RANDOM VIBRATION OF VEHICLE WITHHYSTERETIC NONLINEAR SUSPENSION UNDER ROAD ROUGHNESS EXCITATION

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**ABSTRACT:** The analysis of random vibration of a vehicle with hysteretic nonlinear suspension under road roughness excitation is a fundamental part of evaluation of a vehicle’s dynamic features and design of its active suspension system. The effective analysis method of random vibration of a vehicle with hysteretic suspension springs is presented based on the pseudo-excitation method and the equivalent linearization technique. A stable and efficient iteration scheme is constructed to obtain the equivalent linearized system of the original nonlinear vehicle system. The power spectral density of the vehicle responses (vertical body acceleration, suspension working space and dynamic tyre load) at different speeds and with different nonlinear levels of hysteretic suspension springs are analysed respectively by the proposed method. It is concluded that hysteretic nonlinear suspension influence the vehicle dynamic characteristic significantly, the frequency-weighted root mean square values at the front and rear suspension, and the vehicle’s centre of gravity are reduced greatly with increasing the nonlinear levels of hysteretic suspension springs, resulting in better ride comfort of the vehicle.

**KEY WORDS**: Hysteretic nonlinear; Suspension; Random vibration; Pseudo-excitation method; Equivalent linearization

1. Introduction

In recent years vehicle vibration induced by a rough road profile, including its measurement and elimination, has received significant interest[1-3]. Random vibration is generated when vehicle traverses a rough road, which not only affects driving safety and ride comfort, but also has a great influence on integrity of components and service life of a vehicle. Therefore it is of great significance for characteristic simulation and structural optimal design to study random vibration of a vehicle under irregular excitation from a road surface.

Generally in the vibration analysis of vehicle systems, suspension elements are modelled as linear springs and dampers for simplicity of analysis. Nevertheless, the force-displacement characteristic of springs exhibit nonlinear behaviour especially of hysteretic type in reality, and modelling of deteriorating hysteretic behaviour is increasingly important. Many models have been developed to describe hysteretic restoring force, such as bilinear hysteresis model, Ramberg-Osgood model and Bouc-Wen hysteresis model[4-6]. The Bouc-Wen auxiliary differential equation model is widely used for its good applicability, straightforward description of various hysteretic characteristics by adjusting hysteresis parameters properly, and simple solution of differential equations of motion[7]. And nonlinear suspension elements such as hysteretic stiffness elements were modelled by the Bouc-Wen model in many literatures [8-12].

In cases where the stiffness coefficient varies with responses, the analysis becomes complicated because the resulting equations of motion involve nonlinearities. The study of random vibration of nonlinear systems has always been a subject of great interest for researchers, for example, Fokker-Planck-Kolmogorov (FPK) equation method[13], stochastic averaging methods[14], equivalent linear method[15], equivalent nonlinear system method[16], Monte Carlo method[17,18] were developed in recent decades.

The equivalent linear method substitutes the original nonlinear system with a linear system according to a criterion and can solve the original nonlinear system approximately, which is applied more extensively in engineering. Nevertheless, when the equivalent linearization technique is used to analyse the random vibration of a vehicle with nonlinear hysteretic suspension, the Lyapunov differential equations always need to be solved after linearisation, and the high order covariance matrices of displacement, velocity and hysteretic displacement of each degree of freedom (dof) must be computed, resulting in a more complicated computing process and a higher computing cost consumed in the linearization iteration procedures.

As a highly efficient and accurate algorithm, the pseudo-excitation method (PEM) has been developed to analyse the random vibration of linear time-invariant system in recent two decades, which transforms a stationary random vibration analysis into a deterministic harmonic analysis and has the wider applicability in engineering fields[19-22]. An approach combining the PEM with the equivalent linearization technique is developed to realize the nonlinear random vibration analysis of a vehicle with hysteretic suspension springs in this paper. It is shown that the covariance responses of a small number of dofs need to be computed as the solution of Lyapunov equations is replaced by the PEM, and an appropriate algorithm can be established to achieve equivalent linearization iterative solution for random vibration of a nonlinear vehicle system and meanwhile the solving process is greatly simplified.

