\infty \left(e^{-\frac{\lambda}{c}u} \phi_r(u) \right) \right) = \alpha \lambda^r \phi_r(u).$$

Multiplying both sides by $e^{-\frac{\lambda}{c}u}$ and denoting $g(u) = e^{-\frac{\lambda}{c}u}\phi_r(u)$, one obtains the fractional differential equation for $g(u)$

$$c^{r-1}\lambda \ {}_uD^r_\infty g(u) - c^r \ {}_uD^{r+1}_\infty g(u) + \alpha c^r \ {}_uD^r_\infty g(u) - \alpha\lambda^r g(u) = 0. \tag{4.27}$$

The corresponding characteristic equation for (4.27) is

$$c^r x^r \left(x - \left(\frac{\lambda}{c} + \alpha \right) \right) + \alpha \lambda^r = 0. \tag{4.28}$$

This characteristic equation has an explicit root at $x_1 = \frac{\lambda}{c}$. Solving the equation after differentiation gives the unique stationary point

$$x_s = \frac{r\left(\frac{\lambda}{c} + \alpha\right)}{r+1} > \frac{r\left(\frac{\lambda}{c} + \frac{\lambda}{rc}\right)}{r+1} = \frac{\lambda}{c}$$

on the positive axis, which is strictly larger than the root x_1 due to the net premium condition $\alpha > \frac{\lambda}{rc}$. Since the function on the left-hand side of equation (4.28) is a differentiable in x, and positive when $x = 0$ and $x \to \infty$, one could use the mean value theorem to locate the other real root x_2, on the positive half axis.

x	$\left[0, \frac{\lambda}{c}\right)$	$\frac{\lambda}{c}$	$\left(\frac{\lambda}{c}, x_2\right)$	x_2	(x_2, ∞)
Sign	positive	zero	negative	zero	positive

Table 4.1: The function $c^r x^r \left(x - \left(\frac{\lambda}{c} + \alpha\right)\right) + \alpha\lambda^r$

Consequently, the expression of $g(u)$ has the form

$$K_1 \cdot e^{-\frac{\lambda}{c}u} + K_2 \cdot e^{-x_2 u},$$

which means that the non-ruin probability equals to

$$\phi_r(u) = K_1 + K_2 \cdot e^{\left(\frac{\lambda}{c} - x_2\right)u}. \tag{4.29}$$

The constant K_1 can be obtained from the universal boundary condition of $\phi_r(u)$, which is

$$K_1 = \lim_{u\to\infty} \phi_r(u) = 1.$$

Since another boundary condition in this case is given by

$$\left[c^r e^{\frac{\lambda}{c}u}\ {}_uD^r_\infty \left(e^{-\frac{\lambda}{c}u}\phi_r(u)\right)\right]\Big|_{u=0} = \left[\lambda^r + c^r e^{\frac{\lambda}{c}u} K_2\, x_2^r\, e^{-x_2 u}\right]\Big|_{u=0} = 0,$$

one concludes that $K_2 = -\left(\frac{\lambda}{cx_2}\right)^r$. Thus, the ruin probability of the risk model (4.1) with $\Gamma(r,\lambda)$ distributed inter-arrival times and $\mathrm{Exp}(\alpha)$ distributed claim sizes is

$$\psi_r(u) = 1 - \phi_r(u) = \left(\frac{\lambda}{cx_2}\right)^r e^{\left(\frac{\lambda}{c}-x_2\right)u}.$$

The last step is to verify the assumption made in the beginning of the proof. Indeed, for $j = 0, 1, \ldots, [r] + 1$, all those derivatives are equal to

$$\frac{d^j}{du^j}\left(e^{-\frac{\lambda}{c}u} - e^{-\frac{\lambda}{c}u}\left(\frac{\lambda}{cx_2}\right)^r e^{\left(\frac{\lambda}{c}-x_2\right)u}\right) = \left(-\frac{\lambda}{c}\right)^j e^{-\frac{\lambda}{c}u} + \left(\frac{\lambda}{cx_2}\right)^r (-x_2)^j\, e^{-x_2 u},$$

and go to zero when u tends to infinity. This completes the proof. □

Remark 4.3.2. Let $s = x_2 - \frac{\lambda}{c}$ in the expression (4.24). One has the following calculation

$$\begin{aligned}
&(M_X(s)M_T(-cs))^{-1} - 1 = \left(1 - \frac{s}{\alpha}\right)\left(1 + \frac{cs}{\lambda}\right)^r - 1\\
=&\frac{c^r}{\lambda^r}\left(\left(1 - \frac{x_2 - \frac{\lambda}{c}}{\alpha}\right)\left(\frac{\lambda}{c} + x_2 - \frac{\lambda}{c}\right)^r - \frac{\lambda^r}{c^r}\right)\\
=&\frac{c^r}{\lambda^r\alpha}\left(\left(\alpha + \frac{\lambda}{c} - x_2\right)x_2^r - \frac{\lambda^r}{c^r}\right)\\
=&\frac{-1}{\lambda^r\alpha}\left(c^r x_2^{r+1} - \left(c^{r-1}\lambda + \alpha c^r\right)x_2^r + \alpha\lambda^r\right) = 0,
\end{aligned}$$

which means that $x_2 - \frac{\lambda}{c}$ is the unique positive solution γ of the Lundberg's fundamental equation. This finding coincides with the result from Asmussen and Albrecher (2010) for the renewal risk model with exponential claims.

Example 4.3.2. In the risk model (4.1) with $\Gamma(r,\lambda)$ distributed inter-arrival times and $\Gamma(2,\alpha)$ distributed claim sizes, the ruin probability equals to

$$\psi_r(u) = \frac{\frac{\lambda}{c} - z_3}{z_2 - z_3}\left(\frac{\lambda}{cz_2}\right)^r e^{\left(\frac{\lambda}{c}-z_2\right)u} + \frac{\frac{\lambda}{c} - z_2}{z_3 - z_2}\left(\frac{\lambda}{cz_3}\right)^r e^{\left(\frac{\lambda}{c}-z_3\right)u}, \quad u \geqslant 0,$$

where $z_3 > \frac{\lambda}{c} + \alpha > z_2 > \frac{\lambda}{c}$ are the two larger roots of the equation

$$c^r z^r \left(z - \left(\frac{\lambda}{c} + \alpha \right) \right)^2 - \alpha^2 \lambda^r = 0.$$

Proof. In order to solve the equation

$$\left({}^{\alpha}_{0}R^2_u \left(c^r \cdot {}^{\lambda/c}_{u}R^r_{\infty} \right) \right) [\phi_r](u) = \alpha \lambda^r \cdot \phi_r(u), \tag{4.30}$$

one needs to assume that the non-ruin probability satisfies

$$\left(\frac{d^j}{du^j} \left(e^{-\frac{\lambda}{c}u} \phi_r(u) \right) \right) \Big|_{u \to \infty} = 0, \quad j = 0, 1, \dots, [r] + 1.$$

Under this assumption, one could rewrite equation (4.30) as

$$c^r \left(\frac{d}{du} + \alpha \right)^2 \left(e^{\frac{\lambda}{c}u} \, {}_{u}D^r_{\infty} \left(e^{-\frac{\lambda}{c}u} \phi_r(u) \right) \right) = \alpha^2 \lambda^r \phi_r(u).$$

Analogously, we denote $h(u) = e^{-\frac{\lambda}{c}u} \phi_r(u)$ to obtain the fractional differential equation for $h(u)$

$$c^r \left({}_{u}D^{r+2}_{\infty} h(u) - 2 \left(\frac{\lambda}{c} + \alpha \right) {}_{u}D^{r+1}_{\infty} h(u) + \left(\frac{\lambda}{c} + \alpha \right)^2 {}_{u}D^r_{\infty} h(u) \right) = \alpha^2 \lambda^r h(u)$$

with its characteristic equation

$$c^r z^r \left(z - \left(\frac{\lambda}{c} + \alpha \right) \right)^2 - \alpha^2 \lambda^r = 0. \tag{4.31}$$

By observation, one finds that equation (4.31) has an explicit root $z_1 = \frac{\lambda}{c}$. Taking one derivative of the equation gives two stationary points

$$z_{s1} = \frac{r \left(\frac{\lambda}{c} + \alpha \right)}{r + 2} > \frac{r \left(\frac{\lambda}{c} + \frac{2\lambda}{rc} \right)}{r + 2} = \frac{\lambda}{c} \quad \text{and} \quad z_{s2} = \frac{\lambda}{c} + \alpha > z_{s1} > \frac{\lambda}{c}$$

on the positive axis. One is strictly larger than root z_1 due to the net premium condition $\alpha > \frac{2\lambda}{rc}$. The function on the left-hand side of equation (4.31) is a differentiable function of z, negative when $z = 0$ and positive when $z \to \infty$. In order to verify the existence of other roots, one needs to check the value of the function $c^r z^r \left(z - \left(\frac{\lambda}{c} + \alpha \right) \right)^2 - \alpha^2 \lambda^r$ at $z = z_{s2}$, which is negative

$$c^r \left(\frac{\lambda}{c} + \alpha \right)^r \left(\frac{\lambda}{c} + \alpha - \left(\frac{\lambda}{c} + \alpha \right) \right)^2 - \alpha^2 \lambda^r = -\alpha^2 \lambda^r < 0.$$

z	$\left[0, \frac{\lambda}{c}\right)$	$\frac{\lambda}{c}$	$\left(\frac{\lambda}{c}, z_2\right)$	z_2	(z_2, z_3)	z_3	(z_3, ∞)
Sign	negative	zero	positive	zero	negative	zero	positive

Table 4.2: The function $c^r z^r \left(z - \left(\frac{\lambda}{c} + \alpha\right)\right)^2 - \alpha^2 \lambda^r$

Thus, the mean value theorem can be applied to locate the other real roots $\frac{\lambda}{c} < z_2 < \frac{\lambda}{c} + \alpha$ and $z_3 > \frac{\lambda}{c} + \alpha$ on the positive half axis.

