Lorentzian Kac-Moody algebras with Weyl groups of 2-reflections

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Abstract

We describe a new large class of Lorentzian Kac–Moody algebras. For all ranks, we classify 2-reflective hyperbolic lattices S with the group of 2-reflections of finite volume and with a lattice Weyl vector. They define the corresponding hyperbolic Kac–Moody algebras of restricted arithmetic type which are graded by S. For most of them, we construct Lorentzian Kac–Moody algebras which give their automorphic corrections: they are graded by the S, have the same simple real roots, but their denominator identities are given by automorphic forms with 2-reflective divisors. We give exact constructions of these automorphic forms as Borcherds products and, in some cases, as additive Jacobi liftings.

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1 Introduction

One of the most known examples of Lorentzian Kac–Moody algebras is the Fake Monster Lie algebra defined by R. Borcherds (see [B3]–[B4]) in his solution of Moonshine Conjecture. Lorentzian Kac–Moody (Lie, super) algebras are automorphic corrections of hyperbolic Kac–Moody algebras. In our papers [N7], [N8], [N12], [GN1]–[GN8], we developed a general theory of Lorentzian Kac–Moody algebras (see [GN5] and [GN6] for the most complete exposition) based of the results by Kac [K1]–[K3] and Borcherds [B1]–[B4]. In these our papers (especially see [GN5] and [GN6]), we constructed and classified some of these algebras for the rank 3.

In this paper, we consruct and classify some of Lorentzian Kac–Moody algebras for all ranks ≥ 3. In our papers above and here, we mainly consider and classify Lorentzian Kac–Moody algebras with Weyl groups W of 2-reflections. They are groups generated by reflections in elements with square 2 of hyperbolic (that is of signature $(n,1)$) lattices (that is integral symmetric bilinear forms) S of $\operatorname{rk} S = n+1$.

For an (automorphic) Lorentzian Kac–Moody Lie algebra with a hyperbolic *root* lattice S, the Weyl group W must have the fundamental chamber $\mathcal{M}$ of finite (*elliptic case*) or almost finite (*parabolic case*) volume in the hyperbolic (or Lobachevsky) space $\mathcal{L}(S) = V^+(S)/\mathbb{R}_{++}$ where $V^+(S)$ is a half of the cone $V(S) \subset S \otimes \mathbb{R}$ of elements $x \in S \otimes \mathbb{R}$ with $x^2 < 0$. For parabolic case, there exists a point $c = \mathbb{R}_{++} r \in \mathcal{M}$, $r \in S$, $r \neq 0$ and $r^2 = 0$, at infinity of $\mathcal{M}$ such that $\mathcal{M}$ is finite at any cone in $\mathcal{L}(S)$ with the vertex at c.

We denote by $P = P(\mathcal{M}) \subset S$ the set of *simple real roots* or all elements of S with square 2 which are perpendicular to faces of codimension one of $\mathcal{M}$ and directed outwards. For a Lorentzian Kac–Moody algebra, $P = P(\mathcal{M})$ must have *the lattice Weyl vector* $\rho \in S \otimes \mathbb{Q}$ such that $(\rho, \alpha) = -\alpha^2/2 = -1$ for all $\alpha \in P = P(\mathcal{M})$. For elliptic case, $\rho^2 = (\rho, \rho) < 0$, and $\rho^2 = 0$ for parabolic case where $\mathbb{R}_{++}\rho = c$. For elliptic case, W has finite index in $O(S)$, then S is called *elliptically* 2*-reflective.* For parabolic case, $O^+(S)/W$ is $\mathbb{Z}^m$, up to finite index, for some $m > 0$. We want to construct Lorentzian Kac–Moody algebras with the root lattice S, the set of simple real roots $P = P(\mathcal{M}) \subset S$ and the Weyl group W.

In this paper, we consider the basic case of this problem when the Weyl group W is the full group $W = W^{(2)}(S)$ generated by all reflections in vectors with square 2 of a hyperbolic even lattice S.

All elliptically 2-reflective hyperbolic lattices S when the group $W^{(2)}(S)$

has finite index in $O(S)$ were classified by the second author in [N2] and [N5] for $\operatorname{rk} S \neq 4$, and by E.B. Vinberg [V5] for $\operatorname{rk} S = 4$. Their total number is finite and $\operatorname{rk} S \leq 19$. The number of parabolically 2 reflective hyperbolic lattices S for $W = W^{(2)}(S)$ is also finite by [N8], but their full classification is unknown. Many of them were found in [N2].

In Sect. 3, we give the list of elliptically 2-reflective even hyperbolic lattices S from [N2], [N5] and [V4], and in Theorem 3.1, we find those of them which have the lattice Weyl vector ρ for $P = P(\mathcal{M})$ of $W^{(2)}(S)$. **There are 59 such lattices.** 15 of them are of rank 3 and 44 of rank ≥ 4, and the maximal rank is equal to 19. For all these lattices S, we calculate the set $P = P(\mathcal{M}) \subset S$ of simple real roots and its Dynkin diagram which is equivalent to the generalized Cartan matrix

$$A = ((\alpha_1, \alpha_2))\,, \quad \alpha_1, \alpha_2 \in P = P(\mathcal{M}). \tag{1.1}$$

This matrix defines the usual hyperbolic Kac–Moody algebra $\mathfrak{g}(A)$, see [K1]. We calculate the lattice Weyl vector ρ for $P = P(\mathcal{M})$ for all these cases.

In Sect. 4, for an extended lattice $T = U(m) \oplus S$ of signature $(n+1, 2)$ where U is the even unimodular lattice of signature $(1,1)$, $U(m)$ means that we multiply the pairing of the lattice U by $m \in \mathbb{N}$, and $\oplus$ is the orthogonal sum of lattices, we consider the Hermitian symmetric domain $\Omega(T) \cong S \otimes \mathbb{R} + iV^+(S)$. For all 59 lattices S of Theorem 3, we conjecture existence for some m of so called 2-reflective holomorphic automorphic form $\Phi(z) \in M_k(\Gamma)$ on $\Omega(T)$ of weight $k > 0$ with integral Fourier coefficients, where $\Gamma \subset O(T)$ is of finite index, whose divisor is union of rational quadratic divisors with multiplicity one orthogonal to the elements with square 2 of T. *The Fourier coefficients of $\Phi(z)$ at a 0-dimensional cusp define additional sequence of simple imaginary roots $P'^{im} \subset S$ with non-positive squares.* The sequences of the simple real roots P and the imaginary simple roots P'^{im} define Lorentzian Kac–Moody–Borcherds Lie superalgebra $\mathfrak{g}(P(\mathcal{M}), \Phi)$ by exact generators and defining relations. This superalgebra is the *(automorphic) Lorentzian Kac–Moody algebra* which we want to construct. The Lorentzian Kac–Moody (Lie super) algebra $\mathfrak{g}(P(\mathcal{M}), \Phi)$ is graded by S. *The dimensions* $\dim \mathfrak{g}_\alpha(P(\mathcal{M}), \Phi)$*, $\alpha \in S$, of this grading (equivalently, the multiplicities of all roots of the algebra) are defined by the Borcherds product expansion of the automorphic form $\Phi(z)$ at a zero dimensional cusp.* See Sect. 2 and Sect. 4 for the exact definitions and details of the automorphic correction.

In this paper, we determine automorphic corrections for 36 of 59 lattices of Theorem 3 but we consider here more than 70 reflective modular forms. We are planing to construct automorphic corrections for the rest 10 of 2-reflective lattices of rank 4 and 5 from Theorem 3 in a separate publication. Some of these functions will be modular with respect to congruence subgroups similar to [GN6].

In Sect. 2, we give exact definitions of data (I) – (V) which define the Lorentzian Kac–Moody algebras for the case which we consider. One can find more general definitions in our papers which we mentioned above.

In Sect. 3, we give classification of elliptically 2-reflective hyperbolic lattices S with a lattice Weyl vector for $W^{(2)}(S)$. They give all possible data (I) – (III) for construction of the Lorentzian Kac–Moody algebras which we consider.

In Sections 4–6, we find automorphic forms which finalize the construction of the (automorphic) Lorentzian Kac–Moody algebra $\mathfrak{g}(P(\mathcal{M}),\Phi)$ and give automorphic corrections of the usual Kac–Moody algebra $\mathfrak{g}(A)$ defined in (1.1). We note that $\mathfrak{g}(A)$ might have many automorphic corrections! We give two such examples in Proposition 4.1 and Theorem 6.2.

In Sect. 4, we analyse the quasi pull-backs of the Borcherds modular form Φ_{12} for $\mathrm{O}(II_{26,2},\det)$, construct 34 strongly 2-reflective modular forms which determine the automorphic corrections of 25 hyperbolic lattices from Theorem 3 and of 9 parabolically 2-reflective hyperbolic lattices. We note that the modular objects related to these Lorentzian Kac–Moody algebras are very arithmetic. The 25 modular forms are cusp forms which are new eigenfunctions of all Hecke operators (see Corollary 4.1). *One can consider these cusp forms as generalisations of the Ramanujan Δ-function.* All 34 corresponding modular varieties of orthogonal type are, at least, uniruled (see Corollary 4.3).

In Sect. 5, we describe Borcherds products of Jacobi type of the quasi pull-backs from Sect. 4. Our approach gives an explicite formula for the first two Fourier-Jacobi coefficients of the reflective modular forms and an interesting relation between Lorentzian Kac–Moody algebras and some affine Lie algebras in terms of the denominator functions.

In Sect. 6, we construct automorphic corrections of fourteen hyperbolic root systems of Theorem 3.1 and four automorphic corrections of hyperbolic root systems of parabolic type. Almost all modular forms of Sect. 6 have simple Fourier expansions because they are additive Jacobi liftings of Jacobi forms related to the dual lattices of some root lattices. Some reflective modular forms from Sect. 6 determine automorphic corrections of hyperbolic

Kac–Moody algebras with Weyl groups which are overgroups or subgroups of the Weyl groups of type $W^{(2)}(S)$ considered in this paper.

We note that the denominator functions of the corresponding Lorentzian Kac–Moody algebras are automorphic discriminants of moduli spaces of some $K3$ surfaces with a condition on Picard lattices and they realise the arithmetic mirror symmetry for such $K3$ surfaces (see [GN3], [GN7] and [GN8]).

2 Definition of Lorentzian Kac–Moody algebras corresponding to 2-reflective hyperbolic lattices with a lattice Weyl vector

Here we want to give definition of Lorentzian Kac–Moody algebras which we want to construct and consider in this paper. They are given by data (I) — (V) below. We follow the general theory of Lorentzian Kac–Moody algebras from our papers [GN5], [GN6], [GN8] and [N7], [N8], [N12] where we used ideas and results by Kac [K1]–[K3] and Borcherds [B1]–[B4]. One can find more general definitions and possible data in these our papers.

(I) The datum (I) is given by a *hyperbolic lattice* S of the rank $\operatorname{rk} S \geq 3$.

We recall that a lattice (equivalently, a non-degenerate symmetric bilinear form over $\mathbb{Z}$) M means that M is a free $\mathbb{Z}$-module M of a finite rank with symmetric $\mathbb{Z}$-bilinear non-degenerate pairing $(x, y) \in \mathbb{Z}$ for $x, y \in M$. A lattice M is hyperbolic if the corresponding symmetric bilinear form $M \otimes \mathbb{R}$ over $\mathbb{R}$ has signature $(n, 1)$ where $\operatorname{rk} M = n + 1$.

(II) This datum is given by the *Weyl group* which is the 2-reflection group $W = W^{(2)}(S) \subset O(S)$ of the hyperbolic lattice S from (I). It is generated by 2-reflections s_α in all 2-roots $\alpha \in S$ that is $\alpha^2 = (\alpha, \alpha) = 2$.

We recall that an element α of a lattice M is called *root* if $\alpha^2 > 0$ and $\alpha^2 \; | 2(\alpha, M)$ that is $\alpha^2 | 2(\alpha, x)$ for any $x \in M$. A root $\alpha \in M$ defines the reflection

$$s_\alpha : x \to x - (2(x, \alpha)/\alpha^2)\alpha, \quad \forall x \in M \tag{2.1}$$

which belongs to the automorphism group $O(M)$ of the lattice M. The reflection s_α is characterized by the properties: $s_\alpha(\alpha) = -\alpha$ and $s_\alpha | (\alpha)^\perp_M$ is identity. Any element $\alpha \in M$ with $\alpha^2 = 2$ gives a root of M.

(III) This datum is given by the *set of simple real roots* $P = P(\mathcal{M}) \subset S$ of all 2-roots which are perpendicular and directed outwards to the fundamental chamber $\mathcal{M} \subset \mathcal{L}(S)$ of the Weyl group $W = W^{(2)}(S)$ acting in the hyperbolic space $\mathcal{L}(S)$ defined by S. The set $P = P(\mathcal{M})$ must have the *lattice Weyl vector* $\rho \in S \otimes \mathbb{Q}$ such that

$$(\rho, \alpha) = -1 \ \ \forall \alpha \in P = P(\mathcal{M}). \tag{2.2}$$

The fundamental chamber $\mathcal{M}$ must have either a finite volume (then S is called *elliptically 2-reflective*) and then $\rho^2 < 0$ and $P = P(\mathcal{M})$ is finite (*elliptic case*), or almost finite volume (then S is called *parabolically 2-reflective*) and $\rho^2 = 0$, but $\rho \neq 0$ (*parabolic case*). Here almost finite volume means that $\mathcal{M}$ has finite volume in any cone with the vertex $\mathbb{R}^{++}\rho$ at infinity of $\mathcal{M}$.

We recall that, for a hyperbolic lattice M, we can consider the cone

$$V(M) = \{x \in M \otimes \mathbb{R} \mid x^2 < 0\}$$

of M, and its half cone $V^+(M)$. Any two elements $x, y \in V^+(M)$ satisfy $(x, y) < 0$. The half-cone $V^+(M)$ defines *the hyperbolic space of M,*

$$\mathcal{L}^+(M) = V^+(M)/\mathbb{R}_{++} = \{\mathbb{R}_{++}x \mid x \in V^+(M)\}$$

of the curvature (-1) with the hyperbolic distance

$$\operatorname{ch} \rho(\mathbb{R}_{++}x, \mathbb{R}_{++}y) = \frac{-(x, y)}{\sqrt{x^2y^2}}, \quad x, y \in V^+(M).$$

Here $\mathbb{R}_{++}$ is the set of all positive real numbers, and $\mathbb{R}_+$ is the set of all non-negative real numbers. Any $\delta \in M \otimes \mathbb{R}$ with $\delta^2 > 0$ defines *a half-space*

$$\mathcal{H}^+_\delta = \{\mathbb{R}_{++}x \in \mathcal{L}^+(M) \mid (x, \delta) \leq 0\}$$

of $\mathcal{L}(M)$ bounded by the *hyperplane*

$$\mathcal{H}_\delta = \{\mathbb{R}_{++}x \in \mathcal{L}^+(M) \mid (x, \delta) = 0\}.$$

The δ is called orthogonal to the half-space $\mathcal{H}^+_\delta$ and the hyperplane $\mathcal{H}_\delta$, and it is defined uniquely if $\delta^2 > 0$ is fixed. For a root $\alpha \in M$, the reflection s_α gives the reflection of $\mathcal{L}^+(M)$ with respect to the hyperplane $\mathcal{H}_\alpha$, that is s_α is identity on $\mathcal{H}_\alpha$, and $s_\delta(\mathcal{H}^+_\alpha) = \mathcal{H}^+_{-\alpha}$. It is well-known that the group

$$O^+(S) = \{\phi \in O(S) \mid \phi(V^+(S)) = V^+(S)\}$$

is discrete in $\mathcal{L}^+(S)$ and has a fundamental domain of finite volume. The subgroup $W^{(2)}(S)$ is its subgroup generated by 2-reflections.

The main invariant of the data (I) — (III) is *the generalized Cartan matrix*

$$A=\left(\frac{2(\alpha,\alpha')}{(\alpha,\alpha)}\right)=((\alpha,\alpha'))\,,\quad \alpha,\alpha'\in P=P(\mathcal{M}). \tag{2.3}$$

It is symmetric for the case we consider. It defines the corresponding *hyperbolic Kac–Mody algebra* $\mathfrak{g}(A)$, see [K1]. It has *restricted arithmetic type* and is *graded by the lattice* S. See [N7] and [N8] for details. The next data (IV) and (V) give the *automorphic correction* $\mathfrak{g}$ of this algebra.

(IV) For this datum, we need an extended lattice $T=U(m)\oplus S$ (*the symmetry lattice of the Lie algebra* $\mathfrak{g}$) where

$$U=\begin{pmatrix} 0 & -1\\ -1 & 0\end{pmatrix}, \tag{2.4}$$

$M(m)$ for a lattice M and $m\in\mathbb{Q}$ means that we multiply the pairing of M by m, the orthogonal sum of lattices is denoted by $\oplus$. The lattice T defines the Hermitian symmetric domain of the type IV

$$\Omega(T)=\{\mathbb{C}\omega\subset T\otimes\mathbb{C}\mid (\omega,\omega)=0,\ (\omega,\overline{\omega})<0\}^+ \tag{2.5}$$

where $+$ means a choice of one (from two) connected components. The domain $\Omega(T)$ can be identified with the complexified cone $\Omega(V^+(S))=S\otimes\mathbb{R}+iV^+(S)$ as follows: for the basis e_1,e_2 of the lattice U with the matrix (2.4), we identify $z\in\Omega(V^+(S))$ with $\mathbb{C}\omega_z\in\Omega(T)$ where $\omega_z=(z,z)e_1/2+e_2/m\oplus z\in\Omega(T)^\bullet$ (the corresponding affine cone over $\Omega(T)$). The main datum in (IV) is a *holomorphic automorphic form* $\Phi(z)$, $z\in\Omega(V^+(S))=\Omega(T)$ of some weight $k\in\mathbb{Z}/2$ on the Hermitian symmetric domain $\Omega(V^+(S))=\Omega(T)$ of the type IV with respect to a subgroup $G\subset O^+(T)$ of a finite index (*the symmetry group of the Lie algebra* $\mathfrak{g}$. Here $O^+(T)$ is the index two subgroup of $O(T)$ which preserves $\Omega(T)$.

The automorphic form $\Phi(z)$ must have Fourier expension which gives the denominator identity for the Lie algebra $\mathfrak{g}$:

$$\Phi(z)=\sum_{w\in W}\det(w)(exp(-2\pi i(w(\rho),z))-$$
$$-\sum_{a\in S\cap\mathbb{R}_{++}\mathcal{M}} m(a)exp(-2\pi i(w(\rho+a),z))), \tag{2.6}$$

where all coefficients $m(a)$ must be integral. It also would be nice to calculate the *infinite product expension (the Borcherds product) for the denominator identity of the Lie algebra* $\mathfrak{g}$

$$\Phi(z) = exp(-2\pi i(\rho, z)) \prod_{\alpha\in\Delta_+} (1 - exp(-2\pi i(\alpha, z)))^{mult(\alpha)}\ , \tag{2.7}$$

which gives *multiplicities* $mult(\alpha)$ of roots of the Lie algebra $\mathfrak{g}$. Here $\Delta_+ \subset S$ (see below).

(V) The automorphic form $\Phi(z)$ in $\Omega(V^+(S)) = \Omega(T)$ must be *2-reflective.* It means that the divisor (of zeros) of $\Phi(z)$ is union of rational quadratic divisors which are orthogonal to 2-roots of T. Hear, for $\beta \in T$ with $\beta^2 > 0$ *the rational quadratic divisor which is orthogonal to* β, is equal to

$$D_\beta = \{\mathbb{C}\omega \in \Omega(T) \mid (\omega, \beta) = 0\}.$$

The property (V) is valid in a neighbourhood of the cusp of $\Omega(T)$ where the infinite product (2.7) converges, but we want to have it globally.

For our case, the *Lorentzian Kac–Moody superalgebra* $\mathfrak{g}$ *corresponding to data (I) — (V),* which is a Kac–Moody–Borcherds superablebra or an *automorphic correction* given by $\Phi(z)$ of the Kac–Moody algebra $\mathfrak{g}(A)$ given by the generalized Cartan matrix (2.3) above, is defined by the sequence $P' \subset S$ of *simple roots.* It is divided to the set P'^{re} of *simple real root* (all of them are even) and the set $P'^{im}_{\bar{0}}$ *of even simple imaginary roots* and the set $P'^{im}_{\bar{1}}$ *of odd imaginary roots.* Thus, $P' = P'^{re} \cup P'^{im}_{\bar{0}} \cup P'^{im}_{\bar{1}}$.

For a primitive $a \in S \cap \mathbb{R}_{++}\mathcal{M}$ with $(a, a) = 0$ one should find $\tau(na) \in \mathbb{Z}$, $n \in \mathbb{N}$, from the indentity with the formal variable t:

$$1 - \sum_{k\in\mathbb{N}} m(ka)t^k = \prod_{n\in\mathbb{N}} (1 - t^n)^{\tau(na)}.$$

The set $P'^{re} = P$ where P is defined in (III). The set P'^{re} is even: $P'^{re} = P'^{re}{}_{\bar{0}}$, $P'^{re}{}_{\bar{1}} = \emptyset$. The set

$$P'^{im}{}_{\bar{0}} = \{m(a)a \mid a \in S \cap \mathbb{R}_{++}\mathcal{M},\ (a, a) < 0 \ and\ m(a) > 0\} \cup$$
$$\{\tau(a)a \mid a \in S \cap \mathbb{R}_{++}\mathcal{M},\ (a, a) = 0 \ and\ \tau(a) > 0\}; \tag{2.8}$$

$$P'^{im}_{\overline{1}} = \{-m(a)a \mid a \in S \cap \mathbb{R}_{++}\mathcal{M},\ (a,a) < 0 \ and\ m(a) < 0\} \cup$$

$$\{-\tau(a)a \mid a \in S \cap \mathbb{R}_{++}\mathcal{M},\ (a,a) = 0 \ and\ \tau(a) < 0\} \qquad (2.9)$$

Here, ka for $k \in \mathbb{N}$ means that we repeat a exactly k times in the sequence.

The generalized Kac–Moody superalgebra $\mathfrak{g}$ is the Lie superalgebra with generators h_r, e_r, f_r where $r \in P'$. All generators h_r are even, generators e_r, f_r are even (respectively odd) if r is even (respectively odd).

They have defining relations 1) — 5) of $\mathfrak{g}$ which are given below.

1) The map $r \to h_r$ for $r \in P'$ gives an embedding $S \otimes \mathbb{C}$ to $\mathfrak{g}$ as Abelian subalgebra (it is even).

2) $[h_r, e_{r'}] = (r, r')e_{r'}$ and $[h_r, f_{r'}] = -(r, r')f_{r'}$.

3) $[e_r, f_{r'}] = h_r$ if $r = r'$, and it is 0, if $r \neq r'$.

4) $(ad\ e_r)^{1-2(r,r')/(r,r)}e_{r'} = (ad\ f_r)^{1-2(r,r')/(r,r)}f_{r'} = 0$,
if $r \neq r'$ $(r,r) > 0$ (equivalently, $r \in P'^{\,re}$).

5) If $(r, r') = 0$, then $[e_r, e_{r'}] = [f_r, f_{r'}] = 0$.

The algebra $\mathfrak{g}$ is graded by the lattice S where the generators h_r, e_r and f_r have weights 0, $r \in S$ and $-r \in S$ respectively. We have

$$\mathfrak{g} = \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha = \mathfrak{g}_0 \bigoplus \left(\bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \right) \bigoplus \left(\bigoplus_{\alpha \in -\Delta_+} \mathfrak{g}_\alpha \right), \qquad (2.10)$$

where $\mathfrak{g}_0 = S \otimes \mathbb{C}$, and Δ is the set of roots (that is the set of $\alpha \in S$ with $\dim \mathfrak{g}_\alpha \neq 0$). The root α is positive ($\alpha \in \Delta_+$) if $(\alpha, \mathcal{M}) \leq 0$. By definition, the multiplicity of $\alpha \in \Delta$ is equal to $mult(\alpha) = \dim \mathfrak{g}_{\alpha,\overline{0}} - \dim \mathfrak{g}_{\alpha,\overline{1}}$.

For this definition, we use results by Borcherds, authors, U. Ray.

In Section 3, we give the classification of possible data (I) — (III) of elliptic type. In Sections 4—6, we construct some data (IV) — (V) for these data (I) — (III) of elliptic type and for some data (I) — (III) of parabolic type.

3 Classification of elliptically 2-reflective hyperbolic lattices with lattice Weyl vectors

3.1 Notations

We follow definitions from Sec. 2 of lattices, hyperbolic lattices, roots, 2-roots, reflections in roots, hyperbolic spaces of hyperbolic lattices.

For a lattice M, we denote by $x \cdot y = (x, y)$, $x, y \in M$ the symmetric pairing of M, and $x^2 = x \cdot x$, $x \in M$.

For a hyperbolic lattice S, we denote by $V^+(S)$ the half-cone of S and by $\mathcal{L}^+(S) = V^+(S)/\mathbb{R}_{++}$ the hyperbolic space of S. By $\mathcal{H}_\delta$ and $\mathcal{H}_\delta^+$ we denote the hyperplane and the half-space of $\mathcal{L}^+(S)$ which are orthogonal to $\delta \in S \otimes \mathbb{R}$ where $\delta^2 > 0$.

3.2 Classification of elliptically 2-reflective hyperbolic lattices

Let S be a hyperbolic lattice of the signature $(n, 1)$ where $\operatorname{rk} S = n + 1 \geq 3$.