The main purpose of this paper is to investigate the random vibration response of a half-car modelwith hysteretic nonlinear suspension. The Bouc-Wen model of hysteretic suspension spring and its equivalent linearization model is presented; the dynamic model of a half-car with hysteretic suspension springsis established and its corresponding state space equation is derived; and the general random vibration analysis approach of a vehicle with hysteretic suspension springs is established based on the PEM and the equivalent linearization technique; a sedan is selected for numerical simulation and the power spectral density (PSD) of the vehicle responses (vertical body acceleration, suspension working space and dynamic tyre load) at different speeds and with different nonlinear levels of hysteretic suspension springs are discussed respectively, and the vehicle ride comfort is evaluated in a further discussion. Numerical results show that random vibration analysis of the vehicle with hysteretic nonlinear elements can be carried out effectively using the PEM and the equivalent linearization technique, and hysteretic nonlinear suspension elements influence the vehicle’s dynamic characteristics significantly with the frequency-weighted root mean square (RMS) values of the vehicle body reduced greatly, resulting in better ride comfort of the vehicle.

**2.****The Bouc-Wen Model of Hysteretic Nonlinear Suspension Spring And Its Equivalent Linearization**

The restoring force of a hysteretic nonlinear element can be represented as

|  |  |
| --- | --- |
|  | (1) |

where is restoring force, is the non-hysteretic component, which is a function of displacement and velocity; is the hysteretic component satisfying the following nonlinear differential equation[6]

|  |  |
| --- | --- |
|  | (2) |

where and are parameters controlling the shape of the hysteresis loop; *A* determines the original stiffness of hysteretic displacement; and determine hysteresis strength and stiffness; sets the transitional smoothness from elastic area to plastic area. Selecting these parameters properly, a hysteretic system of different energy consumption level can be expressed by Equation (2).

The hysteretic nonlinear suspension spring in this paper is considered to have a restoring force of the Bouc–Wen type; when and a hysteretic spring subjected to stationary random excitation, the nonlinear differential Equations (2) can be rewritten as

|  |  |
| --- | --- |
|  | (3) |

where and are hysteretic suspension spring parameters; is excitation assumed as a zero mean stationary random process.

An equivalent linear system can be used to represent the original nonlinear system approximately according to a criterion (the mean square value of error, i.e., the difference between nonlinear and linear equations, is minimum) when the hysteretic system is under stationary random load, and the governing differential equation of which is

|  |  |
| --- | --- |
|  | (4) |

where the equivalent damping coefficient and stiffness are[15]

|  |  |
| --- | --- |
|  | (5) |

andwhere is standard deviation of hysteresis displacement, is standard deviation of relative velocity of the node at the two ends of the hysteresis element; is the covariance between hysteresis displacement and relative velocity.

Therefore the dynamic equations of a vehicle can be established according to linear multi-body dynamic theory as the hysteretic suspension spring modelled by the Bouc-Wen model are linearized fromEquations (4) and (5).

**3. Dynamic Model and State Space Equation of Vehicle**



Figure 1. Half-car model with hysteretic suspension springs

A four-dof half-car model is shown in Figure1. As the constitutive relation of the front and rear suspension springs is given by Equation (3), thus the equations of motion of the half-car model can be derived from Lagrange equation as

|  |  |
| --- | --- |
| =  | (6) |
|  |
|  |
|  | (7) |
|  |
|  |
|  | (8) |
|  |
|  | (9) |
|  |
|  | (10) |
|  |
|  |
|  | (11) |
|  |
|  |
| , ,  | (12) |

where and are the front and rear unsprung masses respectively, is the body mass, is the mass moment of inertia of the vehicle about its centre of gravity, and are front and rear suspension damping coefficients, and are the front and rear hysteretic suspension stiffnesses, and are the front and rear tyre stiffnesses,*a* and *b* are the horizontal distances from thefront and rear axles to the centre of gravity.are absolute displacements of the front and rear unsprung masses respectively, is the absolute displacement of the centre of gravity; and are absolute displacements of the front and rear end of the vehicle body; and are hysteretic displacement of the front and rear suspension springs; is angular rotation of the vehicle about its centre of gravity. and are units step road input at front and rear wheel, are thenonlinear level of hysteretic front and rear suspension springs.

The stiffnesses of the front and rear hysteretic suspension springs in the equations of motion are modelled using the Bouc-Wen model according to section 2, and the hysteretic displacements in Equations (6) -(11) satisfy the following differential equations of equivalent linear system

|  |  |
| --- | --- |
|  | (13) |

where and are equivalent damping coefficients corresponding to the front and rear suspension springs, and are equivalent stiffness coefficients of the front and rear suspension respectively, and

|  |  |
| --- | --- |
|  |  |
|  | (14) |

Introducing the state vector

|  |  |
| --- | --- |
|  |  |

where

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| --- | --- |
|  |  |

Combining Equations (6)-(11) and (13), the equations of motion for the vehicle can be rewritten in state space form as

|  |  |
| --- | --- |
|  | (15) |

with

|  |  |
| --- | --- |
| (16) |  |

**4. The PEM based Random Vibration Analysis ofaVehicle withHystereticSuspension Springs**

**4.1. Road roughness**

The displacement PSD (power spectral density) of the road profile is given by [23]

|  |  |
| --- | --- |
|  | (17) |

whereis road irregularity coefficientis spatialfrequency.

