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Stochastic Risk Processes Applied to Insurance Capital Recovery Methods

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To my Nan and Grandad

Abstract

Over recent decades, insurance and financial industries have been affected by the volatility of economic cycles. A severe financial crisis struck the market in the year 2000 and subsequently between 2007 and 2012. During these economic downturns, financial businesses (including insurance companies) experienced technical bankruptcy due to insufficient capital holdings. Therefore, the private sector and, in some cases, national governments were called upon to provide a means of recovery, in terms of capital, since their bankruptcy would cause a serious threat to the economy and community as a whole. In response to this adverse environment, governments and regulators have since drawn up stringent rules and regulations, within the insurance industry, to provide a more prudent risk assessment and, in turn, minimise the possibility of future bankruptcy. These regulations are usually known as 'directives' and have been implemented across the EU, USA, Australia and China, among others.

One of the most efficiently employed capital recovery methods, used in practice, is the provision of capital injections. This injection of capital is usually sourced from a companies shareholders (as long as it is profitable for them to do so) or, in some extreme cases, by the national government. Throughout the majority of this thesis, we employ the classical continuous-time risk model to analyse the financial impact of capital injections under the regulatory constraints of Solvency II and, further, by capturing the realistic procedure of financial and administrative processing linked to raising such funds, consider the risk exposure during the delay between requesting and receiving a capital injection.

In the final chapter, we move to a discrete-time setting and discuss alternative capital recovery methods for a different line of business. In this case, where we consider pharmaceutical and petroleum businesses, the classic insurance risk model of the previous chapters is unsuitable and the so-called dual risk model is analysed. Moreover, it is believed that the fall into deficit (bankruptcy) can be recovered within a given time period from normal trading strategies. That is, capital injections are not required and the company can recover from deficit without financial assistance.

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List of Publications

Parts of this thesis are presented in the following papers, published in peer reviewed journals:

- 2018. Ramsden, L., and Papaioannou, A. D. Ruin probabilities under Solvency II constraints. *Insurance: Mathematics and Economics.* (Under Review)

- 2018. Ramsden, L., and Papaioannou, A. D. On the time in red for a risk process with dependent delayed capital injections. *Journal of Applied Probability.* (Under Review)

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In addition, the following paper has been published during my PhD studies, the content of which is not included in this thesis:

- 2017. Ramsden, L., and Papaioannou, A. D. Asymptotic results for a Markov-modulated risk process with stochastic investment. *Journal of Computational and Applied Mathematics*, 313, 38-53.

Contents

List of Figures

List of Tables

Abbreviations and Notation

c.d.f.	Cumulative distribution function
p.d.f.	Probability density function
d.f.	Distribution function
p.m.f.	Probability mass function
m.g.f.	Moment generating function
p.g.f.	Probability generating function
c.g.f.	Cumulant generating function
w.r.t.	With respect to
i.i.d.	Independent and identically distributed
a.s.	Almost surely
l.h.s.	Left hand side
r.h.s.	Right hand side
IDE	Integro-differential equation
ODE	Ordinary differential equation
SDE	Stochastic differential equation
LT	Laplace transfrom
CTMC	Continuous time Markov chain
$\mathbb{R}$	Set of Real numbers
$\mathbb{R}^+$	Set of strictly positive Real numbers
$\mathbb{N}$	Set of non-negative integers
$\mathbb{N}^+$	Set of strictly positive integers
$\mathbb{I}_{\{A\}}$	Indicator function of a set A
$g(x) = o\big(f(x)\big)$	$f(x) \to 0$ such that $g(x)/f(x) \to 0$ as $x \to \infty$
$g(x) \sim f(x)$	$f(x)/g(x) \to 1$ as $x \to \infty$

$\mathbf{A}^\top$	Transpose of a matrix $\mathbf{A}$
$\det[\mathbf{A}]$	Determinant of the square matrix $\mathbf{A}$
$\mathbf{A}^{-1}$	Inverse of the square matrix $\mathbf{A}$
$F_X(\cdot)$	Cumulative distribution function of X
$f_X(\cdot)$	Probability density function of X
$\overline{F}_X(\cdot)$	Tail function of X
$F_X^s(\cdot)$	Integrated tail function of X
$M_X(\cdot)$	Moment generating function of X
$P_X(\cdot)$	Probability generating function of X
$\widehat{f}(s)$	The Laplace transform of the function $f(x)$ with argument $s \in \mathbb{C}$.
$\mathcal{F}$	Filtration of a stochastic process
$\mathbb{P}(\cdot)$	Probability measure $\mathbb{P} : \mathcal{F}_X \to [0,1]$
$\mathbb{E}(\cdot)$	Expectation operator w.r.t. the measure $\mathbb{P}$, $\mathbb{E}(\cdot) = \int_\Omega x d\mathbb{P}(x)$
$\mu^{(k)}$	k-th moment of a random variable

Chapter 1

An Overview of Risk and Ruin Theory

'Risks' are defined as uncertain events or conditions that, if they occur, have a negative effect on at least one objective and are broadly categorised into four groups: 'Preventable' risks, which are within an individuals power to stop; 'Reducible' risks, the chances of which can be greatly reduced by intervention; 'Avoidable' risks, occurring from situations that one could stay away from and 'Unforeseeable' risks, for which it is out of anybody's power to minimise or prevent. The presence of risks in our daily lives has prompted us to consider ways to mitigate against potential losses and is the cause for development of insurance markets.

The introduction of insurance - the first form of which dates back to Chinese and Babylonian traders in the 3rd and 2nd millennia BC, respectively, where merchants travelling treacherous river rapids would redistribute their wares across many vessels to limit the loss due to any single vessel capsizing - has since seen the emergence of competitive, global insurance markets. In such a vast market, competitive premium pricing is required, whilst ensuring (with maximum probability) that the company stays solvent, to attract customers and increase revenue. Due to the volatile, and perceivably random nature of insurance, there are many risks that are inherent within an insurance firm, which need to be considered when it comes to pricing an individual's policy or managing their entire portfolio, such as: the probability of claim occurrences; the corresponding size of such claims and the financial market as a whole, to name a few. The mathematical area known as 'Risk Theory' has thus been developed in an attempt to quantify these risks, among others, by combining probability theory, statistics, stochastic processes and mathematical finance, as a study of designing and managing the potential liabilities facing a risk enterprise (usually an insurance company). The main goal is to provide a comprehensive understanding of the risks associated with the insurance sector and produce methods to facilitate against potential losses.

Risk theory has been one of the most studied research areas within actuarial science since the beginning of the 20th century due to the emergence of Swedish actuary Filip Lundberg, who established its building blocks, and Harald Cramér who adapted the theory to the study of general stochastic processes. The main purpose of the theory is to analyse the cash flow of an insurance company (over time) and evaluate how the arrival times and claim sizes may affect its surplus. At the core of the work developed by Lundberg (1903) and Cramér (1930) lies a risk model - known as the *Cramér-Lundberg* risk model - defining the evolution of an insurer's surplus, which takes on a rather simple form. Although the model is fundamental in a practical sense, it captures the main features of an insurance business, in a way that has aided a library of extensive mathematical results and provided a foundation for further generalisations. In this classical model, the cash flow of an insurance business assumes income is received, via premiums, at some constant rate, whilst the liabilities (claims) are modelled by random losses which occur at random times. Although this classic model describes the fundamental characteristics of an insurance firms cash flow, in reality, there are many other factors to be considered. Nevertheless, the basic principles and techniques used in the classic risk model are adopted in practise to provide a 'first good view' of the risk insights of a non-life insurance portfolio.

Within the Cramér-Lundberg risk model, one of the key features is that the company is forced to cease trading in the event of negative surplus or, equivalently, when in a deficit. In reality, this assumption is far from valid. Firstly, there exist solvency based regulations which require the company to hold, in excess of some, fixed positive level of capital, at all times, as a means of protection for the policyholders. Moreover, there are many financial instruments and capital raising techniques that allow a company to recover from adverse financial situations, which, if successful, allow the company to continue trading. The focus of this thesis is to generalise the Cramér-Lundberg risk model by incorporating well known capital recovery techniques, used in practice, and investigate their impact on the surplus of an insurance firm and overall solvency position.

In the remainder of this chapter, we will introduce the basic stochastic processes used for modelling an insurer's liabilities and consequently describe the aforementioned Cramér-Lundberg risk model. Furthermore, a survey of known results, tools, basic methodologies and classical/advanced techniques, that form the foundations of the work in this thesis, will be provided. In more detail, we will discuss methods for deriving integro-differential equations (IDEs) and Integral equations for some risk quantities, along with methods to obtain explicit solutions and approximations.

1.1 Introduction to insurance risk models

As with any other financial business, an insurance firms solvency heavily depends on the difference between its income and liabilities (cashflow). In practice, (gross) income is received at fixed times via policyholder premiums, bonds and other equity. On the other hand, the liabilities usually consist of operational costs, which are known and paid at fixed times, and random claim amounts appearing at random times. From this basic insurance structure, it is clear that modelling the surplus of an insurance firm first requires the modelling of the claim arrivals and their corresponding amounts, which are established by the use of stochastic processes.

1.1.1 The arrival of claims via stochastic processes

The randomness of claim arrivals - which can be described by a so-called arrival process - and claim sizes, represent the most basic risks facing an insurance business and are the most important aspects of the classical risk model.

Definition 1 (Arrival process)**.** *An arrival process, $\{\sigma_n\}_{n\in\mathbb{N}}$, is a sequence of increasing random variables $0 = \sigma_0 < \sigma_1 < \sigma_2 < \ldots$, where $\sigma_{i-1} < \sigma_i$ means that the sequence $\{\tau_n\}_{n\in\mathbb{N}}$, such that $\tau_i = \sigma_i - \sigma_{i-1}$, are positive random variables. The process $\{\sigma_n\}_{n\in\mathbb{N}}$*

is commonly known as a point process.

The point process, $\{\sigma_n\}_{n\in\mathbb{N}}$, is a sequence of random variables denoting the time of the n-th claim and is commonly referred to as the sequence of claim epochs, with $\{\tau_n\}_{n\in\mathbb{N}}$ a sequence of random variables denoting the *inter-arrival* time between the $(n-1)$-th and n-th claims. An alternative definition of the arrival process is given by the corresponding 'counting process', denoted $\{N(t)\}_{t\geqslant 0}$, which represents the number of claims up to some deterministic time $t \geqslant 0$, which takes the form

$$N(t) = \sum_{n=0}^{\infty} \mathbb{I}_{\{\sigma_n \leqslant t\}},$$

where $\mathbb{I}_{\{\cdot\}}$ denotes the indicator function, such that

$$\mathbb{I}_{\{A\}}(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$$

Definition 2 (Counting process)**.** *A counting process, $\{N(t)\}_{t\geqslant 0}$, is a stochastic process with values that are positive, integer valued and increasing. That is, for all $t \geqslant 0$, we have*

1. $N(t) \geqslant 0$,
2. $N(t) \in \mathbb{N}$,
3. *If* $s \leqslant t$ *then* $N(s) \leqslant N(t)$.

The definition of an arrival process is rather general and does not provide any constraint on the distributions of the arrival times or, respectively, the inter-arrival times. Such a framework is too broad and would prove almost impossible to implement. If, however, the sequence of inter-arrival times, $\{\tau_n\}_{n\in\mathbb{N}}$ is assumed to be a sequence of independent and identically distributed (i.i.d.) random variables, then the arrival process is known as a 'renewal process' and we are able to exploit its so-called 'renewal property'. In particular, if the inter-arrival time random variables follow an exponential distribution with parameter $\lambda > 0$, the renewal process is known as a Poisson process - the name is given due to the distributional characteristics of the corresponding counting process $\{N(t)\}_{t\geqslant 0}$.

Definition 3 (Poisson process). *The counting process $\{N(t)\}_{t\geqslant 0}$ is called a homogeneous Poisson process, with intensity $\lambda > 0$, if the following conditions hold:*

1. *$N(0) = 0$ almost surely (a.s.),*
2. *$\{N(t)\}_{t\geqslant 0}$ has independent and stationary increments,*
3. *For some small time interval $h > 0$, we have:*

$$\begin{aligned}
&\mathbb{P}(N(h) = 0) \approx 1 - \lambda h + o(h),\\
&\mathbb{P}(N(h) = 1) \approx \lambda h + o(h),\\
&\mathbb{P}(N(h) \geqslant 2) \approx o(h),
\end{aligned}$$

where $o(h)$ is a function of $h > 0$, such that $\lim_{h\to 0} \frac{o(h)}{h} = 0$, which implies that $o(h)$ converges to zero more rapidly than h itself.

The claims arriving via a Poisson process is the most common assumption made throughout the actuarial literature and forms the basis of the vital characteristics in the aforementioned Cramér-Lundberg risk model. That is, the total liability of an insurance firm, up to time $t \geqslant 0$, known as the so-called aggregate claims, is modelled as a compound Poisson process.

Definition 4 (Compound Poisson process). *Assume $\{N(t)\}_{t\geqslant 0}$ is a Poisson process with parameter $\lambda > 0$. Further, let $\{X_k\}_{k\in\mathbb{N}^+}$ be a sequence of i.i.d. random variables denoting the size of the k-th claim with common cumulative distribution function (c.d.f.) $F_X(\cdot)$, corresponding probability density function (p.d.f.) $f_X(\cdot)$ and mean $\mu = \mathbb{E}(X) < \infty$, such that $\{X_k\}_{k\in\mathbb{N}^+}$ is independent of $\{N(t)\}_{t\geqslant 0}$. Then, the aggregate claims up to time $t \geqslant 0$, denoted by $\{S(t)\}_{t\geqslant 0}$, is given by*

$$S(t) = \sum_{i=1}^{\infty} X_i \, \mathbb{I}_{\{\sigma_i \leqslant t\}} = \sum_{i=1}^{N(t)} X_i, \tag{1.1.1}$$

and is known as a compound Poisson process.

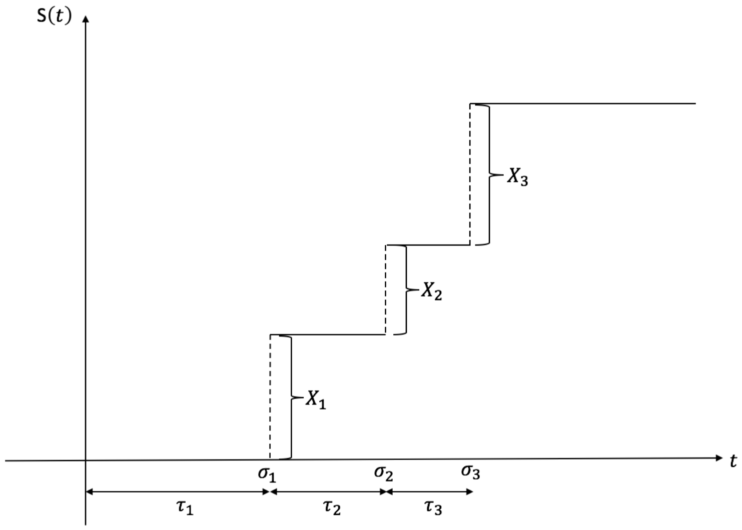


Figure 1.1: Example sample path of the compound Poisson process $S(t)$.

1.1.2 Surplus process for insurance portfolios

Now that the liabilities of an insurer have been defined, using stochastic processes (above), we are in a position to define the entire surplus process (cashflow) in the Cramér-Lundberg risk model.

Definition 5 (Surplus process). *The surplus process of an insurer in the Cramér-Lundberg risk model, denoted by* $\{U(t)\}_{t\geqslant 0}$, *is defined by*

$$U(t) = u + ct - S(t), \qquad U(0) = u, \tag{1.1.2}$$

where $u \geqslant 0$ *represents the insurer's initial capital reserve,* $c > 0$ *is the continuously received premium rate and* $\{S(t)\}_{t\geqslant 0}$ *is a compound Poisson process denoting the aggregate claims up to time* $t \geqslant 0$.

An example of a typical same path for the surplus process can be seen in Fig:1.2.

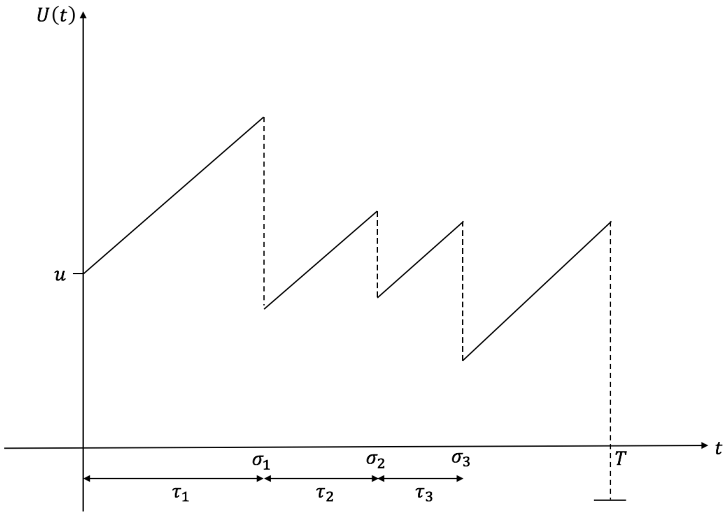


Figure 1.2: Example sample path of the Cramér-Lundberg risk process.

1.2 Ruin probabilities and the integro-differential equation

A crucial requirement for an insurance firm, and other financial businesses, is to remain solvent - in the majority of the risk theory literature, solvency means that the surplus is nonnegative, whereas in practice the threshold is much higher - in order for the company to remain operational and protect their policyholders. If the surplus falls below some pre-determined level (usually zero) in some time interval (finite or infinite), we call this event 'ruin' and the corresponding probability the 'probability of ruin' [see Fig:1.2]. Accurate predictions of such an event, based on current conditions of the business, provides a tool for pricing premiums based on a tolerance to insolvency (ruin). That is, if a company has an insolvency tolerance of 0.5%, they will price premiums such that ruin is acceptable 1 in every 200 years (similar to the current framework within the Solvency II directive implemented throughout the European Union (EU), where Value at Risk is considered with a 99.5% confidence level).

Based on the sample path of the surplus process [see Fig: 1.2], in order to define the probability of ruin, we need first to define the time of ruin.

Definition 6 (Time of ruin)**.** *The time of ruin, denoted by T, is as a non-negative* *random variable, defined as*

$$T = \inf\{t \geqslant 0 : U(t) < 0\}. \tag{1.2.1}$$

Definition 7 (Ruin probability)**.** *For $u \geqslant 0$, the finite-time ruin probability, which is a function of the initial capital, denoted $\psi(u,t)$, is defined by*

$$\psi(u,t) = \mathbb{P}(T < t \big| U(0) = u). \tag{1.2.2}$$

The infinite-time ruin probability, or simply ruin probability, denoted $\psi(u)$, is defined by

$$\psi(u) = \lim_{t\to\infty} \psi(u,t) = \mathbb{P}(T < \infty \big| U(0) = u). \tag{1.2.3}$$

Following from the definition of the ruin probabilities, we define the corresponding finite and infinite-time survival probabilities, i.e. probability of the event $\{T \geqslant t\}$ and $\{T = \infty\}$, respectively, by

$$\phi(u,t) = \mathbb{P}(T \geqslant t \big| U(0) = u) = 1 - \psi(u,t) \tag{1.2.4}$$

and

$$\phi(u) = \mathbb{P}(T = \infty \big| U(0) = u) = 1 - \psi(u), \tag{1.2.5}$$

respectively.

An important criterion for the aforementioned risk model, and business models in general, is that ruin does not occur a.s., i.e. with probability 1. Thus, we want to ensure that the company has a positive cashflow (on average), which is achieved by the assumption of the so called *net profit condition*.

Definition 8 (Net profit condition)**.** *The net profit condition imposes the condition that the continuously received premium rate is greater, per unit time, than the expected losses. That is*

$$c > \lambda\mu.$$

In view of the net profit condition (above), a security loading factor, $\eta > 0$, is implemented to guarantee the premiums of the insurance firm will be sufficient to cover the expected claims per unit time. That is, we define

$$c = (1+\eta)\lambda\mu,$$

where $\eta > 0$ is called the *positive safety loading factor.*

Moreover, if the net profit condition holds, the surplus process, defined in equation (1.1.2), satisfies the following limiting condition which ensures that $\psi(u) < 1$ a.s.

Proposition 1. *Consider the event $\{T = \infty\}$. If the net profit condition holds, then the surplus process $\{U(t)\}_{t\geqslant 0}$ satisfies*

$$\lim_{t\to\infty} U(t) = +\infty.$$

Proof. Consider the limit

$$\begin{aligned}
\lim_{t\to\infty} \frac{U(t)}{t} &= c - \lim_{t\to\infty} \frac{\sum_{i=1}^{N(t)} X_i}{t} \\
&= c - \lim_{t\to\infty} \frac{N(t)}{t} \frac{\sum_{i=1}^{N(t)} X_i}{N(t)} \\
&= c - \lambda\mu,
\end{aligned}$$

where the last equality follows from the limiting behaviour of a renewal process and the strong law of large numbers. Finally, assuming the net profit condition holds, we have $c - \lambda\mu > 0$ and the result follows. □

The infinite-time ruin probability is one of the main measures of interest within the current literature as it provides crucial information about the expected performance of an insurance firm. The most common method, among a few, for obtaining such a quantity is to derive an integro-differential equation (IDE), with respect to $\psi(u)$, and use standard algorithms in an attempt to derive an explicit expression.

Theorem 1 (Integro-differential equation)**.** *Assume the net profit condition holds.*

Then, the ruin probability, $\psi(u)$, satisfies the integro-differential equation

$$c\psi'(u) = \lambda\psi(u) - \lambda\left[\int_0^u \psi(u-x)\, dF_X(x) + \overline{F}_X(u)\right], \tag{1.2.6}$$

along with the boundary condition

$$\lim_{u\to\infty} \psi(u) = 0,$$

where $\overline{F}_X(x) = 1 - F_X(x)$ is the tail distribution of the claim sizes.

The general solution of the IDE, given in equation (1.2.6), does not exist in the actuarial or differential equations literature. Over the years a number of different approaches have been considered in an attempt to derive an explicit solution for the IDE of Theorem 1. The common consensus within each of these approaches is that the IDE is solvable only for some special cases (mainly when the form of the distribution of the claim amounts is known) and therefore, many methods have since been developed to obtain analytic solutions, each time for a wider family of claim size distributions. In the following subsections, we will present a brief outline of the aforementioned methodologies.

1.2.1 Differential approach

The differential approach to solving the IDE, given in equation (1.2.6), is a simple method that works for specific claim size distributions, such as the exponential, Erlang, and mixtures of the two. The idea is that, due to the exponential forms of these distribution functions (d.f.), the integral term on the right hand side (r.h.s.) of equation (1.2.6) can be eliminated by differentiating both sides of the IDE with respect to u and substituting the resulting equation (which will contain an identical integral term) back into equation (1.2.6). This process will result in an ordinary differential equation (ODE) which, along with the boundary conditions of the probability of ruin, can be solved by standard techniques. The simplest case is given when the claim size random variables are exponentially distributed.

Proposition 2 (Exponentially distributed claims)**.** *Let the claim size random variables follow an exponential distribution with parameter $\beta > 0$, i.e. $F_X(x) = 1 - e^{-\beta x}$. Then,* *the ruin probability, $\psi(u)$, is given explicitly by*

$$\psi(u) = \frac{\lambda}{\beta c} e^{-\frac{\lambda \eta}{c} u}, \qquad u \geqslant 0. \tag{1.2.7}$$

1.2.2 Matrix exponential approach

The differential approach described in the previous section is relatively simple, however, solutions can only be obtained via this method for a family of distributions with a very specific form. Furthermore, the method requires individual treatment for all cases and makes the calculations cumbersome.

Asmussen and Rolski (1992) derive a unified approach by considering a claim size distribution that belongs to a wider family of distributions, known as phase-type; a distribution linked to the absorption time of a continuous-time Markov chain (CTMC). This distribution, and thus the results concerned, cover several types of distributions including the exponential, Erlang, Coxian and mixtures of them. The method used to obtain an explicit expression for the ruin probability is known as the exponential matrix technique [see also Neuts (1981)].

Consider a CTMC, $\{X(t)\}_{t \geqslant 0}$, with finite state space, $\mathcal{S}$, consisting of an absorbing state $\mathcal{S}_A = \{0\}$ and transient states $\mathcal{S}_T = \{1, \ldots, n\}$. Furthermore, assume $\{X(t)\}_{t \geqslant 0}$ has an initial probability vector $\boldsymbol{\pi}^* = \{\pi_0, \boldsymbol{\pi}\}$ with $\boldsymbol{\pi} = \{\pi_1, \ldots, \pi_n\}$ (it is usually assumed that $\pi_0 = 0$, i.e. the process can not start in the absorbing state) and infinitesimal generator $\mathbf{Q}$, given by

$$\mathbf{Q} = \left[\begin{array}{c|c} 0 & \vec{\mathbf{0}} \\ \hline \vec{\mathbf{d}} & \mathbf{D} \end{array}\right],$$

where $\vec{\mathbf{0}} = \{0, \ldots, 0\}$, $\vec{\mathbf{d}}$ is an n-dimensional column vector containing the intensity rates from the transient states to the absorbing state and $\mathbf{D}$ is an $n \times n$ sub-intensity matrix containing the intensity rates between transient states.

Definition 9 (Phase-type distribution)**.** *A random variable X is said to have phase-type distribution, $F(x)$, with representation $(\boldsymbol{\pi}, \mathbf{D}, n)$, if it has identical distribution to* *the absorption time of the CTMC,* $\{X(t)\}_{t\geqslant 0}$*. Furthermore, the c.d.f.* $F(x)$*, is given by*

$$F(x) = 1 - \boldsymbol{\pi} e^{\mathbf{D}x} \vec{\mathbf{e}} \quad \textit{for} \quad x \geq 0$$

and its associated p.d.f. by

$$f(x) = \boldsymbol{\pi} e^{\mathbf{D}x} \vec{\mathbf{d}} \quad \textit{for} \quad x \geq 0,$$

where $e^{\mathbf{D}x}$ *is the matrix exponential and* $\vec{\mathbf{e}}$ *is an* n*-dimensional unit column vector.*

Remark 1. *The Erlang distribution, denoted Erlang*(n, β)*, is the distribution of a sum of* n *i.i.d. exponentially distributed random variables with parameter* $\beta > 0$*. The Erlang*(n, β) *distribution can be written in the form of a phase-type distribution by setting* $\mathbf{D}$ *to be an* $n \times n$ *matrix with diagonal elements* $-\beta$*, sub-diagonal elements* β *and the probability of starting in state* n *equal to* 1*, i.e.* $\pi_n = 1$*.*

Using the form of the phase-type d.f., given in Definition 9, Asmussen and Rolski (1992) obtained (by comparison of the ruin problem to queuing theory) the following closed form expression for the probability of ruin under a phase-type claim size distribution.

Theorem 2. *Assume the claim size random variables follow a phase-type distribution with representation* $(\boldsymbol{\pi}, \mathbf{D}, n)$*. Then, for each* $u \geqslant 0$*, the probability of ruin,* $\psi(u)$*, is given by*

$$\psi(u) = \boldsymbol{\pi}_+ e^{\mathbf{T}u} \vec{\mathbf{e}},$$

where $\mathbf{T} = \mathbf{D} + \vec{\mathbf{d}} \boldsymbol{\pi}_+$ *and* $\boldsymbol{\pi}_+ = -\lambda \boldsymbol{\pi} \mathbf{D}^{-1}$*.*

The above result for phase-type distributed claim sizes, as already mentioned, covers a range of distributions. Therefore, Theorem 2 generalises the results of Grandell and Segerdahl (1971) and Thorin (1973) who derived expressions for gamma distributed claims with integer shape parameter (Erlang distributions), which were later generalised by Gerber et al. (1987) who gave a finite series solution for a combination of gamma distributions with integer valued non-scale parameter - for which exponential, Erlang and mixtures of them are special cases.

Recently, the aforementioned results for gamma distributed claim sizes, with integer shape parameter, have been extended in Constantinescu et al. (2017) to the case of general gamma distribution with arbitrary real valued shape parameter. In this work, three equivalent expressions are derived for the probability of ruin by method of Laplace transforms (LTs) and the aid of Mittag Leffler functions.

1.2.3 Laplace transforms for the ruin probability

The main difficulty in deriving explicit expressions from the IDE, given by equation (1.2.6), is the convolution structure of the integral term.

Definition 10 (Convolution)**.** *The convolution of two functions f and g, supported on the interval* $[0,\infty)$*, is defined by*

$$(f * g)(x) = \int_0^x f(x-y)g(y)\,dy.$$

The convolution operation, for d.f's., allows us to compute the distribution of the sum, $X+Y$, of two independent random variables X and Y from their respective distributions F and G and corresponding densities f and g.

The n-fold convolution of a distribution F, defining the distribution of a sum of n i.i.d. random variables with common distribution F, denoted by F^{*n}, is defined iteratively.

Definition 11 (n-fold convolution)**.** *Let $F(x)$ be a d.f. supported on* $[0,\infty)$*. Then, the n-fold convolution of $F(x)$, denoted $F^{*n}(x)$, is defined by*

$$F^{*n}(x) = \left(F^{*(n-1)} * F\right)(x) = (\underbrace{F * \cdots * F}_{n})(x),$$

where $F^{*0}(x) = \mathbb{I}_{\{x\geqslant 0\}}$.

Based on the theory of differential equations, equations involving convolutions are often solved by the use of LTs. Thus, in this subsection we present LT techniques used in the literature to derive a solution for the IDE of Theorem 1.

Definition 12 (Laplace transform)**.** *For $\Re(s) \geqslant 0$, the LTs of the ruin and survival functions are given by*

$$\widehat{\psi}(s) = \int_0^\infty e^{-su}\psi(u)\,du \qquad \text{and} \qquad \widehat{\phi}(s) = \int_0^\infty e^{-su}\phi(u)\,du, \tag{1.2.8}$$

respectively.

Considering the above definition of the LTs for the the ruin quantities, multiplying equation (1.2.6) through by e^{-su}, integrating over the interval $[0, \infty)$ and using the fact (from the LT properties) that the LT of a convolution is simply the product of the LTs, i.e.

$$\int_0^\infty e^{-su}\left(\psi * f_X\right)(u)\,du = \widehat{\psi}(s)\widehat{f}_X(s),$$

where $\widehat{f}_X(s) = \int_0^\infty e^{-su} f_X(u)\,du$ denotes the LT of the density function of the claim size distribution, we have the following theorem.

Theorem 3. *For $\Re(s) \geqslant 0$, the LTs $\widehat{\psi}(s)$ and $\widehat{\phi}(s)$ are given by*

$$\widehat{\psi}(s) = \frac{1}{s} - \frac{c - \lambda\mu}{cs - \lambda\left(1 - \widehat{f}_X(s)\right)}$$

and

$$\widehat{\phi}(s) = \frac{c - \lambda\mu}{cs - \lambda\left(1 - \widehat{f}_X(s)\right)},$$

respectively.

In order to obtain an explicit expression for the ruin (survival) probability, it is necessary to invert the above forms of the corresponding LTs. In general, this inversion proves difficult (even numerically), however, for specific claim size distributions the inversion becomes accessible.

Li and Garrido (2004) show that if the claim size distribution belongs to the family of distributions with a rational LT (i.e. the ratio of two polynomials) which contains a large variety of distributions, then one can employ partial fraction decomposition and Lundberg's fundamental equations (see below), to invert the LTs given in Theorem 3 and obtain an exponential form for the probability of ruin.

Proposition 3. *Let the claim size distribution belong to the family of distributions with a rational LT, i.e.* $\widehat{f}_X(s)$ *is given by*

$$\widehat{f}_X(s) = \frac{Q_{m-1}(s)}{Q_m(s)}, \quad \Re(s) \in (h_X, \infty),$$

with $Q_m(0) = Q_{m-1}(0)$*, for* $m \in \mathbb{N}^+, h_X = \inf\{s \in \mathbb{R} : \mathbb{E}\left(e^{-sX}\right) < \infty\}$*, and* $Q_m(s)$*,* $Q_{m-1}(s)$ *are polynomials of degree* m *and* $m-1$*, respectively. Moreover, define by* $\{-R_i\}_{i=1,\dots,m}$*, with* $\Re(R_i) > 0$*, the (distinct) roots of the Lundberg equation*

$$\left(\frac{\lambda}{c} - s\right) Q_m(s) - \frac{\lambda}{c} Q_{m-1}(s) = 0.$$

Then, the LT of the probability of ruin is given by

$$\widehat{\psi}(s) = \sum_{i=1}^{m} \frac{a_i}{s + R_i},$$

which has an inversion of the form

$$\psi(u) = \sum_{i=1}^{m} a_i e^{-R_i u},$$

where

$$a_i = \frac{Q_m(-R_i)}{Q_m(0)} \prod_{j=1, j \neq i}^{m} \frac{R_j}{(R_j - R_i)}, \quad i = 1, \dots m.$$

1.2.4 The Pollaczek-Khinchin formula

Due to the complexity of the IDE, given by equation (1.2.6), many authors have considered an alternative integral equation. Integrating equation (1.2.6) over the interval $(0, u]$, we arrive at the following integral form for the ruin probability [see Rolski et al. (1999)].

Theorem 4. *The probability of ruin,* $\psi(u)$*, satisfies the following integral equation*

$$\psi(u) = \frac{\lambda}{c} \left(\int_u^\infty \overline{F}_X(x)\, dx + \int_0^u \psi(u - x) \overline{F}_X(x)\, dx \right). \tag{1.2.9}$$

Although this integral form looks simpler than its associated IDE, it remains difficult to solve in a closed form. However, an immediate consequence of the expression given above is the ability to obtain an expression for the ruin probability with zero initial capital, $\psi(0)$. Setting $u = 0$ in equation (1.2.9) yields the following lemma.

Lemma 1. *Assuming the net profit condition holds, the probability of ruin with zero initial capital is given explicitly by*

$$\begin{aligned}\psi(0) &= \frac{\lambda}{c}\int_0^\infty \overline{F}_X(x)\,dy \\ &= \frac{\lambda\mu}{c} < 1.\end{aligned} \tag{1.2.10}$$

For the general case $u \geqslant 0$, we can consider the LT of equation (1.2.9), and use the one-to-one correspondence between functions and their LT, to derive an infinite series solution (in terms of convolutions) for $\psi(u)$. The following theorem is known in risk theory as *Beekman's formula* and is a special case of the *Pollaczek-Khinchin formula* found in queuing theory [see Asmussen (1987)].

Theorem 5 (Pollaczek-Khinchin formula)**.** *For $u \geqslant 0$, the ruin probability, $\psi(u)$, is given by*

$$\psi(u) = \left(1 - \frac{\lambda\mu}{c}\right)\sum_{n=1}^{\infty}\left(\frac{\lambda\mu}{c}\right)^n \overline{(F_X^s)^{*n}}(u), \tag{1.2.11}$$

where

$$F_X^s(x) = \frac{1}{\mu}\int_0^x \overline{F}_X(y)\,dy, \tag{1.2.12}$$

is the integrated-tail distribution of the claim sizes and $\overline{F^{*n}}(x) = 1 - F^{*n}(x)$.

Remark 2. *From the above theorem we see that the probability of ruin, namely $\psi(u)$, is the tail of a compound geometric distribution, with characteristics $\left(\lambda\mu c^{-1}, F_X^s\right)$.*

1.3 Bounds and approximations

As discussed in sections 1.2.1-1.2.4, it is generally difficult to determine an explicit solution for the probability of ruin, $\psi(u)$, from either the IDE, given in equation (1.2.6), or the integral form of equation (1.2.9). Therefore, in this section, we will discuss some of the main results and methodologies used to derive bounds, approximations and asymptotic results, which can be used to analyse the behaviour of the ruin probability in the absence of an explicit expression.

Before we explore these results, let us define a quantity that will play a major role in the derivation of most of the results in the remainder of this section, namely the *adjustment coefficient* or *Lundberg exponent*. Consider the *risk reserve process*, defined by

$$R(t) = \sum_{i=1}^{N(t)} X_i - ct, \tag{1.3.1}$$

which is simply a shifted compound Poisson process and its moment generating function (m.g.f.), denoted by $M_{R(t)}(s)$, is given by

$$M_{R(t)}(s) = e^{t(\lambda(M_X(s)-1)-cs)}, \tag{1.3.2}$$

where $M_X(s) = \mathbb{E}(e^{sX})$ denotes the m.g.f. of the claim sizes. Then, if we define

$$\theta(s) := \lambda(M_X(s) - 1) - cs, \tag{1.3.3}$$

such that the moment generating function $M_{R(t)}(s) = \exp\{t\theta(s)\}$, one can define a useful exponential martingale in terms of the surplus process $\{U(t)\}_{t\geqslant 0}$.

Definition 13 (Martingale)**.** *A real valued stochastic process $\{X(t)\}_{t\geqslant 0}$, adapted to a filtration $\{\mathcal{F}(t)\}_{t\geqslant 0}$, is a martingale if:*

1. $\mathbb{E}\left(\left|X(t)\right|\right) < \infty, \quad \forall t \geqslant 0,$
2. *For all* $0 \leqslant s \leqslant t$, $\mathbb{E}\left(X(t)\middle|\mathcal{F}(s)\right) = X(s) \quad a.s.$

Lemma 2. *Let $s \in \mathbb{R}$ such that $M_X(s) < \infty$. Then, the stochastic process $\{e^{-sU(t)-\theta(s)t}\}_{t\geqslant 0}$ is a martingale.*

Proof. Let $\{\mathcal{F}(\tau)\}_{\tau\geqslant 0}$ be the filtration generated by the process $\{e^{-sU(t)-\theta(s)t}\}_{t\geqslant 0}$.

Then, by using the Markov property, for $\tau \leqslant t$, we have

$$\begin{aligned}
\mathbb{E}\left[e^{-sU(t)-\theta(s)t}\middle|\mathcal{F}(\tau)\right] &= \mathbb{E}\left[e^{-s(U(t)-U(\tau))}\right] e^{-sU(\tau)-\theta(s)t} \\
&= \mathbb{E}\left[e^{s\sum_{i=N(\tau)+1}^{N(t)} X_i}\right] e^{-sU(\tau)-\lambda(M_X(s)-1)t+sc\tau} \\
&= \mathbb{E}\left[e^{s\sum_{i=1}^{N(t-\tau)} X_i}\right] e^{-sU(\tau)-\lambda(M_X(s)-1)t+sc\tau} \\
&= e^{-sU(\tau)-\lambda(M_X(s)-1)\tau+sc\tau} \\
&= e^{-sU(\tau)-\theta(s)\tau}.
\end{aligned}$$

□

The martingale property (above) provides a tool for many interesting results in both the risk and queueing theory settings. However, for simplicity of calculations it would be convenient to have a martingale that does not depend on the function $\theta(s)$ and only on the surplus process $U(t)$. Therefore, we look to eliminate the function $\theta(s)$, and the explicit time dependence, by considering the non-trivial root of the equation $\theta(s) = 0$. Then, since

$$\theta''(s) = \lambda M_X''(s) = \lambda \mathbb{E}\left(X^2 e^{sX}\right) > 0,$$

it follows that $\theta(s)$ is a convex function. For the first derivative, evaluated at $s = 0$, we have

$$\theta'(0) = \lambda M_X'(0) - c = \lambda\mu - c < 0,$$

by the net profit condition and it is easy to see that $\theta(0) = 0$. Therefore, by the above characteristics of the function $\theta(s)$, there may exist a second root of the equation

$$\theta(s) = 0, \tag{1.3.4}$$

known as *Lundberg's fundamental equation.* If such a root exists, then it is unique and strictly positive [see Fig. 1.3]. We call this solution the *adjustment coefficient* or the *Lundberg exponent* and denote it by $\gamma > 0$. The existence of this adjustment coefficient will play an important role in deriving upper and lower bounds, approximations and asymptotic results for the ruin function $\psi(u)$.

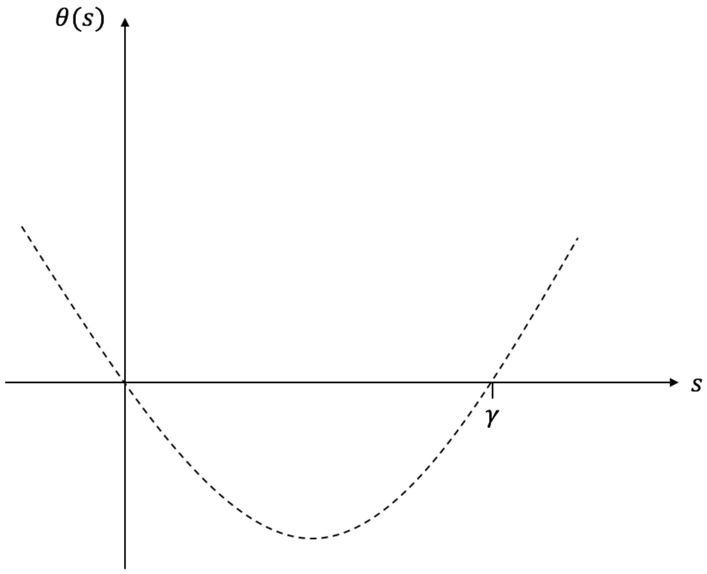


Figure 1.3: Lundberg's fundamental equation, $\theta(s)$.

1.3.1 Lundberg's exponential bound

Let us first look at the importance of the adjustment coefficient in determining an upper bound for the probability of ruin. The first derivation of an upper bound for the ruin probability was given in Lundberg (1926), where he used the method of Weiner-Hopf factorisation. However, over the years, many alternative proofs have been established providing more elegant and simple derivations, such as the martingale approach of Gerber (1979) and an exponential change of measure [see Asmussen and Albrecher (2010)], to name a few. In the following fundamental theorem, we give the proof based on an induction argument.