It is well-known that the group

$$O^+(S) = \{\phi \in O(S) \mid \phi(V^+(S)) = V^+(S)\}$$

is discrete in $\mathcal{L}^+(S)$ and has a fundamental domain of finite volume. The subgroups $W(S)$ and $W^{(2)}(S)$ are its subgroups generated by all and 2-reflections respectively. We denote by $\mathcal{M} \subset \mathcal{L}^+(S)$ and $\mathcal{M}^{(2)} \subset \mathcal{L}^+(S)$ their fundamental chambers respectively.

Definition 3.1. *A hyperbolic lattice S of $\operatorname{rk} S \geq 3$ is called elliptically reflective (respectively elliptically 2-reflective) if $[O(S) : W(S)] < \infty$ (respectively, $[O(S) : W^{(2)}(S)] < \infty$). Equivalently, the fundamental chamber $\mathcal{M} \subset \mathcal{L}^+(S)$ (respectively the fundamental chamber $\mathcal{M}^{(2)} \subset \mathcal{L}^+(S)$) has finite volume.*

In [N2] for $\operatorname{rk} S \geq 5$, [V4] (see also [N6]) for $\operatorname{rk} S = 4$, [N5] for $\operatorname{rk} S = 3$, all elliptically 2-reflective hyperbolic lattices were classified. It is enough to classify even 2-reflective hyperbolic lattices. Indeed, an odd lattice will be 2-reflective if and only if its maximal even sublattice is 2-reflective.

The list of all even elliptically 2-reflective hyperbolic lattices is given below. We use the standard notations. By A_n, $n \geq 1$; D_n, $n \geq 4$; E_n,

$n = 6, 7, 8$, we denote the positive definite root lattices of the corresponding root systems $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_n$ with roots having the square 2. Their standard bases consist of bases of the the root systems. By U, we denote the hyperbolic even unimodular lattices of the rank 2. For the standard basis $\{c_1, c_2\}$, it has the matrix

$$U = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

By $M(n)$ we denote a lattice which is obtained from M by multiplication by $n \in \mathbb{Q}$ of the form of a lattice M. By $\langle A \rangle$ we denote a lattice defined by the symmetric integral matrix A in some (we call it standard) basis. If a lattice M has a standard basis $e_1, \ldots, e_n$, then $M[\alpha_1, \ldots, \alpha_n]$ denotes a lattice which is obtained by adding to M the element $\alpha_1 e_1 + \cdots + \alpha_n e_n$. Here $\alpha_1, \ldots, \alpha_n$ are from $\mathbb{Q}$. By $\oplus$ we denote the orthogonal sum of lattices. If M_1 and M_2 have the standard bases, then the standard basis of $M_1 \oplus M_2$ consists of the union of these bases.

We have the following list of all 2-reflective hyperbolic lattices of rank ≥ 3, up to isomorphisms:

The list of all elliptically 2-reflective even hyperbolic lattices of rank ≥ 3.

If $\mathrm{rk}\, S = 3$, there are 26 lattices which are 2-reflective. They are described in [N5] (We must correct the list of these lattices in [N5]: In notations of [N5], the lattices $S'_{6,1,2} = [3a + c, b, 2c]$ and $S_{6,1,1} = [6a, b, c]$ are isomorphic, they have isomorphic fundamental polygons. See calculations for the proof of Theorem 3.1 below.)

If $\mathrm{rk}\, S = 4$, then $S = \langle -8 \rangle \oplus 3A_1$; $U \oplus 2A_1$; $\langle -2 \rangle \oplus 3A_1$; $U(k) \oplus 2A_1$, $k = 3, 4$; $U \oplus A_2$; $U(k) \oplus A_2$, $k = 2, 3, 6$; $\left\langle \begin{matrix} 0 & -3 \\ -3 & 2 \end{matrix} \right\rangle \oplus A_2$; $\langle -4 \rangle \oplus \langle 4 \rangle \oplus A_2$; $\langle -4 \rangle \oplus A_3$;
$\left\langle \begin{matrix} -2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{matrix} \right\rangle$, $\left\langle \begin{matrix} -12 & -2 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{matrix} \right\rangle$.

If $\mathrm{rk}\, S = 5$, then $S = U \oplus 3A_1$; $\langle -2 \rangle \oplus 4A_1$; $U \oplus A_1 \oplus A_2$; $U \oplus A_3$; $U(4) \oplus 3A_1$; $\langle 2^k \rangle \oplus D_4$, $k = 2, 3, 4$; $\langle 6 \rangle \oplus 2A_2$.

If $\mathrm{rk}\, S = 6$, then $S = U \oplus D_4$, $U(2) \oplus D_4$, $U \oplus 4A_1$, $\langle -2 \rangle \oplus 5A_1$, $U \oplus 2A_1 \oplus A_2$, $U \oplus 2A_2$, $U \oplus A_1 \oplus A_3$, $U \oplus A_4$, $U(4) \oplus D_4$, $U(3) \oplus 2A_2$.

If $\operatorname{rk} S = 7$, then $S = U \oplus D_4 \oplus A_1$, $U \oplus 5A_1$, $\langle -2 \rangle \oplus 6A_1$, $U \oplus A_1 \oplus 2A_2$, $U \oplus 2A_1 \oplus A_3$, $U \oplus A_2 \oplus A_3$, $U \oplus A_1 \oplus A_4$, $U \oplus A_5$, $U \oplus D_5$.

If $\operatorname{rk} S = 8$, then $S = U \oplus D_6$, $U \oplus D_4 \oplus 2A_1$, $U \oplus 6A_1$, $\langle -2 \rangle \oplus 7A_1$, $U \oplus 3A_2$, $U \oplus 2A_3$, $U \oplus A_2 \oplus A_4$, $U \oplus A_1 \oplus A_5$, $U \oplus A_6$, $U \oplus A_2 \oplus D_4$, $U \oplus A_1 \oplus D_5$, $U \oplus E_6$.

If $\operatorname{rk} S = 9$, then $S = U \oplus E_7$, $U \oplus D_6 \oplus A_1$, $U \oplus D_4 \oplus 3A_1$, $U \oplus 7A_1$, $\langle -2 \rangle \oplus 8A_1$, $U \oplus A_7$, $U \oplus A_3 \oplus D_4$, $U \oplus A_2 \oplus D_5$, $U \oplus D_7$, $U \oplus A_1 \oplus E_6$.

If $\operatorname{rk} S = 10$, then $S = U \oplus E_8$, $U \oplus D_8$, $U \oplus E_7 \oplus A_1$, $U \oplus D_4 \oplus D_4$, $U \oplus D_6 \oplus 2A_1$, $U(2) \oplus D_4 \oplus D_4$, $U \oplus 8A_1$, $U \oplus A_2 \oplus E_6$.

If $\operatorname{rk} S = 11$, then $S = U \oplus E_8 \oplus A_1$, $U \oplus D_8 \oplus A_1$, $U \oplus D_4 \oplus D_4 \oplus A_1$, $U \oplus D_4 \oplus 5A_1$.

If $\operatorname{rk} S = 12$, then $S = U \oplus E_8 \oplus 2A_1$, $U \oplus D_8 \oplus 2A_1$, $U \oplus D_4 \oplus D_4 \oplus 2A_1$, $U \oplus A_2 \oplus E_8$.

If $\operatorname{rk} S = 13$, then $S = U \oplus E_8 \oplus 3A_1$, $U \oplus D_8 \oplus 3A_1$, $U \oplus A_3 \oplus E_8$.

If $\operatorname{rk} S = 14$, then $S = U \oplus E_8 \oplus D_4$, $U \oplus D_8 \oplus D_4$, $U \oplus E_8 \oplus 4A_1$.

If $\operatorname{rk} S = 15$, then $S = U \oplus E_8 \oplus D_4 \oplus A_1$.

If $\operatorname{rk} S = 16$, then $S = U \oplus E_8 \oplus D_6$.

If $\operatorname{rk} S = 17$, then $S = U \oplus E_8 \oplus E_7$.

If $\operatorname{rk} S = 18$, then $S = U \oplus 2E_8$.

If $\operatorname{rk} S = 19$, then $S = U \oplus 2E_8 \oplus A_1$.

If $\operatorname{rk} S \geq 20$, there are no such lattices.

Calculations of their fundamental chambers $\mathcal{M}^{(2)}$ and the finite sets $P(\mathcal{M}^{(2)})$ of 2-roots which are perpendicular to codimension one faces of $\mathcal{M}^{(2)}$ and directed outwards are also known. See [N2], [V4] and [N5] for almost all cases. See also below.

Remark 3.1. By global Torelli Theorem for K3 surfaces [PS], ellptically 2-reflective hyperbolic lattices S from the list above, give all Picard lattices $S_X = S(-1)$ of K3 surfaces X over $\mathbb{C}$ with finite automorphism group and $\operatorname{rk} S_X \geq 3$. They have signature $(1, n)$ where $\operatorname{rk} S_X = \operatorname{rk} S(-1) = \operatorname{rk} S = n + 1 \geq 3$. The set $P(\mathcal{M}^{(2)}) \subset S(-1) = S_X$ gives classes of all non-singular rational curves on X. Their number is finite and they generate S_X up to finite index.

Remark 3.2. There are general finiteness results about reflective hyperbolic lattices and arithmetic hyperbolic reflection groups. See [N3], [N4], [N6], [N9] and [V3], [V4].

Classification of maximal reflective (elliptically, parabolically or hyperbolically) hyperbolic lattices of rank 3 was obtained in [N11]. Classification of elliptically reflective hyperbolic lattices of rank 3 was obtained by D. Allcock in [Al1].

3.3 Classification of elliptically 2-reflective even hyperbolic lattices S with lattice Weyl vector for $W^{(2)}(S)$

The particular cases of elliptically 2-reflective hyperbolic lattices will be important for us. They are characterized by the property that they have the lattice Weyl vector.

Let S be an elliptically 2-reflective hyperbolic lattice, $\mathcal{M}^{(2)}(S) \subset \mathcal{L}^{+}(S)$ the fundamental chamber for $W^{(2)}(S)$, and $P(\mathcal{M}^{(2)}(S))$ the set of perpendicular 2-roots to $\mathcal{M}^{(2)}(S))$ directed outwards. That is

$$\mathcal{M}^{(2)}(S) = \{\mathbb{R}_{++}x \in \mathcal{L}^{+}(S) \mid x \cdot P(\mathcal{M}^{(2)}(S)) \leq 0\}$$

and $P(\mathcal{M}^{(2)}(S))$ is minimal with this property.

Definition 3.2. *A 2-reflective hyperbolic lattice S has a lattice Weyl vector for $W^{(2)}(S)$ (equivalently, for $P(\mathcal{M}^{(2)}(S))$) if there exists $\rho \in S \otimes \mathbb{Q}$ such that*

$$\rho \cdot \delta = -1 \quad \forall\ \delta \in P(\mathcal{M}^{(2)}(S)). \tag{3.1}$$

The ρ is called the lattice Weyl vector for $P(\mathcal{M}^{(2)}(S))$.

More generally, a reflective (elliptically or parabolically) hyperbolic lattice S has a lattice Weyl vector for a reflection subgroup $W \subset W(S)$ and a set $P(\mathcal{M})$ of roots of S which are perpendicular to a fundamental chamber $\mathcal{M}$ of W and directed outwards if there exists $\rho \in S \otimes \mathbb{Q}$ such that

$$\rho \cdot \delta = -\frac{\delta^2}{2} \quad \forall\ \delta \in P(\mathcal{M}). \tag{3.2}$$

The ρ is called the lattice Weyl vector for $P(\mathcal{M})$.

Since $\mathcal{M}^{(2)}(S)$ has finite volume, the set $P(\mathcal{M}^{(2)}(S))$ generates $S\otimes\mathbb{Q}$, and the ρ is defined uniquely. The $\mathbb{R}_{++}\rho$ belongs to the interior of $\mathcal{M}^{(2)}(S)$, and $\rho^2<0$. Geometrically, $\mathbb{R}_{++}\rho$ gives a center of a sphere which is inscribed to the fundamental chamber $\mathcal{M}^{(2)}$. Thus, this case is especially special and beautiful.

Using classification of elliptically 2-reflective hyperbolic lattices, we obtain classification of elliptically 2-reflective hyperbolic lattices with lattice Weyl vectors.

Theorem 3.1. *The following and only the following elliptically 2-reflective even hyperbolic lattices S of $\operatorname{rk} S\ge 3$ have a lattice Weyl vector ρ for $W^{(2)}(S)$ (equivalently, for $P(\mathcal{M}^{(2)}(S))$). We order them by the rank and the absolute value of the determinant.*
Rank 3: $S_{3,2}=U\oplus A_1$, $S_{3,8,a}=\langle -2\rangle\oplus 2A_1$, $S_{3,8,b}=(\langle -24\rangle\oplus A_2)[1/3,-1/3,1/3]$, $S_{3,18}=U(3)\oplus A_1$, $S_{3,32,a}=U(4)\oplus A_1$, $S_{3,32,b}=\langle -8\rangle\oplus 2A_1$, $S_{3,32,c}=U(8)[1/2,1/2]\oplus A_1$, $S_{3,72}=\langle -24\rangle\oplus A_2$, $S_{3,128,a}=U(8)\oplus A_1$, $S_{3,128,b}=\langle -32\rangle\oplus 2A_1$, $S_{3,288}=U(12)\oplus A_1$,
anisotropic cases: $S_{3,12}=\langle -4\rangle\oplus A_2$, $S_{3,24}=\langle -6\rangle\oplus 2A_1$, $S_{3,36}=\langle -12\rangle\oplus A_2$, $S_{3,108}=\langle -36\rangle\oplus A_2$.

Rank 4: $S_{4,3}=U\oplus A_2$, $S_{4,4}=U\oplus 2A_1$, $S_{4,12}=U(2)\oplus A_2$, $S_{4,16,a}=\langle -2\rangle\oplus 3A_1$, $S_{4,16,b}=\langle -4\rangle\oplus A_3$, $S_{4,27,a}=U(3)\oplus A_2$, $S_{4,27,b}=\left\langle\begin{array}{cc}0&-3\\-3&2\end{array}\right\rangle\oplus A_2$, $S_{4,64,a}=U(4)\oplus 2A_1$, $S_{4,64,b}=\langle -8\rangle\oplus 3A_1$, $S_{4,108}=U(6)\oplus A_2$,

$$S_{4,28}=\left\langle\begin{array}{cccc}-2&-1&-1&-1\\-1&2&0&0\\-1&0&2&0\\-1&0&0&2\end{array}\right\rangle \text{ (anisotropic case).}$$

Rank 5: $S_{5,4}=U\oplus A_3$, $S_{5,8}=U\oplus 3A_1$, $S_{5,16}=\langle -4\rangle\oplus D_4$, $S_{5,32,a}=\langle -2\rangle\oplus 4A_1$, $S_{5,32,b}=\langle -8\rangle\oplus D_4$, $S_{5,64}=\langle -16\rangle\oplus D_4$, $S_{5,128}=U(4)\oplus 3A_1$.

Rank 6: $S_{6,4}=U\oplus D_4$, $S_{6,5}=U\oplus A_4$, $S_{6,9}=U\oplus 2A_2$, $S_{6,16,a}=U(2)\oplus D_4$, $S_{6,16,b}=U\oplus 4A_1$, $S_{6,64,a}=\langle -2\rangle\oplus 5A_1$, $S_{6,64,b}=U(4)\oplus D_4$, $S_{6,81}=U(3)\oplus 2A_2$.

Rank 7: $S_{7,4}=U\oplus D_5$, $S_{7,6}=U\oplus A_5$, $S_{7,128}=\langle -2\rangle\oplus 6A_1$.

Rank 8: $S_{8,3}=U\oplus E_6$, $S_{8,4}=U\oplus D_6$, $S_{8,7}=U\oplus A_6$, $S_{8,16}=U\oplus 2A_3$, $S_{8,27}=U\oplus 3A_2$, $S_{8,256}=\langle -2\rangle\oplus 7A_1$.

Rank 9: $S_{9,2}=U\oplus E_7$, $S_{9,4}=U\oplus D_7$, $S_{9,8}=U\oplus A_7$, $S_{9,512}=\langle -2\rangle\oplus 8A_1$.

Rank 10: $S_{10,1} = U \oplus E_8$, $S_{10,4} = U \oplus D_8$, $S_{10,16} = U \oplus 2D_4$, $S_{10,64} = U(2) \oplus 2D_4$.

Rank 18: $S_{18,1} = U \oplus 2E_8$.

We shall discuss the proof of Theorem 3.1 in the next section.

3.4 The fundamental chambers $\mathcal{M}^{(2)}$ and the lattice Weyl vectors for lattices of Theorem 3.1

Below, for lattices of Theorem 3.1, we describe the sets $P(\mathcal{M}^{(2)})$ and the Weyl vectors. This describes Gram graphs $\Gamma(P(\mathcal{M}^{(2)}))$ too.

We recall that for $P(\mathcal{M}^{(2)})$ one connects $\delta_1, \delta_2 \in P(\mathcal{M}^{(2)})$ by the edge if $\delta_1 \cdot \delta_2 < 0$. This edge a thin, thick, and broken of the weight $-\delta_1 \cdot \delta_2$ if $\delta_1 \cdot \delta_2 = -1$, $\delta_1 \cdot \delta_2 = -2$, and $\delta_1 \cdot \delta_2 < -2$ respectively.

More generally, for the set $P(\mathcal{M})$ of perpendicular roots to a fundamental chamber $\mathcal{M}$ of a hyperbolic reflection group, one adds weights δ^2 to vertices corresponding to $\delta \in P(\mathcal{M})$ with $\delta^2 \neq 2$ (we draw them black and don't put the weight if $\delta^2 = 4$). The edge for different $\delta_1, \delta_2 \in P(\mathcal{M})$ is thin of the natural weight $n \geq 3$ (equivalently, the $n-2$-multiple thin edge for small n), thick, and broken of the weight $-2\,\delta_1 \cdot \delta_2/\sqrt{\delta_1 \cdot \delta_2}$ if $2\,\delta_1 \cdot \delta_2/\sqrt{\delta_1 \cdot \delta_2} = -2\,\cos(\pi/n)$, $2\,\delta_1 \cdot \delta_2/\sqrt{\delta_1 \cdot \delta_2} = -2$, and $2\,\delta_1 \cdot \delta_2/\sqrt{\delta_1 \cdot \delta_2} < -2$ respectively.

We recall that a lattice M is 2-elementary if its discriminant group M^*/M is 2-elementary, that is $M^*/M \cong (\mathbb{Z}/2\mathbb{Z})^a$.

Cases $S = U \oplus K$ where $K = \oplus_i^n K_i$ is the orthogonal sum of 2-roots lattices A_n, D_n, E_n.

Then $P(\mathcal{M}^{(2)})$ consists of $e = -c_1 + c_2$, bases of root lattices K_i, $c_1 - w_i$ where w_i are the maximal roots of K_i corresponding to the standard bases of K_i, $i = 1, \ldots, n$. The corresponding graph

$$\Gamma = \Gamma(P(\mathcal{M}^{(2)})) = St(\Gamma(\widetilde{K_1}), \ldots, \Gamma(\widetilde{K_n})) \tag{3.3}$$

is called *the Star of the corresponding extended Dynkin diagrams.* Here $e = -c_1 + c_2$ is the center of the Star. The graph $\Gamma - \{e\}$ consists of n connected components $\Gamma(\widetilde{K_i})$ with the bases which are the bases of K_i and $c_1 - w_i$. They give the corresponding extended Dynkin diagrams $\widetilde{\mathbb{A}_n}$, $\widetilde{\mathbb{D}_n}$, and $\widetilde{\mathbb{E}_n}$. Obviously, e is connected (by the thin edge) with $c_1 - w_i$, $i = 1, \ldots, n$, only. See [N2] for details.

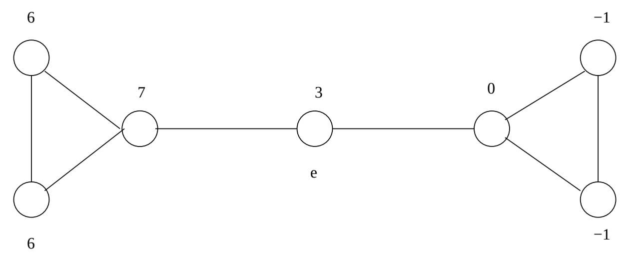

Figure 1: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $U \oplus A_2 \oplus A_2$ is $St(\widetilde{\mathbb{A}_2}, \widetilde{\mathbb{A}_2})$.

Using this description, for all these cases of Theorem 3.1, one can calculate the Weyl vector ρ directly using (3.1), and prove that it does exist. For example, in Figure 1, we draw the graph for the lattice $U \oplus 2A_2$. The rational weights for its vertices show the linear combination of elements of $P(\mathcal{M}^{(2)})$ which gives the lattice Weyl vector ρ. If $n = 1$ (this is valid for the most cases), then $P(\mathcal{M}^{(2)})$ gives the basis of the lattice S, and then ρ exists obviously.

For all remaining similar cases of elliptically 2-reflective lattices of Sect. 3.2, the star (3.3) gives a part of $\Gamma(P(\mathcal{M}^{(2)}))$ (for many cases these graphs coincide, for example if S is not 2-elementary). Calculation of the Weyl vector $\rho \in S \otimes \mathbb{Q}$ satisfying $\rho \cdot \delta = -1$ for all δ of the star (3.3) show that it does not exist for all cases, except $S = U \oplus nA_1$, $5 \le n \le 8$. For these lattices, the full graph $\Gamma(P(\mathcal{M}^{(2)}))$ is calculated in [AN, Table 1], and these calculations show that ρ does not exist in these cases either.

Cases $S = U(2) \oplus D_4$, $U(2) \oplus 2D_4$, $\langle -2 \rangle \oplus nA_1$, $1 \le n \le 8$.

They give remaining 2-elementary cases of Theorem 3.1. All these cases are classical. For example, one can find calculation of the graphs $\Gamma(P(\mathcal{M}^{(2)}))$ in [AN, Table 1].

We choose the standard basis e_1, e_2, e_3, e_4 for D_4 such that $w = e_1 + e_2 + e_3 + 2e_4$ is the maximal root.

Let $S = U(2) \oplus D_4$ with the corresponding standard basis. Then $P(\mathcal{M}^{(2)})$ consists of elements $e_1 = (0, 0, 1, 0, 0, 0)$, $e_2 = (0, 0, 0, 1, 0, 0)$, $e_3 = (0, 0, 0, 0, 1, 0)$, $e_4 = (0, 0, 0, 0, 0, 1)$, $c_1 - w = (1, 0, -1, -1, -1, -2)$, $c_2 - w = (0, 1, -1, -1, -1, -2)$. The Weyl vector $\rho = (3, 3, -3, -3, -3, -5)$. See $\Gamma(P(\mathcal{M}^{(2)}))$ in Figure 2.

Let $S = U(2) \oplus 2D_4$. The same lattice can be written in the form $S = U \oplus (8A_1[1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2])$. We use the standard basis

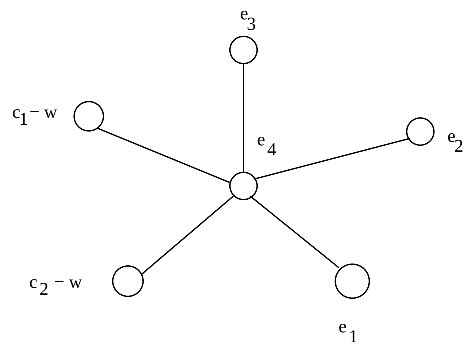

Figure 2: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $U(2) \oplus D_4$.

$c_1, c_2, e_1, \dots, e_8$ for $U \oplus 8A_1$. The set $P(\mathcal{M}^{(2)})$ consists of $e_1, \dots, e_8$, $e_1' = c_1 - e_1$, $e_2' = c_1 - e_2, \dots, e_8' = c_1 - e_8$, $f = -c_1 + c_2$ and $f' = c_1 + c_2 - (e_1 + e_2 + \cdots + e_8)/2$. The Weyl vector $\rho = 3c_1 + 2c_2 - (e_1 + e_2 + \cdots + e_8)/2$. See $\Gamma(P(\mathcal{M}^{(2)}))$ in Figure 3.

Let $S = \langle -2 \rangle \oplus nA_1$, $2 \le n \le 8$. Then calculation of $P(\mathcal{M}^{(2)})$ is equivalent to the calculation of classes of exceptional curves in Picard lattices of the rank $n+1$ for non-singular Del Pezzo surfaces. Then the Weyl vector ρ is equivalent to the anti-canonical class.

For $S = \langle -2 \rangle \oplus nA_1$, $2 \le n \le 8$, with the standard basis $h, e_1, \dots, e_n$, the Weyl vector $\rho = 3h - e_1 - e_2 - \cdots - e_n$, and

$$P(\mathcal{M}^{(2)}) = \{\delta \in S \mid \delta^2 = 2 \ \& \ \delta \cdot \rho = -1\}. \tag{3.4}$$

Then $P(\mathcal{M}^{(2)})$ consists of all elements below which one can get by all permutations of $e_1, \dots, e_n$. They are e_1, $h - e_1 - e_2$ for $n \ge 2$; $2h - e_1 - e_2 - e_3 - e_4 - e_5$ for $n \ge 5$; $3h - 2e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7$ for $n \ge 7$; $4h - 2e_1 - 2e_2 - 2e_3 - e_4 - e_5 - e_6 - e_7 - e_8$, $5h - 2e_1 - 2e_2 - 2e_3 - 2e_4 - 2e_5 - 2e_6 - e_7 - e_8$, $6h - 3e_1 - 2e_2 - \cdots - 2e_8$ for $n = 8$. For example, see [Man, Ch. 4, Sect. 4.2]. Thus, $P(\mathcal{M}^{(2)})$ consists of 240 elements for $n = 8$; 56 elements for $n = 7$; 27 elements for $n = 6$; 16 elements for $n = 5$; 10 elements for $n = 4$; 6 elements for $n = 3$; 3 elements for $n = 2$; 1 element for $n = 1$. It is hard to draw the corresponding graphs for big n. For $n = 2$ and $n = 3$ we draw them in Figures 4 and 5.