After solving equation (4.31), one obtains the expression of $h(u)$, which is

$$K_1 \cdot e^{-\frac{\lambda}{c}u} + K_2 \cdot e^{-z_2 u} + K_3 \cdot e^{-z_3 u}.$$

It implies that the non-ruin probability equals to

$$\phi_r(u) = K_1 + K_2 \cdot e^{\left(\frac{\lambda}{c} - z_2\right)u} + K_3 \cdot e^{\left(\frac{\lambda}{c} - z_3\right)u}. \tag{4.32}$$

The universal boundary condition of $\phi_r(u)$ ensures that $K_1 = 1$, and the other boundary conditions in this case are given by

$$\left[c^r e^{\frac{\lambda}{c}u} \, {}_uD^r_\infty \left(e^{-\frac{\lambda}{c}u} \phi_r(u)\right)\right]\Big|_{u=0} = 0,$$

$$\text{and} \quad \left[\left(\frac{d}{du} + \alpha\right)\left(c^r e^{\frac{\lambda}{c}u} \, {}_uD^r_\infty \left(e^{-\frac{\lambda}{c}u} \phi_r(u)\right)\right)\right]\Big|_{u=0} = 0.$$

Substituting the expression (4.32) into these two boundary conditions

$$\begin{cases} \lambda^r + c^r\, z_2^r\, K_2 + c^r\, z_3^r\, K_3 = 0 \\[2em] \alpha\lambda^r + c^r\, z_2^r \left(\frac{\lambda}{c} + \alpha - z_2\right) K_2 + c^r\, z_3^r \left(\frac{\lambda}{c} + \alpha - z_3\right) K_3 = 0, \end{cases}$$

gives

$$\begin{cases} K_2 = \dfrac{z_3 - \frac{\lambda}{c}}{z_2 - z_3} \left(\dfrac{\lambda}{c z_2}\right)^r \\[2em] K_3 = \dfrac{z_2 - \frac{\lambda}{c}}{z_3 - z_2} \left(\dfrac{\lambda}{c z_3}\right)^r. \end{cases}$$

Hence, the ruin probability of the risk model (4.1) with $\Gamma(r,\lambda)$ distributed inter-arrival times and $\Gamma(2,\alpha)$ distributed claim sizes is

$$\psi_r(u) = \frac{\frac{\lambda}{c} - z_3}{z_2 - z_3}\left(\frac{\lambda}{cz_2}\right)^r e^{\left(\frac{\lambda}{c}-z_2\right)u} + \frac{\frac{\lambda}{c} - z_2}{z_3 - z_2}\left(\frac{\lambda}{cz_3}\right)^r e^{\left(\frac{\lambda}{c}-z_3\right)u}.$$

The last step is to verify the assumption made in the beginning of the proof. In fact, for $j = 0, 1, \ldots, [r] + 1$, all those derivatives are equal to

$$\begin{aligned} &\frac{d^j}{du^j}\left(e^{-\frac{\lambda}{c}u}\left(1 + K_2\, e^{\left(\frac{\lambda}{c}-z_2\right)u} + K_3\, e^{\left(\frac{\lambda}{c}-z_3\right)u}\right)\right) \\ &= \left(-\frac{\lambda}{c}\right)^j e^{-\frac{\lambda}{c}u} + K_2\,(-z_2)^j\, e^{-z_2 u} + K_3\,(-z_3)^j\, e^{-z_3 u}, \end{aligned}$$

and go to zero when u tends to infinity. The completes the proof. □

4.3.2 Fractional Poisson risk models

In this section, we focus on the fractional Poisson risk process. The fractional (compound) Poisson risk model is a special case of the Sparre Anderson model,

$$R_\mu(t) = u + ct - \sum_{i=1}^{N_\mu(t)} X_i, \quad t > 0, \tag{4.33}$$

whose counting process chosen as fractional Poisson process $N_\mu(t)$. The ultimate ruin problem for this model is nontrivial since we have $\mathbb{E}(cT_1 - X_1) = \infty$ almost surely. Recall from Theorem 4.2.1, the ruin probability $\psi_\mu(u)$ of a fractional Poisson risk model satisfies a fractional integro-differential equation

$$c^\mu\, {}^C_u D^\mu_\infty \psi_\mu(u) + \lambda\psi_\mu(u) = \lambda\left(\int_0^u \psi_\mu(u-y)\, dF_X(y) + \int_u^\infty dF_X(y)\right), \quad u \geqslant 0, \tag{4.34}$$

with the universal boundary condition $\lim\limits_{u\to\infty} \psi_\mu(u) = 0$.

Note that the operator ${}^C_u D^\mu_\infty$ tends to the identity operator when $\mu \to 0+$. Thus, we obtain the following result.

Corollary 4.3.1. In a fractional Poisson risk model, the ruin probability $\psi_\mu(u)$ converges to a function $\psi_0(u)$, as $\mu \to 0$. Moreover, the function $\psi_0(u)$ satisfies an integral equation

$$(1+\lambda)\,\psi_0(u) = \lambda \int_0^u \psi_0(u-y)\,dF_X(y) + \lambda \int_u^\infty dF_X(y), \quad \geqslant 0, \tag{4.35}$$

with the universal boundary condition $\lim_{u\to\infty} \psi_0(u) = 0$.

Substituting $u = 0$ into equation (4.35) gives $\psi_0(0) = \frac{\lambda}{\lambda+1}$, which only depends on the value of λ. Taking Laplace transform both sides with respect to u and rearranging leads to

$$\hat{\psi}_0(s) = \frac{1-\hat{f}(s)}{(\lambda+1)s - \lambda s\hat{f}(s)},$$

which can be explicitly inverted back in some cases.

Since when $\mu = 1$, the fractional Poisson process degenerates to a Poisson process, we need the net profit condition to compute the ruin probability. The following examples are under the assumption $0 < \mu < 1$ in the fractional Poisson risk model (4.33) when the net profit condition always holds.

Biard and Saussereau (2014) derived the explicit expression for ruin probability of fractional Poisson risk model with exponential claims. In this section, the same result, as well as other new results will be derived via one fractional differential equation approach.

Example 4.3.3. In a fractional Poisson risk model (4.33) with Exp(α) distributed claim sizes, the ruin probability equals

$$\psi_\mu(u) = \left(1 - \frac{x_2}{\alpha}\right) e^{-x_2 u}, \quad u \geqslant 0, \tag{4.36}$$

where x_2 is the unique positive solution of

$$c^\mu x - \alpha c^\mu + \lambda x^{1-\mu} = 0. \tag{4.37}$$

Proof. In order to solve equation

$${}^{\alpha}_{0}R^1_u \left(c^\mu \cdot {}^C_u D^\mu_\infty + \lambda\right)[\phi_\mu](u) = \lambda\alpha\phi\mu(u) \tag{4.38}$$

one needs to assume that the non-ruin probability satisfies

$$\left(\frac{d^j}{du^j}\left(\phi_\mu(u)\right)\right)\bigg|_{u\to\infty} = 0, \quad j = 0,\, 1,\, 2.$$

Under this assumption, one could rewrite equation (4.38) as

$$\left(\frac{d}{du} + \alpha\right)\left(c^\mu \, {}_uD^\mu_\infty + \lambda\right)\phi_\mu(u) = \lambda\alpha\phi_\mu(u),$$

with the characteristic equation

$$c^\mu x^{\mu+1} - \alpha c^\mu x^\mu + \lambda x = 0. \tag{4.39}$$

We look for all the roots besides the apparent one $x_1 = 0$. Assuming that $x \neq 0$ leads the above equation to

$$c^\mu x - \alpha c^\mu = -\lambda x^{1-\mu}. \tag{4.40}$$

Since the order of left-hand side is higher than the right-hand side, there exists a larger positive real number $M_1 > \alpha$, such that both sides of equation (4.40) are analytic on the closed contour $\Gamma_1 = \{x : |x| = M_1\}$ in $\mathbb{C}$ and $|c^\mu x - \alpha c^\mu| > |-\lambda x^{1-\mu}|$ on the contour Γ_1. Then by Rouché's theorem, both $c^\mu x - \alpha c^\mu = 0$ and equation (4.40) have the same root within the closed contour $\Gamma_1 = \{x : |x| = M_1\}$. This means equation (4.40) has a unique solution. Since the continuous real-valued function $c^\mu x - \alpha c^\mu + \lambda x^{1-\mu}$ is negative when $x \to 0+$ and positive when $x \to \infty$, there must exist a point $x_2 > 0$ such that $c^\mu x_2 - \alpha c^\mu + \lambda x_2^{1-\mu} = 0$.

x	$(0, x_2)$	x_2	(x_2, ∞)
Sign	negative	zero	positive

Table 4.3: The function $c^\mu x - \alpha c^\mu + \lambda x^{1-\mu}$

After finding (numerically or explicitly) the value of x_2, one derives the expression of non-ruin probability as

$$\psi_\mu(u) = K_1 \cdot e^{-x_1 u} + K_2 \cdot e^{-x_2 u} = K_1 + K_2 \cdot e^{-x_2 u}.$$

The universal boundary condition of $\phi_\mu(u)$ ensures $K_1 = 1$. Another boundary condition in this case is

$$[(c^\mu \ {}_uD^\mu_\infty + \lambda)\,\phi_\mu(u)]\big|_{u=0} = \left[\lambda + c^\mu K_2 x_2^\mu e^{-x_2 u} + \lambda K_2 e^{-x_2 u}\right]\Big|_{u=0} = 0.$$

It can be used to obtain the other constant,

$$K_2 = -\frac{\lambda}{(cx_2)^\mu + \lambda} = \frac{x_2}{\alpha} - 1.$$

The last step is to verify the assumption made in the beginning of the proof. In fact, for $j = 0, 1, 2$, all those derivatives are equal to

$$\frac{d^j}{du^j}\left(1 + \left(\frac{x_2}{\alpha} - 1\right)e^{-x_2 u}\right) = \left(\frac{x_2}{\alpha} - 1\right)(-x_2)^j e^{-x_2 u},$$

and go to zero when u tends to infinity. This completes the proof. □

4.4 Numerical calculation and discussion

We have shown that the explicit ruin probabilities can be obtained when the claim sizes have rational Laplace transforms. In this section we will use a mathematical symbolic computation program to implement the numerical calculations for obtained ruin probabilities. A few discussions and comparisons will be considered for each specific model.