Theorem 6. *Assume the adjustment coefficient, $\gamma > 0$, exists. Then, for $u \geqslant 0$, we have*

$$\psi(u) \leqslant e^{-\gamma u}. \tag{1.3.5}$$

Proof. Let us denote by $\psi_n(u)$, the probability that ruin occurs on or before the n-th claim. Then, since $\lim_{n\to\infty} \psi_n(u) = \psi(u)$, it suffices to prove that $\psi_n(u) \leqslant e^{-\gamma u}$, for all $n = 0, 1, \ldots$. Firstly, for $n = 0$, we have $\psi_0(u) = 1$ when $u < 0$ and $\psi_0(u) = 0$ for $u \geqslant 0$

hence, $\psi_0(u) \leqslant e^{-\gamma u}$. Now, for the non-trivial part of the induction step, we assume true for $n-1$ and, by the renewal property, we have $\psi_n(u) = \mathbb{E}\left(\psi_{n-1}(u + c\sigma_1 - X_1)\right)$ [see Feller (1971)], which is equivalent to

$$\begin{aligned}
\psi_n(u) &= \int_0^\infty \lambda e^{-\lambda t} \int_{-\infty}^\infty \psi_{n-1}(u+ct-x)\, dF_X(x)\, dt \\
&\leqslant \int_0^\infty \lambda e^{-\lambda t} \int_{-\infty}^\infty e^{-\gamma(u+ct-x)}\, dF_X(x)\, dt \\
&= \lambda e^{-\gamma u} \int_{-\infty}^\infty e^{\gamma x} \int_0^\infty e^{-(\lambda+c\gamma)t}\, dt\, dF_X(x) \\
&= \frac{\lambda e^{-\gamma u}}{\lambda+\gamma c} \int_{-\infty}^\infty e^{\gamma x}\, dF_X(x).
\end{aligned}$$

Finally, since $\gamma > 0$ was defined as the solution to Lundberg's fundamental equation, i.e. $\theta(\gamma) = 0$, it follows that $\frac{\lambda}{\lambda+\gamma c}\int_{-\infty}^\infty e^{\gamma x}\, dF_X(x) = 1$, which completes our proof by induction. □

1.3.2 Two-sided Lundberg bounds

In addition to the exponential upper bound, given in Theorem 6, it is possible (assuming the Lundberg coefficient $\gamma > 0$ exists) to derive two-sided bounds for the probability of ruin. The two-sided bounds are obtained from the two-sided bounds for the tail of a compound geometric distribution and employing the compound geometric form of $\psi(u)$, given in equation (1.2.11) [see Rolski et al. (1999)].

Theorem 7. *Assume that the adjustment coefficient, $\gamma > 0$, exists. Then, for $u \geqslant 0$, we have*

$$a_- e^{-\gamma u} \leqslant \psi(u) \leqslant a_+ e^{-\gamma u}, \tag{1.3.6}$$

where

$$a_- = \inf_{x\in[0,x_0)} \frac{e^{\gamma x}\int_x^\infty \overline{F}_X(y)\, dy}{\int_x^\infty e^{\gamma x}\overline{F}_X(y)\, dy} \qquad \textit{and} \qquad a_+ = \sup_{x\in[0,x_0)} \frac{e^{\gamma x}\int_x^\infty \overline{F}_X(y)\, dy}{\int_x^\infty e^{\gamma x}\overline{F}_X(y)\, dy}.$$

1.3.3 De Vylder Approximation

As a means to approximate the ruin function, De Vylder (1978) takes advantage of the explicit expression obtained for $\psi(u)$ under the assumption of exponentially distributed claims sizes, given in equation (2.4.14). The main idea behind this approximation method is to replace the surplus process $\{U(t)\}_{t\geqslant 0}$, having general claim size distribution, with a similar process, denoted $\{U'(t)\}_{t\geqslant 0}$, which has exponentially distributed claim sizes, with parameter $\beta' > 0$, such that the first three moments coincide. Then, the *De Vylder approximation* is given by

$$\psi_{app}(u) = \frac{\lambda'}{c'\beta'} e^{-\frac{\lambda'\eta'}{c'}u},$$

where

$$\beta' = \frac{3\mu^{(2)}}{\mu^{(3)}}, \qquad \lambda' = \frac{\lambda\mu^{(2)}(\beta')^2}{2}, \qquad c' = \beta - \lambda\mu + \frac{\lambda'}{\beta'},$$

with $\mu^{(k)} = \mathbb{E}\left(X^k\right)$ denoting the k-th moment of the claim sizes.

1.3.4 Beekman-Bowers approximation

A second method, based on moment fitting and leading to a more accurate approximation, is the so-called *Beekman-Bowers* approximation [see, for example Beekman (1969)].

Consider the distribution function

$$F(x) = 1 - \frac{c\psi(x)}{\lambda\mu}. \tag{1.3.7}$$

Then, by equation (1.2.10) it follows that $F(0) = 0$ and $F(x)$ is the d.f. of some positive random variable, Z, with moment generating function $M_Z(s)$, defined by

$$M_Z(s) = \int_0^\infty e^{sx} \frac{c}{\lambda\mu} \phi'(x)\, dx.$$

Noting that the moment generating function is simply a LT with negative argument, we can use integration by parts and employ the results of Theorem 3 to obtain an expression for $M_Z(s)$. Then, to obtain an approximation for ruin probability, $\psi(u)$, we first consider an approximation to the d.f. $F(x)$ using a gamma d.f., denoted $F'(x)$, with parameters (a', b'), such that the first two moments coincide. Using the moment generating function $M_Z(s)$, we can obtain explicit expressions for the first two moments of the random variable Z and thus, matching these with the corresponding moments of the gamma distribution and using equation (1.3.7), the *Beekman-Bowers approximation* is given by

$$\psi_{app}(u) = \frac{\lambda\mu}{c}\overline{F'}(u),$$

where $\overline{F'}(x) = 1 - F'(x)$ is the tail of the gamma distribution with parameters (a', b'), such that

$$\frac{a'}{b'} = \frac{c\mu^{(2)}}{2\mu(c-\lambda\mu)}, \qquad \frac{a'(a'+1)}{(b')^2} = \frac{c}{\mu}\left(\frac{\mu^{(3)}}{3(c-\lambda\mu)} + \frac{\lambda\left(\mu^{(2)}\right)^2}{2(c-\lambda\mu)^2}\right).$$

1.4 Asymptotic behaviour of the ruin probability (light and heavy tailed)

In the previous sections, we presented approximations to the ruin function, $\psi(u)$, via the method of moments. The results obtained provide an approximation for each value of the initial capital $u \geqslant 0$, and the accuracy depends on the distribution being observed. In this section, we consider a different method for obtaining an approximation by considering the tail (asymptotic) behaviour of the ruin probability, i.e. as $u \to \infty$. Once the tail has been identified, that is, we have obtained a function, $h(u)$, such that $\psi(u) = h(u)$ for $u > u_0$, we approximate $\psi(u)$ by the function $h(u)$ for all $u \geqslant 0$. Clearly, for this method, the error between the approximated value and the true value of $\psi(u)$ decreases as $u \geqslant 0$ increases.

1.4.1 Cramér-Lundberg approximation

Here we will present arguably the most well known and fundamental method for obtaining an approximation for the ruin probability known as the *Cramér-Lundberg approximation*. This approximation will be based on the use of the general theory of defective renewal equations and more specifically, the Key Renewal Theorem.

Theorem 8 (Key Renewal Theorem)**.** *Assume that a function $z_1 : \mathbb{R}^+ \to (0,\infty)$ is increasing and let $z_2 : \mathbb{R}^+ \to \mathbb{R}^+$ be decreasing, such that*

$$\int_0^\infty z_1(x)z_2(x)\,dx < \infty$$

and

$$\lim_{h\to 0} \sup\left\{\frac{z_1(x+y)}{z_1(x)} : x \geqslant 0, 0 \leqslant y \leqslant h\right\} = 1.$$

Then, $z(x) = z_1(x)z_2(x)$ is directly Riemann integrable and for each proper distribution F on $\mathbb{R}^+$, the equation

$$g(u) = z(u) + \int_0^u g(u-v)\,dF(v), \quad u \geqslant 0,$$

admits a unique locally bounded solution such that

$$\lim_{u\to\infty} g(u) = \begin{cases} \mu_F^{-1} \int_0^\infty z(u)\,du & \mu_F < \infty, \\ 0 & \mu_F = \infty. \end{cases}$$

where μ_F is the mean value of the distribution F.

Theorem 9. *Assume the adjustment coefficient, $\gamma > 0$, exists. If $M'_X(\gamma) < \infty$, then*

$$\lim_{u\to\infty} \psi(u)e^{\gamma u} = \frac{c - \lambda\mu}{\lambda M'_X(\gamma) - c}. \tag{1.4.1}$$

If $M'_X(\gamma) = \infty$, then $\lim_{u\to\infty} \psi(u)e^{\gamma u} = 0$.

Proof. Let us begin by recalling the integral equation for the ruin probability, given in equation (1.2.9), that is

$$\psi(u) = \frac{\lambda}{c}\int_u^\infty \overline{F}_X(x)\,dx + \frac{\lambda}{c}\int_0^u \psi(u-x)\overline{F}_X(x)\,dx. \tag{1.4.2}$$

Now, since $\int_0^\infty \frac{\lambda}{c}\overline{F}_X(x)\,dx = \frac{\lambda\mu}{c} < 1$ (by the net profit condition), the above equation is known as a *defective renewal equation* [see Feller (1971)]. The result is obtained by application of the so called Key Renewal Theorem, given in Theorem 8. However, before the theorem can be applied, the renewal equation (1.4.2) must be transformed into a *proper renewal equation.* To do this, we will follow the method as described in Feller (1971, p. 376). That is, assume there exists a constant, $s > 0$, such that

$$
\begin{aligned}
1 = \frac{\lambda}{c}\int_0^\infty e^{sx}\overline{F}_X(x)\,dx &= \frac{\lambda}{c}\int_0^\infty e^{sx}\int_x^\infty dF_X(y)\,dx \\
&= \frac{\lambda}{c}\int_0^\infty \int_0^y e^{sx}\,dx\,dF_X(y) \\
&= \frac{\lambda}{cs}\int_0^\infty (e^{sy}-1)\,dF_X(y) \\
&= \frac{\lambda(M_X(s)-1)}{cs}.
\end{aligned}
\tag{1.4.3}
$$

This condition is equivalent to finding the nonzero solution to Lundberg's fundamental equation, given by equation (1.3.4), i.e. $\theta(s) = 0$. The only non zero solution to this equation (which we will assume exists), as seen previously, is the adjustment coefficient $\gamma > 0$.

It follows, from the above condition, that the function $\frac{\lambda}{c}e^{\gamma x}\overline{F}_X(x)$ forms the density of a proper probability distribution and thus, multiplication of equation (1.4.2) by $e^{\gamma u}$, yields a proper renewal equation of the form

$$
e^{\gamma u}\psi(u) = \frac{\lambda}{c}e^{\gamma u}\int_u^\infty \overline{F}_X(x)\,dx + \frac{\lambda}{c}\int_0^u \psi(u-x)e^{\gamma(u-x)}\overline{F}_X(x)e^{\gamma x}\,dx. \tag{1.4.4}
$$

Using Theorem 8, it is easy to see that for a finite mean $\mu_F < \infty$, i.e.

$$
\int_0^\infty xe^{\gamma x}\frac{\lambda}{c}\overline{F}_X(x)\,dx < \infty
$$

which is equivalent to $M_X'(\gamma) < \infty$ (this, along with the existence of the root $\gamma > 0$, form the so-called Cramér conditions), equation (1.4.4) admits an asymptotic solution, given by

$$
\lim_{u\to\infty} e^{\gamma u}\psi(u) = \mu_F^{-1}\int_0^\infty e^{\gamma u}\int_u^\infty \frac{\lambda}{c}\overline{F}_X(x)\,dx\,du. \tag{1.4.5}
$$

Finally, it remains to evaluate the r.h.s. of the above equation. Firstly,

$$\begin{aligned}
\int_0^\infty e^{\gamma u}\int_u^\infty \frac{\lambda}{c}\overline{F}_X(x)\,dx\,du &= \frac{\lambda}{c}\int_0^\infty \overline{F}_X(x)\int_0^x e^{\gamma u}\,du\,dx \\
&= \frac{\lambda}{c\gamma}\int_0^\infty (e^{\gamma x}-1)\overline{F}_X(x)dx \\
&= \frac{\lambda}{c\gamma}\int_0^\infty e^{\gamma x}\overline{F}_X(x)dx - \frac{\lambda}{c\gamma}\int_0^\infty \overline{F}_X(x)dx \\
&= \frac{1}{\gamma} - \frac{\lambda\mu}{c\gamma} = \frac{1}{c\gamma}(c-\lambda\mu), \qquad (1.4.6)
\end{aligned}$$

by equation (1.4.3) and the fact that the claim sizes are non-negative random variables. The mean value of the distribution, namely μ_F, is given by

$$\begin{aligned}
\mu_F = \int_0^\infty x\frac{\lambda}{c}\overline{F}_X(x)e^{\gamma x}\,dx &= \frac{d}{d\gamma}\left(\frac{\lambda}{c}\int_0^\infty \overline{F}_X(x)e^{\gamma x}\,dx\right) \\
&= \frac{d}{d\gamma}\left(\frac{\lambda(M_X(\gamma)-1)}{c\gamma}\right) \\
&= \frac{\lambda(M_X'(\gamma)c\gamma - c(M_X(\gamma)-1))}{(c\gamma)^2} \\
&= \frac{\lambda M_X'(\gamma)\gamma - \lambda(M_X(\gamma)-1)}{c\gamma^2} \\
&= \frac{\lambda M_X'(\gamma)\gamma - c\gamma}{c\gamma^2} \\
&= \frac{\lambda M_X'(\gamma) - c}{c\gamma}, \qquad (1.4.7)
\end{aligned}$$

if $M_X'(\gamma) < \infty$, and ∞ otherwise. Finally, by combining equations (1.4.5), (1.4.6) and (1.4.7) we obtain the result. □

Remark 3. *Theorem 9 can be written alternatively in its asymptotic form by*

$$\psi(u) \sim \frac{c-\lambda\mu}{\lambda M_X'(\gamma)-c}e^{-\gamma u},$$

and thus, the Cramér-Lundberg approximation is given by

$$\psi_{app}(u) = \frac{c-\lambda\mu}{\lambda M_X'(\gamma)-c}e^{-\gamma u}, \quad u \geqslant 0.$$

Remark 4. *We point out that for exponentially distributed claim sizes, i.e.* $F_X(x) =$

$1 - e^{-\beta x}$, $\beta > 0$, *it follows from the form of the moment generating function of an exponential random variable, given by* $M_X(s) = \frac{\beta}{\beta - s}$, *that*

$$M'_X(\gamma) = \frac{\beta}{(\beta-\gamma)^2} < \infty, \qquad \textit{for} \qquad \gamma > 0.$$

Then, we have

$$\psi_{app}(u) = \frac{c - \frac{\lambda}{\beta}}{\frac{\lambda\beta}{(\beta-\gamma)^2} - c} e^{-\gamma u} = \frac{\lambda}{\beta c} e^{-\gamma u}.$$

Finally, by solving $\theta(s) = 0$ *with exponentially distributed claim sizes, we obtain that* $\gamma = \lambda\eta/c$ *and the Cramér-Lundberg approximation gives*

$$\psi_{app}(u) = \frac{\lambda}{\beta c} e^{-\frac{\lambda\eta}{c} u},$$

which is equivalent to equation (2.4.14)*. That is, the Cramér-Lundberg approximation is exact under exponential claim sizes.*

1.4.2 Asymptotic behaviour for heavy tailed claim size distributions

In reality some claims incurred by an insurance company are ‘extreme’ - a concept that cannot be captured under the assumption of ‘light tailed’ claim size distributions, as considered in the previous section (Cramér conditions hold only for light tailed distributions). A transparent example can be seen in the case of insurance against natural disasters. This type of claim may not appear as frequently as other claims, however, by the nature of such disasters, any claims that are reported would be large in size. Typical examples of such insurance coverage include lines of business concerned with earthquakes, floods or, in general, with CAT bonds. In mathematical terms, these ‘large claims’ are described by *heavy tailed* distributions, which have been shown to more accurately represent insurance risks - since such distributions provide a more appropriate fit to actual claim data. In particular, Pareto, lognormal, log-gamma and Burr distributions are popular in actuarial mathematics. Recently, more and more attention has been directed towards heavy tailed distributions since it has been postulated that only extreme claims dramatically affect the surplus of an insurer and can realistically cause ruin.

There are several mathematical definitions for a heavy tailed distribution, however, the following will be sufficient for the purpose of this thesis.

Definition 14. *Let $F(x)$, $x \geqslant 0$ be a d.f. with support on the positive real line $[0, \infty)$. Then, F is said to be heavy tailed if its moment generating function does not exist, i.e.*

$$\int_0^\infty e^{sx} dF(x) = \infty, \quad \textit{for all} \quad s > 0.$$

The general definition of a heavy tailed distribution is too broad to allow for an effective analysis of their impact in the aforementioned risk model. Thus, the majority of results have been concentrated on a subclass of heavy-tailed distributions known as sub-exponentials, the set of which is denoted by $\mathcal{S}$, for which Embrechts and Veraverbeke (1982) prove that the exponential behaviour of the ruin function, proposed by the Cramér-Lundberg approximation, no longer holds. In more detail, for heavy-tailed distributions, the Cramér conditions no longer hold and the ruin probability behaves asymptotically like the integrated tail of the claim size distribution, which decays slower than that of a light tailed distribution (by definition).

Theorem 10 (Heavy tailed asymptotics)**.** *Suppose that the integrated tail of the claim size distribution is in the class of sub-exponentials i.e. $F_X^s \in \mathcal{S}$. Then, the ruin probability behaves asymptotically as*

$$\psi(u) \sim \frac{\rho}{1-\rho} \overline{F_X^s}(u), \quad u \to \infty,$$

where $\rho = \lambda\mu/c$.

In addition to the general asymptotic expression given above, there exist some explicit results which have been derived for some special cases of heavy tailed distributions. Ramsay (2003) uses LT techniques for a shifted Pareto claim size distribution, with integer valued parameter, to derive an exact expression for the ruin probability containing a single integral, which is later generalised in Ramsay (2007) to a shifted Pareto distribution with non-integer valued parameter. Later, Albrecher and Kortschak (2009) consider a similar method to that of Ramsay [(2003),(2007)] to derive an exact expression for the classic (non-shifted) Pareto distribution.

1.5 The surplus prior and the deficit at ruin

The vast amount of attention spent on deriving results for the ruin probability shows the level of interest such a quantity has created, however, many people argue that determining the probability of such an event does not provide enough practical information to be of value. It is of much more use to consider closely related quantities that allow for a much deeper analysis of an insurer's surplus.

One of the first extensions to the classic ruin quantity was proposed by Gerber et al. (1987), where the severity (deficit) of ruin, as well as its probability, is considered by means of a d.f. $G(u, y)$. This quantity not only captures the probability of ruin, but also by 'how much' deficit the firm is exposed to.

Definition 15 (Deficit at ruin)**.** *Let T be the time of ruin in the Cramér-Lundberg risk model. Then, the joint probability distribution of ruin and the deficit at the time of ruin, denoted by $G(u, y)$ for $y \geqslant 0$, is defined by*

$$G(u, y) = \mathbb{P}\left(T < \infty, |U(T)| \leqslant y \middle| U(0) = u\right), \tag{1.5.1}$$

and $\frac{\partial}{\partial y} G(u, y) = g(u, y)$ denotes its corresponding density function.

We point out that if the net profit condition holds, this quantity is a defective probability d.f., since

$$\lim_{y \to \infty} G(u, y) = \psi(u) < 1. \tag{1.5.2}$$

Therefore, it is sometimes convenient to consider the corresponding conditional distribution function

$$G_u(y) = \mathbb{P}(|U(T)| < y \big| T < \infty, U(0) = u) = \frac{G(u, y)}{\psi(u)}, \tag{1.5.3}$$

which is in the form of a proper d.f.

The deficit at ruin is of particular interest, in practice, due to the unrealistic notion that a company will experience 'ruin' if their surplus drops slightly below zero (a company would have prudent measures in place to help them recover such small losses). Due to the fact that ruin, and thus a deficit, can only occur due to a claim, it is natural to consider that the claim size distribution has a significant impact on the value of this quantity. In fact, it is a well known result [see Bowers et al. (1997)] that the density $g(u, y)$, with zero initial capital, i.e. $g(0, y)$, is proportional to the tail of the claim size distribution.

Theorem 11. *Consider the density function for the probability and deficit of ruin, namely $g(u, y)$, with initial capital $U(0) = 0$. Then, we have*

$$g(0, y) = \frac{\lambda}{c}\overline{F}_X(y). \tag{1.5.4}$$

Using the result of Theorem 11, Gerber et al. (1987) show that the joint d.f., $G(u, y)$, satisfies a familiar renewal equation of the form

$$G(u, y) = \frac{\lambda}{c}\int_0^u G(u - x, y)\,\overline{F}_X(x)\,dx + \frac{\lambda}{c}\int_u^{u+y}\overline{F}_X(x)\,dx, \tag{1.5.5}$$

which is a generalisation of the renewal equation derived for the ruin probability, $\psi(u)$, given in equation (1.2.9). Instead of solving the above renewal equation directly, as in the case of the ruin probability, they first consider a corresponding renewal equation for the density, $g(u, y)$, found by differentiating both sides of equation (1.5.5) with respect to y, from which they derive an explicit expression for the transform $\gamma(r, y) = \int_0^\infty e^{ru} g(u, y)\,dy$. It then remains to invert this form of the transform to obtain explicit expressions for the density $g(u, y)$ and thus $G(u, y)$, which they show can be easily found for a combination of exponentials and a combination of gamma distributions.

Proposition 4 (Exponential claim sizes)**.** *Let the claim sizes be exponentially distributed with parameter $\beta > 0$, i.e. $F_X(x) = 1 - e^{-\beta x}$. Then, the joint d.f. for the deficit* *at ruin, $G(u,y)$, is given by*

$$\begin{aligned} G(u,y) &= \psi(u)\left(1-e^{-\beta y}\right) \\ &= \frac{\lambda}{\beta c} e^{-\frac{\lambda\eta}{c}u}(1-e^{-\beta y}), \end{aligned} \tag{1.5.6}$$

with corresponding density function, $g(u,y)$, given by

$$\begin{aligned} g(u,y) &= \beta\psi(u)e^{-\beta y} \\ &= \frac{\lambda}{c} e^{-\frac{\lambda\eta}{c}u} e^{-\beta y}. \end{aligned} \tag{1.5.7}$$

On the other hand, there exists a unique, locally bounded solution to the renewal equation, given by (1.5.5), for general claim size distribution, in terms of its convolutions [see Lemma 6.1.2, Rolski et al. (1991)], given by

$$G(u,y) = \int_0^u \sum_{n=0}^{\infty} h^{*n}(x) \int_{u-x}^{u-x+y} h(z)\, dz\, dx, \tag{1.5.8}$$

where $h^{*n}(x)$ is the n-fold convolution of the function $h(x) = (\lambda/c)\overline{F}_X(x)$.

Remark 5. *We point out, by taking the limit $y \to \infty$ in the above result, we recover Beekman's formula for the ruin probability, $\psi(u)$, given in equation* (1.2.11).

Following from the introduction of the deficit at ruin, Dufresne and Gerber (1988) investigate the distribution of the amount of surplus immediately prior to the time of ruin and derive a rather surprising symmetry between these two quantities when the initial capital of the insurer is zero. Later, Dickson (1992) provides explicit solutions to the function described in Dufresne and Gerber (1988) when the probability and severity of ruin are known and discusses the analytic properties of their relationship, see also Dickson and Waters (1996), Dickson and Dos Reis (1996), Willmot and Lin (1998) and Schmidli (1999), among others.

A further quantity of interest is the time to ruin, for which Delbaen (1990) considers its distribution and proves that the p-th moment of the time to ruin exists if and only if the $(p+1)$-th moment of the claim sizes exists. Later, Picard and Lefèvre (1998) derive exact expressions for these moments for any arithmetic claim size distribution.

1.6 Expected discounted penalty function

In an attempt to unify these relevant quantities Gerber and Shiu (1997) generalise the results of Dickson (1992) and derive a renewal equation for the joint distribution of the time of ruin (represented by its LT), the surplus immediately prior to ruin and the deficit at ruin. Later, in the seminal paper of Gerber and Shiu (1998), a unified approach to deal with all the ruin related quantities, in the form of one elegant function denoting the expected discounted penalty at ruin (also known as the Gerber-Shiu function), is considered. We point out that the Gerber-Shiu function provides, not only a unified modelling of many ruin based quantities, but also introduces a 'penalty' at ruin, which may be incurred by an insurance firm in practice.

Definition 16 (Gerber-Shiu function). *Let T be the time to ruin and $w(x, y)$ be a non-negative function for $x > 0$ and $y > 0$. Then, the expected discounted penalty function is defined by*

$$m_\delta(u) = \mathbb{E}\left(e^{-\delta T} w(U(T-), |U(T)|)\mathbb{I}_{\{T<\infty\}}|U(0) = u\right), \tag{1.6.1}$$

where $\delta \geqslant 0$ is considered to be a force of interest and $w(x, y)$, $x, y \geqslant 0$ is a non-negative function denoting the penalty at ruin.

We point out that although $\delta > 0$ is interpreted as a force of interest, it can also be considered as a dummy variable in the context of LTs and thus, the Gerber-Shiu function can be used to analyse the time to ruin via its LT.

1.6.1 Integro-differential equation for the Gerber-Shiu function

In a similar way to the probability of ruin, Gerber and Shiu (1998) show that the expected discounted penalty function satisfies an integro-differential equation and a renewal equation which generalise equations (1.2.6) and (1.2.9), respectively.

Theorem 12. *For $\delta \geqslant 0$ and $u \geqslant 0$, the expected discounted penalty function, $m_\delta(u)$, satisfies the integro-differential equation*

$$cm'_\delta(u) = (\delta + \lambda)m_\delta(u) - \lambda \left[\int_0^u m_\delta(u-x)\, dF_X(x) + \omega(u) \right], \tag{1.6.2}$$

where $\omega(u) = \int_u^\infty w(u, y-u)dF_X(y)$, with boundary condition

$$\lim_{u\to\infty} m_\delta(u) = 0.$$

The emergence of this result has seen a shift in the risk theory literature from the ruin function to the newly defined expected discounted penalty function, due to the broader analysis such a quantity provides and also, since the aforementioned results for the ruin related quantities can be recovered as special cases. For example:

- Setting $\delta = 0$ and $w(x, y) \equiv 1$, for all $x, y > 0$, we obtain the probability of ruin, i.e.

  $$\begin{aligned} m_\delta(u) &= \mathbb{E}(\mathbb{I}_{\{T<\infty\}} \big| U(0) = u) \\ &= \mathbb{P}(T < \infty \big| U(0) = u) = \psi(u), \end{aligned}$$

  and the IDE given in equation (1.6.2) reduces to the IDE given for $\psi(u)$ in equation (1.2.6).

- Setting $\delta = 0$ and $w(x, y) \equiv \mathbb{I}_{\{|U(T)|\leqslant y\}}$, for all $x, y > 0$, we obtain the joint d.f. of ruin and the deficit at ruin, i.e.

  $$\begin{aligned} m_\delta(u) &= \mathbb{E}(\mathbb{I}_{\{T<\infty\}}\mathbb{I}_{\{|U(T)|\leqslant y\}} \big| U(0) = u) \\ &= \mathbb{P}(T < \infty, |U(T)| \leqslant y \big| U(0) = u) = G(u, y). \end{aligned}$$

- Setting $\delta = 0$ and $w(x, y) \equiv \mathbb{I}_{\{U(T-)\leqslant x\}}$, for all $x, y > 0$, we get the joint d.f. of ruin and the surplus immediately prior to ruin, i.e.

$$\begin{aligned} m_\delta(u) &= \mathbb{E}(\mathbb{I}_{\{T<\infty\}}\mathbb{I}_{\{U(T-)\leqslant x\}}\big|U(0)=u) \\ &= \mathbb{P}(T<\infty, U(T-)\leqslant x\big|U(0)=u). \end{aligned}$$

- Setting $\delta = 0$ and $w(x, y) \equiv \mathbb{I}_{\{U(T-)\leqslant x\}}\mathbb{I}_{\{|U(T)|\leqslant y\}}$, for all $x, y > 0$, we obtain the joint d.f. of ruin, the surplus immediately prior to ruin and the deficit at ruin, i.e.

$$\begin{aligned} m_\delta(u) &= \mathbb{E}(\mathbb{I}_{\{T<\infty\}}\mathbb{I}_{\{|U(T)|\leqslant y\}}\mathbb{I}_{\{U(T-)\leqslant x\}}\big|U(0)=u) \\ &= \mathbb{P}(T<\infty, |U(T)|\leqslant y, U(T-)\leqslant x\big|U(0)=u). \end{aligned}$$

- For $\delta > 0$ and $w(x, y) \equiv 1$, for all $x, y > 0$, we obtain

$$m_\delta(u) = \mathbb{E}(e^{\delta T}\mathbb{I}_{\{T<\infty\}}\big|U(0)=u),$$

  which denotes the LT of the time to ruin.

- Setting $\delta = 0$ and $w(x, y) = \mathbb{I}_{\{U(T-)+|U(T)|\leqslant x\}}$ for $x \geqslant 0$, we obtain

$$\begin{aligned} m_\delta(u) &= \mathbb{E}(\mathbb{I}_{\{T<\infty\}}\mathbb{I}_{\{U(T-)+|U(T)|\leqslant x\}}\big|U(0)=u) \\ &= \mathbb{P}(T<\infty, U(T-)+|U(T)|\leqslant x\big|U(0)=u). \end{aligned}$$

  which denotes the d.f. of the amount of the claim causing ruin.

As in the case of the ruin probability, the problem remains to solve the IDE, given in equation (1.6.2), to derive explicit results for the Gerber-Shiu function. Similar techniques/methodologies as in subsection 1.2.1-1.2.4 have been applied, as well as alternative methods, in order to derive explicit expressions, see among others Lin and Willmot (1999) and Li and Garrido (2004), (2005). However, as a matter of succinctness, we only present a few results in the following.

1.6.2 Algebraic operator approach

The algebraic operator approach, employed in Albrecher et al. (2010), provides an operator based method of solving the IDE, given in equation (1.6.2). Application of this method results in an explicit solution to the Gerber-Shiu function, in terms of Green's operators, for a general class of claim size distributions with rational LTs, or equivalently density functions satisfying the following linear ordinary differential equation (LODE):

$$p_X\left(\frac{d}{dx}\right) f_X(x) = 0, \tag{1.6.3}$$

with homogeneous boundary conditions

$$\begin{aligned} f_X^{(k)}(0) &= 0, \quad k = 0, 1, \ldots, n-2, \\ f_X^{(n-1)}(0) &= b_0, \end{aligned}$$

where $p_X(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0$ is a polynomial with real coefficients b_j, $j = 0, \ldots, n$ and $b_0 \neq 0$. The solution is derived by reducing the IDE to a linear boundary value problem, with appropriate boundary conditions and employing symbolic methods developed in Rosenkranz (2005) and Rosenkranz and Regensburger (2008) for computing the Green's integral operator that maps the penalty function to the corresponding Gerber-Shiu function. The method relies on the factorisation of the differential operator using the roots of a generalised Lundberg's fundamental equation.

Theorem 13. *Assume the claim size density function, $f_X(x)$, for $x \geqslant 0$, satisfies equation* (1.6.3)*. Then, the Gerber-Shiu function, $m_\delta(u)$, is given by*

$$m_\delta(u) = G_\sigma G_\rho f(u) + m_\delta^p(u),$$

where G_σ, G_ρ denote the Green's operators with respect to the roots $\{\sigma_i\}_{i\in\mathbb{N}}$ and $\{\rho_i\}_{i\in\mathbb{N}}$ of the generalised Lundberg's fundamental equation with negative and positive real parts, respectively. The function $f(u)$ is given by

$$f(u) = \lambda b_0 p_X\left(\frac{d}{dx}\right) \int_x^\infty w(u, y-u)\, dF_X(y),$$

and $m_\delta^p(u)$ is the particular solution of the form

$$m_\delta^p(u) = \sum_{i=1}^{m} a_i e^{\sigma_i u},$$

with co-efficients $\{a_i\}_{i\in\mathbb{N}}$ dependening on the boundary conditions $m_\delta^{(k)}(0)$ for $k = 0, \ldots, n-1$.

Remark 6. *We point out that the algebraic operator results, obtained in Albrecher et al. (2010), are derived for a more general renewal risk model, with inter-arrival distribution satisfying a similar LODE as given in equation* (1.6.3)*. The result stated in Theorem 13 corresponds to the the special case of the compound Poisson risk model.*

1.6.3 Volterra integral equation for the Gerber-Shiu function

As already discussed, as well as satisfying the IDE, given in equation (1.6.2), Gerber and Shiu (1998) show that the expected discounted penalty function also satisfies a renewal equation similar to that given in (1.2.9), known as a Volterra equation.

Theorem 14. *The expected discounted penalty function, $m_\delta(u)$, satisfies the Volterra equation*

$$m_\delta(u) = \frac{\lambda}{c}\left(\int_u^\infty e^{\xi(x-u)}\omega(x)dx + \int_0^u m_\delta(u-x)\int_x^\infty e^{\xi(y-x)}\,dF_X(y)\,dx\right), \qquad (1.6.4)$$

where $\xi \equiv \xi(\delta) \leqslant 0$ is the unique nonpositive solution to the modified Lundberg equation given by

$$\theta(s) - \delta = 0, \qquad (1.6.5)$$

where $\theta(s)$ is given in equation (1.3.4)*.*

Alternatively, if we define the functions

$$h(u) = \frac{\lambda}{c}\int_u^\infty e^{\xi(x-u)}\omega(x)dx$$

and

$$g(u) = \frac{\lambda}{c}\int_u^\infty e^{\xi(y-u)}\,dF_X(y)$$

the Volterra equation, given in (1.6.4), reduces to

$$m_\delta(u) = h(u) + (m_\delta * g)(u), \tag{1.6.6}$$

which can be easily seen, by successive substitution (or by method of LTs), has the so-called *Neumann series* solution [see Gerber and Shiu (1998)], of the form

$$m_\delta(u) = \sum_{n=0}^{\infty} (h * g^{*n})(u).$$

The Neumann series solution, given above, is in fact a generalisation of the convolution series given in equation (1.5.8) and thus a generalisation of Beekman's formula given in equation (1.2.11) for the ruin probability.

Alternatively, Lin and Willmot (1999) provide a solution to a general defective renewal equation (of which equation (1.6.4) is a special case) in terms of the tail of a compound geometric d.f. Consider the defective renewal equation

$$m_\delta(u) = \frac{1}{1+\beta} \int_0^u m_\delta(u-x)\, dG(x) + \frac{1}{1+\beta} H(u), \quad u \geqslant 0, \tag{1.6.7}$$

where $\beta > 0$, $G(x) = 1 - \overline{G}(x)$ is a d.f. with $G(0) = 0$, and $H(u)$ is continuous for $u \geqslant 0$. Now, define the associated compound geometric d.f. $K(u) = 1 - \overline{K}(u)$ by

$$\overline{K}(u) = \sum_{n=1}^{\infty} \frac{\beta}{1+\beta} \left(\frac{1}{1+\beta} \right)^n \overline{G}^{*n}(u), \quad u \geqslant 0,$$

where $\overline{G}^{*n}(u)$ is the tail of the n-th fold convolution of $G(u)$.

Theorem 15. *The solution, $m_\delta(u)$, to equation* (1.6.7) *may be expressed as*

$$m_\delta(u) = \frac{1}{\beta} \int_0^u H(u-x)\, dK(x) + \frac{1}{1+\beta} H(u), \quad u \geqslant 0.$$

Proof. Let $\widehat{g}(s) = \int_0^\infty e^{-su}\, dG(u)$ be the Laplace-Stieltjes transform of $G(x)$. Then, the Laplace-Stieltjes transform of $K(x)$, is given by

$$\begin{aligned}
\widehat{k}(s) &= K(0) + \int_0^\infty e^{-su}\, dK(u) \\
&= K(0) + \frac{\beta}{1+\beta}\sum_{n=1}^{\infty}\left(\frac{1}{1+\beta}\right)^n \int_0^\infty e^{-su}\, dG^{*n}(u) \\
&= K(0) + \frac{\beta}{1+\beta}\sum_{n=1}^{\infty}\left(\frac{1}{1+\beta}\right)^n \int_0^\infty e^{-su} g^{*n}(u)\, du \\
&= K(0) + \frac{\beta}{1+\beta}\sum_{n=1}^{\infty}\left(\frac{\widehat{g}(s)}{1+\beta}\right)^n \\
&= \frac{\beta}{1+\beta+\widehat{g}(s)}.
\end{aligned}$$

Also, let $\widehat{m}_\delta(s) = \int_0^\infty e^{-su} m_\delta(u)\, du$ and $\widehat{H}(s) = \int_0^\infty e^{-su} H(u)\, du$. Then, from equation (1.6.7), we have

$$\widehat{m}_\delta(s) = \frac{\widehat{H}(s)}{1+\beta-\widehat{g}(s)} = \frac{1}{\beta}\widehat{H}(s)\widehat{k}(s),$$

which, upon inversion of the LT, yields the result. □

1.6.4 Asymptotic results for the Gerber-Shiu function

In a similar way as for the ruin probability, Gerber and Shiu (1998) derive the asymptotic behaviour of the expected discounted penalty function, as $u \to \infty$.

Firstly, from the renewal equation given in equation (1.6.4), since $\xi \leqslant 0$, it follows that

$$\begin{aligned}
\int_0^\infty g(x)\, dx &= \frac{\lambda}{c}\int_0^\infty \int_x^\infty e^{\xi(y-x)}\, dF_X(y)\, dx \\
&\leqslant \frac{\lambda\mu}{c} < 1,
\end{aligned}$$

by the net profit condition and thus, equation (1.6.4) is a defective renewal equation. Then, following a similar method to that in the proof of Theorem 9, we seek an $s > 0$, such that

$$
\begin{aligned}
1 = \int_0^\infty e^{sx} g(x)\, dx &= \frac{\lambda}{c} \int_0^\infty e^{sx} \int_x^\infty e^{\xi(y-x)}\, dF_X(y)\, dx \\
&= \frac{\lambda}{c(s-\xi)} \int_0^\infty \left(e^{(s-\xi)y} - 1 \right) e^{\xi y} dF_X(y) \\
&= \frac{\lambda}{c(s-\xi)} \left(M_X(s) - M_X(\xi) \right),
\end{aligned}
$$

which, since $\xi \equiv \xi(\delta) < 0$ is the nonpositive root satisfying $\theta(\xi) - \delta = 0$, reduces to

$$
1 = \frac{\lambda M_X(s) - (\delta + c\xi + \lambda)}{c(s-\xi)}.
$$

If such a solution exists, we call it $\gamma(\delta) > 0$. Note that, for $\delta = 0$, it follows that $\xi(\delta) = 0$ and the above equation reduces to the form of equation (1.4.3), from which it follows that $\gamma(0) = \gamma > 0$ is the Lundberg coefficient discussed previously.

Now, multiplying equation (1.6.4) through by $e^{\gamma(\delta)u}$, yields

$$
\begin{aligned}
e^{\gamma(\delta)u} m_\delta(u) = \frac{\lambda}{c} \Bigg(& e^{\gamma(\delta)u} \int_u^\infty e^{\xi(x-u)} \omega(x) dx \\
& + \int_0^u e^{\gamma(\delta)(u-x)} m_\delta(u-x) \int_x^\infty e^{\gamma(\delta)x} e^{\xi(y-x)}\, dF_X(y)\, dx \Bigg),
\end{aligned}
\tag{1.6.8}
$$

which is a proper renewal equation and thus, by Key Renewal Theorem [see Theorem 8] we have the following theorem, as given in Gerber and Shiu (1998).

Theorem 16. *Assume the adjustment coefficient, $\gamma(\delta) > 0$, exists. Then*

$$
\lim_{u\to\infty} m_\delta(u) e^{\gamma(\delta)u} = \frac{\lambda \int_0^\infty \int_0^\infty w(x,y) \left(e^{\gamma(\delta)x} - e^{\xi x} \right) f_X(x+y)\, dx\, dy}{-\lambda M_X'(\gamma) - c}, \tag{1.6.9}
$$

where $\xi \equiv \xi(\delta) \leqslant 0$ is the nonpositive solution to the modified Lundberg equation $\theta(s) - \delta = 0$.

1.7 Extensions of the classical risk model

In recent years the classical risk model has been the cause of much debate between those studying mathematical theory and practitioners in the actuarial industry - since it does not accurately reflect the reality - which has been the catalyst for the expansion of much more general models, a few of which will be discussed in the following.

1.7.1 Renewal and the Markov-modulated risk model

The first major development to the classic risk model was introduced in Sparre Anderson (1957). In this work, the underlying assumption regarding the distribution of the claim arrivals is weakened such that the inter-arrival times between claims are still i.i.d. but with arbitrary distribution. Then, under this relaxed setting, the counting process, namely $\{N(t)\}_{t\geqslant 0}$, becomes a renewal process, as discussed in Section 1.1. This renewal model, now known as the Sparre Anderson model, allows for a more flexible analysis of an insurer's surplus, with weaker assumptions in terms of the claim arrival distribution and has been extensively studied in the actuarial literature. Detailed studies of the ruin probability and the Gerber-Shiu function can be found, among others in Dickson and Hipp (1998, 2000), Li and Garrido (2004), Gerber and Shiu (2005) and Chadjiconstantinidis and Papaioannou (2009). Although the Sparre Andersen model generalises the classic risk process, the i.i.d. property of the inter-arrival times still limits its capability to accurately reflect the reality.

In practice, there exist periods of time where the arrival intensity may fluctuate due to external factors, such as the weather, natural disasters and economic conditions. Thus, the assumption of i.i.d. inter-arrival times becomes unfavourable and a model which incorporates the volatility of arrival intensities is convenient. Reinhard (1984) presented such a model by introducing a class of semi-Markov risk models, where the claim inter-arrival times are assumed to be exponential, with parameter $\lambda_i > 0$, $i \in E$, when an external environmental Markov process is in some state $i \in E$. Under this setting, Reinhard (1984) obtained explicit expressions for the infinite-time survival probabilities of a special case. The ruin (survival) probabilities and Gerber-Shiu functions have since been extensively studied under this setting, see Asmussen et al. (1995), Bäuerle (1996), Jasiulewicz (2001), Lu and Li (2005), Li and Lu (2008), and references therein.