This proves Theorem 3.1 for $\operatorname{rk} S \ge 7$. Below we consider remaining cases.

Remaining cases of $\operatorname{rk} S = 6$.

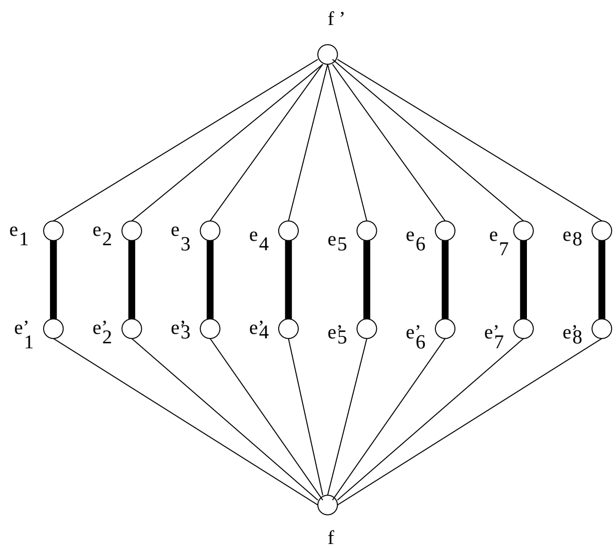

Figure 3: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $U(2) \oplus 2D_4$.

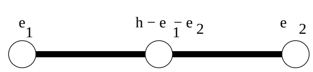

Figure 4: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $\langle -2 \rangle \oplus 2A_1$.

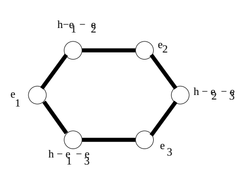

Figure 5: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $\langle -2 \rangle \oplus 3A_1$.

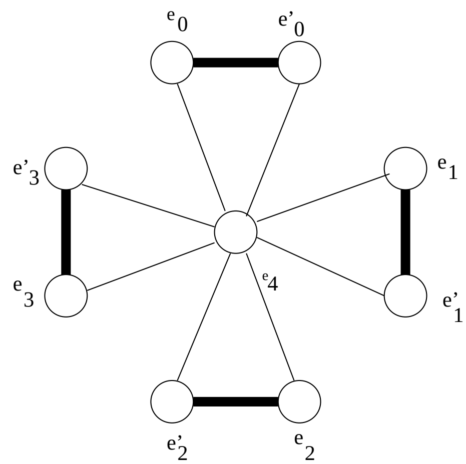

Figure 6: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $U(4) \oplus D_4$.

Let $S = S_{6,64,b} = U(4) \oplus D_4$. See [N2, Sec. 6.4]. Let $c_1, c_2, e_1, e_2, e_3, e_4$ be its standard basis where $w = e_1 + e_2 + e_3 + 2e_4$ is the maximal root of D_4. Then $P(\mathcal{M}^{(2)})$ consists of $e_1, \ldots, e_4$, $e_0 = c_1 - w$, $e_0' = c_2 - w$, and $e_i' = c_1 + c_2 - 2e_1 - 2e_2 - 2e_3 - 4e_4 - e_i$, $i = 1, 2, 3$. The Weyl vector is $\rho = (3c_1 + 3c_2)/2 - 3e_1 - 3e_2 - 3e_3 - 5e_4$. See $\Gamma(P(\mathcal{M}^{(2)}))$ in Figure 6.

Let $S = S_{6,81} = U(3) \oplus 2A_2$. Let $c_1, c_2, e_1, e_2, e_3, e_4$ be its standard basis. The set $P(\mathcal{M}^{(2)})$ consists of e_i, $i = 1, 2, 3, 4$; $c_1 - e_1 - e_2$, $c_1 - e_3 - e_4$, $c_2 - e_1 - e_2$, $c_2 - e_3 - e_4$; $c_1 + c_2 - e_1 - e_2 - e_3 - e_4 - e_i$, $i = 1, 2, 3, 4$. The Weyl vector $\rho = c_1 + c_2 - e_1 - e_2 - e_3 - e_4$. The $P(\mathcal{M}^{(2)})$ has the Gram matrix

$$-\begin{pmatrix}
-2 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 3 & 0 & 1 & 1 \\
1 & -2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 3 & 1 & 1 \\
0 & 0 & -2 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 3 & 0 \\
0 & 0 & 1 & -2 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 3 \\
1 & 1 & 0 & 0 & -2 & 0 & 1 & 3 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & -2 & 3 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 3 & -2 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 3 & 1 & 0 & -2 & 1 & 1 & 0 & 0 \\
3 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 3 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & -2 & 0 & 0 \\
1 & 1 & 3 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -2 & 1 \\
1 & 1 & 0 & 3 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & -2
\end{pmatrix} \quad (3.5)$$

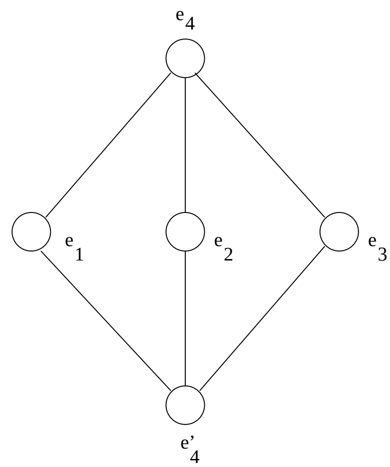

Figure 7: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $\langle -4\rangle \oplus D_4$.

which is very regular.

Thus, we have considered all lattices of the rank 6 of Sect. 3.2, only the lattices of Theorem 3.1 have the lattice Weyl vectors.

Remaining cases of $\operatorname{rk} S = 5$.

Let $S = S_{5,16} = \langle -4\rangle \oplus D_4$. See [N2, Sec. 8.5]. For the standard basis h, e_1, e_2, e_3, e_4 (for D_4, we use the same as above), we get that $P(\mathcal{M}^{(2)})$ consists of e_i, $i = 1, 2, 3, 4$, and $e_4' = h - 2e_1 - 2e_2 - 2e_3 - 3e_4$. The lattice Weyl vector is $\rho = (5/2)h - 3e_1 - 3e_2 - 3e_3 - 5e_4$. See Figure 7 for the Gram graph.

Let $S = S_{5,32,b} = \langle -8\rangle \oplus D_4$. See [N2, Sec. 8.6]. Then $P(\mathcal{M}^{(2)})$ consists of e_i, $i = 1, 2, 3, 4$, and $f_i = h - 2e_1 - 2e_2 - 2e_3 - 4e_4 - e_i$, $i = 1, 2, 3$. The lattice Weyl vector is $\rho = (3/2)h - 3e_1 - 3e_2 - 3e_3 - 5e_4$. See Figure 8 for the Gram graph.

Let $S = S_{5,64} = \langle -16\rangle \oplus D_4$. See [N2, Sec. 8.5]. Then $P(\mathcal{M}^{(2)})$ consists of e_i, $i = 1, 2, 3, 4$; $f_i = h - 3e_1 - 3e_2 - 3e_3 - 5e_4 - e_i$, $i = 1, 2, 3$; $f_4 = h - 3e_1 - 3e_2 - 3e_3 - 6e_4$. The lattice Weyl vector is $\rho = h - 3e_1 - 3e_2 - 3e_3 - 5e_4$. See Figure 9 for the Gram graph.

Let $S = S_{5,128} = U(4) \oplus 3A_1$. See details in [N2, Sec. 8.3]. For the standard basis c_1, c_2, e_1, e_2, e_3, the Weyl vector $\rho = (c_1 + c_2 - e_1 - e_2 - e_3)/2$, and

$$P(\mathcal{M}^{(2)}) = \{\delta \in S \mid \delta^2 = 2 \ \&\ \delta \cdot \rho = -1\}.$$

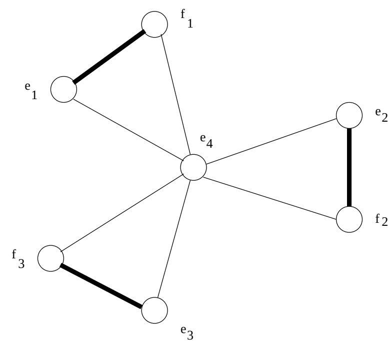

Figure 8: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $\langle -8 \rangle \oplus D_4$.

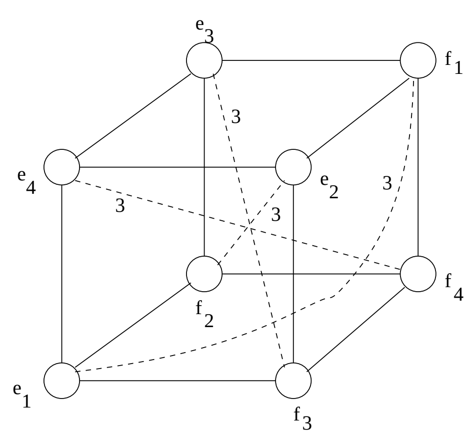

Figure 9: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $\langle -16 \rangle \oplus D_4$.

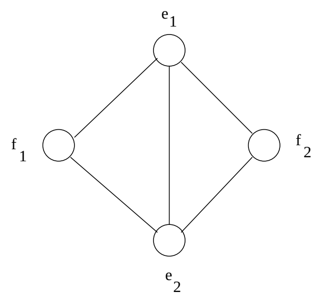

Figure 10: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $U(2) \oplus A_2$.

The set $P(\mathcal{M}^{(2)})$ consists of e_i, $c_1 - e_i$, $c_2 - e_i$, $i = 1, 2, 3$; $c_1 + c_2 - 2e_i - e_j$, $1 \leq i \neq j \leq 3$; $2c_1 + c_2 - 2e_1 - 2e_2 - 2e_3 + e_i$, $c_1 + 2c_2 - 2e_1 - 2e_2 - 2e_3 + e_i$, $i = 1, 2, 3$; $2c_1 + 2c_2 - 2e_1 - 2e_2 - 2e_3 - e_i$, $i = 1, 2, 3$. It has 24 elements, and its Gram graph is very symmetric.

For the remaining lattice $S = \langle 6 \rangle \oplus 2A_2$ of rank 5 of Sect. 3.2, the Gram graph $\Gamma(P(\mathcal{M}^{(2)}))$ is described in [N2, Sec. 8.6], and it has no the lattice Weyl vector.

Remaining cases of $\operatorname{rk} S = 4$. We use Vinberg's algorithm [V2] to calculate $P(\mathcal{M}^{(2)})$, and either to find the lattice Weyl vector or to prove that it does not exist.

Let $S = S_{4,12} = U(2) \oplus A_2$. For the standard basis c_1, c_2, e_1, e_2, the set $P(\mathcal{M}^{(2)})$ consists of e_1, e_2, $f_1 = c_1 - e_1 - e_2$, $f_2 = c_2 - e_1 - e_2$. The lattice Weyl vector is $\rho = (3/2)(c_1 + c_2) - e_1 - e_2$. See Figure 10 for the Gram graph.

Let $S = S_{4,16,b} = \langle -4 \rangle \oplus A_3$. For the standard basis h, e_1, e_2, e_3, the set $P(\mathcal{M}^{(2)})$ consists of e_1, e_2, e_3, $f_1 = h - 2e_1 - 2e_2 - e_3$, $f_3 = h - e_1 - 2e_2 - 2e_3$. The lattice Weyl vector $\rho = (3/2)h - (3/2)e_1 - 2e_2 - (3/2)e_3$. See Figure 11 for the Gram graph.

Let $S = S_{4,27,a} = U(3) \oplus A_2$. For the standard basis c_1, c_2, e_1, e_2, the set $P(\mathcal{M}^{(2)})$ consists of e_1, e_2, $f_1 = c_1 - e_1 - e_2$, $f_2 = c_2 - e_1 - e_2$. The lattice Weyl vector is $\rho = c_1 + c_2 - e_1 - e_2$. See Figure 12 for the Gram graph.

Let $S = S_{4,27,b} = \left\langle \begin{matrix} 0 & -3 \\ -3 & 2 \end{matrix} \right\rangle \oplus A_2$. For the standard basis c, e_1, e_2, e_3, the set $P(\mathcal{M}^{(2)})$ consists of e_1, e_2, e_3, $f_1 = c - e_2 - e_3$, $f_2 = c + e_1 - e_2 - 2e_3$, $f_3 = c + e_1 - 2e_2 - e_3$. The lattice Weyl vector is $\rho = c + e_1 - e_2 - e_3$. See Figure 13 for the Gram graph.

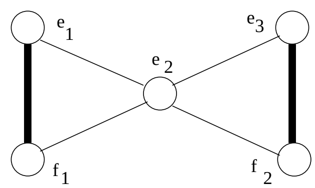

Figure 11: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $\langle -4 \rangle \oplus A_3$.

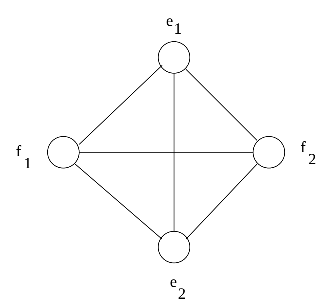

Figure 12: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $U(3) \oplus A_2$.

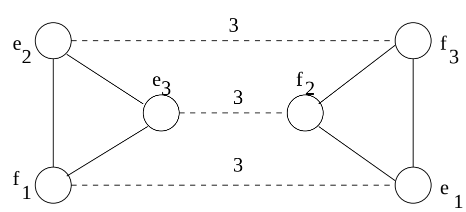

Figure 13: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $S_{4,27,b}$.

Let $S = S_{4,64,a} = U(4) \oplus 2A_1$. For the standard basis c_1, c_2, e_1, e_2, the set $P(\mathcal{M}^{(2)})$ consists of e_1, e_2, $c_1 - e_1$, $c_1 - e_2$, $c_2 - e_1$, $c_2 - e_2$, $c_1 + c_2 - 2e_1 - e_2$, $c_1 + c_2 - e_1 - 2e_2$. The lattice Weyl vector is $\rho = (c_1 + c_2 - e_1 - e_2)/2$. The Gram matrix $\Gamma(P(\mathcal{M}^{(2)}))$ is

$$
-\begin{pmatrix}
-2 & 0 & 2 & 0 & 2 & 0 & 4 & 2 \\
0 & -2 & 0 & 2 & 0 & 2 & 2 & 4 \\
2 & 0 & -2 & 0 & 2 & 4 & 0 & 2 \\
0 & 2 & 0 & -2 & 4 & 2 & 2 & 0 \\
2 & 0 & 2 & 4 & -2 & 0 & 0 & 2 \\
0 & 2 & 4 & 2 & 0 & -2 & 2 & 0 \\
4 & 2 & 0 & 2 & 0 & 2 & -2 & 0 \\
2 & 4 & 2 & 0 & 2 & 0 & 0 & -2
\end{pmatrix} \tag{3.6}
$$

which is very regular.

Let $S = S_{4,64,b} = \langle -8 \rangle \oplus 3A_1$. For the standard basis h, e_1, e_2, e_3, the set $P(\mathcal{M}^{(2)})$ consists of e_1, e_2, e_3, $h - e_1 - 2e_2$, $h - e_1 - 2e_3$, $h - 2e_1 - e_2$, $h - e_2 - 2e_3$, $h - 2e_1 - e_3$, $h - 2e_2 - e_3$, $2h - 3e_1 - 2e_2 - 2e_3$, $2h - 2e_1 - 3e_2 - 2e_3$, $2h - 2e_1 - 2e_2 - 3e_3$. The lattice Weyl vector is $\rho = (h - e_1 - e_2 - e_3)/2$. The Gram matrix $\Gamma(P(\mathcal{M}^{(2)}))$ is

$$
-\begin{pmatrix}
-2 & 0 & 0 & 2 & 2 & 4 & 0 & 4 & 0 & 6 & 4 & 4 \\
0 & -2 & 0 & 4 & 0 & 2 & 2 & 0 & 4 & 4 & 6 & 4 \\
0 & 0 & -2 & 0 & 4 & 0 & 4 & 2 & 2 & 4 & 4 & 6 \\
2 & 4 & 0 & -2 & 6 & 0 & 4 & 4 & 0 & 2 & 0 & 4 \\
2 & 0 & 4 & 6 & -2 & 4 & 0 & 0 & 4 & 2 & 4 & 0 \\
4 & 2 & 0 & 0 & 4 & -2 & 6 & 0 & 4 & 0 & 2 & 4 \\
0 & 2 & 4 & 4 & 0 & 6 & -2 & 4 & 0 & 4 & 2 & 0 \\
4 & 0 & 2 & 4 & 0 & 0 & 4 & -2 & 6 & 0 & 4 & 2 \\
0 & 4 & 2 & 0 & 4 & 4 & 0 & 6 & -2 & 4 & 0 & 2 \\
6 & 4 & 4 & 2 & 2 & 0 & 4 & 0 & 4 & -2 & 0 & 0 \\
4 & 6 & 4 & 0 & 4 & 2 & 2 & 4 & 0 & 0 & -2 & 0 \\
4 & 4 & 6 & 4 & 0 & 4 & 0 & 2 & 2 & 0 & 0 & -2
\end{pmatrix} \tag{3.7}
$$

which is very regular.

Let $S = S_{4,108} = U(6) \oplus A_2$. For the standard basis c_1, c_2, e_1, e_2, the set $P(\mathcal{M}^{(2)})$ consists of e_1, e_2, $f_3 = c_1 - e_1 - e_2$, $f_4 = c_2 - e_1 - e_2$, $f_5 = c_1 + c_2 - 2e_1 - 3e_2$, $f_6 = c_1 + c_2 - 3e_1 - 2e_2$. The lattice Weyl vector is $\rho = (c_1 + c_2)/2 - e_1 - e_2$. See Figure 14 for the Gram graph.

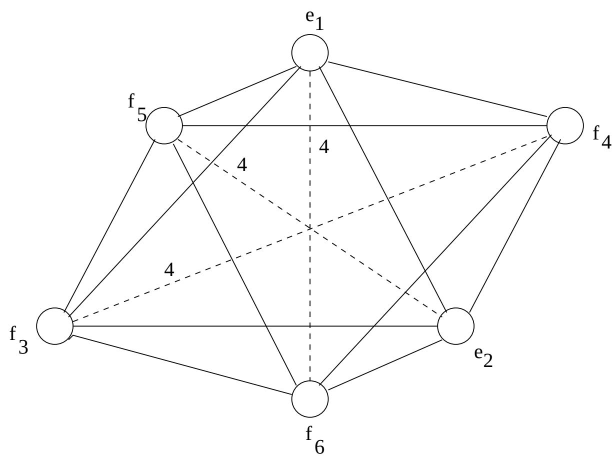

Figure 14: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $U(6) \oplus A_2$.

Let $S = S_{4,28} = \left\langle \begin{array}{cccc} -2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{array} \right\rangle$. For the standard basis h, e_1, e_2, e_3, the set $P(\mathcal{M}^{(2)})$ consists of e_1, e_2, e_3, $f_4 = h - e_1$, $f_5 = h - e_2$, $f_6 = h - e_3$. The lattice Weyl vector $\rho = h$. This case is anisotropic: the polyhedron $\mathcal{M}^{(2)}$ is compact, it has no vertices at infinity. See Figure 15 for the Gram graph.

Let us show that remaining three elliptically 2-reflective hyperbolic lattices of rank 4 of Sec. 3.2 don't have a lattice Weyl vector.

Let $S = U(3) \oplus 2A_1$. For the standard basis c_1, c_2, e_1, e_2, the set $P(\mathcal{M}^{(2)})$ consists of e_1, e_2, $f_3 = c_1 - e_1$, $f_4 = c_1 - e_2$, $f_5 = c_2 - e_1$, $f_6 = c_2 - e_2$, $f_7 = 2c_1 + 2c_2 - 3e_1 - 2e_2$, $f_8 = 2c_1 + 2c_2 - 2e_1 - 3e_2$. These calculations are important as itself.

Considering ρ for first 6 these elements, one can see that $\rho = -2c_1^* - 2c_2^* - e_1^* - e_2^*$. But, then $\rho \cdot f_7 = -3$. Thus, ρ does not exist.

Let $S = \langle -4 \rangle \oplus \langle 4 \rangle \oplus A_2$. For the standard basis h, e, e_1, e_2, the set $P(\mathcal{M}^{(2)})$ consists of e_1, e_2, $f_3 = h - e - e_1 - e_2$, $f_4 = h + e - e_1 - e_2$, $f_5 = h - e_1 - 2e_2$, $f_6 = h - 2e_1 - e_2$. These calculations are important as itself.

Considering ρ for first 4 these elements, one can see that $\rho = (3/4)h - e_1 - e_2$. But, then $\rho \cdot f_5 = 0$. Thus, ρ does not exist.

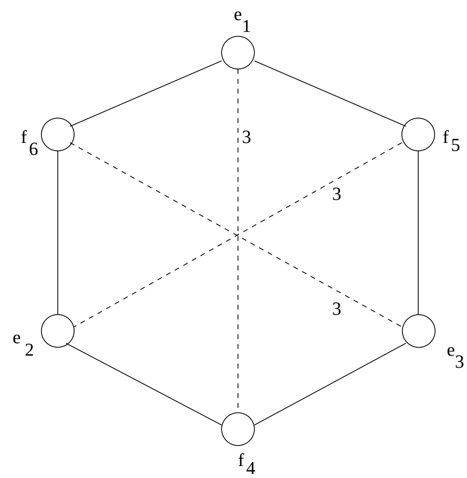

Figure 15: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $S_{4,28}$.

Let

$$S = \left\langle \begin{array}{rrrr} -12 & -2 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{array} \right\rangle .$$

For the standard basis h, e_1, e_2, e_3, the set $P(\mathcal{M}^{(2)})$ consists of e_1, e_2, e_3, $f_4 = h - 2e_1 - 2e_2 - e_3$, $f_5 = h - 2e_2 - 3e_3$, $f_6 = h - e_1 - 3e_2 - 2e_3$, $f_7 = 2h - 2e_1 - 4e_2 - 5e_3$, $f_8 = 2h - 3e_1 - 4e_2 - 4e_3$. This case is anisotropic: the polyhedron $\mathcal{M}^{(2)}$ is compact, it has no vertices at infinity. These calculations are important as itself.

Considering ρ for first 4 these elements, one can see that $\rho = (3h - 3e_1 - 7e_2 - 6e_3)/5$. But, then $\rho \cdot f_6 = 0$. Thus, ρ does not exist.

Remaining cases of $\operatorname{rk} S = 3$.

Firstly, let us consider isotropic cases.

Let $M_0 = U \oplus A_1$ (the lattice $M_0 = S_{3,2}$ in notations of Theorem 3.1). For the standard basis c_1, c_2 for U, and b for A_1, the set $P(\mathcal{M}^{(2)})$ is $a = -c_1 + c_2$, b, $c = c_1 - b$. The lattice Weyl vector $\rho = 3c_1 + 2c_2 - b/2$ and $\rho^2 = -23/2$. The set $P(\mathcal{M}^{(2)})$ has the Gram matrix $\Gamma(P(\mathcal{M}^{(2)}))$ equals to

$$A_{1,0} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix} , \tag{3.8}$$

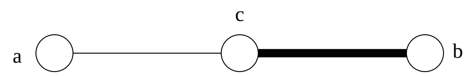

Figure 16: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $U \oplus A_1$.

and the Graph graph which is shown in Figure 16.

All isotropic elliptically reflective hyperbolic lattices of rank 3 are sublattices of $M_0 = U \oplus A_1$ of finite index which are described in [N5]. Let $[v_1, v_2, \ldots, v_n]$ be a sublattice generated by $v_1, \ldots, v_n$. Let $M_{k,l,m} = [ka,\ lb,\ mc] \subset M_0$ where $k,\ l,\ m \in \mathbb{N}$.

By [N5], up to the action of $W^{(2)}(M_0) = O^+(M_0)$, all elliptically reflective sublattices of M_0 are $M_{1,1,m}$, $m = 1,\ 2,\ 3,\ 4,\ 6,\ 8$; $M_{1,l,1}$, $l = 2,\ 3,\ 4,\ 5,\ 6,\ 9$; $M_{k,1,1}$, $k = 4,\ 5,\ 6,\ 7,\ 8,\ 10,\ 12$; $M_{2,1,2}$; $M_{4,1,2}$; $M_{6,1,2}$; $M'_{4,1,2} = [2a + c, b, 2c]$; $M'_{6,1,2} = [3a + c, b, 2c]$ (24 sublattices). See [N5, Table 3]. For these sublattices, only the following are isomorphic as lattices: $M_{4,1,1} \cong M_{2,1,2}$, $M_{8,1,1} \cong M_{4,1,2}$, $M_{12,1,1} \cong M_{6,1,2}$, $M_{6,1,1} \cong M'_{6,1,2}$. Thus, there are 20 isotropic non-isomorphic such lattices. See [N5, Theorem 2.5]. The last isomorphism was missed in this Theorem. For all these 24 sublattices, the fundamental polygons $\mathcal{M}^{(2)}$ and $P(\mathcal{M}^{(2)})$ are calculated in terms of $\Delta^{(2)}(M_0)$ in [N5, Figures 5–10]. For $M'_{6,1,2} = [3a+c, b, 2c]$, the correct polygon will be $PQRT_1$ in Figure 10 (see [N5, Table 3]).

Using these results, one can find all these lattices which have the lattice Weyl vector, and identify them with the isotropic lattices of the rank three of Theorem 3.1. Below, we do these calculations.