Supposing vehicle travels at constant speed Equation (17) can be expressed in time-frequency domain as

|  |  |
| --- | --- |
|  | (18) |

wheredenotes circular frequency.

The amount of road excitation imposed at the vehicle tyres depends on both the road roughness coefficient and vehicle velocity. It is assumed that the rear wheel follows the same path as the front wheel but with a constant time delay so that

|  |  |
| --- | --- |
| ,  | (19) |

**4.2. The PEM of random vibration analysis for vehicle**

Two problems need to be consideredin computational procedure,one is computing efficiency as the conventional linear process requires iterative iteration to compute the equivalent coefficients (parameters in Equation (14)), the other is the coherent effect between the front and rear wheelsas it is assumed that when the vehicle travels at constant speed, the rear wheel follows the same path as the front wheel but with a delay phase.

As the PEM is a highly efficient algorithm for analyzing stochastic vibration of linear system, which transforms a stationary random vibration analysis into a deterministic harmonic analysis, it can easily solve these two problems by constructing the pseudo-excitation as Equation (20) and analyzing the corresponding pseudo-response, and can realize spectrum analysis by vector multiplication calculation of the pseudo-response. Moreover, the PSM can deal with the coherent stochastic loads of the system by constructing proper pseudo-excitation. Combined with the PSM and the equivalent technique, the detailed prodcedure of the stochastic vibration analysis of the nonlinear vehicle system are as follows.

Introducing the pseudo-excitation[19]

|  |  |
| --- | --- |
|  | (20) |

here is a constant phase between the road excitation on the front and rear tyres considering the travelling wave effect, and the phase vector can be expressed as

|  |  |
| --- | --- |
|  | (21) |

Substituting Equation (20) into Equation (15)

|  |  |
| --- | --- |
|  | (22) |

The stable solution of the vehicle system under pseudo harmonic excitation can be obtained as

|  |  |
| --- | --- |
|  | (23) |

For any vehicle response the auto spectrum and cross spectrum based on the PEM are given by

|  |  |
| --- | --- |
|  | (24) |

Accordingly the variance and covariance of the vehicle responses can be expressed as

|  |  |
| --- | --- |
|  | (25) |

**4.3. The equivalent linearization iterative procedures**

The basic idea of the equivalent linearization technique is the mean square value of error, i.e., the difference between the nonlinear and linear equations is minimum, therefore the equivalent linear system approximates the original nonlinear system step by step in terms of a iteration scheme[24]. Based on the foregoing discussion, the implementation process of the random vibration analysis of a vehicle with hysteretic suspension springs using the PEM and the equivalent linearization technique are as follows:

(1) Suppose the initial equivalent coefficient in Equation (13) and the coefficient matrix in Equation (15) can be obtained from Equation (16), and then the initial linear system can be established.

(2)Construct the pseudo-excitation from Equation (20), Equations (22)-(24) are employed to determine the stationary random vibrationpower spectrum of the equivalent linear system, and variance and covariance of the vehicle responses could be calculated from Equation (25).

(3)Calculate the equivalent coefficient Equation (14), and then calculate the coefficient matrix in Equation (15) again.

(4) The norm of coefficients vector and are computed respectively, and the convergence error is defined as

|  |  |
| --- | --- |
|  | (26) |

(5) If (a very small positive number, replace, and iterate the steps (1) to (4) or else stop iteration and output the computing results.

In addition, the following iterativescheme is applied to deal with the problem as the iteration results cannot obtain the stable solution in some condition[16]

|  |  |
| --- | --- |
|  | (27) |

where is iteration implicit function.

**4.4. Vehicle ride comfort analysis**

The ISO2631-1: 1997(E) standard is adopted to evaluate the ride comfort of a vehicle, and the frequency-weighted RMS of body acceleration response can be expressed as

|  |  |
| --- | --- |
|  | (28) |

where*f* denotes the frequency in Hz; is the single-sided PSD of the vertical body acceleration response with ; is the frequency-weighted function satisfying

|  |  |
| --- | --- |
|  | (29) |

**5. Numerical Results**

The main parameters of the half-car model are from Ford Granada sedan[25]. , , , , , , , , , ,. As mentioned earlier in Equation(2), *A* determines the original stiffness of hysteretic displacement, and are parameters controlling the shape of the hysteresis loop. The essential experiment study aimed at the hysteretic suspension system must be carried out to obtain accurate parameters of the Bouc-Wen model. In this paper, the parameters of Bouc-Wen model are refered to Ref.[8] and are assumed to be , , .