1.7.2 Dividends

Over the years the ruin probability has been criticised as being an 'artificial' measure and can cause 'economically strange decisions' [see Eisenberg and Schmidli (2011)]. For example, in the Cramér-Lundberg model, it is assumed that the surplus of an insurance company can increase indefinitely without bound, which is unrealistic. De Finetti (1957) proposed that dividend payments need to be factored into the model and an alternative measure of risk was introduced to reflect the value of a dividend stream in a portfolio. A 'dividend' is a sum of money paid regularly (typically annually) by a company to its shareholders out of its reserves and thus, the success of a company can be measured by the maximal future dividend payments. This naturally leads to the question of what is the optimal strategy to maximise dividend payments whilst minimising the probability of insolvency? De Finetti (1957) found such a strategy to be a *barrier strategy*, where any excess income above the constant *dividend barrier* is paid out continuously to the shareholders, whilst below, the process evolves as in the classical model. The constant dividend barrier problem, in the Poisson process framework, has been studied in Bühlmann(1970), Segerdahl(1970), Paulsen and Gjessing (1997), Lin et al. (2003), Dickson and Waters (2004), Lin and Pavlova (2006) and references therein. Although it was shown that the constant dividend barrier is optimal, De Finetti (1957) also described how such a strategy - even under the net profit condition - causes ruin with probability one, thus, further strategies have been studied to determine their efficiency within an insurance portfolio. Gerber (1981) considers a linear dividend strategy, such that the dividend barrier changes (linearly) with time, i.e. the level of the barrier at time $t \geqslant 0$ is given by $b(t) = b + at$. The actuarial literature has since seen an explosion of research for optimal dividend strategies, which will not be discussed here, but the reader is directed to Avanzi (2009) for a comprehensive review of these results.

1.7.3 Stochastic investment

Within the insurance market, companies not only rely on income from premiums, since a significant percentage of their wealth is gained from returns on investment. Originally, Sergerdahl [(1942), (1959)] and Gerber [(1973), (1979)], among others, considered this important factor and modelled deterministic returns from investment - this type of 'riskless' return comes from derivatives such as bonds. Modelling investment portfolios solely with derivatives of deterministic return is unrealistic. In practice, an insurer's portfolio will comprise of a delicate mixture of both 'riskless' and 'risky' assets, where the term 'risky' refers to the random returns/losses received on stocks in the financial market. The generalisation to stochastic returns on investment was first proposed in Paulsen (1993), as an extension to the previously analysed deterministic returns, as a way of capturing the inherent risks associated with investing in the financial equity markets. In this work, the author initially considers a rather general form of both the risk model and the returns from investment, by proposing both are contained in the broad set of semi-martingales. In order to conduct a more tractable analysis Paulsen restricts the model from this general setting and is able to derive several results.

Since the appearance of this work the risk model with stochastic investment became a 'hot topic' and has produced a library of results. Paulsen and Gjessing (1997) show that, under the restricted setting of Paulsen (1993), the ruin probability can be found, in general, by solving boundary value problems involving IDEs and derive some results for special cases. Frolova et al. (2001) consider a slightly more refined version of the work in Paulsen and Gjessing (1997), and assume that the price of the risky asset follows a geometric Brownian motion - this is one of the main assumptions for the market conditions in Black and Scholes (1973) where the famous Black-Scholes model for option pricing was first derived - and show that, under exponentially distributed claim sizes, the asymptotic behaviour of the ruin probability no longer decays exponentially, but as a power function depending solely on the parameters of the investment, which indicates the potential dangers of investing in financial derivatives.

1.7.4 Capital injections

One fundamental characteristic of the hitherto defined ruin probability is the assumption that the surplus (company) stops once a deficit below zero is realised. All businesses (not just in the insurance market) have stringent measures in place to help protect them against such an event as having to completely cease its operation due to insolvency. One example of such safeguarding measures prevalent in practice, is capital injections. Capital injections are, as the name suggests, an injection of capital into the company (which may appear in several forms) with the main sources usually coming from the shareholders, national government or from a pre-arranged re-insurance agreement. In fact, in connection with dividend payments mentioned above, Dickson and Waters (2004) propose "As the shareholders benefit from the dividends income until ruin, it is reasonable to expect that the shareholders provide the initial surplus u and take care of the deficit at ruin.". In this thesis we will consider that the primary source of capital injections is from such an agreement, but the results and methods applied still make sense under government funding or re-insurance contracts. Pafumi (1998) considers the latter, where the contract is as follows: whenever the surplus is negative, the reinsurer makes the necessary payment to bring the surplus back to zero, instantaneously. In this set up it is clear that 'ruin', in the classical sense, can no longer occur and the company can continue indefinitely, thus it becomes of interest to find the net single premium of this contract, which is the expected sum of discounted future injections. A similar model is considered by Eisenberg and Schmidli (2011), where a retention level reinsurance contract is analysed whilst the insurer is forced to inject capital to keep company solvent. There exists a broad literature on reinsurance within risk theory, however we will not discuss it in details in this thesis [for more details, see Dickson (2005) and references therein]. Although the addition of such injections, which eliminate the event of ruin, seems to be another extreme, it presents a basis for further application and detailed analysis of capital injections. Nie et al. (2011) introduce a model where the capital injections are required to bring the surplus back to some level $k \geqslant 0$, when the surplus falls below this level, and ruin occurs if the surplus ever falls below the zero barrier. In this model the concept of ruin has been reintroduced and provides a more realistic application of capital injections (this two barrier model can be easily adapted such that the ruin barrier is below zero if required). In this work the ruin probability is again the measure of interest and the method for solving is similar to the method of deriving expected discounted capital injections in the aforementioned articles. More details on this derivation and other results regarding capital injections will be given in the proceeding chapter.

1.8 Summary and thesis breakdown

In this thesis, we will consider three separate models analysing the capital recovery plans of insurance firms and other lines of business. Within each model, it is assumed that a company is allowed to continue when in a deficit, however, during this period, the company is required to recover their capital requirements subject to different regulatory constraints. If these regulations are not met, the company experiences ultimate ruin/insolvency and is no longer authorised to continue trading. In more details, within Chapter 2, we generalise the classical risk model to comply with the recently enforced capital requirement regulations under the Solvency II directive. Under this modification, where we introduce three constant barriers to model capital requirement thresholds under the Solvency II framework, we derive an expression for the probability of insolvency in terms of the ruin quantities of the classic Cramér-Lundberg risk model, defined above. Moreover, we show that the moment generating function of a risk quantity related to the accumulated capital injections, required to keep the company solvent, is a mixture of a degenerate and continuous distribution. We further show that the inclusion of a fourth constant dividend barrier produces similar results, where the probability of insolvency is given in terms of ruin quantities for the classic risk model with a constant dividend barrier strategy and is ultimately equal to 1, as to be expected in such a setting. In Chapter 3, we revert back to a classic risk model (without Solvency II constraints) and analyse the ultimate ruin probability for a risk model with capital injections. In this model, it is assumed that the capital injections are received after some time delay, from the moment of a deficit, which depends on the size of the deficit and corresponding capital injection. Under this setting, we show that the ultimate ruin probability, defined in a slightly different way to the classical sense, satisfies an inhomogeneous Fredholm integral equation of the second kind, which under certain dependency structures can be solved explicitly, in terms of classic ruin quantities or given by a Neumann series when a more explicit dependence is assumed. Moreover, we consider two risk related quantities, namely the expected discounted accumulated capital injections and the expected discounted accumulated time in red (deficit) up to the time of ultimate ruin, which are shown to also satisfy a similar Fredholm integral equation and are solved explicitly. Finally, in Chapter 4, we analyse the so-called dual risk model in discrete-time. In this model, which better captures the risk portfolio of different business lines, such as pharmaceutical or petroleum businesses, we assume that the company is allowed to continue trading when in a deficit for a fixed amount of time. If the company is unable to recover from a deficit, within this pre-specified time interval, from normal trading strategies, the company experiences ultimate ruin. This event of ultimate ruin is known in the literature as Parisian ruin. Using the strong Markov property of the risk process, we derive a recursive expression for the finite-time Parisian ruin probability, which can be used to obtain an explicit expression for the corresponding infinite-time case, in terms of the classical dual ruin probability in discrete-time. We further provide an alternate derivation of the classic dual ruin probability, which we use to analyse some specific examples, including an extension to the well known gambler's ruin problem.

Chapter 2

Ruin Probabilities Under Solvency II Constraints

A fundamental characteristic of the classic Cramér-Lundberg risk model is that the surplus of an insurance firm is allowed to evolve freely until the event of down crossing the zero level (theoretical ruin). This (unrestricted) behaviour does not capture the dynamics of insurance undertakings in reality.

In practice, financial institutions such as banks and insurance companies, have to continuously maintain a surplus level, known as capital requirements, subject to a solvency rule. The main objective of the solvency rule is to help avoid insolvency (ruin) and thus create more protection, and confidence, for the consumers and for economic stability. That is, for example, an insurance business must hold a minimum level of available funds to cover any expected future liabilities and a drop (in funds) below such a level may result in the withdrawal of their trading license and liquidation of assets to pay outstanding debts. In the context of risk theory, this 'solvency level' is equivalent to a theoretical ruin barrier much higher than the zero level considered throughout the actuarial literature.

The original European solvency directives were introduced in the 1970's as an initial step towards a single market for insurance throughout the European Union (EU). After a review of those directives, performed by the European Commission in the 1990's, it was identified that some areas were in need of updating, especially those related with capital requirements. A new solvency rule, known as Solvency I, was subsequently implemented in 2004. This directive used a simple robust model to calculate capital requirements using mainly ratios, in a sort of 'one model fits all' plan. These calculations only focused on certain risks and, therefore, the model was not sufficiently risk sensitive and didn't capture new or merged risks. The simple formula approach disregards the complete risk profile of the business, leading to lower capital requirements and in turn a greater chance of insolvency. Therefore, in January 2016, a new legislation known as the Solvency II directive (Directive 2009/138/EC, see EU Commission) was implemented throughout all insurance companies within the EU to create a more sophisticated and harmonised solvency framework.

2.1 Solvency II

Solvency II (SII) is the new prudential regulatory regime with the objectives of: providing an enhanced and more consistent level of protection for policyholders across the EU, improving the firm's understanding and management of its risks (by accurately directing capital throughout the business) and allowing the prudential authorities (regulators), and EIOPA (European Insurance and Occupational Pensions Authority), to effectively monitor the insurance institutions. The Solvency II framework encompasses a three pillar structure:

1. Pillar 1 contains the quantitative capital requirements that the firm will be required to meet based on a market consistent value basis. In more details, Pillar 1 sets two capital requirements providing an upper and lower level of a supervisory intervention ladder. The Solvency Capital Requirement (SCR) forms the upper level, above which the insurance firm is considered to be sufficiently capitalised and no intervention is necessary. The SCR has to be fulfilled by insurance institutions to assure a theoretical ruin probability of 0.005 (ruin occurs on average no more often than once in every 200 years). The Minimum Capital Requirement (MCR) is the lower level below which the regulator's strongest actions are taken

(e.g. removal of the insurer's authorisation).

2. Pillar 2 comprises the qualitative risk management and governance requirements. Within Pillar 2, insurers are required to carry out an Own Risk and Solvency Assessment (ORSA), which is reviewed by the regulator. The main objective of an ORSA is to understand and manage risk exposure that the regulatory capital requirements of Pillar 1 may not capture and are difficult to quantify. For example, some insurance firms may be exposed to long-term effects of climate change which are not necessarily significant over the one year time horizon used in the calibration of the SCR (see below). Thus, the ORSA provides further understanding and helps manage all risks to which the firm may be exposed.

3. Pillar 3 concerns the reporting and disclosure requirements, with the aim of improving the availability of information to the market. This improved market transparency should increase the participants understanding of an insurer's business and risks, therefore strengthening market discipline.

In this chapter, we will be mostly concerned with the capital requirements, and ladder of supervisory intervention, introduced under Pillar 1. The capital requirements, with regards to the required 'eligible' capital that needs to be held by the insurer under SII, are calculated using a layered system (see Figure: 2.1). The bottom layer comprises the Technical Provisions (TP) which is expressed as the sum of the *best estimate liabilities*, valued as the discounted expected future liability cash flow, and a *risk margin* which is the cost to transfer its commitments to another company, that is able to fulfil such obligations, if the insurer cannot continue its business. The top layer corresponds to the aforementioned SCR. The SCR is calibrated using the Value at Risk (VaR) of the basic own funds of an insurance or reinsurance undertaking, subject to a confidence level of 99.5 % over a one-year period. This calibration is applied to each individual risk module and sub-module an insurance firm faces, which is then combined using a specified correlation matrix and matrix multiplication. The same kind of calibration lies in the heart of regulatory regimes for capital requirements applied in the US (Risk Base Capital, RBC, see Cofield et al. (2012)), China (China Risk Oriented Solvency

System, C-ROSS, see The Actuarial Magazine, Feb/April (2014)), and Switzerland (Swiss Solvency Test, see FINMA (2006)). Finally, the MCR, which must lie in between 25%-45% of the SCR, is calculated for each individual business line given by pre-specified (business line dependent) factors, applied to technical provisions and/or written premiums.

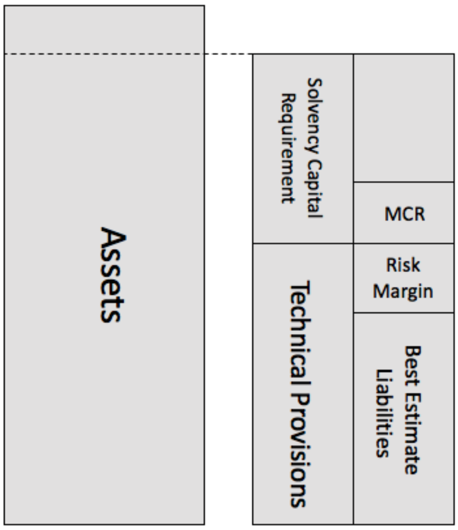


Figure 2.1: Solvency II balance sheet

In spite of its popularity, VaR has been criticised over recent years since it ignores events occurring in the tails, which creates an incentive to take excessive but remote risks [see Einhorn and Brown (2008)]. Moreover, Trufin et al. (2011) argue that since VaR is usually defined in terms of a given time horizon (one year under SII), it does not reflect the possible adverse financial situations between or beyond the specified time interval (as explained above in regards to ORSA). On the other hand, in an insurance context, the ruin probability, which can be interpreted as the continuous alternative to VaR, is considered as a somewhat more robust measure which can reflect the risk in a random environment. The strong connection between VaR and the ruin probability, as a risk measure, has been considered by Cheridito et al. (2006), Trufin et al. (2011), Ren (2012), Loisel and Gerber (2012), Gatto and Baumgartner (2014) and the references therein. According to Loisel and Gerber (2012) “ruin theory provides a more sustainable valuation principle (than the single use of the VaR approach) since it takes into account liquidity constraints and penalises large position sizes”. In view of this connection, we will apply the capital requirement constraints of the SII framework to ruin theory, allowing us to utilise the vast array of mathematical tools within the literature to derive and analyse the performance of the business. In particular, the probability that the surplus of an insurer breaches the MCR level, i.e. experiences insolvency/ruin. Although SII regulation is the framework under which insurance firms now operate, it appears that only a few papers have been written in the risk theory context. Ferriero (2016) derives practical estimators for the capital requirements in a fractional Brownian motion risk model. Floryszczak et al. (2016) confirm that the least-squares Monte-Carlo method is relevant to SII framework, for the capital requirements of an insurance firm, and Asimit et al. (2015) propose optimal allocations for the premium and the liabilities in order to reduce the MCR level.

2.2 Capital Injections

The SII balance sheet (described above) requires that 'eligible' capital is (at all times) in excess of the SCR. If at any point the firm breaches the SCR, it must consider a plan to restore its capital back to the SCR level as soon as possible. If the financial situation of the insurer continues to deteriorate, the level of supervisory intervention will be progressively intensified and, if despite intervention, the available capital falls below the MCR, 'ultimate supervisory action' will be triggered and the insurer's trading license may be revoked (ruin).

There exist many different approaches to re-capitalisation under the SII framework, however, it can be seen from market studies, that one of the most popular, and efficient methods, is provided by capital injections. Capital injections are a re-capitalisation mechanism often implemented by the shareholders, see for example, among others, the report of the ING group insurance in the Netherlands [see Annual Report of (2010)], the case of Liberty Insurance in Ireland [see Insurance Times Report (2017)], or MOODY'S report of April 2016 [see MOODY's Report (2016)], but in some cases can be provided by government or reinsurance contracts.

The introduction of capital injections, within the risk theory context, was first proposed by Pafumi (1998). In a discussion of Gerber and Shiu (1998), Pafumi (1998) proposes a reinsurance type contract, such that whenever the surplus of the insurer is negative, the reinsurer makes the necessary payment (capital injection) to bring the surplus back to the zero level. Under this amended surplus process, ruin can no longer occur and the company continues indefinitely. The question that then arises is; How can the premium of such a reinsurance contract be calculated? Pafumi (1998) considers the answer of such a question to be an (unloaded) net single premium calculated as the expected discounted sum of all future capital injections, which is a function of the initial capital $u \geqslant 0$, denoted by $A(u)$.

For the case $u = 0$, the following simple and explicit formula is obtained.

Proposition 5. *Let $A(u)$, for $u \geqslant 0$, denote the expected discounted sum of future capital injections and let $\xi \equiv \xi(\delta)$ be the non-positive root to the generalised Lundberg equation given by equation* (1.6.5)*. Then, for $u = 0$, we have*

$$A(0) = \frac{1}{\xi} - \frac{c - \lambda\mu}{\delta}.$$

Moreover, for $u > 0$, when the surplus of the insurer is negative for the first time, the reinsurer has to make an immediate capital injection, say Y_1, and reserve the amount $A(0)$ for future payments. Thus, it follows that the single net premium, namely $A(u)$, is a special case of the Gerber-Shiu function, i.e.

$$A(u) = m_\delta(u),$$

with corresponding penalty function $w(x, y) = y + A(0)$, $y > 0$.

An alternative, and more realistic, source of capital injections come from the companies shareholders. Since it is within their interest to keep the company solvent, the shareholders may be willing to raise the necessary capital to keep the firm operating, as long as it remains profitable for them to do so. However, in exchange for this risk exposure, the shareholders may require some financial incentive in the form of dividend payments. Dickson and Waters (2004) propose the counter argument and assume that "As the shareholders benefit from the dividend income until ruin, it is reasonable to expect that the shareholders provide the initial surplus u and take care of the deficit at ruin." In this work, it is considered that in exchange for the dividend payments, which are paid according to a constant barrier strategy, the shareholders cover the initial capital and the deficit at ruin. It turns out that, under such an agreement, the optimal constant dividend barrier, which maximises the expected discounted dividends minus the initial capital and expected discounted capital injections, is at the zero level. That is, the optimal strategy is for the shareholders to take control of the company and act as the insurer directly, i.e. receive all premiums as dividend payments in exchange for covering all claims. For further results on the connection between dividends strategies and capital injections, with particular emphasis on the optimal strategies, see among others Kulenko and Schmidli (2008), Scheer and Schmidli (2011) and Li and Liu (2015) in the Cramér-Lundberg risk model, Dai et al. (2010), Avanzi et al. (2011) and Yao et al. (2011) in the dual risk model and Wu (2013) for the diffusion approximation risk model. We point out that in these papers, the 'value' of the strategy to be optimised is the expected discounted dividends minus penalised discounted capital injections. The reason for the penalty factor is to avoid the trivial solution found in Dickson and Waters (2004).

Eisenberg and Schmidli (2009) consider a model where the capital injections are provided by the shareholders in the absence of dividend payments. Alternatively, their goal is to find a reinsurance strategy, for a diffusion approximation, that minimises the expected discounted future capital injections. In this model, the insurer is allowed to take out a reinsurance contract, $r(X, b)$, with (potentially) dynamic retention level $b \equiv b_t$ to cover some amount of the claim X. For example, $r(X, b) = bX$ corresponds to proportional (quota-share) reinsurance and $r(X, b) = \min(X, b)$ to excess of loss reinsurance. In exchange for this cover, the insurer is required to pay a premium to the reinsurer which is calculated via a (loaded) expected premium principle. It is assumed that the loading factor of the reinsurer, say η_R, is greater than the safety loading of the insurer, η, to avoid trivial solutions. The optimal reinsurance strategy is obtained and is given by a constant retention level, dependent on the contract $r(X, b)$. Later, Eisenberg and Schmidli (2011) consider a similar strategy for the Cramér-Lundberg risk process. As a particular example they consider the case where no reinsurance can be purchased and express the expected discounted capital injections as the linear combination of two Gerber-Shiu functions. For the general case of a (potentially) dynamic retention level $b \equiv b_t$, the optimal strategy is not necessarily a constant strategy, as for the diffusion approximation, and depends heavily on the model parameters. Interestingly, it is shown that the optimal strategy in this model is different to the optimal reinsurance strategy minimising the ruin probability. However, as in the case of the ruin probability, it is shown that for capital close to zero, the optimal strategy is not to purchase reinsurance. For more details on the optimal reinsurance models and a further consideration of optimal investment strategies with capital injections see Eisenberg (2010).

In contrast to capital injections funded by the shareholders, Nie et al. (2011) revisit the problem introduced by Pafumi (1998), and consider a reinsurance type contract that provides the necessary capital to restore the surplus whenever it falls below some constant level, $k \geqslant 0$. However, unlike the model proposed by Pafumi (1998), if the insurer is exposed to a large enough claim, such that the surplus becomes negative, ultimate ruin occurs at this point. The premium issued by the reinsurer, denoted by $Q(u, k)$, is calculated based on the expected (non)discounted capital injections until the time of ruin. It is assumed that the insurer has an initial amount of funds U, which is split to fund an initial capital $u \geqslant k$ and premium $Q(u, k) = U - u$. The aim of the paper is to find the optimal combination of u and k, with $0 \leqslant k \leqslant u$, that minimises the probability of ultimate ruin, denoted $\psi_k(u)$.

In more details, Nie et al. (2011) define $\psi_k(u)$ to be the ultimate ruin probability, for this modified surplus process with capital injection barrier $0 \leqslant k \leqslant u$, with corresponding survival probability defined by $\phi_k(u) = 1 - \psi_k(u)$. Then, by conditioning on the amount of the first drop below the level $k \geqslant 0$, the ultimate ruin probability, for $u = k$, satisfies

$$\psi_k(k) = \int_0^k g(0, y)\psi_k(k)\, dy + \int_k^\infty g(0, y)\, dy,$$

where $g(u, y)$ is the density of the distribution of the deficit at ruin, $G(u, y)$, given by equation (1.5.1). After some algebraic manipulations, an expression for $\psi_k(k)$, in terms of the classic probability of ruin (without capital injections), is given by

$$\psi_k(k) = \frac{\psi(0) - G(0,k)}{1 - G(0,k)}.$$

Then, for the more general situation of $u > k \geqslant 0$, by considering the amount of the first drop below the level $k \geqslant 0$, we have

$$\phi_k(u) = \phi(u-k) + G(u-k,k)\phi_k(k),$$

or equivalently

$$\psi_k(u) = \psi(u-k) - G(u-k,k)\frac{1-\psi(0)}{1-G(0,k)}, \tag{2.2.1}$$

where $\psi(u)$ is the classic ruin probability defined in equation (1.2.3).

Remark 7. *The advantage of expressing the ultimate ruin probability, $\psi_k(u)$, in terms of the classic ruin probability is that the classic quantities have been extensively studied and explicit expressions can be found for many claim size distributions.*

The premium principle for the reinsurance premium, $Q(u,k)$, is calculated based on the moments of the expected (non)discounted capital injections until ruin. Using a similar argument as in Pafumi (1998), Nie et al. (2011) obtain explicit expressions for the first and second moments of the expected (non)discounted capital injections, as follows: Let the total accumulated capital injections, up to time $t \geqslant 0$, be denoted by the pure jump process $\{Z(t)\}_{t\geqslant 0}$ and consider $\mathbb{E}(Z_{u,k})$, where $Z_{u,k} = Z(T)$ is the accumulated capital injections up to the time of ultimate ruin, given the initial capital level $u \geqslant k$. Then, for $u = k$, using a similar argument as in Pafumi (1998), $\mathbb{E}(Z_{k,k})$ satisfies

$$\mathbb{E}(Z_{k,k}) = \int_0^k \left(y + \mathbb{E}(Z_{k,k})\right) g(0,y)\, dy, \tag{2.2.2}$$

which yields the solution

$$\mathbb{E}(Z_{k,k}) = \frac{\int_0^k y g(0,y)\, dy}{1 - G(0,k)}. \tag{2.2.3}$$

Employing a similar argument as above, it is shown that $\mathbb{E}(Z_{u,k})$, for $u > k$ satisfies

$$\begin{aligned}\mathbb{E}(Z_{u,k}) &= \int_0^k (y + \mathbb{E}(Z_{k,k}))\, g(u-k,y)\, dy \\ &= \int_0^k yg(u-k,y)\, dy + \mathbb{E}(Z_{k,k})G(u-k,k). \end{aligned} \tag{2.2.4}$$

The second moment, $\mathbb{E}(Z_{u,k}^2)$ is calculated using a similar methodology, however, for the necessity of this thesis, we omit the calculations. In addition to the explicit expressions for the first and second moments of the expected (non)discounted capital injections, using the renewal property of the process and the independence between successive capital injections (provided they occur), they derive the moment generating function for this quantity. Starting from the case $u = k$, the probability that there exists a capital injection is the probability that the surplus process drops, due to a claim, within the interval $[0, k)$, which happens with probability $G(0, k)$ and the process restarts from the level $k \geqslant 0$. Hence, if we let N denote the number of capital injections up to the time of ultimate ruin, by the above reasoning, N has a geometric distribution with probability mass function (p.m.f.), for $n = 0, 1, 2, \ldots$, given by

$$\mathbb{P}(N = n) = G(0,k)^n \left(1 - G(0,k)\right),$$

and, since the capital injection sizes, denoted by the sequence $\{V_i\}_{i=1}^{\infty}$, are i.i.d. random variables, $Z_{k,k}$ has a compound geometric form given by

$$Z_{k,k} = \sum_{i=1}^{N} V_i,$$

where the common random variable V has p.d.f.

$$f_V(y) = \frac{g(0,y)}{G(0,k)}, \quad 0 \leqslant y < k.$$

Therefore, it follows that the moment generating function of $Z_{k,k}$, denoted by $M_{Z_{k,k}}(s)$, is given by

$$M_{Z_{k,k}}(s) = \frac{1 - G(0,k)}{1 - \int_0^k e^{sx} g(0,x)\, dx}. \tag{2.2.5}$$

Remark 8. *We point out that in the paper of Nie et al. (2011), there is a small error in the remaining calculations. As you will see in the following, for the general case $Z_{u,k}$, for $u > k$, it is necessary to consider a distribution, which is a mixture of a degenerate distribution and a continuous distribution. In their paper, they forget to consider the degenerate part of this distribution. Therefore, in the following, we present the corrected version of their derivation.*

In order to find the moment generating function for the case $u > k$, we first note that $Z_{u,k}$ is equivalent in distribution to $(Y_u + Z_{k,k})\mathbb{I}_{\{A\}}$, where Y_u is the amount of the first capital injection, starting from initial capital $u > k$ and $\mathbb{I}_{\{A\}}$ is the indicator function with respect to the event that a capital injections occurs from initial capital u, denoted A, with probability

$$\mathbb{P}(A) = G(u - k, k).$$

Then, by definition, the p.d.f. of Y_u is given by $f_Y(y) = g(u - k, y)/G(u - k, k)$ and, since Y_u and $Z_{k,k}$ are independent, the moment generating function of $Z_{u,k}$, is given by

$$\begin{aligned} M_{Z_{u,k}}(s) &= \mathbb{E}\left(e^{s\left(Y_u + Z_{k,k}\right)\mathbb{I}_{\{A\}}}\right) \\ &= \mathbb{P}(A)\left(M_{Y_u}(s) M_{Z_{k,k}}(s)\right) + \mathbb{P}(A^c). \end{aligned}$$

Then, since

$$M_{Y_u}(s) = \frac{\int_0^k e^{sy} g(u - k, y)\, dy}{G(u - k, k)},$$

we have

$$M_{Z_{u,k}}(s) = 1 + (1 - G(0, k))\left(\frac{\int_0^k e^{sy} g(u - k, y)\, dy}{1 - \int_0^k e^{sx} g(0, x)\, dx}\right) - G(u - k, k). \tag{2.2.6}$$

More recently, Nie et al. (2015) generalise the capital injection model, considered above, to the Sparre Andersen (renewal) setting. They show that the density of the time to ultimate ruin, in the capital injection amended surplus process, can be expressed in terms of the density of the time to ruin in the ordinary Sparre Andersen risk model. Finally, Dickson and Qazvini (2016) construct a Gerber-Shiu like function for the model proposed in Nie et al. (2011), incorporating the number of claims until ultimate ruin, and derive an explicit expression for its LT. Although the result is not as efficient for finding an expression for the ultimate ruin probability, compared to the derivation in Nie et al. (2011), it provides an effective way to study ruin related quantities in finite time. In particular, they derive a general expression for the joint distribution of the time to ruin and the number of claims until ruin.

In the next section, we introduce a compound Poisson risk model that complies with the SII capital requirements described above, where a graphical interpretation of the model is given and the probability of insolvency is defined and explained.

2.3 The Solvency II risk model

In this section, we will consider the classical compound Poisson risk model, given in equation (1.1.2), amended to incorporate capital recovery plans that are required under the SII regulatory framework. In order to introduce the SII characteristics discussed above, we assume the following:

1. If the surplus falls below the $SCR\,(\equiv k)$ level, due to the occurrence of a claim, then the shareholders in the company inject capital instantaneously to cover this fall, given that the MCR $(\equiv \tilde{b})$ level has not been crossed. The sum of total capital injections, up to time $t \geqslant 0$, is defined by the pure jump process $\{Z(t)\}_{t\geqslant 0}$.

2. Additionally, motivated by practice, we assume that there exists an intermediate capital level barrier, in between the SCR and MCR, which indicates the confidence level for which the shareholders are prepared to inject capital to restore the surplus to the SCR level. If this intermediate level is breached, then the recovery action of the insurance firm is to borrow capital from a third party, which needs to be repaid subject to debit interest, until the confidence level of the shareholders is reached and hence, the SCR can be restored by a capital injection. We call this intermediate barrier the 'confidence' level and denote it by b, where $k \geqslant b \geqslant \tilde{b} \geqslant 0$. That is, if there exists a drop, due to a claim, of the surplus below the confidence level b, the insurance firm is required to borrow an amount of money equal to the

size of the deficit below b, at a debit force $\delta > 0$, given that the (MCR) level $\tilde{b}$ has not been crossed.

3. When the surplus is within the debit interval $(\tilde{b}, b)$, debts are repaid continuously from the premium income. During this period of time, the insurance firm can either recover back to level b (where the shareholders have renewed confidence and will instantaneously inject the amount $k - b$ in order to restore the surplus to the SCR level) or becomes insolvent by falling, due to further claims, below the MCR level [see Fig: 2.2].

Remark 9. *The classic risk model, with the addition of debit interest on a negative surplus, was originally studied in Gerber (1971), with further generalisations considered in Dassios and Embrechts (1989), Embrechts and Schmidli (1994) and Cai (2007) to name a few. Throughout the aforementioned papers, since the classical event of ruin no longer holds, a new definition of ruin is given, known as 'absolute ruin', which occurs at the point where the premiums are no longer sufficient to repay the interest payments on the loan. By similar reasonings, we point out that the value of the confidence level,* b, *must lie in the interval* $[\tilde{b}, \tilde{b} + \frac{c}{\delta}]$, *since the MCR level, namely* $\tilde{b}$, *is fixed. In order to emphasise the effects of the debit environment and for simplicity of calculations, in the remainder of this chapter we consider the case where* $b = \tilde{b} + c/\delta$, *which corresponds to the absolute ruin level defined in the literature.*

Considering the above features, the surplus process under the SII environment, denoted by $\{U_\delta^Z(t)\}_{t \geqslant 0}$, has dynamics of the following form

$$dU_\delta^Z(t) = \begin{cases} cdt - dS(t), & U_\delta^Z(t) \geqslant k, \\ k - \left(U_\delta^Z(t-) - \Delta S(t)\right), & b \leqslant U_\delta^Z(t-) - \Delta S(t) < k, \\ \left[c + \delta(U_\delta^Z(t) - b)\right] dt - dS(t), & \tilde{b} < U_\delta^Z(t) < b, \end{cases} \tag{2.3.1}$$

where $\Delta S(t) = S(t) - S(t-)$ and $\{S(t)\}_{t \geqslant 0}$ is a compound Poisson process as defined in Definition 4.

The crucial features of the proposed risk model, under SII regulations, are the capital management tools employed to reduce the probability of insolvency. Thus, it

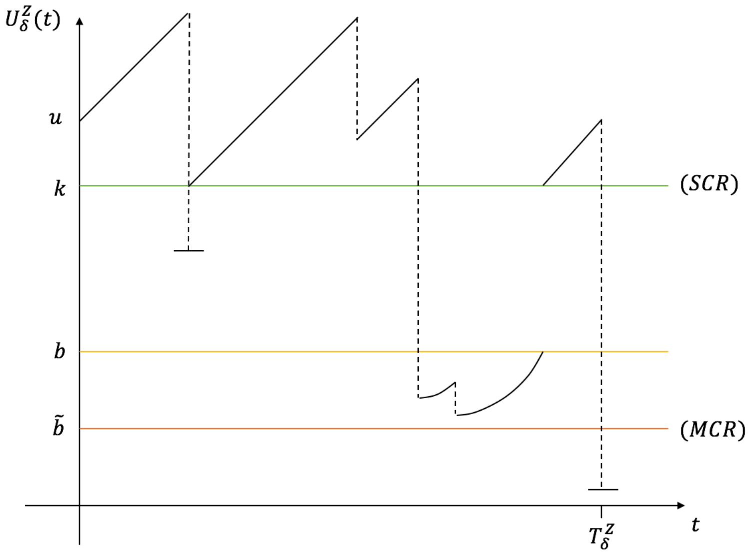


Figure 2.2: Typical sample path of the surplus process under SII constraints.

follows that for the surplus process $\{U_\delta^Z(t)\}_{t\geqslant 0}$, we should define the time to insolvency, denoted by T_δ^Z, as

$$T_\delta^Z = \inf\left\{t \geqslant 0 : U_\delta^Z(t) \leqslant \tilde{b}\right\},$$

with $T_\delta^Z = \infty$ if $U_\delta^Z(t) > \tilde{b}$ for all $t \geqslant 0$. Then, the *probability of insolvency*, which we denote $\psi_{\text{SII}}(u)$, is given by

$$\psi_{\text{SII}}(u) = \mathbb{P}\left(T_\delta^Z < \infty \middle| U_\delta^Z(0) = u\right),$$

with $\psi_{\text{SII}}(u) = 1$ for $u \leqslant \tilde{b}$ and $\phi_{\text{SII}}(u) = 1 - \psi_{\text{SII}}(u)$ is the corresponding solvency (survival) probability, denoting the probability that the insurance firm never experiences insolvency.

Note that for $\tilde{b} = 0$, the time of insolvency, T_δ^Z, corresponds to the classical time of ruin (in a risk model with capital injections and debit interest), which indicates that in reality, the ruin level is greater than zero.

Although the SII regulation employs a one-year VaR risk measure (due to the balance sheet approach), in this paper we analyse the infinite-time probability of insolvency. The main reason for focusing on the analysis of the probability of insolvency, namely $\psi_{\text{SII}}(u)$, is that we establish a closed form expression for the aforementioned risk quantity in terms of the ruin probability of the classical Cramér-Lundberg risk model, for which numerous powerful results exist in the actuarial literature, as seen in Chapter 1.

Finally, we point out, similar to Cai (2007), that $\psi_{\text{SII}}(u)$ has different sample paths for $u \geqslant k$ and $\tilde{b} < u < b$. Therefore, we distinguish between the two situations by denoting $\psi_{\text{SII}}(u) = \psi^{+}_{\text{SII}}(u)$ for $u \geqslant k$ and $\psi_{\text{SII}}(u) = \psi^{-}_{\text{SII}}(u)$ for $\tilde{b} < u < b$. Due to the instantaneous capital injection when the surplus lies within the interval $[b, k)$ we say that for $b \leqslant u < k$, $\psi_{\text{SII}}(u) = \psi^{+}_{\text{SII}}(k)$. It follows that the corresponding solvency (survival) probabilities are given by $\phi_{\text{SII}}(u) = 1 - \psi_{\text{SII}}(u) = \phi^{+}_{\text{SII}}(u)$, for $u \geqslant k$, and $\phi_{\text{SII}}(u) = \phi^{-}_{\text{SII}}(u)$ for $\tilde{b} < u < b$. Finally, in order to ensure that insolvency is not certain, we assume the net profit condition, given in Definition 8, holds.

2.4 Ruin probabilities for the SII risk model

In this section, we derive a closed form expression for the probability of insolvency when $u \geqslant k$, namely $\psi^{+}_{\text{SII}}(u)$, in terms of the infinite-time ruin probability of the classical risk model and an exiting (hitting) probability between two barriers. Note that $\psi^{+}_{\text{SII}}(u)$, is the risk quantity of primary interest as it is assumed (in compliance with SII regulation) that the insurance firm starts from a solvent level, i.e. $u \geqslant k$. Ultimately, we show that the probability of insolvency is proportional to the classical ruin function. Corresponding formulae for $\psi^{-}_{\text{SII}}(u)$, $\tilde{b} < u < b$, are also derived.

Before we proceed, we first define some ruin related quantities that will be extensively used in the following. First, let us define the first time the surplus process crosses the barrier k, for $u \geqslant k$, denoted by T^k, such that

$$T^k = \inf\{t \geqslant 0 : U^Z_\delta(t) < k\}, \tag{2.4.1}$$

with the corresponding probability of down-crossing the barrier k, defined by

$$\xi(u, k) = \mathbb{P}\left(T^k < \infty \middle| U^Z_\delta(0) = u \geqslant k\right).$$

Recalling the behaviour of the surplus process, $\{U_\delta^Z(t)\}_{t\geqslant 0}$, given in equation (2.3.1), it is clear that the dynamics above the barrier k are identical to that of the classical surplus process under a barrier free environment, i.e. for $u \geqslant k$, we have $dU_\delta^Z(t) \equiv d\widetilde{U}(t)$ where

$$\widetilde{U}(t) = \tilde{u} + ct - S(t), \quad t \geqslant 0,$$

with $\widetilde{U}(0) = \tilde{u} = u - k$. Then, it should be clear that T^k, defined by equation (2.4.1), is equivalent to the time of ruin in the classical Cramér-Lundberg risk model with no barrier modification and initial capital $\tilde{u} \geqslant 0$, i.e.

$$T^k = \inf\{t \geqslant 0 : \widetilde{U}(t) < 0 | \, \widetilde{U}(0) = \tilde{u}\},$$

where $T^0 \equiv T$ as defined in equation (1.2.1). Hence, the function $\xi(u,k)$ is identical to the classic ruin probability with initial capital $\tilde{u}$ and can be expressed as $\psi(\tilde{u}) = \mathbb{P}(T^k < \infty) = 1 - \phi(\tilde{u})$.

Extending the arguments of Nie et al. (2011), by conditioning on the occurrence and size of the first drop below k, for $u \geqslant k$, and using the fact that $dU_\delta^Z(t) \equiv d\widetilde{U}(t)$ above the barrier k, we obtain an expression for the solvency probability, $\phi_{\mathrm{SII}}^+(u)$, given by

$$\begin{aligned}
\phi_{\mathrm{SII}}^+(u) &= \phi(\tilde{u}) + \int_0^{k-b} g(\tilde{u},y)\phi_{\mathrm{SII}}^+(k)\, dy + \int_{k-b}^{k-\tilde{b}} g(\tilde{u},y)\phi_{\mathrm{SII}}^-(k-y)\, dy \\
&= \phi(\tilde{u}) + G(\tilde{u}, k-b)\phi_{\mathrm{SII}}^+(k) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u},y)\phi_{\mathrm{SII}}^-(k-y)\, dy, \qquad (2.4.2)
\end{aligned}$$

where

$$G(\tilde{u},y) = \mathbb{P}\left(T^k < \infty, |\widetilde{U}(T^k)| \leqslant y \big| \widetilde{U}(0) = \tilde{u}\right),$$

is the joint distribution of down-crossing the barrier k and experiencing a deficit (below k) of at most y, with $g(\tilde{u},y) = \frac{\partial}{\partial y}G(\tilde{u},y)$ the corresponding density function. This risk quantity is simply the deficit at ruin, given in Definition 15, with a shifted initial capital.

Note that, in the above expression, $\phi_{\mathrm{SII}}^+(u)$ is given in terms of $\phi_{\mathrm{SII}}^-(u)$. In order to derive an analytic expression for $\phi_{\mathrm{SII}}^+(u)$, independent of $\phi_{\mathrm{SII}}^-(u)$, we introduce the following hitting probability.

Let $\chi_\delta(u, b, \tilde{b}) \equiv \chi_\delta(u)$ be the probability that the surplus process hits the upper barrier b, before hitting the lower barrier $\tilde{b}$ from initial capital $\tilde{b} < u < b$, when subject to a debit force $\delta > 0$, defined by

$$\chi_\delta(u) = \mathbb{P}\left(T^b < T_\delta^Z \middle| U_\delta^Z(0) = u\right), \quad \tilde{b} < u < b, \tag{2.4.3}$$

where

$$T^b = \inf\left\{t \geqslant 0 : U_\delta^Z(t) = b\right\}.$$

Then, by conditioning on which of the barriers the surplus hits first, from initial capital $\tilde{b} < u < b$, we are able to express the solvency probability, $\phi_{\text{SII}}^-(u)$, in terms of the hitting probability $\chi_\delta(u)$, however, before we can employ such a conditioning argument, it is necessary to prove that the surplus process hits one of these two barriers a.s.. In order to prove this, we will need the following Lemma.