4.4.1 Discussion on gamma-time risk models

In order to compare the classical and gamma-time risk models, in 4.1 we show numerically obtained ruin probabilities in the case of Example 4.3.1 with different combinations of r and λ_1 such that the mean inter-arrival time is fixed to $r/\lambda_1 = 1$.

Assume that claim sizes are exponentially distributed, also with mean $\alpha = 1$ and that $c = 1.2$ to ensure the net profit condition, then the corresponding ruin probabilities can be obtained by expression (4.24) and (4.25). Many mathematical programming languages can help to solve equation (4.25) numerically. In this section we use Wolfram

Mathematica to implement the numerical calculation step. The resulting ruin probabilities are graphed in Figure 4.1.

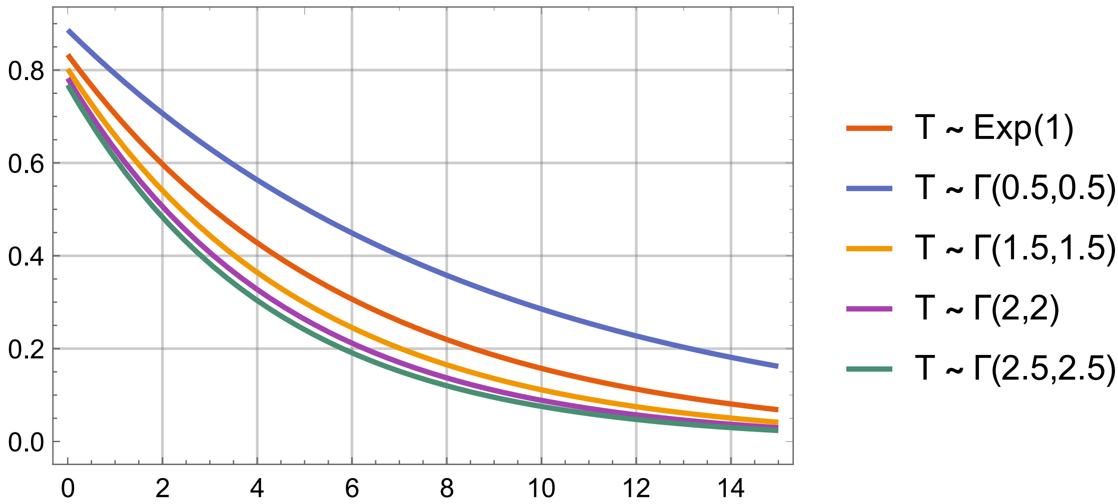

Figure 4.1: Ruin probabilities in the case of Example 4.3.1 for $\lambda = r = 0.5$, 1, 1.5, 2 and 2.5, and claim sizes are taken exponentially distributed with mean $\alpha = 1$ and $c = 1.2$ in order to ensure the net profit condition

Note the substantial impact on $\psi_r(u)$ when changing the Poisson assumption ($r = 1$). Ruin probabilities for gamma-time risk model (inter-arrival times $r > 1$) are relatively smaller, and vice versa. The reason is that in this case, the expected inter-arrival time r/λ_1 is fixed whereas the variance of inter-arrival time r/λ_1^2 decreases as r increases, which means that the chance of having a short waiting period between claims will decrease. Since ruin is usually caused by not enough capital accumulating, the model with a larger shape parameter r is more likely to survive. Figure 4.1 coincides with the finding from Li and Garrido (2004b), which focuses on Erlang(n) risk models.

In Figure 4.2 we illustrate the sensitivity to the parameters r, λ of the ruin probability $\psi_r(u)$ in Example 4.3.1. In order to do this, we define the statistic

$$u_5 := \inf \{u \geq 0 : \psi_r(u) < 0.05\} . \tag{4.41}$$

Namely, u_5 is the minimum capital needed to achieve a ruin probability of 5%. Note that any combinations of r and λ_1 on or above the dashed line marking the net profit condition, will make the ruin happen for sure. The value of u_5 tends to infinity as the parameters approach the dashed line since the safety loading $\frac{c\mathbb{E}(T)}{\mathbb{E}(X)} - 1$ tends to zero. When r takes large enough values or λ_1 take small enough values (in bluer areas), the ruin probability might be less than 5% even with zero initial capital. Note that along contour lines, $d\lambda_1 \approx \frac{1}{c}\,dr$, so the sensitivity of the ruin probabilities to its parameters depends almost exclusively on c.

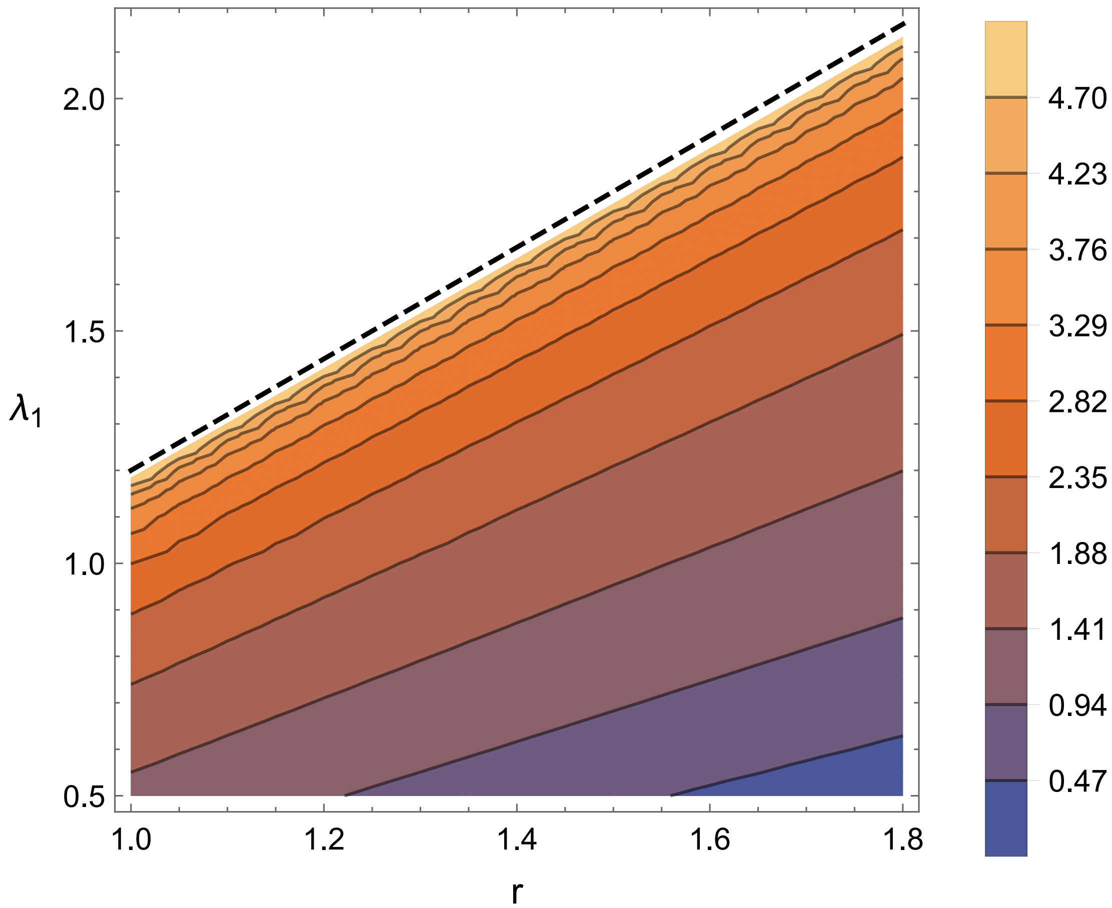

Figure 4.2: Natural log of u_5 (see (4.41)) for the ruin probability in Equation 4.3.1 with continuously varying parameters r, λ, and the claim sizes have fixed exponential distribution with mean $\alpha = 1$ and premium rate $c = 1.2$. The dotted line limits the region where the net profit condition $r/\lambda_1 < c$ holds

4.4.2 Discussion on fractional Poisson risk models

Figure 4.3 shows the ruin probability $\psi_\mu(u)$ for different combinations of the parameters λ_2, μ and fixed exponential claim size distribution.