The case $S = M_{1,1,1} = M_0 = S_{3,2}$ was considered above.

Let $S = M_{1,1,2}$ (equals to $S_{3,8,a} = U(2) \oplus A_1$). Then $P(\mathcal{M}^{(2)})$ is $b + 2c$, a, b with the Gram matrix

$$A_{2,0} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}. \tag{3.9}$$

The lattice Weyl vector is $\rho = a + (5/2)b + 3c$ and $\rho^2 = -7/2$. This is equal to $S_{3,8,a} = U(2) \oplus A_1$. For its standard basis c_1, c_2, e, the set $P(\mathcal{M}^{(2)})$ consists of e, $c_1 - e$, $c_2 - e$ with the Gram matrix $A_{2,0}$, the $\rho = c_1 + c_2 - e/2$ with $\rho^2 = -7/2$. These lattices are isomorphic since they have equal determinants, and their matrices in their generators above are the same: $A_{2,0}$.

Let $S = M_{1,1,3}$. Then $P(\mathcal{M}^{(2)})$ is a, b, $2b+3c$, $2a+3b+6c$. There is no the lattice Weyl vector.

Let $S = M_{1,1,4}$ (equals to $S_{3,32,b} = \langle -8\rangle \oplus 2A_1$). Then $P(\mathcal{M}^{(2)})$ is b, $3b+4c$, $a+2b+4c$, a. with the Gram matrix

$$A_{2,I} = \begin{pmatrix} 2 & -2 & -4 & 0 \\ -2 & 2 & 0 & -4 \\ -4 & 0 & 2 & -2 \\ 0 & -4 & -2 & 2 \end{pmatrix}. \tag{3.10}$$

The lattice Weyl vector is $\rho = (1/2)a+(3/2)b+2c$ and $\rho^2 = -1$. This is equal to $S_{3,32,b} = \langle -8\rangle \oplus 2A_1$. For its standard basis h, e_1, e_2, the set $P(\mathcal{M}^{(2)})$ consists of e_1, $h-e_1-2e_2$, $h-2e_1-e_2$, e_2 with the Gram matrix $A_{2,I}$, the $\rho = (h-e_1-e_2)/2$ with $\rho^2 = -1$.

Let $S = M_{1,1,6}$. Then $P(\mathcal{M}^{(2)})$ is a, b, $5b+6c$, $3a+16b+24c$, $4a+15b+24c$, $2a+3b+6c$. There is no the lattice Weyl vector.

Let $S = M_{1,1,8}$ (equals to $S_{3,128,b} = \langle -32\rangle \oplus 2A_1$). Then $P(\mathcal{M}^{(2)})$ is a, $3a+4b+8c$, $4a+9b+16c$, $4a+15b+24c$, $3a+16b+24c$, $a+12b+16c$, $7b+8c$, b with the Gram matrix

$$A_{2,III} = \begin{pmatrix} 2 & -2 & -8 & -16 & -18 & -14 & -8 & 0 \\ -2 & 2 & 0 & -8 & -14 & -18 & -16 & -8 \\ -8 & 0 & 2 & -2 & -8 & -16 & -18 & -14 \\ -16 & -8 & -2 & 2 & 0 & -8 & -14 & -18 \\ -18 & -14 & -8 & 0 & 2 & -2 & -8 & -16 \\ -14 & -18 & -16 & -8 & -2 & 2 & 0 & -8 \\ -8 & -16 & -18 & -14 & -8 & 0 & 2 & -2 \\ 0 & -8 & -14 & -18 & -16 & -8 & -2 & 2 \end{pmatrix}. \tag{3.11}$$

The lattice Weyl vector is $\rho = (1/4)a + b + (3/2)c$ with $\rho^2 = -1/8$. This is equal to $S_{3,128,b} = \langle -32\rangle \oplus 2A_1$. For its standard basis h, e_1, e_2, the set $P(\mathcal{M}^{(2)})$ consists of e_1, $h-e_1-4e_2$, $2h-4e_1-7e_2$, $3h-8e_1-9e_2$, $3h-9e_1-8e_2$, $2h-7e_1-4e_2$, $h-4e_1-e_2$, e_2 with the Gram matrix $A_{2,III}$, the $\rho = (3/16)h - e_1/2 - e_2/2$ with $\rho^2 = -1/8$.

Let $S = M_{1,2,1}$ (equals to $S_{3,8,b} = (\langle -24\rangle \oplus A_2)[1/3,-1/3,1/3]$). Then $P(\mathcal{M}^{(2)})$ is c, $2b+c$, a with the Gram matrix

$$A_{1,I} = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \tag{3.12}$$

The lattice Weyl vector is $\rho = a + 3b + 3c$ with $\rho^2 = -4$. This is equal to $S_{3,8,b} = (\langle -24\rangle \oplus A_2)[1/3, -1/3, 1/3]$. For the standard basis h, e_1, e_2 of $\langle -24\rangle \oplus A_2$, the set $P(\mathcal{M}^{(2)})$ is e_2, $(h-4e_1-5e_2)/3$, e_1 with the Gram matrix $A_{1,I}$, the $\rho = h/2 - e_1 - e_2$ with $\rho^2 = -4$.

Let $S = M_{1,3,1}$ (equals to $S_{3,18} = U(3) \oplus A_1$). Then $P(\mathcal{M}^{(2)})$ is $3b + 2c$, a, c with the Gram matrix

$$A_{3,0} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix} . \tag{3.13}$$

The lattice Weyl vector is $\rho = (2/3)a+(5/2)b+(7/3)c$ with $\rho^2 = -13/6$. This is equal to $S_{3,18} = U(3)\oplus A_1$. For its standard basis c_1, c_2, e, the set $P(\mathcal{M}^{(2)})$ is e, $c_1 - e$, $c_2 - e$ with the Gram matrix $A_{3,0}$, the $\rho = (2/3)c_1 + (2/3)c_2 - e/2$ with $\rho^2 = -13/6$.

Let $S = M_{1,4,1}$. Then $P(\mathcal{M}^{(2)})$ is c, a, $3a + 12b + 8c$, $4b + 3c$. There is no the lattice Weyl vector.

Let $S = M_{1,5,1}$. Then $P(\mathcal{M}^{(2)})$ is $-5b-4c$, $5a+40b+29c$, $16a+150b+111c$, $4a + 45b + 34c$, $a + 20b + 16c$, $10b + 9c$. There is no the lattice Weyl vector.

Let $S = M_{1,6,1}$ (equals to $S_{3,72} = \langle -24\rangle \oplus A_2$). Then $P(\mathcal{M}^{(2)})$ is a, $a + 6b + 4c$, $6b + 5c$, c with the Gram matrix

$$A_{3,I} = \begin{pmatrix} 2 & -2 & -5 & -1 \\ -2 & 2 & -1 & -5 \\ -5 & -1 & 2 & -2 \\ -1 & -5 & -2 & 2 \end{pmatrix} . \tag{3.14}$$

The lattice Weyl vector is $\rho = (1/3)a + 2b + (5/3)c$ with $\rho^2 = -2/3$. This is equal to $S_{3,72} = \langle -24\rangle \oplus A_2$. For its standard basis h, e_1, e_2, the set $P(\mathcal{M}^{(2)})$ is e_1, $h - 3e_1 - 4e_2$, $h - 4e_1 - 3e_2$, e_2 with the Gram matrix $A_{3,I}$, the $\rho = h/3 - e_1 - e_2$ with $\rho^2 = -2/3$.

Let $S = M_{1,9,1}$. Then $P(\mathcal{M}^{(2)})$ is c, a, $2a + 9b + 6c$, $7a + 54b + 39c$, $8a + 72b + 53c$, $4a + 45b + 34c$, $5a + 72b + 56c$, $3a + 54b + 43c$, $9b + 8c$. There is no the lattice Weyl vector.

Let $S = M_{4,1,1}$ (equals to $S_{3,32,a} = U(4) \oplus A_1$). Then $P(\mathcal{M}^{(2)})$ consists of b, c, $4a + 3b + 4c$ with the Gram matrix

$$A_{1,II} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} . \tag{3.15}$$

The lattice Weyl vector is $\rho = 2a+2b+(5/2)c$ with $\rho^2 = -3/2$. This is equal to $S_{3,32,a} = U(4) \oplus A_1$. For its standard basis c_1, c_2, e, the set $P(\mathcal{M}^{(2)})$ is e, $c_1 - e$, $c_2 - e$ with the Gram matrix $A_{1,II}$, the $\rho = (c_1 + c_2 - e)/2$ with $\rho^2 = -3/2$.

Let $S = M_{5,1,1}$. Then $P(\mathcal{M}^{(2)})$ is b, c, $20a+15b+24c$, $5a+4b+5c$. There is no the lattice Weyl vector.

Let $S = M_{6,1,1}$. Then $P(\mathcal{M}^{(2)})$ is b, c, $12a+9b+14c$, $6a+5b+6c$. There is no the lattice Weyl vector.

Let $S = M_{8,1,1}$ (equals to $S_{3,128,a} = U(8) \oplus A_1$). Then $P(\mathcal{M}^{(2)})$ is b, c, $8a + 6b + 9c$, $8a + 7b + 8c$ with the Gram matrix

$$A_{2,II} = \begin{pmatrix} 2 & -2 & -6 & -2 \\ -2 & 2 & -2 & -6 \\ -6 & -2 & 2 & -2 \\ -2 & -6 & -2 & 2 \end{pmatrix}. \tag{3.16}$$

The lattice Weyl vector is $\rho = 2a + (7/4)b + (9/4)c$ with $\rho^2 = -1/2$. This is equal to $S_{3,128,a} = U(8) \oplus A_1$. For its standard basis c_1, c_2, e, the set $P(\mathcal{M}^{(2)})$ is e, $c_1 - e$, $c_1 + c_2 - 3e$, $c_2 - e$ with the Gram matrix $A_{2,II}$, the $\rho = (c_1 + c_2)/4 - e/2$ with $\rho^2 = -1/2$.

Let $S = M_{10,1,1}$. Then $P(\mathcal{M}^{(2)})$ is b, c, $20a + 15b + 24c$, $30a + 23b + 34c$, $60a + 47b + 66c$, $40a + 32b + 43c$, $40a + 33b + 42c$, $10a + 9b + 10c$. There is no the lattice Weyl vector.

Let $S = M_{12,1,1}$ (equals to $S_{3,288} = U(12) \oplus A_1$). Then $P(\mathcal{M}^{(2)})$ is b, c, $12a + 9b + 14c$, $24a + 19b + 26c$, $24a + 20b + 25c$, $12a + 11b + 12c$ with the Gram matrix

$$A_{3,II} = \begin{pmatrix} 2 & -2 & -10 & -14 & -10 & -2 \\ -2 & 2 & -2 & -10 & -14 & -10 \\ -10 & -2 & 2 & -2 & -10 & -14 \\ -14 & -10 & -2 & 2 & -2 & -10 \\ -10 & -14 & -10 & -2 & 2 & -2 \\ -2 & -10 & -14 & -10 & -2 & 2 \end{pmatrix}. \tag{3.17}$$

The lattice Weyl vector is $\rho = 2a + (5/3)b + (13/6)c$ with $\rho^2 = -1/6$. This is equal to $S_{3,228} = U(12) \oplus A_1$. For its standard basis c_1, c_2, e, the set $P(\mathcal{M}^{(2)})$ is e, $c_2 - e$, $c_1 + 2c_2 - 5e$, $2c_1 + 2c_2 - 7e$, $2c_1 + c_2 - 5e$, $c_1 - e$ with the Gram matrix $A_{3,II}$, the $\rho = (c_1 + c_2)/6 - e/2$ with $\rho^2 = -1/6$.

Let $S = M_{2,1,2}$. Then $P(\mathcal{M}^{(2)})$ is b, $b+2c$, $2a+b+2c$ with the Gram matrix $A_{1,II}$. The lattice Weyl vector is $\rho = a + (3/2)b + 2c$ with $\rho^2 = -1/6$. This case is isomorphic to $M_{4,1,1}$ above.

Let $S = M_{4,1,2}$. Then $P(\mathcal{M}^{(2)})$ is b, $b+2c$, $4a+3b+6c$, $4a+3b+4c$ with the Gram matrix $A_{2,II}$. The lattice Weyl vector is $\rho = a+b+(3/2)c$ with $\rho^2 = -1/2$. This case is isomorphic to $M_{8,1,1}$ above.

Let $S = M_{6,1,2}$. Then $P(\mathcal{M}^{(2)})$ is b, $b+2c$, $6a+5b+10c$, $12a+9b+16c$, $12a+9b+14c$, $6a+5b+6c$ with the Gram matrix $A_{3,II}$. The lattice Weyl vector is $\rho = a+(5/6)b+(4/3)c$ with $\rho^2 = -1/6$. This case is isomorphic to $M_{12,1,1}$ above.

Let $S = M'_{4,1,2} = [2a+c, b, 2c]$ (equals to $S_{3.32,c} = U(8)[1/2,1/2] \oplus A_1$). Then $P(\mathcal{M}^{(2)})$ is b, $b+2c$, $4a+3b+6c$, $4a+3b+4c$ with the Gram matrix $A_{2,II}$ (see (3.16)). The lattice Weyl vector is $\rho = a+b+(3/2)c$ with $\rho^2 = -1/2$. These are the same as for $M_{4,1,2}$, but $M_{4,1,2} \subset M'_{4,1,2}$ is only a sublattice of the index two. The lattice $M'_{4,1,2}$ is not generated by its elements with square 2. This is equal to $S_{3,32,c} = U(8)[1/2,1/2] \oplus A_1$. For the standard basis c_1, c_2, e, of $U(8) \oplus A_1$, the set $P(\mathcal{M}^{(2)})$ is e, $c_1 - e$, $c_1 + c_2 - 3e$, $c_2 - e$ with the Gram matrix $A_{2,II}$, the $\rho = (c_1+c_2)/4 - e/2$ with $\rho^2 = -1/2$.

Let $S = M'_{6,1,2} = [3a+c, b, 2c]$. Then $P(\mathcal{M}^{(2)})$ is b, $b+2c$, $6a+5b+10c$, $12a+9b+16c$, $3a+b+3c$. Their Gram matrix is the same as for $P(\mathcal{M}^2)$ of the lattice $M_{6,1,1}$ above. Thus, these lattices are isomorphic, and there are no the lattice Weyl vector.

Now, let us consider anisotropic cases. According to [N5], there are 6 anisotropic elliptically reflective lattices. For all of them the sets $P(\mathcal{M}^2)$ and their Gram matrices are calculated in [N5]. Using these calculations, one can find the lattice Weyl vector ρ or prove that it does not exist. We give these calculations below. For Gram matrices below we use notations B_i from [GN8]. .

Let $S = S_{3,12} = \langle -4 \rangle \oplus A_2$ (it is S_5 in notations of [N5]). For its standard basis h, e_1, e_2, the set $P(\mathcal{M}^{(2)})$ is e_2, $h-2e_1-e_2$, $h-e_1-2e_2$, e_1 with the Gram matrix

$$B_1 = \begin{pmatrix} 2 & 0 & -3 & -1 \\ 0 & 2 & -1 & -3 \\ -3 & -1 & 2 & 0 \\ -1 & -3 & 0 & 2 \end{pmatrix}. \qquad (3.18)$$

The lattice Weyl vector is $\rho = h - e_1 - e_2$ with $\rho^2 = -2$.

Let $S = S_{3,24} = \langle -6 \rangle \oplus 2A_1$ (it is S_1 in [N5]). For its standard basis h, e_1,

e_2, the set $P(\mathcal{M}^{(2)})$ is e_1, e_2, $h-2e_1$, $2h-3e_1-2e_2$, $2h-2e_1-3e_2$, $h-2e_2$ with the Gram matrix

$$B_3 = \begin{pmatrix} 2 & 0 & -4 & -6 & -4 & 0 \\ 0 & 2 & 0 & -4 & -6 & -4 \\ -4 & 0 & 2 & 0 & -4 & -6 \\ -6 & -4 & 0 & 2 & 0 & -4 \\ -4 & -6 & -4 & 0 & 2 & 0 \\ 0 & -4 & -6 & -4 & 0 & 2 \end{pmatrix}. \tag{3.19}$$

The lattice Weyl vector is $\rho = (h-e_1-e_2)/2$ with $\rho^2 = -1/2$.

Let $S = S_{3,36} = \langle -12\rangle \oplus A_2$ (it is S_3 in [N5]). For its standard basis h, e_1, e_2, the set $P(\mathcal{M}^{(2)})$ is e_1, e_2, $h-3e_1-2e_2$, $h-2e_1-3e_2$ with the Gram matrix

$$B_2 = \begin{pmatrix} 2 & -1 & -4 & -1 \\ -1 & 2 & -1 & -4 \\ -4 & -1 & 2 & -1 \\ -1 & -4 & -1 & 2 \end{pmatrix}. \tag{3.20}$$

The lattice Weyl vector is $\rho = h/2 - e_1 - e_2$ with $\rho^2 = -1$.

Let $S = S_{3,108} = \langle -36\rangle \oplus A_2$ (it is S_2 in [N5]). For its standard basis h, e_1, e_2, the set $P(\mathcal{M}^{(2)})$ is e_1, e_2, $h-5e_1-3e_2$, $2h-9e_1-8e_2$, $2h-8e_1-9e_2$, $h-3e_1-5e_2$ with the Gram matrix

$$B_4 = \begin{pmatrix} 2 & -1 & -7 & -10 & -7 & -1 \\ -1 & 2 & -1 & -7 & -10 & -7 \\ -7 & -1 & 2 & -1 & -7 & -10 \\ -10 & -7 & -1 & 2 & -1 & -7 \\ -7 & -10 & -7 & -1 & 2 & -1 \\ -1 & -7 & -10 & -7 & -1 & 2 \end{pmatrix}. \tag{3.21}$$

The lattice Weyl vector is $\rho = h/4 - e_1 - e_2$ with $\rho^2 = -1/4$.

Let $S = (\langle -60\rangle \oplus A_2)[1/3, -1/3, 1/3]$ (it is S_4 in [N5]). For the standard basis h, e_1, e_2 of $\langle -60\rangle \oplus A_2$, the set $P(\mathcal{M}^{(2)})$ is e_1, e_2, $(h-7e_1-5e_2)/3$, $(2h-8e_1-13e_2)/3$ with the Gram matrix U_4 of the lattice S_4 in [N5, Theorem 1.2]. There is no the lattice Weyl vector.

Let $S = (\langle -132\rangle \oplus A_2)[1/3, -1/3, 1/3]$ (it is S_6 in [N5]). For the standard basis h, e_1, e_2 of $\langle -132\rangle \oplus A_2$, the set $P(\mathcal{M}^{(2)})$ is e_1, e_2, $(h-10e_1-5e_2)/3$, $(2h-17e_1-16e_2)/3$, $h-7e_1-9e_2$, $(2h-11e_1-19e_2)/3$ with the Gram matrix U_6 of the lattice S_6 in [N5, Theorem 1.2]. There is no the lattice Weyl vector.

These completes the proof of Theorem 3.1 with description of the corresponding Gram matrices (equivalently, Gram graphs) $\Gamma(P(\mathcal{M}^{(2)}))$, and the lattice Weyl vectors ρ.

The concluding remark.

By [GN5, Theorems 1.2.1, 1.3.1], there are the only two more Gram matrices $A_{1,III}$ and $A_{3,III}$ for fundamental chambers $\mathcal{M}$ of reflection subgroups $W \subset W^{(2)}(S)$ of finite index for elliptically 2-reflective hyperbolic lattices S of rank 3 with lattice Weyl vectors for $P(\mathcal{M})$. They are as follows.

For the lattice $M_{1,1,3}$ above, let us take

$$P(\mathcal{M}) = \{2a + 3b + 6c,\ a + 6b + 9c,\ 5b + 6c,\ b,\ a\} \tag{3.22}$$

It has the Gram matrix

$$A_{1,III} = \begin{pmatrix} 2 & -2 & -6 & -6 & -2 \\ -2 & 2 & 0 & -6 & -7 \\ -6 & 0 & 2 & -2 & -6 \\ -6 & -6 & -2 & 2 & 0 \\ -2 & -7 & -6 & 0 & 2 \end{pmatrix}, \tag{3.23}$$

and the lattice Weyl vector $\rho = (1/3)a + (7/6)b + (5/3)c$ with $\rho^2 = -7/18$. The polygon $\mathcal{M}$ is obtained from the described above polygon $\mathcal{M}^{(2)}$ for $M_{1,1,3}$ by the reflection at $2b + 3c$. Thus, $[W^{(2)}(M_{1,1,3}) : W] = 2$.

For the lattice $M_{1,6,1}$ above, let us take

$$P(\mathcal{M}) = \{a,\ 3a + 12b + 8c,\ 5a + 30b + 21c,\ 7a + 54b + 39c,\ 8a + 72b + 53c,$$
$$8a + 84b + 63c,\ 7a + 84b + 64c,\ 5a + 72b + 56c,$$
$$3a + 54b + 43c,\ a + 30b + 25c,\ 12b + 11c,\ c\}\ . \tag{3.24}$$

It has the Gram matrix

$$A_{3,III} =$$

$$
-\begin{pmatrix}
-2 & 2 & 11 & 25 & 37 & 47 & 50 & 46 & 37 & 23 & 11 & 1 \\
2 & -2 & 1 & 11 & 23 & 37 & 46 & 50 & 47 & 37 & 25 & 11 \\
11 & 1 & -2 & 2 & 11 & 25 & 37 & 47 & 50 & 46 & 37 & 23 \\
25 & 11 & 2 & -2 & 1 & 11 & 23 & 37 & 46 & 50 & 47 & 37 \\
37 & 23 & 11 & 1 & -2 & 2 & 11 & 25 & 37 & 47 & 50 & 46 \\
47 & 37 & 25 & 11 & 2 & -2 & 1 & 11 & 23 & 37 & 46 & 50 \\
50 & 46 & 37 & 23 & 11 & 1 & -2 & 2 & 11 & 25 & 37 & 47 \\
46 & 50 & 47 & 37 & 25 & 11 & 2 & -2 & 1 & 11 & 23 & 37 \\
37 & 47 & 50 & 46 & 37 & 23 & 11 & 1 & -2 & 2 & 11 & 25 \\
23 & 37 & 46 & 50 & 47 & 37 & 25 & 11 & 2 & -2 & 1 & 11 \\
11 & 25 & 37 & 47 & 50 & 46 & 37 & 23 & 11 & 1 & -2 & 2 \\
1 & 11 & 23 & 37 & 46 & 50 & 47 & 37 & 25 & 11 & 2 & -2
\end{pmatrix} \qquad (3.25)
$$

and the lattice Weyl vector $\rho = (1/6)a + (7/4)b + (4/3)c$ with $\rho^2 = -1/24$. The polygon $\mathcal{M}$ is obtained from the described above polygon $\mathcal{M}^{(2)}$ for $M_{1,6,1}$ by the group D_3 of the order 6 generated by reflections in $a + 6b + 4c$ and $6b + 5c$. Thus, $W^{(2)}(M_{1,6,1})/W \cong D_3$.

Remark 3.3. By Remark 3.1, ellptically 2-reflective hyperbolic lattices S with lattice Weyl vector from the list of Theorem 3.1, give all Picard lattices $S_X = S(-1)$ of K3 surfaces X over $\mathbb{C}$ with finite automorphism group and $\operatorname{rk} S_X \geq 3$ such that all non-singular rational curves E on X have the same degree $E \cdot h$ with respect to an ample element $h = t\rho \in S_X$ for some $t > 0$ from $\mathbb{Q}$ where $\rho \in S_X \otimes \mathbb{Q}$ is the lattice Weyl vector.

See Remark 6.7 below about their arithmetic mirror symmetric K3 surfaces.

Remark 3.4. Finiteness (or almost finiteness) of the set of hyperbolic reflection groups $W \subset W(S)$ of restricted arithmetic type and $P(\mathcal{M})$ for the fundamental chamber $\mathcal{M}$ of W with lattice Weyl vector ρ of elliptic ($\rho^2 < 0$), parabolic ($\rho^2 = 0$) and hyperbolic ($\rho^2 > 0$) types was proved in [N8], [N10].

For $\operatorname{rk} S = 3$, such cases of elliptic type were classified by D. Allcock in [Al2].

3.5 Lorentzian Kac–Moody superalgebras corresponding to lattices of Theorem 3.1

Let S be one of lattices of Theorem 3.1, $\mathcal{M}^{(2)}$ the fundamental chamber for $W^{(2)}(S)$, and $P(\mathcal{M}^{(2)})$ the set of perpendicular vectors to $\mathcal{M}^{(2)}$ with square 2.

The Gram matrix (or the corresponding graph) $\Gamma(P(\mathcal{M}^{(2)}))$ is a hyperbolic symmetric generalized Cartan matrix $A(S)$ with the lattice Weyl vector $\rho(S)$, described in Sec. 3.4. In [GN1] — [GN8], for lattices S of the rank three and with the generalized Cartan matrices $A_{i,j}$, $i=1,\,2,\,3,\,4,\ j=0,\,I,\,II$, and some other parabolic cases below, we additionally constructed appropriate automorphic forms $\Phi(S)$ on appropriate IV type symmetric domains. Together $A(S)$ and $\Phi(S)$ defined the corresponding Lorentzian Kac–Moody Lie superalgebras $g(S)$ which are graded by the hyperbolic lattice S.

They are as follows.

$A_{1,0}$: The lattice is $S_{3,2}=U\oplus A_1$, The automorphic form is $\Phi_{1,0,\overline{0}}=\Delta_{35}$.

$A_{2,0}$: The lattice is $S_{3,8,a}=U(2)\oplus A_1$. The automorphic form is $\Phi_{2,0,\overline{0}}=\Delta_{11}$.