Lemma 3. *(Second Borel-Cantelli Lemma) Let $E_1, E_2, \ldots$ be a sequence of independent events in some probability space. If the infinite sum of the probabilities of E_n diverges, i.e.*

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) = \infty,$$

then, the probability that infinitely many of them occur is 1, that is,

$$\mathbb{P}\left(\limsup_{n\to\infty} E_n\right) = 1.$$

Proposition 6. *For $\tilde{b} < u < b$, the surplus process, $\{U_\delta^Z(t)\}_{t\geqslant 0}$, will hit one of the two barriers $\tilde{b}$ or b, over an infinite-time horizon, a.s..*

Proof. Using similar arguments as in Cai (2007), we first note that when the surplus process is within the interval $(\tilde{b}, b)$, it is driven by the debit interest force $\delta > 0$, until the surplus returns to level b (or experiences insolvency). Therefore, for initial capital $\tilde{b} < u < b$, the process is immediately subject to debit interest on the amount $b - u > 0$ and the evolution of the surplus process (assuming no claims appear up to time $t \geqslant 0$), due to the dynamics of the process below the barrier b, can be expressed by

$$h(t;u,b) = b + (u-b)e^{\delta t} + c\int_0^t e^{\delta s}\,ds, \qquad t \geqslant 0. \tag{2.4.4}$$

Let us further define $t_0 \equiv t_0(u,b)$ to be the solution to $h(t;u,b) = b$. Then

$$t_0 = \ln\left(\frac{c}{\delta(u-b)+c}\right)^{1/\delta} < \infty, \quad \text{for} \quad \tilde{b} < u < b, \tag{2.4.5}$$

is the time taken for the surplus to reach the upper level b, i.e. $h(t_0;u,b) = b$, in the absence of claims and $h(t;u,b) \in (\tilde{b}, b)$ for all $t < t_0$. Therefore, it is clear that the surplus process will recover to the upper level b, if no claims occur before time $0 \leqslant t_0 < \infty$.

Now, consider the events $E_n = \{\tau_n > t_0\}$, where $\{\tau_n\}_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables denoting the inter-arrival time between the $(n-1)$-th and n-th claim and t_0 is defined above. Then, since the inter-arrival times are i.i.d and it is assumed that the claims occur according to a Poisson process, it follows that, for all $n \in \mathbb{N}$, the events E_n are independent and we have

$$\mathbb{P}(E_n) = \mathbb{P}(\tau_n > t_0) = e^{-\lambda t_0} > 0.$$

Therefore, it follows that

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) = \infty,$$

and thus, by Lemma 3, it follows that

$$\mathbb{P}\left(\limsup_{n\to\infty}\{\tau_n > t_0\}\right) = 1.$$

That is, the event $\{\tau_n > t_0\}$ occurs infinitely often with probability 1 (a.s.). □

Now, by conditioning on which of the barriers the surplus hits first, using Proposition 6 and noticing that $\phi^{-}_{\mathrm{SII}}(x) = 0$ for $x \leqslant \tilde{b}$, from the law of total probability it follows that

$$\phi^{-}_{\text{SII}}(u) = \chi_\delta(u)\phi^{+}_{\text{SII}}(k). \tag{2.4.6}$$

Substituting the above expression into equation (2.4.2), we obtain

$$\phi^{+}_{\text{SII}}(u) = \phi(\tilde{u}) + \phi^{+}_{\text{SII}}(k)\left[G(\tilde{u}, k-b) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u}, y)\chi_\delta(k-y)\,dy\right]. \tag{2.4.7}$$

To complete the above expression for $\phi^{+}_{\text{SII}}(u)$, the boundary condition, $\phi^{+}_{\text{SII}}(k)$, and the hitting probability, $\chi_\delta(u)$, need to be determined. Setting $u = k$ in equation (2.4.7), and solving the resulting equation for $\phi^{+}_{\text{SII}}(k)$, yields

$$\phi^{+}_{\text{SII}}(k) = \frac{\phi(0)}{1 - \left(G(0, k-b) + \int_{k-b}^{k-\tilde{b}} g(0, y)\chi_\delta(k-y)\,dy\right)}, \tag{2.4.8}$$

and thus, equation (2.4.7) may be expressed, for $u \geqslant k$, as

$$\phi^{+}_{\text{SII}}(u) = \phi(\tilde{u}) + \frac{\phi(0)\left[G(\tilde{u}, k-b) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u}, y)\chi_\delta(k-y)\,dy\right]}{1 - \left(G(0, k-b) + \int_{k-b}^{k-\tilde{b}} g(0, y)\chi_\delta(k-y)\,dy\right)}.$$

Now, recalling that $\phi^{+}_{\text{SII}}(u) = 1 - \psi^{+}_{\text{SII}}(u)$, for $u \geqslant k$, and using the result of Theorem 11 for the joint density function $g(0, y)$, the probability of insolvency, namely $\psi^{+}_{\text{SII}}(u)$, has the form

$$\psi^{+}_{\text{SII}}(u) = \psi(\tilde{u}) - \frac{\phi(0)\left[G(\tilde{u}, k-b) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u}, y)\chi_\delta(k-y)\,dy\right]}{1 - \frac{\lambda}{c}\left(\mu F^{s}_{X}(k-b) + \int_{k-b}^{k-\tilde{b}} \overline{F}_X(y)\chi_\delta(k-y)\,dy\right)}, \tag{2.4.9}$$

where $F^{s}_{X}(x)$ is the integrated tail distribution defined in equation (1.2.12).

Finally, by considering the proper d.f. $G_{\tilde{u}}(y) = G(\tilde{u}, y)\,(\psi(\tilde{u}))^{-1}$, as defined in equation (1.5.3), with corresponding density $g_{\tilde{u}}(y) = g(\tilde{u}, y)\,(\psi(\tilde{u}))^{-1}$, we obtain the following theorem for the probability of insolvency. Note that similar arguments as above can be applied to obtain an expression for $\psi^{-}_{\text{SII}}(u)$, from equation (2.4.6).

Theorem 17. *For $u \geqslant k$, the probability of insolvency, $\psi^{+}_{SII}(u)$, is given by*

$$\psi^{+}_{SII}(u) = \psi(\tilde{u}) \left[1 - \frac{\phi(0) \left[G_{\tilde{u}}(k-b) + \int_{k-b}^{k-\tilde{b}} g_{\tilde{u}}(y)\chi_{\delta}(k-y)\, dy \right]}{1 - \frac{\lambda}{c}\left(\mu F^{s}_{X}(k-b) + \int_{k-b}^{k-\tilde{b}} \overline{F}_{X}(y)\chi_{\delta}(k-y)\, dy \right)} \right], \tag{2.4.10}$$

where $\psi(\tilde{u})$ is the ruin probability of the classical risk model and $\tilde{u} = u - k$.

For $\tilde{b} < u < b$, $\psi^{-}_{SII}(u)$ is given by

$$\psi^{-}_{SII}(u) = 1 - \frac{\phi(0)\chi_{\delta}(u)}{1 - \frac{\lambda}{c}\left(\mu F^{s}_{X}(k-b) + \int_{k-b}^{k-\tilde{b}} \overline{F}_{X}(y)\chi_{\delta}(k-y)\, dy \right)}. \tag{2.4.11}$$

From equations (2.4.10) and (2.4.11), it follows that the two types of insolvency probabilities are given in terms of the (shifted) ruin probability and deficit of the classical risk model, as well as the probability of exiting between two barriers. Thus, $\psi^{+}_{\mathrm{SII}}(\cdot)$ and $\psi^{-}_{\mathrm{SII}}(\cdot)$ can be calculated by employing the well known results, with respect to $G_{\tilde{u}}(\cdot)$ and $\psi(\cdot)$ [see Chapter 1 and the references therein], whilst the latter exiting probability, $\chi_{\delta}(u)$, can be determined from the following proposition:

Proposition 7. *For $\tilde{b} < u < b$, the probability of the surplus process, $\{U^{Z}_{\delta}(t)\}_{t \geqslant 0}$, hitting the upper barrier, b, before hitting the lower barrier, $\tilde{b}$, (under a debit force $\delta > 0$), denoted $\chi_{\delta}(u)$, satisfies the following integro-differential equation*

$$(\delta(u-b) + c)\chi'_{\delta}(u) = \lambda\chi_{\delta}(u) - \lambda \int_{0}^{u-\tilde{b}} \chi_{\delta}(u-x)\, dF_{X}(x), \tag{2.4.12}$$

with boundary conditions

$$\lim_{u \uparrow b} \chi_{\delta}(u) = 1,$$
$$\lim_{u \downarrow \tilde{b}} \chi_{\delta}(u) = 0.$$

Proof. Recalling the definition of the function $h(t; u, b)$ and the hitting time t_0, defined in equations (2.4.4) and (2.4.5) respectively, and noting that the claims occur according to a Poisson process, by conditioning on the time and amount of the first claim, it follows that

$$\chi_\delta(u) = e^{-\lambda t_0} + \int_0^{t_0} \lambda e^{-\lambda t} \int_0^{h(t;u,b)-\tilde{b}} \chi_\delta\big(h(t;u,b) - x\big)\, dF_X(x)\, dt.$$

Using the change of variable $y = h(t;u,b)$ and the form of t_0 given in equation (2.4.5), we have that

$$\chi_\delta(u) = \left(\frac{\delta(u-b)+c}{c}\right)^{\frac{\lambda}{\delta}} + \lambda\left(\delta(u-b)+c\right)^{\frac{\lambda}{\delta}} \left[\int_u^b \left(\delta(y-b)+c\right)^{-\frac{\lambda}{\delta}-1} \times \int_0^{y-\tilde{b}} \chi_\delta(y-x)\, dF_X(x)\, dy\right]. \tag{2.4.13}$$

Differentiating the above equation, with respect to u, and combining the resulting equation with equation (2.4.13), we obtain equation (2.4.12). Moreover, the first boundary condition can be found by letting $u \to b$ in equation (2.4.13).

Now, for the second boundary condition, one can see that if

$$\lim_{u \downarrow \tilde{b}} \int_u^b \left[\big(\delta(y-b)+c\big)^{-\frac{\lambda}{\delta}-1} \int_0^{y-\tilde{b}} \chi_\delta(y-x)\, dF_X(x)\right] dy < \infty,$$

then

$$\lim_{u \downarrow \tilde{b}} \lambda\big(\delta(u-b)+c\big)^{\frac{\lambda}{\delta}} \int_u^b \left[\big(\delta(y-b)+c\big)^{-\frac{\lambda}{\delta}-1} \int_0^{y-\tilde{b}} \chi_\delta(y-x)\, dF_X(x)\right] dy = 0,$$

since $b = \tilde{b} + \frac{c}{\delta}$. Alternatively, if

$$\lim_{u \downarrow \tilde{b}} \int_u^b \left[\big(\delta(y-b)+c\big)^{-\frac{\lambda}{\delta}-1} \int_0^{y-\tilde{b}} \chi_\delta(y-x)\, dF_X(x)\right] dy = \infty,$$

then, by L'Hopital's rule, we have

$$\lim_{u \downarrow \tilde{b}} \lambda\big(\delta(u-b)+c\big)^{\frac{\lambda}{\delta}} \int_u^b \left[\big(\delta(y-b)+c\big)^{-\frac{\lambda}{\delta}-1} \int_0^{y-\tilde{b}} \chi_\delta(y-x)\, dF_X(x)\right] dy = 0.$$

Hence, using the above limiting results and taking the limit $u \to \tilde{b}$, in equation (2.4.13), we obtain the second boundary condition. □

Recalling the forms of the insolvency probabilities, given by Theorem 17, it can be seen that both depend heavily on the solution of the IDE (2.4.12), which is discussed in the next subsection.

2.4.1 Explicit expressions for exponential claim size distribution

In this subsection, we derive exact expressions for the two types of insolvency probabilities, given in Theorem 17, under the assumption of exponentially distributed claim sizes. Then, by comparing these expressions with the classical ruin probability under exponentially distributed claims, we identify that the probability of insolvency is proportional to the probability of ruin in the classical model.

Let us assume the claim sizes are exponentially distributed with parameter $\beta > 0$ i.e. $F_X(x) = 1 - e^{-\beta x}$, $x \geqslant 0$. Then, equation (2.4.12) can be written as

$$(\delta(u-b)+c)\chi_\delta'(u) = \lambda\chi_\delta(u) - \lambda\int_{\tilde{b}}^{u} \beta e^{-\beta(u-x)}\chi_\delta(x)\,dx, \qquad \tilde{b} < u < b.$$

The above IDE can be solved as a boundary value problem, since, from Proposition 7, the boundary conditions at $\tilde{b}$ and b are given. Differentiating the above equation with respect to u, yields a second-order homogeneous ODE of the form

$$\chi_\delta''(u) + p(u)\chi_\delta'(u) = 0, \tag{2.4.14}$$

where

$$p(u) = \frac{\delta - \lambda + \beta(\delta(u-b)+c)}{\delta(u-b)+c} = \frac{\delta-\lambda}{\delta(u-b)+c} + \beta. \tag{2.4.15}$$

Employing the general theory of differential equations, the above ODE has a general solution of the form

$$\chi_\delta'(u) = Ce^{-\int p(u)\,du},$$

where C is an arbitrary constant that needs to be determined. Recalling the form of

$p(u)$, given in equation (2.4.15), the above solution reduces to

$$\chi'_\delta(u) = Ce^{-\beta u}\left(\delta(u-b)+c\right)^{\frac{\lambda}{\delta}-1}.$$

Then, integrating the above equation from $\tilde{b}+\epsilon$ to u, for some small $\epsilon > 0$ and $\tilde{b} < u < b$, we have that

$$\chi_\delta(u) - \chi_\delta(\tilde{b}+\epsilon) = C\int_{\tilde{b}+\epsilon}^{u} e^{-\beta w}\left(\delta(w-b)+c\right)^{\frac{\lambda}{\delta}-1}\,dw,$$

which, after letting $\epsilon \to 0$ and using the second boundary condition of Proposition 7, the general solution of equation (2.4.14) is given by

$$\begin{aligned}\chi_\delta(u) &= C\int_{\tilde{b}}^{u} e^{-\beta w}\left(\delta(w-b)+c\right)^{\frac{\lambda}{\delta}-1}\,dw\\ &= Cc^{\frac{\lambda}{\delta}-1}\int_{\tilde{b}}^{u} e^{-\beta w}\left(\frac{\delta(w-b)}{c}+1\right)^{\frac{\lambda}{\delta}-1}\,dw. \qquad (2.4.16)\end{aligned}$$

Finally, in order to complete the solution we need to determine the constant C, which can be obtained by using the first boundary condition for $\chi_\delta(u)$ of Proposition 7, i.e. $\lim_{u\to b}\chi_\delta(u) = 1$. Letting $u \to b$ in equation (2.4.16), we obtain

$$\begin{aligned}C^{-1} &= c^{\frac{\lambda}{\delta}-1}\int_{\tilde{b}}^{b} e^{-\beta w}\left(\frac{\delta(w-b)}{c}+1\right)^{\frac{\lambda}{\delta}-1}\,dw\\ &= c^{\frac{\lambda}{\delta}-1}C_1^{-1},\end{aligned}$$

where $C_1^{-1} = \int_{\tilde{b}}^{b} e^{-\beta w}\left(\frac{\delta(w-b)}{c}+1\right)^{\frac{\lambda}{\delta}-1}\,dw.$

Proposition 8. *For $\tilde{b} < u < b$ and exponentially distributed claim sizes with parameter $\beta > 0$, the probability of the surplus process, $\{U_\delta^Z(t)\}_{t\geqslant 0}$, hitting the upper barrier b, before hitting the lower barrier $\tilde{b}$, under a debit force $\delta > 0$, is given by*

$$\chi_\delta(u) = C_1\int_{\tilde{b}}^{u} e^{-\beta w}\left(\frac{\delta(w-b)}{c}+1\right)^{\frac{\lambda}{\delta}-1}\,dw, \qquad (2.4.17)$$

where

$$C_1^{-1} = \int_{\tilde{b}}^{b} e^{-\beta w} \left(\frac{\delta(w-b)}{c} + 1 \right)^{\frac{\lambda}{\delta}-1} dw. \tag{2.4.18}$$

Using Theorem 17 and Proposition 8, the two types of insolvency probabilities, namely $\psi^+_{\mathrm{SII}}(u)$ and $\psi^-_{\mathrm{SII}}(u)$, under exponentially distributed claim amounts, are given in the following theorem.

Theorem 18. *Let the claim amounts be exponentially distributed with parameter $\beta > 0$. Then, for $u \geqslant k$, the probability of insolvency, $\psi^+_{SII}(u)$, is given by*

$$\psi^+_{SII}(u) = \frac{(1+\eta)e^{\frac{\lambda\eta}{c}k}}{1 + \frac{\lambda\eta}{c} C_1^{-1} e^{\beta k}} \psi(u), \tag{2.4.19}$$

where $\psi(u)$ is the classic ruin probability and, for $\tilde{b} < u < b$, $\psi^-_{SII}(u)$ is given by

$$\psi^-_{SII}(u) = 1 - \frac{\frac{\lambda\eta}{c} e^{\beta k} \int_{\tilde{b}}^{u} e^{-\beta w} \left(\frac{\delta(w-b)}{c} + 1 \right)^{\frac{\lambda}{\delta}-1} dw}{1 + \frac{\lambda\eta}{c} C_1^{-1} e^{\beta k}}, \tag{2.4.20}$$

where C_1^{-1} is given in Proposition 8.

Proof. Considering the numerator of equation (2.4.10), which is of the form

$$\phi(0) \left[G_{\tilde{u}}(k-b) + \int_{k-b}^{k-\tilde{b}} g_{\tilde{u}}(y) \chi_\delta(k-y)\, dy \right].$$

Assuming that the claim amounts are exponentially distributed, employing the corresponding forms for $G_{\tilde{u}}(y)$ and $g_{\tilde{u}}(y)$, which can be found from equation (1.5.3) and Proposition 4, and using equation (2.4.17) of Proposition 8, it follows that the above equation may be written as

$$\phi(0) \left[\left(1 - e^{-\beta(k-b)}\right) + C_1 \beta \int_{k-b}^{k-\tilde{b}} e^{-\beta y} \int_{\tilde{b}}^{k-y} e^{-\beta w} \left(\frac{\delta(w-b)}{c} + 1 \right)^{\frac{\lambda}{\delta}-1} dw dy \right]. \tag{2.4.21}$$

Changing the order of integration, evaluating the resulting inner integral and after some algebraic manipulations, equation (2.4.21) can be re-written in the form

$$\phi(0)\left[1-e^{-\beta(k-b)}\left(1-C_1\int_{\tilde{b}}^{b}e^{-\beta w}\left(\frac{\delta(w-b)}{c}+1\right)^{\frac{\lambda}{\delta}-1}dw\right)-C_1\frac{c}{\lambda}e^{-\beta k}\right],$$

which, after recalling the definition of the constant C_1 given in Proposition 8, reduces to the concise form

$$\phi(0)\left[1-C_1\frac{c}{\lambda}e^{-\beta k}\right]. \tag{2.4.22}$$

By considering a similar methodology as above, the corresponding denominator of equation (2.4.10) reduces to the form

$$1-\frac{1}{1+\eta}\left(1-C_1\frac{c}{\lambda}e^{-\beta k}\right). \tag{2.4.23}$$

Now, replacing the numerator and denominator, in equation (2.4.10) by equations (2.4.22) and (2.4.23), respectively, it follows that the insolvency probability, for $u \geqslant k$, is given by

$$\psi^{+}_{\mathrm{SII}}(u)=\psi(\tilde{u})\left(1-\frac{\phi(0)A}{1-\frac{1}{1+\eta}A}\right), \tag{2.4.24}$$

where

$$A=\left(1-C_1\frac{c}{\lambda}e^{-\beta k}\right).$$

Re-arranging equation (2.4.24), substituting $\phi(0)=\eta(1+\eta)^{-1}$ [see Lemma 1] and noticing that $\psi(\tilde{u})=\psi(u-k)=e^{\frac{\lambda\eta}{c}k}\psi(u)$, by equation (2.4.14), we obtain our result. For $\psi^{-}_{\mathrm{SII}}(u)$, given by equation (2.4.20), one can apply similar arguments and thus the proof is omitted. □

Remark 10. *(i) From equation (2.4.19), we conclude that the constant* $\frac{(1+\eta)e^{\frac{\lambda\eta}{c}k}}{1+\frac{\lambda\eta}{c}C_1^{-1}e^{\beta k}}$ *plays the role of a 'measurement of protection' for the insurer. Thus, given a set of parameters, the above factor could lead to a lower/higher value of* $\psi^{+}_{SII}(u)$*, compared to the classical ruin probability* $\psi(u)$*, in the sense that the insurer is more/less protected by the SII regulations.*

(ii) *Under SII, the SCR is calibrated to ensure a theoretical ruin of* 0.5% *[see Section 2.1]. Therefore, since equation* (2.4.19) *can be written as*

$$\psi^{+}_{SII}(u) = \frac{1}{1 + \frac{\lambda\eta}{c} C_1^{-1} e^{\beta k}} e^{-\frac{\lambda\eta}{c}(u-k)}, \tag{2.4.25}$$

it follows that, by setting $\psi^{+}_{SII}(u) = 0.5\%$*, and given a set of parameters, we can obtain the value of the SCR, simply by solving equation* (2.4.25) *with respect to* k*. This provides a quick and convenient approximation to the SCR under SII requirements.*

Remark 11. *If we set* $k = b = 0$ *such that* $\tilde{b} = -\frac{c}{\delta}$*, then equation* (2.4.19) *becomes*

$$\psi^{+}_{SII}(u) = \frac{e^{-\frac{\lambda\eta}{c}u}}{1 + \frac{\lambda\eta}{c} C_1^{-1}} \quad u \geqslant 0,$$

where $C_1^{-1} = \int_{-\frac{c}{\delta}}^{0} e^{-\beta w} \left(\frac{\delta w}{c} + 1\right)^{\frac{\lambda}{\delta}-1} dw$ *and thus we retrieve Theorem 12 of Dassios and Embrechts (1989) for the ruin probability in the classical risk model with debit interest, under exponentially distributed claim sizes.*

Example (Comparison of SII insolvency versus the classical ruin probability)**.** *In order to compare the insolvency probability* $\psi^{+}_{SII}(u)$*,* $u \geqslant k$*, with the classical ruin probability,* $\psi(u)$*, recall that under exponentially distributed claim sizes,* $\psi(u)$ *is given by*

$$\psi(u) = \frac{1}{1+\eta} e^{-\frac{\lambda\eta}{c}u}, \quad u \geqslant 0.$$

In addition, we consider the following set of parameters $\lambda = \beta = 1$*,* $\eta = 5\%$*, which due to the net profit condition, fixes our premium rate at* $c = 1.05$*. We further set the debit force* $\delta = 0.05$ *and the fixed MCR barrier* $\tilde{b} = 3$*, which in turn gives* $b = 24$*, since* $b = \tilde{b} + \frac{c}{\delta}$*. Table 2.1 (below) shows the comparison of the classical ruin probability and the SII insolvency probability for several values of* u *and SCR level* k *such that* $u \geqslant k > b = 24$*.*

	$k = 25$		$k = 30$		$k = 50$	
u	$\psi(u)$	$\psi^+_{\text{SII}}(u)$	$\psi(u)$	$\psi^+_{\text{SII}}(u)$	$\psi(u)$	$\psi^+_{\text{SII}}(u)$
k	0.290	0.509	0.228	6.933×10^{-3}	0.088	1.439×10^{-11}
$k+5$	0.228	0.401	0.180	5.464×10^{-3}	0.069	1.134×10^{-11}
$k+10$	0.180	0.316	0.142	4.306×10^{-3}	0.055	8.938×10^{-12}
$k+15$	0.142	0.249	0.112	3.394×10^{-3}	0.043	7.044×10^{-12}
$k+20$	0.112	0.196	0.088	2.675×10^{-3}	0.034	5.552×10^{-12}

Table 2.1: Classical ruin against SII insolvency probabilities, exponential claims.

Furthermore, in Table 2.2 (below), numerics for the required initial capital are given in the case of a fixed probability of insolvency and SCR level.

	u		
$\psi^+_{\text{SII}}(u)$	$k = 25$	$k = 26$	$k = 27$
0.1	59.17	47.32	31.34
0.05	73.72	61.87	45.90
0.025	88.28	76.43	60.46
0.01	107.52	95.67	79.70

Table 2.2: Initial capital required for varying insolvency probabilities and SCR levels.

Remark 12. *Numerical results for the ruin probability $\psi^-_{SII}(u)$ are not given, for reasons explained in Section 3.*

2.4.2 Asymptotic results for the probability of insolvency

In the previous section, we derived an expression for the insolvency probability, $\psi^+_{\text{SII}}(u)$, in terms of the classic ruin quantities. Therefore, this result provides us with an explicit expression, provided explicit expressions exist for the corresponding classic ruin quantities for different claim size distributions. However, we can exploit the form of this expression, using the fact that $\psi^+_{\text{SII}}(u)$ is given in terms of $\psi(\cdot)$ and $G.(\cdot)$, to derive the asymptotic behaviour, for the probability of insolvency, under a general claim size distribution. Note that an asymptotic expression for the ruin probability $\psi^-_{\text{SII}}(u)$ cannot be considered since the initial capital is bounded, i.e. $\tilde{b} < u < b$.

In order to do this, we first need to obtain the asymptotic behaviour of the functions $G_{\tilde{u}}(y)$ and $g_{\tilde{u}}(y)$. Recall, from equation (1.5.5), that the deficit at ruin function, $G(u, y)$, satisfies the defective renewal equation

$$G(u,y) = \frac{\lambda}{c}\int_0^u G(u-x,y)\overline{F}_X(x)\,dx + \frac{\lambda}{c}\int_u^{u+y}\overline{F}_X(x)\,dx. \tag{2.4.26}$$

Then, following a similar methodology as in Section 1.4.1, we assume there exists a constant $\gamma > 0$, known as the adjustment coefficient, such that

$$\frac{\lambda}{c}\int_0^\infty e^{\gamma x}\overline{F}_X(x)\,dx = 1,$$

and it follows, from algebraic manipulations and application of the Key Renewal Theorem [see Theorem 8], that

$$\lim_{u\to\infty} e^{\gamma u}G(u,y) = \frac{\int_0^\infty e^{\gamma t}\int_t^{t+y}\overline{F}_X(x)\,dxdt}{\int_0^\infty te^{\gamma t}\overline{F}_X(t)\,dt}.$$

Now, using the expression for the asymptotic behaviour of the ruin function, $\psi(u)$, given by equation (1.4.5), and since $G_u(y) = \frac{G(u,y)}{\psi(u)}$, we have

$$\lim_{u\to\infty} G_u(y) = \frac{\int_0^\infty e^{\gamma t}\int_t^{t+y}\overline{F}_X(x)\,dxdt}{\int_0^\infty e^{\gamma t}\int_t^\infty\overline{F}_X(x)\,dxdt},$$

from which it follows, by differentiating the above equation with respect to y, that

$$\lim_{u\to\infty} g_u(y) = \frac{\int_0^\infty e^{\gamma t}\overline{F}_X(t+y)dt}{\int_0^\infty e^{\gamma t}\int_t^\infty\overline{F}_X(x)\,dxdt}.$$

Finally, by combining the above asymptotic expressions for $G_u(y)$, $g_u(y)$ and the form of the insolvency probability, $\psi^+_{\mathrm{SII}}(u)$, given in equation (2.4.10), the asymptotic behaviour of $\psi^+_{\mathrm{SII}}(u)$, as $u \to \infty$, is given by the following Proposition.

Proposition 9. *The probability of insolvency, $\psi^+_{SII}(u)$, behaves asymptotically as*

$$\psi^+_{SII}(u) \sim K\psi(u), \quad u\to\infty,$$

where $\psi(u)$ is the classical ruin probability of the Cramér-Lundberg risk model and K is a constant given by

$$K = 1 - \frac{\phi(0)\left[\int_0^\infty e^{\gamma t}\int_t^{t+(k-b)}\overline{F}_X(x)\,dxdt + \int_{k-b}^{k-\tilde{b}}\int_0^\infty e^{\gamma t}\overline{F}_X(t+y)\chi_\delta(k-y)\,dt\,dy\right]}{\frac{\mu\eta}{\gamma}\left(1-\frac{\lambda}{c}\left(\mu F_X^s(k-b)+\int_{k-b}^{k-\tilde{b}}\overline{F}_X(y)\chi_\delta(k-y)\,dy\right)\right)}.$$

2.5 Probability characteristics of the accumulated capital injections

In this section we analyse the probabilistic characteristics of the accumulated capital injections up to the time of insolvency, where we derive an explicit expression for the first moment and an analytic form of the moment generating function. For the latter, we show that the distribution of the accumulated capital injections up to the time of insolvency is a mixture of a degenerate distribution at zero and a continuous distribution.

2.5.1 Expected accumulated capital injections up to the time of insolvency

The main source of capital injections, in reality, comes from the companies shareholders who are willing to inject capital up to some level. This confidence level could correspond to a maximum limit for any one transaction, or alternatively, could be determined depending on their expected risk exposure and performance of the business. In the latter case, the shareholders require information about their expected liabilities in terms of capital injections contributed to the company. That is, the accumulated capital injections up to the time of insolvency. An alternative motivation to analyse such a quantity is discussed in Nie et al. (2011), where the injections are provided via a reinsurance agreement, for which the premium principle is based on the accumulated capital injections up to the time of ruin [see Section 2.2].

We point out that in the above set up, the confidence level of the shareholders has been fixed at $b = \tilde{b} + \frac{c}{\delta}$, however, in general, this level can vary within the interval $[\tilde{b}, \tilde{b} + \frac{c}{\delta}]$ for which the following analysis holds.

Let us denote the accumulated capital injections up to the time of insolvency, from initial capital $u \geqslant 0$, by $Z_{u,k} = Z(T_\delta^Z)$. Then, we are interested in the quantity $\mathbb{E}(Z_{u,k})$. Due to similar reasons as for the insolvency probability, it is necessary to decompose $\mathbb{E}(Z_{u,k})$ depending on the size of the initial capital. Therefore, we define $\mathbb{E}(Z_{u,k}) = \mathbb{E}(Z^+_{u,k})$ when $u \geqslant k$ and $\mathbb{E}(Z_{u,k}) = \mathbb{E}(Z^-_{u,k})$, when $\tilde{b} < u < b$. Using a similar argument as in the previous section (conditioning on the amount of the first drop below the SCR barrier k), it follows that $\mathbb{E}(Z^+_{u,k})$, for $u \geqslant k$, satisfies

$$\begin{aligned}
\mathbb{E}(Z^+_{u,k}) &= \int_0^{k-b} \left(y + \mathbb{E}(Z^+_{k,k})\right) g(\tilde{u}, y)\, dy + \int_{k-b}^{k-\tilde{b}} \left((k-b) + \mathbb{E}(Z^+_{k,k})\right) g(\tilde{u}, y)\chi_\delta(k-y)dy \\
&= \int_0^{k-b} y g(\tilde{u}, y)\, dy + (k-b) \int_{k-b}^{k-\tilde{b}} g(\tilde{u}, y)\chi_\delta(k-y)dy \\
&\qquad + \mathbb{E}(Z^+_{k,k}) \left[G(\tilde{u}, k-b) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u}, y)\chi_\delta(k-y)dy\right],
\end{aligned} \tag{2.5.1}$$

where $\chi_\delta(u)$, is given by Proposition 7. In order to complete the calculation for $\mathbb{E}(Z^+_{u,k})$, given by the above expression, we need to determine the boundary value $\mathbb{E}(Z^+_{k,k})$, which represents the expected accumulated capital injections up to the time of insolvency from initial capital $u = k$ and can be obtained by setting $u = k$ in equation (2.5.1). Then, we have

$$\begin{aligned}
\mathbb{E}(Z^+_{k,k}) = \int_0^{k-b} y g(0, y)\, dy + (k-b) \int_{k-b}^{k-\tilde{b}} g(0, y)\chi_\delta(k-y)dy \\
+ \mathbb{E}(Z^+_{k,k}) \left[G(0, k-b) + \int_{k-b}^{k-\tilde{b}} g(0, y)\chi_\delta(k-y)dy\right],
\end{aligned}$$

from which, it follows that $\mathbb{E}(Z^+_{k,k})$ is given explicitly by

$$\mathbb{E}(Z^+_{k,k}) = \frac{\int_0^{k-b} y g(0, y)\, dy + (k-b) \int_{k-b}^{k-\tilde{b}} g(0, y)\chi_\delta(k-y)dy}{1 - \left(G(0, k-b) + \int_{k-b}^{k-\tilde{b}} g(0, y)\chi_\delta(k-y)dy\right)}. \tag{2.5.2}$$

On the other hand, to compute $\mathbb{E}(Z^-_{u,k})$, for $\tilde{b} < u < b$, we first note that for there to exist a first capital injection (which will be of size $k - b$) the surplus has to hit the upper confidence level b, before experiencing insolvency. Thus, it follows that $\mathbb{E}(Z^-_{u,k})$, satisfies

$$\mathbb{E}(Z^-_{u,k}) = \chi_\delta(u)\left((k-b) + \mathbb{E}(Z^+_{k,k})\right), \quad \tilde{b} < u < b, \tag{2.5.3}$$

with $\mathbb{E}(Z^+_{k,k})$ given by equation (2.5.2). Combining equations (2.5.1), (2.5.2) and (2.5.3) leads to the following lemma.

Lemma 4. *Let $Z_{u,k} = Z(T^Z_\delta)$ denote the accumulated capital injections up to the time of insolvency. Then, the expected value $\mathbb{E}(Z_{u,k})$, for $u \geqslant k$, denoted $\mathbb{E}(Z^+_{u,k})$, is given by*

$$\mathbb{E}(Z^+_{u,k}) = \int_0^{k-b} y\zeta(\tilde{u},k,y)\,dy + (k-b)\int_{k-b}^{k-\tilde{b}} \zeta(\tilde{u},k,y)\chi_\delta(k-y)dy, \tag{2.5.4}$$

where

$$\zeta(\tilde{u},k,y) = g(\tilde{u},y) + \frac{g(0,y)\left(G(\tilde{u},k-b) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u},x)\chi_\delta(k-x)dx\right)}{1-\left(G(0,k-b) + \int_{k-b}^{k-\tilde{b}} g(0,x)\chi_\delta(k-x)dx\right)}$$

and, for $\tilde{b} < u < b$, denoted $\mathbb{E}(Z^-_{u,k})$, by

$$\mathbb{E}(Z^-_{u,k}) = \chi_\delta(u)\left((k-b) + \frac{\int_0^{k-b} yg(0,y)\,dy + (k-b)\int_{k-b}^{k-\tilde{b}} g(0,y)\chi_\delta(k-y)dy}{1-\left(G(0,k-b) + \int_{k-b}^{k-\tilde{b}} g(0,y)\chi_\delta(k-y)dy\right)}\right). \tag{2.5.5}$$

To illustrate the applicability of the results for $\mathbb{E}(Z^+_{u,k})$ and $\mathbb{E}(Z^-_{u,k})$, given in the above lemma, we consider the case of exponentially distributed claim sizes and present the explicit results in the following proposition.

Proposition 10. *Let the claim amounts be exponentially distributed with parameter $\beta > 0$, i.e. $F_X(x) = 1 - e^{-\beta x}$, $x \geqslant 0$. Then, the expected accumulated capital injections, $\mathbb{E}(Z^+_{u,k})$ for $u \geqslant k$, is given by*

$$\mathbb{E}(Z^+_{u,k}) = K_1\psi^+_{SII}(u), \tag{2.5.6}$$

where

$$K_1 = \frac{1}{1+\eta}\left(\frac{\lambda}{c\beta}C_1^{-1}e^{\beta k}\left(1-e^{-\beta(k-b)}\right)-(k-b)\right),$$

and $\psi^{+}_{SII}(u)$ *is the probability of insolvency, for* $u \geqslant k$*, given in Theorem 18.*

For $\tilde{b} < u < b$, $\mathbb{E}(Z^{-}_{u,k})$ *is given by*

$$\mathbb{E}(Z^{-}_{u,k}) = K_2\phi^{-}_{SII}(u) \tag{2.5.7}$$

where

$$K_2 = \frac{1}{\beta\eta}\left(1-e^{-\beta(k-b)}\right)+(k-b),$$

and $\phi^{-}_{SII}(u)$ *is the solvency (survival) probability, for* $\tilde{b} < u < b$*, which can be obtained from equation* (2.4.20) *of Theorem 18.*

Proof. Substituting the forms for $G(\cdot,\cdot)$, $g(\cdot,\cdot)$ and $\chi_\delta(\cdot)$, under exponentially distributed claim amounts [see Section 2.4.1 and equation (2.4.17), respectively], into the results of Lemma 4, then, after some algebraic manipulations and recalling the forms of $\psi^{+}_{\mathrm{SII}}(u)$ and $\psi^{-}_{\mathrm{SII}}(u) = 1-\phi^{-}_{\mathrm{SII}}(u)$, from Theorem 18, the results follow. □

2.5.2 The distribution of the accumulated capital injections up to the time of insolvency

In practice, the expected value alone does not provide enough information to make financial and strategical decisions. Usually the second moments, and hence the variance, or higher moments are important in determining premium principles or evaluating risk exposure. Therefore, in this subsection, we derive the moment generating function of the risk quantity introduced in the previous section, namely $Z_{u,k}$, and show that its distribution is a mixture of a degenerative distribution at zero and a continuous distribution.

Extending the arguments of Nie et al. (2011), we first consider the case where $u = k$. The event of a first capital injection can be seen as the union of the event that the surplus process drops, due to a claim, in the interval $[b, k)$, which occurs with probability

$G(0,k-b)$, and the event that the surplus process drops, due to a claim, in the interval $(\tilde{b}, b)$ and then recovers back up to the level b before crossing $\tilde{b}$, which occurs with probability $\int_{k-b}^{k-\tilde{b}} g(0,y)\chi_\delta(k-y)\,dy$.

Given that there exists a first capital injection, the process restarts from the level k. Hence, if we let N denote the number of capital injections up to the time of insolvency, by the above reasoning, N has a geometric distribution with p.m.f., for $n = 0, 1, 2, \ldots$, given by

$$\mathbb{P}(N=n) = \left(G(0,k-b) + \int_{k-b}^{k-\tilde{b}} g(0,y)\chi_\delta(k-y)\,dy \right)^n \times \left(1 - \left[G(0,k-b) + \int_{k-b}^{k-\tilde{b}} g(0,y)\chi_\delta(k-y)\,dy \right] \right),$$

with a probability generating function of the form

$$\mathbb{E}(z^N) = P_N(z) = \frac{1 - \left(G(0,k-b) + \int_{k-b}^{k-\tilde{b}} g(0,y)\chi_\delta(k-y)\,dy \right)}{1 - z\left(G(0,k-b) + \int_{k-b}^{k-\tilde{b}} g(0,y)\chi_\delta(k-y)\,dy \right)}.$$

Then, the accumulated capital injections up to the time of insolvency, with initial capital $u = k$, namely $Z^+_{k,k}$, has the compound geometric form

$$Z^+_{k,k} = \sum_{i=1}^{N} V_i,$$

where N is the geometric random variable defined above and $\{V_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables, denoting the size of the i-th injection, with p.d.f. given by

$$f_V(y) = \begin{cases} \frac{g(0,y)}{G(0,k-b) + \int_{k-b}^{k-\tilde{b}} g(0,x)\chi_\delta(k-x)\,dx} & 0 < y < k-b, \\ \frac{\int_{k-b}^{k-\tilde{b}} g(0,x)\chi_\delta(k-x)\,dx}{G(0,k-b) + \int_{k-b}^{k-\tilde{b}} g(0,x)\chi_\delta(k-x)\,dx} & y = k-b. \end{cases}$$

It follows, from independence, that the moment generating function of $Z^+_{k,k}$ can be expressed as

$$M_{Z^+_{k,k}}(z) = P_N(M_V(z)),$$

where

$$M_V(z) = \mathbb{E}(e^{zV}) = \frac{\int_0^{k-b} e^{zy} g(0,y)\,dy + e^{z(k-b)}\int_{k-b}^{k-\tilde{b}} g(0,x)\chi_\delta(k-x)\,dx}{G(0,k-b) + \int_{k-b}^{k-\tilde{b}} g(0,x)\chi_\delta(k-x)\,dx}.$$

In order to find the moment generating functions of the accumulated capital injections up to the time of insolvency with general initial capital, namely $Z^+_{u,k}$, when $u \geqslant k$, and $Z^-_{u,k}$, when $\tilde{b} < u < b$, we first note that $Z^+_{u,k}$ and $Z^-_{u,k}$ are equivalent in distribution to $(Y^+_u + Z^+_{k,k})\mathbb{I}_{\{A^+\}}$ and $(Y^-_u + Z^+_{k,k})\mathbb{I}_{\{A^-\}}$, respectively, where Y^+_u is the amount of the first capital injection, starting from initial capital $u > k$, Y^-_u from initial capital $\tilde{b} < u < b$ and $\mathbb{I}_{\{\cdot\}}$ is the indicator function with respect to the event that a capital injections occurs from initial capital u. Note that the event that a capital injections occurs from initial capital u can be decomposed to the sub events depending on the value of the initial capital and thus we denote by A^+ and A^- the events that a capital injections occurs from initial capital $u > k$ and $\tilde{b} < u < b$, respectively, with probabilities

$$\mathbb{P}(A^+) = G(\tilde{u}, k-b) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u}, y)\chi_\delta(k-y)\,dy,$$

and

$$\mathbb{P}(A^-) = \chi_\delta(u).$$

Based on the above notation, for $\tilde{u} = u - k$, the density of Y^+_u is given by

$$f_{Y^+_u}(y) = \begin{cases} \frac{g(\tilde{u},y)}{G(\tilde{u},k-b)+\int_{k-b}^{k-\tilde{b}} g(\tilde{u},x)\chi_\delta(k-x)\,dx} & 0 < y < k-b, \\ \frac{\int_{k-b}^{k-\tilde{b}} g(\tilde{u},x)\chi_\delta(k-x)\,dx}{G(\tilde{u},k-b)+\int_{k-b}^{k-\tilde{b}} g(\tilde{u},x)\chi_\delta(k-x)\,dx} & y = k-b, \end{cases}$$

while Y_u^- has a p.m.f. of the form

$$\mathbb{P}(Y_u^- = i) = \begin{cases} 1, & i = k - b \\ 0 & \text{otherwise.} \end{cases}$$

Then, since Y_u^+ and $Z_{k,k}^+$ are independent, the moment generating function of $Z_{u,k}^+$ is given by

$$M_{Z_{u,k}^+}(z) = \left(M_{Y_u^+}(z) M_{Z_{k,k}^+}(z) \right) \mathbb{P}(A^+) + \mathbb{P}((A^+)^c), \tag{2.5.8}$$

where

$$M_{Y_u^+}(z) = \mathbb{E}(e^{zY_u^+}) = \frac{\int_0^{k-b} e^{zy} g(\tilde{u}, y)\, dy + e^{z(k-b)} \int_{k-b}^{k-\tilde{b}} g(\tilde{u}, x) \chi_\delta(k-x)\, dx}{G(\tilde{u}, k-b) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u}, x) \chi_\delta(k-x)\, dx},$$

whilst, following a similar argument as above, the moment generating function of $Z_{u,k}^-$ is given by

$$M_{Z_{u,k}^-}(z) = \left(M_{Y_u^-}(z) M_{Z_{k,k}^+}(z) \right) \mathbb{P}(A^-) + \mathbb{P}((A^-)^c), \tag{2.5.9}$$

where

$$M_{Y_u^-}(z) = \mathbb{E}(e^{zY_u^-}) = e^{z(k-b)}.$$

From equations (2.5.8) and (2.5.9), it should be clear that the distribution of the accumulated capital injections up to the time of insolvency, is mixture of a degenerative distribution at zero and a continuous distribution.