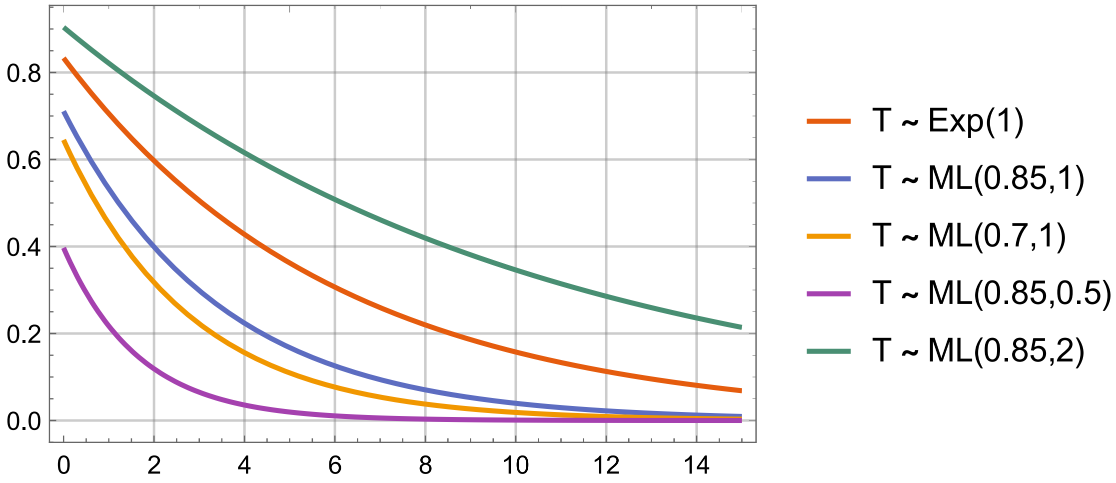

Figure 4.3: Ruin probabilities in the case of Example 4.3.3 for different combinations of λ_2, μ. Claim sizes are taken exponentially distributed with mean $\alpha = 1$ and $c = 1.2$

Note the substantial impact on $\psi_\mu(u)$ when changing the Poisson assumption ($\mu = 1$). Increasing λ_2 or μ increases the chances for ruin to happen. The reason is that, the expected number of jumps before time t in the fractional Poisson process (see equation (2.20)) is a monotone increasing function of $\lambda_2 > 0$. For argument μ, the derivative of the expected number of jumps before time t with respect to μ equals to

$$\frac{d}{d\mu}\frac{\lambda t^\mu}{\Gamma(\mu+1)} = \frac{\lambda t^\mu}{\Gamma(\mu+1)}\left(\ln(t) - \Psi^{(0)}(1+\mu)\right), \tag{4.42}$$

where meromorphic function $\Psi^{(0)}$ refers to the digamma function, defined as the logarithmic derivative of the gamma function:

$$\Psi^{(0)}(x) = \frac{d}{dx}\ln\left(\Gamma(x)\right) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Since we are considering the ultimate ruin probabilities, number of claims happen in small time intervals are not interested. Thus, we only focus on the case when t is large enough. In this case, equation (4.42) is positive when $0 < \mu \leqslant 1$, which means the expected number of jumps before time t in the fractional Poisson process is also a monotone increasing function of μ.

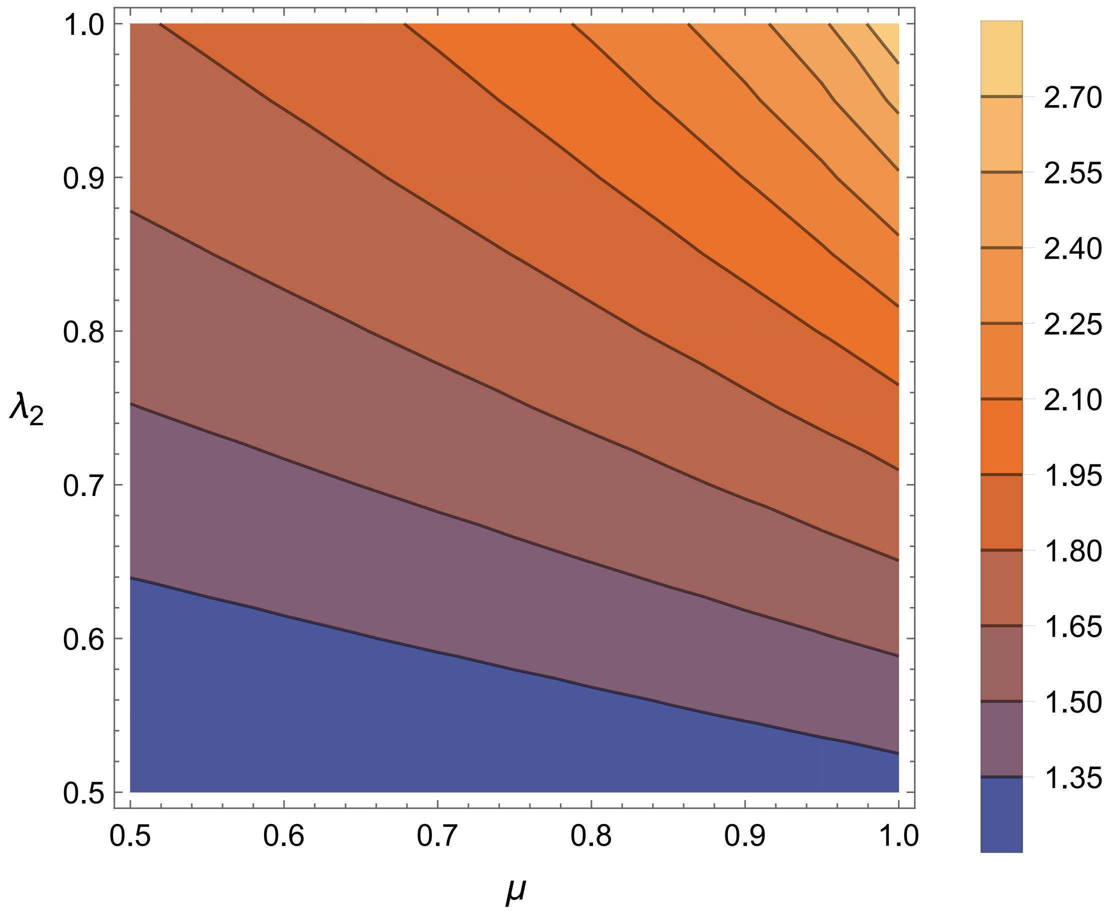

Figure 4.4: Natural log of u_5 (see (4.41)) for the ruin probability in Example 4.3.3 with continuously varying parameters μ, λ_2, and the claim sizes have fixed exponential distribution with mean $\alpha = 1$ and premium rate $c = 1.2$

Moreover, Figure 4.4 shows the values of natural logarithm of u_5 computed from (4.41) with ψ_μ as a function of μ and λ_2. Note that the contour lines in this plot are not parallel to each other. As the value of μ decreases, the parameter λ_2 plays a less significant role in the ruin probability function.

Chapter 5

Concluding Remarks and Future Work

This thesis mainly considers ruin probabilities in the classical risk model and the renewal risk models. Various equations and explicit expressions for ruin probabilities are obtained. By using properties of Mittag-Leffler functions, the Laplace transforms of ruin probabilities in the classical risk model when gamma claims can be inverted back. Independently, a new family of fractional differential operators is defined, which can be used to construct fractional integro-differential equations for ruin probabilities in collective risk and dual risk models. Classical results can be retrieved by setting appropriate parameter values. This brand new approach is based on constructing and solving fractional differential equations. It gives more analytical information on ruin probabilities and other related functions in risk theory.

5.1 Conclusions

Using a shift argument in Laplace transform, Theorem 3.1.1 obtains an explicit expression for non-ruin probabilities the in the classical risk model with rational shape gamma claims, which is a finite sum of Mittag-Leffler functions. Further, applying geometric expansions or iterative expressions, Theorem 3.1.2, 3.1.3 and 3.1.4 manage to extend the result to a more general case, where real-valued shape gamma claims are assumed in the classical risk model. Three formulas, all in infinite series forms, two involving Mittag-Leffler functions and a third one involving moments of the claims distribution are presented. It has been shown that these expressions can be reduced to existing results when Erlang claims. This thesis also considers the ruin problem when discrete claim sizes are assumed. Using a martingale approach, Theorem 3.2.1 obtains an explicit ruin probability expression in classical risk model with geometric claims.

In the case of renewal ruin theory, this thesis follows the idea from Albrecher et al. (2010), of an algebraic approach to study the ruin problem in both collective risk models and dual risk models. The inter-arrival times in such renewal models are assumed to be the sum of gamma and Mittag-Leffler random variables. Theorem 4.1.1 shows that the corresponding time density function solves a fractional differential equation equipped with the left Rock differential operator (defined in Definition 4.1.1). Theorem 4.2.1 and 4.2.3 apply the Rock operators in renewal risk models and dual risk models respectively. Fractional integro-differential equations for (non-)ruin probabilities in these models are obtained. Theorem 4.2.2 tells that when the claim sizes in renewal risk models are distributed as the sum of gamma random variables, the problem of getting ruin probabilities is transformed into solving fractional differential equations with appropriate initial-value boundary conditions. Especially when the claim size distributions have rational Laplace transforms, all initial-value boundary conditions have analytic forms. In this case, the fractional differential equations for ruin probabilities can be solved explicitly and the solutions are always sums of exponential functions. Specific models, gamma-time risk model and fractional Poisson risk model are analysed in detail.

5.2 Future research

There are four main venues of future research to pursue.

1. All existed literature (including this project) in risk theory can only obtain ex-

plicit expressions for ruin probabilities when claim sizes have rational Laplace transform. The main reason is that a classical method to solve integro-differential equations is via Laplace transforms. In this thesis, fractional differential equations for ruin probabilities are constructed, which gives an alternative direction to derive explicit/numerical ruin probabilities. One of the future projects would focus on solving these fractional differential equations numerically, when analytical solutions do not exist. One can attempt to derive numerical solutions/approximations by means of the partial differential equation parametrix method. This is one of the methods of studying boundary value problems, for differential equations with variable coefficients. Using integral equations, a parametrix is an approximation of a fundamental solution of a differential equation, which is essentially an approximate inverse to a differential operator. In the case of ruin probabilities, the fractional integro-differential equations can be transformed (back) to double-integral equations of Fredholm type, by an operator expansion, which can be understood as a parametrix of a heat equation of a certain type. As an approximation, a parametrix of a differential operator is often easier to construct than a fundamental solution, and for most purposes is almost as good. A sufficiently good parametrix can often be used to construct an exact fundamental solution by a convergent iterative procedure. Recently, the research group of Ritsumeikan University is working on exact/unbiased simulation of stochastic differential equations, where the key idea is lying in understanding the parametrix as a ruin probability of certain type. The link or the duality may be trivial when we deal with exponential waiting time but for the fractional case (gamma or Mittag-Leffler inter-arrivals) we do not have any intuition to support it. Once the link is established, we can on one hand export our results to stochastic calculus. The explicit solutions of the above case will establish an interesting formula for the theory of stochastic differential equations. On the other hand, the vast literature of the stochastic calculus, like the martingale theory of Kunita-Watanabe, Yamada-Watanabe theory for stochastic differential equations, Malliavin calculus, could be made available for ruin theory. Ruin probability acts as a timely measure of risk for insurance companies and in general for risk management purposes. Better understanding of the ruin probability could improve the capital management of the insurance company and their premium policies. In the meantime, introducing the Malliavin calculus in the analysis would bring a new dimension to the risk theory literature, by accounting for more sophisticated models.