The PSD curves of the vehicle responses (vertical body acceleration, suspension working space and dynamic tyre load) at different speeds with different nonlinear levels of hysteretic suspension springs are discussed respectively, the RMS values of the vehicle responses, the frequency- weighted RMS values of the front and rear suspensions, and the vehicle’s centre of gravity are also calculated separately.

5.1. **Vehicle model with linear suspension springs at different speeds**

Firstly, the random vibration responses of the half-car model with linear suspensionat , 20 and are computed. and in Equations (6)-(11) equal to 1.0. The road roughness coefficient is selected. Figure 2 illustrates the PSD curves of vertical body acceleration, suspension working space and dynamic tyre load at different speeds. Table 1 is the main PSD peak frequencies of the vehicle responses with the PEM and the conventional randon vibration analyais method in Ref.[25], it can be seen that the resonant frequencies of the vehicle body and the bounce frequencies of the front and rear wheelswith the two methods are remarkable close, which also verified the correctness of the PEM.Meanwhile, as the PEM transforms a stationary random vibration analysis of time invariant linear system into a deterministic harmonic analysis, no matter how complex the road excitation is, the PSD of the vehicle responses could be obtained provided with the PSD of the surface roughness, resulting in less computation and high effective random vibration analysis of vehicle.

Table 1. PSD peak frequencies of the vehicle responses (Hz)

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | Front body | Rear body | Front wheel | Rear wheel |
| PEM | 1.04 | 1.30 | 10.45 | 10.70 |
| Ref.[25] | 1.01 | 1.27 | 10.43 | 10.88 |

|  |  |
| --- | --- |
| C:\Users\klxtd\Desktop\Graph1.tif(a) | C:\Users\klxtd\Desktop\Graph2.tif(b) |
| C:\Users\klxtd\Desktop\Graph1.tif(c) | M:\hcr2015\2016论文工作(O)\2016-6非线性悬架随机振动分析\图形处理2016-07-05\线性\后悬SWS.tif(d) |
| C:\Users\klxtd\Desktop\Graph1.tif(e) | C:\Users\klxtd\Desktop\Graph1.tif(f) |
| Figure 2. PSD of vehicle responses: (a, b) vertical body acceleration (BA) at front and rear suspension; (c, d) front and rear suspension working spaces(SWS); (e, f) front and rear dynamic tyre loads (DTL) (the black, red and blue line corresponding to, 20 , respectively) |

Table 2 gives the RMS values of the vehicle responses,it shows that the RMS values of vertical body acceleration, suspension working space and dynamic tyre load all increase with vehicle speeds. Similarly the frequency-weighted RMS values of vertical acceleration at the front and rear suspension and the vehicle’scentre of gravity all increase as the vehicle speed increases.

Table2. RMS values of linear vehicle’s responses

|  |  |  |  |
| --- | --- | --- | --- |
| Speed(m/s) | Body acceleration (m/s2) | Suspension working space (mm) | Dynamic tyre load (kN) |
| Front  | Rear | Front | Rear | Front | Rear |
| 15 | 1.411 | 1.821 | 14.01 | 13.50 | 0.925 | 0.969 |
| 20 | 1.634 | 2.103 | 16.29 | 15.56 | 1.080 | 1.130 |
| 30 | 1.988 | 2.555 | 20.09 | 18.94 | 1.336 | 1.394 |

5.2. **Vehicle model with different nonlinear levels of hysteretic suspension springs**

The vehicle speed is now selected as , and the road roughness coefficient . The nonlinear level of the front and rear hystereric suspension springs are the same and areset as 0.9, 0.7, 0.5. The random vibration responses of the vehicle with hysteretic suspension springs are investigated. Figure 3 depicts the PSD curves of vertical body acceleration, suspension working space and dynamic tyre load for vehicle with different nonlinear values of hysteretic suspension springs. It shows that the hysteretic nonlinar suspension influences the vehicle dynamic responses greatly, especially in the frequency range close to the bounce frequencies of the vehicle body, as the nonlinear level increases, the front and rear vertical body acceleration are decreased significantly compared with the same kinds of responses of the vehicle with linear suspension springs; it can be seen from Figure 3 (d) that the suspension working space of the rear suspension are decreased to some degree, whileFigure 3 (c) and (e) indicates that the working space and dynamic tyre load of the front suspension are decreased firstly, and increased, and then decreased sharply with the increase of nonlinear level of hysteretic suspension springs.