2.6 Constant dividend barrier strategy with SII constraints

As discussed at the start of this chapter, the shareholders of the company will only inject capital if it is profitable for them to do so and, in exchange for bearing some of the risks, they expect to receive some financial incentive in the form of dividend payments.

Dividend strategies have been extensively studied, within risk theory, since their introduction by De Finetti (1957) [see Section 1.7.2], with a main focus on optimisation of the companies utility. In the next section we will provide some of the basic models and results for dividend strategies, within the ruin theory context, that will be used in the final section of this chapter.

2.6.1 Dividend barrier strategies in risk theory

Throughout the risk theory literature, a number of different dividend strategies have been considered, with the aim of maximising the expected discounted dividend payments up to the time of ruin. For example, De Finetti (1957) proposed a constant barrier strategy, Gerber (1981) a linear barrier strategy and, more recently, Albrecher and Hartinger (2007) considered a multilayered strategy, where the dividend rate varies depending on the surplus level, to name a few. For a comprehensive review of the dividend strategies within risk theory see Avanzi (2009) and the references therein.

The constant dividend barrier strategy, as was first shown in De Finetti (1957), provides the optimal distribution of dividends with regards to maximising the expected discounted dividend payments until ruin. In such a model, any excess income above the *dividend barrier* is paid out continuously to the shareholders, whilst below, the process evolves as in the classical model. That is, if we let $d \in [0, \infty)$ denote the level of a constant dividend barrier, then the surplus process with dividend payments under a constant dividend barrier strategy, denoted $\{U^d(t)\}_{t \geqslant 0}$, has dynamics

$$dU^d(t) = \begin{cases} -dS(t), & U^d(t) \geqslant d \\ cdt - dS(t), & 0 \leqslant U^d(t) < d, \end{cases} \tag{2.6.1}$$

where $\{S(t)\}_{t \geqslant 0}$ is a compound Poisson process as described in equation (1.1.2). The constant dividend barrier problem, in the compound Poisson framework, has been studied in Bühlmann (1970), Segerdahl (1970), Paulsen and Gjessing (1997), Lin et al. (2003), Dickson and Waters (2004), Lin and Pavlova (2006) and references therein. Although it was shown that the constant dividend barrier is optimal, De Finetti (1957) also described how such a strategy, even under the net profit condition, causes ruin with probability one. That is, if we define the time to ruin in the model described above, denoted by T^d, such that

$$T^d = \inf\{t \geqslant 0 : U^d(t) < 0\},$$

then, the probability of infinite-time ruin, denoted by $\psi_d(u) = \mathbb{P}(T^d < \infty \big| U^d(0) = u)$, satisfies

$$\psi_d(u) = 1, \quad \text{for all} \quad u \geqslant 0.$$

Although the event of ruin is certain under a constant dividend barrier strategy, the properties of the event itself still contain information of interest, e.g. the deficit at ruin, the surplus prior to ruin and the event causing ruin, among others. Lin et al. (2003), show that the well known Gerber-Shiu function, for which many risk quantities, including the ruin probability and deficit at ruin, are special cases (see Section 1.6.1 of Chapter 1), under a constant divided barrier strategy, denoted by $m_\delta^d(u)$, satisfies an IDE, from which the general solution can be expressed as a linear combination of the corresponding Gerber-Shiu function without the presence of dividends and a secondary function $v(u)$. That is, the Gerber-Shiu function under a constant dividend barrier strategy, namely $m_\delta^d(u)$, with initial capital $0 \leqslant u \leqslant d$, can be expressed as

$$m_\delta^d(u) = m_\delta(u) - \frac{m'_\delta(d)}{v'(d)} v(u), \quad 0 \leqslant u \leqslant d, \tag{2.6.2}$$

where $m_\delta(u)$ is the classic Gerber-Shiu function without dividend constraints, given by equation (1.6.1), and $v(u)$ is a function satisfying a homogeneous IDE, from which the general solution is given by

$$v(u) = \frac{1 - \Psi(u)}{1 - \Psi(0)},$$

for some auxiliary function $\Psi(u)$, the details of which are not needed for this thesis. However, we point out that when the Gerber-Shiu function is reduced to the special cases of the ruin probability or the deficit at ruin, for which equation (2.6.2) holds, the auxiliary function above is equivalent to the classic ruin function, i.e. $\Psi(u) = \psi(u)$.

In the remainder of this chapter, we will consider the SII risk model, proposed in the previous sections, with the addition of a constant dividend barrier $d \geqslant k$, such that when the surplus reaches the level d, dividends are paid continuously at rate c until a new claim appears (see Fig: 2.3). Under this modified setting, we will show that the probability of insolvency can be expressed in terms of the classical risk quantities with a constant dividend barrier strategy and, as in the case of the classic ruin probability, is certain.

2.6.2 The Solvency II risk model with a constant dividend barrier strategy

The surplus process of a SII risk model under a constant dividend barrier strategy, denoted $\{U^Z_{\delta,d}(t)\}_{t\geqslant 0}$, has dynamics of the following form

$$dU^Z_{\delta,d}(t) = \begin{cases} -dS(t), & U^Z_{\delta,d}(t) \geqslant d, \\ cdt - dS(t), & k \leqslant U^Z_{\delta,d}(t) < d, \\ k - \left(U^Z_{\delta,d}(t-) - \Delta S(t)\right), & b \leqslant U^Z_{\delta,d}(t-) - \Delta S(t) < k, \\ \left[c + \delta(U^Z_{\delta}(t) - b)\right] dt - dS(t), & \tilde{b} < U^Z_{\delta,d}(t) < b. \end{cases}$$

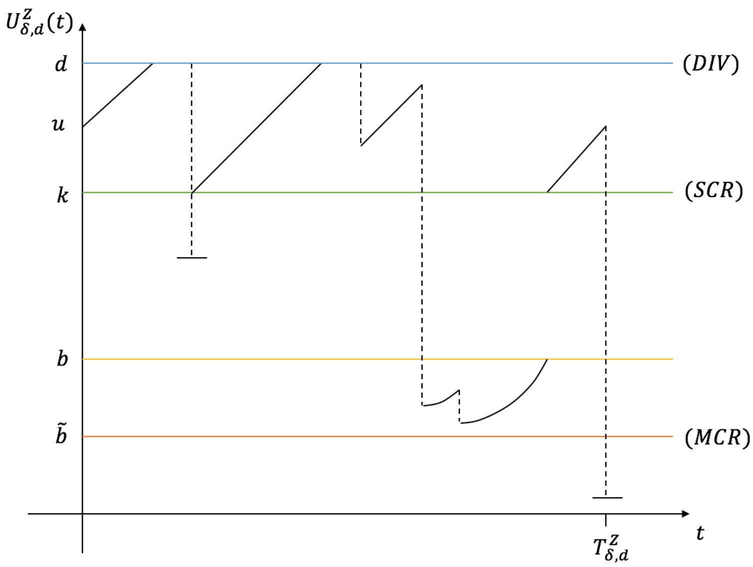


Figure 2.3: Typcial sample path of the surplus process under SII constraints with a constant dividend barrier.

The time to insolvency, in the dividend amended model, can be defined by

$$T^{Z}_{\delta,d} = \inf\left\{t \geqslant 0 : U^{Z}_{\delta,d}(t) \leqslant \tilde{b}\right\}$$

and the probability of insolvency, which we denote by $\psi_{\text{SII},d}(u)$, is defined as

$$\psi_{\text{SII},d}(u) = \mathbb{P}\left(T^{Z}_{\delta,d} < \infty \middle| U^{Z}_{\delta,d}(0) = u\right),$$

with the corresponding solvency probability defined by $\phi_{\text{SII},d}(u) = 1 - \psi_{\text{SII},d}(u)$.

We once again note that the insolvency probability, as in the previous sections, can be decomposed for $k \leqslant u \leqslant d$ and $\tilde{b} < u < b$, for which we define $\psi_{\text{SII},d}(u) = \psi^{+}_{\text{SII},d}(u)$ and $\psi_{\text{SII},d}(u) = \psi^{-}_{\text{SII},d}(u)$, for the two separate cases with corresponding solvency probabilities $\phi^{+}_{\text{SII},d}(u)$ and $\phi^{-}_{\text{SII},d}(u)$, respectively.

In order to derive an expression for the solvency probability for $k \leqslant u \leqslant d$, namely $\phi^{+}_{\text{SII},d}(u)$, (or equivalently the insolvency probability $\psi^{+}_{\text{SII},d}(u)$) we will need to define the crossing probability of the surplus below the SCR level k (as in Section 3), given by

$$\xi_d(u,k) = \mathbb{P}(T^{k,d} < \infty \big| \, k \leqslant U^{Z}_{\delta,d}(0) = u \leqslant d),$$

where $T^{k,d} = \inf\{t \geqslant 0 : U^{Z}_{\delta,d}(t) < k\}$ is the first time the process down crosses the barrier k.

Using a similar argument as in Section 3, it follows that the dynamics of the surplus process $U^{Z}_{\delta,d}(t)$ above the SCR level are equivalent to that of the classic surplus process with a constant dividend barrier $\tilde{d} = d - k$ (i.e. no capital injections or debit borrowing barriers). That is, for $k \leqslant U^{Z}_{\delta,d}(t) \leqslant d$, we have $dU^{Z}_{\delta,d}(t) \equiv d\widetilde{U}_{\tilde{d}}(t)$ where

$$\widetilde{U}_{\tilde{d}}(t) = \tilde{u} + ct - S(t), \qquad 0 \leqslant \widetilde{U}_{\tilde{d}}(0) = \tilde{u} \leqslant \tilde{d},$$

with dynamics

$$d\widetilde{U}_{\tilde{d}}(t) = \begin{cases} -dS(t), & \widetilde{U}_{\tilde{d}}(t) \geqslant \tilde{d}, \\ cdt - dS(t), & 0 \leqslant \widetilde{U}_{\tilde{d}}(t) < \tilde{d}. \end{cases}$$

Therefore, it is clear that $T^{k,d}$, defined above, is equivalent to the time of ruin in the classical risk model with a constant dividend barrier strategy and initial capital $0 \leqslant \tilde{u} \leqslant \tilde{d}$, given by

$$T^{k,d} = T^{\tilde{d}} = \inf\{t \geqslant 0 : \widetilde{U}_{\tilde{d}}(t) < 0\},$$

and the probability $\xi_d(u,k)$ is identical to the probability of ruin, namely $\psi_{\tilde{d}}(\tilde{u}) = \mathbb{P}(T^{\tilde{d}} < \infty \big| 0 \leqslant \widetilde{U}_{\tilde{d}}(0) = \tilde{u} \leqslant \tilde{d}) = 1 - \phi_{\tilde{d}}(\tilde{u})$, for the classical risk model with a constant dividend barrier strategy.

To obtain an expression for the insolvency probability under a constant dividend barrier strategy, we use the fact that $dU^{Z}_{\delta,d}(t) \equiv d\widetilde{U}_{\tilde{d}}(t)$, when the surplus is above the level k, and condition on the occurrence and amount of the first drop below the SCR barrier, k. Then for $k \leqslant u \leqslant d$, the respective solvency probability $\phi^{+}_{\mathrm{SII},d}(u)$, satisfies

$$\begin{aligned}
\phi^{+}_{\mathrm{SII},d}(u) &= \phi_{\tilde{d}}(\tilde{u}) + \int_{0}^{k-b} g_{\tilde{d}}(\tilde{u},y)\phi^{+}_{\mathrm{SII},d}(k)\,dy + \int_{k-b}^{k-\tilde{b}} g_{\tilde{d}}(\tilde{u},y)\phi^{-}_{\mathrm{SII},d}(k-y)\,dy \\
&= \phi_{\tilde{d}}(\tilde{u}) + G_{\tilde{d}}(\tilde{u},k-b)\phi^{+}_{\mathrm{SII},d}(k) + \int_{k-b}^{k-\tilde{b}} g_{\tilde{d}}(\tilde{u},y)\phi^{-}_{\mathrm{SII},d}(k-y)\,dy,
\end{aligned}$$

where

$$G_{\tilde{d}}(\tilde{u},y) = \mathbb{P}\left(T^{\tilde{d}} < \infty, |\widetilde{U}_{\tilde{d}}(T^{\tilde{d}})| \leqslant y \big| 0 \leqslant \widetilde{U}_{\tilde{d}}(0) = \tilde{u} \leqslant \tilde{d}\right),$$

is the distribution of the deficit below k at the time of crossing the barrier, under the constant dividend barrier strategy, and $g_{\tilde{d}}(\tilde{u},y) = \frac{\partial}{\partial y}G_{\tilde{d}}(\tilde{u},y)$ its corresponding density.

For $\tilde{b} < u < b$, we have

$$\phi^{-}_{\mathrm{SII},d}(u) = \chi_{\delta}(u)\phi^{+}_{\mathrm{SII},d}(k),$$

where $\chi_{\delta}(u)$ is the probability of hitting the upper barrier b before the lower barrier $\tilde{b}$, in a debit environment, as studied in Section 3. We point out that the function $\chi_{\delta}(u)$ is unaffected by the addition of the dividend barrier and therefore the IDE given in Proposition 7 still holds, along with the corresponding boundary conditions. Following similar arguments as in Section 2.4 we obtain the following theorem.

Theorem 19. *For $k \leqslant u \leqslant d$, the probability of insolvency under a constant dividend barrier strategy, $\psi^{+}_{SII,d}(u)$, satisfies*

$$\psi^{+}_{SII,d}(u) = \psi_{\tilde{d}}(\tilde{u}) - \frac{\phi_{\tilde{d}}(0)\left[G_{\tilde{d}}(\tilde{u}, k-b) + \int_{k-b}^{k-\tilde{b}} g_{\tilde{d}}(\tilde{u}, y)\chi_{\delta}(k-y)\,dy\right]}{1 - \left(G_{\tilde{d}}(0, k-b) + \int_{k-b}^{k-\tilde{b}} g_{\tilde{d}}(0, y)\chi_{\delta}(k-y)\,dy\right)}. \tag{2.6.3}$$

For $\tilde{b} < u < b$, $\psi^{-}_{SII,d}(u)$ is given by

$$\psi^{-}_{SII,d}(u) = 1 - \frac{\phi_{\tilde{d}}(0)\chi_{\delta}(u)}{1 - \left(G_{\tilde{d}}(0, k-b) + \int_{k-b}^{k-\tilde{b}} g_{\tilde{d}}(0, y)\chi_{\delta}(k-y)\,dy\right)}. \tag{2.6.4}$$

We again point out, from equations (2.6.3) and (2.6.4) that the two types of insolvency probabilities, for the risk model under SII constraint with the addition of a constant dividend barrier, are given in terms of the (shifted) ruin probability and the deficit at ruin of the classical risk model with constant dividend barrier, as well as the probability of exiting between two barriers. Thus, $\psi^{+}_{\mathrm{SII},d}(\cdot)$ and $\psi^{-}_{\mathrm{SII},d}(\cdot)$ can be calculated by employing known results, with respect to $G_d(\cdot,\cdot)$ and $\psi_d(\cdot)$ (see Lin et al. (2003), among others), whilst the latter exiting probability, $\chi_{\delta}(u)$, can be evaluated by Proposition 7.

Finally, by considering the forms of the insolvency probabilities given in the above theorem and recalling that the infinite-time ruin probability for the classical risk model with a constant dividend barrier strategy, namely $\psi_d(u) = 1$, for all $u \geqslant 0$, it follows that the ultimate-time survival probability $\phi_d(u) = 0$, for all $u \geqslant 0$ and we have the following Corollary.

Corollary 1. *For $u \in \mathbb{R}$, the probability of insolvency under a constant dividend barrier strategy, namely $\psi_{SII,d}(u)$, satisfies*

$$\psi_{SII,d}(u) = 1, \qquad a.s. \tag{2.6.5}$$

In this chapter, we considered the constraints of current insurance legislation, with particular emphasis on the capital requirement regulations under SII, on the theoretical Cramér-Lundberg risk model. It was assumed that upon breaching the SCR level, at which point the company is forced to re-capitalise under the SII directive, the firm looks to its shareholders for a capital injection necessary to keep the business from insolvency. In the proposed SII risk model, and within the majority of capital injection models in the risk theory literature, the receipt of the capital injections is assumed to occur instantaneously from the moment of a deficit below a pre-specified level. In the next chapter, we revert back to a more theoretical risk model (without SII constraints) and analyse how the introduction of a delayed receipt of capital injections, which occurs naturally in practice, impacts the performance of an insurance firm.

Chapter 3

Capital Injections with Deficit Dependent Delayed Receipt

An important assumption made throughout the previous chapter, and in the majority of literature dealing with capital injections, is their instantaneous receipt. However, in the real world markets, when an insurance firm is required to raise capital after a fall below the SCR level, by means of capital injections (as seen in Chapter 2), they are not usually received instantaneously. Time delays for the capital injections may occur naturally in insurance business due to decision-making problems or regulatory delays (for example, preparatory and administrative work) and must be taken into account since, during this delay time, the company remains subject to risk and may experience further losses before being recapitalised.

The concept of delayed capital injections has been introduced in Jin and Yin (2014), for a pure diffusion risk model without jumps. In the aforementioned work, the authors study optimal dividend strategies by means of a stochastic control problem, with mixed singular and delayed impulse controls, assuming that random injections occur at random stopping times throughout the time horizon.

In this chapter, we generalise the present capital injection risk models by introducing a time delay from the moment of a deficit below zero (or below the SCR barrier) to the time when the capital injection is received. Additionally, it is assumed that the delay time is dependent on the size of the deficit, and thus the corresponding capital injection size, to reflect the increase in time required to raise a larger amount of funds. Initially, we will propose a relatively simple dependence structure, based on a single critical level $k \geqslant 0$, which enables us to derive explicit expressions for numerous risk related quantities and is described in the next section.

3.1 Delayed capital injections under a single critical value

Consider the Cramér-Lundberg risk model, $\{U(t)\}_{t\geqslant 0}$, defined in equation 1.1.2. At the time of ruin T (assuming it occurs), the surplus process, $\{U(t)\}_{t\geqslant 0}$, experiences a deficit below zero of size $|U(T)|$ and we assume a capital injection, equal to the size of the deficit, is required to restore the surplus back to the zero level. If the deficit and thus the required capital injection, is less than a critical value $k \geqslant 0$, i.e. $|U(T)| \leqslant k$, it is assumed that the shareholders are in a position to inject the required capital from readily available funds and thus, the injection is received instantaneously (similar to the models of Nie et al. (2011), (2015) and Dickson and Qazvini (2016)). On the other hand, if the deficit of the insurance firm is larger than the critical value, i.e. $|U(T)| > k$, then the shareholders need time to raise the required capital for an injection of amount $|U(T)|$. Therefore, there exists a dependency between the magnitude of the deficit and the time delay between the moment of deficit and the receipt of the required capital injection. Intuitively, the critical value can be interpreted as the size of the deficit below which the injection is considered small enough to be covered by available funds and thus received instantaneously, whilst a deficit greater than the critical value requires time for the firm to raise the necessary funds and thus, a delay for financial processing is required.

Note that, throughout this chapter we assume that the critical value $k \geqslant 0$ is connected with the deficit below zero, i.e. when the surplus process becomes negative, however, for an environment with capital requirement regulations (see Chapter 2), $k \geqslant 0$ may be associated with the deficit below the SCR of an insurance firm.

Under the delay time setting, described above, there exist two different possibilities at the moment the surplus process, namely $\{U(t)\}_{t\geqslant 0}$, first becomes negative (which occurs at time T):

a) The deficit is at most $k \geqslant 0$, i.e. $|U(T)| \leqslant k$, which occurs with probability $G(u,k)$. Then, a capital injection of size $|U(T)| \leqslant k$ is required to restore the surplus back to the zero level which occurs instantaneously, since the amount of the capital injection is of a size that can be covered by readily available funds.

b) The deficit is larger than the critical value $k \geqslant 0$, i.e. $|U(T)| > k$, which occurs with probability

$$\overline{G}(u,k) = \psi(u) - G(u,k). \tag{3.1.1}$$

In this case, the available funds are unable to cover the required capital injection of size $|U(T)| > k$ and thus, the injection is received after some delay time, denoted by the random variable L, having d.f. $F_L(\cdot)$, to account for the administration and processing time (see Fig: 3.1 for the two cases, respectively).

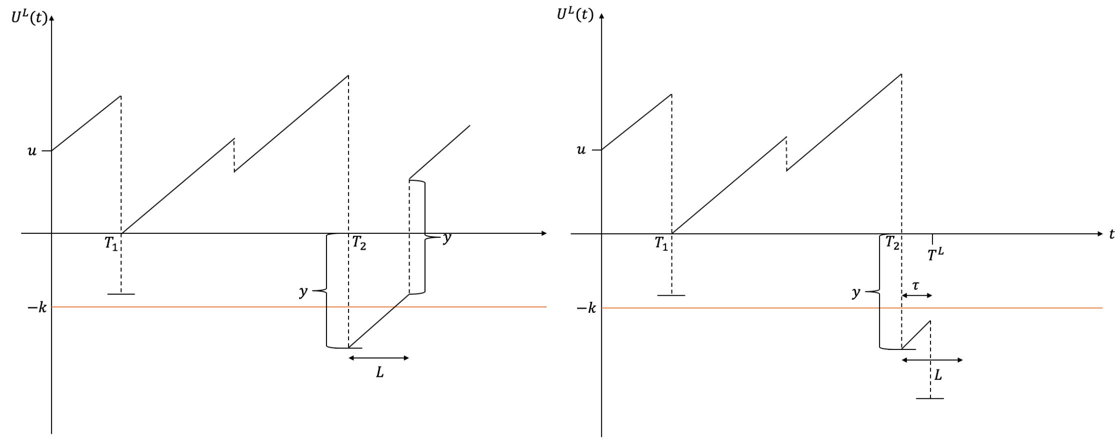
Based on the above set up, it is clear that the company is allowed to continue when in deficit and will receive premium income during this time. However, it is assumed that if a subsequent claim occurs before the capital injection is received, i.e. $\tau < L$, where τ denotes the common inter-arrival time under a Poisson process, then the company is exposed to too much risk at any one time and is declared as 'ruined'. We call this time 'ultimate ruin' to distinguish from the classical ruin time defined in equation (1.2.1). We can now consider the surplus process under a delayed capital injection setting, denoted by $\{U^L(t)\}_{t\geqslant 0}$, defined by

$$U^L(t) = U(t) + \sum_{i=1}^{\infty} |U^L(T_i)| \mathbb{I}_{(\{|U^L(T_i)| \leqslant k\} \cup \{(|U^L(T_i)| > k) \cap (T_i + L \leqslant t)\})}, \tag{3.1.2}$$

where

$$T_i = \inf\{t > T_{i-1} : U^L(t) < 0, U^L(t-) \geqslant 0\}, \qquad i = 1, 2, \ldots,$$

is the i-th time the surplus falls below zero, due to a claim, with $T_0 = 0$. Note that $T_1 = T$ is the classic ruin time defined in equation (1.2.1). We can now define the time



(a) Delayed capital injection arriving before subsequent claim in deficit.

(b) Subsequent claim arriving before delayed capital injection, resulting in ultimate ruin.

Figure 3.1: Possible cases following a fall into deficit.

of ultimate ruin by

$$T^L = \inf\left\{\sigma_i > 0 : U^L(\sigma_{i-1}) < -k, \sigma_i < \sigma_{i-1} + L\right\}, \tag{3.1.3}$$

for some $i = 1, 2, \ldots$, where $\{\sigma_i\}_{i\in\mathbb{N}}$ is the sequence of claim arrival epochs for the Poisson process, as defined in Definition 1. Then, it follows that the ultimate ruin probability can be defined by

$$\psi_L(u) = \mathbb{P}(T^L < \infty \big| U^L(0) = u), \qquad u \geqslant 0,$$

with the corresponding ultimate survival probability, given by

$$\phi_L(u) = 1 - \psi_L(u).$$

3.1.1 Ultimate ruin probabilities for a single critical value

In this section, we consider three separate cases for the delay time variable, L, and in each case, by using a conditioning argument and the Markov property, we derive and solve integral equations to obtain explicit expressions for the ultimate ruin probability, $\psi_L(u)$, for $u \geqslant 0$.

Capital injections with discrete time random delays

Let us first consider the case where the capital injection delay time random variable, namely L, can take finitely many discrete values. That is, $L \in \{m_1, \ldots, m_N\}$, for $N \in \mathbb{N}^+$, with probability $p_i = \mathbb{P}(L = m_i) > 0$, where $m_i \geqslant 0$, for all $i = 1, \ldots, N$ and $\sum_{i=1}^N p_i = 1$. Then, by conditioning on the amount of the first drop below zero, the delay time random variable, L, and the subsequent claim inter-arrival time, the law of total probability gives

$$\phi_L(u) = \phi(u) + G(u,k)\phi_L(0) + \int_k^\infty g(u,y) \int_0^\infty f_\tau(s) \sum_{i=1}^N p_i \phi_L(cm_i) \mathbb{I}_{\{m_i < s\}} \, ds \, dy, \tag{3.1.4}$$

where $\phi(u)$ is the survival probability for the Cramér-Lundberg risk process, defined in equation (1.2.5), and $f_\tau(\cdot)$ is the p.d.f. of the claim inter-arrival time. Following from the definition of an indicator function, the above equation can be re-written as

$$\begin{aligned}
\phi_L(u) &= \phi(u) + G(u,k)\phi_L(0) + \int_k^\infty g(u,y) \sum_{i=1}^N p_i \int_{m_i}^\infty f_\tau(s) \phi_L(cm_i) \, ds \, dy \\
&= \phi(u) + G(u,k)\phi_L(0) + \overline{G}(u,k) \sum_{i=1}^N p_i \overline{F}_\tau(m_i) \phi_L(cm_i),
\end{aligned} \tag{3.1.5}$$

where, since the claims arrive according to a Poisson process, $\overline{F}_\tau(t) = 1 - F_\tau(t) = e^{-\lambda t}$, for $t \geqslant 0$. Thus, equation (3.1.5) reduces to the form

$$\phi_L(u) = \phi(u) + G(u,k)\phi_L(0) + \overline{G}(u,k) \sum_{i=1}^N p_i e^{-\lambda m_i} \phi_L(cm_i). \tag{3.1.6}$$

In order to complete the expression for $\phi_L(u)$, in equation (3.1.6), (since the risk quantities $\phi(u)$ and $G(u,y)$ are known for the Cramér-Lundberg risk model under certain claim size distributions) we need to determine the boundary value $\phi_L(0)$ and particular values $\phi_L(cm_i)$, for $i = 1, \ldots, N$.

Setting $u = 0$, in equation (3.1.6), and solving with respect to $\phi_L(0)$, yields

$$\phi_L(0) = \frac{\phi(0) + \overline{G}(0,k) \sum_{i=1}^N p_i e^{-\lambda m_i} \phi_L(cm_i)}{1 - G(0,k)}, \tag{3.1.7}$$

which, after substituting this form for $\phi_L(0)$ back into equation (3.1.6) and re-arranging, yields

$$\phi_L(u) = w(u,k) + v(u,k)\sum_{i=1}^{N} p_i e^{-\lambda m_i}\phi_L(cm_i), \tag{3.1.8}$$

where

$$w(u,k) = \phi(u) + \frac{G(u,k)\phi(0)}{1-G(0,k)} > 0, \tag{3.1.9}$$

and

$$\begin{aligned} v(u,k) &= \frac{G(u,k)\overline{G}(0,k)}{1-G(0,k)} + \overline{G}(u,k), \\ &= \psi(u) - \frac{G(u,k)\phi(0)}{1-G(0,k)} < 1, \end{aligned} \tag{3.1.10}$$

such that $w(u,k) + v(u,k) = 1$, for all $u,k \geqslant 0$. The strict inequalities in equations (3.1.9) and (3.1.10), for the functions $w(u,k)$ and $v(u,k)$, follow from the fact that, under the net profit condition, the classical ruin function $\psi(u) < 1$, for all $u \geqslant 0$ [see Chapter 1].

Remark 13. *The function $w(u,k) > 0$, defined in equation* (3.1.9)*, corresponds to the infinite-time survival probability in the capital injection risk model without delays of Nie et al. (2011). Moreover, the function $v(u,k) = 1 - w(u,k) < 1$ is equivalent to the corresponding ruin probability, defined in equation* (2.2.1).

Now, in order to uniquely determine $\phi_L(u)$ in equation (3.1.8), it remains to determine the values $\phi_L(cm_i)$, for $i = 1, \dots, N$.

To do this, we will construct and solve N linear simultaneous equations. Setting $u = cm_j$, for $j = 1, \dots, N$, in equation (3.1.8), results in the simultaneous equation system

$$\phi_L(cm_j) = w(cm_j,k) + v(cm_j,k)\sum_{i=1}^{N} p_i e^{-\lambda m_i}\phi_L(cm_i), \quad \text{for} \quad j = 1, \dots, N,$$

or equivalently

$$\left(1 - v(cm_j, k)p_j e^{-\lambda m_j}\right) \phi_L(cm_j) = w(cm_j, k) + v(cm_j, k) \sum_{i=1, i\neq j}^{N} p_i e^{-\lambda m_i} \phi_L(cm_i),$$

which can be written as a first order matrix equation, of the form

$$\mathbf{A}\vec{\phi}^* = \vec{w},$$

where

$$\mathbf{A} = \begin{pmatrix} \left(1 - v(cm_1, k)p_1 e^{-\lambda m_1}\right) & -v(cm_1, k)p_2 e^{-\lambda m_2} & \cdots & -v(cm_1, k)p_N e^{-\lambda m_N} \\ -v(cm_2, k)p_1 e^{-\lambda m_1} & \left(1 - v(cm_2, k)p_2 e^{-\lambda m_2}\right) & \cdots & -v(cm_2, k)p_N e^{-\lambda m_N} \\ \vdots & \vdots & \ddots & \vdots \\ -v(cm_N, k)p_1 e^{-\lambda m_1} & -v(cm_N, k)p_2 e^{-\lambda m_2} & \cdots & \left(1 - v(cm_N, k)p_M e^{-\lambda m_N}\right) \end{pmatrix}, \tag{3.1.11}$$

is an N-dimensional square matrix, $\vec{\phi}^* = (\phi_L(cm_1), \ldots, \ \phi_L(cm_N))^\top$ and $\vec{w} = (w(cm_1, k), \ldots, w(cm_N, k))^\top$, are both N-dimensional column vectors, where $(\cdot)^\top$ denotes the transpose of a vector/matrix. In order to evaluate the vector of unknowns, namely $\vec{\phi}^*$, we will show, in the following lemma, that the matrix $\mathbf{A}$ is non-singular and thus invertible.

Lemma 5. *For $u \geqslant 0$, $0 < p_i \leqslant 1$, for $i = 1, \ldots, N$ and $\sum_{j=1}^{N} p_j = 1$, the matrix $\mathbf{A}$ is non-singular.*

Proof. To show that $\mathbf{A}$ is a non-singular matrix, it suffices to prove, by the Lévy-Desplanques Theorem [see Horn and Johnson (1990)], that $\mathbf{A}$ is a strictly diagonally dominant matrix.

Definition 17 (Strictly diagonally dominant)**.** *An N-dimensional square matrix, $\mathbf{A} = \{a_{ij}\}_{i,j=1}^{N}$, is called strictly diagonally dominant if $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ for all $i \in N$.*

It follows from the form of the matrix $\mathbf{A}$, given in equation (3.1.11), and the definition of strict diagonal dominance, that we need to prove

$$|1 - v(cm_i, k)p_i e^{-\lambda m_i}| > \sum_{j \neq i} | - v(cm_i, k)p_j e^{-\lambda m_j}|,$$

for all $i = 1, \ldots, N$, or equivalently

$$1 - v(cm_i, k)p_i e^{-\lambda m_i} > v(cm_i, k) \sum_{j \neq i} p_j e^{-\lambda m_j},$$

since, from equation (3.1.10), we have $0 \leqslant v(u, k) < 1$, for all $u \geqslant 0$, which guarantees that $0 \leqslant v(u, k)p_j e^{-\lambda m_j} \leqslant 1$ for every $j = 1, \ldots, N$ and $u \geqslant 0$.

Employing the fact that $v(u, k) < 1$, for all $u \geqslant 0$ (under the net profit condition), we have that

$$1 > v(cm_i, k) = v(cm_i, k) \sum_{j=1}^{N} p_j \geqslant v(cm_i, k) \sum_{j=1}^{N} p_j e^{-\lambda m_j}, \quad \text{for all} \quad i = 1, \ldots, N,$$

from which it follows that $\mathbf{A}$ is strictly diagonally dominant and thus, the result follows.

□

Now, since the matrix $\mathbf{A}$ is non-singular, and thus invertible, the forms of $\phi_L(cm_i)$, $i = 1, \ldots, N$, can be determined by

$$\vec{\phi}_L = \mathbf{A}^{-1} \vec{w},$$

where $\mathbf{A}^{-1}$ is the inverse of the matrix $\mathbf{A}$. Finally, from equation (3.1.8), the ultimate survival probability for capital injections with a discrete random time delay is given by the linear expression

$$\phi_L(u) = w(u, k) + v(u, k) \sum_{i=1}^{N} p_i e^{-\lambda m_i} \left[\mathbf{A}^{-1} \vec{w}\right]_i$$

where $\left[\mathbf{A}^{-1} \vec{w}\right]_i$ is the i-th element of the vector $\mathbf{A}^{-1} \vec{w}$ and we have the following theorem.

Theorem 20. *For $u \geqslant 0$, the ultimate ruin probability for capital injections with discrete time random delays, namely $\psi_L(u)$, is given by*

$$\psi_L(u) = v(u,k)\left(1 - \sum_{i=1}^{N} p_i e^{-\lambda m_i}\left[\mathbf{A}^{-1}\vec{w}\right]_i\right), \tag{3.1.12}$$

where

$$v(u,k) = \psi(u) - \frac{\eta G(u,k)}{1+\eta-F_X^s(k)},$$

with $F_X^s(x)$ the integrated tail distribution of the claim sizes, defined in equation (1.2.12).

Capital injections with deterministic delay times

In practice, market studies indicate that the delay times for capital injections may not be random, but instead a fixed amount of time, i.e. the number of days, or months, required to gather the necessary funds due to financial or regulatory purposes. Thus, a natural consideration is to consider the case of deterministic delay times. Let the delay time $L = \rho \geqslant 0$. Note that this is equivalent to the discrete time case with $N = 1$ and random time delay $m_1 = \rho$, with $p_1 = 1$. Thus, equation (3.1.8) reduces to

$$\phi_L(u) = w(u,k) + v(u,k)e^{-\lambda\rho}\phi_L(c\rho). \tag{3.1.13}$$

and from Theorem 20, we have the following corollary.

Corollary 2. *For $u \geqslant 0$, the ultimate ruin probability under capital injections with deterministic time delay $L = \rho \geqslant 0$, namely $\psi_L(u)$, is given by*

$$\psi_L(u) = v(u,k)\left(\frac{1-e^{-\lambda\rho}}{1-v(c\rho,k)e^{-\lambda\rho}}\right), \tag{3.1.14}$$

where

$$v(u,k) = \psi(u) - \frac{\eta G(u,k)}{1+\eta-F_X^s(k)}.$$

Remark 14 ($\rho \to \infty$)**.** *Consider the case where $\rho \to \infty$. Then, for a deficit larger than the critical value $k \geqslant 0$, a subsequent claim will appear before the capital injec-*

tion is received a.s. (since $F_\tau(\cdot)$ is a proper distribution function) and ultimate ruin is certain. This scenario reduces the model to one similar to Nie et al. (2011), where an instantaneous capital injection is received for a deficit less than $k \geqslant 0$ but we experience ultimate ruin if the deficit is larger than $k \geqslant 0$. Then, since $\lim_{\rho\to\infty} e^{-\lambda\rho} = 0$, equation (3.1.14) *reduces to*

$$\begin{aligned}\psi_L(u) &= v(u,k)\\ &= \psi(u) - G(u,k)\frac{\phi(0)}{1-G(0,k)},\end{aligned}$$

which is a shifted analogue of the results given in equation (2.2.1).

Remark 15 ($\rho \to 0$). *Now, consider the opposing case that $\rho \to 0$. Then, the capital injection is received instantaneously, regardless of the size of the deficit and the subsequent claim after a fall into a deficit greater than $k \geqslant 0$ will never occur before the capital injection a.s. In this case, ultimate ruin is never experienced and the surplus continues indefinitely, as in the models of Pafumi (1998) and Eisenberg and Schmidli (2011), among others. From equation* (3.1.14), *and recalling that $v(u,k) < 1$, for all $u \geqslant 0$, since $\lim_{\rho\to 0} e^{-\lambda\rho} = 1$, we have $\psi_L(u) = 0$, for all $u \geqslant 0$, as expected.*

Capital injections with continuous time random delays

Finally, we will consider the case where the delay time random variable, L, is a continuous time random variable having p.d.f. $f_L(\cdot)$ and finite mean $\mathbb{E}(L) < \infty$. If we apply a similar conditioning argument as in the discrete time case, i.e. conditioning on the amount of the first drop below zero, the delay time and the subsequent claim inter-arrival time, we obtain the continuous time analogue of equation (3.1.4), given by

$$\begin{aligned}\phi_L(u) &= \phi(u) + G(u,k)\phi_L(0) + \int_k^\infty g(u,y)\int_0^\infty f_L(t)\int_0^\infty f_\tau(s)\phi_L(ct)\mathbb{I}_{\{t<s\}}\,ds\,dt\,dy\\ &= \phi(u) + G(u,k)\phi_L(0) + \overline{G}(u,k)\int_0^\infty f_L(t)\overline{F}_\tau(t)\phi_L(ct)\,dt,\end{aligned}$$

or equivalently, since $\overline{F}_\tau(t) = e^{-\lambda t}$, for $t \geqslant 0$, by

$$\phi_L(u) = \phi(u) + G(u,k)\phi_L(0) + \overline{G}(u,k) \int_0^\infty f_L(t) e^{-\lambda t} \phi_L(ct)\, dt. \tag{3.1.15}$$

Now, as in the discrete case, in order to complete the expression for $\phi_L(u)$, in equation (3.1.15), we first need to determine the boundary value $\phi_L(0)$.

Setting $u = 0$, in equation (3.1.15), and solving with respect to $\phi_L(0)$, we have that

$$\phi_L(0) = \frac{\phi(0) + \overline{G}(0,k) \int_0^\infty f_L(t)\, e^{-\lambda t} \phi_L(ct)\, dt}{1 - G(0,k)},$$

which is simply the continuous analogue of the expression given in equation (3.1.7). Substituting this form of the boundary value, $\phi_L(0)$, into equation (3.1.15), yields

$$\phi_L(u) = w(u,k) + v(u,k) \int_0^\infty f_L(t) e^{-\lambda t} \phi_L(ct)\, dt, \tag{3.1.16}$$

where $w(u,k)$ and $v(u,k)$ are defined as in equations (3.1.9) and (3.1.10), respectively. At this point, we cannot employ the same methods as in the discrete case, however, by using a change of variables, the above equation can be written as

$$\phi_L(u) = w(u,k) + \frac{1}{c} v(u,k) \int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} \phi_L(t)\, dt, \tag{3.1.17}$$

which is the form of an inhomogeneous Fredholm integral equation of the second kind over a semi-infinite interval, with degenerate kernel

$$K(u,t) = v(u,k) f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}}, \tag{3.1.18}$$

[see Definition 4.5.2 of Appendix].

Following the general theory of integral equations for deriving a closed form expression for the inhomogeneous Fredholm equation with degenerate kernel [see Polyanin and Manzhirov (2008)], we point out that the integral in equation (3.1.17) evaluates to a constant, say C_1, provided it exists.

Proposition 11. *The constant* $C_1 = \int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} \phi_L(t)\, dt$ *is finite and bounded by* *the premium rate $c > 0$.*

Proof. The function $\phi_L(\cdot)$ is a probability measure, hence $e^{-\frac{\lambda t}{c}}\phi_L(t) \leqslant 1$, for all $t \geqslant 0$. Therefore, it follows that

$$C_1 = \int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} \phi_L(t)\, dt \leqslant \int_0^\infty f_L\left(\frac{t}{c}\right) dt = c,$$

since $f_L(\cdot)$ is a proper density function. $\square$

Then, it follows that the general solution to equation (3.1.17) is given by the linear combination

$$\phi_L(u) = w(u,k) + \frac{C_1}{c} v(u,k), \tag{3.1.19}$$

where C_1 is some constant to be determined.