2. The Rock operators defined in this project have played an important role in the generalisation step. The motivation of using such operators comes from the ordinary differential equations for Erlang density functions. Following the same idea of Babenko's symbolic method, we would like to find a more generalised form of the Rock operators. Consider a positive random variable T with density function f_T. Suppose the Laplace transform of f_T is given by

$$\hat{f}_T(s) = \left(\frac{\lambda}{s^\mu + \lambda}\right)^r,$$

where $\mu \in (0, 1]$, λ and r are positive real numbers. When r takes integer value, the random variable T can be interpreted as the sum of i.i.d. Mittag-Leffler random variables. By using one property of Mittag-Leffler function, the expression of density function f_T can be obtained in terms of the fractional derivatives of Mittag-Leffler functions. When $\mu = 1$, the random variable T becomes a gamma random variable. The purpose of this research direction is to find the corresponding fractional differential equation that f_T satisfies. Intuitively, we need to use such a differential operator

$$({}_0D_t^\mu + \lambda)^r \tag{5.1}$$

to construct the desired fractional differential equation. Once the proper definition of (5.1) is found, most results derived in this thesis can be smoothly extended to a more general risk model.

3. In Gerber and Shiu (1998) the expected discounted penalty at ruin function $\Phi(u)$, also known as Gerber-Shiu function, is introduced,

$$\Phi(u) = \mathbb{E}\left[w(R(\tau_u-), |R(\tau_u)|)e^{-\delta\tau_u}\mathbb{1}(\tau_u < \infty) \middle| R(0) = u \right],$$

where τ_u is the time of ruin, $R(\tau_u-)$ represents the surplus immediately before ruin and $|R(\tau_u)|$ the deficit at ruin, often called the severity of ruin. Albrecher et al. (2010) considered a renewal risk model when inter-arrival time distribution with rational Laplace transform and transformed the usual integral equation into a boundary value problem, which can be solved by symbolic techniques. We would like to derive fractional integro-differential equations for Gerber-Shiu functions in more general renewal risk models, where the Rock operators might appear.

4. Waters and Papatriandafylou (1985) used martingale techniques to derive upper bounds for the probability of ruin for a risk process explicitly which allows for delays in the claims settlement. After that the issues around ruin problems involving delayed claim settlements have been studied (Boogaert and Haezendonck, 1989; Klüppelberg and Mikosch, 1995; Brémaud, 2000; Macci and Torrisi, 2004; Yuen et al., 2005; Albrecher and Asmussen, 2006; Trufin et al., 2011). We would like to advocate an approach to study the ruin problem in a renewal risk model with delayed claims by applying the Rock operators defined in this thesis. Fractional delayed-integro-differential equations might be constructed and further analysed.

Appendix

Proof of Remark 3.1.5

Since the Mittag-Leffler function is related to the error function, for special parameter values as follows

$$
\begin{aligned}
E_{\frac{1}{2},\frac{1}{2}}\left(s_k u^{\frac{1}{2}}\right) &= \sum_{i=0}^{\infty} \frac{\left(s_k u^{\frac{1}{2}}\right)^i}{\Gamma\left(\frac{i+1}{2}\right)} = \frac{1}{\sqrt{\pi}} + s_k u^{\frac{1}{2}} E_{\frac{1}{2},1}\left(s_k u^{\frac{1}{2}}\right) \\
=& \frac{1}{\sqrt{\pi}} + s_k u^{\frac{1}{2}} e^{s_k^2 u} \operatorname{erfc}\left(-s_k u^{\frac{1}{2}}\right) = \frac{1}{\sqrt{\pi}} + s_k u^{\frac{1}{2}} e^{s_k^2 u} \frac{2}{\sqrt{\pi}} \int_{-s_k u^{\frac{1}{2}}}^{\infty} e^{-t^2}\, dt,
\end{aligned}
$$

where the error function and the complementary error function defined by

$$
\operatorname{erf}(x) = 1 - \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,
$$

one can express the non-ruin probability as

$$
\phi(u) = e^{-\alpha u} u^{-\frac{1}{2}} (m_0 + m_1 + m_2) \frac{1}{\sqrt{\pi}} + \sum_{k=0}^{2} s_k m_k e^{\left(s_k^2 - \alpha\right) u} \frac{2}{\sqrt{\pi}} \int_{-s_k u^{\frac{1}{2}}}^{\infty} e^{-t^2}\, dt.
$$

Note that in this case

$$
\begin{aligned}
m_1 + m_2 &= \frac{\phi(0)}{s_1 - s_2} \left(\frac{s_1}{s_0 - s_1} + \frac{s_2}{s_0 - s_2} \right) \\
&= \frac{\phi(0)}{s_1 - s_2} \frac{-s_1 s_0 + s_2 s_0}{(s_0 - s_1)(s_0 - s_2)} \\
&= -\frac{s_0 \phi(0)}{(s_0 - s_1)(s_0 - s_2)} = -m_0,
\end{aligned}
$$

and

$$
m_0 = \frac{s_0 \phi(0)}{(s_1 - s_0)(s_2 - s_0)} = \frac{\alpha^{\frac{1}{2}} \left(1 - \frac{\lambda}{2c\alpha}\right)}{\frac{9}{4}\alpha - \frac{1}{4}\alpha - \frac{\lambda}{c}} = \frac{1}{2\sqrt{\alpha}},
$$

one can obtain by simple calculation that

$$\begin{aligned}\phi(u) =& \frac{1}{\sqrt{\pi}} \int_{-(\alpha u)^{\frac{1}{2}}}^{\infty} e^{-t^2}\, dt \\ &+ \frac{s_1^2\left(1-\frac{\lambda}{2c\alpha}\right)}{(s_0-s_1)(s_2-s_1)} e^{\left(s_1^2-\alpha\right)u} \frac{2}{\sqrt{\pi}} \int_{-s_1 u^{\frac{1}{2}}}^{\infty} e^{-t^2}\, dt \\ &+ \frac{s_2^2\left(1-\frac{\lambda}{2c\alpha}\right)}{(s_0-s_2)(s_1-s_2)} e^{\left(s_2^2-\alpha\right)u} \frac{2}{\sqrt{\pi}} \int_{-s_2 u^{\frac{1}{2}}}^{\infty} e^{-t^2}\, dt.\end{aligned}$$

Moreover, one can show that $R = \alpha - s_1^2$. Hence, the non-ruin probability can be expressed as

$$\begin{aligned}\phi(u) =& \frac{2s_1^2\left(1-\frac{1}{c}\right)}{(s_0-s_1)(s_2-s_1)} e^{-Ru} + 1 - \frac{1}{2}\mathrm{erfc}\left(s_0 u^{\frac{1}{2}}\right) \\ &- \frac{s_1^2\left(1-\frac{1}{c}\right)}{(s_0-s_1)(s_2-s_1)} \mathrm{erfc}\left(s_1 u^{\frac{1}{2}}\right) e^{\left(s_1^2-\frac{1}{2}\right)u} \\ &+ \frac{s_2^2\left(1-\frac{1}{c}\right)}{(s_0-s_2)(s_1-s_2)} \mathrm{erfc}\left(-s_2 u^{\frac{1}{2}}\right) e^{\left(s_2^2-\frac{1}{2}\right)u},\end{aligned}$$

which is equivalent to say the ruin probability equals to

$$\begin{aligned}\psi(u) =& -\frac{2s_1^2\left(1-\frac{1}{c}\right)}{(s_0-s_1)(s_2-s_1)} e^{-Ru} + \frac{1}{2}\mathrm{erfc}\left(s_0 u^{\frac{1}{2}}\right) \\ &+ \frac{s_1^2\left(1-\frac{1}{c}\right)}{(s_0-s_1)(s_2-s_1)} \mathrm{erfc}\left(s_1 u^{\frac{1}{2}}\right) e^{\left(s_1^2-\frac{1}{2}\right)u} \\ &- \frac{s_2^2\left(1-\frac{1}{c}\right)}{(s_0-s_2)(s_1-s_2)} \mathrm{erfc}\left(-s_2 u^{\frac{1}{2}}\right) e^{\left(s_2^2-\frac{1}{2}\right)u}. \qquad (5.2)\end{aligned}$$

Here is the proof that expression (3.6) and (5.2) are the same. Indeed, the first term of (5.2) equals to

$$\begin{aligned}&-\frac{2s_1^2\left(1-\frac{1}{c}\right)}{(s_0-s_1)(s_2-s_1)} e^{-Ru} = \frac{(c-1)2s_1^2}{-c(s_0 s_2 - (s_0+s_2)s_1 + s_1^2)} e^{-Ru} \\ =& \frac{(c-1)2s_1^2}{1+c(\frac{1}{2}-3s_1^2)} e^{-Ru} = \frac{(c-1)(1-2R)}{1+c(3R-1)} e^{-Ru}\end{aligned}$$

and the second term of (3.6) can be rewritten as

$$\frac{2(c-1)}{\pi c^2} \int_0^{\infty} \frac{\sqrt{x} e^{-(x+1)u/2}}{(x+1)(x+a)(x+b)}\, dx = \frac{4(c-1)}{\pi c^2} \int_0^{\infty} \frac{t^2 e^{-(t^2+1)u/2}}{(t^2+1)(t^2+a)(d^2+b)}\, dt$$

where

$$a = \frac{1}{2} + \frac{2}{c} - \sqrt{\frac{1}{4} + \frac{2}{c}} = 2s_1^2, \quad b = \frac{1}{2} + \frac{2}{c} + \sqrt{\frac{1}{4} + \frac{2}{c}} = 2s_2^2.$$