$A_{3,0}$: The lattice is $S_{3,18}=U(3)\oplus A_1$. The automorphic form is $\Phi_{3,0,\overline{0}}=D_6\Delta_1$.

$A_{4,0,\overline{0}}=A_{1,II}$: The lattice is $S_{3,32,a}=U(4)\oplus A_1$. The automorphic form is $\Phi_{4,0,\overline{0}}=\Delta_5^{(4)}$.

$A_{1,I}$: The lattice is $S_{3,8,b}=(\langle -24\rangle\oplus A_2)[1/3,-1/3,1/3]$. The automorphic form is $\widetilde{\Phi}_{1,I,\overline{0}}=\Phi_{1,0,\overline{0}}(Z)/\Phi_{1,II,\overline{0}}(2Z)=\Delta_{35}(Z)/\Delta_5(2Z)$.

$A_{2,I}$: The lattice $S_{3,32,b}=\langle -8\rangle\oplus 2A_1$. The automorphic form is $\widetilde{\Phi}_{2,I,\overline{0}}=\Phi_{2,0,\overline{0}}(Z)/\Phi_{2,II,\overline{0}}(2Z)=\Delta_{11}(Z)/\Delta_2(2Z)$.

$A_{3,I}$: The lattice is $S_{3,72}=\langle -24\rangle\oplus A_2$. The automorphic form is $\widetilde{\Phi}_{3,I,\overline{0}}=\Phi_{3,0,\overline{0}}(Z)/\Phi_{3,II,\overline{0}}(2Z)=D_6(Z)\Delta_1(Z)/\Delta_1(2Z)$.

$A_{4,I}=A_{2,II}$: The lattice is $S_{3,128,a}=U(8)\oplus A_1$ (or $S_{3,32,c}=U(8)[1/2,1/2]$ $\oplus A_1$ which has the same elements of square 2). The automorphic form is $\widetilde{\Phi}_{4,I,\overline{0}}=\Phi_{4,0,\overline{0}}(Z)/\Phi_{4,II,\overline{0}}(2Z)=\Delta_5^{(4)}(Z)/\Delta_{1/2}(2Z)$.

$A_{1,II}$: The lattice is $S_{3,32,a}=U(4)\oplus A_1$. The automorphic form is $\Phi_{1,II,\overline{0}}=\Delta_5$.

$A_{2,II}$: The lattice is $S_{3,128,a}=U(8)\oplus A_1$. The automorphic form is $\Phi_{2,II,\overline{0}}=\Delta_2$.

$A_{3,II}$: The lattice is $S_{3,288}=U(12)\oplus A_1$. The automorphic form is $\Phi_{3,II,\overline{0}}=\Delta_1$.

Additional parabolic cases:

$A_{4,II}$ is $\Gamma(P(\mathcal{M}^{(2)}))$ for $U(16)\oplus A_1$. The lattice is $U(16)\oplus A_1$. The automorphic form is $\Phi_{4,II,\overline{0}}=\Delta_{1/2}$.

$A=\Gamma(P(\mathcal{M}^{(2)}))$ for $U\oplus\langle 4\rangle$ (the $\mathcal{M}^{(2)}$ is infinite polygon with angles $\pi/2$). The lattice is $U\oplus\langle 4\rangle$. The automorphic form is $\Phi_{2,\overline{1}}=\Psi_{12}^{(2)}$.

$A=\Gamma(P(\mathcal{M}^{(2)}))$ for $U\oplus\langle 6\rangle$ (the $\mathcal{M}^{(2)}$ is infinite polygon with angles $\pi/3$). The lattice is $U\oplus\langle 6\rangle$. The automorphic form is $\Phi_{3,\overline{1}}=\Psi_{12}^{(3)}$.

$A=\Gamma(P(\mathcal{M}^{(2)})$ for $U\oplus\langle 8\rangle$ (the $\mathcal{M}^{(2)}$ is infinite polygon with angles 0). The lattice is $U\oplus\langle 8\rangle$. The automorphic form is $\Phi_{4,\overline{1}}=\Psi_{12}^{(4)}$.

Using these basic automorphic forms, in [GN1] — [GN8], we constructed many other Lorentzian Kac–Moody superalgebras. Roughly speaking, they are obtained by products and quotients of these basic forms.

We want to extend these results to other lattices of Theorem 3.1, especially to the higher ranks ≥ 4. In Sections 4–6, we construct 2-reflective automorphic forms for 2-reflective hyperbolic lattices of Theorem 3.1. Using different base functions, we get six series of such automorphic forms.

1) For the lattices $U\oplus K$,

$$K=A_1,\,2A_1,A_2;\;3A_1,\,A_3;\;4A_1,\,2A_2,\,A_4,\,D_4;\;A_5,D_5;$$

$$3A_2,\,2A_3,\,A_6,\,D_6,\,E_6;\;A_7,\,D_7,\,E_7;\;2D_4,\,D_8,\,E_8,\,2E_8$$

and $U(2)\oplus 2D_4$, it is done in Theorem 4.3 and Theorem 5.1.

2) For the lattices $\langle -2\rangle\oplus kA_1$, $2\leq k\leq 9$ (the case $k=9$ is parabolic), it is done in Theorem 6.1.

3) For the lattices $U(4)\oplus kA_1$, $1\leq k\leq 4$ (the case $k=4$ is parabolic), it is done in Theorem 6.4.

4) For the lattices $U(3)\oplus A_2$, $U(3)\oplus 2A_2$, $U(3)\oplus 3A_2$ (the last case is parabolic), it is done in Theorem 6.5.

5) For the lattices $U(2)\oplus D_4$ and $U(4)\oplus D_4$, it is done in Theorem 6.2 and Theorem 6.3.

6) For the 2-reflective lattices of parabolic type $U\oplus K$,

$$K=A_1(2),\;A_1(3),\;A_1(4),\;D_2(2),\;A_2(2),\;A_2(3),\;A_3(2),\;D_4(2),\;E_8(2),$$

it is done in Theorem 4.4.

4 The strongly reflective modular forms

Lorentzian Kac-Moody algebras give automorphic corrections of hyperbolic Kac-Moody algebras since their Kac-Weyl-Borcherds denominator functions are automorphic forms with respect to arithmetic orthogonal groups of signature $(n, 2)$ (see Section 2). Here we give the general set-up for construction of corresponding automorphic forms which we shortly call as *automorphic corrections* of the hyperbolic root systems. We note that the signature $(2, n)$ is usually used in algebraic geometry and the theory of automorphic forms. The signatures $(n, 0)$, $(n, 1)$ and $(n, 2)$ are natural in the theory of Lie algebras.

Let T be an integral lattice with a quadratic form of signature $(n, 2)$ and let

$$\Omega(T) = \{[Z] \in \mathbb{P}(T \otimes \mathbb{C}) \mid (Z, Z) = 0,\ (Z, \bar{Z}) < 0\}^+ \qquad (4.1)$$

be the associated n-dimensional Hermitian domain of type IV (here $+$ denotes one of its two connected components) and $\Omega(T)^\bullet$ its affine cone. We denote by $\mathrm{O}^+(T)$ the index 2 subgroup of the integral orthogonal group $\mathrm{O}(T)$ preserving $\Omega(T)$.

Definition 4.1. *Suppose that T has signature $(n, 2)$, with $n \geq 3$. Let $k \in \mathbb{Z}$ and let $\chi\colon \Gamma \to \mathbb{C}^*$ be a character of a subgroup $\Gamma \subset \mathrm{O}^+(T)$ of finite index. A holomorphic function $F\colon \Omega(T)^\bullet \to \mathbb{C}$ on the affine cone $\Omega(T)^\bullet$ over $\Omega(T)$ is called a* modular form *of* weight *k and character χ for the group Γ if*

$$F(tZ) = t^{-k} F(Z) \quad \forall\, t \in \mathbb{C}^*,$$

$$F(gZ) = \chi(g) F(Z) \quad \forall\, g \in \Gamma.$$

A modular form is called a cusp form *if it vanishes at every cusp.*

We denote the linear spaces of modular and cusp forms of weight k and character χ by $M_k(\Gamma, \chi)$ and $S_k(\Gamma, \chi)$ respectively. We recall that a cusp is defined by an isotropic line or plane in T. For applications, one of the most important subgroups of $\mathrm{O}^+(T)$ is the stable orthogonal group

$$\widetilde{\mathrm{O}}^+(T) = \{g \in \mathrm{O}^+(T) \mid g|_{T^*/T} = \mathrm{id}\} \qquad (4.2)$$

where T^* is the dual lattice of T.

For any $v \in L \otimes \mathbb{Q}$ such that $v^2 = (v, v) > 0$ we define the *rational quadratic divisor*

$$\mathcal{D}_v = \mathcal{D}_v(T) = \{[Z] \in \Omega(T) \mid (Z, v) = 0\} \cong \Omega(v_T^\perp) \qquad (4.3)$$

where $v_T^\perp$ is an even integral lattice of signature $(n-1,2)$. Therefore, $\mathcal{D}_v$ is also a homogeneous domain of type IV. We note that $\mathcal{D}_v(T)=\mathcal{D}_{tv}(T)$ for any $t\neq 0$. The theory of automorphic Borcherds products (see [B4]–[B5] and [GN6], [CG1], [G4] for the Jacobi variant of these products) gives a method of constructing automorphic forms with rational quadratic divisors.

The reflection with respect to the hyperplane defined by a non-isotropic vector $v\in T^*$ is given by

$$\sigma_v\colon l\longmapsto l-\frac{2(l,v)}{(v,v)}v. \tag{4.4}$$

If $v\in T^*$ and $(v,v)>0$, the divisor $\mathcal{D}_v(T)$ is called a *reflective divisor* if $\sigma_v\in \mathrm{O}(T)$. In what follows we consider the divisor of a modular form F as a divisor of $\Omega(T)$ since F is homogeneous on $\Omega(T)^\bullet$.

Definition 4.2. *A modular form $F\in M_k(\Gamma,\chi)$ is called* **reflective** *if*

$$\mathrm{Supp}(\mathrm{div}_{\Omega(T)} F)\subset \bigcup_{\substack{\pm v\in T\\ v\ \text{is primitive}\\ \sigma_v\in\Gamma\ \textit{or}\ -\sigma_v\in\Gamma}} \mathcal{D}_v(T). \tag{4.5}$$

We call F 2-reflective if all v above are of square 2. We call F strongly reflective if multiplicity of any irreducible component of $\mathrm{div}\, F$ *is equal to one. We say that a strongly reflective modular form F is a modular form with the complete 2-divisor if*

$$\mathrm{div}_{\Omega(T)} F=\sum_{v\in R_2(T)/\{\pm1\}}\mathcal{D}_v(T) \tag{4.6}$$

where $R_2(T)$ is the set of 2-vectors (roots) in T.

Our main goal is to construct *strongly reflective* modular forms with the *complete 2-divisor* related to the hyperbolic root systems described in Section 3.

Example 4.1. *The Borcherds modular form Φ_{12}* (see [B4]). This is the unique, up to a constant, modular form of the singular (i.e. the minimal possible) weight 12 and character det with respect to $\mathrm{O}^+(II_{26,2})$

$$\Phi_{12}\in M_{12}(\mathrm{O}^+(II_{26,2}),\det)$$

where $II_{26,2}$ is the unique (up to an isomorphism) even unimodular lattice of signature $(26,2)$. It was proved in [B4] that

$$\operatorname{div}_{\Omega(II_{26,2})}\Phi_{12} = \sum_{v\in R_2(II_{26,2})/\{\pm 1\}} \mathcal{D}_v(II_{26,2}).$$

We note that all 2-vectors in $II_{26,2}$ form only one orbit with respect to $\mathrm{O}^+(II_{26,2})$.

Example 4.2. If

$$T^{(5)}_{2t} = 2U \oplus \langle 2t\rangle \qquad \text{where} \quad U \cong \begin{pmatrix} 0 & -1\\ -1 & 0\end{pmatrix}, \quad t\in\mathbb{N},$$

of signature $(3,2)$, then the modular forms with respect to $\widetilde{\mathrm{SO}}^+(T^{(5)}_{2t})$ coincide with Siegel modular forms of genus two with respect to the paramodular group $\Gamma_t \subset \mathrm{Sp}_2(\mathbb{Q})$ (see [G2], [GN6]). In particular, if $t = 1$ we obtain the Siegel modular forms with respect to $\mathrm{Sp}_2(\mathbb{Z})$. A large well defined class of strongly reflective modular forms for Γ_t was described in [GN1]–[GN8]. See also [CG1] where all Siegel modular forms with the simplest diagonal divisor were classified for all Hecke congruence subgroups of all paramodular groups.

All reflective modular forms have a Borcherds product expansion. It follows from the results of J.H. Bruinier who proved existence of a Borcherds product expansion for modular forms with a divisor which is sum of rational quadratic divisors if the lattice is not very exotic (see [Bru]). To construct strongly reflective modular forms for the reflective hyperbolic lattices with a lattice Weyl vector we use the method of quasi pull-back of the Borcherds form Φ_{12} which was proposed in [B1, pp. 200–201]. It was successfully applied to the theory of moduli spaces in [BKPS], [GHS1]–[GHS4]. See [GHS4, §8] on the detailed description of this construction.

The statements of the next theorem were proved in [BKPS, Theorem 1.2] and [GHS4, Theorems 8.3 and 8.18].

Theorem 4.1. *Let $T \hookrightarrow II_{26,2}$ be a primitive sublattice of signature $(n,2)$, $n\ge 3$, and let $\Omega(T)\hookrightarrow \Omega(II_{26,2})$ be the corresponding embedding of the homogeneous domains. The set of 2-roots*

$$R_2(T^\perp) = \{v\in II_{26,2} \mid v^2 = 2,\ (v,T) = 0\}$$

in the orthogonal complement is finite. We put $N(T^\perp) = \#R_2(T^\perp)/2$. Then the function

$$\Phi_{12}|_T = \frac{\Phi_{12}(Z)}{\prod_{v\in R_2(T^\perp)/\pm 1}(Z,v)}\Bigg|_{\Omega(T)^\bullet} \in M_{12+N(T^\perp)}(\widetilde{\mathrm{O}}^+(T), \det), \tag{4.7}$$

where in the product over v we fix a finite system of representatives in $R_2(T^\perp)/\pm 1$. The modular form $\Phi_{12}|_T$ vanishes only on rational quadratic divisors of type $\mathcal{D}_u(T)$ where $u \in T^*$ is the orthogonal projection of a 2-root $r \in II_{26,2}$ to T^* satisfying $0 < (u,u) \le 2$. If the set $R_2(T^\perp)$ of 2-roots in $T^\perp$ is non-empty then the quasi pull-back $\Phi_{12}|_T \in S_{12+N(T^\perp)}(\widetilde{\mathrm{O}}^+(T), \det)$ is a cusp form.

In [G4] we proposed twenty four Jacobi type constructions of the Borcherds function Φ_{12} based on the twenty four one dimensional boundary components of the Baily-Borel compactification of the modular variety $\mathrm{O}^+(II_{26,2}) \setminus \Omega(II_{26,2})$. These components correspond exactly to the classes of positive definite even unimodular lattices of rank 24. They are the 23 Niemeier lattices $N(R)$ uniquely determined by their root sublattices R of rank 24

$$\begin{gathered}3E_8,\ E_8\oplus D_{16},\ D_{24},\ 2D_{12},\ 3D_8,\ 4D_6,\ 6D_4,\\ A_{24},\ 2A_{12},\ 3A_8,\ 4A_6,\ 6A_4,\ 8A_3,\ 12A_2,\ 24A_1,\\ E_7\oplus A_{17},\ 2E_7\oplus D_{10},\ 4E_6,\ E_6\oplus D_7\oplus A_{11},\\ A_{15}\oplus D_9,\ 2A_9\oplus D_6,\ 2A_7\oplus D_5,\ 4A_5\oplus D_4\end{gathered}$$

and the Leech lattice $\Lambda_{24} = N(\emptyset)$ without roots (see [CS, Chapter 18]). We note that $II_{26,2} \cong 2U \oplus N(R)$. The quasi pull-backs of Φ_{12} considered in the different one-dimensional boundary components give the first series of strongly reflective modular forms which determine the Lorentzian Kac-Moody algebras of some reflective lattices considered in Sect. 3.

The next theorem is a particular case of a more general result proved in [G6] and a generalisation of [GH, Theorem 3.4].

Theorem 4.2. *Let K be a primitive sublattice of $N(R)$ containing a direct summand of the same rank of a root lattice R of a Niemeier lattice $N(R)$ or a primitive sublattice of the Leech lattice $N(\emptyset) = \Lambda_{24}$. We assume that K satisfies the following condition:*

$$(\mathrm{Norm}_2) \qquad \forall\, \bar c \in K^*/K \ \ (\bar c^2 \not\equiv 0 \bmod 2\mathbb{Z})\ \ \exists\, h_c \in \bar c \,:\, 0 < h_c^2 < 2.$$

We consider $T = 2U \oplus K$ as a sublattice of the corresponding model of $II_{26,2} = 2U \oplus N(R)$. Then $\Phi_{12}|_T$ is a strongly reflective modular form with the complete 2-divisor. More exactly

$$\Phi_{12}|_T \in M_k(\widetilde{\mathrm{O}}^+(T), \det)$$

where $k = 12 + |R_2(K^\perp)|/2$ and

$$\operatorname{div} \Phi_{12}|_T = \sum_{v \in R_2(T)/\pm 1} \mathcal{D}_v(T). \tag{4.8}$$

Remark 4.1. In the discriminant group $A_K = K^*/K$, if $h \in \bar{c} \in K^*/K$ then $(h,h) \equiv (\bar{c},\bar{c}) = q_K(\bar{c}) \bmod 2\mathbb{Z}$ is well defined modulo 2. The condition (Norm$_2$) claims that there exists an element h_c in every $\bar{c}$ with the smallest possible norm.

Proof. The quasi pull-back $\Phi_{12}|_T$ is a modular form with respect to the character det. For any 2-vector $v \in T$ the reflection σ_v is in $\widetilde{\mathrm{O}}^+(T)$. Therefore $\Phi_{12}|_T$ vanishes on the walls of all 2-reflections in T.

For any 2-vector $v \in II_{26,2}$ we write $v = \alpha + \beta$ where

$$\alpha = \mathrm{pr}_{T^*}(v) \in T^*,\ \beta \in (T^\perp)^* = (K_N^\perp)^* \quad \text{and} \quad \alpha^2 + \beta^2 = 2, \quad \beta^2 \geq 0.$$

Then we have

$$\Omega(T) \cap \mathcal{D}_v(II_{26,2}) = \begin{cases} \mathcal{D}_\alpha(T), & \text{if } \ \alpha^2 > 0, \\ \emptyset, & \text{if } \ \alpha^2 \leq 0,\ \alpha \neq 0, \\ \Omega(T), & \text{if } \ \alpha = 0, \text{ i.e. } v \in T^\perp. \end{cases}$$

We note that if $\beta^2 = 0$ then $v \in T$ because T is primitive in $II_{26,2}$. In this case, we get the divisor $\mathcal{D}_v(T)$ in $\Omega(T)$.

Let $0 < \alpha^2 < 2$. Since K satisfies (Norm$_2$)-condition and $K^*/K = T^*/T$, there exists $h \in K^*$ such that $h \in \alpha + K$ and $h^2 = \alpha^2$. We have $h + \beta \in v + K \subset II_{26,2}$. Therefore,

$$h + \beta \in \big(K^* \oplus (K_N^\perp)^*\big) \cap \big(2U \oplus N(R)\big) = N(R)$$

and $(h+\beta)^2 = 2$. It follows that $h + \beta$ is a 2-root in $N(R)$ which does not belong to $K \oplus K_N^\perp$. This contradicts to the condition on the roots in $K \subset N(R)$. □

In order to apply the last theorem, we have to fix models of irreducible 2-roots lattices R.

$$D_n = \{\sum_{i=1}^{n} x_i e_i \in \bigoplus_{i=1}^{n} \mathbb{Z}e_i = \mathbb{Z}^n \mid x_1 + \cdots + x_n \in 2\mathbb{Z},\ (e_i, e_j) = \delta_{i,j}\} \quad (4.9)$$

is the maximal even sublattice of the odd unimodular lattice $\mathbb{Z}^n$. Then

$$A_n = \{\sum_{i=1}^{n+1} x_i e_i \in \mathbb{Z}^{n+1} \,|\, x_1 + \cdots + x_{n+1} = 0\} \subset D_{n+1}.$$

In particular, $A_1 \cong \langle 2\rangle$, $A_1 \oplus A_1 \cong D_2$ and $A_3 \cong D_3$. We note that $D_1 \cong \langle 4\rangle = A_1(2)$ is not a root lattice. $A_n = (1, \ldots, 1)^{\perp}_{\mathbb{Z}^{n+1}}$, therefore A_n^*/A_n is the cyclic group C_{n+1} of order $n+1$. D_n^*/D_n is isomorphic to C_4 for odd n and to $C_2 \times C_2$ for even n. The classes of the discriminant group R^*/R of these root lattices are generated by the following elements having *the minimal possible norm* in the corresponding classes modulo A_n or D_n:

$$D_n^*/D_n = \{\, 0,\ e_n,\ (e_1 + \cdots + e_n)/2,\ (e_1 + \cdots + e_{n-1} - e_n)/2 \,\} + D_n,$$

$$A_n^*/A_n =$$
$$\{\, \varepsilon_i = \frac{1}{n+1}(\,\underbrace{i, \ldots, i,}_{n+1-i}\ \underbrace{i-n-1, \ldots, i-n-1}_{i}\,),\ 1 \le i \le n+1 \,\} + A_n.$$

Then the nontrivial classes of D_n^*/D_n have representatives of norm 1 and $\frac{n}{4}$. For A_n we see that if $n \le 7$ then $(\varepsilon_i, \varepsilon_i) \le 2$ and $(\varepsilon_i, \varepsilon_i) = 2$ only for $n = 7$ and $i = 4$.

Example 4.3. The following even lattice of determinant 2^6 is important for $K3$ surfaces with symplectic involutions:

$$N_8 = \langle 8A_1, h = (a_1 + ... + a_8)/2\rangle \cong D_8^*(2), \quad (a_i, a_j) = 2\delta_{i,j},\ h^2 = 4 \quad (4.10)$$

which is usually called *Nikulin's lattice*. Then $\bar{h} = h + 8A_1$ is an isotropic element of the discriminant group $(8A_1)^*/(8A_1)$

$$8A_1 \subset N_8 \subset N_8^* \subset 8A_1^*, \quad \det N_8 = 2^6 \quad \text{and} \quad N_8^*/N_8 \cong \bar{h}^{\perp}/\bar{h}.$$

From the last representation of N_8^*/N_8 it follows that the lattice N_8 satisfies the condition (Norm_2) of Theorem 4.2.

Theorem 4.3. *For the lattice $T = 2U \oplus K$ where K is one of the following 24 lattices*

$$\begin{array}{l} A_1,\ 2A_1,\ 3A_1,\ 4A_1,\ N_8;\ \ A_2,\ 2A_2,\ 3A_2;\ \ A_3,\ 2A_3;\ \ A_4,\ A_5,\ A_6,\ A_7;\\ k = 35,\ 34,\ 33,\ 32,\ 28;\ \ 45,\ 42,\ 39;\ \ 54,\ 48;\ \ 62,\ 69,\ 75,\ 80;\end{array}$$

$$\begin{array}{l} D_4,\ 2D_4,\ D_5,\ D_6,\ D_7,\ D_8;\ \ E_6,\ E_7,\ E_8,\ 2E_8;\\ k = 72,\ 60,\ 88,\ 102,\ 114,\ 124;\ 120,\ 165,\ 252,\ 132\end{array}$$

there exists a strongly reflective modular form

$$\Phi_{k,K} = \Phi_{12}|_{2U\oplus K\hookrightarrow 2U\oplus N_K} \in S_k(\widetilde{\mathrm{O}}^+(2U \oplus K), \det)$$

with the complete 2-divisor where N_K is the Niemeier lattice whose root system R contains K as a direct summand. All these functions $\Phi_{k,K}$ are cusp forms.

The Köcher principle together with the standard divisors argument give the next result.

Corollary 4.1. *For the lattices K of Theorem 4.3 the reflective modular form $\Phi_{k,K}$ is the only, up to a constant, cusp form in $S_k(\widetilde{\mathrm{O}}^+(K), \det)$. In particular, $\Phi_{k,K}$ is a (new) eigenfunction of all Hecke operators acting on modular forms.*

Remark 4.2. 1) The cusp forms in Corollary 4.1 are generalisations of the Ramanujan delta-function $\Delta(\tau)$. The strongly reflective modular form for $K = A_1$ is, in fact, the Igusa modular form $\Delta_{35} \in S_{35}(\mathrm{Sp}_2(\mathbb{Z}))$ which is the only genus 2 Siegel modular form of odd weight up to a factor. The corresponding Lorentzian Kac-Moody algebra, the algebra with the smallest possible Cartan matrix, was defined in [GN4]. We can say that all 2-reflective modular forms of Theorem 4.3 are of Δ_{35}-type.

2) In the next section we show that all quasi pull-backs of Theorem 4.3 have integral Fourier coefficients and we show how to describe their Borcherds products and multiplicities of imaginary roots of the corresponding Lorentzian Kac-Moody algebras.

Proof. Any root lattice K, mentioned in Theorem 4.3, satisfies (Norm_2) of Theorem 4.2. This follows from the description of the discriminant forms of the irreducible root lattices given above. Moreover there exists a Niemeier lattice $N(R)$ such that K is a direct summand of the root lattice R. The lattice $4A_1$, $3A_2$, $2A_3$, A_7, $2D_4$ and D_8 are not maximal but their even extensions contain a new root which is not possible in $N(R)$. Therefore the embedding $K \hookrightarrow N(R)$ is primitive sublattice for all root lattices K of the theorem.