To complete the solution for $\phi_L(u)$, in equation (3.1.19), it remains to calculate explicitly the constant C_1. In order to do this, let us first replace the variable u, in equation (3.1.19), by t, then multiply through by $f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}}$ and integrate over the interval $[0,\infty)$, to obtain the expression

$$\int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} \phi_L(t)\, dt = \int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t,k)\, dt + \frac{C_1}{c}\int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v(t,k)\, dt.$$

Note that, the left hand side (l.h.s.) of the above equality is simply the constant C_1 and thus, can be re-written in the form

$$C_1 = \int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t,k)\, dt + \frac{C_1}{c}\int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v(t,k)\, dt. \tag{3.1.20}$$

Moreover, since we have that $w(u,k) \leqslant 1$ and $v(u,k) < 1$, from equations (3.1.9) and (3.1.10), we can use a similar argument as in the proof of Proposition 11 to show that both $\int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t,k)\, dt$ and $\int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v(t,k)\, dt$ exist and are bounded by $c > 0$. Therefore, we can solve equation (3.1.20), with respect to C_1, to obtain

$$C_1 = \frac{\int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t,k)\, dt}{1 - \frac{1}{c}\int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v(t,k)\, dt},$$

where $\frac{1}{c}\int_0^\infty f_L\left(\frac{t}{c}\right)e^{-\frac{\lambda t}{c}}v(t,k)\,dt \neq 1$, since $v(u,k) < 1$, for all $u \geqslant 0$.

Substituting this form of C_1 back into equation (3.1.17), we obtain an explicit expression for the survival probability, of the form

$$\phi_L(u) = w(u,k) + \frac{\int_0^\infty f_L\left(\frac{t}{c}\right)e^{-\frac{\lambda t}{c}}w(t,k)\,dt}{c - \int_0^\infty f_L\left(\frac{t}{c}\right)e^{-\frac{\lambda t}{c}}v(t,k)\,dt}v(u,k). \tag{3.1.21}$$

Finally, defining the Laplace-Stieltjes transform of the delay time distribution by $\widehat{f}_L(s) = \int_0^\infty e^{-sx}\,dF_L(x)$ and recalling that $w(u,k) = 1 - v(u,k)$, we have the following theorem.

Theorem 21. *For all $u \geqslant 0$, the ultimate ruin probability for capital injections with continuous time delays, namely $\psi_L(u)$, is given by*

$$\psi_L(u) = v(u,k)\left(\frac{1-\widehat{f}_L(\lambda)}{1-\int_0^\infty f_L(t)v(ct,k)e^{-\lambda t}\,dt}\right), \tag{3.1.22}$$

where $\widehat{f}_L(s)$ is the Laplace-Stieltjes transform of the delay time distribution and

$$v(u,k) = \psi(u) - \frac{\eta G(u,k)}{1+\eta-F_X^s(k)}. \tag{3.1.23}$$

In order to illustrate the applicability of Theorem 21, in the next proposition we give an exact expression for the ultimate ruin probability, namely $\psi_L(u)$, in the case where both the delay time of the capital injections and the individual claim sizes follow an exponential distribution with parameters $\alpha > 0$ and $\beta > 0$, respectively.

Proposition 12. *Assume that the delay time, L, follows an exponential distribution with parameter $\alpha > 0$. Further, assume that the claim sizes also follow an exponential distribution with parameter $\beta > 0$. Then, the probability of ultimate ruin for delayed capital injections is given by*

$$\psi_L(u) = Ke^{-\frac{\lambda\eta}{c}u}, \qquad u \geqslant 0, \tag{3.1.24}$$

where K is a constant given by

$$K = \frac{\lambda(\alpha+\beta c)}{(\alpha+\lambda)\left(\beta c + (\alpha+\beta c)\eta e^{\beta k}\right)}.$$

Proof. For a delay time, L, which is exponentially distributed with parameter $\alpha > 0$, we have that $F_L(x) = 1 - e^{-\alpha x}$, with corresponding density $f_L(x) = \alpha e^{-\alpha x}$ and LT $\widehat{f}_L(s) = \frac{\alpha}{\alpha+s}$. In addition, the forms of the quantities $G(u, y)$ and $\overline{G}(u, y)$, for the classical Cramér-Lundberg risk model, are known explicitly for the case of exponentially distributed claim sizes, i.e. when $F_X(x) = 1 - e^{-\beta x}$, $\beta > 0$, and are given in Proposition 4. Thus, from equation (3.1.23), it follows that

$$v(u, k) = e^{-\frac{\lambda\eta}{c}u} \left(\frac{1}{1 + \eta e^{\beta k}} \right),$$

and

$$\int_0^\infty f_L(t)\, v(ct, k) e^{-\lambda t}\, dt = \frac{\alpha}{(1 + \eta e^{\beta k})(\alpha + \beta c)}.$$

Substituting these expressions into equation (3.1.22) of Theorem 21 and after some algebraic manipulations, the result follows. □

In the rest of this chapter, we consider further generalisations to the aforementioned risk model, by considering stricter dependency structures between the amount of the deficit and the corresponding delay time and studying other risk related quantities. We point out that all of the following results are given for the case of continuous delay times, however, the methodologies presented can be easily adapted to the discrete and deterministic time cases as well.

3.2 Extension to a model with N critical values

In this section, we generalise the dependence structure of the previous section to allow for $N \geqslant 1$ independent deficit critical values, introducing a stricter dependence between the size of the deficit and the corresponding delay time of the capital injections.

Let k_i, $i = 0, 1, \dots, (N + 1)$, be ordered, positive constants denoting the critical values, between which the magnitude of the deficit may lie (deficit thresholds) such that $0 = k_0 < k_1 < \dots < k_N < k_{N+1} = \infty$. Then, we can define the joint probability function of ruin and a deficit size within the critical value interval $(k_i, k_{i+1}]$ by $G_i(u) = \mathbb{P}(T < \infty, k_i < |U(T)| \leqslant k_{i+1} \big| U(0) = u)$ which can be expressed in terms of the classic deficit at ruin functions, $G(u, y)$, since

$$\begin{aligned} G_i(u) &= \int_{k_i}^{k_{i+1}} g(u, y)\, dy \\ &= G(u, k_{i+1}) - G(u, k_i), \end{aligned}$$

with $G_0(u) = G(u, k_1)$ and $G_N(u) = \overline{G}(u, k_N) = \mathbb{P}(T < \infty, |U(T)| > k_N \big| U(0) = u)$ being the probability that ruin occurs with a deficit larger than the greatest deficit critical value, namely k_N.

In a similar way to the previous section, we assume that if ruin occurs with a deficit less than the smallest barrier k_1, which occurs with probability $G(u, k_1)$, then the required capital injection can be covered by available funds and is received instantaneously. On the other hand, if ruin occurs and the deficit has magnitude $|U(T)| = y \in (k_i, k_{i+1}]$, $i = 1, 2, \ldots, N$, which occurs with probability $G_i(u)$, then the capital injection (of size y) is received after some random time delay, L_i, having d.f. $F_{L_i}(\cdot)$ and corresponding density $f_{L_i}(\cdot)$. Finally, it is assumed that the delay time random variable L_i is 'less than' the time delay random variable L_{i+1}, in the sense of stochastic ordering, i.e. $L_i \leqslant_{st} L_{i+1}$, such that there exists a positive correlation between the size of the required injection and the corresponding delay time.

Using a similar conditioning argument as in Section 3.1.1, i.e. conditioning on the amount of the first drop below zero, the corresponding delay time and the subsequent inter-arrival time of a claim, we obtain an integral equation for the ultimate survival probability, under $N \geqslant 1$ deficit threshold barriers and continuous delay times, given by

$$\begin{aligned} \phi_L(u) = \phi(u) + G(u, k_1)\phi_L(0) + \sum_{i=1}^{N} \int_{k_i}^{k_{i+1}} g(u, y) \int_0^\infty f_{L_i}(t) \\ \times \int_0^\infty f_\tau(s)\phi_L(ct)\mathbb{I}_{\{t<s\}}\, ds\, dt\, dy \\ = \phi(u) + G(u, k_1)\phi_L(0) + \sum_{i=1}^{N} G_i(u) \int_0^\infty f_{L_i}(t)\overline{F}_\tau(t)\phi_L(ct)\, dt, \end{aligned}$$

or equivalently

$$\phi_L(u) = \phi(u) + G(u, k_1)\phi_L(0) + \sum_{i=1}^{N} G_i(u) \int_0^\infty f_{L_i}(t)e^{-\lambda t}\phi_L(ct)\, dt. \tag{3.2.1}$$

To complete the solution for $\phi_L(u)$, in equation (3.2.1), as in the previous sections, we need to determine the boundary value $\phi_L(0)$. Setting $u = 0$, in the above equation, and solving with respect to $\phi_L(0)$, yields

$$\phi_L(0) = \frac{\phi(0) + \sum_{i=1}^{N} G_i(0) \int_0^\infty f_{L_i}(t)e^{-\lambda t}\phi_L(ct)\, dt}{1 - G(0, k_1)},$$

which, after substitution back into equation (3.2.1), gives

$$\phi_L(u) = w(u, k_1) + \sum_{i=1}^{N} v_i(u) \int_0^\infty f_{L_i}(t)e^{-\lambda t}\phi_L(ct)\, dt, \tag{3.2.2}$$

where $w(u, k)$ is defined as in equation (3.1.9) and $v_i(u)$, for $i = 1, 2, \ldots, N$, is defined as

$$v_i(u) = \frac{G(u, k_1)G_i(0)}{1 - G(0, k_1)} + G_i(u), \tag{3.2.3}$$

with $\sum_{i=1}^{N} v_i(u) = 1 - w(u, k_1)$.

It now remains for us to solve the integral equation (3.2.2). Employing a change of variables, equation (3.2.2) can be written in the form of an inhomogeneous Fredholm equation of the second kind, given by

$$\phi_L(u) = w(u, k_1) + \frac{1}{c}\sum_{i=1}^{N} v_i(u) \int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}}\phi_L(t)\, dt, \tag{3.2.4}$$

with degenerate kernel of the form

$$K(u, t) = \sum_{i=1}^{N} v_i(u) f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}}.$$

Then, following similar arguments as in Section 3.1.1 and the proof of Proposition 11, we note that the integral terms on the right hand side of the Fredholm integral equation, given in equation (3.2.4), evaluate to some constants, say $C_i = \int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} \phi_L(t)\, dt < \infty$. Thus, the general solution to equation (3.2.4) is given by the linear combination

$$\phi_L(u) = w(u, k_1) + \frac{1}{c}\sum_{i=1}^{N} C_i v_i(u). \tag{3.2.5}$$

In order to calculate explicitly the constants C_i, for $i = 1, 2, \ldots, N$, similarly to Section 3.1.1, we can replace the variable u, in equation (3.2.5), by t, multiply through by $f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}}$, for $j = 1, 2, \ldots, N$, and integrate over the interval $[0, \infty)$, to obtain a system of N simultaneous equation, given by

$$\begin{aligned}\int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} \phi_L(t)\, dt = & \int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t, k_1)\, dt \\ & + \frac{1}{c}\sum_{i=1}^{N} C_i \int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_i(t)\, dt,\end{aligned}$$

which, after recalling the definition of the constants C_i, $i = 1, 2, \ldots, N$, reduces to the form

$$C_j = \int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t, k_1)\, dt + \frac{1}{c}\sum_{i=1}^{N} C_i \int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_i(t)\, dt, \quad j = 1, 2, \ldots, N,$$

or equivalently

$$\begin{aligned}\int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t, k_1)\, dt = & \left(1 - \frac{1}{c}\int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_j(t)\, dt\right) C_j \\ & - \frac{1}{c}\sum_{i \neq j}^{N} C_i \int_0^\infty f_{L_j}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_i(t)\, dt, \quad j = 1, 2, \ldots, N.\end{aligned}$$

In a more concise matrix form, the above linear system of equation for C_j, $j = 1, 2, \ldots, N$, can be expressed by

$$\mathbf{M}\vec{C} = \vec{w},$$

where $\mathbf{M}$ is an N-dimensional square matrix given by

$$\mathbf{M} = \begin{pmatrix} 1 - \frac{1}{c}\int_0^\infty f_{L_1}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_1(t)\,dt & \cdots & -\frac{1}{c}\int_0^\infty f_{L_1}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_N(t)\,dt \\ \vdots & \ddots & \vdots \\ -\frac{1}{c}\int_0^\infty f_{L_N}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_1(t)\,dt & \cdots & 1 - \frac{1}{c}\int_0^\infty f_{L_N}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_N(t)\,dt \end{pmatrix},$$

$\vec{C} = (C_1, \ldots, C_N)^\top$ and $\vec{w} = \left(\int_0^\infty f_{L_1}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t, k_1)\,dt, \ldots, \int_0^\infty f_{L_N}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} w(t, k_1)\,dt\right)^\top$ are both N-dimensional column vectors. In order to evaluate the vector of unknowns, $\vec{C}$, we will show in the following lemma that the matrix $\mathbf{M}$ is non-singular and thus invertible.

Lemma 6. *The N-dimensional square matrix $\mathbf{M}$ is non-singular.*

Proof. As in the proof of Lemma 5, in order to prove the matrix $\mathbf{M}$ is non-singular, it suffices to prove it is a strictly diagonally dominant matrix [see Definition 17]. That is, the i-th diagonal element of $\mathbf{M}$, for all $i = 1, \ldots, N$, satisfies

$$\left|1 - \frac{1}{c}\int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_i(t)\,dt\right| > \sum_{j \neq i} \left|-\frac{1}{c}\int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_j(t)\,dt\right|,$$

or equivalently

$$1 - \frac{1}{c}\int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_i(t)\,dt > \sum_{j \neq i} \frac{1}{c}\int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_j(t)\,dt,$$

since (similarly to the proof of Lemma 5) $v_i(u) < 1$, for $u \geqslant 0$, which guarantees that $0 \leqslant \frac{1}{c}\int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_i(t)\,dt < 1$, for all $i = 1, \ldots, N$.

Now, since $\sum_{i=1}^{N} v_i(u) = 1 - w(u, k_1) < 1$, for all $u \geqslant 0$, we have that

$$\begin{aligned}1 = \int_0^\infty f_{L_i}(t)\, dt &> \int_0^\infty f_{L_i}(t)(1 - w(ct, k_1))\, dt \\ &> \int_0^\infty f_{L_i}(t) e^{-\lambda t} \sum_{j=1}^{N} v_j(ct)\, dt \\ &= \sum_{j=1}^{N} \frac{1}{c} \int_0^\infty f_{L_i}\left(\frac{t}{c}\right) e^{-\frac{\lambda t}{c}} v_j(t)\, dt,\end{aligned}$$

which completes the proof. □

Using the results of Lemma 6, it follows that the matrix $\mathbf{M}$ is invertible and the constants C_i, for $i = 1, 2 \ldots, N$, can be evaluated by

$$\vec{C} = \mathbf{M}^{-1} \vec{w},$$

where $\mathbf{M}^{-1}$ is the inverse of the matrix $\mathbf{M}$. Now, since the constants C_i, for $i = 1, \ldots, N$, are uniquely determined, we can employ the form of the general solution to the Fredholm integral equation, given by equation (3.2.5), to obtain the following theorem for the corresponding probability of ultimate ruin.

Theorem 22. *For $u \geqslant 0$, the ultimate ruin probability for capital injections with continuous-time random delays and $N \geqslant 1$ critical values, namely $\psi_L(u)$, is given by*

$$\psi_L(u) = \frac{1}{c} \sum_{i=1}^{N} \left(c - \left[\mathbf{M}^{-1} \vec{w}\right]_i\right) v_i(u), \tag{3.2.6}$$

where $\left[\mathbf{M}^{-1} \vec{w}\right]_i$ is the i-th element of the vector $\mathbf{M}^{-1} \vec{w}$ and

$$v_i(u) = \frac{G(u, k_1) G_i(0)}{1 - G(0, k_1)} + G_i(u).$$

3.3 Further risk related quantities

In the previous chapter, we discussed the necessity of analysing further risk related quantities, namely the accumulated capital injections up to the time of insolvency (see

Section 2.5) to provide an indication of the risk profile of the insurance firm and/or related parties contributing to its solvency via capital injections, i.e. the shareholders and reinsurance firms. Therefore, in this section, we consider the effects of a delay in the receipt of capital injections on a discounted version of this quantity, namely the expected discounted accumulated capital injections up to the time of ultimate ruin, which gives an indication of the (discounted) amount of funds needed to keep the company solvent during its lifetime.

In addition, since the company is allowed to continue whilst in deficit (during the delay time of a capital injection), we consider the expected discounted overall time in red (deficit), up to the time of ultimate ruin. This is a natural consideration, since the firm may be subject to some penalty during the time in which it is in a deficit and thus, the expected discounted overall time in red up to the time of ultimate ruin, provides the present value of this penalised time in red, allowing the company to more accurately calculate its capital requirements. There exist many papers concerned with the time/duration in a negative surplus, see for example Dos Reis (1993), (2000) and Dickson and Dos Reis (1996), among others, however, such a quantity has yet to be considered in connection with capital injections.

For simplicity of calculations, we revert back to the simplest model of a single critical value, given by $k \geqslant 0$ as in Section 3.1, but point out that the following results hold for the N barrier setting, by employing a similar method to that discussed in Section 3.2.

3.3.1 The expected discounted accumulated capital injections up to the time of ultimate ruin

Let $\{Z^L(t)\}_{t\geqslant 0}$ be a pure jump process denoting the accumulated capital injections in a continuous delay time setting, up to time $t \geqslant 0$, for the risk process $\{U^L(t)\}_{t\geqslant 0}$, defined in equation (3.1.2). We are interested in the expected discounted accumulated capital injections up to the time of ultimate ruin, from initial capital $u \geqslant 0$, denoted $z_\delta^L(u) = \mathbb{E}\left(e^{-\delta T^L} Z^L(T^L) \middle| U^L(0) = u\right)$, where $\delta \geqslant 0$ is a constant discount rate and T^L is the time of ultimate ruin, defined in equation (3.1.3).

In order to derive the discounted value $z_\delta^L(u)$, we introduce the joint probability d.f. of the time to ruin and the deficit at ruin, denoted $W(u, y, t)$.

Definition 18. *Let T be the time to ruin for the Cramér-Lundberg risk process, $\{U(t)\}_{t\geqslant 0}$, defined in equation 1.1.2. Then, the joint probability function of the time to ruin and the deficit at ruin, with initial capital $u \geqslant 0$, denoted by $W(u, y, t)$, is defined by*

$$W(u, y, t) = \mathbb{P}\left(T \leqslant t, |U(T)| \leqslant y \middle| U(0) = u\right),$$

having corresponding joint density function, denoted by $w(u, y, t)$, given by

$$w(u, y, t) = \frac{\partial^2}{\partial t \partial y} W(u, y, t).$$

The joint probability function $W(u, y, t)$ is a generalisation to the deficit at ruin function, defined in equation (1.5.1), where $\lim_{t\to\infty} W(u, y, t) = G(u, y)$, which has been studied in Dickson and Drekic (2006), Landriault and Willmot (2009) and Nie et al. (2011), (2015) and explicit expressions exist for certain claim size distributions. Moreover, we define by

$$g_\delta(u, y) = \int_0^\infty e^{-\delta t} w(u, y, t)\, dt, \tag{3.3.1}$$

and

$$G_\delta(u, y) = \int_0^y g_\delta(u, x)\, dx, \tag{3.3.2}$$

the (defective) discounted density function and distribution function, respectively, of the deficit at ruin, with initial surplus $u \geqslant 0$ and force of interest $\delta \geqslant 0$, with $g_0(u, y) = g(u, y)$ and $G_0(u, y) = G(u, y)$.

Now, using a similar conditioning argument as in the previous sections, that is by conditioning on the time and amount of the first fall into deficit and the subsequent delay and claim inter-arrival times, it follows that $z_\delta^L(u)$ satisfies an integral equation of the form

$$z_\delta^L(u) = \int_0^\infty \int_0^k e^{-\delta t} w(u,y,t)[y + z_\delta^L(0)]\, dy + \int_0^\infty \int_k^\infty e^{-\delta t} w(u,y,t) \int_0^\infty e^{-\delta s} f_L(s) \int_0^\infty f_\tau(v)[y + z_\delta^L(cs)] \mathbb{I}_{\{s<v\}}\, dv\, ds\, dy\, dt. \tag{3.3.3}$$

Employing the notation introduced in equations (3.3.1) and (3.3.2), for the discounted deficit at ruin functions, the above equation can be re-written as

$$\begin{aligned} z_\delta^L(u) &= \int_0^k y g_\delta(u,y)\, dy + G_\delta(u,k) z_\delta^L(0) \\ &\qquad + \int_k^\infty g_\delta(u,y) \int_0^\infty e^{-s(\delta+\lambda)} f_L(s)[y + z_\delta^L(cs)]\, ds\, dy \\ &= \int_0^k y g_\delta(u,y)\, dy + G_\delta(u,k) z_\delta^L(0) + \int_k^\infty y g_\delta(u,y) \int_0^\infty e^{-s(\delta+\lambda)} f_L(s)\, ds\, dy \\ &\qquad + \overline{G}_\delta(u,k) \int_0^\infty e^{-s(\delta+\lambda)} f_L(s) z_\delta^L(cs)\, ds. \end{aligned} \tag{3.3.4}$$

To complete the solution for $z_\delta^L(u)$, in equation (3.3.4), we need to determine an explicit expression for the boundary value $z_\delta^L(0)$. Setting $u = 0$, in equation (3.3.4), and solving with respect to $z_\delta^L(0)$, yields

$$\begin{aligned} z_\delta^L(0) = \frac{1}{1 - G_\delta(0,k)} \Bigg(&\int_0^k y g_\delta(0,y)\, dy + \int_k^\infty y g_\delta(0,y) \int_0^\infty e^{-s(\delta+\lambda)} f_L(s)\, ds\, dy \\ &+ \overline{G}_\delta(0,k) \int_0^\infty e^{-s(\delta+\lambda)} f_L(s) z_\delta^L(cs)\, ds \Bigg), \end{aligned}$$

and thus, equation (3.3.4), can be written in the form

$$z_\delta^L(u) = h_\delta(u,k) + v_\delta(u,k) \int_0^\infty e^{-(\delta+\lambda)t} f_L(t)\, z_\delta^L(ct)\, dt, \tag{3.3.5}$$

where

$$
\begin{aligned}
h_\delta(u,k) = &\int_0^k y g_\delta(u,y)\,dy + \int_k^\infty y g_\delta(u,y) \int_0^\infty e^{-s(\delta+\lambda)} f_L(s)\,ds\,dy \\
&+ \frac{G_\delta(u,k)}{1-G_\delta(0,k)} \left(\int_0^k y g_\delta(0,y)\,dy + \int_k^\infty y g_\delta(0,y) \int_0^\infty e^{-s(\delta+\lambda)} f_L(s)\,ds\,dy \right),
\end{aligned}
\tag{3.3.6}
$$

and

$$
v_\delta(u,k) = \frac{G_\delta(u,k)\overline{G}_\delta(0,k)}{1-G_\delta(0,k)} + \overline{G}_\delta(u,k) < 1, \tag{3.3.7}
$$

such that, when $\delta = 0$, we have $v_0(u,k) = v(u,k)$ given by equation (3.1.10).

Note that, equation (3.3.5) is of a similar form to equation (3.1.16). Thus, by a change of variable in the integral term, we have that

$$
z_\delta^L(u) = h_\delta(u,k) + \frac{1}{c} v_\delta(u,k) \int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) z_\delta^L(t)\,dt, \tag{3.3.8}
$$

which is an inhomogeneous Fredholm equation of the second kind and of a similar form to equation (3.1.17). Hence, provided that both $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) z_\delta^L(t)\,dt$ and $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) h_\delta(t,k)\,dt$ exist and are finite, the general solution of equation (3.1.17), given by equation (3.1.21), can be employed to solve equation (3.3.8).

Proposition 13. *Let $g(x)$ be a continuous function defined on the positive half line $[0,\infty)$, which is bounded by its finite maximum $M = \max_{x\in[0,\infty)}\{g(x)\} < \infty$. Then, $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) g(t)\,dt$ is finite and we have $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) g(t)\,dt < cM$.*

Proof. Firstly, by dividing $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) g(t)\,dt$ through by M, we obtain the normalised integral $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) \omega(t)\,dt$, where $\omega(t) = \frac{g(t)}{M} \leqslant 1$ for all $t \geqslant 0$. Now, applying similar arguments as the proof of Proposition 11, we have

$$
\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) \omega(t)\,dt < c.
$$

The result follows by multiplying the above inequality through by the maximum value $M < \infty$. □

Then, under the assumption that the expected deficit at ruin is finite, i.e. $\int_0^\infty y g(u,y)\,dy <$

∞, it can be seen from equation (3.3.6) that $h_\delta(u,k)$, and thus $z_\delta^L(u)$, are finite, for all $u \geqslant 0$. Hence, by Proposition 13, we have the following theorem.

Theorem 23. *Let $z_\delta^L(u)$ denote the expected discounted accumulated capital injections, in the continuous-time delayed capital injection setting, up to the time of ultimate ruin with initial capital $U^L(0) = u \geqslant 0$. Then, if $\int_0^\infty y g(u,y)\,dy < \infty$, the solution to the Fredholm integral equation* (3.3.8) *is given by*

$$z_\delta^L(u) = h_\delta(u,k) + \frac{\int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{(\delta+\lambda)t}{c}} h_\delta(t,k)\,dt}{c - \int_0^\infty f_L\left(\frac{t}{c}\right) e^{-\frac{t(\delta+\lambda)}{c}} v_\delta(t,k)\,dt} v_\delta(u,k), \tag{3.3.9}$$

where $h_\delta(u,k)$ and $v_\delta(u,k)$ are given by equation (3.3.6) *and* (3.3.7)*, respectively.*

3.3.2 Expected overall time in red up to the time of ultimate ruin

The expected discounted time in red, which reflects the expected discounted duration in deficit up to the time of ultimate ruin, can also be obtained using similar methodologies as above. That is, let $\{V^L(t)\}_{t\geqslant 0}$ be a stochastic process denoting the overall time in red up to time $t \geqslant 0$, defined by

$$V^L(t) = \int_0^\infty \mathbb{I}_{\{U^L(s)<0\}}\,ds.$$

Then, we are interested in the expected discounted overall time in red up to the time of ultimate ruin, from initial capital $u \geqslant 0$, denoted $\nu_\delta^L(u) = \mathbb{E}\left(e^{-\delta T^L} V^L(T^L) \middle| U^L(0) = u\right)$. Using a similar conditioning argument to the previous subsection, that is conditioning on the time and amount of the first fall into deficit, the subsequent delay and claim inter-arrival time, we have the following possibilities:

1. The deficit is less than the critical value, i.e. $|U(T)| \leqslant k$, and thus the capital injection is received instantaneously. Then, the duration of time in a deficit is zero, and the process renews from initial capital $u = 0$,
2. The deficit is larger than the critical value, i.e. $|U(T)| > k$ and:
   (a) The inter-arrival time, $w > 0$, of the subsequent claim occurs before the

capital injection is received. Then, our overall duration in deficit up to the time of ultimate ruin is $w > 0$, or,

(b) The delay time, $s > 0$, of the capital injection is smaller than the subsequent inter-arrival time. Thus, the duration of time in deficit is $s > 0$ and the process renews from initial capital $cs > 0$.

Considering the above possibilities, after a fall into deficit, we have

$$\begin{aligned}\nu_\delta^L(u) &= \int_0^\infty \int_0^k e^{-\delta t} w(u,y,t) \nu_\delta^L(0)\, dy dt + \int_0^\infty \int_k^\infty e^{-\delta t} w(u,y,t) \int_0^\infty f_L(s) \int_0^\infty f_\tau(w) \\ &\qquad \times \left[e^{-\delta w} w \mathbb{I}_{\{w<s\}} + e^{-\delta s} \left(s + \nu_\delta^L(cs) \right) \mathbb{I}_{\{s<w\}} \right] dw\, ds\, dy\, dt \\ &= G_\delta(u,k)\nu_\delta^L(0) + \overline{G}_\delta(u,k) \left(\int_0^\infty s \left[\lambda \overline{F}_L(s) + f_L(s) \right] e^{-(\delta+\lambda)s}\, ds \right. \\ &\qquad \left. + \int_0^\infty e^{-\delta s} f_L(s) \overline{F}_\tau(s) \nu_\delta^*(cs)\, ds \right), \end{aligned} \tag{3.3.10}$$

where

$$\overline{G}_\delta(u,k) = \int_k^\infty g_\delta(u,y)\, dy.$$

To complete the solution for $\nu_\delta^L(u)$, in equation (3.3.10), we need to determine an explicit expression for the boundary value $\nu_\delta^L(0)$. Setting $u = 0$, in the above equation, and solving with respect to $\nu_\delta^L(0)$, yields

$$\begin{aligned}\nu_\delta^L(0) = \frac{\overline{G}_\delta(0,k)}{1 - G_\delta(0,k)} &\left(\int_0^\infty s \left[\lambda \overline{F}_L(s) + f_L(s) \right] e^{-(\delta+\lambda)s}\, ds \right. \\ &\qquad \left. + \int_0^\infty e^{-\delta s} f_L(s) \overline{F}_\tau(s) v_\delta^L(cs)\, ds \right), \end{aligned}$$

and thus, equation (3.3.10), can be written in the form

$$\nu_\delta^L(u) = b_\delta(u,k) + v_\delta(u,k) \int_0^\infty e^{-(\delta+\lambda)t} f_L(t) \nu_\delta^L(ct)\, dt, \tag{3.3.11}$$

where

$$b_\delta(u,k) = v_\delta(u,k) \int_0^\infty s \left[\lambda \overline{F}_L(s) + f_L(s) \right] e^{-(\delta+\lambda)s}\, ds, \tag{3.3.12}$$

and $v_\delta(u, k)$ is defined in equation (3.3.7).

Now, equation (3.3.11) is again of a similar form to equation (3.1.16) and thus the general solution of equation (3.1.16) can be employed to solve the Fredholm integral equation in equation (3.3.11), provided both $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) \nu_\delta^L(t)\, dt$ and $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) b_\delta(t, k)\, dt$ exist and are finite.

In order to show that these conditions are satisfied, let us consider the behaviour of the function $b_\delta(u, k)$, given by equation (3.3.12) and recall that the function $v_\delta(u, k) < 1$, for all $u \geqslant 0$. Then, we have

$$\begin{aligned} b_\delta(u,k) &= v_\delta(u,k) \int_0^\infty s\left[\lambda \overline{F}_L(s) + f_L(s)\right] e^{-(\delta+\lambda)s}\, ds \\ &< \int_0^\infty s\left[\lambda \overline{F}_L(s) + f_L(s)\right] e^{-(\delta+\lambda)s}\, ds \\ &\leqslant \lambda \int_0^\infty s e^{-\lambda s}\, ds + \int_0^\infty s f_L(s)\, ds \\ &= 1 + \mathbb{E}(L) < \infty, \end{aligned}$$

since it is assumed that the delay time distribution has finite mean $\mathbb{E}(L) < \infty$ [see Section 3.1.1]. Using this result, the fact that the function $\nu_\delta^L(u)$ is finite (by the net profit condition) and applying the result of Proposition 13 to show the two integrals $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) \nu_\delta^L(t)\, dt$ and $\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) b_\delta(t, k)\, dt$ are finite, we have the following theorem.

Theorem 24. *Let $\nu_\delta^L(u)$ denote the expected discounted time in red, in the continuous time delayed capital injection setting, up to the time of ultimate ruin with initial capital $U^L(0) = u \geqslant 0$. Then, the solution to the Fredholm integral equation* (3.3.11) *is given by*

$$\nu_\delta^L(u) = b_\delta(u,k) + \frac{\int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) b_\delta(t,k)\, dt}{c - \int_0^\infty e^{-\frac{(\delta+\lambda)t}{c}} f_L\left(\frac{t}{c}\right) v_\delta(t,k)\, dt} v_\delta(u,k), \tag{3.3.13}$$

where $v_\delta(u, k)$ and $b_\delta(u, k)$ are given by equations (3.3.7) *and* (3.3.12)*, respectively.*

3.4 Capital injections with explicit delay time dependence

In this chapter, the dependency structure between the deficit at ruin and corresponding delay time of the capital injection has been hitherto based on a deficit falling between certain threshold barriers. In this section, we generalise the dependence such that, when the deficit is greater than the critical value $k \geqslant 0$, the random delay time depends explicitly on the size of the deficit ($y > 0$).

Let the delay time be denoted by a continuous random variable, L, which depends on the size of the deficit via the conditional distribution $F_{L|Y=y}(\cdot) =: F_{L|Y}(\cdot; y)$ and corresponding density $f_{L|Y}(\cdot; y)$, where $Y = |U(T)|$ is a random variable denoting the size of the deficit having d.f. $G(\cdot, \cdot)/\psi(\cdot)$. Intuitively, if the insurance company experiences a deficit of $Y = y > k$, then the delay time, L, increases as Y increases (the more capital the firm requires through a capital injection, the more time that will be needed to gather and process the funds), hence it is assumed that the conditional distribution, $F_{L|Y}(\cdot; y)$, is a decreasing function of $y > 0$.

Then, conditioning on the size of the deficit, the subsequent delay time and claim inter-arrival time, the ultimate survival probability satisfies an integral equation of the form

$$\begin{aligned} \phi_L(u) &= \phi(u) + G(u,k)\phi_L(0) + \int_k^\infty g(u,y) \int_0^\infty \int_0^\infty f_{L|Y}(t;y) f_\tau(s) \phi_L(ct) \mathbb{I}_{\{t<s\}} \, ds \, dt \, dy \\ &= \phi(u) + G(u,k)\phi_L(0) + \int_k^\infty g(u,y) \int_0^\infty e^{-\lambda t} f_{L|Y}(t;y) \phi_L(ct) \, dt \, dy. \end{aligned} \tag{3.4.1}$$

In order to determine the boundary value, $\phi_L(0)$, we set $u = 0$, in equation (3.4.1), and solve for $\phi_L(0)$, to obtain

$$\phi_L(0) = \frac{\phi(0) + \int_k^\infty g(0,y) \int_0^\infty e^{-\lambda t} f_{L|Y}(t;y) \phi_L(ct) \, dt \, dy}{1 - G(0,k)}.$$

Substituting this form of $\phi_L(0)$, into equation (3.4.1), and changing the order of integration in the resulting integral, yields

$$\phi_L(u) = w(u,k) + \int_0^\infty e^{-\lambda t} \left(\int_k^\infty z(u,k,y) f_{L|Y}(t;y) \, dy \right) \phi_L(ct) \, dt, \tag{3.4.2}$$

where $w(u,k)$ is given by equation (3.1.9) and

$$z(u,k,y) = \frac{G(u,k)g(0,y)}{1-G(0,k)} + g(u,y). \tag{3.4.3}$$

We note that, since $\int_k^\infty z(u,k,y)\,dy = v(u,k)$, defined in equation (3.1.10), it is not difficult to show that the right hand side of equation (3.4.2) is less than equal to 1 and thus, the integral equation is well defined.

Now, using a change of variables, equation (3.4.2) can be transformed to

$$\phi_L(u) = w(u,k) + \frac{1}{c}\int_0^\infty e^{-\frac{\lambda t}{c}} \left(\int_k^\infty z(u,k,y) f_{L|Y}\left(\frac{t}{c};y\right) dy \right) \phi_L(t)\,dt, \tag{3.4.4}$$

which is an inhomogeneous Fredholm integral equation of the second kind with kernel

$$K(u,t) = e^{-\frac{\lambda t}{c}} \left(\int_k^\infty z(u,k,y) f_{L|Y}\left(\frac{t}{c};y\right) dy \right). \tag{3.4.5}$$

The kernel $K(u,t)$, given above, is non-degenerate and a closed form solution is no longer obtainable. However, it is possible to derive a solution in terms of the Neumann series. For details of the following method of solution see Zemyan (2012).

To derive the Neumann series solution, let us first rewrite equation (3.4.4) in the following form

$$\phi_L(u) = w(u,k) + \alpha \int_0^\infty K(u,t)\phi_L(t)\,dt, \tag{3.4.6}$$

where $\alpha = c^{-1} > 0$ and $K(u,t)$ is given in equation (3.4.5). Then, by the method of successive substitution (see Chapter 2 of Zemyan (2012)), i.e. substituting the form of $\phi_L(u)$, given in equation (3.4.6), back into the integral itself, we obtain

$$\begin{aligned} \phi_L(u) &= w(u,k) + \alpha \int_0^\infty K(u,t) \left[w(t,k) + \alpha \int_0^\infty K(t,s)\phi_L(s)\,ds \right] dt \\ &= w(u,k) + \alpha \int_0^\infty K(u,t) w(t,k)\,dt + \alpha^2 \int_0^\infty \int_0^\infty K(u,t)K(t,s)\phi_L(s)\,ds\,dt, \end{aligned}$$

which, after changing the order of integration in the last term, yields

$$\phi_L(u) = w(u,k) + \alpha \int_0^\infty K(u,t) w(t,k)\,dt + \alpha^2 \int_0^\infty K_2(u,t)\phi_L(t)\,dt,$$

where

$$K_2(u,t) = \int_0^\infty K(u,s)K(s,t)\,ds.$$

Repeating the above iterative process, n times, yields

$$\phi_L(u) = w(u,k) + \sum_{m=1}^{n} \alpha^m \int_0^\infty K_m(u,t)w(t,k)\,dt + \alpha^{n+1} \int_0^\infty K_{n+1}(u,t)\phi_L(t)\,dt,$$

where $K_1(u,t) = K(u,t)$ and

$$K_m(u,t) = \int_0^\infty K_{m-1}(u,s)K(s,t)\,ds,$$

or equivalently

$$\phi_L(u) = w(u,k) + \alpha\Gamma_n(u) + \rho_n(u), \tag{3.4.7}$$

with

$$\Gamma_n(u) = \sum_{m=1}^{n} \alpha^{m-1} \left(\int_0^\infty K_m(u,t)w(t,k)\,dt \right) \tag{3.4.8}$$

and

$$\rho_n(u) = \alpha^{n+1} \int_0^\infty K_{n+1}(u,t)\phi_L(t)\,dt. \tag{3.4.9}$$

Following the theory of Fredholm integral equations of the second kind with general kernels, equation (3.4.7) has a unique solution as long as the sequence $\{\Gamma_n(u)\}_{n\in\mathbb{N}^+}$ of continuous functions converges uniformly to a continuous limit function on the interval $[0,\infty)$, and the sequence $\rho_n(u) \to 0$, as $n \to \infty$ [see Zemyan (2012) for more details].

Definition 19 (Uniform convergence)**.** *Let D be a subset of $\mathbb{R}$ and let f_n be a sequence of real valued functions defined on D. Then, f_n is said to converge uniformly to f if, given any $\epsilon > 0$, there exists a natural number $N = N(\epsilon)$, such that*

$$|f_n(x) - f(x)| < \epsilon, \quad \textit{for every} \quad n > N \quad \textit{and for every} \quad x \in D.$$

In order to prove that the sequence of functions, $\{\Gamma_n(u)\}_{n\in\mathbb{N}^+}$, converges uniformly to some limit function, say $\Gamma(u)$, for $u \geqslant 0$, it suffices to show that it is a uniformly Cauchy sequence [see Theorem 25 below].

Definition 20 (Uniformly Cauchy sequence)**.** *A sequence of real valued functions $f_n : D \to \mathbb{R}$ is uniformly Cauchy on D if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$n, p > N \quad \textit{implies that} \quad |f_n(x) - f_p(x)| < \epsilon \quad \textit{for all} \quad x \in D.$$

Theorem 25. *A sequence of functions $f_n : D \to \mathbb{R}$ converges uniformly on D if and only if it is uniformly Cauchy on D.*

To show that the sequence $\{\Gamma_n(u)\}_{n\in\mathbb{N}^+}$ is a uniformly Cauchy sequence on $[0,\infty)$, let $M = \max\{f_{L|Y}(x; y) : x \in [0,\infty), y \in [k,\infty)\}$ be the maximum value of the conditional delay time density, for all $y \geqslant k$. Then, it follows that

$$\begin{aligned}
|K(u,t)| &= e^{-\frac{\lambda t}{c}} \int_k^\infty z(u,k,y) f_L\left(\frac{t}{c}; y\right) dy \\
&\leqslant M e^{-\frac{\lambda t}{c}} \int_k^\infty z(u,k,y)\, dy, \quad \text{for all} \quad t \geqslant 0, \\
&= M e^{-\frac{\lambda t}{c}} v(u,k) \\
&< M e^{-\frac{\lambda t}{c}}, \quad \text{for all} \quad u \geqslant 0,
\end{aligned}$$

since $v(u,k) < 1$. Now, using the bound for $K(u,t) = K_1(u,t)$, we can determine a bound for $|K_2(u,t)|$, given by

$$\begin{aligned}
|K_2(u,t)| &= \int_0^\infty K(u,s)K(s,t)\, ds \\
&< M^2 e^{-\frac{\lambda t}{c}} \int_0^\infty e^{-\frac{\lambda s}{c}}\, ds \\
&= \frac{cM^2}{\lambda} e^{-\frac{\lambda t}{c}}.
\end{aligned}$$

Repeating this argument it is not hard to show that

$$|K_m(u,t)| < \left(\frac{cM}{\lambda}\right)^{m-1} M e^{-\frac{\lambda t}{c}},$$

for each $m \in \mathbb{N}^+$ and all $u \geqslant 0$. Now, by recalling the form of the functions $\Gamma_n(u)$, for $n \in \mathbb{N}^+$, given in equation (3.4.8), and using the bound for $|K_m(u,t)|$ given above, it follows that each summand within the summation of $\Gamma_n(u)$, satisfies the inequality

$$\begin{aligned}\left|\alpha^{m-1}\left(\int_0^\infty K_m(u,t)w(t,k)\,dt\right)\right| &< \left(\frac{\alpha c M}{\lambda}\right)^{m-1} M\int_0^\infty e^{-\frac{\lambda t}{c}}w(t,k)\,dt\\ &\leqslant \left(\frac{\alpha c M}{\lambda}\right)^{m-1}\frac{cM}{\lambda}, \qquad \text{(since } w(u,k)\leqslant 1\text{)}\\ &= c\left(\frac{M}{\lambda}\right)^{m},\end{aligned}$$

since $\alpha = c^{-1}$. Therefore, provided $\lambda > M$, we have

$$\begin{aligned}|\sigma_n(x)-\sigma_p(x)| &< c\sum_{m=p+1}^{n}\left(\frac{M}{\lambda}\right)^m\\ &< \frac{c(M/\lambda)^p}{1-(M/\lambda)}\\ &< \epsilon,\end{aligned}$$

for large enough p and hence, by Definition 20, the sequence $\{\Gamma_n(u)\}_{n\in\mathbb{N}^+}$ is a uniformly Cauchy sequence on $[0,\infty)$ and, by Theorem 25, the sequence converges uniformly to a continuous limit function, $\Gamma(u)$, given by

$$\Gamma(u) = \sum_{m=1}^{\infty}\alpha^{m-1}\left(\int_0^\infty K_m(u,t)w(t,k)\,dt\right).$$

Finally, it follows that

$$\begin{aligned}|\rho_n(u)| &= \alpha^{n+1}\int_0^\infty K_{n+1}(u,t)\phi_L(t)\,dt\\ &< \alpha M\left(\frac{\alpha c M}{\lambda}\right)^n\int_0^\infty e^{-\frac{\lambda t}{c}}\phi_L(t)\,dt,\\ &\leqslant (M/\lambda)^{n+1}\to 0, \quad \text{as} \quad n\to\infty,\end{aligned}$$

if $\lambda > M$, and we have the following theorem.