It also shows from Table 3 that the bounce frequencies of the vehicle body attenuated greatly with increasing the nonlinear level of hysteretic suspension springs, and the bounce frequencies of the front and rear wheels change slightly.

Table 3. PSD peak frequencies of the vehicle responses (Hz)

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | Front body | Rear body | Front wheel | Rear wheel |
| 1.0 | 1.04 | 1.30 | 10.45 | 10.70 |
| 0.9 | 0.90 | 1.12 | 10.48 | 10.72 |
| 0.7 | 0.48 | 0.61 | 10.55 | 10.79 |
| 0.5 | 0.50 | 0.64 | 10.63 | 10.88 |

|  |  |
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| M:\hcr2015\2016论文工作(O)\2016-6悬架非线性\图形处理2016-07-05\NewFig-0708\NewFig-0708\SWS1.tif(e) | M:\hcr2015\2016论文工作(O)\2016-6悬架非线性\图形处理2016-07-05\NewFig-0708\NewFig-0708\SWS2.tif(f) |
| Figure 3. PSD of vehicle responses: (a, b) vertical body acceleration (BA) at front and rear suspension; (c, d) front and rear suspension working spaces(SWS); (e, f) front and rear dynamic tyre loads(DTL) (the purple, blue, red and black line corresponding to ==1.0,0.9,0.7,0.5 respectively) |

Table 4 gives the RMS values of the vehicle responses with different nonlinear levels of hysteretic suspension springs. As can be seen, the RMS values of the vehicle responses are more or less reducedas the nonlinear levels of hysteretic suspension springsincreases except that the RMS value of the front suspension working space is increased slightly when It is observed from Table5 that the frequency-weighted RMS values at the front and rear suspension, and the vehicle’s centre of gravity are reducedapparently with the increase of the nonlinear levels of hysteretic suspension springs, resulting in better vehicle ride comfort.

Table 4. RMS values of nonlinear vehicle’s responses

|  |  |  |  |
| --- | --- | --- | --- |
|  | Body acceleration (m/s2) | Suspension working space (mm) | Dynamic tyre load (kN) |
| Front  | Rear | Front | Rear | Front | Rear |
| 0.9 | 1.537 | 1.927 | 16.18 | 15.29 | 1.077 | 1.124 |
| 0.7 | 1.408 | 1.696 | 16.30 | 15.07 | 1.072 | 1.116 |
| 0.5 | 1.368 | 1.630 | 11.11 | 10.12 | 1.038 | 1.070 |

 Table5. Frequency-weighted RMS values of body acceleration (m/s2)

|  |  |  |
| --- | --- | --- |
| Position | Linear | Nonlinear() |
| 0.9 | Reduction | 0.7 | Reduction  | 0.5 | Reduction |
| At front suspension | 1.411 | 1.366 | 3.19% | 1.304 | 7.58% | 1.284 | 9.00% |
| At rear suspension | 1.779 | 1.692 | 4.89% | 1.579 | 11.24% | 1.545 | 13.15% |
| Center of gravity | 1.040 | 0.988 | 5.00% | 0.919 | 11.63% | 0.899 | 13.56% |

**6. Conclusions**

The pseudo-excitation method is an effective method to analyse the random vibration of a system, which transforms a stationary random vibration analysis of time invariant linear system into a deterministic harmonic analysis. The random vibration analysis on a half-car model with hysteretic suspension springs under road irregularity is made using the PEM and the equivalent linearization technique.

(1)The hysteretic characteristic of the nonlinear suspension spring is represented by the Bouc-Wen model, and a general random vibration analysis of a vehicle with hysteretic nonlinear suspension springs is carried out.

(2)The random vibration of a half-car model is analysed, and the PSD of the vehicle responses (vertical body acceleration, suspension working space and dynamic tyre load) at different speeds and with different nonlinear levels of hysteretic suspension springs are discussed, and the vehicle ride comfort is also evaluated. The numerical results show that the PSD of a road surface can be used to evaluate the PSD of vehicle responses accurately and efficiently.

(3)Hysteretic nonlinear suspension springs influence the vehicle dynamic responses significantly, and the frequency-weighted RMS values of the front and rear body, and the vehicle’s center of gravity are reduced remarkably with increasing the nonlinear levels of hysteretic suspension springs, resulting in better ride comfort of the vehicle.

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