The lattice $N_8 \hookrightarrow N(24A_1)$ is the extension of $8A_1$ in $N(24A_1)$ by one octave of the Golay code. Therefore, N_8 is primitive in $N(24A_1)$ and it satisfies the condition (Norm_2).

The quasi pull-back $\Phi_{12}|_{2U\oplus K}$ is a strongly reflective modular form with the complete 2-divisor according to Theorem 4.2. All these functions are cusp forms according to [GHS4, Theorem 8.18]. □

The last theorem gives a nice example of two different automorphic corrections of the same hyperbolic root system.

Proposition 4.1. *The hyperbolic Kac-Moody algebra defined by the 2-root system of the lattice $U \oplus D_4$ has two different automorphic corrections, i.e. there are two 2-reflective modular forms with this lattice as the hyperbolic lattice of a zero dimensional cusp.*

Proof. These two modular forms are related to the different non isomorphic extensions $U \oplus (U \oplus D_4)$ and $U(2) \oplus (U \oplus D_4)$ of the given hyperbolic lattice. The first Lorentzian Kac-Moody algebra is defined by the modular form from Theorem 4.3

$$\Phi_{72,D_4} = \Phi_{12}|_{2U\oplus D_4 \hookrightarrow 2U\oplus N(6D_4)}.$$

To define the second Lorentzian Kac-Moody algebra we use the isomorphism

$$U \oplus N_8 \cong U(2) \oplus D_4 \oplus D_4.$$

(These lattices are indefinite, 2-elementary and have isomorphic discriminant forms, see [N2].) We consider the embedding

$$U(2) \oplus U \oplus D_4 \hookrightarrow \big(U(2) \oplus U \oplus D_4\big) \oplus D_4 \cong 2U \oplus N_8.$$

The arguments of the proof of Theorem 4.2 show that

$$\Phi_{40,\,U\oplus U(2)\oplus D_4} = \Phi_{28}^{(N_8)}|_{U(2)\oplus U\oplus D_4 \hookrightarrow 2U\oplus N_8} \qquad (4.11)$$

is a strongly 2-reflective modular form of weight 40 from the space of cusp forms $S_{40}(\widetilde{\mathrm{O}}^+(U(2) \oplus U \oplus D_4), \det)$. □

In Theorem 4.3, we used 23 Niemeier lattices with non-trivial root systems. We can construct nine strongly reflective modular forms using the Leech lattice. The next result is proved in [G6].

Theorem 4.4. *For the lattice $T = 2U \oplus K$, where K is one of the following nine sublattices of the Leech lattice Λ_{24}*

$A_1(2),\ A_1(3),\ A_1(4),\ D_2(2) = \langle 4\rangle \oplus \langle 4\rangle,\ A_2(2),\ A_2(3),\ A_3(2),\ D_4(2),\ E_8(2)$

the corresponding pull-back for $T = 2U \oplus K \hookrightarrow 2U \oplus \Lambda_{24}$

$$\Phi_{12}|_T = P_{12} \in M_{12}(\widetilde{\mathrm{O}}^+(T), \det)$$

is a strongly reflective (non cusp) modular form of weight 12 *with the complete* 2*-divisor.*

Remark 4.3. The reflective Siegel modular forms corresponding to the lattices $K = A_1(2)$, $A_1(3)$ and $A_1(4)$ were constructed in [GN4, §4.2] by another method. Theorem 4.4 can be considered as a generalisation of the fact indicated in [GN4, §4.2, Remark 4.4].

The last theorem implies

Corollary 4.2. *Let K be one of the positive definite lattices of Theorem 4.4. Then the* 2*-root system of the hyperbolic lattice $U \oplus K$ is reflective with a lattice Weyl vector of norm* 0*, i.e. it has parabolic type.*

We plan to consider in more details the corresponding Lorentzian Kac-Moody algebras in a separate paper.

In this section we constructed 33 reflective modular forms with respect to the groups of type $\widetilde{\mathrm{O}}^+(2U \oplus K)$ and one modular form for $\widetilde{\mathrm{O}}^+(U \oplus U(2) \oplus D_4)$. (We note that in many cases $\widetilde{\mathrm{O}}^+(2U \oplus K)$ is not the maximal modular group of the reflective modular form.) The main theorem of [GH] gives a result on the geometric type of the corresponding modular varieties.

Corollary 4.3. *For all* 34 *lattices T of signature $(n, 2)$ from Theorem 4.3, Proposition 4.1 and Theorem 4.4, the modular variety $\widetilde{\mathrm{O}}^+(T) \setminus \Omega(T)$ is at least uniruled. The same is true for the modular variety $\widetilde{\mathrm{SO}}^+(T) \setminus \Omega(T)$ if the rank of T is odd.*

5 Jacobi type representation of Borcherds products and the lattice Weyl vector

It is known that one can consider the Kac–Weyl denominator function of the affine Lie algebra $\hat{\mathfrak{g}}(K)$ with positive part K of the root system as a product of eta- and theta-functions (see [K1], [KP]) or as a Jacobi form. The Borcherds product of 34 reflective modular forms constructed in Sect. 4 is equal to the right hand side of the Kac-Weyl-Borcherds denominator formula of the corresponding Lorentzian Kac-Moody algebra (see Sect. 2). In this section we consider a Jacobi type representation of the Borcherds products of the reflective modular forms. This gives a description of the multiplicities of imaginary roots of the corresponding Lorentzian Kac-Moody algebras as Fourier coefficients of some Jacobi forms of weight 0.

It is hard to get explicit formulae for the Fourier expansion of the quasi pull-backs constructed in Theorem 4.3, Proposition 4.1 and Theorem 4.4. In [G4] we proposed twenty four Jacobi type constructions of the Borcherds modular form Φ_{12}. This approach gives similar formulae for the Borcherds products of all modular forms of Sect. 4. In particular we give simple explicit formulae for the first two Fourier-Jacobi coefficients of these reflective forms.

Theorem 5.1. *Let K be one of lattices of Theorem 4.3.*

1) All Fourier coefficients of the reflective form $\Phi_{k,K}$ are integral.

2) $\Phi_{k,K}$ has the Borcherds product described in (5.6) and (5.7) below.

3) The lattice Weyl vector of the Lorentzian Kac-Moody algebra with the hyperbolic 2-root system of $U\oplus K$ and the denominator function $\Phi_{k,K}$ is given by the formula

$$\rho_{U\oplus K} = (A,B,C) = \big(1+h(K),\, -\frac{1}{2}\sum_{v\in R_2(K)_{>0}} v,\, h(K)\big) \tag{5.1}$$

where $h(K)$ is the Coxeter number of K and $B = -\frac{1}{2}\sum_{v\in R_2(K)_{>0}} v$ is the direct sum of the Weyl vectors of the irreducible components of the root system of the positive definite lattice K.

4) The first non-zero Fourier-Jacobi coefficient of $\Phi_{k,K}$ has index $h(K)$ and it is equal to

$$\eta(\tau)^{(h(K)+1)(24-\mathrm{rk}(K))}\cdot \eta(\tau)^{\mathrm{rk}\,K}\prod_{v\in R(K)_{>0}}\frac{\vartheta(\tau,(v,\mathfrak{z}))}{\eta(\tau)}$$

where $\mathfrak{z} \in K \otimes \mathbb{C}$, $\vartheta(\tau, z)$ is the Jacobi theta-series defined in (5.8) and (6.2). The second Fourier-Jacobi coefficient of index $h(K)+1$ is given in (5.10).

Remark 5.1. We note that for a root lattice $K > 0$

$$\eta(\tau)^{\operatorname{rk} K} \prod_{v \in R(K)_{>0}} \frac{\vartheta(\tau, (v, \mathfrak{z}))}{\eta(\tau)}$$

is *the Kac–Weyl denominator function of the affine Lie algebra $\hat{\mathfrak{g}}(K)$ where $\tau \in \mathbb{H}^+$, $q = \exp(2\pi i \tau)$,*

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n)$$

is the Dedekind eta-function.

In order to define Fourier and Fourier-Jacobi expansions of modular forms, we have to fix a tube realisation of the homogeneous domain $\Omega(T)$ related to boundary components of its Baily–Borel compactification. In this paper, we shall use automorphic forms mainly for the lattices T of signature $(n_0+2, 2)$ of the simplest possible type

$$T = U' \oplus (U \oplus K) = U \oplus S$$

where $U' \cong U = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ is the unimodular hyperbolic plane and K is a positive definite even integral lattice of rank n_0 and $S = U \oplus K$ is a hyperbolic lattice of signature $(n_0+1, 1)$.

Let $[\mathcal{Z}] \in \Omega(T)$. Using the basis $\langle e', f' \rangle_{\mathbb{Z}} = U = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ we write $\mathcal{Z} = z'e' + \widetilde{Z} + zf'$ with $\widetilde{Z} \in S \otimes \mathbb{C}$. We note that $z \neq 0$. (If $z = 0$, the real and imaginary parts of $\widetilde{Z}$ form two orthogonal vectors of negative norm in the hyperbolic lattice $S \otimes \mathbb{R}$.) Thus $[\mathcal{Z}] = [\frac{1}{2}(Z, Z)e' + Z + f']$. Using the similar basis $\langle e, f \rangle_{\mathbb{Z}} = U$ of the second hyperbolic plane in T, we see that $\Omega(T)$ is isomorphic to the tube domains

$$\mathcal{H}(S) = \{Z \in S \otimes \mathbb{C} \mid -(\operatorname{Im} Z, \operatorname{Im} Z) > 0\}^+$$

and

$$\mathcal{H}(K) = \{Z = \omega e + \mathfrak{z} + \tau f \in S \otimes \mathbb{C} \mid \tau, \omega \in \mathbb{H}^+, \ \mathfrak{z} \in K \otimes \mathbb{C}, \ 2 \operatorname{Im} \tau \cdot \operatorname{Im} \omega - (\operatorname{Im} \mathfrak{z}, \operatorname{Im} \mathfrak{z})_K > 0\}. \quad (5.2)$$

We fix the isomorphism $[\mathrm{pr}]: \mathcal{H}(K) \to \Omega(T)$ defined by the 1-dimensional cusp $\langle e', e\rangle$ fixed above

$$Z = (\omega e + \mathfrak{z} + \tau f) \mapsto \mathrm{pr}(Z) = (\tfrac{(Z,Z)}{2} e' + \omega e + \mathfrak{z} + \tau f + f') \;\mapsto\; [\mathrm{pr}(Z)]\,.$$

For a primitive isotropic vector $c \in T$ and any $a \in c_T^{\perp}$, one defines the Eichler transvection

$$t(c,a)\colon v \longmapsto v + (a,v)c - (c,v)a - \frac{1}{2}(a,a)(c,v)c \in \widetilde{\mathrm{SO}}^+(T).$$

If $Z \in \mathcal{H}(S)$ and $l \in S = (e')_T^{\perp}/\mathbb{Z}e'$, then $t(e',l)(\mathrm{pr}[Z]) = \mathrm{pr}[Z+l]$ is a translation in $\mathcal{H}(S)$. Therefore, any modular form $F \in M_k(\widetilde{\mathrm{SO}}^+(T))$ is periodic, i.e. $F(Z+l) = F(Z)$ for any $l \in S$. One defines the Fourier expansion of F at the zero-dimensional cusp $\langle e'\rangle$

$$F(Z) = \sum_{l \in S^*,\, -(l,l) \geq 0} f(l) \exp\left(-2\pi i\,(l,Z)\right) \tag{5.3}$$

and its Fourier-Jacobi expansion at the one-dimensional cusp $\langle e', e\rangle$

$$F(\tau,\mathfrak{z},\omega) = \sum_{m \geq 0} \varphi_m(\tau,\mathfrak{z}) \exp\left(2\pi i\, m\omega\right). \tag{5.4}$$

(See a general description of a Fourier expansion at an arbitrary cusp in [GN3, §2.3] and [GHS4, §8.2-8.3].) The Fourier-Jacobi coefficients $\varphi_m(\tau,\mathfrak{z})$ of $F \in M_k(\widetilde{\mathrm{SO}}^+(T))$, where $T = 2U \oplus K$, are examples of holomorphic Jacobi forms of weight k and index m for the even integral lattice $K > 0$. We note that we use the positive orientation of the indices defined by (5.3) in the Fourier expansion of the Jacobi forms $\varphi_m(\tau,\mathfrak{z})$.

Definition 5.1. *(See [G2], [CG2].) A holomorphic (respectively, weak or nearly holomorphic) Jacobi form of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{N}$ for an even integral positive definite lattice K is a holomorphic function*

$$\phi : \mathbb{H} \times (K \otimes \mathbb{C}) \to \mathbb{C}$$

satisfying the functional equations

$$\varphi\Big(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}\Big) = (c\tau+d)^k \exp\Big(\pi i \frac{cm(\mathfrak{z},\mathfrak{z})}{c\tau+d}\Big)\varphi(\tau,\,\mathfrak{z}),$$
$$\varphi(\tau,\mathfrak{z}+\lambda\tau+\mu) = \exp\big(-\pi i m\big((\lambda,\lambda)\tau + 2(\lambda,\mathfrak{z})\big)\big)\varphi(\tau,\mathfrak{z})$$

for any $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, λ, $\mu \in K$ and having a Fourier expansion

$$\varphi(\tau, \mathfrak{z}) = \sum_{n\in\mathbb{Z},\ \ell\in K^*} f(n,\ell)\, \exp\bigl(2\pi i(n\tau - (\ell,\mathfrak{z})\bigr)$$

where $f(n,l) \neq 0$ implies $N_m(n,\ell) := 2nm - (\ell,\ell) \geq 0$ for holomorphic, $n \geq 0$ for weak and $N_m(n,\ell) = 2nm - (\ell,\ell) >> -\infty$ for nearly holomorphic Jacobi forms. We denote the space of holomorphic (reps. weak or nearly-holomorphic) Jacobi forms by $J_{k,K;m} \subset J^{w}_{k,K;m} \subset J^{!}_{k,K;m}$. If $m = 1$, we write simply $J_{k,K}$, etc.

In [G4], we showed that any Jacobi form of weight 0 in $J^{!}_{0,K}$ with integral Fourier coefficients defines an automorphic Borcherds product which is a (meromorphic) automorphic form with respect to $\widetilde{\mathrm{O}}^+(2U \oplus K)$ with a character.

A Niemeier lattice $N(R)$ is defined by its non-empty root system $R = R_1 \oplus \cdots \oplus R_m$. All components R_i have the same Coxeter number $h(R) = h(R_i)$. In Theorem 4.3 above, the lattice K is a direct component of R and we put $h(K) = h(R)$.

We introduce the Jacobi theta-series ϑ_N of the even unimodular positive definite lattice $N = N(R)$ of rank 24 (see [G2])

$$\vartheta_N(\tau,\mathfrak{z}) = \sum_{\ell\in N} \exp\left(\pi i(\ell,\ell)\tau - 2\pi i(\ell,\mathfrak{z})\right) \in J_{12,N}$$

and a nearly holomorphic Jacobi form of weight 0 with integral Fourier coefficients

$$\varphi_{0,N}(\tau,\mathfrak{z}) = \frac{\vartheta_N(\tau,\mathfrak{z})}{\Delta(\tau)} = \sum_{\substack{n\geq -1,\ \ell\in N\\ 2n-\ell^2\geq -2}} a_N(n,\ell) q^n r^\ell \in J^{!}_{0,N} \tag{5.5}$$

where $q = \exp(2\pi i\tau)$, $r^\ell = \exp(-2\pi i(\ell,\mathfrak{z}))$ and $\Delta(\tau) = q\prod_{n\geq 1}(1-q^n)^{24}$ is the Ramanujan delta-function. The Fourier expansion starts with

$$\varphi_{0,N}(\tau,\mathfrak{z}) = q^{-1} + 24 + \sum_{v\in R_2(N)} e^{2\pi i(v,\mathfrak{z})} + q(\dots),$$

where $R_2(N) = \{v \in N,\ v^2 = 2\}$ is the set of roots of the Niemeier lattice. One has similar formula for the Jacobi theta-series $\vartheta_{\Lambda_{24}}(\tau,\mathfrak{z})$ of the Leech lattice, but, in this case, the Fourier expansion of $\varphi_{0,\Lambda_{24}}$ does not contain the sum with respect to the roots in q^0-term.

We define the pullback of $\varphi_{0,N}$ on the lattice $K \hookrightarrow N = N(R)$. In other words, we write $\mathfrak{z}_N = \mathfrak{z} + \mathfrak{z}'$ with $\mathfrak{z}_N \in N \otimes \mathbb{C}$, $\mathfrak{z} \in K \otimes \mathbb{C}$, $\mathfrak{z}' \in K_N^{\perp} \otimes \mathbb{C}$ and we put

$$\varphi_{0,K}(\tau,\mathfrak{z}) = \varphi_{0,N}|_K = \varphi_{0,N}(\tau,\mathfrak{z}_N)|_{\mathfrak{z}'=0} = \sum_{n\ge -1,\ \ell\in K^*} a_K(n,\ell) q^n r^\ell \qquad (5.6)$$

$$= q^{-1} + 24 + h(K)(24 - \operatorname{rk} K) + \sum_{v\in R_2(K)} e^{2\pi i (v,\mathfrak{z})} + q(\dots) \in J^!_{0,K}$$

where

$$a_K(n,\ell) = \sum_{\substack{\ell_1\in (K_N^{\perp})^*,\ \ell+\ell_1\in N \\ 2n-\ell^2-\ell_1^2\ge -2}} a_N(n,\ell+\ell_1).$$

We note that $h(K)(24-\operatorname{rk} K) = |R_2(K_N^{\perp})|$ since all irreducible components of the root system of the Niemeier lattice N have the same Coxeter number.

In [G4, Theorem 3.1], we proved that any Jacobi form of weight 0

$$\psi(\tau,\mathfrak{z}) = \sum_{n\in\mathbb{Z},\, \ell\in K^*} f(n,\ell) q^n r^\ell \in J^!_{0,K}$$

with integral Fourier coefficients $f(n,\ell)$ with indices (n,ℓ) of negative hyperbolic norm $N(n,\ell) = 2n-(\ell,\ell) < 0$ determines the (meromorphic) Borcherds product

$$\mathcal{B}_\psi(\tau,\mathfrak{z},\omega) = q^A r^B s^C \prod_{\substack{n,m\in\mathbb{Z},\, \ell\in K^* \\ (n,\ell,m)>0}} \big(1 - q^n r^\ell s^m\big)^{f(nm,\ell)} \qquad (5.7)$$

where $Z = (\tau,\mathfrak{z},\omega) \in \mathcal{H}(K) \cong \Omega(T)$, $q = \exp(2\pi i\tau)$, $s = \exp(2\pi i\omega)$ and $r^\ell = \exp(-2\pi i(\ell,\mathfrak{z}))$, $(n,\ell,m) > 0$ means that $m>0$, or $m=0$ and $n>0$, or $m=n=0$ and $\ell>0$ (in the sense of the root system in K) and

$$A = \frac{1}{24}\sum_{\ell\in K^*} f(0,\ell), \quad B = -\frac{1}{2}\sum_{\ell>0} f(0,\ell)\ell \in \frac{1}{2}K^*,$$

$$C = \frac{1}{2\operatorname{rk} K}\sum_{\ell\in K^*} f(0,\ell)(\ell,\ell).$$

We note that for the formula of the Borcherds product given above, we fix an ordering in K. A positive vector $u \in K^*$ has a positive scalar product with a fixed vector in $T \otimes \mathbb{R}$ which is not orthogonal to the vectors $\ell \in K^*$ such that $f(0, \ell) \neq 0$. We can fix such a vector once at a boundary component, for example in $2U \oplus \Lambda_{24}$.

The Borcherds product $\mathcal{B}_\psi$ has also a Jacobi type representation

$$\mathcal{B}_\psi(Z) = \widetilde{\psi}_{K;C}(Z) \exp\big(-\sum_{m \geq 1} m^{-1} \tilde{\psi}|T_-(m)(Z)\big)$$

where $\tilde{\psi}(Z) = \psi(\tau, \mathfrak{z}) e^{2\pi i \omega}$, $T_-(m)$ is a Hecke operator of type (6.6) below,

$$\widetilde{\psi}_{K;C}(Z) = \eta(\tau)^{f(0,0)} \prod_{\ell > 0} \bigg(\frac{\vartheta(\tau, (\ell, \mathfrak{z}))}{\eta(\tau)} \bigg)^{f(0,\ell)} e^{2\pi i C \omega}$$

and

$$\vartheta(\tau, z) = q^{1/8} (\zeta^{1/2} - \zeta^{-1/2}) \prod_{n \geq 1} (1 - q^n \zeta)(1 - q^n \zeta^{-1})(1 - q^n) \tag{5.8}$$

is a Jacobi theta-series of characteristic 2 with $q = e^{2\pi i \tau}$ and $\zeta = e^{2\pi i z}$ (see (6.2) for its Fourier expansion). In particular, we have 23 formulae for the Borcherds modular forms

$$\Phi_{12}(Z) = B_{\varphi_{0,N(R)}}(Z). \tag{5.9}$$

See [G4] for more details where we used the signature $(2, n)$ typical in the theory of moduli spaces of $K3$ surfaces.

According to (5.6), (5.7) and (5.9) the quasi pull-back $\Phi_{k,K}$ is written as the Borcherds product defined by $\varphi_{0,K}(\tau, \mathfrak{z})$

$$\Phi_{k,K} = \Phi_{12}|_{2U \oplus K}(\tau, \mathfrak{z}, \omega) = B_{\varphi_{0,K}}(\tau, \mathfrak{z}, \omega) \in S_k(\widetilde{\mathrm{O}}^+(2U \oplus K), \det).$$

All Fourier coefficients of the Borcherds product $\Phi_{k,K}$ are integral since all Fourier coefficients $a(n, \ell)$ of the Jacobi form $\varphi_{0,K}$ are integral. Moreover, for $\varphi_{0,K}$ the first factor $q^A r^B s^C$ of the Borcherds product $B_{\varphi_{0,K}}$ has a very simple expression. We have

$$(A, B, C) = \Big(1 + h(K),\ -\frac{1}{2} \sum_{v \in R_2(K)_{>0}} v,\ h(K)\Big)$$

where $B = -\frac{1}{2}\sum_{v\in R_2(K)_{>0}} v$ is the direct sum of the Weyl vectors of the irreducible components of the root systems of K.

The Jacobi type representation of the Borcherds product $B_{\varphi_{0,K}}$ gives its first two non-zero Fourier-Jacobi coefficients of indices $h(K)$ and $h(K)+1$

$$\mathcal{B}_{\varphi_{0,K}}(\tau,\mathfrak{z},\omega) = \eta(\tau)^{h(K)(24-\operatorname{rk} K)} \prod_{v\in R_2(K)_{>0}} \frac{\vartheta(\tau,(v,\mathfrak{z}))}{\eta(\tau)}\, e^{2\pi i\, h(K)\,\omega} \times \tag{5.10}$$

$$\times\big(\Delta(\tau) - \vartheta_{N(R)}|_K(\tau,\mathfrak{z})\, e^{2\pi i\,\omega} + \dots\big)$$

where the product is taken over all positive roots v of K. In order to finish the proof of Theorem 5.1 we have to use that $\Delta(\tau) = \eta(\tau)^{24}$.

6 Reflective towers of Jacobi liftings

6.1 The Jacobi lifting and Fourier coefficients of modular forms

In Sect. 5, we calculated the Borcherds products of the reflective modular forms of Theorem 4.3. Using the differential operators (see [GHS4, §8]), we can write some expressions for their Fourier coefficients in terms of Fourier coefficients of Φ_{12} but such formulae are not, in fact, explicit because they contain rather complicated summations. In this section, we consider reflective modular forms related to the seventeen lattices

$$\langle -2\rangle \oplus k\langle 2\rangle\ (2\le k\le 9),\quad U(2)\oplus D_4,\quad U(4)\oplus D_4,$$

$$U(4)\oplus kA_1\ (1\le k\le 4)\quad \text{and}\quad U(3)\oplus kA_2\ (1\le k\le 3).$$

The corresponding strongly 2-reflective modular forms have rather simple Fourier expansions. We construct them using the Jacobi (additive) lifting of holomorphic Jacobi forms. This construction was proposed in [G1]-[G2] and was extended to the Jacobi modular forms of half-integral index in [GN6] and [CG2].

We take a holomorphic Jacobi form of integral weight k for an arbitrary even positive definite lattice K

$$\varphi_k(\tau,\mathfrak{z}) = \sum_{n\in\mathbb{N},\,\ell\in K^*} f(n,\ell)q^n r^\ell \in J_{k,K}$$

with $f(0,0)=0$. Then the lifting of φ_k is defined as

$$\operatorname{Lift}(\varphi_k)(\tau,\mathfrak{z},\omega)=\sum_{\substack{n,m>0,\,\ell\in K^*\\ 2nm-(\ell,\ell)\ge 0}}\ \sum_{d|(n,\ell,m)} d^{k-1} f\big(\tfrac{nm}{d^2},\tfrac{\ell}{d}\big)\, e^{2\pi i(n\tau+(\ell,\mathfrak{z})+m\omega)} \tag{6.1}$$

where $d|(n,\ell,m)$ denotes a positive integral divisor of the vector in $U\oplus K^*$. Then

$$\operatorname{Lift}(\varphi_k)\in M_k(\widetilde{\mathrm{O}}^+(2U\oplus K)).$$

We note that *the Fourier coefficients of the lifting are integral if all Fourier coefficients of the holomorphic Jacobi form φ_k are integral.* There is a natural sufficient condition in terms of the discriminant form of the lattice K (see [GHS1, Theorem 4.2]) which implies that the lifting of a Jacobi cusp form is a cusp form. In the last case *the norm of the Weyl vector of the automorphic correction will be automatically positive.*

Example 6.1. 1-reflective modular form of singular weight for $2U\oplus D_8$. (See [G3] and [CG2].) We can consider the Jacobi theta-series (5.8) having the following Fourier expansion

$$\vartheta(z)=\vartheta(\tau,z)=\sum_{m\in\mathbb{Z}}\left(\frac{-4}{m}\right)q^{m^2/8}\,\zeta^{m/2}\in J_{\frac12,\frac12}(v_\eta^3\times v_H), \tag{6.2}$$

as a Jacobi form of weight $\frac12$ and index $\frac12$. In the last formula $\big(\frac{-4}{m}\big)$ denotes the quadratic Kronecker symbol. The Jacobi forms of half-integral indices were introduced in [GN6]. (See [CG2] for the lattice case.)