Theorem 26. *Assume that the conditional delay time density $f_{L|Y}(\cdot;y)$ is bounded for all $y \geqslant k$ and let $M = \max\{f_{L|Y}(x;y) : x\in[0,\infty), y\in[k,\infty)\}$ be its maximum value. Then, the ruin probability under an explicit delay time dependence, namely $\psi_L(u)$, is* *given by*

$$\psi_L(u) = v(u,k) - \sum_{m=1}^{\infty} c^{-m} \left(\int_0^{\infty} K_m(u,t) w(t,k)\, dt \right), \tag{3.4.10}$$

provided

$$\lambda > M,$$

where $w(u,k)$ *and* $v(u,k)$ *are given by equations* (3.1.9) *and* (3.1.10)*, respectively, and* $K_n(u,t)$ *is the* n*-th iterated kernel of* $K(u,t)$*, given in equation* (3.4.5)*.*

Example (Conditional exponential delay time)**.** *Assume that the conditional distribution of the delay time random variable, given a deficit size* $|U(T)| = y$*, follows an exponential distribution, with parameter* y^{-1}*, i.e.* $f_{L|Y}(x;y) = y^{-1}e^{-\frac{x}{y}}, y \geqslant k$*. Then, since a delay occurs only when the deficit is larger than* $k \geqslant 0$*, we have that*

$$\begin{aligned} M &= \max\{y^{-1}e^{-\frac{x}{y}} : x \in [0,\infty), y \in [k,\infty)\} \\ &= k^{-1}. \end{aligned}$$

Then, by Theorem 26, the ultimate ruin probability is given by

$$\psi_L(u) = v(u,k) - \sum_{m=1}^{\infty} c^{-m} \left(\int_0^{\infty} K_m(u,t) w(t,k)\, dt \right), \tag{3.4.11}$$

as long as $\lambda k > 1$*.*

Chapter 4

Parisian Ruin for the Dual Risk Process in Discrete-Time

In the previous chapters we have seen that deriving, and solving, differential/integral equations for the ruin probability, and other ruin related quantities, proves difficult and requires complex methodologies in continuous time. Therefore, in this final chapter, we will move away from risk models in the continuous time setting and concern ourselves with an analogous discrete-time model, where the probabilistic reasoning and methodologies for analysis are more attainable.

The discrete-time risk model was first proposed by Gerber (1988), with the introduction of the compound binomial risk process, which is a discrete-time analogue of the classic Cramér-Lundberg risk model, given by equation (1.1.2), and provides a more realistic interpretation of the cash flows for an insurance firm. In this compound binomial risk model, with discrete time periods $n \in \mathbb{N}$, it is assumed that income is received via a periodic premium of size one, whilst the initial reserve and the claim amounts are assumed to be non-negative integer valued. We point out that a unit premium does not restrict the model assumptions, since a general premium can be considered by scaling the time periods.

Definition 21 (Discrete-time compound binomial process)**.** *The surplus process of an insurer in the discrete-time compound binomial risk model, denoted* $\{U(n)\}_{n\in\mathbb{N}}$*, is* *defined by*

$$U(n) = u + n - \sum_{i=1}^{n} Y_i, \tag{4.0.1}$$

where $u \in \mathbb{N}$ is the insurers initial capital and the sequence of random, non-negative claim amounts, namely $\{Y_i\}_{i\in\mathbb{N}^+}$, are i.i.d. random variables with p.m.f. $p_k = \mathbb{P}(Y_1 = k)$, for $k \in \mathbb{N}$, and finite mean $\mathbb{E}(Y_1) < \infty$.

In the compound binomial risk model, defined in Definition 21, it is assumed that in any period of time, no claim appears with probability $p_0 = (1-q) \in [0,1]$, or a claim appears with probability $q \in [0,1]$, where the occurrence of claims in different time periods are independent events. Thus, it follows that the sequence of i.i.d. claim inter-arrival times, denoted $\{\tau_i\}_{i\in\mathbb{N}^+}$, follows a geometric distribution with parameter $q \in [0,1]$ and the corresponding counting process, $\{N(n)\}_{n\in\mathbb{N}}$, is a binomial process, with parameter $n \in \mathbb{N}$ and $q \in [0,1]$, with $N(0) = 0$. Moreover, the generic claim size random variables, Y_i, for $i \in \mathbb{N}^+$, have the form $Y_i = I_i \cdot X_i$ and their p.m.f. can be written as

$$p_k = \begin{cases} 1-q, & k = 0 \\ qf_k, & k \in \mathbb{N}^+, \end{cases} \tag{4.0.2}$$

where $f_k = \mathbb{P}(X = k) = p_k q^{-1}$ is the p.m.f. of the i.i.d. strictly positive claim size random variables $\{X_i\}_{i\in\mathbb{N}^+}$. Thus, the compound binomial process can be expressed alternatively in the form

$$U(n) = u + n - \sum_{i=1}^{N(n)} X_i, \qquad n \in \mathbb{N}. \tag{4.0.3}$$

Finally, it is assumed that the premiums contain a safety loading, where

$$\mathbb{E}(Y_1) = q\mathbb{E}(X_1) < 1,$$

such that the event of ruin (defined below) is not certain. Note that, since the geometric distribution of the claim inter-arrival times has the memoryless property, it is considered the discrete analogue of the exponential distribution and thus, it is easy to see the similarities between the compound binomial process, given in equation (4.0.3), and the compound Poisson process in the continuous time Cramér-Lundberg risk model, given in equation (1.1.2). Such a model is of independent interest, due to its more intuitive reflection of the trading periods within in an insurance firm and due to the ability to obtain recursive formulas without assuming a claim severity distribution. On the other hand, it can also be used as an approximation to the continuous time compound Poisson model [see Dickson (1994)].

Early works in the risk theory literature, concerning the binomial risk process, cover several different probabilistic methods to obtain a common expression for the probability of ruin, the definition of which alters slightly between different authors, and other ruin related quantities, such as those described in Chapter 1. To explain these results in some more detail, we need first to define the event of ruin in the discrete-time model.

Definition 22 (Time to ruin in discrete-time). *The time to ruin in the discrete-time risk model, denoted by T, is a non-negative random variable, defined as*

$$T = \inf\{n \in \mathbb{N}^+ : U(n) \leqslant 0\},$$

where $T = \infty$ if $U(n) > 0$, for all $n \in \mathbb{N}^+$.

Note that this definition is consistent with Gerber (1988) and alters from the classical ruin time in the continuous-time model by the inclusion of zero, whilst other authors define the ruin time when the reserve takes strictly negative values [see Shiu (1989) and Willmot (1993) among others].

Definition 23 (Discrete-time ruin probability). *For $u \in \mathbb{N}$, the finite-time ruin probability, denoted $\psi(u,t)$, is defined by*

$$\psi(u,t) = \mathbb{P}(T < t \big| U(0) = u), \qquad t \in \mathbb{N},$$

with finite-time survival probability, denoted $\phi(u,t)$, defined as

$$\begin{aligned}\phi(u,t) &= \mathbb{P}(T \geqslant t \big| U(0) = u) \\ &= 1 - \psi(u,t). \end{aligned} \tag{4.0.4}$$

The corresponding infinite-time ruin probability for the discrete time risk model, denoted by $\psi(u)$, is defined by

$$\psi(u) = \lim_{t\to\infty} \psi(u,t) = \mathbb{P}(T < \infty \big| U(0) = u),$$

with infinite time survival probability $\phi(u) = 1 - \psi(u)$.

We point out that although the ruin time is defined at the first time of a non-positive surplus, ruin does not occur at time zero for an initial capital $u = 0$. In this case, the ruin time is defined as the first time the surplus revisits the zero level, or drops below, and $\psi(0,t) \neq 1$, for any $t \in \mathbb{N}$. Then, using the definition of the infinite-time ruin probability (above) and conditioning on the possible events in the first time period, from the law of total probability, we have

$$\psi(0) = (1-q)\psi(1) + q, \tag{4.0.5}$$

and

$$\psi(u) = (1-q)\psi(u+1) + q\sum_{j=1}^{u} \psi(u+1-j)f_j + q\sum_{j=u+1}^{\infty} f_j, \quad \text{for} \quad u \in \mathbb{N}^+. \tag{4.0.6}$$

Notice that, if the value of the ruin probability with zero initial surplus, namely $\psi(0)$, is known, then the above equation can be used to calculate the ruin probabilities $\psi(u)$, for $u \in \mathbb{N}^+$, recursively.

In the original paper of Gerber (1988), explicit expressions are derived for the infinite-time ruin probability and other ruin related quantities such as the the number of visits to a given level, the probability of reaching a given level and the surplus immediately prior and deficit at ruin, to name a few, when the initial capital $u =$

0. In fact, all these expressions follow readily from a key result (Theorem 1, Gerber (1988)), where an expression involving the expected value of the aggregate claims can be expressed as a simple function of the mean claim size, $\mathbb{E}(Y_1)$, and the claim occurrence probability, namely $q \in (0, 1)$. The infinite-time probability of ruin with initial capital $u = 0$, for the compound binomial risk process, is given in the following lemma.

Lemma 7. *For $u = 0$, the discrete-time ruin probability for the compound binomial risk process is given by*

$$\psi(0) = q\mathbb{E}(X_1). \tag{4.0.7}$$

Remark 16. *Comparing the result of Lemma 7 with Lemma 1 of Chapter 1, we can see further evidence of the similarities between the discrete-time binomial model and the continuous-time Poisson risk model.*

Finally, using a similar argument as for the case of zero initial capital, i.e. $u = 0$, Gerber (1988) derives an explicit expression for the ruin probability with general initial capital $u \in \mathbb{N}$.

Shortly after the original work of Gerber (1988), Shiu (1989) considers a similar model and derives an equivalent expression to that of Gerber (1988), in terms of the infinite-time survival probability $\phi(u)$, for general $u \in \mathbb{N}$, using alternative methods. That is, by considering the definition of discrete convolutions, he is able to re-write the recursive equation for the ultimate-time survival probability, corresponding to equation (4.0.6), in the form of a Volterra equation of the second kind, obtaining a Neumann series solution, which is equivalent to the result found in Gerber (1988).

Later, Dickson (1994) proposes a third method and derives a simple recursive equation for the infinite-time survival probability, in terms of the tail d.f. of the claim sizes and, by using the discrete version of the deficit at ruin function, obtains the initial value, $\psi(0)$, given in Lemma 7. In order to obtain an explicit expression for the ruin probability, with general initial capital $u \in \mathbb{N}$, Dickson (1994) considers the so-called Binomial/Geometric model, which is the discrete analogue of the Poisson/Exponential risk model considered in Proposition 2. In this model, it is assumed that the positive claim size distribution, namely f_k, is geometrically distributed with parameter

$(1-\alpha) \in (0,1)$, such that

$$f_k = (1-\alpha)\alpha^{k-1}, \qquad k \in \mathbb{N}^+,$$

or equivalently, the p.m.f. of the generic claim size random variable Y_1, namely p_k, for $k \in \mathbb{N}^+$, is given by

$$p_k = q(1-\alpha)\alpha^{k-1}, \qquad k \in \mathbb{N}^+,$$

where $q/(1-\alpha) < 1$, by the net profit condition. Employing these forms of the probability functions, Dickson (1994) shows that the recursive expression satisfied by the survival probability, reduces to a difference equation (of order 1) in terms of the ruin probability, the solution of which is given in the following proposition.

Proposition 14. *Let the individual claim size random variable be geometrically distributed with parameter* $(1-\alpha) \in (0,1)$*. Then, the ruin probability for the compound binomial risk model, namely* $\psi(u)$*, is given by*

$$\psi(u) = \psi(0)\left(\frac{\alpha}{1-q}\right)^u, \qquad u \in \mathbb{N},$$

where

$$\psi(0) = \frac{q}{1-\alpha} < 1.$$

In addition to obtaining a recursive expression for the ruin probability, Dickson (1994) investigates the method for approximating the continuous-time compound Poisson model by the compound binomial model, as discussed in Gerber (1988). The main idea behind this approximation is to first approximate a discrete-time compound Poisson with the compound binomial, then employ the approximation algorithm of Dickson and Waters (1991), which approximates the continuous-time compound Poisson with a discrete-time compound Poisson under which the Poisson parameter and the mean individual claim amounts are unitary. For further references on ruin related results, such as; the discounted probability of ruin, the deficit and surplus prior to ruin and the well known Gerber-Shiu function, to name a few, see Cheng et al. (2000), Cossette et al. (2003, 2004), Boudreault et al. (2006), Dickson (1994), Li and Garrido (2002), Pavlova and Willmot (2004), Wu and Li (2009), Yuen and Guo (2006), and references therein.

The finite-time ruin probability for the discrete-time risk model was first studied in Willmot (1993), where an explicit formula is derived for the finite-time survival function, $\phi(u, t)$, by deriving a bivariate difference equation, which is solved using bivariate generating functions and analytical techniques such as Lagrange's expansion. Later, Lefèvre and Loisel (2008) derive a seal-type formula based on the ballot theorem [see Takács (1962)] and a Picard-Lefèvre-type formula for the corresponding finite-time survival probability. More recently, Li and Sendova (2013) derived a technical result connected to generating functions, by which the inverse of a generating function, with a particular argument, can be obtained if we know the inverse of the same generating function when the argument is the solution to the discrete-Lundberg equation (see below). Using this result, they derive an expression for the first hitting time for the surplus process, $\{U(n)\}_{n\in\mathbb{N}}$ with initial capital $u = 0$, of a pre-specified level $x \in \mathbb{N}$.

Proposition 15 (First hitting time)**.** *Let $T^x = \inf\{n \in \mathbb{N}^+ : U(n) = x\}$ be the first time the surplus process $\{U(n)\}_{n\in\mathbb{N}}$ hits the level $x \in \mathbb{N}^+$. Then, the probability mass function of the hitting time T^x, with initial capital $U(0) = 0$, is given by*

$$\mathbb{P}(T^x = n \big| U(0) = 0) = \frac{x}{n} p^{*n}_{n-x}, \quad n \geqslant x,$$

where $\{p^{*n}_k\}_{n\in\mathbb{N}}$ denotes the n-th fold convolution of Y_1.

In addition, they employ this relationship between generating function to derive the probability function of the time to ruin and the duration of the negative surplus. For a comprehensive review of the earlier results on the discrete-time risk model, we refer the reader to Li et al. (2009), and references therein.

4.1 Parisian ruin for the dual risk model in discrete-time

The compound binomial risk model, defined in (4.0.1), and the corresponding continuous-time compound Poisson risk model, are well suited for describing the cash flows of an non-life insurance firm which receives constant premium as income and incurs losses due to random claims. However, as pointed out by Avanzi et al. (2007) for the continuous-time model, depending on the line of business there exist companies for which an alternative model may be better suited. For instance, pharmaceutical or petroleum companies receive income as random gains from new invention or discoveries, whilst facing continuous/constant expenses in terms of labour and utilities. For this case, the classical risk models are no longer sufficient and an alternative model is required. In discrete-time, this alternative model, denoted $\{U^*(n)\}_{n\in\mathbb{N}}$, can be defined by

$$U^*(n) = u - n + \sum_{i=1}^{n} Y_i, \tag{4.1.1}$$

where $u \in \mathbb{N}$ is the initial capital and the sequence of non-negative random variables $\{Y_i\}_{i\in\mathbb{N}^+}$, denoting the random income gains, has p.m.f. p_k, for $k \in \mathbb{N}$, as defined in Definition 21. This model is known as the *dual compound binomial risk model*. The continuous analogue of the dual risk model has been considered by various authors, with the majority of focus in dividend problems [see Avanzi et al. (2007), Bergel et al. (2016), Cheung and Drekic (2008), Ng (2009) and references therein]. Additionally, Albrecher et al. (2008) considered the continuous-time dual risk model under a loss-carry forward tax system, where, in the case of exponentially distributed jump sizes, the infinite-time ruin probability is derived in terms of the ruin probability without taxation. However, the dual risk problem in discrete-time remains to be studied.

The finite-time ruin probability, for the dual risk process given in equation (4.1.1), is defined in a similar way to that of the compound binomial risk model, defined in equation (4.0.1). That is, the finite-time ruin probability is the probability that the dual risk process $\{U^*(n)\}_{n\in\mathbb{N}}$ attains a non-positive level before some pre-specified time horizon $t \in \mathbb{N}$, from initial capital $u \in \mathbb{N}$. Note that, since the dual risk model is downward skip free, in terms of claims, and experiences deterministic losses of one per period, it follows that the probability of experiencing a non-positive level is equivalent to the probability of hitting the zero level. Thus, let us denote the time to ruin for the dual risk model, given in equation (4.1.1), by T^*, which is defined by

$$T^* = \inf\{n \in \mathbb{N} : U^*(n) = 0\}.$$

Then, the finite-time dual ruin probability from initial capital $u \in \mathbb{N}$, denoted by $\psi^*(u,t)$, is defined by

$$\psi^*(u,t) = \mathbb{P}(T^* < t \big| U^*(0) = u), \qquad t \in \mathbb{N}, \tag{4.1.2}$$

with $\psi^*(0,t) = 1$, for all $t \in \mathbb{N}$, and corresponding finite-time dual survival probability, denoted $\phi^*(u,t)$, given by

$$\phi^*(u,t) = \mathbb{P}(T^* \geqslant t \big| U^*(0) = u), \tag{4.1.3}$$

with $\phi^*(0,t) = 0$, for all $t \in \mathbb{N}$.

Remark 17. *We point out that our boundary condition, $\psi^*(0,t) = 1$, for $t \in \mathbb{N}^+$, differs slightly from that of the compound binomial risk process - in the compound binomial definition, ruin does not occur immediately, for initial capital $u = 0$, but is defined as the first strictly positive time that the process attains the zero level - however, the reason for this deviation will become apparent in the following.*

The infinite-time dual ruin probability, denoted by $\psi^*(u)$, is defined as the limiting case of the finite-time dual ruin probability as $t \to \infty$, for $t \in \mathbb{N}$, i.e.

$$\begin{aligned}\psi^*(u) &= \lim_{t\to\infty} \psi^*(u,t) \\ &= \mathbb{P}(T^* < \infty \big| U^*(0) = u),\end{aligned} \tag{4.1.4}$$

with corresponding infinite-time dual survival probability

$$\begin{aligned}\phi^*(u) &= \lim_{t\to\infty} \phi^*(u,t) \\ &= 1 - \psi^*(u).\end{aligned} \tag{4.1.5}$$

Remark 18. *From the definition of the time to ruin, for the dual risk model, and the fact that the process is downward skip free, i.e. downward movements only occur due to the deterministic losses of one per period, it is clear that from initial capital* $U^*(0) = u$, *we have* $T^* \geqslant u$.

Finally, similarly to the compound binomial risk model, it is assumed that a net profit condition holds. That is, $\mathbb{E}(Y_1) > 1$, such that $U^*(n) \to +\infty$ as $n \to \infty$ and ruin does not occur a.s.

In the remainder of this chapter, for convenience, we will employ the notation $\mathbb{P}_u(\cdot) := \mathbb{P}(\cdot \big| U^*(0) = u)$.

In the previous sections, we discussed capital injections as a recapitalisation method of recovery for an insurance firm in deficit, which usually has an associated cost, e.g. premiums for a reinsurance contract, dividends to the shareholders in exchange for a capital injections etc. On the other hand, if there is confidence within a company that this fall into deficit can be recovered quickly, by its usual trading strategy, a more economic alternative would be to simply allow the company to continue when in a deficit. However, if the company fails to recover before some pre-specified time period $r \in \mathbb{N}^+$, then the confidence is lost and ultimate ruin occurs at this point. That is, the time of ultimate ruin is no longer defined as the time of falling into a deficit, as in the classical sense, but if the surplus experiences a continuous excursion below zero for some fixed time period. This is known in the literature as Parisian ruin and follows from Parisian stock options, where prices are activated or cancelled when underlying assets stay above or below a barrier long enough [see Chesney et al. (1997) and Dassios and Wu (2008)]. The time of Parisian ruin, in the discrete-time dual risk model, denoted T^r with $r \in \mathbb{N}^+$, is defined as

$$T^r = \inf\{n \in \mathbb{N} : U^*(n) < 0, n - \sup\{s < n : U^*(s) = -1, U^*(s-1) = 0\} = r \in \mathbb{N}^+\}, \tag{4.1.6}$$

with finite and infinite-time Parisian ruin probabilities defined by

$$\psi_r^*(u,t) = \mathbb{P}_u(T^r < t), \qquad t \in \mathbb{N}, \tag{4.1.7}$$

and

$$\psi_r^*(u) = \lim_{t\to\infty} \psi_r^*(u,t), \tag{4.1.8}$$

respectively. We further define the corresponding finite and infinite-time Parisian survival probabilities by

$$\begin{aligned}\phi_r^*(u,t) &= \mathbb{P}_u(T^r \geqslant t)\\ &= 1-\psi_r^*(u,t), \qquad t\in\mathbb{N},\end{aligned} \tag{4.1.9}$$

and

$$\phi_r^*(u) = 1-\psi_r^*(u). \tag{4.1.10}$$

The generalisation from classical ruin to Parisian ruin was first proposed, in a continuous time setting, by Dassios and Wu (2008) for the compound Poisson risk process with exponential claim sizes. In this setting, they derive expressions for the LT of the time and probability of Parisian ruin. In a more general setting, Czarna and Palmowski (2011) and Loeffen et al. (2013) derived results for the Parisian ruin probability for spectrally negative Lévy processes. More recently, Czarna et al. (2016) adapted the Parisian ruin problem to a discrete-time risk model, as in equation (4.0.1), where finite and infinite-time expressions for the ruin probability are derived, along with the light and heavy-tailed asymptotic behaviour. To obtain such results, the authors first derive the following joint probability function of the time to ruin and the deficit at ruin for the discrete model defined in Definition 21, using a generalised ballot type theorem of Lefèvre and Loisel (2008).

Lemma 8. *For $s\in\mathbb{N}^+$, we have*

$$\begin{aligned}\mathbb{P}(T=s,-U(T)=z\big|U(0)=u) &= \sum_{k=0}^{u+s-2}\mathbb{P}\left(T>s-1,\sum_{i=1}^{s-1}Y_i=k\middle|U(0)=u\right)p_{u+s-k+z}\\ &= \sum_{k=0}^{u+s-2}p_k^{*(s-1)}p_{u+s-k+z}-\sum_{k=u+1}^{u+s-2}\sum_{j=u+1}^{k}\frac{s-1+u-k}{s-1+u-j}p_{k-j}^{*(s-1+u-j)}p_j^{*(j-u)}p_{u+s-k+z}.\end{aligned} \tag{4.1.11}$$

In this chapter, we consider the alternative dual model of that in Czarna et al. (2016), for which the results and analysis differs due to the lack of downward jumps.

4.2 Finite-time Parisian ruin probability

In this section, we derive an expression for the finite-time Parisian survival probability, $\phi_r^*(u,t)$, for the dual risk model given in equation (4.1.1), with general initial capital $u \in \mathbb{N}$.

First note that, since the dual risk process, $\{U^*(n)\}_{n\in\mathbb{N}}$, experiences only positive random gains and losses occur at a rate of one per period, it follows that $\phi_r^*(u,t) = 1$, when $t \leqslant u + r + 1$. Now, for $t > u + r + 1$, by conditioning on the time to ruin for a dual risk process, namely T^*, we have

$$\begin{aligned}
\phi_r^*(u,t) &= \sum_{k=0}^{\infty} \mathbb{P}_u(T^* = k)\phi_r^*(0, t-k) \\
&= \sum_{k=u}^{t-r-2} \mathbb{P}_u(T^* = k)\phi_r^*(0, t-k) + \sum_{k=t-r-1}^{\infty} \mathbb{P}_u(T^* = k) \\
&= \sum_{k=u}^{t-r-2} \mathbb{P}_u(T^* = k)\phi_r^*(0, t-k) + \phi^*(u, t-r-1), \qquad (4.2.1)
\end{aligned}$$

since $\mathbb{P}_u(T^* = k) = 0$, for $k < u$ and $\sum_{k=t}^{\infty} \mathbb{P}_u(T^* = k) = \phi^*(u,t)$ is the finite-time survival probability (non-Parisian) of the dual risk process with initial capital $u \in \mathbb{N}$. Then, from the form of equation (4.2.1), in order to obtain an explicit expression for the Parisian survival probability, $\phi_r^*(u,t)$, it suffices to derive expressions for $\mathbb{P}_u(T^* = k)$ and the Parisian survival probability with zero initial capital, namely $\phi_r^*(0,t)$.

Lemma 9. *In the discrete-time dual risk model, the probability of hitting the zero level from initial capital $u \in \mathbb{N}$, in $n \in \mathbb{N}$ periods, namely $\mathbb{P}_u(T^* = n)$, is given by*

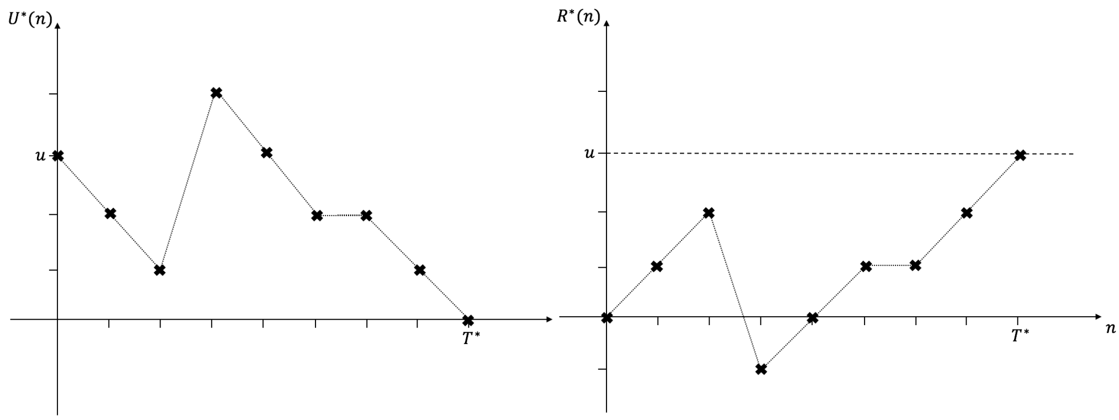
$$\mathbb{P}_u(T^* = n) = \frac{u}{n} p_{n-u}^{*n}, \quad n \geqslant u. \qquad (4.2.2)$$

Proof. Consider the discrete-time dual risk process, $\{U^*(n)\}_{n\in\mathbb{N}}$, defined in equation

(4.1.1), where

$$U^*(n) = u - R^*(n), \tag{4.2.3}$$

with $R^*(n) = n - \sum_{i=1}^{n} Y_i$. The 'increment' process, $\{R^*(n)\}_{n\in\mathbb{N}}$, is equivalent to a discrete-time risk process, given by equation (4.0.1), with initial capital zero. Therefore, it follows that the dual ruin time, namely T^*, is equivalent to the first hitting time for the increment process, $\{R^*(n)\}_{n\in\mathbb{N}}$, of the level $u \in \mathbb{N}$ (see Fig:4.1). That is, $T^* \equiv T^u$ of Proposition 15, from which the result follows. $\square$



(a) Typical sample path of surplus process $U^*(n)$, with initial capital $u \in \mathbb{N}$.

(b) Corresponding sample path of the increment process $R^*(n)$ with initial capital 0.

Figure 4.1: Equivalence between dual risk process and classic risk process.

Now that we have an expression for $\mathbb{P}_u(T^* = k)$ and consequently for the finite-time dual survival probability, $\phi^*(u,t)$, given by

$$\phi^*(u,t) = \sum_{k=t}^{\infty} \mathbb{P}_u(T^* = k) = \sum_{k=t}^{\infty} \frac{u}{k} p_{k-u}^{*k}, \tag{4.2.4}$$

it remains to derive an expression for the finite-time Parisian survival probability with zero initial reserve, i.e. $\phi_r^*(0,t)$. Before we begin with deriving an expression for $\phi_r^*(0,t)$, note that in order to avoid Parisian ruin, once the reserve process becomes negative, it is necessary to return to the zero level (or above) in $r \in \mathbb{N}^+$ time periods or less. Considering this observation, we will introduce another random stopping time, denoted by τ^-, which we name 'recovery time', that measures the number of periods it takes to recover from a deficit to a non-negative reserve and is defined by

$$\tau^- = \inf\{n \in \mathbb{N}^+ : U^*(n) \geqslant 0, U^*(s) < 0, \forall\, 0 \leqslant s < n\}.$$

Moreover, since the recovery time only occurs at the time of an upward jump, we introduce the joint probability distribution of the time of recovery and the overshoot at recovery from initial capital $x < 0$, denoted by $\mathbb{P}_x\left(\tau^- = n, U^*(\tau^-) = z\right)$, which will prove convenient for the subsequent derivations.

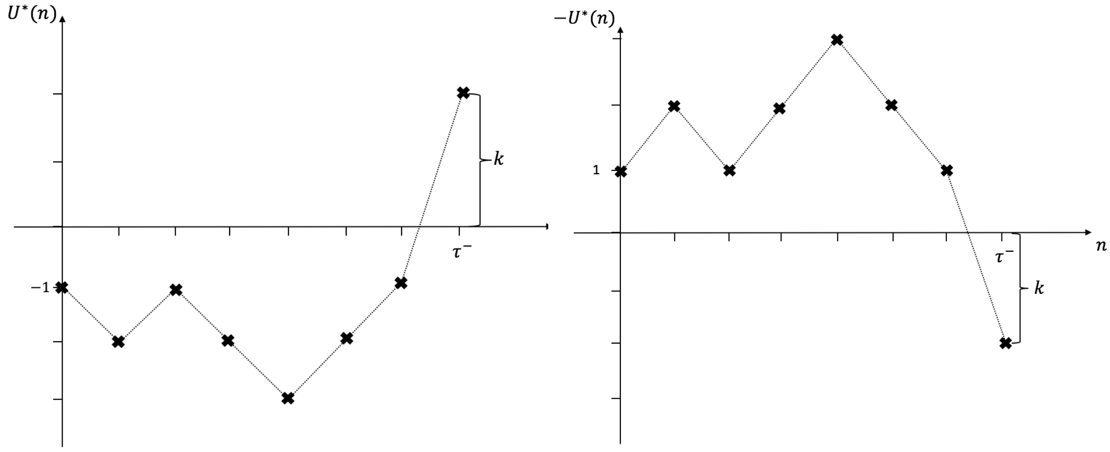
Now, consider the dual risk process defined in equation (4.1.1), with initial capital $u = 0$. If no gain occurs in the first period of time, the surplus becomes $U^*(1) = -1$ at the end of the period. On the other hand, if there is a random gain of amount $k \in \mathbb{N}^+$, in the first period, the surplus becomes $U^*(1) = k - 1$. Hence, by the law of total probability, we obtain a recursive equation for the finite-time Parisian survival probability, with initial capital zero, i.e. $\phi_r^*(0, n)$, for $n > r + 1$ (with $\phi_r^*(0, n) = 1$ for $n \leqslant r + 1$), of the form

$$\begin{aligned}\phi_r^*(0, n) &= p_0\, \phi_r^*(-1, n-1) + \sum_{k=1}^{\infty} p_k \phi_r^*(k-1, n-1) \\ &= p_0 \sum_{s=1}^{r} \sum_{z=0}^{\infty} \mathbb{P}_{-1}\left(\tau^- = s, U^*(\tau^-) = z\right) \phi_r^*(z, n-s-1) + \sum_{k=0}^{\infty} p_{k+1} \phi_r^*(k, n-1),\end{aligned} \tag{4.2.5}$$

where $\mathbb{P}_{-1}(\tau^- = \cdot, U^*(\tau^-) = \cdot)$ is the joint probability of the recovery time and the size of the overshoot at recovery, given initial capital $x = -1$.

By exploiting the distributional similarities between the reflected dual risk process and the compound binomial model, we obtain an explicit expression for this joint probability, given in the following lemma.

Lemma 10. *For, $n \in \mathbb{N}^+$ and $k \in \mathbb{N}$, the joint distribution of the recovery time and*



(a) Typical sample path of risk reserve process $U^*(n)$ with initial capital $u = -1$.

(b) Sample path of the reflected risk reserve process $-U^*(n)$ with initial capital $u = 1$.

Figure 4.2: Equivalence between dual and the compound binomial risk processes.

the overshoot at recovery, from initial capital $u = -1$*, is given by*

$$\mathbb{P}_{-1}(\tau^- = n, U^*(\tau^-) = k) = \sum_{j=0}^{n-1} p_j^{*(n-1)} p_{1+n-j+k} - \sum_{j=2}^{n-1} \sum_{i=2}^{j} \frac{n-j}{n-i} p_{j-i}^{*(n-i)} p_i^{*(i-1)} p_{1+n-j+k}, \tag{4.2.6}$$

Proof. Consider the dual risk process, $\{U^*(n)\}_{n \in \mathbb{N}}$ defined in equation (4.1.1), with initial capital $u = -1$. Then, the corresponding 'reflected' dual risk process, denoted $\{-U^*(n)\}$, is given by

$$-U^*(n) = 1 + n - \sum_{i=1}^{n} Y_i,$$

which is equivalent to the compound binomial risk process, $\{U(n)\}_{n \in \mathbb{N}}$, defined in equation (4.0.1), with initial capital $u = 1$. Therefore, it follows that the distribution of the time to cross the time axis and the overshoot of the process at this hitting time are equivalent for both processes (see Fig: 4.4). It follows that the joint distribution, $\mathbb{P}_{-1}(\tau^- = n, U^*(\tau^-) = k)$, for the dual risk model can be found by employing the discrete ruin related quantity for the compound binomial risk model of Lemma 8. That is, by setting $u = 1$ in equation (4.1.11), the result follows.

□

Finally, in order to make equation (4.2.5) dependent only on an unknown in terms of the boundary value, $\phi_r^*(0,n)$, we substitute the form of $\phi_r^*(u,t)$, given in equation (4.2.1), into equation (4.2.5), to obtain

$$\begin{aligned}\phi_r^*(0,n) &= p_0\sum_{s=1}^{r}\sum_{z=0}^{\infty}\mathbb{P}_{-1}(\tau^- = s, U^*(\tau^-) = z)\phi^*(z, n-s-r-2)\\ &\quad + p_0\sum_{s=1}^{r}\sum_{z=0}^{\infty}\sum_{i=z}^{n-s-r-3}\mathbb{P}_{-1}(\tau^- = s, U^*(\tau^-) = z)\mathbb{P}_z(T^* = i)\phi_r^*(0, n-s-i-1)\\ &\quad + \sum_{k=0}^{\infty}p_{k+1}\phi^*(k, n-r-2) + \sum_{k=0}^{\infty}\sum_{i=k}^{n-r-3}p_{k+1}\mathbb{P}_k(T^* = i)\phi_r^*(0, n-i-1).\end{aligned} \tag{4.2.7}$$

Remark 19. *An explicit expression for* $\phi_r^*(0,n)$*, based on equation* (4.2.7)*, proves difficult to obtain. However, due to the form of equation* (4.2.7)*, a recursive calculation for* $\phi_r^*(0,n)$ *can be employed and is given by the following algorithm:*

Step 1. *Substituting* $n = r+2$*, in equation* (4.2.7)*, and using the fact that* $\phi^*(u,t) = 1$ *for* $t \leqslant u$*, we have that*

$$\begin{aligned}\phi_r^*(0, r+2) &= p_0\sum_{s=1}^{r}\sum_{z=0}^{\infty}\mathbb{P}_{-1}(\tau^- = s, U^*(\tau^-) = z) + 1 - p_0\\ &= 1 - p_0\left(1 - \sum_{z=0}^{\infty}\mathbb{P}_{-1}(\tau^- \leqslant r, U^*(\tau^-) = z)\right)\\ &= 1 - p_0\phi(1, r+1),\end{aligned}$$

where $\phi(1,t)$ *is the classic finite-time survival probability in the compound binomial risk model, with initial capital* $u = 1$*, which has been extensively studied in the literature, [see Li and Sendova (2013) and references therein] and alternatively can be evaluated using the results of Lemma 10, since*

$$\phi(1,t) = \sum_{n=t}^{\infty}\sum_{k=0}^{\infty}\mathbb{P}_{-1}\left(\tau^- = n, U^*(\tau^-) = k\right).$$

Step 2. *Based on the recursive nature of equation* (4.2.7) *and using the result of Step*

1, we can compute $\phi_r^*(0,n)$, for $n = r+3$, since

$$\begin{aligned}\phi_r^*(0,r+3) &= p_0 \sum_{s=1}^{r}\sum_{z=0}^{\infty}\mathbb{P}_{-1}(\tau^- = s, U^*(\tau^-) = z) + \sum_{k=1}^{\infty} p_{k+1} + p_1\phi_r^*(0,r+2)\\ &= 1-(1+p_1)p_0\phi(1,r+1).\end{aligned}$$

Step 3. *Using similar arguments as in the previous steps, for $n = r+4$, we have*

$$\begin{aligned}\phi_r^*(0,r+4) = p_0 &\left(\sum_{z=1}^{\infty}\mathbb{P}_{-1}(\tau^- = 1, U^*(\tau^-) = z) + \sum_{s=2}^{r}\sum_{z=0}^{\infty}\mathbb{P}_{-1}(\tau^- = s, U^*(\tau^-) = z)\right)\\ &+ p_0\mathbb{P}_{-1}(\tau^- = 1, U^*(\tau^-) = 0)\phi_r^*(0,r+2) + p_2\phi^*(1,2)\\ &\quad + \sum_{k=2}^{\infty} p_{k+1} + p_1\phi_r^*(0,r+3) + p_2\mathbb{P}_1(T^* = 1)\phi_r^*(0,r+2)\\ = p_0 &\left(\psi(1,r+1) - \mathbb{P}_{-1}(\tau^- = 1, U^*(\tau^-) = 0)\right)\\ &+ p_0\mathbb{P}_{-1}(\tau^- = 1, U^*(\tau^-) = 0)\phi_r^*(0,r+2) + p_2\left(1-p_0\right)\\ &\quad + 1 - (p_0+p_1+p_2) + p_1\phi_r^*(0,r+3) + p_2p_0\phi_r^*(0,r+2).\end{aligned}$$

Employing the results of steps 1 and 2 and using the fact that $\mathbb{P}_{-1}(\tau^- = 1, U^*(\tau^-) = 0) = p_2$, by Lemma 10, after some algebraic manipulations we obtain

$$\phi_r^*(0,r+4) = 1 - \left[1 + 2p_0p_2 + p_1(1+p_1)\right]p_0\phi(1,r+1).$$

Thus, based on the above steps, $\phi_r^*(0,r+k)$, for $k = 2,3,\ldots$, can be evaluated recursively for each value of k in terms of the mass functions, p_n, and the classic ruin quantity $\phi(1,r+1)$.

Theorem 27. *For $u \in \mathbb{N}$, the finite-time Parisian ruin probability $\psi_r^*(u,t) = 0$ for $t \leqslant u+r+1$ and for $t > u+r+1$, is given by*

$$\psi_r^*(u,t) = \sum_{k=u}^{t-r-2}\frac{u}{k}p_{k-u}^{*k}\psi_r^*(0,t-k), \tag{4.2.8}$$

where $\psi_r^*(0,n)$ can be found recursively from equation (4.2.7).

In the next section, we use the above expressions to derive results for the infinite-time Parisian ruin probabilities, for which, as will be seen, a more analytic expression can be found.