Using partial fraction decomposition, the integral becomes

$$\frac{4(c-1)}{\pi c^2}\left(A\int_0^\infty \frac{e^{-(t^2+1)u/2}}{t^2+1}\,dt + B\int_0^\infty \frac{e^{-(t^2+1)u/2}}{t^2+a}\,dt + C\int_0^\infty \frac{e^{-(t^2+1)u/2}}{t^2+b}\,dt\right),$$

where

$$A=\frac{-1}{(a-1)(b-1)},\quad B=\frac{-a}{(1-a)(b-a)},\quad C=\frac{-b}{(1-b)(a-b)}.$$

Denoting $f(\theta)=\int_0^\infty \frac{e^{-(t^2+1)\theta}}{t^2+\varepsilon}\,dt$, one has

$$\begin{aligned}
e^{-(\varepsilon-1)\theta}f(\theta) &= \int_0^\infty \frac{e^{-(t^2+\varepsilon)\theta}}{t^2+\varepsilon}\,dt = \int_0^\infty\int_\theta^\infty e^{-(t^2+\varepsilon)r}\,dr\,dt \\
&= \int_\theta^\infty e^{-\varepsilon r}\left(\int_0^\infty e^{-rt^2}\,dt\right)dr = \int_\theta^\infty e^{-\varepsilon r}\frac{1}{2}\sqrt{\frac{\pi}{r}}\,dr = \sqrt{\frac{\pi}{\varepsilon}}\int_{\sqrt{\theta\varepsilon}}^\infty e^{-s^2}\,ds \\
&= \frac{\pi}{\sqrt{2\varepsilon}}\operatorname{erfc}\left(\sqrt{\theta\varepsilon}\right),
\end{aligned}$$

which leads to

$$\int_0^\infty \frac{e^{-(t^2+1)u/2}}{t^2+\varepsilon}dt = \frac{\pi}{\sqrt{2\varepsilon}}e^{(\varepsilon-1)u/2}\operatorname{erfc}\left(\sqrt{\varepsilon u/2}\right).$$

Substituting the three integral terms into the special case of Thorin (1973) expression (3.6), one has

$$\begin{aligned}
&\frac{2(c-1)}{c^2}\left(\frac{-\operatorname{erfc}\left(\sqrt{u/2}\right)}{(a-1)(b-1)} + e^{(a-1)u/2}\frac{-\sqrt{a}\operatorname{erfc}\left(\sqrt{au/2}\right)}{(1-a)(b-a)} + e^{(b-1)u/2}\frac{-\sqrt{b}\operatorname{erfc}\left(\sqrt{bu/2}\right)}{(1-b)(a-b)}\right) \\
=&\frac{2(c-1)}{c^2}\left(\frac{-\operatorname{erfc}\left(s_0\sqrt{u}\right)}{(a-1)(b-1)} + e^{(a-1)u/2}\frac{-\sqrt{a}\operatorname{erfc}\left(s_1\sqrt{u}\right)}{(1-a)(b-a)} + e^{(b-1)u/2}\frac{-\sqrt{b}\operatorname{erfc}\left(-s_2\sqrt{u}\right)}{(1-b)(a-b)}\right) \\
=&\frac{1}{2}\operatorname{erfc}\left(s_0u^{\frac{1}{2}}\right) + \frac{2(1-c)\left(-\sqrt{2}s_1\right)e^{\left(s_1^2-\frac{1}{2}\right)u}\operatorname{erfc}\left(s_1\sqrt{u}\right)}{\left(1-2s_1^2\right)\left(2s_2^2-2s_1^2\right)} \\
&-\frac{2(1-c)\left(-\sqrt{2}s_2\right)e^{\left(s_2^2-\frac{1}{2}\right)u}\operatorname{erfc}\left(-s_2\sqrt{u}\right)}{\left(1-2s_2^2\right)\left(2s_1^2-2s_2^2\right)} \\
=&\frac{1}{2}\operatorname{erfc}\left(s_0u^{\frac{1}{2}}\right) + \frac{s_1^2\left(1-\frac{1}{c}\right)}{(s_0-s_1)(s_2-s_1)}\operatorname{erfc}\left(s_1u^{\frac{1}{2}}\right)e^{\left(s_1^2-\frac{1}{2}\right)u} \\
&-\frac{s_2^2\left(1-\frac{1}{c}\right)}{(s_0-s_2)(s_1-s_2)}\operatorname{erfc}\left(-s_2u^{\frac{1}{2}}\right)e^{\left(s_2^2-\frac{1}{2}\right)u},
\end{aligned}$$

which coincides with our expression (5.2).

Proof of Remark 3.1.9

When $r = 2$,the n-fold convolution in expression (3.12) becomes

$$(1+\alpha u)^{*n} = \sum_{i=0}^{n} \binom{n}{i} \frac{\alpha^i u^{n+i-1}}{(n+i-1)!}, \tag{5.3}$$

which needs to be further convolved with $e^{\alpha u}$. Recall that the convolution of an exponential function and a power function is given by

$$e^{\alpha u} * u^k = \int_0^u e^{\alpha(u-s)} s^k \, ds = \frac{k!}{\alpha^{k+1}} e^{\alpha u} - \sum_{j=0}^{k} \frac{k! \, u^j}{\alpha^{k+1-j} \, j!}. \tag{5.4}$$

Using the linearity of the convolution, one may conclude from identities (5.3) and (5.4) that

$$\begin{aligned} e^{\alpha u} * (1+\alpha u)^{*n} &= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{e^{\alpha u}}{\alpha^n} - \sum_{j=0}^{n+i-1} \frac{u^j}{\alpha^{n-j} \, j!} \right) \\ &= \frac{2^n e^{\alpha u}}{\alpha^n} - \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{j=0}^{n+i-1} \frac{u^j}{\alpha^{n-j} \, j!} \right), \end{aligned}$$

which leads to the non-ruin probability

$$\begin{aligned} \phi(u) &= e^{-\alpha u} \phi(0) \sum_{n=0}^{\infty} \left(\frac{\lambda}{c} \right)^n \left[\frac{2^n e^{\alpha u}}{\alpha^n} - \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{j=0}^{n+i-1} \frac{u^j}{\alpha^{n-j} \, j!} \right) \right] \\ &= \phi(0) \frac{1}{1-\frac{2\lambda}{c\alpha}} - e^{-\alpha u} \phi(0) \sum_{n=0}^{\infty} \left(\frac{\lambda}{c} \right)^n \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{j=0}^{n+i-1} \frac{u^j}{\alpha^{n-j} \, j!} \right) \\ &= 1 - \left(1 - \frac{2\lambda}{\alpha c} \right) e^{-\alpha u} \sum_{n=0}^{\infty} \left(\frac{\lambda}{c} \right)^n \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{j=0}^{n+i-1} \frac{u^j}{\alpha^{n-j} \, j!} \right). \end{aligned}$$

To deal with the infinite series term

$$\sum_{n=0}^{\infty} \left(\frac{\lambda}{c} \right)^n \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{j=0}^{n+i-1} \frac{u^j}{\alpha^{n-j} \, j!} \right)$$

in the above expression, first take its Laplace transform to obtain the following expression for $s > \alpha$,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{\lambda}{c}\right)^n \sum_{i=0}^{n}\binom{n}{i}\left(\sum_{j=0}^{n+i-1}\frac{\alpha^j}{\alpha^n\, s^{j+1}}\right)\\
&=\frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{\lambda}{\alpha c}\right)^n \sum_{i=0}^{n}\binom{n}{i}\frac{1-(\alpha/s)^{n+i}}{1-\alpha/s}\\
&=\frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{\lambda}{\alpha c}\right)^n \left(\frac{2^n}{1-\alpha/s}-\left(\frac{\alpha}{s}\right)^n\frac{(1+\alpha/s)^n}{1-\alpha/s}\right),
\end{aligned}$$

where one detects a sum of two geometric series with general terms $\frac{2\lambda}{\alpha c}$ and $\frac{\lambda}{cs}\left(1+\frac{\alpha}{s}\right)$ respectively. Therefore, the term of infinite series can be further expressed as

$$\begin{aligned}
&\frac{1}{(s-\alpha)\left(1-\frac{2\lambda}{\alpha c}\right)}-\frac{1}{(s-\alpha)\left(1-\frac{\lambda}{cs}\left(1+\frac{\alpha}{s}\right)\right)}\\
=&\frac{\alpha c}{\alpha c-2\lambda}\left(\frac{\left(1-\frac{\lambda}{cs}\left(1+\frac{\alpha}{s}\right)\right)-\left(1-\frac{2\lambda}{\alpha c}\right)}{(s-\alpha)\left(1-\frac{\lambda}{cs}\left(1+\frac{\alpha}{s}\right)\right)}\right)\\
=&\frac{2\lambda}{\alpha c-2\lambda}\,\frac{s+\frac{\alpha}{2}}{s^2-\frac{\lambda}{c}(s+\alpha)}=\frac{2\lambda}{\alpha c-2\lambda}\left(\frac{m_1}{s-s_1}+\frac{m_1}{s-s_2}\right),
\end{aligned}$$

where the last step involves a partial fraction decomposition, with

$$s_{1,2}=\frac{\lambda\pm\sqrt{\lambda^2+4\lambda\alpha c}}{2c}.$$

One can invert the Laplace transform back to obtain

$$\sum_{n=0}^{\infty}\left(\frac{\lambda}{c}\right)^n \sum_{i=0}^{n}\binom{n}{i}\left(\sum_{j=0}^{n+i-1}\frac{u^j}{\alpha^{n-j}\, j!}\right)=\frac{2\lambda}{\alpha c-2\lambda}\left(\delta(u)+m_1e^{s_1u}+m_2e^{s_2u}\right),$$

and so the non-ruin probability for $r = 2$ is

$$\begin{aligned}
\phi(u)&=1-\left(1-\frac{2\lambda}{\alpha c}\right)e^{-\alpha u}\frac{2\lambda}{\alpha c-2\lambda}\left(m_1e^{s_1u}+m_2e^{s_2u}\right)\\
&=1-\frac{2\lambda}{\alpha c}\left(m_1e^{(s_1-\alpha)u}+m_2e^{(s_2-\alpha)u}\right),
\end{aligned}$$

where $s_{1,2}$ are given above, and $m_{1,2}$ can be calculated from the fraction decomposition step.