We define the following Jacobi form of singular weight 4 for D_8

$$\vartheta_{D_8}(\tau,\mathfrak{z})=\vartheta(z_1)\cdot\ldots\cdot\vartheta(z_8)\in J_{4,D_8}$$

where $\mathfrak{z}=(z_1,\ldots,z_8)\in D_8\otimes\mathbb{C}$ where the coordinates z_i correspond to the euclidean basis of the model (4.9) of D_n. For any $\ell\in D_8^*\subset\frac12\mathbb{Z}^8$, we put $\big(\frac{-4}{2\ell}\big)=\prod_{i=1}^8\big(\frac{-4}{2l_i}\big)$. Then

$$\vartheta_{D_8}(\tau,\mathfrak{z})=\sum_{\ell\in\frac12\mathbb{Z}^8}\left(\frac{-4}{2\ell}\right)\exp\big(\pi i((\ell,\ell)\tau+2(\ell,\mathfrak{z}))\big). \tag{6.3}$$

According to (6.1), we have

$$\operatorname{Lift}(\vartheta_{D_8})=\sum_{\substack{n,\,m\in\mathbb{N},\,\ell\in\frac12\mathbb{Z}^8\\ 2nm-(\ell,\ell)=0}}\ \sum_{d|(n,\,\ell,\,m)} d^3\left(\frac{-4}{2\ell/d}\right)e^{2\pi i(n\tau+(\ell,\mathfrak{z})+m\omega))}$$

where d is a divisor of (n,ℓ,m) in $U\oplus D_8^*$. (See a more detailed formula in [G3, (17)].) This lifting is invariant with respect to the permutations of $z_1,\dots,z_8$ and anti-invariant with respect to the reflections σ_{e_i}. Therefore,

$$\Delta_{4,D_8}=\mathrm{Lift}(\vartheta(z_1)\cdot\ldots\cdot\vartheta(z_8))\in M_4(\mathrm{O}^+(2U\oplus D_8),\chi_2) \tag{6.4}$$

where χ_2 is a character of order 2. It was proved in [G3] that Δ_{4,D_8} is *strongly reflective with the divisor determined by all* 1*-reflections in* $2U\oplus D_8^*$. In fact, Δ_{4,D_8} gives a simple additive construction of the Borcherds-Enriques modular form $\Phi_4^{(BE)}\in M_4(\mathrm{O}^+(U\oplus U(2)\oplus E_8(2)),\chi_2)$ from [B5]. Moreover, this automatically gives the explicit description of the character.

We have the following Jacobi type Borcherds product $\Delta_{4,D_8}=B_{\varphi_{0,D_8}}$ (see [G3, §3] and (5.7) above) with

$$\varphi_{0,D_8}=2^{-1}(\vartheta_{D_8}|T_-(2))/\vartheta_{D_8}=r_1+r_1^{-1}+\dots+r_8+r_8^{-1}+8+q(\dots) \tag{6.5}$$

$$=8\prod_{i=1}^{8}\frac{\vartheta(2\tau,2z_i)}{\vartheta(\tau,z_i)}+\frac{1}{2}\prod_{i=1}^{8}\frac{\vartheta(\frac{\tau}{2},z_i)}{\vartheta(\tau,z_i)}+\frac{1}{2}\prod_{i=1}^{8}\frac{\vartheta(\frac{\tau+1}{2},z_i)}{\vartheta(\tau,z_i)}$$

where $r_i=\exp(2\pi i z_i)$. We recall that for a Jacobi form of weight k we have by definition

$$\psi_k|T_-(m)(\tau,\mathfrak{z})=\sum_{\substack{ad=m\\ b \bmod d}}a^k\psi_k\big(\frac{a\tau+b}{d},\ a\mathfrak{z}\big). \tag{6.6}$$

In the rest of the section, we analyse the towers of strongly reflective modular forms based on three modular forms of singular weight for the lattices $2U\oplus D_8$, $2U\oplus 4A_1$ and $2U\oplus 3A_2$ constructed in [G3].

6.2 The singular modular form for $2U\oplus D_8$ and the elliptic 2-reflective lattices $\langle -2\rangle\oplus k\langle 2\rangle$, $2\le k\le 8$.

In this subsection we construct a tower of eight reflective modular forms. In Theorem 3.1 of Section 3, we have the series of 2-reflective lattices $\langle -2\rangle\oplus k\langle 2\rangle$ where $2\le k\le 8$. The corresponding automorphic corrections will be defined by the reflective modular forms with respect to the orthogonal groups of $U(2)\oplus\langle -2\rangle\oplus k\langle 2\rangle$.

Theorem 6.1. *The automorphic correction of the* 2*-root system of* $\langle -2\rangle\oplus(k+1)\langle 2\rangle$ $(1\le k\le 7)$ *is defined by* $U(2)\oplus\langle -2\rangle\oplus(k+1)\langle 2\rangle$ $(1\le k\le 7)$ *and by the modular form*

$$\Delta_{12-k,D_k}=\mathrm{Lift}(\psi_{12-k,\,D_k})\in S_{12-k}(\mathrm{O}^+(U(2)\oplus\langle -2\rangle\oplus(k+1)\langle 2\rangle),\chi_2)$$

where

$$\psi_{12-k,\,D_k}(\tau,\mathfrak{z})=\eta(\tau)^{24-3k}\ \vartheta(z_1)\cdot\ldots\cdot\vartheta(z_k)\quad(2\le k\le 7)\tag{6.7}$$

and

$$\psi_{11,\,D_1}=\eta(\tau)^{21}\ \vartheta(2z).$$

Remark 6.1. We want to remark about $\Delta_{11,\,D_1}$. The last case of Δ_{12-k,D_k} for $k=1$ is special because $D_1=\langle 4\rangle$. One gets it as a degeneration of D_2. The function $\Delta_{11,D_1}=\mathrm{Lift}(\eta(\tau)^{21}\vartheta(\tau,2z))$ is one the basic reflective Siegel modular forms $\Delta_{11}\in S_{11}(\Gamma_2)$ with respect to the paramodular group Γ_2 in the classification of Lorentzian Kac-Moody algebras of rank 3 in [GN6], [GN8]. Therefore, the D_8-tower of reflective modular forms considered in this subsection starts with the reflective Siegel form Δ_{11}.

In this subsection, we show how to calculate the Fourier expansions of the reflective modular forms from Theorem 6.1, and propose three ways to write the Borcherds products of them. In the next lemma, we describe a trick, *the duality argument*, which is very useful for our considerations.

Lemma 6.1. *Let* $1\le k\le 8$.

1) The next three groups are canonically isomorphic if $k\ne 4$

$$\mathrm{O}(\langle -2\rangle\oplus(k+1)\langle 2\rangle)=\mathrm{O}(U\oplus k\langle 1\rangle)=\mathrm{O}(U\oplus D_k).$$

For $k=4$, $\mathrm{O}(\langle -2\rangle\oplus 5\langle 2\rangle)$ *is isomorphic to a double extension of* $\widetilde{\mathrm{O}}^+(U\oplus D_4)$.

2) The reflections with respect to 2*-vectors of* $\langle -2\rangle\oplus(k+1)\langle 2\rangle$ *correspond to the reflections with respect to* 4*-reflective vectors of* $U\oplus D_k$ *or* 1*-vectors of* $U\oplus D_k^*$. *If* $k\ne 4$, *then all* 1*-reflective vectors of* $2U\oplus D_k^*$ *belong to the unique* $\widetilde{\mathrm{O}}^+(2U\oplus D_k)$*-orbit which is equal to the set of* 1*-vectors in* $2U\oplus k\langle 1\rangle$.

3) If $k=4$, *then there are three such* $\widetilde{\mathrm{O}}^+(2U\oplus D_4)$*-orbits, and one of them coincides with the* 1*-vectors in* $2U\oplus k\langle 1\rangle$.

Proof. Let M be an integral quadratic lattice. Then

$$\mathrm{O}(M) = \mathrm{O}(M^*) = \mathrm{O}(M^*(n)), \qquad n \in \mathbb{N}. \tag{6.8}$$

By $M^*(n)$ we denote the renormalisation by the factor n of the quadratic form of the dual lattice M^*. If M is odd, we denote by M_{ev} the maximal even sublattice of M. Then $\mathrm{O}(M)$ can be considered as a subgroup of $\mathrm{O}(M_{ev})$. The following isomorphism is valid

$$\langle -2\rangle \oplus (k+1)\langle 2\rangle \cong U(2) \oplus k\langle 2\rangle, \qquad k \geq 1,$$

since for $k = 1$ one has $\langle a\rangle \oplus \langle b\rangle \oplus \langle c\rangle = \langle a+b, a+c\rangle \oplus \langle a+b+c\rangle$ where $a^2 = -2$ and $b^2 = c^2 = 2$. Thus, for $M = \langle -2\rangle \oplus (k+1)\langle 2\rangle$ we get

$$\mathrm{O}(M) = \mathrm{O}(M^*(2)) = \mathrm{O}(U \oplus k\langle 1\rangle) \subset \mathrm{O}(U \oplus D_k)$$

because D_k is the maximal even sublattice of $k\langle 1\rangle$. For the discriminant group we have

$$D_k^*/D_k \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } k \equiv 0 \mod 2, \\ \mathbb{Z}/4\mathbb{Z} & \text{if } k \equiv 1 \mod 2 \end{cases}$$

and D_k for $k \leq 8$ has the unique extension to the odd unimodular lattice $\mathbb{Z}^k$ if and only if $k \neq 4$. This proves the first isomorphism of the lemma. The renormalisation $M^*(2)$ explains the relation between the reflective vectors of the lattices.

Let us assume that $u \in M$, $u^2 = \pm 4$ and $\sigma_u \in O(M)$. In this case, $v = u/2 \in M^*$. The 1-vectors v in $2U \oplus D_k^*$ are classified by their images in the discriminant group D_k^*/D_k (see the Eichler criterion in [G2] and [GHS4]). It follows that there exists only one $\widetilde{\mathrm{O}}^+(2U \oplus D_k)$-orbit of such vectors if $k \neq 4$.

If $k = 4$, then all classes e_1, $(e_1+e_2+e_3 \pm e_4)/2 \mod D_4$ of the discriminant group contain 1-vector. This gives three orbits. A permutation of e_i and $(e_1 + e_2 + e_3 \pm e_4)/2$ could be realised in $\mathrm{O}(2U \oplus 4\langle 1\rangle)$, and the reflection σ_4 permutes the last two vectors. $\square$

According to the last lemma, the problem of construction of automorphic corrections of the hyperbolic root systems $\langle -2\rangle \oplus (k+1)\langle 2\rangle$, $1 \leq k \leq 8$ from Theorem 3.1 is reduces to construction of a 1-reflective (or equivalently

4-reflective) modular form for the lattice $U \oplus D_k$ which vanishes along the walls of all reflections σ_v for $v \in 2U \oplus D_k^*$, $v^2 = 1$, $v \equiv e_1 \bmod 2U \oplus D_k$. We obtain the D_8-tower of the reflective modular forms by taking the quasi pull-backs of Δ_{4,D_8} (see (6.4)) for $D_k \oplus D_{8-k} \hookrightarrow D_8$.

For $k = 9$, the lattice $\langle -2\rangle \oplus 9\langle 2\rangle$ *is a 2-reflective lattice of parabolic type with a lattice Weyl vector of norm zero, and* Δ_{4,D_8} *is its automorphic correction.*

We put $\mathfrak{z}_k = \sum_{i=1}^k z_i e_i \in D_k \otimes \mathbb{C}$, $2 \le k \le 8$ (see (4.9)). A cusp form of the D_8-tower is the quasi pull-back of $\Delta_{4,D_8} = \Phi_4^{(BE)}$ for $z_k = \cdots = z_8 = 0$

$$\Delta_{12-k,D_k} = \Delta_{4,D_8}|_{z_k=\cdots=z_8=0}.$$

All of them are 1-reflective modular forms, and

$$\Delta_{12-k,D_k} = \operatorname{Lift}(\eta(\tau)^{24-3k}\ \vartheta(z_1)\cdot\ldots\cdot\vartheta(z_k)) \in S_{12-k}(\widetilde{\mathrm{O}}^+(2U \oplus D_k))$$

(see [G3, §3]). If $k \ne 4$, then there is an extension of order 2

$$\widetilde{\mathrm{O}}^+(2U \oplus D_k) \subset \mathrm{O}^+(2U \oplus D_k).$$

$\Delta_{12-k,D_k}(\tau, z_1, \ldots, z_k, \omega)$ is invariant with respect to the permutations of the variables $(z_1, \ldots, z_k)$, and anti-invariant with respect to the reflection $z_i \to -z_i$. It follows that

$$\Delta_{12-k,D_k} \in S_{12-k}(\mathrm{O}^+(2U \oplus D_k), \chi_2) \tag{6.9}$$

where $\chi_2 : \mathrm{O}^+(2U \oplus D_k) \to \{\pm 1\}$ is defined by the relation

$$\chi(g) = 1 \ \Leftrightarrow\ g|_{A_{2U\oplus D_k}} = \mathrm{id}\,.$$

The form $\Delta_{12-k,D_k}(\tau, \mathfrak{z}_k, \omega)$ $(2 \le k \le 8)$ vanishes on $z_i = 0$ $(1 \le i \le k)$ as the lifting of the product of the Jacobi theta-series by a power of $\eta(\tau)$. This is exactly the union of walls which we are looking for. The Borcherds product of Δ_{12-k,D_k} is defined by a Jacobi form which can be written in two ways, as

$$\varphi_{0,D_k} = \varphi_{0,D_8}|_{z_{k+1}=\cdots=z_8=0} = r_1 + r_1^{-1} + \cdots + r_k^{-1} + (24-2k) + q(\ldots)$$

or using the quotient $2^{-1}(\psi_{12-k,\,D_k}|T_-(2))/\psi_{12-k,\,D_k}$ (see (6.5)). In the proof of Proposition 6.1 below, we give the third formula for the same Jacobi form.

According to Lemma 6.1, we have

$$\Delta_{12-k,D_k}(\tau, z_1, \ldots, z_k, \omega) \in S_{12-k}(\mathrm{O}^+(2U \oplus k\langle 1\rangle), \chi_2) = S_{12-k}(\mathrm{O}^+(U(2) \oplus \langle -2\rangle \oplus (k+1)\langle 2\rangle), \chi_2)$$

is strongly 1-reflective with respect to $\mathrm{O}^+(2U \oplus k\langle 1\rangle)$ and is strongly 2-reflective with respect to $\mathrm{O}^+(U(2) \oplus \langle -2\rangle \oplus k\langle 2\rangle)$. Theorem 6.1 is proved.

We study the first cusp form of the modular D_8-tower in more details. We note that

$$(2\pi i)^{-1}\frac{\partial \vartheta(\tau, z)}{\partial z}\Big|_{z=0} = \sum_{n>0}\left(\frac{-4}{n}\right) n q^{n^2/8} = \eta(\tau)^3.$$

Using the exact formula for the Fourier coefficients of the Jacobi form, we find the Fourier expansion of the Jacobi lifting (see (6.1))

$$\Delta_{5,D_7} = \mathrm{Lift}(\eta^3\vartheta(z_1)\cdot\ldots\cdot\vartheta(z_7)) = \sum_{\substack{n,m,N\in\mathbb{N},\,\ell\in D_7^*\\ 8nm-(2\ell,2\ell)=N^2}}$$

$$N\sum_{d|(n,\ell,m)} d^3\left(\frac{-4}{N/d}\right)\left(\frac{-4}{2\ell/d}\right)\exp\big(2\pi i(n\tau + (\ell, \mathfrak{z}_7) + m\omega)\big) \qquad (6.10)$$

where $\ell/d \in D_7^* = \langle \mathbb{Z}^7, (\frac{1}{2}, \ldots, \frac{1}{2})\rangle$ and $\left(\frac{-4}{2\ell}\right) = \left(\frac{-4}{2l_1}\right)\ldots\left(\frac{-4}{2l_7}\right)$. One can find the Fourier expansion of other functions of the D_8-tower in terms of Fourier coefficients of $\eta(\tau)^{3(8-k)}$.

Now we give a new product construction of Δ_{5,D_7} using the strongly reflective forms

$$\Phi_{124,D_8} = \Phi_{12}|_{D_8\hookrightarrow N(3D_8)} \quad \text{and} \quad \Phi_{114,D_7} = \Phi_{12}|_{D_7\hookrightarrow N(D_7\oplus E_6\oplus A_{11})}$$

of Theorem 4.3.

Proposition 6.1.

$$\Delta_{5,D_7}^2 = \frac{\Phi_{124,D_8}|_{D_7\hookrightarrow D_8}}{\Phi_{114,D_7}}.$$

Proof. We consider the quasi pull-back

$$\Phi_{124,D_8}|_{D_7\hookrightarrow D_8} \in S_{124}(\widetilde{\mathrm{O}}^+(2U\oplus D_7), \det)$$

where $D_7 = \langle e_8\rangle_{D_8}^{\perp} \hookrightarrow D_8$. The arguments of the proof of Theorem 4.2 give

$$\operatorname{div}\Phi_{124,D_8}|_{D_7\hookrightarrow D_8} = \sum_{\substack{\pm u\in 2U\oplus D_7\\ u^2=2}} \mathcal{D}_u + \sum_{\substack{\pm v\in 2U\oplus D_7\\ v^2=4,\, v/2\in D_7^*}} 2\mathcal{D}_v.$$

(For a general result of this type see [G6].) Then

$$\operatorname{div}\frac{\Phi_{124,D_8}|_{D_7\hookrightarrow D_8}}{\Phi_{114,D_7}} = \sum_{\substack{\pm v\in 2U\oplus D_7\\ v^2=4,\, v/2\in D_7^*}} 2\mathcal{D}_v.$$

For the reflective modular forms of Theorem 4.3, we found the Jacobi type Borcherds products in Sect. 5. We get the following weak Jacobi form of weight 0 with integral Fourier coefficients

$$\begin{aligned}2\phi_{0,D_7}(\tau,\mathfrak{z}_7) &= \Delta(\tau)^{-1}\big(\vartheta_{N(3D_8)}(\tau,\mathfrak{z})|_{\mathfrak{z}\in D_7\hookrightarrow D_8} - \vartheta_{N(D_7\oplus E_6\oplus A_{11})}(\tau,\mathfrak{z})|_{\mathfrak{z}\in D_7}\big)\\ &= 2(r_1+r_1^{-1}+\cdots+r_7+r_7^{-1})+20+q(\dots)\in J_{0,D_7}^{weak}\end{aligned}$$

where $r_i=\exp(2\pi i z_i)$. The formula for the divisor of the quotient shows that the last expansion contains all Fourier coefficients with negative hyperbolic norms of their indices. According to (5.7), $B_{\phi_{0,D_7}}$ is strongly 4-reflective of weight 5. Using the Köcher principle, we obtain

$$\Delta_{5,D_7} = \operatorname{Lift}(\eta^3\vartheta(z_1)\cdot\ldots\cdot\vartheta(z_7)) = B_{\phi_{0,D_7}} = \sqrt{\frac{\Phi_{124,D_8}|_{D_7\hookrightarrow D_8}}{\Phi_{114,D_7}}}.$$

We can find similar expressions for all functions Δ_{12-k,D_k}. □

Remark 6.2. The modular form $\Delta_{4+\deg V, D_{8-\deg V}}$ is equal to the automorphic dicriminant of the moduli spaces of the Kähler moduli of a Del Pezzo surfaces V of degree $1\le \deg V\le 6$ (see [Y], [G3]). The explicit formula of type (6.10) gives us the generating function of the imaginary simple roots of the corresponding Lorentizian Kac-Moody algebra defined by this automorphic dicriminant.

Remark 6.3. Some other reflective modular forms of singular weight similar to Borcherds forms Φ_{12} and Φ_4^{BE} above were found by N. Scheithauer (see [Sch]). The reflective forms of singular weight in his class are modular with respect to congruence subgroups.

6.3 The D_8-tower of Jacobi liftings and $U(2) \oplus D_4$

In this subsection, we construct three new reflective modular forms. In Proposition 4.1, we proved that the 2-root system of $U \oplus D_4$ has two different automorphic corrections. The second example of this type is given in the next theorem.

Theorem 6.2. *The hyperbolic 2-root system of $U(2) \oplus D_4$ has two different automorphic corrections. One of them has (nearly) a Jacobi lifting construction.*

Proof. We construct two 2-reflective modular forms of different weights with respect to the stable orthogonal groups of the lattices

$$U \oplus U(2) \oplus D_4 \quad \text{and} \quad U(2) \oplus U(2) \oplus D_4.$$

The first automorphic correction was given in Proposition 4.1:

$$\Phi_{40,\,U(2)\oplus D_4} = \Phi_{28,N_8}|_{U\oplus U(2)\oplus D_4} \in S_{40}(\widetilde{\mathrm{O}}^+(U \oplus U(2) \oplus D_4), \det).$$

The second automorphic correction is related to the D_4-modular form

$$\Delta_{8,D_4} = \mathrm{Lift}(\eta^{12}(\tau)\vartheta(z_1)\vartheta(z_2)\vartheta(z_3)\vartheta(z_4)) \in S_8(\mathrm{O}^+(2U \oplus D_4), \chi_2)$$

which is a 4-reflective modular form in the reflective D_8-tower of Theorem 6.1. It vanishes along the 4-vectors of the orbit $\widetilde{\mathrm{O}}^+(2U \oplus D_4)(2e_1)$ (see (4.9)). We proved in Lemma 6.1 that the lattice $2U \oplus D_4$ contains three $\widetilde{\mathrm{O}}^+(2U \oplus D_4)$-orbits of 4-vectors. The trick of Lemma 6.1 shows that

$$\mathrm{O}(2U(2) \oplus D_4) = \mathrm{O}(2U \oplus D_4^*(2)) = \mathrm{O}(2U \oplus D_4)$$

since $D_4^*(2) \cong D_4$. The 2-vectors of $2U(2) \oplus D_4$ correspond to all 4-vectors of $2U \oplus D_4$. We get them from the first orbit $\widetilde{\mathrm{O}}^+(2U \oplus D_4)(2e_1)$ using the reflections

$$\sigma_1 = \sigma_{(-e_1-e_2-e_3+e_4)/2}, \quad \sigma_2 = \sigma_{(e_1+e_2+e_3-e_4)/2} \in \mathrm{O}^+(2U \oplus D_4)$$

with respect to 1-vectors in D_4^*. We have

$$\sigma_1(e_4) = (e_1 + e_2 + e_3 - e_4)/2, \quad \sigma_2(e_4) = (e_1 + e_2 + e_3 + e_4)/2.$$

Therefore, the product of three Jacobi liftings is a strongly reflective modular form with the complete 4-divisor

$$F_{24,U(2)\oplus D_4} = \Delta_{8,D_4} \cdot (\Delta_{8,D_4}|\sigma_1) \cdot (\Delta_{8,D_4}|\sigma_2) \in S_{24}(\mathrm{O}^+(2U \oplus D_4), \chi_2). \quad (6.11)$$

We note that $\Delta_{8,D_4}|\sigma_1 = \mathrm{Lift}(\varphi^{(1)}_{8,D_4})$ and $\Delta_{8,D_4}|\sigma_2 = \mathrm{Lift}(\varphi^{(2)}_{8,D_4})$ are 4-reflective where

$$\varphi^{(1)}_{8,D_4} = \eta^{12}(\tau)\vartheta(\tfrac{-z_1+z_2+z_3+z_4}{2})\vartheta(\tfrac{z_1-z_2+z_3+z_4}{2})\vartheta(\tfrac{z_1+z_2-z_3+z_4}{2})\vartheta(\tfrac{z_1+z_2+z_3-z_4}{2}),$$

$$\varphi^{(2)}_{8,D_4} = \eta^{12}(\tau)\vartheta(\tfrac{z_1+z_2+z_3+z_4}{2})\vartheta(\tfrac{z_1+z_2-z_3-z_4}{2})\vartheta(\tfrac{z_1-z_2-z_3+z_4}{2})\vartheta(\tfrac{z_1-z_2+z_3-z_4}{2})$$

(see [CG2, Example 1.8]). The Jacobi type Borcherds product for $F_{24,U(2)\oplus D_4}$ can be easily constructed from the corresponding product for Δ_{8,D_4}. $\square$

Remark 6.4. The products of reflective forms of D_m-type from Theorem 4.3 and the functions constructed in the subsection 6.2

$$\Phi_{k_m,D_m} \cdot \Delta_{12-m,D_m} \qquad (2 \le m \le 8)$$

are strongly reflective modular forms with the complete reflective divisor determined by all reflections in $2U \oplus D_m$ ($m \ne 4$). These modular forms determine Lorentzian Kac–Moody algebras with the maximal Weyl groups generated by all 2- and 4-reflections of the hyperbolic lattices $2U \oplus D_m$ ($m \ne 4$).