4.3 Infinite-time Parisian ruin probability

In this section, we derive an explicit expression for the infinite-time Parisian survival (ruin) probabilities using the arguments of the previous section. First, let us recall that the infinite-time Parisian survival probability is defined as $\phi_r^*(u) = \lim_{t\to\infty} \phi_r^*(u,t)$, with the infinite-time dual ruin quantities being defined in a similar way, i.e. $\phi^*(u) = \lim_{t\to\infty} \phi^*(u,t)$. Then, it follows, by taking the limit $t \to \infty$ with $t \in \mathbb{N}$, equation (4.2.1) reduces to

$$\phi_r^*(u) = \psi^*(u)\phi_r^*(0) + \phi^*(u), \tag{4.3.1}$$

where $\phi_r^*(0)$ is the infinite-time Parisian survival probability with initial capital $u = 0$, defined by $\phi_r^*(0) = \lim_{t\to\infty} \phi_r^*(0,t)$ and $\phi_r^*(0,t)$ satisfies equation (4.2.5). Therefore it follows, by taking the limit $n \to \infty$ in equation (4.2.5), that $\phi_r^*(0)$ satisfies

$$\phi_r^*(0) = p_0 \sum_{z=0}^{\infty} \mathbb{P}_{-1}(\tau^- \leqslant r, U^*(\tau^-) = z)\phi_r^*(z) + \sum_{j=0}^{\infty} p_{j+1}\phi_r^*(j),$$

or equivalently

$$\phi_r^*(0) = \sum_{k=0}^{\infty} a_k \phi_r^*(k), \tag{4.3.2}$$

where

$$a_k := \left(p_0\mathbb{P}(\tau^- \leqslant r, U^*(\tau^-) = k) + p_{k+1}\right), \tag{4.3.3}$$

can be obtained from the result of Lemma 10, since

$$\mathbb{P}(\tau^- \leqslant r, U^*(\tau^-) = k) = \sum_{s=1}^{r} \mathbb{P}(\tau^- = s, U^*(\tau^-) = k).$$

Considering the first term of the summation in the r.h.s. of equation (4.3.2) and solving with respect to $\phi_r^*(0)$, we get an explicit representation for $\phi_r^*(0)$, of the form

$$\phi_r^*(0) = C^{-1} \sum_{k=1}^{\infty} a_k \phi_r^*(k), \tag{4.3.4}$$

where

$$C = 1 - p_0 \mathbb{P}_{-1}(\tau^- \leqslant r, U^*(\tau^-) = 0) - p_1. \tag{4.3.5}$$

Finally, substituting the general form of the infinite-time Parisian survival probability, $\phi_r^*(u)$, given by (4.3.1), into equation (4.3.4) and solving with respect to $\phi_r^*(0)$, we obtain

$$\phi_r^*(0) = \frac{C^{-1} \sum_{k=1}^{\infty} a_k \phi^*(k)}{1 - C^{-1} \sum_{k=1}^{\infty} a_k \psi^*(k)}. \tag{4.3.6}$$

Note that, the above equation provides an explicit expression for the boundary condition, $\phi_r^*(0)$, unlike the finite-time case, which is given in terms of the infinite-time dual ruin probabilities $\phi^*(\cdot)$ and $\psi^*(\cdot)$. Combining equations (4.3.1) and (4.3.6) and after some algebraic manipulations, we obtain an explicit expression for the infinite-time Parisian survival probability, with general initial reserve $u \in \mathbb{N}$, which is given in the following theorem.

Theorem 28. *For $u \in \mathbb{N}$, the infinite-time Parisian ruin probability, $\psi_r^*(u)$, is given by*

$$\psi_r^*(u) = \psi^*(u) \left(\frac{C - \sum_{k=1}^{\infty} a_k}{C - \sum_{k=1}^{\infty} a_k \psi^*(k)} \right), \tag{4.3.7}$$

where the coefficients a_k, for $k \in \mathbb{N}^+$, are given by

$$a_k = p_0 \mathbb{P}(\tau^- \leqslant r, U^*(\tau^-) = k) + p_{k+1}, \tag{4.3.8}$$

and

$$C = 1 - p_0 \mathbb{P}_{-1}(\tau^- \leqslant r, U^*(\tau^-) = 0) - p_1.$$

4.4 Alternative methods for deriving the infinite-time dual ruin probability

In Lemma 9, we derived an expression for the p.m.f. of the dual ruin time, namely $\mathbb{P}_u(T^* = k)$, in terms of convolutions of the claim size distribution. This result, as discussed previously, can be used to obtain an expression for both the finite-time dual ruin probability and consequently, the infinite-time dual ruin probability, since

$$\begin{aligned}\psi^*(u) &= \sum_{k=0}^{\infty} \mathbb{P}_u(T^* = k) \\ &= \sum_{k=u}^{\infty} \mathbb{P}_u(T^* = k) = \sum_{k=u}^{\infty} \frac{u}{k} p_{k-u}^{*k}.\end{aligned}$$

Although the above expression is explicit, this representation does not give much insight into the behaviour of the dual ruin probability and thus a closed form expression would be more favourable.

In this section, we implement alternative methods for deriving an explicit expression for the infinite-time dual ruin probability.

Difference equation

The first method is based on the fact that the dual ruin probability, $\psi^*(u)$, satisfies a difference equation, for which a particular form of the solution is employed. In the following, we show that this solution is indeed an analytical solution for $\psi^*(u)$ and is unique.

Consider the dual risk process given in equation (4.1.1), with initial reserve $u + 1$, for $u \in \mathbb{N}$. Then, conditioning on the possible events in the first period of time and using the law of total probability, we obtain a difference equation for the infinite-time dual ruin probability, namely $\psi^*(\cdot)$, of the form

$$\begin{aligned}\psi^*(u+1) &= p_0\psi^*(u) + \sum_{j=1}^{\infty} p_j\psi^*(u+j) \\ &= \sum_{j=0}^{\infty} p_j\psi^*(u+j), \end{aligned} \tag{4.4.1}$$

with boundary conditions $\psi^*(0) = 1$ and $\lim_{u\to\infty}\psi^*(u) = 0$.

Equation (4.4.1) is in the form of an infinite-order difference (recursive) type equation. Thus, by adopting the general methodology for solving difference equations, we search for a solution of the form

$$\psi^*(u) = cA^u,$$

where c and A are constants to be determined. Using the boundary conditions $\psi^*(0) = 1$ and $\lim_{u\to\infty}\psi^*(u) = 0$, it follows that the constant $c = 1$ and $0 \leqslant A < 1$. That is, the general solution to the recursive equation (4.4.1) is of the form

$$\psi^*(u) = A^u, \tag{4.4.2}$$

for some $0 \leqslant A < 1$, which, after substitution into equation (4.4.1), yields

$$A^{u+1} = \sum_{j=0}^{\infty} p_j A^{u+j}, \qquad u \in \mathbb{N}. \tag{4.4.3}$$

Dividing the above equation through by A^u and defining the probability generating function (p.g.f.) of Y_1 by $P_Y(z) = \sum_{i=0}^{\infty} p_i z^i$, yields

$$A = P_Y(A), \qquad 0 \leqslant A < 1. \tag{4.4.4}$$

That is, $0 \leqslant A < 1$ is a solution (if it exists) to the discrete-time dual analogue of Lundberg's fundamental equation, given by

$$\theta(z) = 0, \tag{4.4.5}$$

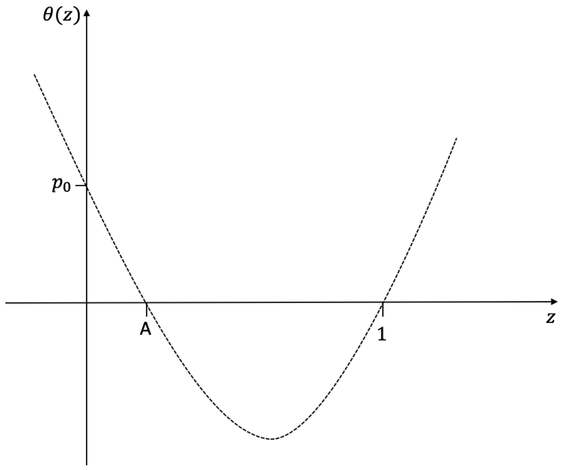


Figure 4.3: Graph of the function $\theta(z) := P_Y(z) - z$.

where $\theta(z) := P_Y(z) - z$.

Proposition 16. *In the interval* $[0, 1)$*, there exists a unique solution to the equation* $P_Y(z) - z = 0$.

Proof. It follows from the properties of a p.g.f. that

$$
\begin{aligned}
\theta(0) &= p_0 \geqslant 0, \\
\theta'(0) &= p_1 - 1 \leqslant 0, \\
\theta(1) &= 0, \\
\theta'(1) &= \mathbb{E}(Y_1) - 1 > 0, \\
\theta''(z) &> 0, \quad \text{for all} \quad z \in [0, 1).
\end{aligned}
$$

From the above conditions, which show the characteristics of the function $\theta(z) := P_Y(z) - z$ (see Fig: 4.3), it follows that there exists a root of $\theta(z) = 0$ at $z = 1$ and a second solution $z = A$, which is unique in the interval $[0, 1)$.

□

Finally, from equations (4.4.2), (4.4.4) and Proposition 16, we obtain a closed form expression for the infinite-time dual ruin probability, given by the following lemma.

Lemma 11. *The infinite-time dual probability of ruin, namely* $\psi^*(u)$ *for* $u \in \mathbb{N}$*, is* *given by*

$$\psi^*(u) = A^u, \tag{4.4.6}$$

where A is the unique solution in the interval $[0, 1)$ to the Lundberg equation $P_Y(z)-z = 0$, with $P_Y(z)$ the p.g.f. of Y_1.

Remark 20. *We note that the p.g.f., $P_Y(z)$, converges for all $|z| \leqslant 1$ and thus, in the interval $z \in [0, 1]$ the p.g.f. and thus A, exists (finite) for all probability distributions, i.e. light and heavy-tailed. Therefore, it follows that Theorem 11 holds for a general gain size distribution.*

Exponential martingale and random walks

The second method is to consider the hitting probability of the increment process (random walk), $R^*(n) = n - \sum_{i=1}^{n} Y_i$, using exponential martingales and Optional Stopping Theorem. That is, we want to find an $s > 0$, such that the sequence $\{e^{sR^*(n)}\}_{n\in\mathbb{N}}$ forms a discrete-time martingale.

Definition 24 (Discrete-time martingale)**.** *A discrete-time stochastic process $X = \{X_n\}_{n\in\mathbb{N}}$ is called a martingale (with respect to some filtration $\mathcal{F} = \{\mathcal{F}_n\}_{n\in\mathbb{N}}$) if for each $n \in \mathbb{N}$, we have*

1. *X is adapted to $\mathcal{F}$; that is, X_n is $\mathcal{F}_n$-measurable,*

2. *X_n is in L^1; that is, $\mathbb{E}(|X_n|) < \infty$, and*

3. *$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ a.s.*

From Definition 24, it follows that we require an $s > 0$, such that

$$\begin{aligned} 1 = \mathbb{E}\left(e^{sR^*(0)}\right) &= \mathbb{E}\left(e^{sR^*(1)}\right) \\ &= e^s \mathbb{E}\left(e^{-sY_1}\right) = e^s M_{Y_1}(-s), \end{aligned}$$

or equivalently

$$e^{-s} = M_{Y_1}(-s).$$

Setting $z = e^{-s}$, in the above equation, yields $z = P_Y(z)$, as in the previous derivation and thus, it follows that the solution to the above Lundberg equation, namely the adjustment coefficient, denoted by $\gamma > 0$, is given by $\gamma = \ln(A^{-1}) > 0$.

Now that we have found an exponential martingale (with respect to the natural filtration), given by $\{e^{\gamma R^*(n)}\}_{n\in\mathbb{N}}$, we want to define the hitting time for the random walk, $\{R^*(n)\}_{n\in\mathbb{N}}$, of the level $u \in \mathbb{N}$, denoted T^u as in Proposition 15, i.e. $R^*(T^u) = u$, and is defined by

$$T^u = \inf\{n \in \mathbb{N} : R^*(n) = u\} = \inf\{n \in \mathbb{N} : U^*(n) = 0\} = T^*. \tag{4.4.7}$$

Finally, if the hitting time T^u, defined above, is a stopping time with respect to the natural filtration, we can employ Optional Stopping Theorem from which, as will be seen, the result follows.

Definition 25 (Stopping time)**.** *A random variable τ, with support on the non-negative integers, is called a stopping time if*

$$\{\tau \leqslant n\} \in \mathcal{F}_n, \quad \textit{for every} \quad n \in \mathbb{N}.$$

Theorem 29 (Optional stopping)**.** *Let $X = \{X_n\}_{n\in\mathbb{N}}$ be a discrete-time martingale and τ a stopping time with values in the non-negative integers, both with respect to a filtration $\mathcal{F}$. Further, assume that there exists a constant c such that $|X_{\tau\wedge n}| \leqslant c$ a.s. for all $n \in \mathbb{N}$, where $\tau \wedge n$ denotes the minimum of τ and n. Then, X_τ is an almost surely well defined random variable and*

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0].$$

It is clear to see, by definition, that the hitting time T^u is indeed a stopping time, with respect to the natural filtration and, since

$$\left|e^{\gamma R^*(T^u\wedge n)}\right| \leqslant e^{\gamma u}, \quad \text{for all} \quad n \in \mathbb{N},$$

by the Optional Stopping Theorem, we have

$$\begin{aligned}1 = \mathbb{E}\left(e^{\gamma R^*(0)}\right) &= \mathbb{E}\left(e^{\gamma R^*(T^u)}\right) \\ &= e^{\gamma u}\mathbb{P}\left(T^u < \infty\right),\end{aligned}$$

from which, by recalling that the hitting time T^u for the increment process $\{R^*(n)\}_{n\in\mathbb{N}}$ of the level $u \in \mathbb{N}$, is equivalent to the time of ruin in the dual risk model, we have the following lemma.

Lemma 12. *The infinite-time dual probability of ruin, namely $\psi^*(u)$ for $u \in \mathbb{N}$, is given by*

$$\psi^*(u) = e^{-\gamma u}, \tag{4.4.8}$$

where $\gamma > 0$ is the non-trivial solution of the Lundberg equation $\mathbb{E}\left(e^{s(1-Y_1)}\right) = 1$.

Exponential change of measure

For the final method, we derive a similar result as above, using an exponential change of measure which provides the discrete analogue of Theorem 2.1, Chapter VI, of Asmussen and Albrecher (2010).

Definition 26 (Exponential change of measure)**.** *Let X be a discrete random variable with probability measure $\mathbb{P}$ and cumulant generating function (c.g.f.), denoted by $\kappa(\alpha)$, given by*

$$\kappa(\alpha) = \ln\mathbb{E}\left(e^{\alpha X}\right) = \ln\left(\sum_{k=-\infty}^{\infty} e^{\alpha k}\mathbb{P}(X = k)\right) = \ln\left(M_X(\alpha)\right).$$

Then, a change of measure via the exponential family, denoted $\mathbb{P}_\theta$ is given by

$$\mathbb{P}_\theta = e^{\theta x - \kappa(\theta)}\mathbb{P},$$

or equivalently, in terms of the c.g.f. of $\mathbb{P}_\theta$ denoted $\kappa_\theta(\alpha)$, by

$$\kappa_\theta(\alpha) = \kappa(\alpha + \theta) - \kappa(\theta),$$

where θ is any number such that $\kappa(\theta)$ is well defined.

Consider the dual risk process, defined in equation (4.2.3), i.e. $U^*(n) = u - R^*(n)$, where $R^*(n) = n - \sum_{i=1}^{n} Y_i$, and denote the c.g.f. of $R^*(1)$, by $\kappa^*(\alpha)$, with $\alpha \geqslant 0$, where

$$\kappa^*(\alpha) = \ln \mathbb{E}\left(e^{\alpha R^*(1)}\right) = \alpha + \ln \mathbb{E}\left(e^{-\alpha Y_1}\right). \tag{4.4.9}$$

Now, if $\mathbb{E}(Y_1) < 1$, then by the strong law of large numbers we have that the incremental process, $R^*(n) \to +\infty$ and it follows that $\psi^*(u) = \mathbb{P}(T^u < \infty) = 1$, for all $u \in \mathbb{N}$. Moreover, if $\mathbb{E}(Y_1) = 1$, such that the random walk $\{R^*(n)\}_{n\in\mathbb{N}}$ has mean zero, i.e. $\mathbb{E}(R^*(n)) = 0$, for all $n \in \mathbb{N}$, then the process $\{R^*(n)\}_{n\in\mathbb{N}}$ will visit every accessible integer a.s. and thus, it follows that if $p_0 > 0$, we have $\psi^*(u) = 1$, for all $u \in \mathbb{N}$. Assume now that $\mathbb{E}(Y_1) > 1$. Then, the c.g.f., $\kappa^*(\alpha)$, has the following characteristics (for details see Lemmas 4.5.2 and 4.5.2 of Appendix):

$$\begin{aligned}
\kappa^*(0) &= 0, \\
\kappa^{*\prime}(0) &= 1 - \mathbb{E}(Y_1) < 0, \\
\kappa^{*\prime\prime}(\alpha) &> 0, \quad \text{for all} \quad \alpha > 0, \\
\kappa^*(\alpha) &\to \infty, \quad \text{as} \quad \alpha \to \infty \quad \text{iff} \quad p_0 > 0,
\end{aligned}$$

which implies a typical shape of Fig: 4.4a. Hence, there exists a $\gamma > 0$, known as the adjustment coefficient, such that $\kappa^*(\gamma) = 0$, which is unique. Note that, the adjustment coefficient here is equivalent to that of the previous method, hence $\gamma = \ln(A^{-1}) > 0$.

Recall that the probability measure governing the claim sizes $\{Y_i\}_{i\in\mathbb{N}}$, and thus the entire increment process $\{R^*(n)\}_{n\in\mathbb{N}}$, is denoted by $\mathbb{P}$ and consider an exponential change of measure, as defined in Definition 26, $\mathbb{P}_\theta$ with corresponding expectation operator $\mathbb{E}_\theta$, such that

$$\mathbb{P}_\theta(R^*(1) = k) = e^{\theta k - \kappa^*(\theta)}\mathbb{P}(R^*(1) = k).$$

Then, it follows that the c.g.f. of $R^*(1)$ under $\mathbb{P}_\theta$, denoted $\kappa_\theta(\alpha)$, is given by $\kappa_\theta(\alpha) = \kappa^*(\alpha + \theta) - \kappa^*(\theta)$ and we have the following theorem [see Asmussen and Albrecher

(2010)].

Theorem 30. *Let τ be a stopping time and let $G \in \mathcal{F}_\tau$, $G \subseteq \{\tau < \infty\}$. Then,*

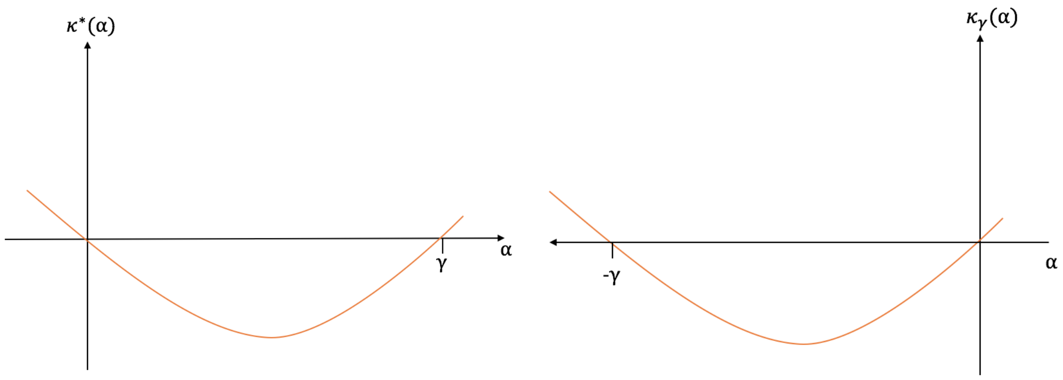
$$\mathbb{P}(G) = \mathbb{E}_\theta \left(e^{-\theta R^*(\tau) + \tau\kappa^*(\theta)} \mathbb{I}_{\{G\}} \right).$$

If we let $\theta = \gamma > 0$, be the adjustment coefficient, then by definition, the c.g.f. $k_\gamma(\alpha) = \kappa^*(\alpha + \gamma)$ and has typical shape given in Fig: 4.4b. From Fig: 4.4b, it can be easily seen $\kappa'_\gamma(0) = \mathbb{E}_\gamma(R^*(1)) > 0$ and it follows that, under the probability measure $\mathbb{P}_\gamma$, we have $R^*(n) \to +\infty$ and hence, the hitting time T^u is finite a.s., i.e. $\mathbb{P}_\gamma(T^u < \infty) = 1$, for all $u \in \mathbb{N}$.

Finally, by noting that T^u is indeed a stopping time with respect to the natural filtration [see Definition 25], we can employ Theorem 30 with the event $G = \{T^u < \infty\}$, which yields

$$\begin{aligned}
\psi^*(u) = \mathbb{P}(T^u < \infty) &= \mathbb{E}_\gamma \left(e^{-\gamma R^*(T^u)} \mathbb{I}_{\{T^u < \infty\}} \right) \\
&= e^{-\gamma u} \mathbb{P}_\gamma(T^u < \infty) \\
&= e^{-\gamma u},
\end{aligned}$$

and we have the following lemma.



(a) Typical shape of c.g.f. $\kappa^*(\alpha)$. (b) Typical shape of c.g.f. $\tilde{\kappa}(\alpha)$.

Figure 4.4: Relationship of c.g.f.'s under original measure and exponential change of measure with parameter $\gamma > 0$.

Lemma 13. *For $\mathbb{E}(Y_1) \leqslant 1$, we have $\psi^*(u) = 1$ for all $u \in \mathbb{N}$. For $\mathbb{E}(Y) > 1$, we have*

$$\psi^*(u) = e^{-\gamma u}, \tag{4.4.10}$$

where $\gamma > 0$ is the unique solution of

$$\kappa^*(\alpha) = \log \mathbb{E}\left(e^{\alpha R^*(1)}\right) = \alpha + \log \mathbb{E}\left(e^{-\alpha Y_1}\right) = 0. \tag{4.4.11}$$

Remark 21. *From Lemmas 11, 12 and 13 we see that the infinite-time dual ruin probability, $\psi^*(u)$, decays exponentially fast for any gain size distribution. Hence, we cannot expect to observe the classical heavy-tailed asymptotic behaviour, by which we mean a power law decay, of the dual ruin probability or, by the result of Theorem 28, the Parisian ruin probability, $\psi_r^*(u)$.*

4.5 Examples

4.5.1 Binomial/Geometric model

In this section, we consider the Binomial/Geometric model as studied in Dickson (1994), among others, and we derive an exact expression for the infinite-time dual probability of ruin, namely $\psi^*(u)$. Consequently, from Theorem 28 we can obtain an expression for the corresponding infinite-time Parisian ruin probability, $\psi_r^*(u)$.

We recall that in the Binomial/Geometric model, it is assumed that the gain size random variables, $\{Y_i\}_{i\in\mathbb{N}^+}$, have the form $Y_i = I_i \cdot X_i$, where I_i, for $i \in \mathbb{N}^+$, are i.i.d. random variables following a Bernoulli distribution with parameter $q \in [0,1]$, i.e. $\mathbb{P}(I_1 = 1) = 1 - \mathbb{P}(I_1 = 0) = q$ and the sequence of random gain amounts, $\{X_k\}_{k\in\mathbb{N}^+}$, are i.i.d. random variables following a geometric distribution f_k, with parameter $(1-\alpha) \in [0,1]$, i.e. $\mathbb{P}(Y_1 = 0) = p_0 = 1 - q$ and $\mathbb{P}(Y_1 = k) = p_k = q f_k = q(1-\alpha)\alpha^{k-1}$ for $k \in \mathbb{N}^+$.

Proposition 17. *For $u \in \mathbb{N}$, the infinite-time dual ruin probability, $\psi^*(u)$, in the Binomial/Geometric model, with parameters $q \in [0,1]$ and $(1-\alpha) \in [0,1]$ such that $q + \alpha > 1$, is given by*

$$\psi^*(u) = \left(\frac{1-q}{\alpha}\right)^u. \tag{4.5.1}$$

Proof. From Lemma 11, the infinite-time dual ruin probability, $\psi^*(u)$, has the form $\psi^*(u) = A^u$, where $0 \leqslant A < 1$ is the solution to $\theta(z) := P_Y(z) - z = 0$, where

$$P_Y(z) = 1 - q + qP_X(z), \tag{4.5.2}$$

and $P_X(z)$ is the p.g.f. of a geometric distribution, which takes the form

$$P_X(z) = \frac{(1-\alpha)z}{1-\alpha z}, \qquad |z| \leqslant 1. \tag{4.5.3}$$

Combining equation (4.5.2) and (4.5.3) and after some algebraic manipulations, Lundberg's fundamental equation, $\theta(z) = 0$, yields a quadratic equation of the form

$$z^2 + k_1 z + k_2 = 0,$$

where

$$\begin{aligned} k_1 &= \frac{q-1}{\alpha} - 1, \\ k_2 &= \frac{1-q}{\alpha}. \end{aligned}$$

The above quadratic equation has two roots $z_1 = (1-q)/\alpha$ and $z_2 = 1$. Finally, from the positive drift assumption in the the model set up, we have that $\mathbb{E}(Y_1) = q/(1-\alpha) > 1$, from which it follows that $q + \alpha > 1$ and the root $z_1 \in [0, 1)$. Thus, we have $A = z_1$, since this root is unique in the interval $[0, 1)$ (see Proposition 16). $\square$

In order to illustrate the behaviour of the dual ruin probability under geometric claim sizes, given in Proposition 17 and the corresponding Parisian ruin probability of Theorem 28, we consider the set of parameters, $q = 0.3$, $\alpha = 0.9$. Then, the dual ruin probability and the Parisian ruin probabilities, for $r = 1, 2, 3, 4$, are given in the following plot.

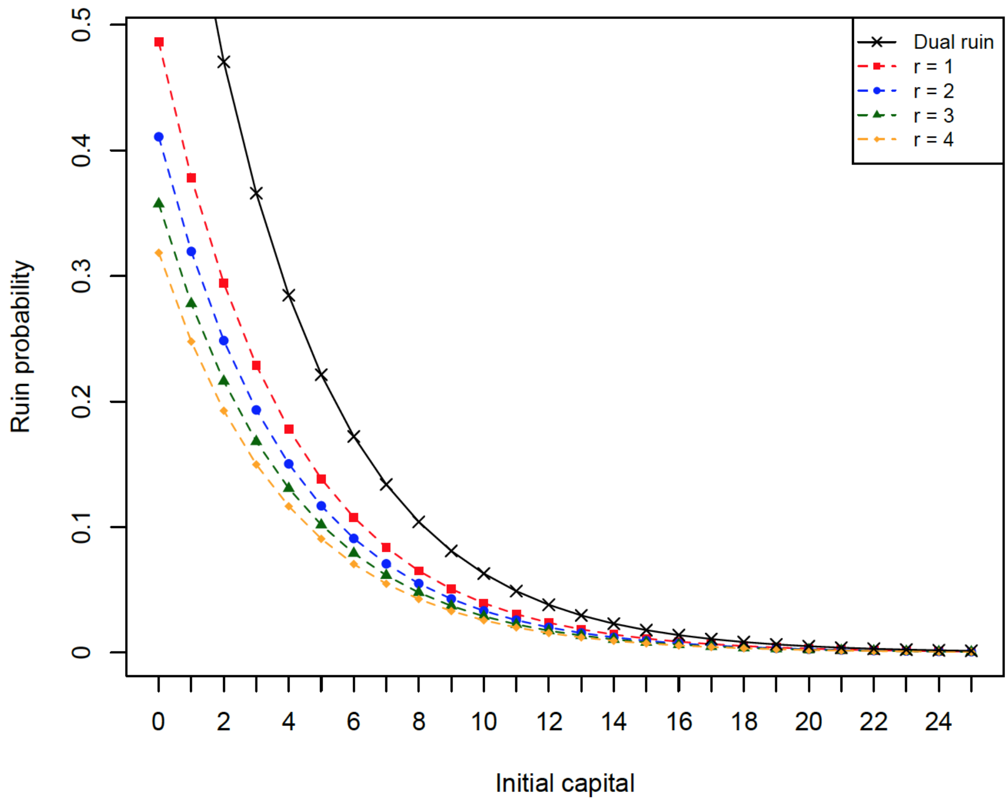


Figure 4.5: Plot of dual ruin and Parisian ruin probabilities under geometric claim size distribution, for different values of r.

4.5.2 Parisian ruin for the gambler's ruin problem

Finally, in this section, we derive an exact expression for the infinite-time Parisian ruin probability for one of the most fundamental ruin problems, namely the gambler's ruin problem. In this model a player makes a bet on the outcome of a random game, with a chance to double their bet with probability $q \in [0, 1]$. Ruin in this model is defined as being the event that the player runs out of money (bankrupt) at some point in the future [see Feller (1968)].

Mathematically, the gambler's ruin model can be described by the discrete-time compound binomial process or equivalently by the discrete dual risk model, considered in the previous sections, with a loss probability $p_0 = 1 - q$ and corresponding win probability $p_2 = q$, with $p_k = 0$ otherwise. Further, in order to satisfy the net profit condition, and consequently avoid definite ruin over an infinite-time horizon, it follows that $q > 1/2$. Under these assumptions Lundberg's fundamental equation, namely

$\theta(z) = 0$, produces a quadratic equation of the form

$$z^2 - \frac{1}{q}z + \frac{1-q}{q} = 0,$$

which has solutions $z_1 = 1$ and $z_2 = \frac{1-q}{q}$. From the net profit condition, i.e. $q > 1/2$, it follows that $z_2 = \frac{1-q}{q} < 1$. Thus, from Lemma 11, we have that $A = \frac{1-q}{q}$ and the classic gambler's ruin probability is given by

$$\psi^*(u) = \left(\frac{1-q}{q}\right)^u, \tag{4.5.4}$$

as seen in Feller (1968).

Now, let us assume that the gambler is allowed to continue playing after going bankrupt (he borrows money from another player, or friend, which is not subject to interest), but is declared ultimately ruined if he does not recover to a positive surplus in $r \in \mathbb{N}^+$ bets, or less, after bankruptcy. Then, since the definition of Parisian ruin is defined by the number of periods with a strictly negative surplus (where as for the gambler, ultimate ruin is defined by the number of periods with non-positive surplus), the probability that the gambler experiences ultimate ruin, from initial capital $u \in \mathbb{N}^+$, is equivalent to the infinite-time Parisian ruin probability of Theorem 28, with initial capital $(u-1) \in \mathbb{N}$ and is given explicitly by the following Proposition.

Proposition 18. *The infinite-time Parisian ruin probability to the Gambler's ruin problem for initial capital $u \in \mathbb{N}^+$, with win probability $q > 1/2$, is given by*

$$\psi_r^*(u-1) = \frac{1 - qC_1}{1-(1-q)C_1}\left(\frac{1-q}{q}\right)^u, \tag{4.5.5}$$

where

$$C_1 = \sum_{n=1}^{r} p_{n-1}^{*(n-1)} - \sum_{n=1}^{r}\sum_{i=2}^{n-1} \frac{1}{n-i} p_{n-1-i}^{*(n-i)} p_i^{*(i-1)}. \tag{4.5.6}$$

Proof. Using the result of Theorem 28, and the form of the classic gambler's ruin problem given by equation (4.5.4), it remains to find explicit expressions for the coefficients a_k, for $k = 1, \ldots, \infty$ and the constant C^{-1}. Let us first consider the coefficients a_k, given by equation (4.3.8), of the form

$$a_k = \left(p_0 \mathbb{P}_{-1}(\tau^- \leqslant r, U^*(\tau^-) = k) + p_{k+1}\right).$$

Recalling that in the gambler's ruin problem the p.m.f's of the random gain sizes, $p_k = 0$ for $k \neq 0, 2$, it follows that only positive jumps of size $Y_i = 2$, for $i \in \mathbb{N}^+$, can occur (with probability q) and thus, the joint distribution of recovery and the overshoot at the time of recovery, namely $\mathbb{P}_{-1}(\tau^- \leqslant r, U^*(\tau^-) = k) = 0$, for all $k \neq 0$. Then, it follows that $a_k = p_{k+1}$, for $k = 1, \ldots, \infty$, which yields

$$a_k = \begin{cases} q, & k = 1, \\ 0 & otherwise. \end{cases}$$

Substituting this into the result of Theorem 28 and after some algebraic manipulations, we obtain

$$\psi_r^*(u-1) = \frac{C-q}{C-(1-q)} \left(\frac{1-q}{q}\right)^{u-1},$$

where $C = 1 - (1-q)\mathbb{P}_{-1}(\tau^- \leqslant r, U^*(\tau^-) = 0)$. Finally, by setting $k = 0$ in equation (4.2.6) and noticing that, since $p_k = 0$, for $k = 3, 4, \ldots$, only the term $j = n-1$ remains in both summation terms, we obtain

$$\begin{aligned} \mathbb{P}_{-1}(\tau^- \leqslant r, U^*(\tau^-) = 0) &= \sum_{n=1}^{r} \mathbb{P}_{-1}(\tau^- = n, U^*(\tau^-) = 0) \\ &= q\left(\sum_{n=1}^{r} p_{n-1}^{*(n-1)} - \sum_{n=1}^{r}\sum_{i=2}^{n-1} \frac{1}{n-i} p_{n-1-i}^{*(n-i)} p_i^{*(i-1)}\right), \end{aligned}$$

and it follows that $C = 1 - q(1-q)C_1$, where C_1 is given by equation (4.5.6). Finally, the result follows after some algebraic manipulations. $\square$

Summary

The classical event of ruin is defined as the moment an insurer's surplus process, defined as the difference between incoming premium and losses due to claims, drops below zero or, equivalently, the surplus becomes negative. This notion of ruin is unrealistic in practice, since insurance companies (and other financial businesses) are subject to capital requirement legislation (Solvency II for insurance companies within the EU) which requires the classic risk model to be updated.

In order to provide adequate protection for the policy holders, the Solvency II framework imposes a so-called 'ladder of supervisory intervention'. This supervisory ladder provides an early indication of any deterioration, in terms of capital, to the regulators. If the company falls below a fixed level, known as the Solvency Capital Requirement, the company is not considered as ruined, as in the classical model, and is allowed to continue trading under the agreement that they develop a capital recovery plan. If the company continues to deteriorate, following the implementation of the recovery plan, and falls below the Minimum Capital Requirement, the company is defined as insolvent and the regulators strongest actions are enforced, resulting in potential withdrawal of the companies trading license.

In this thesis, we considered three separate risk models analysing the potential capital recovery plans of insurance firms (and other lines of business) and considered their effect on the ruin probability and other risk related quantities. Within each model, it is assumed that the company is allowed to continue below the SCR, however, during this period, the company is required to recover their capital requirements subject to

different regulatory constraints.

In more details, in Chapter 2, we introduced two constant barriers to model the capital requirement thresholds under the Solvency II framework (SCR and MCR), where the companies recovery method is to request a capital injection from the shareholders or, if the shareholders are not willing to inject due to a lack of confidence in the company, take out a loan from a lender subject to some debit interest. Under this setting, we derived an explicit expression for the probability of insolvency (dropping below the MCR) in terms of the ruin quantities of the classic Cramér-Lundberg risk model. By doing so, under the assumption of exponentially distributed claim sizes, we showed that depending on the model parameters, the capital requirements under SII may in fact provide less protection to the policyholders and increase the probability of insolvency/ruin compared to the classical case. In addition, we obtained an explicit expression for the moment generating function of the expected accumulated capital injections up to the time of insolvency, from which we determined is a mixture of a degenerate distribution at zero and a continuous distribution. In the last section of this chapter, we incorporated the dividend payments to the shareholders, by means of a constant barrier strategy, and proved that the probability of insolvency under this modification can also be given in terms of classic ruin quantities.

In Chapter 3, we reverted back to a classic risk model (without Solvency II constraints) and analysed the ultimate ruin probability for a risk model with capital injections. We point out that the results obtained in this section can easily be applied to the SII model of the previous chapter. In this model, it is assumed that the capital injections are no longer received instantaneously, but received after some time delay from the moment of a deficit, which depends on the size of the deficit and corresponding capital injection. Under this setting, we showed that the ultimate ruin probability, defined in a slightly different way to the classical sense, satisfies an inhomogeneous Fredholm integral equation of the second kind, which under certain dependency structures can be solved explicitly, in terms of classic ruin quantities, or given by a Neumann series when a more general dependence is assumed. Moreover, we considered two risk related quantities, namely the expected discounted accumulated capital injections and the expected discounted accumulated time in red (deficit) up to the time of ultimate ruin, which were also shown to satisfy a similar Fredholm integral equation and were solved explicitly.

Finally, in Chapter 4, we analysed the so-called dual risk model in discrete-time. In this model, which better captures the risk portfolio of different business lines, such as pharmaceutical or petroleum businesses, we considered that the fall into deficit could be recovered by means of normal trading strategies. That is, the recovery plan is simply to allow the business to continue as usual. This could be the case if the fall into deficit (below the SCR) is due to a 'one off' large claim. This recovery plan is preferable for the companies shareholders as it doesn't require any capital injections, however, they would only allow the company a fixed amount of time to recover their surplus to a non-negative level before their confidence is lost. This event of ultimate ruin is known in the literature as Parisian ruin. Using the strong Markov property of the risk process, we derived a recursive expression for the finite-time Parisian ruin probability, in terms of classic dual ruin quantities, which was then used to obtain an explicit expression for the corresponding infinite-time case. Moreover, since the known result for the classical dual ruin probability (in discrete-time) is given in terms of convolutions of the gain size probability function, we derived three equivalent exponential expressions for this quantity, providing a more analytic interpretation. Using these results, we obtained an explicit expression for a generalisation to the famous gambler's ruin problem.

The Cramér-Lundberg risk model has received a lot of criticism over the years for its simplicity and model assumptions. However, this 'simplicity' has allowed for a vast library of results, in terms of the ruin probability and other risk related quantities, and provides a foundation to understanding the key risks associated with the insurance sector. All of the models discussed in this thesis generalise this classic model, by incorporating market legislation and more realistic trading strategies, however, it is shown that all the results related to these more realistic models are given in terms of the classical risk quantities. Therefore, although the Cramér-Lundberg risk model is thought to be theoretical and somewhat unrealistic, it is fundamental to understanding the risk profile of the global insurance markets.

Appendix

Definition A1 (Fredholm integral equation)**.** *An equation of the form*

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\phi(t)\, dt,$$

is known as a Fredholm integral equation of the second kind, where

- *The unknown function, $\phi(x)$, is assumed to be integrable in the sense of Riemann so that the integral equation itself makes sense,*
- *The free term, $f(x)$, is assumed to be complex-valued and continuous on the interval $[a,b]$,*
- *The complex constant $\lambda(\neq 0)$ is a parameter that should not be absorbed into the kernel $K(x,t)$,*
- *The kernel $K(x,t)$ is assumed to be complex-valued and continuous on the square $Q(a,b) = \{(x,t) : a \leqslant x \leqslant b, a \leqslant t \leqslant b\}$.*

Moreover, a kernel is called separable or degenerate, if it assumes the specific form

$$K(x,t) = \sum_{i=1}^{n} a_i(x)b_i(t),$$

where the functions $a_i(x)$ and $b_i(t)$ are complex valued and continuous on the interval $[a,b]$.

Lemma A1. *The c.g.f. $\kappa^*(\alpha)$ is a convex function for all $\alpha \geqslant 0$.*

Proof. The second derivative of the c.g.f. $\kappa^*(\alpha)$, as defined in equation (4.4.11), is given by

$$\kappa''(\alpha) = \frac{\mathbb{E}\left(Y_1^2 e^{-\alpha Y_1}\right)\mathbb{E}\left(e^{-\alpha Y_1}\right) - \mathbb{E}^2\left(Y_1 e^{-\alpha Y_1}\right)}{\mathbb{E}^2\left(e^{-\alpha Y_1}\right)},$$

which is positive if and only if

$$\mathbb{E}\left(Y_1^2 e^{-\alpha Y_1}\right)\mathbb{E}\left(e^{-\alpha Y_1}\right) - \mathbb{E}^2\left(Y_1 e^{-\alpha Y_1}\right) > 0.$$

Recalling that the p.m.f. of Y_1 is given by the sequence $\{p_k\}_{k\in\mathbb{N}}$, then the l.h.s. of the above equation can be re-written as

$$\left(\sum_{k=0}^{\infty} k^2 e^{-\alpha k} p_k\right)\left(\sum_{n=0}^{\infty} e^{-\alpha n} p_n\right) - \left(\sum_{k=0}^{\infty} k e^{-\alpha k} p_k\right)^2,$$

which, by Cauchy product rule [see Riley et al. (2006)], is equivalent to

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k} j^2 e^{-\alpha k} p_j p_{k-j} - \sum_{k=0}^{\infty}\sum_{j=0}^{k} j(k-j) e^{-\alpha k} p_j p_{k-j},$$

or alternatively, by combining both terms

$$\sum_{k=0}^{\infty} e^{-\alpha k} \sum_{j=0}^{k} \left(j^2 - j(k-j)\right) p_j p_{k-j}.$$

At this point we note that within the inner sum, the product $p_j p_{k-j}$ will result in repeating terms (due to commutativity of p_k) and thus the above equation can be re-written as

$$\sum_{k=0}^{\infty} e^{-\alpha k} \sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor} \left(j^2 - j(k-j) + (k-j)^2 - (k-j)j\right) p_j p_{k-j} = \sum_{k=0}^{\infty} e^{-\alpha k} \sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor} (2j-k)^2\, p_j p_{k-j},$$

which is clearly positive for all $\alpha \geqslant 0$. The result follows from the definition of convexity. □

Lemma A2. *The c.g.f. $\kappa^*(\alpha) \to \infty$ as $\alpha \to \infty$ if and only if $\mathbb{P}(Y_1 = 0) = p_0 > 0$.*

Proof. We note that if the exponential of a function $f(x)$, i.e. $\exp\{f(x)\}$, tends to infinity as $x \to \infty$, then it follows necessarily that the function $f(x) \to \infty$ as $x \to \infty$. Therefore, it suffices to prove that the exponential of the c.g.f., i.e. $\exp(\kappa^*(\alpha)) \to \infty$ as $\alpha \to \infty$. The exponential of the c.g.f. $\kappa^*(\alpha)$, as defined in equation (4.4.11), has the form

$$\begin{aligned} e^{\kappa^*(\alpha)} = \mathbb{E}\left(e^{\alpha R^*(1)}\right) &= \sum_{k=0}^{\infty} e^{\alpha(1-k)} p_k \\ &= p_0 e^{\alpha} + p_1 + \sum_{k=1}^{\infty} e^{-\alpha k} p_{k+1}. \end{aligned}$$

Thus, it follows that $e^{\kappa^*(\alpha)} \to \infty$ as $\alpha \to \infty$ provided $p_0 > 0$. □

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