Proof of Remark 3.1.10

When $r = 1$, the non-ruin probability expression obtained by the second method can be written as

$$\begin{aligned}\phi(u) &= e^{-\alpha u}\phi(0)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\lambda\alpha}{c}\right)^k u^{2k}\sum_{j=0}^{\infty}\frac{(j+k)!\left((\alpha+\frac{\lambda}{c})u\right)^j}{j!\Gamma(2k+1+j)}\\ &= e^{-\alpha u}\phi(0)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-\frac{\lambda\alpha}{c}u^2)^k}{k!}\frac{(j+k)!\left((\alpha+\frac{\lambda}{c})u\right)^j}{j!\Gamma(2k+1+j)}.\end{aligned}$$

Denote $p = \frac{\lambda u}{c}$ and $q = \alpha u$, then

$$\begin{aligned}\phi(u) &= e^{-\alpha u}\phi(0)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(j+k)!}{j!k!(2k+j)!}(-pq)^k(p+q)^j\\ &= e^{-\alpha u}\phi(0)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(j+k)!}{j!k!(2k+j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}(-1)^k p^{k+i}q^{k+j-i}.\end{aligned}$$

After furthermore setting $k+i = m$, $k+j-i = n$ and $k+j = l$, then $i = l-n$, $j-i = l-m$ and $k = m+n-l$, one has

$$\begin{aligned}\phi(u) &= e^{-\alpha u}\phi(0)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}p^m q^n\sum_{l=\min(m,n)}^{m+n}\frac{(-1)^{m+n-l}l!}{(m+n-l)!(m+n)!(l-m)!(l-n)!}\\ &= e^{-\alpha u}\phi(0)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{p^m q^n}{(m+n)!}\left(\sum_{l=\min(m,n)}^{m+n}\frac{(-1)^{m+n-l}l!}{(m+n-l)!(l-m)!(l-n)!}\right).\end{aligned}$$

Now let us focus on the last summation term in the above expression, denoting by

$$
\begin{aligned}
f(m,n) = &\sum_{l=\min(m,n)}^{m+n} \frac{(-1)^{m+n-l} l!}{(m+n-l)!(l-m)!(l-n)!} \\
= &\sum_{l=\min(m,n)}^{m+n} \frac{(-1)^{m+n-l}(l-1)![(l-m)+(l-n)+(m+n-l)]}{(m+n-l)!(l-m)!(l-n)!} \\
= &\sum_{l=\min(m+1,n)}^{m+n} \frac{(-1)^{m+n-l}(l-1)!}{(m+n-l)!(l-m-1)!(l-n)!} \\
&+ \sum_{l=\min(m,n+1)}^{m+n} \frac{(-1)^{m+n-l}(l-1)!}{(m+n-l)!(l-m)!(l-n-1)!} \\
&+ \sum_{l=\min(m,n)}^{m+n-1} \frac{(-1)^{m+n-l}(l-1)!}{(m+n-l-1)!(l-m)!(l-n)!} \\
= &\sum_{l=\min(m,n-1)}^{m+n-1} \frac{(-1)^{m+n-1-l}(l)!}{(m+n-l-1)!(l-m)!(l-n+1)!} \\
&+ \sum_{l=\min(m-1,n)}^{m+n-1} \frac{(-1)^{m+n-l-1}(l-1)!}{(m+n-l-1)!(l-m+1)!(l-n)!} \\
&+ \sum_{l=\min(m-1,n-1)}^{m+n-2} \frac{-(-1)^{m+n-l}(l)!}{(m+n-l-2)!(l-m+1)!(l-n+1)!} \\
= &f(m-1,n) + f(m.n-1) - f(m-1,n-1),
\end{aligned}
$$

with boundary values $f(0,0) = f(1,0) = f(0,1) = 1$. By induction, we have $f(m,n) = 1$. Thus, the non-ruin probability equals

$$
\begin{aligned}
\phi(u) &= e^{-\alpha u}\phi(0) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{p^m q^n}{(m+n)!} = e^{-\alpha u}\phi(0) \sum_{m=0}^{\infty} \frac{p^m}{m!} \sum_{n=0}^{m} \left(\frac{q}{p}\right)^n \\
&= e^{-\alpha u}\phi(0) \sum_{m=0}^{\infty} \frac{p^m}{m!} \frac{(q/p)^{m+1}-1}{q/p-1} = e^{-\alpha u}\phi(0) \sum_{m=0}^{\infty} \frac{1}{q-p} \frac{q^{m+1}-p^{m+1}}{m!} \\
&= e^{-\alpha u}\phi(0) \frac{qe^q - pe^p}{q-p} = e^{-\alpha u}\left(1 - \frac{\lambda}{\alpha c}\right) \frac{\alpha e^{\alpha u} - \frac{\lambda}{c} e^{\frac{\lambda u}{c}}}{\alpha - \frac{\lambda}{c}} \\
&= 1 - \frac{\lambda}{\alpha c} e^{\left(\frac{\lambda}{c}-\alpha\right)u},
\end{aligned}
$$

as desired.

Proof of Theorem 3.1.4

We start by proving that the sequence $\big(b_i(F_X),\ i = 1, 2\ldots\big)$ defined in (3.20) has the property that b_m is independent of $n \geq m$. Since the statement is clear for $m = 1$, we proceed by induction. Assume that $b_k(F_X)$ is independent of $n \geqslant k$ for all $k \leqslant m$, and let $n \geqslant m + 1$. We have:

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{j=1}^{n+1} X_j\right)^m - \sum_{i=1}^{m}\binom{m}{i-1} b_i(F_X)\mathbb{E}\left(\sum_{j=1}^{n+1-i} X_j\right)^{m+1-i} \\
&- \mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^m + \sum_{i=1}^{m}\binom{m}{i-1} b_i(F_X)\mathbb{E}\left(\sum_{j=1}^{n-i} X_j\right)^{m+1-i} \\
=&\sum_{k=1}^{m}\binom{m}{k}\mathbb{E}\left(X^k\right)\mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^{m-k} \\
&- \sum_{i=1}^{m}\binom{m}{i-1} b_i(F_X)\sum_{k=1}^{m+1-i}\binom{m+1-i}{k}\mathbb{E}(X^k)\mathbb{E}\left(\sum_{j=1}^{n-i} X_j\right)^{m+1-i-k} \\
=&\sum_{k=1}^{m}\binom{m}{k}\mathbb{E}\left(X^k\right)\mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^{m-k} \\
&- \sum_{k=1}^{m}\mathbb{E}(X^k)\sum_{i=1}^{m+1-k}\frac{m!}{(i-1)!k!(m+1-i-k)!}b_i(F_X)\mathbb{E}\left(\sum_{j=1}^{n-i} X_j\right)^{m+1-i-k}
\end{aligned}
$$

Recognizing the binomial coefficients, we carry on the calculation

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{j=1}^{n+1} X_j\right)^m - \sum_{i=1}^{m}\binom{m}{i-1} b_i(F_X)\mathbb{E}\left(\sum_{j=1}^{n+1-i} X_j\right)^{m+1-i} \\
=&\sum_{k=1}^{m}\binom{m}{k}\mathbb{E}\left(X^k\right)\cdot\left[\mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^{m-k} - \sum_{i=1}^{m+1-k}\binom{m-k}{i-1}b_i(F_X)\mathbb{E}\left(\sum_{j=1}^{n-i} X_j\right)^{m-k+1-i}\right] \\
=&\sum_{k=1}^{m}\binom{m}{k}\mathbb{E}\left(X^k\right)\cdot \\
&\left[\mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^{m-k} - \sum_{i=1}^{m-k}\binom{m-k}{i-1}b_i(F_X)\mathbb{E}\left(\sum_{j=1}^{n-i} X_j\right)^{m-k+1-i} - b_{m-k+1}\right] \\
=&\sum_{k=1}^{m}\binom{m}{k}\mathbb{E}\left(X^k\right)\left(b_{m-k+1} - b_{m-k+1}\right) = 0
\end{aligned}
$$

by (3.20). This completes the induction step and, hence, proves that b_m is independent of $n \geq m$.

In order to prove the representation (3.19) we start from the cases $n = 2$ and $n = 3$, checking the structure of the formula in those cases and then proceed by induction. For $n = 2$ we have

$$
\begin{aligned}
g^{*2}(x) = &\int_0^x g(y)g(x-y)dy = \int_0^x dy \int_y^\infty f(v)dv \int_{x-y}^\infty f(w)dw \\
= &\int\int_{v+w>x} (\min(v,x) - (x-w))\, f(v)f(w)\, dv\, dw \\
= &\int_{v>x} f(v)dv \int_0^\infty w f(w)dw + \int\int_{v\leqslant x,\, v+w>x} (v+w-x) f(v)f(w)\, dv\, dw \\
=&\mathbb{P}(X>x)\mathbb{E}(X) + \mathbb{E}\left[(X_1+X_2-x)\mathbb{1}(X_1+X_2>x)\right] \\
&- \int\int_{v>x} (v+w-x)f(v)f(w)\, dv\, dw \\
=&\mathbb{P}(X>x)\mathbb{E}(X) + \mathbb{E}\left[(X_1+X_2-x)\mathbb{1}(X_1+X_2>x)\right] \\
&- \mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right] - \mathbb{P}(X>x)\mathbb{E}(X) \\
=&\mathbb{E}\left[(X_1+X_2-x)\mathbb{1}(X_1+X_2>x)\right] - \mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right],
\end{aligned}
$$

which coincides with (3.19) for $n = 2$ with $b_1(F_X) = 1$.