For $m = 4$ we get more complicated formula for the root system for the strongly reflective modular form with the complete reflective divisor "of type" F_4

$$\Phi_{72,D_4} \cdot \Delta_{8,D_4} \cdot (\Delta_{8,D_4}|\sigma_1) \cdot (\Delta_{8,D_4}|\sigma_2).$$

6.4 The singular modular form for $2U \oplus 4A_1$ and $U(4) \oplus D_4$.

In this subsection, we construct four reflective modular forms, in particular, the automorphic correction of the 2-root system of $U(4) \oplus D_4$.

Theorem 6.3. *There is a strongly 2-reflective modular form*

$$F_6 \in M_6(\mathrm{O}^+(2U(4) \oplus D_4), \chi_2)$$

with complete 2-divisor.

We construct this reflective form using the singular modular form for $2U \oplus 4A_1$ (see [G3, §5]). As in Lemma 6.1,

$$\mathrm{O}(2U(4) \oplus D_4) = \mathrm{O}(2U \oplus D_4^*(4)) = \mathrm{O}(2U \oplus D_4(2))$$

since $D_4^*(2) = D_4$.

Lemma 6.2. *The 2-vectors of $2U(4) \oplus D_4$ correspond to $\frac{1}{2}$-vectors of the dual lattice $2U \oplus D_4(2)^*$.*

Proof. Any 2-vector $v \in M = 2U(4) \oplus D_4 \subset M^*$ is primitive in M^* since D_4^* is odd integral. After renormalisation by 4, we have $\frac{v}{4} \in (M^*(4))^* \cong 2U \oplus D_4(2)^*$, $(\frac{v}{4})^2 = \frac{1}{2}$. Any $\frac{1}{2}$-vector in $2U \oplus D_4(2)^*$ is primitive in this lattice since the minimal possible norm in $D_4(2)^*$ is equal to $\frac{1}{2}$. $\square$

By definition, $D_4 \subset \mathbb{Z}^4$. Therefore, $2U \oplus D_4(2) \subset 2U \oplus 4A_1$ of index 2 and

$$\widetilde{\mathrm{O}}^+(2U \oplus D_4(2)) \subset \widetilde{\mathrm{O}}^+(2U \oplus 4A_1).$$

Lemma 6.3. *We put $D_4(2) \subset 4A_1 = \oplus_{i=1}^4 \mathbb{Z}a_i$. There are twenty four $\widetilde{\mathrm{O}}^+(2U \oplus D_4(2))$-orbits of $\frac{1}{2}$-vectors in the dual lattice $2U \oplus D_4(2)^*$ and four $\widetilde{\mathrm{O}}^+(2U \oplus 4A_1)$-orbits of $\frac{1}{2}$-vectors in $2U \oplus 4A_1$.*

Proof. We proved above that all $\frac{1}{2}$-vectors are primitive in the corresponding dual lattices. According to the Eichler criterion (see [G2], [GHS4]), the orbit of a $\frac{1}{2}$-vector with respect to the stable orthogonal group is defined by its image in the discriminant group. It is clear that there are four such orbits $\frac{a_i}{2}$ $(1 \le i \le 4)$ in $2U \oplus 4A_1$.

We have

$$D_4(2)^*/D_4(2) \cong \frac{1}{2}D_4^*/D_4 \cong D_4^*/2D_4 \cong (D_4^*/D_4)/(D_4/2D_4).$$

Analysing the last quotient, we see that the discriminant groups $D_4(2)^*/D_4(2)$ contains 24 (respectively 4, 12, 24) elements of norm $\frac{1}{2}$ mod 2 (respectively of norms $0, 1, \frac{3}{2}$ mod 2). In the case of norm $\frac{1}{2}$, their representatives are $\pm a_i/2$, $(\pm a_1 \pm a_2 \pm a_3 \pm a_4)/4$. $\square$

The product of Jacobi theta-series $\vartheta(z_1)\cdots\vartheta(z_n)$ can be considered as a Jacobi form for D_n (see Example 3.1) or a Jacobi form of half-integral index for nA_1. For example,

$$\psi_{2,4A_1}(\tau, \mathfrak{z}_4) = \vartheta(\tau, z_1)\vartheta(\tau, z_2)\vartheta(\tau, z_3)\vartheta(\tau, z_4) \in J_{2,D_4}(v_\eta^{12})$$

is a D_4-Jacobi form with character $v_\eta^{12} : SL_2(\mathbb{Z}) \to \{\pm 1\}$. The same product

$$\psi_{2,4A_1}(\tau, \mathfrak{z}_4) = \vartheta(\tau, z_1)\vartheta(\tau, z_2)\vartheta(\tau, z_3)\vartheta(\tau, z_4) \in J_{2,4A_1;\frac{1}{2}}(v_\eta^{12} \times v_H)$$

is a Jacobi form of index $\frac{1}{2}$ with respect to the lattice $4A_1$ where v_H is the unique non-trivial binary character of the Heisenberg group $H(4A_1)$ (see [CG2]). For such Jacobi forms, the corresponding lifting contains only Hecke operators of indices congruent to a constant modulo the conductor of the character. According to the lifting constructions (see [GN4, Theorem 1.12] and [CG2, Theorem 2.2]), the following function is defined:

$$\Delta_{2,4A_1} = \mathrm{Lift}(\psi_{2,\,4A_1}) \in M_2(\mathrm{O}^+(2U \oplus 4A_1), \chi_2). \tag{6.12}$$

All Fourier coefficients with primitive indices of this lifting are equal to ± 1 or 0

$$\Delta_{2,4A_1} = \sum_{\ell=(l_1,\dots,l_4)\in\mathbb{Z}^4,\ l_i\equiv 1 \bmod 2}$$

$$\sum_{\substack{n,\,m\in\mathbb{N}\\ n\equiv m\equiv 1 \bmod 2\\ 4nm-l_1^2-l_2^2-l_3^2-l_4^2=0}} \sigma_1((n,\ell,m))\left(\frac{-4}{\ell}\right)\exp(\pi i(n\tau + l_1z_1 + \cdots + l_4z_4 + m\omega))$$

where (n, ℓ, m) is the greatest common divisor and $\left(\frac{-4}{\ell}\right) = \left(\frac{-4}{l_1l_2l_3l_4}\right)$ is the Kronecker symbol. It was proved in [G3, Theorem 5.1] that

$$\mathrm{div}\,\Delta_{2,\,4A_1} = \sum_{\pm v\in 2U\oplus 4A_1^*,\ v^2=\frac{1}{2}} \mathcal{D}_v(2U \oplus 4A_1).$$

In other words, $\Delta_{2,4A_1}$ is a strongly 2-reflective modular form which vanishes along the divisors defined by one of two $\mathrm{O}^+(2U \oplus 4A_1)$-orbits of 2-vectors in $2U \oplus 4A_1$ ($(2v)^2 = 2$).

The Borcherds product of this modular form is defined by the Jacobi form (see (6.5) and (6.6))

$$\varphi_{0,\,4A_1}(\tau, \mathfrak{z}_4) = 3^{-1}\frac{\psi_{2,\,4A_1}|\,T_-(3)}{\psi_{2,\,4A_1}} \in J^{(weak)}_{0,4A_1}.$$

We can consider $\Delta_{2,\,4A_1}$ as a modular form with respect to $\widetilde{\mathrm{O}}^+(2U \oplus D_4(-2))$. We note that $\psi_{2,4A_1}(\tau, -\mathfrak{z}_4) = \psi_{2,4A_1}(\tau, \mathfrak{z}_4)$, and the same property has its lifting. Therefore

$$\Delta_{2,\,4A_1} \in M_2(\widetilde{\mathrm{O}}^+(2U \oplus D_4(2)), \chi_2).$$

More exactly, $\Delta_{2,\,4A_1}$ is anti-invariant under the action of 4 reflections with respect to $\pm a_i/2$ and invariant with respect to any permutation of a_i. Therefore, $\Delta_{2,\,4A_1}$ vanishes along the devisors defined by only 8 of 24 vectors of square $\frac{1}{2}$ from Lemma 6.3. As in (4.11), we put

$$F_6 = \Delta_{2,4A_1} \cdot (\Delta_{2,4A_1}|\sigma_1) \cdot (\Delta_{2,4A_1}|\sigma_2) \in M_6(\mathrm{O}^+(2U \oplus D_4(2)), \chi_2) \qquad (6.13)$$

where $\sigma_1 = \sigma_{(a_1+a_2+a_3-a_4)/4}, \sigma_2 = \sigma_{(a_1+a_2+a_3+a_4)/4} \in \mathrm{O}^+(2U \oplus D_4(2))$. We note that $\Delta_{2,4A_1}|\sigma_1$ and $\Delta_{2,4A_1}|\sigma_2$ are reflective. (Compare with the functions from the previous subsection.) The product of three Jacobi liftings is a strongly reflective modular form with the complete $\frac{1}{2}$-divisor. Moreover, F_6 is anti-invariant with respect to twenty four $\frac{1}{2}$-reflections in $D_4(2)$ and invariant with respect to all permutations of a_i. Therefore, F_6 is modular with respect to the full orthogonal group $\mathrm{O}^+(2U \oplus D_4(2))$ since $\mathrm{O}(D_4(2)) = \mathrm{O}(D_4)$. Theorem 6.3 is proved.

6.5 The reflective tower of Jacobi liftings for $U(4) \oplus kA_1$, $k \le 3$.

In this subsection, we construct seven reflective modular forms. This $4A_1$-tower is based on the reflective modular form of singular weight $\Delta_{2,4A_1}$ (see (6.12)) and starts with the Igusa modular form Δ_5 considered as Borcherds product in [GN1].

Theorem 6.4. *The automorphic correction of the* 2*-root system* $U(4) \oplus kA_1$ ($1 \le k \le 3$) *is given by*

$$\Delta_{6-k,kA_1} = \mathrm{Lift}(\eta^{3k}\vartheta(z_1)\vartheta(z_2)\vartheta(z_3)) \in S_{6-k}(\mathrm{O}(2U \oplus kA_1)). \qquad (6.14)$$

Similar to Lemma 6.1 and Lemma 6.2, we get

Lemma 6.4. *Assume* $1 \le k \le 4$. *Then*

$$\mathrm{O}(2U(4) \oplus kA_1) = \mathrm{O}(2U \oplus kA_1).$$

The 2*-reflective vectors of* $2U(4) \oplus kA_1$ *correspond to* $\frac{1}{2}$*-reflective vectors of* $2U \oplus kA_1^*$. *These vectors belong to* k *different* $\widetilde{\mathrm{O}}^+(2U \oplus kA_1)$*-orbits or one* $\mathrm{O}^+(2U \oplus kA_1)$*-orbit.*

We note that the basic reflective modular form of this tower $\Delta_{2,4A_1} = \mathrm{Lift}(\vartheta(z_1)\vartheta(z_2)\vartheta(z_3)\vartheta(z_4))$ defines *an automorphic correction of the hyperbolic* 2*-root system* $U(4)\oplus 4A_1$ *of parabolic type with a lattice Weyl vector of norm* 0. We take three consecutive quasi pull-backs of this modular form for $z_4=0$, $z_3=0$ and $z_2=0$ and get three strongly reflective cusp forms for $k=1,\,2,\,3$

$$\mathrm{Lift}(\eta^{3k}\vartheta(z_1)\cdots\vartheta(z_{4-k}))\in S_{2+k}(\widetilde{\mathrm{O}}^+(2U\oplus(4-k)A_1))$$

with the complete $\frac{1}{2}$-divisor described in Lemma 6.4 (see [G3, §5]). The Fourier expansion of the quasi pull-backs can be written in terms of the Fourier coefficients of $\eta(\tau)^{3k}$. Here we give the formula for the first cusp modular form of this tower which contains only elementary functions:

$$\Delta_{3,3A_1}(\tau,\mathfrak{z}_3,\omega)=\mathrm{Lift}(\eta^3\vartheta(z_1)\vartheta(z_2)\vartheta(z_3))=$$

$$\sum_{\substack{n\equiv m\equiv 1 \bmod 2\\ l_i\equiv 1 \bmod 2,\\ 4nm-l_1^2-l_2^2-l_3^2=N^2}} N\left(\frac{-4}{Nl_1l_2l_3}\right)\sigma_1((n,\ell,m))\exp(\pi i(n\tau+l_1z_1+l_2z_2+l_3z_3+m\omega)).$$

As in the case the reflective modular D_8-tower (see (6.5) and Proposition 6.1), we give different constructions of the Borcherds product for $\Delta_{3,3A_1}$. First (see (6.6) and [G3, §5]), we have two formulae for the weak Jacobi form

$$\varphi_{0,3A_1}(\tau,\mathfrak{z}_3)=3^{-1}\frac{\psi_{5,\,3A_1}\,|\,T_-(3)}{\psi_{5,\,3A_1}}=\varphi_{0,4A_1}(\tau,\mathfrak{z}_4)|_{z_4=0}\in J^{weak}_{0,3A_1}$$

such that $\Delta_{3,3A_1}=B_{\phi_{0,3A_1}}$, see (5.10). Similar formulae are valid for $\phi_{0,2A_1}$ and ϕ_{0,A_1}.

Second, we can construct the modular forms of the reflective $4A_1$-tower using some 2-reflective modular forms of Theorem 4.3.

Proposition 6.2.

$$\Delta^2_{3,3A_1}=\frac{\Phi_{39,3A_2}|_{3A_1\hookrightarrow 3A_2}}{\Phi_{33,3A_1}},\qquad \Delta^2_{4,2A_1}=\frac{\Phi_{42,2A_2}|_{2A_1\hookrightarrow 2A_2}}{\Phi_{34,2A_1}},$$

$$\Delta^2_{5,A_1}=\frac{\Phi_{45,A_2}|_{A_1\hookrightarrow A_2}}{\Phi_{35,A_1}}.$$

Proof. We consider the case $k = 3$. We embed $A_1 = \langle u \rangle$ in $A_2\langle u, v\rangle$ where $u^2 = v^2 = 2$. Then $A_1 \perp \langle 6 \rangle \subset A_2$, and two pairs $\pm v$ and $\pm(u+v)$ of A_2-roots have "small" orthogonal projections of norm 1 on u. We take this embedding for 3 copies $3A_1 \hookrightarrow 3A_2$. We note that the lattices $3A_2$ and $3A_1$ satisfy the (Norm_2) condition of Theorem 4.2. Therefore, the arguments from the proof of Theorem 4.2 show that the pull-back $\Phi_{39,3A_2}|_{3A_1 \hookrightarrow 3A_2}$ has weight 39 and vanishes along all 2-divisors and additionally along 1-divisors corresponding to 1-vectors of $2U \oplus 3A_1^*$. The 1-divisors have multiplicity 2. Dividing this pull-back by the strongly reflective form $\Phi_{33,3A_1}$, we get $\Delta^2_{3,3A_1}$ according to the Köcher principle. ☐

Remark 6.5. 1) We note that Δ^2_{5,A_1} is equal to the Igusa modular form $\Psi_{10} \in S_{10}(\mathrm{Sp}_2(\mathbb{Z}))$ which is the first Siegel cusp form of weight 10. The Borchers product formula for Ψ_{10} was constructed in [GN1] (see also [GN6] for other constructions of the Igusa modular form). Proposition 6.2 gives a new model of this very important function. The function Δ_{5,A_1} is the automorphic correction of the hyperbolic root system with Cartan matrix

$$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

(see [GN1]).

2) The quotient

$$\frac{\Phi_{32+k,(4-k)A_1}}{\Delta_{2+k,\,(4-k)A_1}}$$

for $0 \le k \le 3$ is a holomorphic strongly reflective modular form. It defines a Lorentzian Kac–Moody algebra of a hyperbolic root system of $U \oplus (4-k)A_1$ whose Weyl group is smaller than the full Weyl group generated by all 2-reflections in $U \oplus (4-k)A_1$. The Cartan matrix of such Lorentzian Kac–Moody algebra for $U \oplus A_1$ was found in [GN6].

6.6 The reflective tower $U(3) \oplus kA_2$, $1 \le k \le 3$.

In this subsection, we define six reflective modular forms using a modular form of singular weight for the lattice $2U \oplus 3A_2$ proposed in [G3, §4]. This modular form also gives interesting series of canonical differential forms on Siegel modular three-folds constructed with theta-blocks (see [GPY]).

Lemma 6.5. *The function*

$$\sigma_{A_2}(\tau,\mathfrak{z})=\sigma_{A_2}(\tau,z_1,z_2)=\frac{\vartheta(\tau,z_1)\vartheta(\tau,z_2)\vartheta(\tau,z_1+z_2)}{\eta(\tau)}\in J_{1,A_2}(v_\eta^8) \quad (6.15)$$

is a holomorphic Jacobi form of singular weight which is anti-invariant with respect to the 6*-reflections from* $\mathrm{O}(A_2)$*.*

Proof. By construction, σ_{A_2} is a weak Jacobi form. See [CG2, Corollary 3.4] for a modular proof that this is a holomorphic Jacobi form.

One can see the same using the theory of affine Lie algebras. σ_{A_2} is a dual variant of the denominator function of the affine Lie algebra $\hat{\mathfrak{g}}(A_2)$ (see the remark after Theorem 5.1) which is a holomorphic Jacobi form. Let v_1 and v_2 be the simple roots of A_2 ($v_1^2=v_2^2=2$ and $(v_1,v_2)=-1$) and $\lambda_1=(2v_1+v_2)/3$ and $\lambda_2=(v_1+2v_2)/3$ is the corresponding dual basis of A_2^*, that is $(v_i,\lambda_j)=\delta_{i,j}$). Then $\pm 3\lambda_1$, $\pm 3\lambda_2$ and $\pm 3(\lambda_1-\lambda_2)$ are reflective vectors of square 6 in A_2, and

$$\sigma_{A_2}(\tau,\mathfrak{z})=-\frac{\vartheta(\tau,(\mathfrak{z},\lambda_1))\vartheta(\tau,(\mathfrak{z},\lambda_2))\vartheta(\tau,(\mathfrak{z},\lambda_1-\lambda_2))}{\eta(\tau)} \qquad (\mathfrak{z}\in A_2\otimes\mathbb{C}).$$

We see that this product is anti-invariant with respect to the 6-reflections. □

The direct product of three copies of $\sigma_{A_2}(\tau,\mathfrak{z})$ is a Jacobi form of singular weight

$$\sigma_{3A_2}(\tau,\mathfrak{z}_1+\mathfrak{z}_2+\mathfrak{z}_3)=\prod_{i=1}^{3}\sigma_{A_2}(\tau,\mathfrak{z}_i)\in J_{3,3A_2}$$

with trivial character. It was proved in [G3, Theorem 4.2] that its lifting

$$\Delta_{3,3A_2}=\mathrm{Lift}(\sigma_{3A_2})\in M_3(\widetilde{\mathrm{O}}^+(2U\oplus 3A_2))$$

is *a strongly reflective modular form with the complete* 6*-divisor*. We note that all Fourier coefficients of σ_{3A_2} and $\Delta_{3,3A_2}$ are integral.

The quotient group $\mathrm{O}^+(2U\oplus 3A_2)/\widetilde{\mathrm{O}}^+(2U\oplus 3A_2)$ is isomorphic to the orthogonal group $\mathrm{O}(q_{3A_2})$ of the discriminant form of $(3A_2)^*/(3A_2)\cong(\mathbb{Z}/3\mathbb{Z})^3$. We have $\mathrm{O}(q_{3A_2})\cong S_3\times C_2^3$ where S_3 is the group of permutations of three copies of A_2, and the cyclic group C_2 of order 2 is generated by a 6-reflection in A_2. The Jacobi form σ_{3A_2} is invariant with respect to the permutations of the copies of A_2 and anti-invariant with respect to 6-reflections due to

Lemma 6.5. The same properties are valid for the lifting of σ_{3A_2}. Therefore, we prove that

$$\Delta_{3,3A_2} = \mathrm{Lift}(\sigma_{3A_2}) \in M_3(\mathrm{O}^+(2U \oplus 3A_2), \chi_2)$$

where χ_2 is a binary character of the full orthogonal group which is trivial on the stable orthogonal group $\widetilde{\mathrm{O}}^+(2U \oplus 3A_2)$. For two quasi pull-backs of $\mathrm{Lift}(\sigma_{3A_2})$ on the homogeneous domains of $2U \oplus 2A_2$ and $2U \oplus A_2$ we get

$$\Delta_{6,2A_2} = \mathrm{Lift}(\eta^8 \cdot \sigma_{2A_2}) \in S_6(\mathrm{O}^+(2U \oplus 2A_2), \chi_2),$$

$$\Delta_{9,A_2} = \mathrm{Lift}(\eta^{16} \cdot \sigma_{A_2}) \in S_9(\mathrm{O}^+(2U \oplus A_2), \chi_2).$$

The quasi pull-backs are also strongly reflective modular forms with the complete 6-divisor according to the arguments of Theorem 4.2.

The Borcherds product of $\Delta_{3,3A_1}$ is defined by the weak Jacobi form of weight 0 with integral Fourier coefficients (see (6.6))

$$\varphi_{0,\,3A_2}(\tau, \mathfrak{z}) = 2^{-1}\frac{\sigma_{3,\,3A_2}|_3T_-(2)}{\sigma_{3,\,3A_2}}.$$

Then

$$\varphi_{0,\,3A_2}(\tau, \mathfrak{z}) = 6 + \sum_{i=1,3,5}(r_i + r_i^{-1} + r_{i+1} + r_{i+1}^{-1} + r_i r_{i+1}^{-1} + r_i^{-1} r_{i+1}) + q(\dots)$$

where $r_i = \exp(2\pi i(\mathfrak{z}, \lambda_i))$, $\mathfrak{z} \in (3A_2) \otimes \mathbb{C}$ and λ_i $(i = 1, \dots, 6)$ give the dual bases to simple roots of the corresponding copies of A_2. The Borcherds products of the reflective modular forms $\Delta_{6,2A_1}$ and Δ_{9,A_1} are defined by the corresponding pull-backs of $\varphi_{0,3A_2}$.

Lemma 6.6. *We have that* $\mathrm{O}^+(2U(3) \oplus kA_2) = \mathrm{O}^+(2U \oplus kA_2)$*, and the* 2*-reflections of* $2U(3) \oplus kA_2$ *(*$k = 1, 2, 3$*) correspond to the* 6*-reflections of the lattice* $2U \oplus kA_2^*(-3) \cong 2U \oplus kA_2$*.*

Proof. The isomorphism of the lemma follows from (6.8) and the fact that $A_2^*(3) \cong A_2$. The 2-reflections of $2U(3) \oplus kA_2$ are the 6-reflections in $(2U(3) \oplus kA_2)^*(3)$. We recall that $[\mathrm{O}(A_2) : W_2(A_2)] = 2$ where $\mathrm{O}(A_2)$ is the integral orthogonal group of the lattice A_2. $\mathrm{O}(A_2)$ contains reflections with respect to the 2- and 6-vectors. All these roots form the root system G_2, and $\mathrm{O}(A_2) = W(G_2)$. See [GHS3] for the root systems G_2 and F_4 in the context of automorphic forms. $\square$

The results, proved above, give

Theorem 6.5. *For $k = 1$, 2 or 3 the modular form*

$$\Delta_{12-3k,\,kA_2} \in M_{12-3k}(\mathrm{O}^+(2U(3) \oplus kA_2), \chi_2)$$

is strongly 2-reflective modular form with the complete 2-divisor where χ_2 is a binary character of the orthogonal group. For $k = 1$ and 2 they are cusp forms.

Remark 6.6. 1) The Fourier expansions of reflective forms $\Delta_{12-3k,\,kA_2}$ ($k = 1, 2, 3$) are given by formula (6.1). All their Fourier coefficients are integral. These modular forms determine three Lorentzian Kac–Moody algebras related to the hyperbolic lattices $U(3) \oplus kA_2$. The lattice Weyl vector of the hyperbolic 2-root system of $U(3) \oplus 3A_2$ has norm 0 since the modular form $\Delta_{3,\,3A_2}$ is of singular weight. *The corresponding hyperbolic root system of signature* $(7, 1)$ *is of the parabolic type.*

2) The products of reflective forms $\Phi_{45,A_2} \cdot \Delta_{9,\,A_2}$, $\Phi_{42,2A_2} \cdot \Delta_{6,\,2A_2}$ and $\Phi_{39,3A_2} \cdot \Delta_{3,\,3A_2}$ are strongly reflective modular forms with the complete reflective divisor determined by all reflections in the corresponding lattices. These modular forms determine three Lorentzian Kac–Moody algebras with the maximal Weyl groups generated by all 2- and 6-reflections of the hyperbolic lattices (compare with the root system G_2). For these more general cases of reflections in roots with arbitrary squares, one should follow general definitions from [GN5], [GN6], [GN8], [N12] of data (I)—(V) and Lorentzian Kac–Moody algebras instead of given in Sec. 2.

Remark 6.7. K3 surfaces X with transcendental lattices $T_X = T(-1)$ for lattices T of signature $(n, 2)$ with strongly 2-reflective modular forms Φ constructed above, have discriminants which are divisors of Φ, and these divisors are rational quadratic divisors orthogonal to all (-2)-roots of $T_X = T(-1)$ and of multiplicity one.

These K3 surfaces can be considered as mirror symmetric (by the arithmetic mirror symmetry defined by the corresponding Lorentzian Kac–Moody algebras) to K3 surfaces with Picard lattices S_X from Remark 3.3. See [GN3], [GN7] and [GN8] about the arithmetic mirror symmetry.

Remark 6.8. We expect some finiteness results about the set of reflective modular forms. The main reason is the Köcher principle. See [N9] about first observations. Recently, Sh. Ma (see [Ma]) obtained some finiteness results about 2-reflective modular forms.

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