For a generic random variable Y with a finite mean consider the function

$$
h_1(x) = \mathbb{E}\left((Y-x)\mathbb{1}(Y>x)\right), \quad x > 0.
$$

Note the appearance of such functions in the above expression for g^{*2}. We proceed with calculating the convolution of this function with g. The notation in the following calculation assumes that X and Y are defined on the same probability space and are independent.

$$\begin{aligned}
g * h_1(x) =& \int_0^x g(y)\mathbb{E}\left((Y-(x-y))\mathbb{1}(Y>x-y)\right)\, dy \\
=& \int_0^\infty \int_0^\infty f_X(v)\, dv f_Y(w)\, dw \int_0^\infty (w-x+y)\mathbb{1}(x-w \leqslant y \leqslant \min(x,v))\, dy \\
=& \frac{1}{2}\int\int_{v+w>x} (\min(w, v+w-x))^2 f_X(v) f_Y(w)\, dv\, dw \\
=& \frac{1}{2}\mathbb{P}(X>x)\mathbb{E}(Y^2) + \frac{1}{2}\mathbb{E}\left((X+Y-x)^2\mathbb{1}(X+Y>x)\right) \\
& - \frac{1}{2}\int\int_{v>x} (v+w-x)^2 f_X(v) f_Y(w)\, dv\, dw \\
=& \frac{1}{2}\mathbb{E}\left((X+Y-x)^2\mathbb{1}(X+Y>x)\right) - \frac{1}{2}\mathbb{E}\left[(X-x)^2\mathbb{1}(X>x)\right] \\
& - \mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right]\mathbb{E}(Y),
\end{aligned}$$

with the last step following by simple algebraic manipulations.

Applying this result, first with $Y = X_1 + X_2$ and then with $Y = X$, we obtain the following expression for g^{*3}:

$$\begin{aligned}
g^{*3}(x) =& g * g^{*2}(x) \\
=& \frac{1}{2}\mathbb{E}\left[(X_1+X_2+X_3-x)^2\mathbb{1}(X_1+X_2+X_3>x)\right] - \frac{1}{2}\mathbb{E}\left[(X-x)^2\mathbb{1}(X>x)\right] \\
& - 2\mathbb{E}(X)\mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right] - \frac{1}{2}\mathbb{E}\left[(X_1+X_2-x)^2\mathbb{1}(X_1+X_2>x)\right] \\
& + \frac{1}{2}\mathbb{E}\left[(X-x)^2\mathbb{1}(X>x)\right] + \mathbb{E}(X)\mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right] \\
=& \frac{1}{2}\mathbb{E}\left[(X_1+X_2+X_3-x)^2\mathbb{1}(X_1+X_2+X_3>x)\right] \\
& - \frac{1}{2}\mathbb{E}\left[(X_1+X_2-x)^2\mathbb{1}(X_1+X_2>x)\right] - \mathbb{E}(X)\mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right].
\end{aligned}$$

This coincides with (3.19) for $n = 3$ with $b_1(F_X) = 1$, $b_2(F_X) = EX$. Accordingly, we are led to introduce, for a generic random variable Y, and $n \geq 1$, the function

$$h_n(x) = \mathbb{E}\left[(Y-x)^n\mathbb{1}(Y>x)\right], \quad x > 0,$$

and calculate its convolution with g. Once again, in the following calculation we assume that X and Y are defined on the same probability space and are independent.

$$
\begin{aligned}
g * h_n(x) = &\int_0^x g(y)\mathbb{E}\left[(Y-(x-y))^n \mathbb{1}(Y>(x-y))\right] dy \qquad (5.5)\\
=&\frac{1}{n+1}\int\int_{v+w>x} (\min(w, v+w-x))^{n+1} f_X(v) f_Y(w)\, dv\, dw\\
=&\frac{1}{n+1}\mathbb{P}(X>x)\mathbb{E}(Y^{n+1}) + \frac{1}{n+1}\mathbb{E}\left[(X+Y-x)^{n+1}\mathbb{1}(X+Y>x)\right]\\
&-\frac{1}{n+1}\int\int_{v>x} (v+w-x)^{n+1} f_X(v) f_Y(w)\, dv\, dw\\
=&\frac{1}{n+1}\mathbb{P}(X>x)\mathbb{E}(Y^{n+1}) + \frac{1}{n+1}\mathbb{E}\left[(X+Y-x)^{n+1}\mathbb{1}(X+Y>x)\right]\\
&-\frac{1}{n+1}\sum_{j=0}^{n+1}\binom{n+1}{j}\mathbb{E}(Y^{n+1-j})\mathbb{E}\left[(X-x)^j\mathbb{1}(X>x)\right]\\
=&\frac{1}{n+1}\mathbb{E}\left[(X+Y-x)^{n+1}\mathbb{1}(X+Y>x)\right]\\
&-\frac{1}{n+1}\sum_{j=1}^{n+1}\binom{n+1}{j}\mathbb{E}(Y^{n+1-j})\mathbb{E}\left[(X-x)^j\mathbb{1}(X>x)\right].
\end{aligned}
$$

Assume now that the statement (3.19) holds for g^{*k} with all $k \leq n$ for some $n \geq 3$. We will establish the validity of this formula for $k = n+1$. We have by (5.5):

$$
\begin{aligned}
g^{*(n+1)}(x) = &\frac{1}{(n-1)!}\frac{1}{n}\Bigg\{\mathbb{E}\left[\left(\sum_{j=1}^{n+1} X_j - x\right)^n \mathbb{1}\left(\sum_{j=1}^{n+1} X_j > x\right)\right]\\
&-\sum_{i=1}^{n}\binom{n}{i}\mathbb{E}\left[\left(\sum_{j=1}^{n} X_j\right)^{n-i}\right]\mathbb{E}\left[(X-x)^i\mathbb{1}(X>x)\right]\Bigg\}\\
&-\frac{1}{(n-1)!}\sum_{k=1}^{n-1}\binom{n-1}{n-k-1} b_{n-k}(F_X)\frac{1}{k+1}\Bigg\{\mathbb{E}\left[\left(\sum_{j=1}^{k+1} X_j - x\right)^{k+1}\mathbb{1}\left(\sum_{j=1}^{k+1} X_j > x\right)\right]\\
&-\sum_{i=1}^{k+1}\binom{k+1}{i}\mathbb{E}\left[\left(\sum_{j=1}^{k} X_j\right)^{k+1-i}\right]\mathbb{E}\left[(X-x)^i\mathbb{1}(X>x)\right]\Bigg\}.
\end{aligned}
$$

By further simplification, we get

$$\begin{aligned}
g^{*(n+1)}(x) =& \frac{1}{n!}\mathbb{E}\left[\left(\sum_{j=1}^{n+1} X_j - x\right)^n \mathbb{1}\left(\sum_{j=1}^{n+1} X_j > x\right)\right] \\
& - \frac{1}{n!}\sum_{k=2}^{n} \frac{n}{k}\binom{n-1}{n-k} b_{n-k+1}(F_X)\mathbb{E}\left[\left(\sum_{j=1}^{k} X_j - x\right)^k \mathbb{1}\left(\sum_{j=1}^{k} X_j > x\right)\right] \\
& - \sum_{i=2}^{n} \mathbb{E}\left[(X-x)^i \mathbb{1}(X>x)\right] \left[\frac{1}{n!}\binom{n}{i}\mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^{n-i}\right. \\
& \left. -\frac{1}{(n-1)!}\sum_{k=i-1}^{n-1}\binom{n-1}{n-k-1} b_{n-k}(F_X)\frac{1}{k+1}\binom{k+1}{i}\mathbb{E}\left(\sum_{j=1}^{k} X_j\right)^{k+1-i}\right] \\
& + \theta_n(F_X)\mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right] \\
=& \frac{1}{n!}\mathbb{E}\left[\left(\sum_{j=1}^{n+1} X_j - x\right)^n \mathbb{1}\left(\sum_{j=1}^{n+1} X_j > x\right)\right] \\
& - \frac{1}{n!}\sum_{k=2}^{n} \frac{n}{k}\binom{n-1}{n-k} b_{n-k+1}(F_X)\mathbb{E}\left[\left(\sum_{j=1}^{k} X_j - x\right)^k \mathbb{1}\left(\sum_{j=1}^{k} X_j > x\right)\right] \\
& + \theta_n(X)\mathbb{E}\left[(X-x)\mathbb{1}(X>x)\right],
\end{aligned}$$

with the cancellation due to the defining expression (3.20). Here

$$\begin{aligned}
\theta_n(F_X) =& -\frac{n}{n!}\mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^{n-1} \\
& + \frac{1}{(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{n-k-1} b_{n-k}(F_X)\frac{1}{k+1}\mathbb{E}\left(\sum_{j=1}^{k} X_j\right)^k (k+1) \\
=& -\frac{1}{(n-1)!} b_1(F_X),
\end{aligned}$$

once again by the defining Proposition (3.20). Therefore,

$$\begin{aligned}
g^{*(n+1)}(x) =& \frac{1}{n!}\mathbb{E}\left[\left(\sum_{j=1}^{n+1} X_j - x\right)^n \mathbb{1}\left(\sum_{j=1}^{n+1} X_j > x\right)\right] \\
& - \frac{1}{n!}\sum_{i=1}^{n}\binom{n}{n-i} b_{n+1-i}(F_X)\mathbb{E}\left[\left(\sum_{j=1}^{i} X_j - x\right)^i \mathbb{1}\left(\sum_{j=1}^{i} X_j > x\right)\right].
\end{aligned}$$

This completes the induction step.

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